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Manifolds with Non-Smooth Boundaries and Asymptotics of the Graph Laplacian

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Abstract

This work studies the asymptotic behavior of discrete graph Laplacians constructed from random samples on Riemannian manifolds whose boundaries may exhibit geometric irregularities. We introduce the class of *manifolds with kinks* (MFK)—a broad generalization of smooth manifolds with boundaries and corners—and establish convergence results of the graph Laplacian at interior, border, and cusp points. The results unify earlier analyses on smooth domains (Belkin–Niyogi, Hein–Luxburg, Peoples–Harlim) and extend them to non-smooth geometries that frequently occur in data analysis. We also discuss applications to edge detection in image processing and possible extensions to curvature-dependent asymptotics.

1. From Data Clouds to Geometry

In manifold learning and geometric data analysis, one observes independent samples. Denote them by $X_1, \dots, X_n$. These are drawn from a distribution supported on a metric measure space (M, d, μ) embedded in $\mathbb{R}^D$. The goal is to recover analytic structures—in particular the Laplace operator—from data.

For a general metric measure space (M, d, μ) of Hausdorff dimension d, and a kernel $k : [0, \infty) \to [0, \infty)$ integrable and non-increasing (e.g. $k(r) = e^{-r}$), the **continuous graph Laplacian** acting on $f \in L^1(M, \mu)$ is defined by

$$L_t f(x) := \frac{1}{t^{d/2+1}} \int_M k\left(\frac{d(x,y)^2}{t}\right)\big(f(x) - f(y)\big)\, d\mu(y), \qquad t > 0.$$

The discrete empirical version based on samples $X_1, \dots, X_n$ is

$$L_{n,t} f(x) := \frac{1}{n\, t^{d/2+1}} \sum_{j=1}^{n} k\left(\frac{d(x,X_j)^2}{t}\right)\big(f(x) - f(X_j)\big).$$

Above, $d(x, y)$ denotes the Euclidean distance $\|x - y\|_{\mathbb{R}^D}$ when (M, d, μ) is isometrically embedded in $\mathbb{R}^D$, but otherwise $d(x, y)$ denotes the intrinsic distance. When (M, g) is a Riemannian manifold, $\mu = \mathrm{vol}_g$. For smooth compact M without boundary, $L_{n,t}$ converges (Belkin and Niyogi, 2006, 2008) to the Laplace–Beltrami operator Δ_g both pointwise and spectrally. However, for domains with boundaries or singularities, convergence behavior near those points was less understood.

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2. From Smooth to Non-Smooth Boundaries

Manifolds with smooth boundaries are locally modeled on $[0,\infty)\times\mathbb{R}^{d-1}$; manifolds with corners admit local models $[0,\infty)^k\times\mathbb{R}^{d-k}$. We generalize these to allow "kinks"—boundaries that are merely Lipschitz and continuously directionally differentiable (LCDD). This category, denoted by **Smooth Manifolds with Kinks (MFK)**, has local models given by the epigraph of a **L**ipschitz **C**ontinuous and **D**irectionally **D**ifferentiable (LCDD) function, and it includes smooth boundaries, corners, and cusp-like points such as those appearing in $\{(x,y) : y \le \sqrt{x}, x \ge 0\}$ near the origin.

From tangent space to tangent cone: Every point $x \in M$ admits a tangent space T_xM defined via derivations on $C^\infty(M)$. At border points, an *inward sector* $I(T_xM) \subset T_xM$ captures admissible tangent directions. This inward sector almost everywhere equals the Bouligand cone $T_x^B\overline{M} := \{v : \exists x_n \in \overline{M}, (x_n - x)/\|x_n - x\| \to v/\|v\|\}$, whose complement has zero d-dimensional Lebesgue measure.

3. Main Results

The following theorems summarize our analytic and probabilistic asymptotics.

Theorem 1.1 (Taylor expansion of the *continuous* graph Laplacian). *Let M be a d-dimensional Riemannian MFK endowed with a C^2 metric g. Let $x \in M$ be either an interior point, an LCDD border point, or a cusp. For a density $p \in C^2_{\ge 0}(M)$ and any $f \in C^3(M) \cap L^1(M, p\,\mathrm{vol}_g)$, one has as $t \downarrow 0$:*

$$\begin{aligned} L_t f(x) = &- c_d\, t^{-1/2}\big(p(x)\partial_{v_g(x)} f(x) + o(1)\big) \\ &- c_{d+1}\big(p(x) A_g f(x) + r_{p,f}(x)\big) + O(\sqrt{t}), \end{aligned}$$

where $v_g(x)$ denotes the generalized normal,

$$v_g(x) := \int_{S_g I_x M} \theta \, d\sigma(\theta), \qquad \partial_{v_g(x)} f(x) := d_x f(v_g(x)),$$

and

$$\begin{aligned} A_g f(x) &:= \tfrac{1}{2} \int_{S_g I_x M} d_x^2 f(\theta,\theta)\, d\sigma(\theta), \\ r_{p,f}(x) &:= \int_{S_g I_x M} d_x f(\theta)\, d_x p(\theta)\, d\sigma(\theta), \\ c_\ell &:= \frac{1}{2}\Gamma\left(\frac{d+1}{2}\right) \end{aligned}$$

Theorem 1.2 (*Discrete* graph Laplacian: convergence in probability). *Let M be a smooth d-dimensional MFK with a C^2 metric g, and let $X_1, \dots, X_n$ be i.i.d. random* *variables on M with density $p \in C^2(M)$. For $f \in C^3(M) \cap L^1(M, p\,\mathrm{vol}_g)$ assume $f(X)$ is α-subexponential, $\alpha \in (0,2]$. If $\sqrt{nt_n^{(d+1)/2}} \to \infty$, then*

$$\sqrt{t_n}\, L_{n,t_n} f(x) \xrightarrow{P} -c_d\, p(x)\, \partial_{v_g} f(x), c_d := \frac{1}{2}\Gamma\left(\frac{d+1}{2}\right).$$

Theorem 1.3 (*Discrete* graph Laplacian: almost-sure convergence). *Under the assumptions of Theorem 1.2, if additionally $\left(\sqrt{nt_n^{(d+1)/2}}\right)^{\alpha} / \ln n \to \infty$, then*

$$\sqrt{t_n}\, L_{n,t_n} f(x) \xrightarrow{\text{a.s.}} -c_d\, p(x)\, \partial_{v_g} f(x), c_d := \frac{1}{2}\Gamma\left(\frac{d+1}{2}\right).$$

For $\alpha \leq 1$, the factor $\sqrt{n}$ may be replaced by n.

These theorems reveal that the rescaled graph Laplacian $\sqrt{t}\, L_{n,t}$ approximates a first-order boundary operator governed by the inward normal $v_g(x)$, generalizing the classical Neumann condition to non-smooth boundaries.

Proof The complete proof is in our paper (Pal and Tewodrose, 2025), see Theorem 1.3 and its proof. We arrive at the proof at the end of Section 7. But the proof of the asymptotics of the discrete graph Laplacian also needs us to first go through the proof of Theorem 1.2 in that paper, which is the corresponding continuous graph Laplacian. ■

4. Numerical Experiments and Image-Processing Application

4.1 Numerical verification on geometric domains

We tested the asymptotic behavior on planar domains including triangles (smooth edges and corners) and the cusp region $\{(x,y) : y \leq \sqrt{x}, x \geq 0\}$ as well as on 3D domains e.g. 3D balls. For $n = 10^8$ points and $t_n \in [0.001, 0.01]$ (so that the hypotheses of Theorems 1.2, 1.3 are satisfied), we computed $\sqrt{t_n}\, L_{n,t_n} f(0)$ for $f(x,y) = x + y$, observing convergence to a constant matching the theoretical coefficient $-c_d p(x) \partial_{v_g} f(x)$. Below is the demonstration for the scaled graph Laplacian on a 3D ball. We observe that $\sqrt{t_n} L_{n,t_n} f(0)$ converges to a constant.

4.2 Edge detection via graph Laplacian

We model an RGB image as a point cloud in $\mathbb{R}^3$ (grayscale) or $\mathbb{R}^5$ (RGB), where each pixel i has coordinates (x_i, y_i, I_i) or $(x_i, y_i, R_i, G_i, B_i)$. We build a weighted graph with Gaussian kernel

$$W_{ij} = \exp\left(-\frac{\|X_i - X_j\|^2}{t}\right), \qquad L = D - W.$$

We pick t_n satisfying the hypotheses of Theorem 1.2, 1.3. For linear probe functions f (coordinate components), the scores $S_i = \sqrt{t_n}\, L_{n,t_n} f(X_i)$ satisfy: interior points $X_i, \Rightarrow S_i \approx 0$, boundary/edge points $X_i, \Rightarrow |S_i|$ noticeably larger.

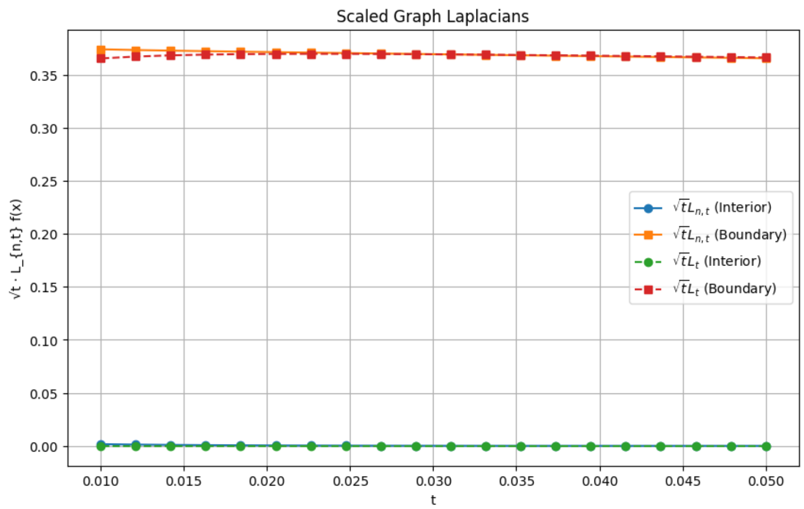

Figure 1: Simulation of $\sqrt{t}\,L_{n,t}f(x)$ for a 3D ball: Observed $\sqrt{t}$-scaling agrees with Theorem 1.1.

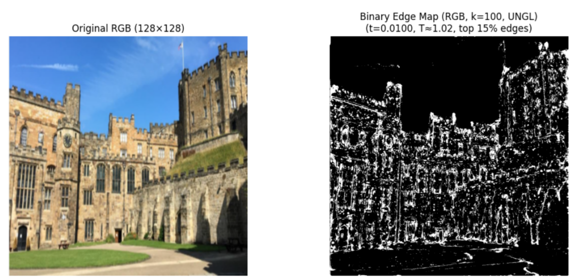

Figure 2: Edge detection using $\sqrt{t_n}L_{n,t_n}f(x)$ scores. Left: grayscale flower; Right: RGB Durham Castle. White = edges.

5. Historical Context and Outlook

Early consistency results by Belkin and Niyogi (2006, 2008) and Hein et al. (2007) established convergence of $L_{n,t}$ to the weighted Laplace-Beltrami operator on smooth compact manifolds for uniform and non-uniform densities. Hein et al. (2007) also worked on the asymptotics of the random walk and normalized graph Laplacians. Peoples and Harlim (2026) extended spectral convergence to manifolds with smooth boundaries, showing convergence to the Neumann Laplacian. Our work provides the first analysis for *non-smooth* boundaries, where the limit operator exhibits an explicit $t^{-1/2}$ boundary term.

Ongoing efforts include:

- deriving curvature-dependent corrections to the bias term, i.e. the difference between the continuous graph Laplace operator and its limit as the bandwidth $t \downarrow 0$.
- using asymptotics to estimate the "index" of a manifold with corners (number of orthogonal boundary directions),
- applying the theory to data-driven geometry reconstruction,
- and establishing convergence results for the random-walk graph Laplacian.

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