The embedded homology of hypergraphs on manifolds and configuration spaces

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Editors: Michael Bleher, Freya Jensen, Levin Maier, Diaaeldin Taha, and Anna Wienhard

Abstract

In this article, we give a short survey on the embedded homology of hypergraphs, random hypergraphs, and hypergraphs on manifolds by S. Bressan, J. Li, S. Ren, C. Wu, J. Wu, M. Zhang et al. from 2019 to 2025. We review the motivating problems as well as the meanings for the embedded homology of hypergraphs, the map algebra of hypergraphs, the relations between random hypergraphs and random simplicial complexes, and the double complexes for hypergraphs on manifolds. Besides, we give some further discussions about potential applications of hypergraphs on manifolds in the r-ball packing problems and the r-ball covering problems.

This article is mainly a short survey of the embedded homology of hypergraphs (Bressan et al., 2019; Ren et al., 2022b), the map algebra of hypergraphs and the applications in random hypergraphs (Ren et al., 2022a, 2023), and the double complexes for hyper(di)graphs on manifolds from the perspective of configuration spaces (Ren, 2025b). Besides, the article gives some further discussions prospectively and propose a related open problem. The remaining part of this article consists of four parts, where each part is motivated by a problem.

Part I. Firstly, we review the motivations and the meanings of the embedded homology of hypergraphs.

Problem 1. Scientists care about homological tools for hypergraphs which are compatible with the usual homology theory for simplicial complexes. A practical problem in topological data analysis is that if a network data is only a hypergraph but is not a simplicial complex, then how to apply the persistent homology method to analyze the data.

In 2019, Bressan et al. (2019) introduced the embedded homology of hypergraphs as an answer of Problem 1. The embedded homology is a natural generalization of the simplicial homology theory to hypergraphs and its naturality admits persistence. The key insights of the embedded homology of hypergraphs is that the largest chain complex contained in the free module spanned by the hyperedges and the smallest chain complex containing the free module spanned by the hyperedges have isomorphic homology groups, and this isomorphism is naturally induced by the canonical inclusion.

The embedded homology of hypergraphs as well as its persistence has applications in data science, higher networks and bioinformatics. For example, Liu et al. (2021b) and Liu et al. (2021a) applied the persistent barcodes of the embedded homology of hypergraphs to study the protein-ligand binding network; and Bick et al. (2023) considered the paper of Bressan et al. (2019) as an approach to study the topology of higher-order networks.

Besides its convenience and practicality for the applications in topological data analysis, the embedded homology for hypergraphs matters for the following reasons:

Reason 1: the quasi-isomorphism between the infimum chain complex and the supremum chain complex gives a natural way to define the embedded homology;

Reason 2: the canonical inclusion of the lower-associated simplicial complex into a hypergraph as well as the canonical inclusion of a hypergraph into the associated simplicial complex gives homomorphisms between the embedded homology of a hypergraph and the usual homology of the (lower-)associated simplicial complex. In particular, if a hypergraph is a simplicial complex, then the hypergraph equals to its (lower-)associated simplicial complex, which implies that the embedded homology of the hypergraph equals to the usual homology of the (lower-)associated simplicial complex;

Reason 3: The usual simplicial homology is invariant under homotopy equivalence, while the embedded homology of hypergraphs is not invariant under the homotopy equivalence of the geometric realizations. In fact, the embedded homology detects the simplicial structures, which are more precise than homotopy equivalences. For example, in (Ren, 2026, Section 2), the homotopy type of the independence complex gives obstructions for the existence of regular embeddings of graphs. However, these obstructions are not sufficiently enough. A simple case to see this is that even if the dimension of the independence complex is large and cannot have geometric realizations in a low-dimensional vector space, it could still be homotopy equivalent to a sphere of a lower dimension or be contractible. By taking sub-hypergraphs of the independence complex and considering the embedded homology, we will get more precise algebraic measurements for the simplicial structure of the independence complex. Hence the embedded homology of sub-hypergraphs of the independence complex is hopefully to detect more precise obstructions for the existence of regular embeddings.

Let V be a discrete set. A hypergraph $\mathcal{H}$ on V is a collection of certain non-empty subsets of V. Parks and Lipscomb (1991) defined the associated simplicial complex $\mathcal{K}_{\mathcal{H}}$ to be the smallest simplicial complex on V such that each hyperedge of $\mathcal{H}$ is a simplex of $\mathcal{K}_{\mathcal{H}}$ and studied the homology of $\mathcal{K}_{\mathcal{H}}$. Bressan et al. (2019) defined the embedded homology $H_*(\mathcal{H})$ and proved the Mayer-Vietoris sequence for the pairs $\mathcal{H}, \mathcal{H}'$ of hypergraphs such that $\sigma \cap \sigma' \in \mathcal{H} \cap \mathcal{H}'$ or equals to the emptyset for any $\sigma \in \mathcal{H}$ and any $\sigma' \in \mathcal{H}'$. Ren et al. (2022b) studied the relative embedded homology $H_*(\mathcal{H}, \mathcal{A})$ for a hypergraph $\mathcal{H}$ and a subhypergraph $\mathcal{A}$. They used $\Delta\mathcal{H}$ to denote the associated simplicial complex $\mathcal{K}_{\mathcal{H}}$ and used $\delta\mathcal{H}$, called the lower-associated simplicial complex, to denote the largest simplicial complex whose simplices are hyperedges of $\mathcal{H}$. A morphism of hypergraphs $\varphi : \mathcal{H} \longrightarrow \mathcal{H}'$ induces simplicial maps $\Delta\varphi : \Delta\mathcal{H} \longrightarrow \Delta\mathcal{H}'$ and $\delta\varphi : \delta\mathcal{H} \longrightarrow \delta\mathcal{H}'$. Some commutative diagrams of multiple long exact sequences of homology groups concerning $\mathcal{H}$, $\Delta\mathcal{H}$ and $\delta\mathcal{H}$ were proved in (Ren et al., 2022b), which would reduce to the usual Mayer-Vietoris sequence if $\mathcal{H}$ is a simplicial complex. The stability of the persistent embedded homology and the persistent homology of the (lower-)associated simplicial complexes for hypergraphs was given by Ren and Wu (2026). The Hodge-Laplacians for the infimum chain complex and the supremum chain complex for hypergraphs was studied by Ren et al. (2018b). The discrete Morse theory for hypergraphs was studied by Ren et al. (2018a).

Part II. Secondly, we review the motivations and the meanings of the map algebra of hypergraphs, random hypergraphs and random simplicial complexes.

Problem 2. Random higher-order networks are generalizations of Erdös-Rényi random graphs, where relations among multiple vertices are generated at random. In various scenarios of complex networks, percolations, diffusions, etc., scientists use random hypergraphs and random simplicial complexes as higher-order analogues of the Erdös-Rényi random graphs. It is important to investigate the mathematical relations between random hypergraphs and random simplicial complexes. The topology of random hypergraphs and random simplicial complexes can detect global geometric information of random higher-order networks.

In 2022-2023, Ren et al. (2022a, 2023) introduced the map algebra and used it to detect the relations between random hypergraphs and random simplicial complexes systematically, as an answer of Problem 2. By calculating the exact parameters for the randomness, the map algebra shows that the (lower-)associated simplicial complex of an Erdös-Rényi random hypergraph is a Erdös-Rényi random simplicial complex or the complement of an Erdös-Rényi random independence hypergraph. Some Künneth-type formulae are obtained for random hypergraphs and random simplicial complexes.

An independence hypergraph on V is the complement of a simplicial complex in $\Delta[V]$. In Ren (2023), certain differential calculus on the vertices was investigated with some rough dualities detected between the chain complexes for simplicial complexes and the cochain complexes for independence hypergraphs. Consequently, the constrained homology was constructed for simplicial complexes and the constrained cohomology was constructed for independence hypergraphs in Ren (2023). A systematic development of the constrained (co)homology for simplicial complexes and independence hypergraphs from the perspective of homological algebra was given by Ren (2025a). Drawing on (Ren, 2023, 2025a) and (Ren et al., 2023, Section 6), the embedded cohomology for hypergraphs is a roughly dual version of the embedded homology for hypergraphs, in the sense of the differential calculus of simplicial complexes and independence hypergraphs.

Given a hypergraph $\mathcal{H}$ on V, we have its associated simplicial complex $\Delta\mathcal{H}$, its lower-associated simplicial complex $\delta\mathcal{H}$ and its complement $\gamma\mathcal{H}$ in the clique complex $\Delta[V]$. Ren et al. (2022a, 2023) studied the semigroup, which is called the map algebra, generated by the operators Δ, δ and γ. They applied the map algebra to study the Erdös-Rényi-type models of random hypergraphs and random simplicial complexes. As shown in (Ren et al., 2022a; Farber et al., 2022; Ren et al., 2023), the relations among random hypergraphs, random simplicial complexes and random independence hypergraphs of Erdös-Rényi-types could be described in a concise form in terms of the map algebra. Some Künneth-type formulae for the embedded (co)homology of joins of random hypergraphs, the usual homology and the constrained homology of joins of random simplicial complexes and the constrained cohomology of joins of random independence hypergraphs were proved in (Ren et al., 2023).

Part III. Thirdly, we review the motivations and the meanings of hypergraphs on metric spaces, in particular, hypergraphs on manifolds and sub-hypergraphs of the independence complexes.

Problem 3. In the capacity problems of communication channels, robotic motion plannings and various optimization scenarios, it is important to find optimizations of r-ball covers, r-ball packings or other positions of r-balls with certain constraints, for a metric space (X, d). These problems are usually difficult. In order to analyze these problems carefully, it is important to investigate the topological combinatorics of the space of all the r-ball covers, the space of all the r-ball packings or the space of all the positions of r-balls with certain constraints in (X, d).

In order to give some partial answer for Problem 3, we let the vertices of a hypergraph move continuously in (X, d). We extend our study of the embedded homology to hypergraphs on (X, d). The k-th ordered configuration space $\mathrm{Conf}_k(X)$ consists of all the ordered k-tuples $(x_1, \ldots, x_k)$ of distinct points in X. The k-th symmetric group Σ_k acts on $\mathrm{Conf}_k(X)$ by permuting the coordinates. The k-th unordered configuration space $\mathrm{Conf}_k(X)/\Sigma_k$ is the orbit space, which consists all the sets $\{x_1, \ldots, x_k\}$ of distinct k-points in X. We refer to Bödigheimer et al. (1989) and Cohen (2010) for an introduction of the homology and the homotopy of configuration spaces. A hyperdigraph $\boldsymbol{\mathcal{H}}(X)$ on X is a graded subspace of $\bigcup_{k\geq 1} \mathrm{Conf}_k(X)$ and a hypergraph $\mathcal{H}(X)$ on X is a graded subspace of $\bigcup_{k\geq 1} \mathrm{Conf}_k(X)/\Sigma_k$. The space of all the finite r-ball covers is the hypergraph on X whose hyperedges are $\{x_1, \ldots, x_k\}$ such that for any $x \in X$, there exists some x_i satisfying $d(x_i, x) \leq r$. The space of all the finite r-ball packings is the hypergraph on X whose hyperedges are $\{x_1, \ldots, x_k\}$ such that $d(x_i, x_j) \geq 2r$ for any $i \neq j$.

Here are two particular cases:

Case 1: hypergraphs on manifolds.

Many constraints in engineering are given by equations of multiple variables. The solutions of these constraints are submanifolds of the Euclidean space, which is a Riemannian manifold. Hence we let (X, d) be a Riemannian manifold (M, g). Then the space of all the positions of r-balls with certain constraints in (M, g) is a hyperdigraph $\boldsymbol{\mathcal{H}}(M)$ on M. If we do not distinguish the orders of the positions, then the space is a hypergraph $\mathcal{H}(M)$ on M. For example, the space of all the finite r-ball packings of M is a simplicial complex on M, and the space of all the finite r-ball covers of M is an independence hypergraph on M.

In Ren (2025b), hypergraphs on manifolds are regarded as graded submanifolds of configuration spaces. The infimum chain complexes and the supremum chain complexes of hyper(di)graphs are generalized to infimum double complexes and supremum double complexes of hyper(di)graphs on manifolds. By using the commutative differential graded algebra of differential forms, it is feasible to construct the infimum double complexes and the supremum double complexes with rational coefficients. The double homology and the Dolbeault homology for hyper(di)graphs are feasible to be studied.

When we generalize hypergraphs on a discrete set V to hypergraphs on a manifold M, some new phenomenon appears when $\dim M \geq 2$. Any discrete set V has a total order $\prec$ thus any simplicial complex with vertices in V has a boundary map with respect to $\prec$. However, if the vertices of a simplicial complex move continuously on M where $\dim M \geq 2$, then the total order of the vertices cannot be preserved continuously. Intuitively, two points can exchange their positions by moving continuously on M if and only if they are in the same connected component of M when $\dim M \geq 2$. Thus the boundary maps of simplicial complexes on M cannot be defined by the previous method continuously with respect to a total order. This argument applies for hypergraphs on M as well.

Case 2: hypergraphs on graphs.

In the Shannon capacity problem of communication channels, the metric space (X, d) is the vertex set V of a graph G with the geodesic distance d_G in G. An independent set of G is a finite set of mutually non-adjacent vertices. The independence complex $\mathrm{Ind}(G)$ is the simplicial complex whose simplices are independent sets. The topology and combinatorics of the independence complex have attracted attentions and been investigated in mathematics and computer science.

The k-regular embedding problem, which is motivated by the k-interpolation problem of functions, is equivalent to the geometric realizations of the independence complex. The homotopy type of $\mathrm{Ind}(G)$ can give topological obstructions for the existence of geometric realizations of $\mathrm{Ind}(G)$ thus give topological obstructions for the existence of k-regular embeddings. However, the geometric realizations depend on more precise structures rather than the homotopy type. Consequently, the above homotopy obstructions are not sufficient. Hypergraphs on (V, d_G) are sub-hypergraphs of $\mathrm{Ind}(G)$. The embedded homology of such sub-hypergraphs will detect more precise structures rather than the homotopy type itself hence it would hopefully give more precise obstructions for the existence of k-regular embeddings (Ren, 2026).

Part IV. Fourthly, we give some further discussions about the topology of hypergraphs on (X, d), as a continuation of Problem 3.

Given a metric space (X, d), there are difficult problems: How to find the smallest number $p(r)$ of r-balls that can cover X? How to find the largest number $q(r)$ of r-balls that can be packed in X? As r goes from 0 to $+\infty$, $p(r)$ and $q(r)$ are integer valued non-increasing functions of r. Moreover, it is also difficult to characterize the space of all the possible positions of minimal r-ball covers as well as the space of all the possible positions of maximal r-ball packings, in (X, d).

From a topological point of view, we consider a hypergraph $\mathcal{H}(X)$ on X. We give a filtration $\mathcal{H}(X, r)$ for $\mathcal{H}(X)$ where $\mathcal{H}(X, r)$ consists of the hyperedges whose vertices have mutual distances greater than $2r$. The double complexes of $\mathcal{H}(X, r)$ give persistence double complexes and the double homology of $\mathcal{H}(X, r)$ gives persistent double homology. We want to find the precise values of r such that $p(r)$ or $q(r)$ is discontinuous. We also want to study the topology of $\mathcal{H}(X, r)$ where r takes the above values.

In the case that (X, d) is (V, d_G), the problem reduces to investigations of the independence complexes of the distance powers of G. The discontinuous values of r are integers and tools about the independence complexes could be applied. However, in the case that (X, d) is a Riemannian manifold (M, g), the discontinuous values of r could be any positive numbers and the problem becomes more difficult.

The next is our open problem.

Problem 4. Whether could the computational tools of persistent homology, computational Morse theory, etc. in topological data analysis be helpful for the r-ball packing problem and the r-ball covering problem in a metric space (X, d), especially when (X, d) is a Riemannian manifold?

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