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Learning-Based Resilient Interval Observers for Nonlinear Discrete-Time Bounded-Error Systems

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Abstract

This paper develops a unified framework for the synthesis of interval observers for nonlinear discrete-time systems with partially unknown dynamics and bounded noise. The proposed approach enables simultaneous state estimation and model identification by embedding a learning-based data-driven abstraction mechanism within an interval-observer structure. Specifically, the method integrates Jacobian sign-stable (JSS) decompositions and tight mixed-monotone decomposition functions with recursive data-driven over-approximations of the unknown dynamics. This integration yields tractable closed-form bounds for the learned models, which are iteratively refined using past interval framers, therefore ensuring both correctness and adaptivity. In addition, observer gains are synthesized via a semidefinite programming (SDP) formulation that guarantees input-to-state stability and $\mathcal{H}_\infty$-optimality. Comprehensive simulations confirm that the proposed learning-augmented observer achieves accurate state and model estimation with significantly reduced computational complexity compared to previous optimization-based approaches.

Keywords: Data-Driven Estimation; Bounded-Error Systems; Model-Resilient Interval Observer

1. Introduction

In response to the increasing demand for safe and reliable operation in safety-critical engineering systems ranging from fault detection and urban transportation to attack mitigation and detection in cyber-physical systems, researchers have recently developed a variety of robust set-valued estimation algorithms capable of providing consistent and bounded estimates of system states and unknown inputs under diverse uncertainties Ren et al. (2023); Liu and Hwang (2011); Yong (2018).

In addition, the dynamic models of many real-world systems are typically partially known or subject to structural uncertainties, which limits the effectiveness of model-based estimation techniques. So, the integration of data-driven model learning with set-membership or interval-based estimation frameworks has emerged as a critical research direction. Such hybrid approaches aim to exploit physical knowledge while adapting to unknown or time-varying dynamics through learning mechanisms, thereby enhancing both robustness and adaptability Disarò and Valcher (2024).

Literature review. Extensive research efforts have been directed toward developing set-valued and interval observers across diverse system classes Jaulin (2002); Kieffer and Walter (2004); Moisan et al. (2007); Bernard and Gouzé (2004); Raïssi et al. (2010, 2011); Mazenc and Bernard (2011); Mazenc et al. (2013); Wang et al. (2015); Efimov et al. (2013); Zheng et al. (2016); Mazenc et al. (2014); Ellero et al. (2019); Yong (2018); Khajenejad and Yong (2019a,b, 2022). These works encompass linear time-invariant (LTI) systems Mazenc and Bernard (2011), linear parameter-varying

(LPV) systems, Wang et al. (2015); Ellero et al. (2019), Metzler and partially linearizable systems Raïssi et al. (2011); Mazenc et al. (2013), cooperative systems Raïssi et al. (2010, 2011), Lipschitz nonlinear systems Efimov et al. (2013), monotone nonlinear systems Moisan et al. (2007); Bernard and Gouzé (2004), and uncertain nonlinear systems Zheng et al. (2016). Despite their strong theoretical foundations, most fail to explicitly handle unknown inputs, which can stem from external disturbances, unmodeled dynamics, cyber attacks, or unobserved exogenous signals—and thus often assume complete or nearly complete system knowledge. To address this limitation, several studies have introduced set-valued observers capable of simultaneously estimating both system states and unknown inputs, particularly for LTI Yong (2018), LPV Khajenejad and Yong (2019a), switched linear Khajenejad and Yong (2019b), and nonlinear systems Khajenejad and Yong (2022, 2020), often under bounded-norm noise assumptions.

In scenarios where precise analytical models of the system are unavailable or unreliable, researchers have increasingly focused on data-driven set-valued estimation techniques Milanese and Novara (2004); Canale et al. (2014); Zabinsky et al. (2003); Beliakov (2006); Calliess (2014). Instead of depending on exact system equations, these data-driven approaches utilize measured input–output trajectories to construct uncertainty-bounded representations of the system's evolution. The objective is to identify a set of dynamics that effectively frame or bracket the true system behavior Milanese and Novara (2004); Canale et al. (2014). Depending on how smooth or regular the unknown mapping is assumed to be, different formulations have been proposed—ranging from scalar Lipschitz continuity Zabinsky et al. (2003) and multivariate Lipschitz regularity Beliakov (2006) to Hölder-continuous dynamics Calliess (2014). Such data-driven formulations serve as a bridge between classical model-based estimation and emerging learning-based paradigms, enabling observer design even when only partial or implicit system knowledge is available.

In our earlier work Khajenejad et al. (2021), we previously introduced a framework that integrated model abstraction techniques Jin et al. (2020) with tailored decomposition-based formulations Khajenejad and Yong (2020) to construct interval framers for systems with partially known dynamics and bounded disturbances. This formulation was capable of handling a wide class of nonlinear state and measurement functions while representing the influence of unknown inputs through a fully unspecified mapping. Despite its generality, the approach in Khajenejad et al. (2021) relied on computationally demanding online optimization at every time step to derive affine over-approximations of nonlinear dynamics, which limited its real-time applicability. Moreover, the absence of an explicit stabilizing gain synthesis rendered the performance highly case-dependent, causing the stability of the estimated intervals to vary with the underlying system characteristics. To overcome these drawbacks, the present work introduces a learning-enhanced and structurally systematic framework that preserves the rigor of interval-based reasoning while achieving robust stability and computational scalability through adaptive data-informed modeling.

Contributions. This work integrates model-based set-membership observer design with data-driven function approximation into a unified framework that achieves computational efficiency and provable stability in simultaneous state and model estimation. The principal innovation lies in the synergistic integration of Jacobian Sign-Stable (JSS) decompositions with data-driven over-approximation mechanisms within a single recursive observer capable of concurrently estimating the system states and identifying the unknown dynamics. This synthesis is far from straightforward and leads to several noteworthy advances. First, it provides a closed-form and tractable procedure for deriving tight interval bounds on the learned abstraction, thereby eliminating the need for iterative online optimization previously required in Jin et al. (2020). Second, it introduces a semidefinite programming (SDP)–based gain synthesis formulation that ensures both Input-to-State Stability (ISS) and $\mathcal{H}_\infty$-

optimal performance for the overall estimation process. Finally, the proposed method constitutes a formally correct, computationally lightweight algorithm that achieves a marked improvement in real-time feasibility and robustness, making it well suited for online and safety-critical applications.

2. Preliminaries

Notation. $\mathbb{N}_n$, $\mathbb{N}$, $\mathbb{R}^{n\times p}$, $\mathbb{R}^n$, and $\mathbb{R}^n_{>0}$ represent, respectively, the natural numbers up to n, the set of natural numbers, matrices of size n by p, n-dimensional Euclidean space, and positive vectors of dimension n. For a vector $v \in \mathbb{R}^n$, the p-norm is defined as $\|v\|_p \triangleq (\sum_{i=1}^n |v_i|^p)^{\frac{1}{p}}$. For simplicity, the 2-norm of v is denoted by $\|v\|$. For a matrix $M \in \mathbb{R}^{n\times p}$, the i-th row and j-th column entry is denoted by M_{ij}. We define $M^\oplus \triangleq \max(M, \mathbf{0}_{n\times p})$, $M^\ominus \triangleq M^\oplus - M$, and $|M| \triangleq M^\oplus + M^\ominus$, which represents the element-wise absolute value of M. Additionally, we use $M \succ 0$ and $M \prec 0$ (or $M \succeq 0$ and $M \preceq 0$) to indicate that M is positive definite and negative definite (or positive semi-definite and negative semi-definite), respectively. All vector and matrix inequalities are element-wise inequalities. The zero vectors in $\mathbb{R}^n$, and zero matrices of size $n \times p$ are denoted by $\mathbf{0}_{n\times p}$ and $\mathbf{0}_n$, respectively. Moreover, a continuous function $\alpha : [0, a) \to \mathbb{R}_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing with $\alpha(0) = 0$. Lastly, a continuous function $\lambda : [0, a) \times [0, \infty) \to \mathbb{R}_{\geq 0}$ is considered to be in class $\mathcal{KL}$ if, for every fixed $t \geq 0$, the function $\lambda(s, t)$ is in class $\mathcal{K}$; for each fixed $s \geq 0$, $\lambda(s, t)$ decreases with respect to t and satisfies $\lim_{t\to\infty} \lambda(s, t) = 0$.

Definition 1 (Interval) *An n-dimensional interval is denoted by $\mathcal{I} \triangleq [\underline{z}, \overline{z}] \subset \mathbb{R}^n$ and represents the set of all vectors $z \in \mathbb{R}^n$ satisfying $\underline{z} \leq z \leq \overline{z}$. Interval matrices can be similarly defined.*

Proposition 2 ((Efimov et al., 2013, Lemma 1)) *Let $A \in \mathbb{R}^{m\times n}$ and $\underline{x} \leq x \leq \overline{x}$ with $\underline{x}, \overline{x} \in \mathbb{R}^n$. Then $A^\oplus \underline{x} - A^\ominus \overline{x} \leq Ax \leq A^\oplus \overline{x} - A^\ominus \underline{x}$. As a corollary, if A is nonnegative, then $A\underline{x} \leq Ax \leq A\overline{x}$.*

Definition 3 (Lipschitz Continuity) *A mapping $q : \mathbb{R}^n \to \mathbb{R}^m$ is said to be κ^q-Lipschitz continuous on $\mathcal{Z} \subseteq \mathbb{R}^n$ if there exists a constant $\kappa^q > 0$ such that $\|q(z_1) - q(z_2)\| \leq \kappa^q \|z_1 - z_2\|$ for all $z_1, z_2 \in \mathcal{Z}$.*

Definition 4 (Jacobian Sign-Stability) *A mapping $q : \mathcal{Z} \to \mathbb{R}^m$ is said to be* Jacobian sign-stable *(JSS) if the sign of each entry of its Jacobian matrix $J_q(z)$ remains constant for all $z \in \mathcal{Z}$.*

Proposition 5 (JSS Decomposition) *(Khajenejad et al., 2022, Proposition 2) If the Jacobian of q is bounded, i.e., it satisfies $\underline{J}_q \leq J_q(z) \leq \overline{J}_q, \forall z \in \mathcal{Z}$, then q can be decomposed as: $q(z) = Az + \mu(z)$, where the $(i, j)^{th}$ element of $A \in^{n\times n}$ satisfies $A_{ij} = (\underline{J}_f)_{ij}$ or $A_{ij} = (\overline{J}_f)_{ij}$, and the function $\mu(z) = f(z) - Az$ is JSS in $\mathcal{Z}$.*

Definition 6 ((Yang et al., 2019, Definition 4)) *A function $q_d : \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}^n$ is called a* mixed-monotone decomposition function *for q if (i) $q_d(z, z) = q(z)$, and (ii) q_d is monotonically increasing in its first argument and monotonically decreasing in its second argument.*

It follows directly from Definition 6 that

$$q_d(\underline{z}, \overline{z}) \leq q(z) \leq q_d(\overline{z}, \underline{z}) \quad \forall z \in [\underline{z}, \overline{z}] \subseteq \mathcal{Z}. \tag{1}$$

Proposition 7 (Tight Mixed-Monotone Decomposition Functions) *(Khajenejad et al., 2022, Proposition 4 & Lemma 3) Suppose q is JSS and has a bounded Jacobian. Then, the i^{th} component of a mixed-monotone decomposition function q_d is given by $q_{d,i}(z_1, z_2) \triangleq q_i(D^i z_1 + (I_m - D^i)z_2)$, with $D^i = \mathrm{diag}(\max(\mathrm{sgn}((\overline{J}_q)_i), \mathbf{0}_{1\times m}))$. Additionally, for any interval $[\underline{z}, \overline{z}] \subseteq \mathcal{Z}$, it holds that $\delta^q_d \leq F_q e^z$, where $F_q \triangleq \overline{J}^\oplus_q + \underline{J}^\ominus_q$, $e^z \triangleq \overline{z} - \underline{z}$, and $\delta^q_d \triangleq q_d(\underline{z}, \overline{z}) - q_d(\overline{z}, \underline{z})$.*

3. Problem Formulation

System Assumptions. Consider a nonlinear discrete-time system with partially unknown dynamics and bounded process and measurement noise

$$x_k^+ = f(x_k, d_k) + W w_k, \quad y_k = g(x_k, d_k) + V v_k, \quad d_k^+ = h(x_k, d_k), \tag{2}$$

where $x_k^+ \triangleq x_{k+1}$, $x_k \in \mathcal{X} \subset \mathbb{R}^n$ is the state vector at time $k \in \mathbb{N}$, $y_k \in \mathbb{R}^l$ is the measurement vector and $d_k \in \mathcal{D} \subset \mathbb{R}^p$ is an unknown (dynamic) input vector whose dynamics is governed by the *unknown*[1] *mapping* $h : \mathbb{R}^{(n+p)} \to \mathbb{R}^p$. We refer to $z_k \triangleq \begin{bmatrix} x_k^\top & d_k^\top \end{bmatrix}^\top \in \mathcal{Z} \subseteq \mathbb{R}^{n_z}$ as the augmented state, where $n_z \triangleq n + p$. The process noise $w_k \in [\underline{w}, \overline{w}] \subset \mathbb{R}^{n_w}$ and the measurement noise $v_k \in [\underline{v}, \overline{v}] \subset \mathbb{R}^{n_v}$ are assumed to be bounded. Moreover, we assume the following.

Assumption 1 *The bounds for disturbances and noises $\underline{w}$, $\overline{w}$, $\underline{v}$, and $\overline{v}$, along with the output signals y_k are known at every time instant. Additionally, the initial condition x_0 lies in $\mathcal{X}_0 = [\underline{x}_0, \overline{x}_0] \subseteq \mathcal{X}$ with known limits $\underline{x}_0$ and $\overline{x}_0$, while the lower and upper bounds, $\underline{z}_0$ and $\overline{z}_0$, for the initial augmented state $z_0 \triangleq \begin{bmatrix} x_0^\top & d_0^\top \end{bmatrix}^\top$ are available, i.e., $\underline{z}_0 \le z_0 \le \overline{z}_0$.*

Assumption 2 *The mappings $f : \mathbb{R}^{n_z} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n_z} \to \mathbb{R}^l$ are known, locally Lipschitz, and differentiable over their domains. The function $h = [h_1^\top \ \dots \ h_p^\top]^\top : \mathbb{R}^{n_z} \to \mathbb{R}^p$ is* unknown*, yet each component $h_j : \mathbb{R}^{n_z} \to \mathbb{R}$, $j \in \{1, \dots, p\}$, is assumed to be Lipschitz continuous. Without loss of generality, a known (possibly conservative) upper bound κ_j^h for the Lipschitz constant of each h_j is available, which is a standard assumption in set-membership estimation. If such bounds are unavailable, they can be estimated to arbitrary precision following the procedure in (Jin et al., 2020, Eq. (12), Prop. 3). Furthermore, the Jacobian matrices of f and g are bounded by known interval matrices satisfying $\underline{J}_s(z) \le J_s(z) \le \overline{J}_s(z)$ for all $s \in \{f, g\}$ and all $z \in \mathcal{Z} \triangleq \mathcal{X} \times \mathcal{D}$.*

It should be emphasized that assuming a known (upper bound on the) Lipschitz constant for an unknown mapping, or equivalently, imposing a continuity or regularity condition, is a common and well-established practice in the literature on data-driven set-membership estimation and interval observer design for partially unknown systems Milanese and Novara (2004); Canale et al. (2014); Calliess (2014); Khajenejad et al. (2021). The Lipschitz constant characterizes the maximum rate of variation of the function, which is crucial for constructing reliable over-approximations in the presence of measurement noise.

Definition 8 (Correct Interval Framers and Model Resilient $\mathcal{H}_\infty$-Optimal Interval Observer) *The signals/sequences $\overline{z}, \underline{z} : \mathbb{K} \to \mathbb{R}^{n_z}$ are called upper and lower framers for the system augmented states $z_k \triangleq [x_k^\top \ d_k^\top]^\top$ of the nonlinear partially known system in (2) if $\underline{z}_k \le z_k \le \overline{z}_k$, $\forall k \ge 0, \forall w_k \in \mathcal{W} \triangleq [\underline{w}, \overline{w}], \forall v_k \in \mathcal{V} \triangleq [\underline{v}, \overline{v}]$. Further, $\varepsilon_k \triangleq \overline{z}_k - \underline{z}_k$ is called the* observer error *at time k. Any dynamical system whose states are correct framers for the augmented states of (2), i.e., with $\varepsilon_k \ge 0, \forall k \in \mathbb{K}$, is called a* correct *interval framer for (2). Moreover, the interval framer is $\mathcal{H}_\infty$-optimal (and if so is called interval observer), if the $\mathcal{H}_\infty$-gain of the framer error system $\tilde{\mathcal{G}}$ is minimized: $\|\tilde{\mathcal{G}}\|_{\ell_2} \triangleq \sup_{\|\delta\|_{\ell_2}=1} \|\varepsilon_k\|_{\ell_2}$, where $\|\nu\|_{\ell_2} \triangleq \sum_0^\infty \sqrt{\|\nu_k\|^2}$ denotes the ℓ_2 signal norm for $\nu \in \{\varepsilon, \delta\}$, respectively, while $\delta_k = \delta \triangleq [\delta_w^\top \delta_v^\top]^\top$, $\delta_w \triangleq \overline{w} - \underline{w}$ and $\delta_v \triangleq \overline{v} - \underline{v}$.*

1. Note that if the mapping h is partially known (i.e., consists of the sum of a known component $\hat{h}$ and an unknown component $\tilde{h}$), we can simply consider $d_{k+1} - \hat{h}$ as the output data for the model learning procedure to learn a model of the (completely) unknown function $\tilde{h}$.

The state and model estimation problem can be stated through a model resilient observer design:

Problem 1 (Improved State & Model Estimation) *Given the partially known nonlinear system (2), design a model resilient correct $\mathcal{H}_\infty$-optimal interval observer. such that its framer error is input-to-state stable (ISS), that is, $\|\varepsilon_k\|_2 \leq \beta(\|\varepsilon_0\|_2, t) + \rho(\|\delta\|_{L_\infty}), \forall k \geq 0$, where β and ρ are class $\mathcal{KL}$ and $\mathcal{K}$ functions, respectively, with the L_∞ signal norm given by $\|\delta\|_{L_\infty} = \sup_{t\in[0,\infty)} \|\delta_k\|_2 = \|\delta\|_2$ and δ_k defined as in Definition 8.*

4. Model Resilient Interval Observers

To address Problem 1, we first augment x_k and d_k in (2) to obtain the following augmented system:

$$z_k^+ = \mu(z_k) + \hat{W}w_k = \hat{A}z_k + \hat{\phi}(z_k) + \hat{W}w_k,\ y_k = g(z_k) + Vv_k = Cz_k + \psi(z_k) + Vv_k, \quad (3)$$

where $z \triangleq [x^\top\ d^\top]^\top$, $\hat{W} \triangleq [W^\top\ \mathbf{0}_{n_w\times p}]^\top$, $\mu(z) \triangleq [f^\top(z)\ h^\top(z)]^\top = [(Az+\phi(z))^\top\ h^\top(z)]^\top = \hat{A}z_k + \hat{\phi}(z_k)$, $\hat{A} \triangleq [A^\top\ 0]^\top$, $\hat{\phi}(z_k) \triangleq [\phi^\top(z_k)\ h^\top(z_k)]^\top$, and $g(z) = Cz + \psi(z)$, while A and C are computed offline through applying Proposition 5, using Jacobian bounds of f and g, respectively, such that ϕ and ψ become JSS mappings.

Our goal is to design an ISS interval observer that outputs correct framers for the augmented states $\{z_k \triangleq [x_k^\top\ d_k^\top]\}_{k\geq 0}$ of (3). The proposed observer will leverage decomposition functions (cf. Definition 6 and Proposition 7) to bound the known components of the system dynamics (3). However, the mapping h, as part of the augmented system, is unknown. So, we first apply the approach in to obtain a data-driven over-approximation model of h, followed by a new result on how to tractably compute bounds for the data-driven over-approximation model, as a function of the state framers, to tractably include them into the observer structure with to-be-designed gains.

4.1. Data-Driven Abstractions & Their Tight Bounds

Building upon our previous result in (Jin et al., 2020, Theorem 1), given the history of obtained compatible intervals up to the current time, $\{[\underline{z}_j, \overline{z}_j]\}_{j=0}^k$ (satisfying $\underline{z}_j \leq z_j \leq \overline{z}_j$) as the noisy input data and the compatible interval of unknown inputs, $\{[\underline{d}_j, \overline{d}_j]\}_{j=0}k$ (satisfying $\underline{d}_j \leq d_j \leq \overline{d}_j$) as the noisy output data, we can recursively construct a sequence of *abstraction/over-approximation models* $\{\overline{h}_k, \underline{h}_k\}_{k=1}^\infty$ for the unknown input function h (recall that $z \triangleq [x^\top\ d^\top]^\top$):

$$\underline{h}_{k,j}(z_k) = \max_{t\in\{0,\ldots,T-1\}} (\underline{d}_{k-t,j} - \kappa_j^h \|z_k - \tilde{z}_{k-t}\|) - \epsilon_{k,j}, \quad (4a)$$

$$\overline{h}_{k,j}(z_k) = \min_{t\in\{0,\ldots,T-1\}} (\overline{d}_{k-t,j} + \kappa_j^h \|z_k - \tilde{z}_{k-t}\|) + \epsilon_{k,j}, \quad (4b)$$

where $j \in \{1\ldots p\}$, $\{\tilde{z}_{k-t} = \frac{1}{2}(\overline{z}_{k-t} + \underline{z}_{k-t})\}_{t=0}^k$ and $\{\overline{d}_{k-t}, \underline{d}_{k-t}\}_{t=0}^k$ are the *augmented* input-output data set[2]. Moreover, $\epsilon_{k,j}$ is the bounded abstraction/over-approximation noise added to guarantee that all possible realizations of the true function h_j are within the provided bounds by the approximation model (cf. for more details). Furthermore, the parameter T is a design choice that sets the size of the moving data window used for the abstraction. A larger T may yield tighter

2. Note that The bounds for the unknown input, $\overline{d}_{k-t}$ and $\underline{d}_{k-t}$, are not obtained from direct measurement but are the "outputs" of the observer itself from previous time steps. This creates a recursive "bootstrapping" process: the estimates $\underline{d}_j, \overline{d}_j$ for $j > 0$ are generated by an observer (cf. (11)) at each time step. These estimates are then stored and used to build the data-driven abstraction for subsequent steps.

bounds but increases computational load. The data points $\tilde{z}_{k-t}$ are the midpoints of the state intervals estimated by the observer at previous time steps. These are stored in a *fixed-length buffer of size* T, which is updated recursively at each time step. This ensures the abstraction uses the most recent T data points, maintaining computational tractability even as k becomes very large. Defining $\underline{h}_k(z_k) \triangleq [\underline{h}_{k,1}(z_k) \dots \underline{h}_{k,p}(z_k)]^\top$ and $\overline{h}_k(z_k) \triangleq [\overline{h}_{k,1}(z_k) \dots \overline{h}_{k,p}(z_k)]^\top$, as a result of the data-driven over-approximation satisfies:

$$\underline{h}_k(z_k) \le h(z_k) \le \overline{h}_k(z_k). \tag{5}$$

The following Lemma provides an approach to computing tractable bounds for the over-approximation functions $\{\overline{h}_k, \underline{h}_k\}_{k=0}^{\infty}$ using upper and lower framers.

Lemma 9 (Tractable & Tight Bounds for Data-Driven Abstractions) *Khajenejad and Jin (2025) The data-driven over-approximation/ abstraction model in* (4a)–(4b) *satisfies the tight bounds:*

$$\begin{aligned} \underline{h}^*_{k,j} &\triangleq \max_{z_k \in [\underline{z}_k, \overline{z}_k]} \underline{h}_{k,j}(z_k) = \max_{t \in \{0,\dots,T-1\}} (\underline{d}_{k-t,j} - \kappa^h_j \| z^*_{k,t} - \tilde{z}_{k-t} \|) - \epsilon_{k,j}, \\ \overline{h}^*_{k,j} &\triangleq \min_{z_k \in [\underline{z}_k, \overline{z}_k]} \overline{h}_{k,j}(z_k) = \min_{t \in \{0,\dots,T-1\}} (\overline{d}_{k-t,j} + \kappa^h_j \| z^*_{k,t} - \tilde{z}_{k-t} \|) + \epsilon_{k,j}, \end{aligned} \tag{6}$$

where for $i = 1, \dots, n_z$, $z^*_{k,t,i} = \underset{\zeta,\ \zeta_i \in \{\underline{z}_{k,i}, \overline{z}_{k,i}\},\ i=1,\dots,n_z}{\arg\max} \|\zeta - \tilde{z}_{k-t}\|.$

Remark 10 *The results in Lemma 9 can be interpreted as an implicit data-driven computation of decomposition functions. In particular, when a mapping* h *is known, then it can bounded in its interval domain through the mix-monotone decomposition functions described in Propositions 5 and 7. As a counterpart, when* h *is unknown and satisfies a continuity assumption, then "all possible realizations of* h*" can be bounded through the data-driven scheme in* (6)*. It is worth emphasizing that the core of our data-driven approach is not to discover the Lipschitz constant but to use its known (or conservatively estimated) value to construct a valid over-approximation of the unknown function* h *from data. Knowing* κ^h_j *allows us to build the abstraction in* (4) *with a guaranteed correctness property. Alternatively, this could be relaxed to a bounded Jacobian with unknown sign structure, though this would require a different decomposition strategy and likely result in more conservative intervals. The data-driven aspect is crucial for obtaining tighter, state-dependent bounds that improve over time, as opposed to a single, worst-case global bound.*

4.2. Model-Resilient $\mathcal{H}_\infty$-Optimal Interval Observer Design

We start by deriving an equivalent system representation that helps us include multiple degrees of freedom, i.e., observer gains in our design. To do this, we introduce an auxiliary state variable:

$$\xi_k = z_k - N(y_k - V v_k) = z_k - N(C z_k - \psi(z_k) = (I_{n_z} - NC) z_k - N\psi(z_k) = T z_k - N\psi(z_k), \tag{7}$$

where $T = [T_x\ T_d], T_x \in \mathbb{R}^{n_z \times n}, T_d \in \mathbb{R}^{p \times n}, N$ are observer gains satisfying $T + NC = I_{n_z}$.

Lemma 11 *Suppose that Assumptions 1–2 hold. Let* $L, N \in \mathbb{R}^{n_z \times l}$ *and* $T \in \mathbb{R}^{n_z \times n_z}$ *be arbitrary matrices satisfying* $T + NC = I_{n_z}$*. Then, the dynamics* (3) *can be equivalently written as*

$$\begin{aligned} \xi^+_k &= M_z \xi_k + M_w w_k - M_v v_k + M_y y_k + T\hat{\phi}(z_k) - L\psi(z_k) - N\rho(z_k, w_k), \\ z_k &= \xi_k + N y_k - N V v_k, \end{aligned} \tag{8}$$

where $M_z \triangleq T\hat{A} - LC - NA_2$, $M_w \triangleq T\hat{W} - NW_2$, $M_v \triangleq (M_zN + L)V$, *and* $M_y = M_zN + L$ *with* $A_2 \in \mathbb{R}^{l\times n_z}$, *and* $W_2 \in \mathbb{R}^{l\times n_w}$ *chosen such that the following decompositions hold:*

$$\psi^+(z,w) = A_2 z + W_2 w + \rho(z,w), \text{ where } \rho \text{ is JSS (cf. Lemma 5).} \tag{9}$$

Proof Deriving the dynamics of the auxiliary state ξ_k starting from (7), and using (3), we obtain:

$$\begin{aligned}\xi_k^+ &= Tz_k^+ - N\psi^+(z_k,w_k) = T(\hat{A}z_k + \hat{W}w_k) - N(A_2z_k + W_2w_k + \rho(z_k,w_k)) \\ &= (T\hat{A} - A_2)z_k + (T\hat{W} - NW_2)w_k - N\rho(z_k,w_k).\end{aligned} \tag{10}$$

Next, from the second equations in (2) and (9), we have $L(y_k - Cz_k - \psi(z_k) - Vv_k) = 0$, for any to-be-designed observer gain matrix L with appropriate dimensions. Adding this 'zero term' to the right-hand side of (10) and plugging in z_k from (8), yields the equivalent system. ■

Given the equivalent dynamics in (8) and equipped with the results in Lemma 9, we propose the following dynamic system to provide framers for the sate and unknown input signals in (2) (recall that $z_k = [x_k^\top \ d_k^\top]^\top$ and similarly for their framers $\underline{z}_k \triangleq [\underline{x}_k^\top \ \underline{d}_k^\top]^\top, \overline{z}_k \triangleq [\overline{x}_k^\top \ \overline{d}_k^\top]^\top$):

$$\begin{aligned}\underline{\xi}_k^+ &= M_z^\oplus\underline{\xi}_k - M_z^\ominus\overline{\xi}_k + M_y y_k + T^\oplus\underline{\phi}_k - T^\ominus\overline{\phi}_k + L^\ominus\psi_d(\underline{z}_k,\overline{z}_k) - L^\oplus\psi_d(\overline{z}_k,\underline{z}_k) \\ &\quad + M_w^\oplus\underline{w}_k - M_w^\ominus\overline{w}_k + M_v^\ominus\underline{v}_k - M_v^\oplus\overline{v}_k + N^\ominus\rho_d(\underline{z}_k,\underline{w}_k,\overline{z}_k,\overline{w}_k) - N^\oplus\rho_d(\overline{z}_k,\overline{w}_k,\underline{z}_k,\underline{w}_k), \\ \overline{\xi}_k^+ &= M_z^\oplus\overline{\xi}_k - M_z^\ominus\underline{\xi}_k + M_y y_k + T^\oplus\overline{\phi}_k - T^\ominus\underline{\phi}_k + L^\ominus\psi_d(\overline{z}_k,\underline{z}_k) - L^\oplus\psi_d(\underline{z}_k,\overline{z}_k) \\ &\quad + M_w^\oplus\overline{w}_k - M_w^\ominus\underline{w}_k + M_v^\ominus\overline{v}_k - M_v^\oplus\underline{v}_k + N^\ominus\rho_d(\overline{z}_k,\overline{w}_k,\underline{z}_k,\underline{w}_k) - N^\oplus\rho_d(\underline{z}_k,\underline{w}_k,\overline{z}_k,\overline{w}_k), \\ \underline{z}_k &= \underline{\xi}_k + (NV)^\ominus\underline{v}_k - (NV)^\oplus\overline{v}_k + Ny_k, \quad \overline{z}_k = \overline{\xi}_k + (NV)^\ominus\overline{v}_k - (NV)^\oplus\underline{v}_k + Ny_k,\end{aligned} \tag{11}$$

where $\underline{\phi}_k \triangleq [\phi_d^\top(\underline{z}_k,\overline{z}_k) \ (\underline{h}_k^*)^\top]^\top, \overline{\phi}_k \triangleq [\phi_d^\top(\overline{z}_k,\underline{z}_k) \ (\overline{h}_k^*)^\top]^\top$, while ϕ_d, ψ_d, and ρ_d are tight remainder-from decomposition functions of the JSS mappings ϕ, ψ, and ρ respectively, computed through Proposition 7. Moreover, L, T and N are to-be-designed observer gains, and $\underline{h}_k^*, \overline{h}_k^*$ are given in Lemma 9.

The following Lemma shows that the proposed system in (11) is indeed a correct framer for the partially known system (2), and that the abstraction/over-approximation model $\{\underline{h}_k, \overline{h}_k\}_{k\geq 0}$ becomes tighter or more precise with time.

Lemma 12 (Correctness & Model Improvement) *Suppose the nonlinear partially known system* (2) *(equivalently the augmented system* (3)*) satisfies Assumptions 1 and 2. Then, for all* $k \geq 0$ *and any* $w_k \in \mathcal{W}$, $v_k \in \mathcal{V}$, *we have* $\underline{z}_k \leq z_k \leq \overline{z}_k$. *Here,* z_k *denotes the state vector of* (3) *at time* $k \geq 0$, *and* $[\underline{z}_k^\top \ \overline{z}_k^\top]^\top$ *represents the state vector of* (11) *at the same time. Thus, the dynamical system given by* (11) *is a correct interval* framer *for the nonlinear system* (3) *(equivalently for* (2)*). Moreover, the following holds:*

$$\underline{h}_0(z) \leq \ldots \leq \underline{h}_k(z) \leq \ldots \leq \lim_{k\to\infty}\underline{h}_k(z) \leq h(z), \text{ and } h(z) \leq \lim_{k\to\infty}\overline{h}_k(z) \leq \ldots \leq \overline{h}_k(z) \leq \ldots \leq \overline{h}_0(z),$$

for any realization of the augmented state z. *In other words, the unknown input model estimations/abstractions are correct and become more precise or tighter with time.*

Proof The correctness (framer) property follows from applying Propositions 2 and 7 to (8) to find upper and lower bounds for the known linear and nonlinear terms by the inequality in (1), respectively, as well as applying Lemma 9 to find $\underline{h}_k^*, \overline{h}_k^*$ that bound the unknown function h from below and above, respectively. Furthermore, by (5) $\forall k \in \{0\ldots\infty\} : \underline{h}_k(z) \leq h(z) \leq \overline{h}_k(z)$.

Moreover, considering the data-driven abstraction procedure in (4a)–(4b), note that by construction, the data set used at time step k is a subset of the one used at time $k+1$. Hence, by (Jin et al., 2020, Proposition 2) the abstraction model satisfies *monotonicity*, i.e., the inequalities above hold. ■

4.3. Input-to-State Stability

Beyond the framer property, we provide sufficient conditions to design/synthesize observer gains satisfying input-to-state stability of the interval framers. This is an improvement to the approach in Khajenejad et al. (2021), where a gain design procedure was lacking.

Theorem 13 ($\mathcal{H}_\infty$-Optimal and ISS Observer Design) *Suppose Assumptions 1 and 2 hold for the nonlinear partially known system in (2). Then, the correct interval framer proposed in (11) is $\mathcal{H}_\infty$-optimal with $\mathcal{H}_\infty$ system gain γ if The following optimization program with linear matrix inequalities (LMI) is feasible:*

$$\begin{array}{l}\min\limits_{\{\alpha,\gamma,Q,\Omega,\tilde{L}_1,\tilde{L}_2,L_1^p,L_2^p,L_{1v}^n,L_{1v}^p,L_{2v}^n,L_{2v}^p,L_{2c}^p,L_{2c}^n,T^p,T^n,Q_1,Q_2\}} \gamma \\ s.t. \begin{bmatrix}(1-\kappa^2)Q-\alpha I & 0 & \Gamma^\top \\ 0 & \gamma I & \Omega^\top \\ \Gamma & \Omega & Q\end{bmatrix} \succ 0,\ Q \in \mathbb{D}^{n_z}_{>0},\ \alpha,\gamma>0, \tilde{T}+\tilde{N}C=Q, \\ \Gamma = \tilde{M}_z^p+\tilde{M}_z^n+(\tilde{L}^p+\tilde{L}^n)F_\psi+(\tilde{N}^p+\tilde{N}^n)F_\rho^z+(\tilde{T}^p+\tilde{T}^n)\hat{F}_\phi, \\ \Omega = \begin{bmatrix}\tilde{M}_w^p+\tilde{M}_w^n+(\tilde{N}^p+\tilde{N}^n)F_\rho^w & \tilde{L}_v^p+\tilde{L}_v^n+\tilde{N}_v^p+\tilde{N}_v^n & 2(\tilde{T}^p+\tilde{T}^n)\end{bmatrix}, \\ \tilde{S}^p,\tilde{S}^n \geq 0, \tilde{S}^p-\tilde{S}^n=\tilde{S}, \forall S\in\{L,N,T,M_z\}, \tilde{L}_v^p-\tilde{L}_v^n=\tilde{L}V, \tilde{N}_v^p-\tilde{N}_v^n=\tilde{N}V, \\ \tilde{M}_z^p-\tilde{M}_z^n=\tilde{T}\hat{A}-\tilde{L}C-\tilde{N}A_2, \tilde{M}_w^p-\tilde{M}_w^n=\tilde{T}\hat{W}-\tilde{N}W_2, \tilde{L}_v^p,\tilde{N}_v^n,\tilde{N}_v^p,\tilde{N}_v^n \geq 0,\end{array} \tag{12}$$

where $\hat{F}\phi \triangleq \begin{bmatrix} F_\phi \\ \begin{bmatrix}\mathbf{0}_{p\times n} & I_p\end{bmatrix}\end{bmatrix}$, *while* $F_\phi, F_\psi, F_\rho^z, F_\rho^w$ *are computed by applying Proposition 7 to the JSS mappings* ϕ,ψ, *and* ρ, *and* $\kappa=\sqrt{\sum_{j=1}^p(\kappa_j^h)^2}$ *is the Lipschitz constant of the mapping* h.

Finally, the $\mathcal{H}_\infty$-robust observer gains L,N, and T can be computed as $Y=Q^{-1}\tilde{Y}$, *where* $Q\in(\tilde{L},\tilde{N},\tilde{T})$ *is an optimal solution to the program in (12).*

Remark 14 *In case the Lipschitz constant is estimated via Jin et al. (2020) with some error* $\delta\kappa_j>0$, *the abstraction model (4a)-(4b) can be modified to use* $\kappa_j^h+\delta\kappa_j$ *to ensure (5) is still maintained, albeit with potentially looser initial bounds. The stability analysis would then use this "inflated" constant. This "robustifies" the method against the Lipschitz constant approximation errors.*

Proof Defining $\varepsilon_k \triangleq \overline{z}_k-\underline{z}_k$, $\delta\nu \triangleq \overline{\nu}-\underline{\nu}, \forall\nu\in\{d_k,w,v\}$, $\delta_k^s \triangleq s_d(\overline{z}_k,\underline{z}_k)-s_d(\underline{z}_k,\overline{z}_k), \forall s\in\{\phi,\psi\}$, $\delta_k^\rho \triangleq \rho_d(\overline{z}_k,\overline{w}_k,\underline{z}_k,\underline{w}_k)-\rho_d(\underline{z}_k,\underline{w}_k,\overline{z}_k,\overline{w}_k)$, and starting from the framer system in (11), the observer error dynamics, which by construction is a non-negative system, can be obtained:

$$\begin{aligned}\varepsilon_k^+ &= |M_z|\varepsilon_k+|T|\delta_k^{\hat{\phi}}+|L|\delta_k^\psi+|N|\delta_k^\rho+|M_w|\delta_w+(|M_v|-|M_z||NV|+|NV|)\delta_v \\ &\leq |M_z|\varepsilon_k+|T|\delta_k^{\hat{\phi}}+|L|\delta_k^\psi+|N|\delta_k^\rho+|M_w|\delta_w+(|LV|+|NV|)\delta_v,\end{aligned} \tag{13}$$

where $\delta_k^{\hat{\phi}}=[(\delta_k^\phi)^\top\ \delta_k^h]^\top$, $\delta_k^h \triangleq \overline{h}_k^*-\underline{h}_k^*=[\delta_{k,1}^h\ldots\delta_{k,p}^h]^\top=[(\overline{h}_{k,1}^*-\underline{h}_{k,1}^*)\ldots(\overline{h}_{k,p}^*-\underline{h}_{k,p}^*)]^\top$. On the other hand, from (6) in Lemma 6, for $j=1,\ldots,p$, we derive

$$\delta_{k,j}^h \leq \overline{d}_{k,j}-\underline{d}_{k,j}+2\kappa_j^h\|z_{k,0}^{j*}-\tilde{z}_k\|+2\epsilon_{k,j}=\delta_{k,j}^d+\kappa_j^h\|\delta_{k,j}^z\|+2\epsilon_{k,j}, \tag{14}$$

where the inequality holds since taking "min" and "$-\max$" of finite number of values (6) (corresponding to $t = 0, \ldots, T-1$) is less than or equal to each of the values, including the one corresponding to $t = 0$. Moreover, the equality follows from the fact that $z^*_{k,0,j} = \overline{z}_{k,j}$ or $z^*_{k,0,j} = \underline{z}_{k,j}$, and $\tilde{z}_{k,j}$ is the midpoint of the interval $[\underline{z}_{k,j}, \overline{z}_{k,j}]$, and hence their distance (in 2-norm) is half of the distance between $\underline{z}_{k,j}$ and $\overline{z}_{k,j}$, i.e., $\delta^z_{k,j}$. By augmenting all the inequalities in (14), together with (13) and non-negativity of the error dynamics, and given the results in Proposition 7 in addition to the fact that $\delta^d_k = \begin{bmatrix}\mathbf{0}_{p\times n} & I_p\end{bmatrix} \varepsilon_k$, we obtain the following comparison system (with $\kappa^h \triangleq [\kappa^h_1 \ldots \kappa^h_p]^\top$ denoting the vector of Lipschitz constants):

$$\varepsilon^+_k \leq A_z \varepsilon_k + B_z \delta\tilde{w}_k + f_\ell(\varepsilon_k). \tag{15}$$

where $\tilde{w}_k \triangleq \begin{bmatrix}\delta w^\top & \delta v^\top & 2\epsilon^\top_k\end{bmatrix}^\top$ is the augmented noise vector, $\hat{F}\phi \triangleq \begin{bmatrix} F_\phi \\ \begin{bmatrix}\mathbf{0}_{p\times n} & I_p\end{bmatrix}\end{bmatrix}$, and

$$A_z \triangleq \begin{bmatrix}|M_z| + |L|F_\psi + |N|F^z_\rho + |T|\hat{F}\phi\end{bmatrix}, B_z \triangleq \begin{bmatrix}|M_w| + |N|F^w_\rho & |LV| + |NV| & 2|T|\end{bmatrix}.$$

Furthermore, $f_\ell(\varepsilon) \triangleq \|\varepsilon\| \begin{bmatrix}\mathbf{0}^\top_n & (\kappa^h)^\top\end{bmatrix}^\top$ is a Lipschitz continuous nonlinear mapping with the Lipschitz constant $\kappa = \sum_{j=1}^p (\kappa^h_j)^2$ (since 2-norm is a Lipschitz function with its Lipschitz constant being one). By (De Oliveira et al., 2002, Lemma 2) the comparison system in (15) is ISS (and consequently A_z is Schur stable), if the following matrix inequalities are feasible:

$$\begin{aligned}&\begin{bmatrix}(1-\kappa^2)Q - \alpha I - A^\top_z Q A_z & -A^\top_z Q B_z \\ -B^\top_z Q A_z & \gamma I - B^\top_z Q B_z\end{bmatrix} \succ 0, \\ &Q \succ 0,\ \gamma, \alpha > 0.\end{aligned} \tag{16}$$

By applying Schur complement, choosing a positive diagonal Q that implies $Q|Y| = \tilde{Y} \triangleq |QY|$, for all $Y \in \{L, N, NV, M_z, LV, M_w\}$, as well as the fact that we can equivalently replace $|\tilde{Y}|$ with the sum of two non-negative matrix variables Y^p and Y^n such that $\tilde{Y} = \tilde{Y}^p - \tilde{Y}^n$ (cf. (Pati et al., 2025, Lemma 2 and the proof of Theorem 2)), for the matrix inequalities in (16) to hold it suffices that the program in (12) be feasible. ■

5. Illustrative Example

In order to demonstrate the effectiveness of our new approach, we consider a slightly modified version of the continuous-time predator-prey system in Pylorof et al. (2019):

$$\dot{x}_{1,t} = -x_{1,t}x_{2,t} - x_{2,t} + d_t + w_{1,t},\ \dot{x}_{2,t} = x_{1,t}x_{2,t} + x_{1,t} + w_{2,t},\ \dot{d}_t = 0.1(\cos(x_{1,t}) - \sin(x_{2,t})) + w_{d,t},$$

where (unknown input) dynamics $\dot{d}$ is an unknown function, and the output equations are given by:

$$y_{1,t} = x_{1,t} + v_{1,t}, y_{2,t} = x_{2,t} + v_{2,t}, y_{3,t} = \sin(d_t) + v_{3,t}.$$

By applying the forward Euler method to discretize the system, it can be described in the form (2) with the following parameters: $n = l = p = 2$, $f = \begin{bmatrix}f_1 & f_2\end{bmatrix}^\top$, $g = \begin{bmatrix}g_1 & g_2 & g_3\end{bmatrix}^\top$, $w_k = [w_{1,k}\ w_{2,k}\ w_{d,k}]^\top$, $v_k = [v_{1,k}\ v_{2,k}\ v_{3,k}]^\top$, $\overline{v} = -\underline{v} = \overline{w} = -\underline{w} = \begin{bmatrix}0.1 & 0.1 & 0.1\end{bmatrix}^\top$, $\overline{x}_0 = \begin{bmatrix}0 & 0.6\end{bmatrix}^\top$, $\underline{x}_0 = \begin{bmatrix}-0.35 & -0.1\end{bmatrix}^\top$, where

$$\begin{gathered}f_1(z_k) = x_{1,k} + \delta_t(-x_{1,k}x_{2,k} - x_{2,k} + d_k + w_{1,k}),\ f_2(z_k) = x_{2,k} + \delta_t(x_{1,k}x_{2,k} + x_{1,k} + w_{2,k}) \\ h(z_k) = d_k + \delta_t(0.1(\cos(x_{1,k}) - \sin(x_{2,k})) + w_{d,k}), \\ g_1(z_k) = x_{1,k} + v_{1,k},\ g_2(z_k) = x_{2,k} + v_{2,k},\ g_3(z_k) = \sin(d_k) + v_{3,k},\end{gathered}$$

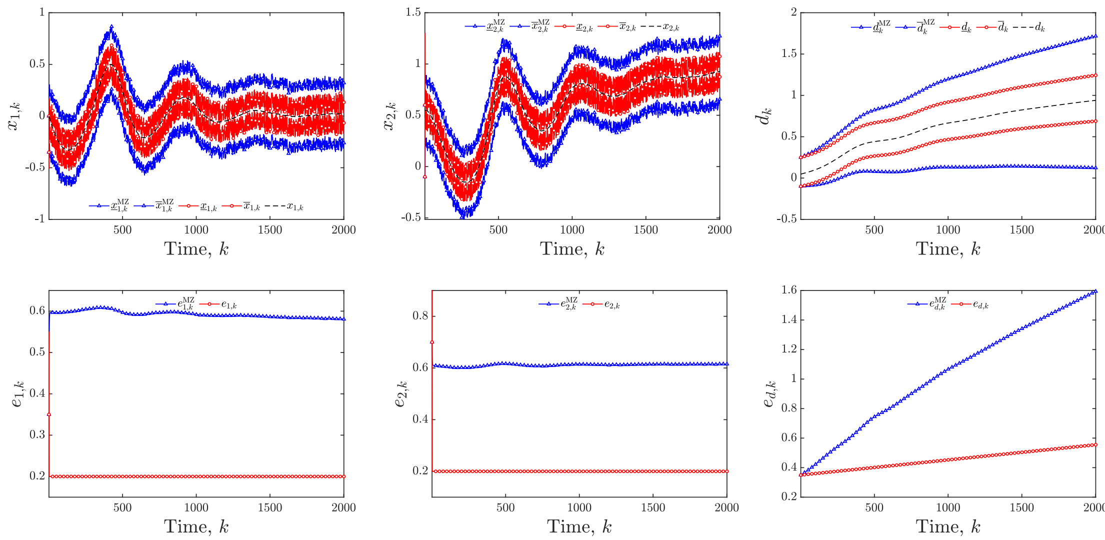

Figure 1: **Up:** learning-based state and unknown input framers; **Bottom:** state and unknown input errors; Obtained using the proposed approach (red), $\overline{x}_k, \underline{x}_k, \overline{d}_k, \underline{d}_k, e_k, e_{d,k}$, versus the benchmark method in Khajenejad and Jin (2025) (blue), $\overline{x}_k^{\text{MZ}}, \underline{x}_k^{\text{MZ}}, \overline{d}_k^{\text{MZ}}, \underline{d}_k^{\text{MZ}}, e_k^{\text{MZ}}, e_{d,k}^{\text{MZ}}$.

with sampling time $\delta_t = 0.01s$. Moreover, by (Khajenejad et al., 2021, Proposition 2) (with abstraction slopes set to zero) we can obtain finite-valued upper and lower bounds (horizontal abstractions) for the partial derivatives of f as: $\underline{J}_f = \begin{bmatrix} 0.994 & -0.01 & 0.99 \\ 0.009 & 0.9965 & -0.01 \end{bmatrix}, \overline{J}_f = \begin{bmatrix} 1.006 & -0.0065 & 1.01 \\ 0.016 & 1 & 0.01 \end{bmatrix}$. Solving the LMIs in (12) through Yalmip Löfberg (2004), the stabilizing gains are computed.[3] As we can see in Figure 1, the obtained interval estimates framers using the proposed approach in a horizon of $K = 2000$ time steps are tighter than those returned by our previous benchmark method Khajenejad and Jin (2025).

6. Conclusion & Future Work

This paper introduced a unified framework for interval observer synthesis for nonlinear systems with partially unknown dynamics and bounded noise. By integrating Jacobian sign-stable (JSS) decompositions and mixed-monotone formulations with a learning-based data-driven abstraction mechanism, the proposed approach enables simultaneous state and model estimation within a computationally tractable and theoretically sound framework. The method eliminates the need for repeated online optimization by generating closed-form, recursively updated bounds on the unknown dynamics, while guaranteeing input-to-state stability and $\mathcal{H}_\infty$-optimality through semidefinite programming (SDP)-based gain synthesis. Simulation results demonstrated that the proposed learning-augmented observer achieves high estimation accuracy and robustness with substantially reduced computational complexity compared to existing optimization methods.

3. $L = 10^{-10} \times \begin{bmatrix} 19.99 & -9.69 & 0 \\ 9.30 & 20.81 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $N = \begin{bmatrix} 0.99 & 0 & 0 \\ 0 & 0.99 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $T = 10^{-7} \times \begin{bmatrix} 385.19 & 1.21 & 0 \\ -1.15 & 261.33 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

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