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On the Exponential Stability of Koopman Model Predictive Control

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Abstract

Koopman Model Predictive Control (MPC) uses a lifted linear predictor to efficiently handle constrained nonlinear systems. While constraint satisfaction and (practical) asymptotic stability have been studied, explicit guarantees of local exponential stability seem to be missing. This paper revisits the exponential stability for Koopman MPC. We first analyze a Koopman LQR problem and show that 1) with zero modeling error, the lifted LQR policy is globally optimal and globally asymptotically stabilizes the nonlinear plant, and 2) with the lifting function and one-step prediction error both Lipschitz at the origin, the closed-loop system is locally exponentially stable. These results facilitate terminal cost/set design in the lifted Koopman space. Leveraging linear-MPC properties (boundedness, value decrease, recursive feasibility), we then prove local exponential stability for a stabilizing Koopman MPC under the same conditions as Koopman LQR. Experiments on an inverted pendulum show better convergence performance and lower accumulated cost than the traditional Taylor-linearized MPC approaches.

Keywords: Model Predictive Control; Koopman Operator; Closed-loop Stability

1. Introduction

Model Predictive Control (MPC) is a well-established feedback strategy where, at each sampling instant, a finite-horizon optimal control problem is solved using the current state, and only the first control input is applied before repeating the process (Rawlings et al., 2017). Its explicit handling of input and state constraints has led to success across a wide range of applications (Mayne et al., 2000; Qin and Badgwell, 2003; Zheng et al., 2016). For nonlinear systems, however, the MPC problem becomes nonlinear and nonconvex, making exact solutions difficult. In practice, *suboptimal yet feasible* implementations are often adopted, which can still guarantee closed-loop performance with careful design (Scokaert et al., 2002). This *feasibility-implies-stability* principle underpins many real-time methods, such as successive linearization (Diehl et al., 2005).

To overcome the computational challenges of nonlinear MPC, recent work has explored alternative linearization strategies beyond first-order Taylor expansions. One promising approach is *Koopman linearization*, which lifts the nonlinear dynamics into a higher-dimensional space where they evolve approximately linearly. Originally introduced for autonomous systems (Koopman, 1931), the Koopman operator framework has been extended to controlled systems (Korda and Mezić, 2018; Haseli and Cortés, 2025). This enables the design of Koopman MPC, where each step solves a convex program using the Koopman linear model as a predictor (Korda and Mezić, 2018). By combining the expressiveness of nonlinear modeling with computational benefits of convex optimization, Koopman MPC has attracted a growing interest for its practical scalability and performance (Haggerty et al., 2023; Mamakoukas et al., 2019; Shang et al., 2025a).

Koopman MPC fits within a sequential learning-and-control framework: a Koopman-based model is first learned from data via Extended Dynamic Mode Decomposition (EDMD) (Williams et al., 2015) or Deep Neural Networks (DNN) (Shi and Meng, 2022), and then used for controller design. However, general controlled nonlinear systems do not admit exact finite-dimensional Koopman linear or bilinear models (Haseli and Cortés, 2025), and the learned model typically introduces modeling error. This has motivated recent work on investigating the closed-loop performance of Koopman MPC for nonlinear systems utilizing such inaccurate learned models. For instance, Zhang et al. (2022) proposed a robust tube-based Koopman MPC strategy, which relies on tightened constraints in the Koopman space to ensure *constraint satisfaction* in the presence of small modeling errors; similarly, Mamakoukas et al. (2022) enforced constraint satisfaction via conservative surrogate constraints using a Hankel-Koopman model. More recently, de Jong et al. (2024) established *input-to-state stability* for a Koopman MPC variant with an interpolated initial condition, while Worthmann et al. (2024) proved *practical asymptotic stability* using terminal ingredients designed from the original nonlinear system. A terminal-free variant was analyzed in Bold et al. (2025); Schimperna et al. (2025), where *(practical) asymptotic stability* is obtained under a cost-controllability assumption on the original nonlinear dynamics.

As discussed above, several notions of closed-loop stability for Koopman MPC have been established under various assumptions. Despite the progress, to our best knowledge, the basic question of *local exponential stability* seems to have been overlooked. In particular, local exponential stability cannot be implied or directly derived by the results in (Zhang et al., 2022; Mamakoukas et al., 2022; Worthmann et al., 2024; Bold et al., 2025; de Jong et al., 2024). In this paper, we revisit this fundamental question: we design practical terminal ingredients in the lifted Koopman space, and provide clear conditions under which Koopman MPC ensures *local exponential stability* of the nonlinear closed-loop system. For brevity, we refer to this scheme as stabilizing Koopman MPC (`S-KMPC`).

Our technical results are as follows. We begin with a Koopman linear quadratic regulator (LQR), closely linked to `S-KMPC`. This is an *unconstrained, infinite-horizon* optimal control problem. With no Koopman modeling error, a globally optimal policy can be obtained by solving a standard LQR in the lifted space (Lemma 2); under mild assumptions, this policy *globally asymptotically* stabilizes the original nonlinear dynamics (Theorem 1). If both the lifting function and the one-step Koopman prediction error are Lipschitz around the origin, the nonlinear closed loop is *locally exponentially stable* (Theorem 2). These results guide the terminal design in `S-KMPC`: we construct the terminal cost and terminal set *in the lifted space* via the Koopman-LQR. Since `S-KMPC` is designed based on a lifted linear model, it naturally inherits several key properties of linear MPC (Rawlings et al., 2017), including boundedness, decrease of the value function, and recursive feasibility of the Koopman update (Propositions 1 to 3). Together with *continuous* lifting, these yield *local asymptotic stability* of the nonlinear closed loop (Theorem 3). Strengthening the assumptions to the Lipschitz lifting function and one-step Koopman prediction error around the origin, `S-KMPC` guarantees the *local exponential stability* of the nonlinear closed-loop system (Theorem 4). We provide a detailed comparison with existing Koopman stability results later in Remark 2.

The rest of this paper is structured as follows. Section 2 provides preliminaries on the Koopman MPC. The Koopman LQR problem is discussed in Section 3 and the `S-KMPC` design is provided in Section 4. Numerical results are shown in Section 5 and we gather our conclusions in Section 6.

2. Preliminaries and Problem Statement

We consider a discrete-time nonlinear system of the form

$$x_{t+1} = f(x_t, u_t), \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the system sate, $u_t \in \mathbb{R}^m$ is the system input at time t, and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ denotes the system dynamics. We make the following standard assumption.

Assumption 1 *The function f is continuously differentiable and satisfies $f(\mathbb{0}, \mathbb{0}) = \mathbb{0}$.*

2.1. Exponential stability

A basic objective is to design a feedback policy $u_t = \pi(x_t)$ that exponentially stabilizes (1).

Definition 1 (Local exponential stability) *The origin of system (1) with a control law $u_t = \pi(x_t)$ is* locally exponentially stable *if there exist $c > 0$, $\rho \in (0, 1)$, and a neighborhood $\mathcal{N}$ of the origin, such that the closed-loop dynamics satisfies $\|x_t\| \leq c\, \rho^t\, \|x_0\|$, for all $x_0 \in \mathcal{N}, t \in \mathbb{Z}_{\geq 0}$.*

Lemma 1 *(Rawlings et al., 2017, Theorem B.19) Consider the autonomous system $x_{t+1} = g(x_t)$ with $g(\mathbb{0}) = \mathbb{0}$. Suppose there exist constants $\alpha_1, \alpha_2, \alpha_3 > 0$, an invariant set $\mathcal{D} \subseteq \mathbb{R}^n$ with $\mathbb{0} \in \operatorname{int}(D)$, and a function $V : \mathcal{D} \to \mathbb{R}_+$ satisfying*

$$\alpha_1 \|x\|^2 \leq V(x) \leq \alpha_2 \|x\|^2, \quad \forall x \in \mathcal{D}, \tag{2a}$$

$$V\big(g(x)\big) - V(x) \leq -\alpha_3 \|x\|^2, \quad \forall x \in \mathcal{D}. \tag{2b}$$

Then, the system is locally exponentially stable, and $\mathcal{D}$ is a region of attraction (ROA).

This standard result is used throughout the paper; for completeness, we provide a brief proof in Appendix A.1. The function V in Lemma 1 is called a Lyapunov function. Both the quadratic upper and lower bounds in (2a) are important, and the decrease condition (2b) ensures stability. Notably, there exist $c \coloneqq \sqrt{\alpha_2/\alpha_1} \geq 1, \rho \coloneqq \sqrt{1 - \alpha_3/\alpha_2} \in (0, 1)$ and $r > 0$ such that every trajectory with $\|x_0\| < r$ is well-defined for all $k \geq 0$ and satisfies $\|x_t\| \leq c\, \rho^t\, \|x_0\|, t \in \mathbb{Z}_{\geq 0}$.

2.2. Nonlinear MPC basics and Koopman linearization

At each time t, the MPC algorithm solves the nonlinear problem (3) with initial state $x_t = x \in \mathbb{R}^n$ and horizon $N \in \mathbb{N}$. In (3), $\mathcal{X}$ and $\mathcal{U}$ are the state and input constraints, and $\tilde{V}_\mathrm{f}$ and $\tilde{\mathcal{X}}_\mathrm{f}$ denote suitable terminal cost and set. We assume both $\mathcal{X}$ and $\mathcal{U}$ are convex, and $\mathcal{U}$ is also bounded. We choose $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ as a quadratic stage cost:

$$\tilde{V}_N^*(x) = \min_{\mathbf{u}} \quad \sum_{k=0}^{N-1} l(x_{t+k}, u_{t+k}) + \tilde{V}_\mathrm{f}(x_{t+N}) \tag{3a}$$

$$\text{subject to} \quad x_{t+k+1} = f(x_{t+k}, u_{t+k}), \tag{3b}$$

$$u_{t+k} \in \mathcal{U},\ x_{t+k} \in \mathcal{X},\ k \in \mathbb{Z}_{[0,N-1]}, \tag{3c}$$

$$x_t = x,\ x_{t+N} \in \tilde{\mathcal{X}}_\mathrm{f}. \tag{3d}$$

$$\ell(x, u) := \|x\|_Q^2 + \|u\|_R^2, \tag{4}$$

with Q and R being two positive definite matrices. If (3) is feasible at x, then the optimal value function $\tilde{V}_N^*(x)$ is finite and we say $x \in \operatorname{dom}(\tilde{V}_N^*)$. In this case, let $\tilde{\mathbf{u}}_x^*$ be an optimal input sequence to (3). The MPC feedback is the first control action $\tilde{\kappa}(x) = \tilde{\mathbf{u}}_x^*(0),\ x \in \operatorname{dom}(\tilde{V}_N^*)$.

With appropriately chosen terminal ingredients $\tilde{V}_\mathrm{f}$ and $\tilde{\mathcal{X}}_\mathrm{f}$, the closed-loop dynamics with the MPC law, *i.e.*, $x_{t+1} = f(x_t, \tilde{\kappa}(x_t))$, is *locally exponentially stable*, and the value function $\tilde{V}_N^*$ serves as a Lyapunov function; see Rawlings et al. (2017, Ch. 2) for details. A well-known challenge is that (3) is generally *nonconvex* and thus hard to solve for global optimality. We next introduce a popular *Koopman linearization* strategy which is increasingly used in applications; see *e.g.*, Shi et al. (2024).

The key idea is to approximate the state sequence in (3b) using a high-dimensional linear predictor in the Koopman framework (Koopman, 1931). Define the lifted state

$$z_t = \Psi(x_t) := \operatorname{col}(\psi_1(x_t), \ldots, \psi_{n_z}(x_t)), \tag{5}$$

where $n_z \geq n$ and each $\psi_i : \mathbb{R}^n \to \mathbb{R}$ is a chosen *observable*. The dictionary Ψ typically includes the identity mapping so that x_t can be reconstructed from z_t (*i.e.*, $x_t = C z_t$) (Korda and Mezić, 2018; Strässer et al., 2024; Mamakoukas et al., 2022). We can represent the nonlinear dynamics (1) as

$$z_{t+1} = A z_t + B u_t + e(C z_t, u_t), \qquad x_t = C z_t, \tag{6}$$

where A, B, C are matrices with compatible size, and the one-step modeling error is defined as

$$e(Cz_t, u_t) = e(x_t, u_t) := \Psi(f(x_t, u_t)) - A\Psi(x_t) - Bu_t. \tag{7}$$

The modeling error depends on the choice of observables (5) and the matrices A, B. If $e \equiv 0$, we say (1) admits an *exact Koopman linear embedding* (Shang et al., 2024), leading to an exact linear predictor in the lifted Koopman space.

$$\begin{aligned} V_N^*(z) = \min_{\mathbf{u}} \quad & \sum_{k=0}^{N-1} l(Cz_{t+k}, u_{t+k}) + V_{\mathrm{f}}(z_{t+N}) & \text{(8a)} \\ \text{subject to} \quad & z_{t+k+1} = Az_{t+k} + Bu_{t+k}, & \text{(8b)} \\ & u_{t+k} \in \mathcal{U},\ Cz_{t+k} \in \mathcal{X},\ k \in \mathbb{Z}_{[0,N-1]}, & \text{(8c)} \\ & z_t = z,\ z_{t+N} \in \mathcal{Z}_{\mathrm{f}}. & \text{(8d)} \end{aligned}$$

In practice, if e is "small" locally, we can handle it within the MPC design. At each time t with an initial state $x_t = x$, we lift this initial state to the Koopman space as $z = \Psi(x)$, and replace the nonlinear optimization (3) with problem (8), where V_{f} and $\mathcal{Z}_{\mathrm{f}} \subseteq \mathbb{R}^{n_z}$ are appropriate terminal cost and terminal set to be designed. Let $\mathbf{u}_z^*$ be one optimal solution for feasible z. The Koopman MPC law is

$$\kappa(z) = \mathbf{u}_z^*(0), \quad \forall z \in \mathrm{dom}(V_N^*), \tag{9}$$

and the closed-loop dynamics of the original nonlinear system (1) with the control law (9) becomes

$$x_{t+1} = f(x_t, \kappa(\Psi(x_t))). \tag{10}$$

The basic Koopman MPC formulation was first proposed in Korda and Mezić (2018), without discussing the terminal ingredients and the closed-loop stability.

2.3. Problem statement

The Koopman MPC problem (8) is computationally efficient because the predictor (8b) is *linear* in $\mathbf{u}$. The lifted linear model matrices A, B, C in (6) can be identified from data via EDMD and related methods (Williams et al., 2015). Some recent works (de Jong et al., 2024; Zhang et al., 2022; Mamakoukas et al., 2022; Worthmann et al., 2024) have addressed constraint satisfaction and (practical) asymptotic stability of the closed-loop system (10).

In this paper, we revisit the *exponential stability* of (10) with the Koopman MPC law, and discuss the design of the terminal cost V_{f} and terminal set $\mathcal{Z}_{\mathrm{f}}$ in (8). Unlike de Jong et al. (2024); Zhang et al. (2022); Mamakoukas et al. (2022); Worthmann et al. (2024), we assume the lifting observable (5) is given and a linear model A, B, C in (6) has been estimated from data. This isolates the core stability questions of the Koopman MPC from the complexity arising in the identification procedure. Our idea follows the standard stabilizing MPC for linear systems (Rawlings et al., 2017, Ch. 2), adapted carefully to the Koopman-lifted space while accounting for the error in (6). Besides the terminal ingredients V_{f} and $\mathcal{Z}_{\mathrm{f}}$, the lifting function (5) and the modeling error (6) also affects the closed-loop stability in (10). We will compare with the existing Koopman stability results in Remark 2. We note that we relax the state constraint $\mathcal{X}$ in (8) for simplicity in the remainder of this work. When $\mathcal{X}$ contains a neighborhood of the origin, satisfaction of the state constraint can be established through a more involved discussion, which we defer to future work.

Throughout this paper, for a state x and policy $u = \pi(x)$, we denote the successor of system (1) by x^+, with lifted state $z^+ := \Psi(x^+)$. The one-step Koopman prediction is denoted by $\bar{z}^+ := A\,\Psi(x) + B\,\pi(x)$. Note that $z^+ \neq \bar{z}^+$ unless $e \equiv 0$ in (7). We make another assumption, similar to (de Jong et al., 2024; Zhang et al., 2022; Mamakoukas et al., 2022).

Assumption 2 *The lifting function $\Psi : \mathbb{R}^n \to \mathbb{R}^{n_z}$ is continuous, with $\Psi(\mathbb{0}) = \mathbb{0}$, and $x = C\Psi(x)$ where $C \in \mathbb{R}^{n \times n_z}$. In (6), the pair (A, B) is stabilizable and the pair (A, C) is observable.*

The conditions on the lifting function can typically be enforced by construction. For the Koopman model (*i.e.*, (A, B, C)), existing work has proposed a data-driven identification procedure that enforces open-loop stability of A (Mamakoukas et al., 2023), while guaranteeing the stabilizability of (A, B) with potentially unstable A remains an open problem. We note that assuming (A, C) to be observable is without loss of generality, as any unobservable modes can be removed through an observability decomposition (Shang et al., 2026).

3. Infinite-horizon Koopman LQR

This section discusses an infinite-horizon Koopman LQR problem. The results will facilitate the design of the terminal set and terminal cost for stabilizing Koopman MPC in the next section.

3.1. Koopman optimal control

$$\begin{aligned}\min_{\mathbf{u}_\infty} \quad & \sum_{k=0}^{\infty} l(x_{t+k}, u_{t+k}) \\ \text{subject to} \quad & x_{t+k+1} = f(x_{t+k}, u_{t+k}), \\ & x_t = x, k \in \mathbb{Z}_{\geq 0}.\end{aligned} \tag{11}$$

$$\begin{aligned}\min_{\mathbf{u}_\infty} \quad & \sum_{k=0}^{\infty} l(Cz_{t+k}, u_{t+k}) \\ \text{subject to} \quad & z_{t+k+1} = Az_{t+k} + Bu_{t+k}, \\ & z_t = z, k \in \mathbb{Z}_{\geq 0}.\end{aligned} \tag{12}$$

Consider the infinite-horizon optimal control problem (11). We denote $\mathbf{u}_\infty := (u_t, u_{t+1}, \ldots)$ as the control sequence, $x_t = x$ as the initial state at time t. The stage cost l is defined in (4). Compared with the MPC (3), this formulation (11) has an infinite horizon and no state/input constraints.

This nonlinear optimal control problem (11) is generally hard to solve. We can utilize a Koopman linear model to approximate the nonlinear dynamics in (11), leading to the Koopman LQR problem (12), where $z = \Psi(x)$. When the Koopman linear model is exact, we can obtain an *explicit* optimal policy for (11).

Lemma 2 *Suppose there exists a Koopman lifting (5) such that $e(Cz_t, u_t) \equiv 0$ in (7), and Assumptions 1 and 2 hold. Then, problem (11) has a globally optimal feedback policy*

$$u_{t+k} = Kz_{t+k} = K\Psi(x_{t+k}), \quad k \in \mathbb{Z}_{\geq 0} \tag{13}$$

where K is the optimal LQR feedback gain for (12) associated with A, B, C, Q, R, i.e., $K = -(R + B^\top PB)^{-1}B^\top PA$, *with P be the unique positive definite solution to the Riccati equation*

$$P = C^\top QC + A^\top PA - A^\top PB(R + B^\top PB)^{-1}B^\top PA. \tag{14}$$

The proof is not difficult as (12) is a standard LQR problem, and we provide some details in Appendix A.2. Note that the optimal policy (13) is linear in the lifted Koopman space, but remains nonlinear in the original state space. The associated optimal value function for (12) is $V_\infty^*(z) = \|z\|_P^2$. While Lemma 2 is not difficult to establish, it actually gives a globally optimal (nonlinear) policy to a class of nonlinear control problems (11). We can further show that the nonlinear feedback law (13) *globally asymptotically* stabilizes the nonlinear system (1) if there is no modeling error.

Theorem 1 *Under the same conditions of Lemma 2, consider the feedback law (13). The closed-loop system $x_{t+1} = f(x_t, K\Psi(x_t))$ of (1) is globally asymptotically stable.*

The proof needs to establish both globally attractive and locally Lyapunov stable properties. Establishing the global attractiveness is easy, but the stability in the sense of Lyapunov requires additional arguments. Due to page limit, we present the proof details in Appendix A.3.

Theorem 1 guarantees only *global asymptotic* stability of the nonlinear closed loop. Although the lifted linear system $z_{t+1} = (A + BK)z_t$ is globally exponentially stable, this does *not* in general imply global exponential stability of the physical state x_t. The reason is that the exponential decay of z_t does not automatically transfer to $x_t = Cz_t$ unless the lifting Ψ satisfies additional regularity properties. For example, $x_t = Cz_t = 0$ does not imply $z_t = 0$; some components of z_t in $\ker C$ may be nonzero yielding $x_{t+1} \neq 0$ with the propagation of the dynamics.

3.2. Local exponential stability of Koopman LQR

Under additional conditions, the closed-loop system with the controller (13) is *locally exponentially stable* even with Koopman modeling error. In the following, we denote $\mathcal{B}_r = \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$.

Assumption 3 *There exists constants $r_\psi > 0$ and $L_\psi > 0$ such that the lifting function $\Psi : \mathbb{R}^n \to \mathbb{R}^{n_z}$ satisfies $\|\Psi(x)\| \le L_\psi \|x\|, \ \forall x \in \mathcal{B}_{r_\psi}$.*

Assumption 4 *There exist constants $r > 0$ and $L > 0$ such that the closed-loop one-step prediction error $\bar{e}_{\text{cl}}(x_t) := \Psi(f(x_t, K\Psi(x_t)) - A\Psi(x_t) - BK\Psi(x_t)$ satisfies $\|\bar{e}_{\text{cl}}(x)\| \le L\|x\|, \ \forall x \in \mathcal{B}_r$.*

Assumption 3 means that the lifting function is locally bounded with respect to the physical state x, which holds for all locally Lipschitz continuous functions (note that $\Psi(\mathbb{0}) = \mathbb{0}$ in Assumption 2). Assumption 4 requires that the one-step closed-loop prediction is sufficiently accurate.

Theorem 2 *Suppose Assumptions 1 to 4 hold. Consider the feedback law (13). There exists $\delta > 0$, such that if L in Assumption 4 satisfies $L < \delta$, the closed-loop system $x_{t+1} = f(x_t, K\Psi(x_t))$ of (1) is locally exponentially stable.*

The key idea is to show that the optimal value function for (12) *i.e.*, $\tilde{V}(x) := \Psi(x)^{\mathsf{T}} P\Psi(x)$, is a valid Lyapunov function for the nonlinear system (1). In other words, we show that $\tilde{V}(x)$ satisfies Lemma 1. To prove this, one key step is to decompose the one-step Lyapunov change $\tilde{V}(x^+) - \tilde{V}(x)$ as a *Koopman decrease* induced by the Koopman linear model and a *perturbation* resulting from the Koopman prediction error. The Riccati identity ensures a substantial Koopman decrease, while Assumptions 3 and 4 together guarantee the perturbation can be made strictly smaller than the Koopman decrease. The proof details are given in Appendix A.4.

We conclude this section with a simple example of Koopman LQR applied to a nonlinear system.

Example 1 *Consider the nonlinear system*

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ -5x_1^2 \end{bmatrix}.$$

We compare LQR controllers designed from (i) the exact lifted Koopman model and (ii) the local linearization, using $Q = \operatorname{diag}(1,1)$ and $R = 1$. The exact *lifted-Koopman linear representation with the lifting function $\Psi(x) = \operatorname{col}(x_1, x_2, x_1^2)$ and the first-order (Taylor) linearization at the origin are*

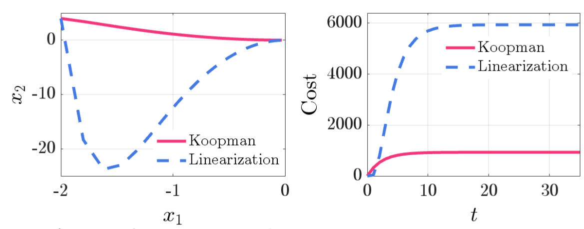

Figure 1: LQR performance: exact Koopman model vs. first-order (Taylor) linearization. Left: closed-loop trajectory; right: accumulated cost.

$$\text{Koopman: } \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix}^+ = \begin{bmatrix} 0.9 & 0 & 0 \\ 0 & 1.5 & -5 \\ 0 & 0 & 0.81 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u, \quad \text{Taylor: } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} 0.9 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

As illustrated in Fig. 1, the Koopman LQR yields faster convergence with much better transient behavior than the linearized LQR. This is because the lifted model captures the quadratic coupling exactly; by Lemma 2, the Koopman LQR gives the optimal solution of (11) in this instance. □

4. Stabilizing Koopman MPC

It is well known that *terminal ingredients* are crucial for closed-loop stability in MPC. In this section, we design the terminal ingredients for Koopman MPC using the Koopman LQR results of Section 3.

4.1. Design of the terminal ingredients

We construct the terminal cost and terminal set *in the lifted space* using the Koopman LQR. In the following, we assume Assumptions 1 and 2 hold unless stated otherwise.

- **Terminal cost**: Let K be the optimal Koopman LQR gain from (13) and $A_K := A + BK$. We choose matrix $\hat{Q} \succ 0$ such that $\hat{Q} \succeq C^{\mathsf{T}}QC + K^{\mathsf{T}}RK$. We design the terminal cost
$$V_{\mathrm{f}}(z) := z^{\mathsf{T}}\hat{P}z \leq \sigma_{\hat{P}}\|z\|^2, \tag{15}$$
where $\hat{P} \succ 0$ is the solution to the Lyapunov equation $A_K^{\mathsf{T}}\hat{P}A_K - \hat{P} + \hat{Q} = 0$ (note that A_K is Shur stable) and $\sigma_{\hat{P}}$ is its maximum eigenvalue.
- **Terminal set**: Let σ_R be the maximum eigenvalue of R. The terminal set is designed as
$$\mathcal{Z}_{\mathrm{f}} := \{z \in \mathbb{R}^{n_z} \mid V_{\mathrm{f}}(z) \leq \tau\}, \tag{16}$$
where $\tau > 0$ is chosen such that $\sqrt{\frac{\tau}{\sigma_R}}\mathcal{B}_1 \subseteq \mathcal{U}$. The existence of τ is guaranteed as $\mathbb{0} \in \mathrm{int}(\mathcal{U})$.

With these terminal ingredients, the Koopman LQR controller $u_t = Kz_t = K\Psi(x_t)$ can ensure a sufficient decrease in terminal cost at each step and guarantee that the terminal set $\mathcal{Z}_{\mathrm{f}}$ is invariant for the nominal Koopman linear model.

Proposition 1 (Terminal Controller): *Consider the terminal cost V_{f} in (15) and terminal set $\mathcal{Z}_{\mathrm{f}}$ in (16). For the terminal controller $\kappa_{\mathrm{f}}(z) := Kz$ with K from (13), we have $\kappa_{\mathrm{f}}(z) \in \mathcal{U}$, $\forall z \in \mathcal{Z}_{\mathrm{f}}$, and*
$$V_{\mathrm{f}}(\bar{z}^+) - V_{\mathrm{f}}(z) \leq -l(Cz, \kappa_{\mathrm{f}}(z)), \quad \forall z \in \mathcal{Z}_{\mathrm{f}}, \tag{17}$$
where $\bar{z}^+ = Az + B\kappa_{\mathrm{f}}(z)$ is the one-step Koopman prediction.

This is a standard result in linear MPC, and we provide a proof in Appendix A.5 for completeness.

4.2. Asymptotic stability for an exact Koopman model

Since the Koopman MPC problem (8) is designed using the Koopman linear model $z^+ = Az + Bu$, it naturally inherits key properties of standard MPC for linear systems (Rawlings et al., 2017), such as boundedness, decrease of the value function, and recursive feasibility of the Koopman update.

At time step t, given a sequence of control inputs $\mathbf{u} := (u_t, u_{t+1}, \ldots, u_{t+N-1})$ and an initial state $z_t = z$, we denote the nominal Koopman state prediction at time $t + k$ as $\phi(k; z, \mathbf{u}) := z_{t+k}$, where $k \in \mathbb{Z}_{[0,N]}$. We also denote the objective value of (8) with initial condition z and input sequence $\mathbf{u}$ as $V_N(z, \mathbf{u})$ and its feasible region as $\mathcal{Z}_N := \{z \in \mathbb{R}^{n_z} | \exists \mathbf{u} \in \mathcal{U}^N$ such that $\phi(N; z, \mathbf{u}) \in \mathcal{Z}_{\mathrm{f}}$ and $C\phi(k; z, \mathbf{u}) \in \mathcal{X}, k = 0, \ldots, N-1\}$, where $\mathcal{U}^N := \mathcal{U} \times \cdots \times \mathcal{U}$. The following properties are standard in stabilizing linear MPC; see (Rawlings et al., 2017, Ch. 2.4) for details.

Proposition 2 (Continuity and boundedness): *Consider the Koopman MPC problem (8) with terminal design $V_{\mathrm{f}}(\cdot)$ in (15) and $\mathcal{Z}_{\mathrm{f}}$ in (16). Let V_N^* denote its optimal value function. Then, we have:*

1. *$\mathbb{0} \in \mathrm{int}(\mathcal{Z}_N)$ and V_N^* is continuous on the interior of its domain $\mathcal{Z}_N$;*
2. *V_N^* is bounded with respect to z, Cz, and the optimal control sequence $\mathbf{u}_z^*$:*
$$\lambda_Q\|Cz\|^2 \leq V_N^*(z) \leq c_z\|z\|^2, \quad \lambda_R\|\mathbf{u}_z^*(0)\|^2 \leq \lambda_R\|\mathbf{u}_z^*\|^2 \leq V_N^*(z), \quad \forall z \in \mathcal{Z}_N,$$
where c_z is a positive constant and $\lambda_Q, \lambda_R \in \mathbb{R}$ are the minimum eigenvalues of Q and R.

Proposition 3 (Recursive feasibility and cost decrease): *Consider the one-step Koopman prediction $\bar{z}^+ = Az + B\kappa(z)$, where $\kappa(z)$ is the Koopman MPC law (9). Then, we have:*

1. *$\bar{z}^+ \in \mathcal{Z}_N$,* i.e., *there exists a feasible input sequence to (8) with initial state $\bar{z}^+$. One such feasible choice is the one-step shifted input sequence $\bar{\mathbf{u}} := (\mathbf{u}_z^*(1 : N-1), \kappa_{\mathrm{f}}(\phi(N; z, \mathbf{u}_z^*)))$;*
2. *The optimal value function V_N^* decreases at the next Koopman prediction $\bar{z}^+$,* i.e.,
$$V_N^*(\bar{z}^+) \leq V_N(\bar{z}^+, \bar{\mathbf{u}}) \leq V_N^*(z) - \lambda_Q\|Cz\|^2, \quad \forall z \in \mathcal{Z}_N. \tag{18}$$

These properties are closely related to the requirements on a Lyapunov function in Lemma 1. However, we note that the optimal value function V_N^* only has a decrease in terms of $\|Cz\|^2$, instead of $\|z\|^2$. If the Koopman model is exact, the next result shows that the Koopman MPC asymptotically stabilizes the physical state x of the original nonlinear system (1).

Theorem 3 *Suppose* $e(Cz_t, u_t) \equiv \mathbb{0}$ *in* (7) *and Assumptions 1 and 2 hold. Consider the Koopman MPC* (8) *with terminal design* $V_\mathrm{f}(\cdot)$ *in* (15) *and* $\mathcal{Z}_\mathrm{f}$ *in* (16)*. Then,*

1. *The entire feasible region of the Koopman MPC* (8)*,* $\mathcal{X}_N := \{x \in \mathbb{R}^n \mid \Psi(x) \in \mathcal{Z}_N\}$*, is an ROA of the closed-loop system* $x_{t+1} = f(x_t, \kappa(\Psi(x_t)))$*;*
2. *The closed-loop system is also locally asymptotically stable at the origin.*

We prove that $\mathcal{X}_N$ is a region of attraction via showing it is invariant and the running cost of MPC with the optimal Koopman control law is finite. For the Lyapunov stability part, the proof of Theorem 3 is similar to that in Theorem 1. The detailed proof is shown in Appendix A.6.

Analogously to Theorem 1, Assumptions 1 and 2 do not guarantee *local exponential stability* of the closed-loop Koopman MPC system, and additional regularity of the lifting Ψ is required.

Remark 1 (Enlarging ROA via Koopman MPC): *With state and input constraints, the MPC* N*-step feasible set* $\mathcal{X}_N := \{x \in \mathbb{R}^n \mid \Psi(x) \in \mathcal{Z}_N\}$ *includes the ROA from the Koopman LQR control law because* $\mathcal{Z}_\mathrm{f} \subseteq \mathcal{Z}_N$*. Consequently, Koopman-MPC naturally* enlarges the ROA *of the nonlinear closed-loop system while enforcing constraints. This is in direct analogy with standard stabilizing MPC for linear systems; see (Rawlings et al., 2017, Ch. 2.5).* □

4.3. Local exponential stability of stabilizing Koopman MPC

We show the Koopman MPC locally exponentially stabilizes the nonlinear system under suitable assumptions, even when the Koopman model is inaccurate. Similar to Assumption 4, we make the following assumption for the prediction error.

Assumption 5 *There exists constants* $r > 0$ *and* $L > 0$*, such that the closed-loop one-step prediction error* $e_\mathrm{cl}(x)$ *in* (7) *with the Koopman MPC law* (9) *satisfies*

$$\|e_\mathrm{cl}(x)\| \le L\|x\|, \quad \forall x \in \mathcal{B}_r. \tag{19}$$

Let $\rho \in (0, \lambda_Q \hat{r}^2]$ where $\hat{r} := \min\{r_\psi, r\}$ with r_ψ and r from Assumption 3 and Assumption 5 respectively. We choose the sublevel set

$$\mathcal{S} := \{x \in \mathcal{X}_N \mid V_N^*(\Psi(x)) < \rho\}. \tag{20}$$

This set $\mathcal{S}$ is bounded because of $\lambda_Q\|x\|^2 < \rho$ from Proposition 2, which also implies $\mathcal{S} \subseteq \mathcal{B}_{\hat{r}} \subseteq \mathcal{B}_{r_\psi} \cap \mathcal{B}_r$. Furthermore, $\mathbb{0}$ is an interior point of $\mathcal{S}$ because of $V_N^*(\Phi(\mathbb{0})) = V_N^*(\mathbb{0}) = 0$ and $V_N^*(\Psi(x))$ is continuous (see Assumption 2 and Proposition 2). We will prove that $\mathcal{S}$ is an ROA for the closed-loop system when the Koopman prediction error is sufficiently small.

We first prove the one-step feasibility of the Koopman MPC (8) over the entire $\mathcal{S}$.

Lemma 3 *Suppose Assumptions 1 to 3 and 5 hold. Fix* $\rho \in (0, \lambda_Q \hat{r}^2]$*. Then, there exists* $\delta_1 > 0$ *such that, if* L *in* (19) *satisfies* $L \le \delta_1$*, the Koopman MPC* (8) *is feasible at* $\Psi(x^+)$ *for all* $x \in \mathcal{S}$*.*

Similar to Proposition 3, this result guarantees the feasibility of (8) for the Koopman lifting of the true physical state x^+. The proof is constructive by showing that $\bar{\mathbf{u}}$ in Proposition 3 is a feasible solution with sufficiently small δ_1. The details are given in Appendix A.7.

Lemma 3 guarantees $V_N^*(\Psi(x^+))$ is well-defined for $x \in \mathcal{S}$. We next prove that $V_N^*(\Psi(x^+)) < V_N^*(\Psi(x))$ when $x \in \mathcal{S} \setminus \{0\}$ and the constant L in (19) is sufficiently small (see Appendix A.8).

Lemma 4 *Suppose Assumptions 1 to 3 and 5 hold. Fix* $\rho \in (0, \lambda_Q \hat{r}^2]$*. Then, there exists* $c, \delta_2 > 0$*, such that if* L *in* (19) *satisfies* $L < \delta_2 \le \delta_1$*, we have*

$$V_N^*(\Psi(x^+)) - V_N^*(\Psi(x)) \le -c\|x\|^2, \ \forall x \in \mathcal{S}.$$

The proof of this result is similar to the decomposition of the Lyapunov change in Theorem 2. The one-step feasibility in Lemma 3 and the descent property in Lemma 4 can hold for any state in $\mathcal{S}$. This leads to the recursive feasibility and local exponential stability of the closed-loop system.

Theorem 4 *Suppose Assumptions 1 to 3 and 5 hold. Fix a* $\rho \in (0, \lambda_Q \hat{r}^2]$ *and define the sublevel set* $\mathcal{S}$ *in (20). There exists* $\delta > 0$ *such that if* L *in (19) satisfies* $L \leq \delta$*, the closed-loop Koopman MPC system (10) is locally exponentially stable and the set* $\mathcal{S}$ *is an ROA of it.*

Proof Consider the optimal value function $V_N^*(\Psi(\cdot))$ as a Lyapunov candidate. We next prove that 1) the set $\mathcal{S}$ is invariant, and 2) $V_N^*(\Psi(x))$ satisfies the conditions (2a) and (2b) over $\mathcal{S}$ in Lemma 1.

We can choose $\delta < \delta_2 \leq \delta_1$, where δ_1 and δ_2 come from Lemma 3 and Lemma 4. Then, from Lemma 4, we have the following decent property

$$V_N^*(\Psi(x^+)) - V_N^*(\Psi(x)) \leq -c\|x\|^2, \ \forall x \in \mathcal{S}, \tag{21}$$

which implies $V_N^*(\Psi(x^+)) \leq V_N^*(\Psi(x)) < \rho$ and $x^+ \in \mathcal{S}$. This guarantees that 1) the set $\mathcal{S}$ is invariant and 2) the Koopman MPC (8) is recursive feasible.

Furthermore, for any x in $\mathcal{S}$, we have

$$\lambda_Q\|x\|^2 \leq V_N^*(\Psi(x)) \leq c_z\|\Psi(x)\|_2^2 \leq c_z L_\psi^2 \|x\|_2^2, \tag{22}$$

where the last inequality comes from Assumption 3. Inequalities (21) and (22) confirm that the value function $V_N^*(\Psi(x))$ satisfies (2a) and (2b) in Lemma 1. This completes the proof. ■

The proof of Theorem 4 closely parallels that of Theorem 2. The theoretical upper bound δ on the error coefficient L provides a sufficient condition on the model mismatch for guaranteeing stability. It depends on all problem data in the Koopman MPC (8) as well as the set $\mathcal{S}$. While L can be evaluated *a posteriori* once the Koopman model and $\mathcal{S}$ are fixed, enforcing $L \leq \delta$ during identification (*e.g.*, as an explicit constraint in EDMD) is generally nontrivial. A convenient sufficient surrogate is a Taylor-like condition on the one-step prediction error $e(\tilde{x})$, with $\tilde{x} = \mathrm{col}(x, u)$ as in (7): $\lim_{\|\tilde{x}\| \to 0} \|e(\tilde{x})\|/\|\tilde{x}\| = \mathbb{0}$. This ensures that the constant L can be chosen arbitrarily close to zero when decreasing r in Assumption 5. Thus, by shrinking the neighborhood, one can ensure the bound in Assumption 5 with arbitrarily small L, and Theorem 4 applies.

Remark 2 (Comparison with existing Koopman MPC) *Our proof strategies for Theorems 3 and 4 are similar to standard stabilizing MPC for linear systems (Rawlings et al., 2017, Chap 2). The Koopman MPC (8) can naturally stabilize the Koopman linear model* $z^+ = Az + Bu$*. With Assumption 3 and Assumption 5 on the lifting function and the one-step prediction error, the Koopman MPC (8) can also exponentially stabilize the original nonlinear system. We here compare with some existing results. Zhang et al. (2022); Mamakoukas et al. (2022) focused on ensuring constraint satisfaction and their settings are closer to the robust MPC framework. In Bold et al. (2025); Schimperna et al. (2025), a variant of Koopman MPC is designed without terminal ingredients, but based on a cost controllability assumption of the nonlinear system. The closest studies in the literature are Worthmann et al. (2024); de Jong et al. (2024), where the Koopman MPC formulations also include suitable terminal ingredients. In particular, Worthmann et al. (2024) mainly focused on a Koopman bilinear model and its terminal ingredients are constructed based on the original nonlinear system. In de Jong et al. (2024), the initial condition is interpolated, which may lead to an unbounded prediction error. Both Worthmann et al. (2024) and de Jong et al. (2024) only show practical asymptotic stability. Instead, our Theorem 4 establishes the exponential stability of* `S-KMPC`*. This result only requires one assumption for the actual nonlinear system in Assumption 1 and several mild assumptions for the Koopman linear model in Assumptions 2 to 5.* □

5. Numerical experiments

In this section, we illustrate the performance of `S-KMPC` on a standard inverted pendulum (Strässer et al., 2024; Zhang et al., 2022). We compare `S-KMPC` with a standard linear MPC (`L-MPC`) based on a first-order Taylor linearization at the origin.

5.1. Experiment setup

We consider an inverted pendulum with dynamics

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^+ = \begin{bmatrix} 1 & T_s \\ 0 & 1 - \frac{bT_s}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{T_s}{ml^2} \end{bmatrix} u + \begin{bmatrix} 0 \\ \frac{gT_s}{l} \sin(x_1) \end{bmatrix} \tag{23}$$

where the parameters are $m = 1\,\text{kg}$, $l = 1\,\text{m}$, $b = 0.2$, $g = 9.81\,\text{m/s}^2$, and $T_s = 0.02\,\text{s}$.

A Koopman predictor is learned via EDMD using the dictionary $\Psi(x) = \text{col}(x_1, x_2, \sin x_1)$. Specifically, we generate 200 trajectories of length 1000; initial states and inputs are sampled from $[-2, 2] \times [-8, 8]$ and $[-40, 40]$, respectively. The identified model that is projected to satisfy Assumptions 2 and 5 which makes the derivative of the one-step prediction error in (7) become $\mathbb{0}$ at the origin (see the discussion after Theorem 4), and the Taylor linearization at the origin are

$$\textbf{Koopman: } A = \begin{bmatrix} 1 & 0.02 & 0 \\ 0 & 0.996 & 0.1962 \\ 0.002 & 0.02 & 0.998 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.02 \\ 0 \end{bmatrix}, \textbf{ Taylor: } A = \begin{bmatrix} 1 & 0.02 \\ 0.1962 & 0.996 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 0.02 \end{bmatrix}.$$

Both controllers use the stage cost (4) with $Q = 10I$ and $R = I$, and a prediction horizon $N = 20$.

5.2. Closed-loop performance

We compare S-KMPC and L-MPC on multiple initial conditions. Both controllers employ terminal ingredients designed according to Section 4.1. Figure 2 shows phase portraits and accumulated costs. For all initial states, both controllers can stabilize the pendulum, and the accumulated costs converge to steady values. However, as the initial state moves farther from the origin, L-MPC exhibits larger state deviation than S-KMPC and consequently higher accumulated cost. Relative to S-KMPC, the cost increase of L-MPC is approximately 0.13%, 2.2%, and 9.2% for trajectories 1-3, growing with the initial distance to the origin.

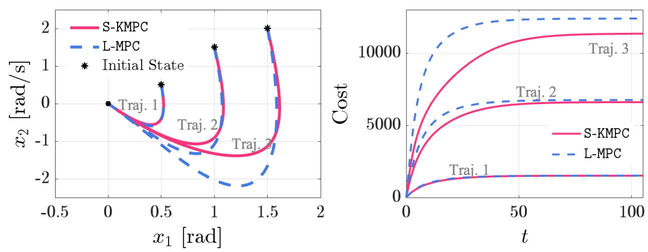

Figure 2: MPC performance for inverted pendulum: approximated Koopman model vs. first-order (Taylor) linearization. Left: closed-loop trajectory; right: accumulated cost.

The rationale for this behavior is intuitive. For small deviations, both the Koopman model and the Taylor model are relatively accurate, leading to similar performance. As the initial state moves away from the linearization point, the Taylor predictor incurs larger modeling error, degrading the performance L-MPC. In principle, the EDMD-based Koopman predictor may remain reliable over a larger region, and thus S-KMPC maintains better trajectories in the transient and lower accumulated costs. These results are consistent with the simulation in Example 1, cf. Figure 1.

6. Conclusions

We have revisited the local exponential stability of Koopman MPC. In particular, we have presented a stabilizing Koopman MPC variant, where the terminal cost and terminal set are designed based on Koopman LQR. We have shown that the nonlinear closed-loop is locally asymptotically stable when the lifting function and one-step Koopman prediction error are both Lipschitz at the origin, with the Lipschitz constants being small. Numerical simulations confirm the superior performance of the S-KMPC, which has a faster convergence rate and a lower accumulated cost. Some future directions include developing data-driven identification methods that can guarantee the satisfaction of the required assumptions, extending the stability analysis to systems subject to external disturbances, and validating the performance of S-KMPC in practical nonlinear systems (*e.g.*, mixed traffic systems (Shang et al., 2025b) and robotic systems (Haggerty et al., 2023)).

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Appendix A. Technical proofs

We here present technical proofs for Lemma 1 and the results in Sections 3 and 4.

A.1. Proof of Lemma 1

Following Definition 1, we here demonstrate that the Lyapunov function V satisfying the conditions in Lemma 1 leads to local exponential stability.

Let us recall standard definitions for an invariant set and region of attraction.

Definition 2 (Positive invariant set and region of attraction Rawlings et al. (2017)) *Consider the autonomous system $x_{t+1} = g(x_t)$.*

1. *A set $\mathcal{D}$ is positive invariant if $x_t \in \mathcal{D}$ implies $x_{t+1} \in \mathcal{D}$.*
2. *A set $\mathcal{D}$ of initial states x_0 with $\lim_{t\to\infty} x_t = \mathbb{0}$ is a region of attraction for the origin $\mathbb{0}$.*

Proof of Lemma 1: From the condition (2b), we can obtain

$$\begin{aligned} V(x_{t+1}) &\leq V(x_t) - \alpha_3 \|x_t\|^2 \\ &\leq (1 - \alpha_3/\alpha_2) V(x_t), \quad \forall x_t \in \mathcal{D}, \end{aligned}$$

which implies $V(x_t) \leq (1-\alpha_3/\alpha_2)^t V(x_0)$. Since the set $\mathcal{D}$ is invariant, the function value $V(x_{t+1})$ is always well-defined when $x_t \in \mathcal{D}$.

Then, using condition (2a), we can write

$$\begin{aligned} \|x_t\| &\leq \sqrt{V(x_t)/\alpha_1} \\ &\leq \sqrt{(1-\alpha_3/\alpha_2)^t/\alpha_1 V(x_0)} \\ &\leq \sqrt{\alpha_2/\alpha_1 (1-\alpha_3/\alpha_2)^t} \|x_0\|, \quad \forall x_0 \in \mathcal{D}. \end{aligned}$$

This implies that states in $\mathcal{D}$ converge to the origin exponentially and $\mathcal{D}$ is an ROA.

Let $c := \sqrt{\alpha_2/\alpha_1}$ and $\rho := \sqrt{1-\alpha_3/\alpha_2}$. We note that there exists a neighborhood $\mathcal{N} \subseteq \mathcal{D}$ as $\mathbb{0} \in \mathrm{int}(\mathcal{D})$. Thus, we have $\|x_t\| \leq c\rho^t \|x_0\|, \forall x_0 \in \mathcal{N}, t \in \mathbb{Z}_{\geq 0}$, which completes the proof.

A.2. Proof of Lemma 2

Thanks to the exact Koopman linear embedding, we can equivalently rewrite the nonlinear dynamics in (11) in the Koopman-lifted space as (we restate (12) here)

$$\begin{aligned} \min_{\mathbf{u}_\infty} \quad & \sum_{k=0}^{\infty} l(Cz_{t+k}, u_{t+k}) \\ \text{subject to} \quad & z_{t+k+1} = Az_{t+k} + Bu_{t+k}, \\ & z_t = z, k \in \mathbb{Z}_{\geq 0}, \end{aligned}$$

where $z = \Psi(x)$ and we have $x_{t+k} = Cz_{t+k}, k \in \mathbb{Z}_{\geq 0}$. For a fixed $x \in \mathbb{R}^n$, the cost values in (11) and (12) are the same given the same input sequence $\mathbf{u}_\infty$.

Since (12) can be viewed as a standard LQR problem with an initial condition $z_t = \Psi(x_t)$ under Assumption 2, the controller (13) is an optimal feedback policy to (12). Therefore, (13) is also a globally optimal feedback controller to the original problem (11).

A.3. Proof of Theorem 1

We need to show that: 1) the system is globally attractive, that is, $\lim_{t\to\infty} x_t = 0,\ \forall x_0 \in \mathbb{R}^n$ and 2) the system is locally Lyapunov stable, that is, for any $\epsilon > 0$, there exists $\delta > 0$, such that $\|x_0\| < \delta$ implies $\|x_t\| < \epsilon$, for all $t \in \mathbb{Z}_{\geq 0}$.

We recall the associated optimal value function for (12) is $V_\infty^*(z) = \|z\|_P^2 := z^\mathsf{T} P z$ and we have the following lower and upper bounds:

$$\lambda_Q \|Cz\|^2 \leq V_\infty^*(z) \leq \sigma_P \|z\|^2, \tag{24}$$

where λ_Q is the minimum eigenvalue of Q, and σ_P is the maximum eigenvalue of P (see (14)).

The globally attractive property is easy to show. Since the optimal LQR gain can stabilize the exact Koopman linear model, we have $\lim_{t\to\infty} z_t = 0, \forall z_0 \in \mathbb{R}^{n_z}$ which implies $\lim_{t\to\infty} \Psi(x_t) = 0, \forall x_0 \in \mathbb{R}^n$. As $x_t = C\Psi(x_t)$ by Assumption 2, the physical state x_t of the nonlinear system converges to the origin asymptotically.

We next show the local stability. For any fixed $\epsilon > 0$, let $\rho = \lambda_Q \epsilon^2$. Consider the set

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid V_\infty^*(\Psi(x)) < \rho\}.$$

Using the Riccati equation, for any $x \in \mathcal{S}$, we have

$$V_\infty^*(\Psi(x^+)) - V_\infty^*(\Psi(x)) = -\Psi(x)^\mathsf{T}(C^\mathsf{T} Q C + K^\mathsf{T} R K)\Psi(x) \leq 0.$$

This implies that $V_\infty^*(\Psi(x^+)) \leq V_\infty^*(\Psi(x)) < \rho$, and thus we have $x^+ \in \mathcal{S}$. Therefore, $\mathcal{S}$ is an invariant set.

On the other hand, from (24), we have

$$\lambda_Q \|x\|^2 \leq V_\infty^*(\Psi(x)) < \rho \qquad \Rightarrow \qquad \|x\| < \epsilon, \forall x \in \mathcal{S},$$

meaning that $\mathcal{S} \subseteq \mathcal{B}_\epsilon$. Since $V_\infty^*(\Psi(x))$ is continuous, we know that the set $\mathcal{S}$ is open. In addition, we have $V(\Psi(\mathbb{0})) = V(\mathbb{0}) = \mathbb{0}$, implying $\mathbb{0} \in \mathcal{S}$.

Thus, there exists a neighbor $\mathcal{N} \subset \mathcal{S}$ of the origin such that any trajectory starting from $\mathcal{N}$ remains in $\mathcal{S}$. It is also contained in $\mathcal{B}_\epsilon$. This establishes the stability in the sense of Lyapunov. We now complete the proof.

A.4. Proof of Theorem 2

Consider a Lyapunov candidate

$$\tilde{V}(x) = V_\infty^*(\Psi(x)) := \Psi(x)^\mathsf{T} P \Psi(x), \tag{25}$$

where $P \succ 0$ is the unique solution to (14). We show that $\tilde{V}$ satisfies (2a) and (2b) in Lemma 1 over an invariant set S.

Let $\rho \in (0, \lambda_Q \hat{r}^2]$, where $\hat{r} := \min\{r_\psi, r\}$ with r_ψ and r from Assumption 3 and Assumption 4 respectively. We choose the sublevel set

$$\mathcal{S} := \{x \in \mathbb{R}^n \mid \tilde{V}(x) < \rho\}.$$

Note that $\mathbb{0} \in \mathcal{S}$ and $\mathcal{S}$ is open as $\tilde{V}$ is continuous. We will prove that S is bounded and invariant.

We first verify the lower bound in (2a). This directly comes from (24) as we have $x = C\Psi(x)$ by Assumption 2:

$$\lambda_Q \|x\|^2 \leq V_\infty^*(\Psi(x)) = \tilde{V}(x), \quad \forall x \in \mathcal{S}. \tag{26}$$

By our choice $\rho \leq \lambda_Q \hat{r}^2$, we have $\lambda_Q \|x\|^2 \leq \tilde{V}(x) < \rho \Rightarrow \|x\| < \hat{r}$, for all $x \in \mathcal{S}$. Thus, we have $\mathcal{S} \subseteq \mathcal{B}_{\hat{r}}$, which is bounded.

The upper bound in (2a) is ensured by Assumption 3. As $\mathcal{S} \subseteq \mathcal{B}_{\hat{r}} \subseteq \mathcal{B}_{r_\psi}$, we have

$$\tilde{V}(x) \leq \sigma_P \|\Psi(x)\|^2 \leq \sigma_P L_\psi^2 \|x\|^2, \quad \forall x \in \mathcal{S}. \tag{27}$$

Combining (26) with (27), our Lyapunov candidate $\tilde{V}$ satisfies (2a) with $\alpha_1 = \lambda_Q$ and $\alpha_2 = \sigma_P L_\psi^2$.

We next verify the decrease condition (2b). We see that

$$\begin{aligned}\tilde{V}(x^+)-\tilde{V}(x) &= V^*_\infty(\Psi(x^+))-V^*_\infty(\Psi(x)) \\ &= \underbrace{V^*_\infty(\Psi(x^+))-V^*_\infty(\bar{z}^+)}_{\text{Koopman error}}+\underbrace{V^*_\infty(\bar{z}^+)-V^*_\infty(z)}_{\text{Koopman decrease}},\end{aligned} \tag{28}$$

where we denote z as $\Psi(x)$ and $\bar{z}^+ = (A + BK)z$ is the one-step ahead Koopman prediction. The Koopman decrease term in (28) can be bounded as

$$\begin{aligned}V^*_\infty(\bar{z}^+) - V^*_\infty(z) &= -z^\top(C^\top QC + K^\top RK)z \\ &\le -\lambda_Q\|x\|^2.\end{aligned} \tag{29}$$

where we have used the Riccati equation (14) and $x = Cz$.

Meanwhile, consider the Koopman error term in (28), for any $x \in \mathcal{S}$, we have

$$\begin{aligned}\|V^*_\infty(\Psi(x^+)) - V^*_\infty(\bar{z}^+)\| &= \|\bar{e}_{\text{cl}}(x)^\top P\bar{e}_{\text{cl}}(x) + 2\bar{e}_{\text{cl}}(x)^\top P\bar{z}^+\| \\ &\le \sigma_P\|\bar{e}_{\text{cl}}(x)\|^2+2\sigma_P\|\bar{z}^+\|\|\bar{e}_{\text{cl}}(x)\| \\ &\le (\sigma_P L + 2\bar{\sigma}L_\psi\sigma_P)L\|x\|^2,\end{aligned} \tag{30}$$

where the last inequality uses Assumptions 3 and 4 (note $\mathcal{S} \subseteq \mathcal{B}_{\hat{r}} \subseteq \mathcal{B}_{r_\psi} \cap \mathcal{B}_r$), and $\bar{\sigma}$ denotes the maximum singular value of $A + BK$. We can now find $\delta > 0$ such that, when $L < \delta$, we have

$$\alpha_3 := \lambda_Q - (\sigma_P L + 2\bar{\sigma}L_\psi\sigma_P)L > 0.$$

Substituting (29) and (30) into (28), we have

$$\tilde{V}(x^+) - \tilde{V}(x) \le -\alpha_3\|x\|^2, \quad \forall x \in \mathcal{S}, \tag{31}$$

which implies $x^+ \in \mathcal{S}$ and $\mathcal{S}$ is invariant. Since the Lyapunov candidate (25) satisfies the conditions in Lemma 1, the result follows.

Remark 3 *A key step is the decomposition of the one-step Lyapunov change in (28). This is a standard strategy in showing the inherent robustness of MPC that uses a nominal model with the modelling error or external disturbance (Allan et al., 2017; Kuntz and Rawlings, 2025).*

Specifically, we first demonstrate the MPC controller can stabilize the nominal model (Koopman decrease) and then treat the modelling error as a perturbation (Koopman error). The Koopman decrease *term is certified by the Riccati identity together with the state inclusion $x = Cz$, yielding (29). The Koopman error depends on both the lifting function and the one-step prediction error, thus, we make Assumptions 3 and 4 on them to ensure the Koopman error is sufficiently small. Assumption 3 (local bound of the lifting function) is used to ensure $\|\Psi(x)\|$ is bounded by the actual state x locally, which (i) gives the* upper quadratic bound *on $\tilde{V}(x)$ and (ii) ensures $\|\bar{z}^+\| = \|(A+BK)\Psi(x)\| \le \bar{\sigma}L_\psi\|x\|$. Assumption 4 (local closed-loop prediction accuracy) bounds the* Koopman prediction-error *term as in (30), which can be made strictly smaller than the Koopman decrease by choosing L sufficiently small. Thus $L < \delta$ ensures the quadratic decay in (31) and, in turn, local exponential stability by Lemma 1.* □

A.5. Proof of Proposition 1

We recall that, from the construction of the terminal cost and terminal set, we have

$$\hat{Q} \succeq C^\top QC + K^\top RK, \tag{32a}$$

$$A_K^\top\hat{P}A_K - \hat{P} + \hat{Q} = 0. \tag{32b}$$

We first show the inequality (17) is satisfied with the proposed terminal controller. We can write

$$\begin{aligned} V_{\rm f}(\bar{z}^+) - V_{\rm f}(z) &= z^\mathsf{T}(A_K^\mathsf{T}\hat{P}A_K - \hat{P})z = -z^\mathsf{T}\hat{Q}z \\ &\leq -z^\mathsf{T}(C^\mathsf{T}QC + K^\mathsf{T}RK)z \\ &= -l(Cz, \kappa_{\rm f}(z)), \end{aligned}$$

where the second equality and the third inequality come from (32b) and (32a), respectively.

As $l(Cz, \kappa_{\rm f}(z)) \geq 0$, we obtain

$$V_{\rm f}(\bar{z}^+) \leq V_{\rm f}(z) \leq \tau.$$

Thus, $\bar{z}^+ \in \mathcal{Z}_{\rm f}$ and the terminal set is control invariant.

We then present that the input given by the terminal controller satisfies the input constraint. From the inequality (17) and the designed terminal set (16), we have

$$\begin{aligned} V_{\rm f}(\bar{z}^+) + l(Cz, \kappa_{\rm f}(z)) =&\ V_{\rm f}(\bar{z}^+) + z^\mathsf{T}C^\mathsf{T}QCz + \kappa_{\rm f}(z)^\mathsf{T}R\kappa_{\rm f}(z) \\ \leq&\ V_{\rm f}(z) \leq \tau, \ \forall z \in \mathcal{Z}_{\rm f}. \end{aligned}$$

This implies that

$$\|\kappa_{\rm f}(z)\| \leq \sqrt{\frac{\tau}{\sigma_R}}, \ \forall z \in \mathcal{Z}_{\rm f},$$

where σ_R is the maximum eigenvalue of R. Since $\sqrt{\frac{\tau}{\sigma_R}}\mathcal{B}_1 \subseteq \mathcal{U}$ from the construction, we have $\kappa_{\rm f}(z) \in \mathcal{U}, \forall z \in \mathcal{Z}_{\rm f}$. This completes the proof.

A.6. Proof of Theorem 3

We first establish $\mathcal{X}_N$ is an ROA, *i.e.*, the Koopman MPC problem (8) is recursively feasible and the resulting closed-loop states converge to zero asymptotically from any initial state $x_t = x \in \mathcal{X}_N$.

Since the Koopman linear model is exact by assumption, we have

$$x^+ = C\bar{z}^+.$$

Meanwhile, we know $\bar{z}^+ \in \mathcal{Z}_N$ by Proposition 3. Thus, the Koopman MPC problem (8) is feasible with the initial state x^+. We can then denote the resulting optimal Koopman control sequence as

$$\mathbf{u}^{\rm opt} := (\kappa(\Psi(x_t)), \kappa(\Psi(x_{t+1})), \ldots, \kappa(\Psi(x_{t+\infty}))).$$

We prove $\lim_{t\to\infty} x_t = 0$ by showing the running MPC cost that $\sum_{k=0}^{\infty} \|x_{t+k}\|_Q^2 + \|\kappa(\Psi(x_{t+k}))\|_R^2$ is finite (recall that Q, R are positive definite). From (18), we have

$$V_N^*(\Psi(x_t)) \geq V_N^*(\Psi(x_{t+1})) + \|x_t\|_Q^2 + \|\kappa(\Psi(x_t))\|_R^2 \geq \sum_{k=0}^{\infty} \|x_{t+k}\|_Q^2 + \|\kappa(\Psi(x_{t+k}))\|_R^2.$$

As $V_N^*(\Psi(x_t))$ is upper bounded, cf. Proposition 2, we know $\sum_{k=0}^{\infty} \|x_{t+k}\|_Q^2 + \|\kappa(\Psi(x_{t+k}))\|_R^2$ is finite. Thus, we have $\lim_{t\to\infty} x_t = 0$, i.e., all states in $\mathcal{X}_N$ converge to the origin asymptotically.

We then prove the closed-loop system is locally asymptotically stable. This is equivalent to showing that 1) the system is locally attractive and 2) stable in the sense of Lyapunov. Point 1) is obviously true as $\mathcal{X}_N$ is an ROA and $\mathbb{0} \in \text{int}(\mathcal{X}_N)$.

For point 2), fix any $\epsilon > 0$, let $\rho = \lambda_Q \epsilon^2$, and consider $\mathcal{S} := \{x \in \mathcal{X}_N \mid V_N^*(\Psi(x)) < \rho\}$. From (18), we have

$$V_N^*(\Psi(x^+)) \leq V_N^*(\Psi(x)) < \rho \Rightarrow x^+ \in \mathcal{S}, \ \forall x \in \mathcal{S}.$$

Thus, $\mathcal{S}$ is an invariant set. From the boundedness property in Proposition 2, we have

$$\lambda_Q \|x\|^2 \leq V_N^*(\Psi(x)) < \rho \Rightarrow \|x\| < \epsilon, \ \forall x \in \mathcal{S},$$

indicating $\mathcal{S} \subseteq \mathcal{B}_\epsilon$. Since $V_N^*(\Psi(x))$ is continuous and $\mathbb{0} \in \mathcal{S}$, the origin is an interior point of $\mathcal{S}$ with the associated neighborhood $\mathcal{N} \subseteq \mathcal{S}$. Thus, any trajectory starting in $\mathcal{N}$ remains in $\mathcal{S} \subseteq \mathcal{B}_\epsilon$. This completes the proof.

A.7. Proof of Lemma 3

We show that $\bar{\mathbf{u}}$ (see the construction in Proposition 3) is a feasible solution for the initial state $z^+ = \Psi(x^+)$, which guarantees the one-step feasibility. We need to prove $\phi(N; z^+, \bar{\mathbf{u}}) \in \mathcal{Z}_\mathrm{f}$, which is equivalent to show $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}})) \leq \tau$. Thus, our goal in this part is to derive an upper bound for $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}}))$. The key idea for the derivation is to first bound the difference between $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}}))$ and $V_\mathrm{f}(\phi(N; \bar{z}^+, \bar{\mathbf{u}}))$, then derive an upper bound for $V_\mathrm{f}(\phi(N; \bar{z}^+, \bar{\mathbf{u}}))$, and finally combine both to obtain the upper bound of $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}}))$.

We first derive upper bounds for the optimal control input $\bar{\mathbf{u}}_z^*(0)$, the input sequence $\bar{\mathbf{u}}$, and the predicted state $\bar{z}^+$ of the nominal Koopman linear embedding. We will utilize these bounds later in bounding the difference between $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}}))$ and $V_\mathrm{f}(\phi(N; \bar{z}^+, \bar{\mathbf{u}}))$. Using the bound of z (*i.e.*, $\|z\| = \|\Psi(x)\| \leq L_\psi \|x\|$ from Assumption 3) and the property of Koopman MPC (see Proposition 2 and Proposition 3), the upper bound of $\mathbf{u}_z^*(0)$ and $\bar{\mathbf{u}}$ can be derived as

$$\|\mathbf{u}_z^*(0)\|^2 \leq \frac{V_N^*(z)}{\lambda_R} \leq \frac{c_z \|z\|^2}{\lambda_R} \leq c_1 \|x\|^2, \tag{33}$$

$$\|\bar{\mathbf{u}}\|^2 \leq \frac{V_N(\bar{z}^+, \bar{\mathbf{u}})}{\lambda_R} \leq \frac{V_N^*(z)}{\lambda_R} \leq \frac{c_z \|z\|^2}{\lambda_R} \leq c_1 \|x\|^2, \tag{34}$$

where $c_1 = \frac{c_z L_\psi^2}{\lambda_R} \in \mathbb{R}$ is a finite constant. Combining both the bound of z from Assumption 3 and the bound of $\mathbf{u}_z^*(0)$ (33), we obtain the bound of $\|\bar{z}^+\|$, that is:

$$\begin{aligned}\|\bar{z}^+\| &= \|Az + B\mathbf{u}_z^*(0)\| \\ &\leq \|A\|\|z\| + \|B\|\|\mathbf{u}_z^*(0)\| \\ &\leq \sigma_A L_\psi \|x\| + \sigma_B \sqrt{c_1}\|x\| = c_2 \|x\|,\end{aligned}$$

where σ_A, σ_B are maximum singular values of A, B, respectively and $c_2 = \sigma_A L_\psi + \sigma_B \sqrt{c_1}$.

We then bound the difference between $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}}))$ and $V_\mathrm{f}(\phi(N; \bar{z}^+, \bar{\mathbf{u}}))$. We can write the terminal state $\phi(N; z, \mathbf{u})$ with given z and $\mathbf{u}$ as

$$\phi(N; z, \mathbf{u}) = \bar{O}_N z + \bar{T}_N \mathbf{u},$$

where, following the propagation of the nominal Koopman linear embedding, we have

$$\bar{O}_N = A^N, \quad \bar{T}_N = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \ldots & B \end{bmatrix}.$$

Thus, we can derive the difference between $V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}}))$ and $V_\mathrm{f}(\phi(N; \bar{z}^+, \bar{\mathbf{u}}))$ as

$$\begin{aligned}&\|V_\mathrm{f}(\phi(N; z^+, \bar{\mathbf{u}})) - V_\mathrm{f}(\phi(N; \bar{z}^+, \bar{\mathbf{u}}))\| \\ &= \|(\bar{O}_N z^+ + \bar{T}_N \bar{\mathbf{u}})^\mathsf{T} \hat{P} (\bar{O}_N z^+ + \bar{T}_N \bar{\mathbf{u}}) - (\bar{O}_N \bar{z}^+ + \bar{T}_N \bar{\mathbf{u}})^\mathsf{T} \hat{P} (\bar{O}_N \bar{z}^+ + \bar{T}_N \bar{\mathbf{u}})\| \\ &\leq 2\sigma_1 \|\bar{z}^+\| \|z^+ - \bar{z}^+\| + \sigma_1 \|z^+ - \bar{z}^+\|^2 + 2\sigma_2 \|\bar{\mathbf{u}}\| \|z^+ - \bar{z}^+\| \\ &\leq 2\sigma_1 c_2 L \|x\|^2 + \sigma_1 L^2 \|x\|^2 + 2\sigma_2 \sqrt{c_1} L \|x\|^2 = c_3 \|x\|^2,\end{aligned} \tag{35}$$

in which σ_1, σ_2 are maximum singular values of $\bar{O}_N^\mathsf{T} \hat{P} \bar{O}_N$ and $\bar{O}_N^\mathsf{T} \hat{P} \bar{T}_N$ and we have $c_3 = \sigma_1 L^2 + 2(\sigma_1 c_2 + \sigma_2 \sqrt{c_1}) L$. We note that the difference between z^+ and $\bar{z}^+$ is the closed-loop one-step prediction error in Assumption 5:

$$\|z^+ - \bar{z}^+\| = \|\Psi(f(x, \mathbf{u}_z^*(0))) - Az - B\mathbf{u}_z^*(0)\| = \|e_\mathrm{cl}(x)\| \leq L\|x\|.$$

We next derive the upper bound of $V_{\mathrm{f}}(\phi(N;\bar{z}^+,\bar{\mathbf{u}}))$. We consider two cases: 1) $0 \le V_{\mathrm{f}}(\phi(N;z,\mathbf{u}_z^*)) < \frac{\tau}{2}$; 2) $\frac{\tau}{2} \le V_{\mathrm{f}}(\phi(N;z,\mathbf{u}_z^*)) \le \tau$. For case I, we have

$$V_{\mathrm{f}}(\phi(N;\bar{z}^+,\bar{\mathbf{u}})) - V_{\mathrm{f}}(\phi(N;z,\mathbf{u}_z^*)) \le 0$$
$$\Rightarrow V_{\mathrm{f}}(\phi(N;\bar{z}^+,\bar{\mathbf{u}})) \le \frac{\tau}{2}.$$

For case II, from the design of the terminal cost (see the Lyapunov equation of $\hat{P}$ in (15)) and its upper bound $V_{\mathrm{f}}(z) \le \sigma_{\hat{P}}\|z\|^2$, we have

$$\begin{aligned}&V_{\mathrm{f}}(\phi(N;\bar{z}^+,\bar{\mathbf{u}})) - V_{\mathrm{f}}(\phi(N;z,\mathbf{u}_z^*))\\ =&-\phi(N;z,\mathbf{u}_z^*)^{\mathsf{T}}\hat{Q}\phi(N;z,\mathbf{u}_z^*)\\ \le&-\lambda_{\hat{Q}}\|\phi(N;z,\mathbf{u}_z^*)\|^2 \le -\frac{\lambda_{\hat{Q}}\tau}{2\sigma_{\hat{P}}},\end{aligned}$$

where $\lambda_{\hat{Q}} > 0$ is the minimum eigenvalue of $\hat{Q} \succ 0$. Thus, we have

$$V_{\mathrm{f}}(\phi(N;\bar{z}^+,\bar{\mathbf{u}})) \le \tau - \gamma, \tag{36}$$

where $\gamma = \min\{\frac{\tau}{2}, \frac{\lambda_{\hat{Q}}\tau}{2\sigma_{\hat{P}}}\} > 0$.

Combing (35) and (36), we finally have

$$\begin{aligned}V_{\mathrm{f}}(\phi(N;z^+,\bar{\mathbf{u}})) &\le V_{\mathrm{f}}(\phi(N;\bar{z}^+,\bar{\mathbf{u}})) + c_3\|x\|^2\\ &\le \tau - \gamma + c_3\|x\|^2.\end{aligned}$$

We can consider c_3 as a function of L such that $c_3 = \alpha_1(L) := \sigma_1 L^2 + 2(\sigma_1 c_2 + \sigma_2\sqrt{c_1})L \in \mathcal{K}$. Thus, if $L \le \delta_1 := \alpha_1^{-1}(\frac{\gamma}{c_x})$ where $c_x = \sup_{x\in\mathcal{S}}\|x\|^2$, we have $c_3\|x\|^2 \le \gamma$ which means $V_{\mathrm{f}}(\phi(N;z^+,\bar{\mathbf{u}})) \le \tau$.

A.8. Proof for Lemma 4

Our goal in this part is to obtain an upper bound of the difference between $V_N^*(\Psi(x^+))$ and $V_N^*(\Psi(x))$ (*i.e.*, $V_N^*(z^+)$ and $V_N^*(z)$) and present it is a negative quadratic function under sufficiently small δ_2. From the descent property of the Koopman MPC (see Proposition 3), we can obtain an upper bound for the difference between $V_N(\bar{z}^+,\bar{\mathbf{u}})$ and $V_N^*(z)$. We can then bridge $V_N^*(z^+)$ and $V_N^*(z)$ by bounding the difference between $V_N^*(z^+)$ and $V_N(\bar{z}^+,\bar{\mathbf{u}})$.

We first bound the difference between $V_N(z^+,\bar{\mathbf{u}})$ and $V_N(\bar{z}^+,\bar{\mathbf{u}})$, which is also an upper bound for the difference between $V_N^*(z^+)$ and $V_N(\bar{z}^+,\bar{\mathbf{u}})$ from the principle of optimality. Given z and $\mathbf{u}$, we can express the explicit form of $V_N(z,\mathbf{u})$ as

$$V_N(z,\mathbf{u}) = (O_N z + T_N\mathbf{u})^{\mathsf{T}}\bar{Q}(O_N z + T_N\mathbf{u}) + \mathbf{u}^{\mathsf{T}}\bar{R}\mathbf{u}$$

where, following the propagation of the nominal Koopman linear embedding, we have

$$O_N = \begin{bmatrix} I & A & A^2 & \dots & A^N\end{bmatrix}^{\mathsf{T}}, \quad T_N = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0\\ B & 0 & 0 & \cdots & 0\\ AB & B & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ A^{N-1}B & A^{N-2}B & A^{N-3}B & \cdots & B\end{bmatrix},$$

and $\bar{Q} := \mathrm{diag}(\tilde{Q},\tilde{Q},\ldots,\tilde{Q},\hat{P}) \in \mathbb{R}^{(N+1)n_z\times(N+1)n_z}$, $\tilde{Q} = C^{\mathsf{T}}QC \in \mathbb{R}^{n_z\times n_z}$ and $\bar{R} := \mathrm{diag}(R, R,\ldots,R)\in\mathbb{R}^{Nm\times Nm}$.

The difference between $V_N(z^+, \bar{\mathbf{u}})$ and $V_N(\bar{z}^+, \bar{\mathbf{u}})$ can be bounded as

$$\begin{aligned} &\|V_N(z^+, \bar{\mathbf{u}}) - V_N(\bar{z}^+, \bar{\mathbf{u}})\| \\ &= \|(O_N z^+ + T_N \bar{\mathbf{u}})^\mathsf{T} \bar{Q} (O_N z^+ + T_N \bar{\mathbf{u}}) - (O_N \bar{z}^+ + T_N \bar{\mathbf{u}})^\mathsf{T} \bar{Q} (O_N \bar{z}^+ + T_N \bar{\mathbf{u}})\| \\ &\leq 2\sigma_3 \|\bar{z}^+\| \|z^+ - \bar{z}^+\| + \sigma_3 \|z^+ - \bar{z}^+\|^2 + 2\sigma_4 \|\bar{\mathbf{u}}\| \|z^+ - \bar{z}^+\| \\ &\leq 2\sigma_3 c_2 L \|x\|^2 + \sigma_3 L^2 \|x\|^2 + 2\sigma_4 \sqrt{c_1} L \|x\|^2 = c_4 \|x\|^2, \end{aligned} \tag{37}$$

in which σ_3, σ_4 are maximum singular values of $O_N^\mathsf{T} \bar{Q} O_N$ and $O_N^\mathsf{T} \bar{Q} T_N$ and we have $c_4 = \sigma_3 L^2 + 2(\sigma_3 c_2 + \sigma_4 \sqrt{c_1}) L$. Thus, (37) implies

$$V_N^*(z^+) - V_N(\bar{z}^+, \bar{\mathbf{u}}) \leq V_N(z^+, \bar{\mathbf{u}}) - V_N(\bar{z}^+, \bar{\mathbf{u}}) \leq c_4 \|x\|^2. \tag{38}$$

We then bridge $V_N^*(z^+)$ and $V_N^*(z)$ via $V_N(\bar{z}^+, \bar{\mathbf{u}})$. From Proposition 3, we have

$$V_N(\bar{z}^+, \bar{\mathbf{u}}) \leq V_N^*(z) - \lambda_Q \|x\|^2. \tag{39}$$

Combing (38) and (39), we finally have

$$V_N^*(z^+) - V_N^*(z) \leq -\lambda_Q \|x\|^2 + c_4 \|x\|^2.$$

Again, we treat c_4 as $c_4 = \alpha_2(L) := \sigma_3 L^2 + 2(\sigma_3 c_2 + \sigma_4 \sqrt{c_1}) L \in \mathcal{K}$, thus, there exists $\delta_2 := \alpha_2^{-1}(\lambda_Q)$ and c such that $c := \lambda_Q - c_4 > 0$ for $L < \delta_2$ which implies

$$V_N^*(z^+) - V_N^*(z) \leq -c \|x\|^2, \; \forall x \in \mathcal{S}.$$