

Characterizing Online and Private Learnability under Distributional Constraints via Generalized Smoothness

Moïse Blanchard*

Georgia Institute of Technology

MBLANCHARD41@GATECH.EDU

Abhishek Shetty*

Massachusetts Institute of Technology and Georgia Institute of Technology

SHETTY@MIT.EDU

Alexander Rakhlin

Massachusetts Institute of Technology

RAKHLIN@MIT.EDU

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Abstract

Understanding minimal assumptions that enable learning and generalization is perhaps the central question of learning theory. Several celebrated results in statistical learning theory, such as the VC theorem and Littlestone’s characterization of online learnability, establish conditions on the hypothesis class that allow for learning under independent data and adversarial data, respectively. Building upon recent work bridging these extremes, we study sequential decision making under *distributional adversaries* that can adaptively choose data-generating distributions from a fixed family \mathcal{U} and ask when such problems are learnable with sample complexity that behaves like the favorable independent case. We provide a near complete characterization of families \mathcal{U} that admit learnability in terms of a notion known as *generalized smoothness* i.e. a distribution family admits VC-dimension-dependent regret bounds for every finite-VC hypothesis class if and only if it is generalized smooth. Further, we give universal algorithms that achieve low regret under any generalized smooth adversary without explicit knowledge of \mathcal{U} . Finally, when \mathcal{U} is known, we provide refined bounds in terms of a combinatorial parameter, the fragmentation number, that captures how many disjoint regions can carry nontrivial mass under \mathcal{U} . These results provide a nearly complete understanding of learnability under distributional adversaries. In addition, building upon the surprising connection between online learning and differential privacy, we show that the generalized smoothness also characterizes private learnability under distributional constraints.

Keywords: Online learning, distribution shift, smoothed adversaries, VC dimension, universal learning

1. Introduction

The grand challenge of statistics and learning theory is to understand when data can be used to make accurate predictions. Various formalizations of this question have been studied over the decades, leading to a rich theory of learnability under different data-generating assumptions on the covariates and the labels, performance metrics, feedback models, and computational constraints.

One prominent framework is *PAC learning* (Valiant, 1984; Vapnik and Chervonenkis, 1974) which supposes that data are drawn independently and identically from a distribution. In this setting, the celebrated VC theorem (Vapnik and Chervonenkis, 1971, 1974) shows that a hypothesis class \mathcal{F} is learnable with sample complexity depending on its VC dimension (Vapnik and Chervonenkis, 1971). On the other extreme of the data-generating spectrum lies *online learning* (Littlestone, 1988),

* Equal contribution.

which models data as being completely arbitrarily chosen (modeled as being chosen by an adversary). Here, Littlestone’s seminal work (Littlestone, 1988) established that a hypothesis class \mathcal{F} is learnable with regret depending on its Littlestone dimension (Littlestone, 1988). Though these two frameworks provide a solid understanding of learnability under independent data (which we view as “benign” or well-behaved in our present context but consider a strong and perhaps unreasonable modelling assumption) and adversarial data (which is a minimal assumption that treats the data as malicious and ill-behaved), real-world applications typically involve data-generating processes that lie between these two extremes. Further, and perhaps unsurprisingly, the characterization of learnability under the fully adversarial model of Littlestone precludes learnability even for simple hypothesis classes such as linear thresholds (Littlestone, 1988).

This motivates the study of structured data-generating processes that interpolate between the two extremes of i.i.d. and fully adversarial data. A natural formulation is to consider a process where the choice of data at each round is constrained to lie within a certain family of distributions. This framework is referred to as *distributionally constrained online learning*. Perhaps the most well-studied special case is that of *smoothed adversaries* (Rakhlin et al., 2011; Haghtalab et al., 2024; Shetty, 2024; Block et al., 2022), where the adversary is constrained to choose distributions that have bounded density ratio with respect to a fixed reference measure. This line of work, which can be also be seen as a model for distribution shift (See (Hanneke and Kpotufe, 2025) and references therein), has established that, under smoothness, sequential decision-making has the same qualitative learnability guarantees as in the i.i.d. setting (Haghtalab et al., 2024). Though this gives a satisfactory answer under smoothness, perhaps the condition is too strong, leading to the following fundamental question:

What are minimal conditions on a distribution family \mathcal{U} that ensure any form of learnability for low-complexity function classes (e.g., finite VC dimension), against adaptive \mathcal{U} -distributionally constrained adversaries?

Surprisingly, this question is intimately connected to a seemingly unrelated problem of learning under *privacy* constraints. Differential privacy (Dwork et al., 2006) is a rigorous framework for ensuring that learning algorithms do not leak sensitive information about individual data points. Unfortunately, differential privacy, being a stringent condition often imposes significant limitations on what can be learned, perhaps most notably exemplified by the fact that for binary classification, only classes with finite Littlestone dimension are privately learnable (Alon et al., 2019b). As with online learning, this negative result has motivated the study of structured data-generating processes that allow for private learnability of richer classes (Haghtalab et al., 2020; Alon et al., 2019a; Bassily et al., 2020; Beimel et al., 2014). We ask whether there is a characterization of distribution families that admit private learnability with finite sample complexity for VC hypothesis classes.

1.1. Our Contribution.

We address this question by providing a complete characterization of distribution families \mathcal{U} that admit learning for VC classes. To do this, we introduce a structural condition on \mathcal{U} that generalizes the classical smoothness condition from smoothed online learning. Recall that in smoothed online learning, the adversary is constrained to choose distributions that have bounded density ratio with respect to a fixed reference measure μ_0 ; that is, there exists $\sigma > 0$ such that for every $\mu \in \mathcal{U}$ and measurable set A , $\mu(A) \leq \mu_0(A)/\sigma$. *Generalized Smoothness* relaxes this linear relationship to

allow for more general tolerance functions. That is, a distribution family \mathcal{U} is said to be (ρ, μ_0) -*generalized smooth* if for every $\mu \in \mathcal{U}$, we have

$$\mu(A) \leq \rho(\mu_0(A)) \quad \text{for all measurable sets } A. \quad (1)$$

This condition implies that the adversary cannot place significant mass on sets that are small under the base measure—but the relationship need not be linear as in classical smoothness. Our main result ([Theorem 9](#)) shows that this condition is necessary for any form of learnability against every finite-VC class \mathcal{F} . In fact, we show a stronger result: generalized smoothness is necessary even to achieve asymptotic consistency against thresholds, the simplest non-trivial function classes.

To complete the characterization, we build upon recent algorithmic techniques for smoothed online learning ([Shetty, 2024](#); [Haghtalab et al., 2024](#); [Block et al., 2024b](#); [Blanchard, 2025](#)) to show that generalized smoothness is sufficient to achieve VC-dimension-dependent regret bounds for every finite-VC class \mathcal{F} , even when \mathcal{U} is not explicitly known to the learner ([Theorems 13](#) and [14](#)). These results show that the complex interplay of combinatorial structure and regret bound for distributionally constrained adversaries can be fully captured by a single structural condition on the distribution family \mathcal{U} . Additionally, and perhaps surprisingly, this also shows that merely asking for asymptotic consistency on simple VC classes further implies non-asymptotic regret bounds. Note that asymptotic and non-asymptotic learnability are, in general, qualitatively different notions with distinct characterizations.

Our main technical contribution is establishing a connection between regret bound against threshold classes to a uniform notion of compactness of the distribution family \mathcal{U} , which we capture via the existence of a continuous dominating submeasure ([Definition 6](#)) for \mathcal{U} . Further, we show that the existence of such a continuous dominating submeasure is equivalent to generalized smoothness. The relation between continuity of submeasures and uniform domination by a measure (corresponding to generalized smoothness) is well-studied in measure theory, with celebrated results such as the *Kalton-Roberts theorem* ([Kalton and Roberts, 1983](#))¹ and (Talagrand’s resolution of) *Maharam’s problem* ([Talagrand, 2008](#)). In our setting, with the extra structure of our submeasure (arising as a supremum over a family of probability measures), we are able to establish a clean equivalence between continuity and uniform domination directly without relying on opaque application of this heavy machinery but developing a deeper connection to these classical results is an interesting direction for future work.

We complement the characterization with algorithms ([Theorems 13](#) and [14](#)) that achieve VC-dimension-dependent regret bounds when \mathcal{U} is generalized smooth, even when \mathcal{U} is unknown to the learner. These can be viewed as the generalized smoothness analogues of the algorithms for smoothed online learning with unknown base measure from [Block et al. \(2024b\)](#); [Blanchard \(2025\)](#). Finally, when \mathcal{U} is known to the learner, we provide refined regret bounds ([Theorems 16](#) and [18](#)) that depend on a combinatorial parameter of \mathcal{U} that we term the *fragmentation number* ([Definition 15](#)). While having finite fragmentation numbers is equivalent to generalized smoothness, the precise dependence of regret bounds on fragmentation numbers provides a more refined understanding of the difficulty of learning under a given distribution family \mathcal{U} , corresponding to the different rates in the *known-base-measure* smoothed online learning setting ([Haghtalab et al., 2024](#)) and the *unknown-base-measure* setting ([Block et al., 2024b](#); [Blanchard, 2025](#)).

In addition to the characterization of online learnability in the distribution constrained setting, generalized smoothness serves as a characterization of private learnability under the distributionally

1. Interestingly, this result is perhaps the first significant example of the use of expander graphs outside graph theory

constrained settings. This can be seen as the distributionally constrained analogue of the characterization of private learnability via the Littlestone dimension (Alon et al., 2019b). We show that a distribution family \mathcal{U} admits private learnability with sample complexity depending only on the VC dimension of the hypothesis class if and only if \mathcal{U} is generalized smooth (Theorems 20 and 21).

2. Preliminaries

Let \mathcal{X} be a separable measurable instance space and denote by \mathcal{B} the collection of measurable sets of \mathcal{X} . Let $\mathcal{Y} = \{0, 1\}$ denote the label space. Let $\mathcal{F} \subseteq \mathcal{B}$ be a family of measurable binary classifiers, associated with indicators over sets, and use the indicator loss $\mathbb{1}\{f(x) \neq y\}$. While we focus on the case of binary classification and the indicator loss for concreteness, extending ideas proposed here to multiclass classification and regression is an interesting direction for future work. The central combinatorial quantity in the binary classification setting is the VC dimension (Vapnik and Chervonenkis, 1971).

Definition 1 (VC dimension) *The VC dimension of a function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ is defined as*

$$\text{VC}(\mathcal{F}) = \sup \left\{ d \in \mathbb{N} : \exists x_1, \dots, x_d \in \mathcal{X} \text{ s.t. } |\{(f(x_1), \dots, f(x_d)) : f \in \mathcal{F}\}| = 2^d \right\}.$$

A set $\{x_1, \dots, x_d\} \subseteq \mathcal{X}$ satisfying the above condition is said to be shattered by \mathcal{F} .

We next define the online learning protocol under distributional constraints. Let \mathcal{U} be a family of probability distributions over \mathcal{X} . Consider the following T -round online protocol. For each round $t = 1, \dots, T$:

1. The adversary selects a distribution $\mu_t \in \mathcal{U}$, either *obliviously* (the sequence $(\mu_t)_{t=1}^T$ is fixed in advance) or *adaptively* based on the history.
2. The learner chooses $f_t \in \mathcal{B}$ (possibly randomized).
3. A sample $x_t \sim \mu_t$ is drawn; then a label $y_t \in \mathcal{Y}$ is revealed. The learner incurs loss $\mathbb{1}\{f_t(x_t) \neq y_t\}$ (and, in the full-information setting, observes (x_t, y_t)). Note that μ_t is not observed.

The objective of the learner is to minimize regret against the comparator class \mathcal{F} , that is, the difference between the cumulative loss of the learner and that of the best fixed function in \mathcal{F} in hindsight. Formally, define the regret after T rounds for an algorithm alg as

$$\text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U}) = \mathbb{E} \left[\sum_{t=1}^T \mathbb{1}\{f_t(x_t) \neq y_t\} - \inf_{f \in \mathcal{F}} \sum_{t=1}^T \mathbb{1}\{f(x_t) \neq y_t\} \right].$$

The setup described above generalizes both the classical adversarial online learning setting (when \mathcal{U} contains all distributions over \mathcal{X}) and the (sequential analogue of the) standard i.i.d. statistical learning setting (when \mathcal{U} is a single fixed distribution). We denote the best achievable regret against the worst-case \mathcal{U} -constrained adversary by $\text{Reg}_T(\mathcal{F}, \mathcal{U})$. Note that when \mathcal{U} is a singleton, from the VC theory of statistical learning (Vapnik and Chervonenkis, 1971), we have $\text{Reg}_T(\mathcal{F}, \mathcal{U}) \lesssim \sqrt{\text{VC}(\mathcal{F})T}$. Further, when \mathcal{U} contains all distributions over \mathcal{X} , from the classical theory of adversarial online learning (Littlestone, 1988; Ben-David et al., 2009), we have that

$\text{Reg}_T(\mathcal{F}, \mathcal{U}) \lesssim \sqrt{\text{LD}(\mathcal{F})T}$, where $\text{LD}(\mathcal{F})$ is the Littlestone dimension of \mathcal{F} (Littlestone, 1988)². The key issue we aim to address is the fact that $\text{LD} \gg \text{VC}$ for most natural function classes, with LD typically being infinite. With this in mind, the aim of this work is to characterize conditions on \mathcal{U} that allow achieving regret bounds that depend on the VC dimension of \mathcal{F} . In order to capture this, we introduce two notions of learnability for distribution families.

Definition 2 (Weak VC Learnability) *A distribution class \mathcal{U} on \mathcal{X} is weakly VC learnable if for any function class \mathcal{F} with finite VC dimension, there exists an online algorithm alg such that for any \mathcal{U} -constrained adversary generating a data sequence $(x_t, y_t)_{t \geq 1}$ with the label y_t consistent with some function in \mathcal{F} , we have*

$$\lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \sum_{t=1}^T \mathbb{1}[\hat{y}_t \neq y_t] \right] = 0.$$

Note that the definition above merely asks for *asymptotic* consistency on realizable data: convergence rates are allowed to potentially depend on the adversary’s behavior. Arguably, this is the weakest meaningful learning guarantee. To emphasize the difference between asymptotic consistency and finite-regret bounds, we note that any countable function class \mathcal{F} —in particular, even with infinite VC dimension—admits asymptotically consistent learners even when the distribution class \mathcal{U} is fully unrestricted, i.e., even for adversarial data (this is for instance a direct consequence of Hanneke (2021, Lemma 34)). We can strengthen the notion of learnability by requiring *non-asymptotic* regret bounds that depend on the VC dimension of the comparator class. We will refer to this as *strong VC learnability*.

Definition 3 (Strong VC Learnability) *A distribution class \mathcal{U} on \mathcal{X} is strongly VC learnable if for any function class \mathcal{F} with finite VC dimension, for any $\epsilon > 0$ there exists $m_{\mathcal{U}}(\epsilon, d)$ and an online algorithm alg such that for any \mathcal{U} -constrained adversary, and $T \geq m_{\mathcal{U}}(\epsilon, d)$,*

$$\text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U}) \leq \epsilon T.$$

The key structural condition that captures learnability is *generalized smoothness*.

Definition 4 (Generalized smoothed distribution class) *Let \mathcal{X} be a measurable space. Fix a measure μ_0 on \mathcal{X} and a non-decreasing function $\rho : [0, 1] \rightarrow \mathbb{R}_+$ with $\lim_{\epsilon \rightarrow 0} \rho(\epsilon) = 0$. A family of distributions \mathcal{U} on \mathcal{X} is said to be (μ_0, ρ) -generalized smoothed if for any distribution $\mu \in \mathcal{U}$ and measurable set $A \subseteq \mathcal{X}$,*

$$\mu(A) \leq \rho(\mu_0(A)).$$

One can see that generalized smoothness implies that the distributions in \mathcal{U} are absolutely continuous with respect to the base measure μ_0 , with ρ acting as a uniform control on this continuity. Note that this definition generalizes classical smoothness, which corresponds to the special case $\rho(\epsilon) = \epsilon/\sigma$ for some constant $\sigma \leq 1$. Further, this notion also generalizes smoothness notions based on f -divergences studied in Block and Polyanskiy (2023). Last, the notion of generalized smoothness is closely related to the notion of coverage studied by Chen et al. (2025). All these notions control the mass in the tails of density ratios between distributions in \mathcal{U} and a base measure μ_0 and can be seen as different parameterizations of the same underlying concept. However, the notion of generalized smoothness is the most convenient for our purposes here and most directly addresses the underlying question of learnability.

2. Since we do not directly work with the Littlestone dimension, we defer the definition to the appendix (Definition 22)

3. Learnability and Generalized Smoothness

In this section, we characterize weak and strong VC learnability as defined in [Definitions 2 and 3](#). Specifically, we show that both are equivalent to generalized smoothness: in particular, weak VC learnability is—despite its asymptotic nature by definition—sufficient to recover non-asymptotic regret bounds. In fact, it turns out that weak learnability with respect to simple VC=1 classes is already sufficient for non-asymptotic regret bounds.

To formalize this result, we begin by introducing the family of *generalized thresholds*, which are the prototypical example of VC one classes and conceptually, the simplest non-trivial function classes for learning.

Definition 5 (Generalized thresholds) *A function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ is a generalized threshold function class if there exists a total preorder³ \preceq on \mathcal{X} such that $\mathcal{F} \subseteq \{\mathbb{1}[\cdot \preceq x_0] : x_0 \in \mathcal{X}\}$.*

These function classes naturally generalize the standard thresholds— $\{\mathbb{1}[\cdot \leq x_0] : x_0 \in [0, 1]\}$ —on the interval $\mathcal{X} = [0, 1]$ to the case when one may use potentially different orderings than \leq on the domain $[0, 1]$. Note that the VC dimension of any generalized threshold class is at most 1. Due to their simplicity, we use generalized thresholds as “test classes” for learnability. We specialize the notion of weak learnability ([Definition 2](#)) and strong learnability ([Definition 3](#)) to generalized thresholds and refer to these as *weak threshold learnability* and *strong threshold learnability*, respectively.

Surprisingly, we show in this section that weak threshold learnability is equivalent to generalized smoothness and hence sufficient to recover non-asymptotic regret-bounds for all function classes with finite VC dimension. In turn, this will imply the equivalence of weak and strong VC learnability to generalized smoothness.

As the first step of this proof, we study links between weak threshold learnability of a distribution class \mathcal{U} and properties of its envelope functional $\mu_{\mathcal{U}}$ defined via

$$\mu_{\mathcal{U}}(A) := \sup_{\mu \in \mathcal{U}} \mu(A), \quad A \in \mathcal{B}.$$

A key property will be whether $\mu_{\mathcal{U}}$ is additionally a continuous submeasure, as defined below.

Definition 6 (Continuous Submeasure) *A function $\hat{\mu} : \mathcal{B} \rightarrow [0, 1]$ is a continuous submeasure if:*

1. **Monotonicity:** For any $A, B \in \mathcal{B}$ such that $A \subseteq B$, $\hat{\mu}(A) \leq \hat{\mu}(B)$.
2. **Sub-additivity:** For any $A, B \in \mathcal{B}$, $\hat{\mu}(A \cup B) \leq \hat{\mu}(A) + \hat{\mu}(B)$.
3. **Continuity from above:** For any decreasing sequence of measurable sets $(A_i)_{i \geq 1}$ with $\bigcap_{i \geq 1} A_i = \emptyset$, we have $\lim_{n \rightarrow \infty} \hat{\mu}(A_n) = 0$.

Of course, the functional $\mu_{\mathcal{U}}$ always satisfies monotonicity and sub-additivity for any distribution class \mathcal{U} . More importantly, we show that $\mu_{\mathcal{U}}$ must also satisfy the continuity from above property if the distribution class \mathcal{U} is weakly threshold learnable.

Lemma 7 *Let \mathcal{U} be a weakly threshold learnable distribution class on \mathcal{X} . Then, $\mu_{\mathcal{U}}$ is a continuous submeasure.*

3. Preorders satisfy the same properties as orders except for antisymmetry: \preceq is a preorder if it is reflexive and transitive.

As a brief overview of the proof, if $\mu_{\mathcal{U}}$ fails to be continuous, then there exist infinitely-many (countably) disjoint sets each carrying at least ϵ mass under some distribution in \mathcal{U} for some parameter $\epsilon > 0$. We index these sets by rationals and construct a threshold class where each function f_r is the indicator of the union of sets indexed by rationals at most r . By construction, the adversary can place ϵ mass under each of these sets, which allows the adversary to simulate an unconstrained adversary on the threshold function class. This yields an expected lower bound $\Omega(\epsilon T)$ on the number of mistakes following classical lower bound arguments for adversarial online learning (Littlestone, 1988), up to minor modifications to ensure realizability. Essentially, the adversary hides the true threshold r^* through a binary search. We defer the full proof to [Appendix A](#).

The main step of the proof is to show that if $\mu_{\mathcal{U}}$ is a continuous submeasure, then the distribution class \mathcal{U} corresponds to generalized smoothed classes.

Lemma 8 *Let \mathcal{U} be a family of distributions on \mathcal{X} such that $\mu_{\mathcal{U}}$ is a continuous submeasure. Then, \mathcal{U} is (μ_0, ρ) -generalized smooth for some distribution μ_0 on \mathcal{X} and a non-decreasing function $\rho : [0, 1] \rightarrow \mathbb{R}_+$ with $\lim_{\epsilon \rightarrow 0} \rho(\epsilon) = 0$.*

The proof is constructive. At the high-level, if $\mu_{\mathcal{U}}$ is continuous then for any scale $\epsilon > 0$ then we can construct a finite cover $\{\mu_1, \dots, \mu_N\} \in \mathcal{U}$ of the distribution class \mathcal{U} for any measurable set $A \in \mathcal{B}$ on which some distribution $\mu \in \mathcal{U}$ places ϵ mass. Taking an appropriate mixture of these finite covers for decreasing scales $\epsilon \rightarrow 0$ then yields a distribution μ_0 with the desired generalized-smoothness property on \mathcal{U} .

Proof Fix \mathcal{U} satisfying the assumptions and any $\epsilon > 0$. We iteratively construct a sequence of distributions $(\mu_i^\epsilon)_{i \geq 1}$ together with measurable sets $(A_i^\epsilon)_{i \geq 1}$ as follows. Suppose we have constructed $\mu_j^\epsilon, A_j^\epsilon$ for all $j < i$ for some $i \geq 1$. If there exist a distribution $\mu \in \mathcal{U}$ and a measurable set A such that

$$\mu(A) \geq \frac{2}{\epsilon} \sum_{j < i} 2^{j-i} \mu_j^\epsilon(A) \quad \text{and} \quad \mu(A) \geq \epsilon,$$

we fix $\mu_i^\epsilon, A_i^\epsilon$ satisfying this property. Otherwise, we end the recursive construction.

Suppose by contradiction that this construction does not terminate. Define $B_i^\epsilon := A_i^\epsilon \setminus \bigcup_{j > i} A_j^\epsilon$ for all $i \geq 1$. Then,

$$\mu_{\mathcal{U}}(B_i^\epsilon) \geq \mu_i^\epsilon(B_i^\epsilon) \geq \mu_i^\epsilon(A_i^\epsilon) - \sum_{j > i} \mu_i^\epsilon(A_j^\epsilon) \geq \epsilon - \sum_{j > i} \frac{\epsilon}{2^{j-i+1}} \mu_j^\epsilon(A_j^\epsilon) \geq \frac{\epsilon}{2}.$$

Since the sets B_i^ϵ are disjoint, this contradicts the continuous submeasure assumption on $\mu_{\mathcal{U}}$. Specifically, we have $\mu(\bigcap_{j \geq i} B_j) \geq \epsilon/2$ while $\bigcap_{j \geq i} B_j \downarrow \emptyset$. As a result, the construction terminates. Let N_ϵ be the last constructed index. Then, by construction, the finite measure $\mu^\epsilon := \frac{2}{\epsilon} \sum_{j \leq N_\epsilon} \mu_j^\epsilon$ satisfies for any measurable set $A \subseteq \mathcal{X}$,

$$\mu_{\mathcal{U}}(A) = \sup_{\mu \in \mathcal{U}} \mu(A) \leq \epsilon + \mu^\epsilon(A). \quad (2)$$

Now fix any positive decreasing sequence $(\epsilon_i)_{i \geq 1}$ in $[0, 1]$ with $\lim_{i \rightarrow \infty} \epsilon_i = 0$. For each $i \geq 1$, since μ^{ϵ_i} is a finite measure, we can write it as $\mu^{\epsilon_i} = C_i \nu_i$ where ν_i is a probability measure and $C_i = \mu^{\epsilon_i}(\mathcal{X}) \geq 0$. Without loss of generality, we may assume that $(C_i)_{i \geq 0}$ is non-decreasing while

preserving Eq. (2). In particular, the sequence formed by $\delta_i := \frac{\epsilon_i}{2^i C_i}$ is non-increasing in $i \geq 1$. We then define the probability measure

$$\mu_0 := \sum_{i \geq 1} 2^{-i} \nu_i,$$

and we fix a non-decreasing function $\rho : [0, 1] \rightarrow \mathbb{R}_+$ by setting $\rho(z) = 2\epsilon_i$ for all $z \in (\delta_{i+1}, \delta_i]$ for all $i \geq 1$, $\rho(0) = 0$ and $\rho(z) = 2$ for $z \in (\delta_1, 1]$.

Now fix a measurable set $A \subseteq \mathcal{X}$ and denote $z = \mu_0(A)$. If $z > \delta_1$ then we immediately have $\mu_{\mathcal{U}}(A) \leq 1 = \rho(\mu_0(A))$. Next, if $z \in (\delta_{i+1}, \delta_i]$ for some $i \geq 1$, from Eq. (2) we have

$$\mu_{\mathcal{U}}(A) \leq \epsilon_i + \mu^{\epsilon_i}(A) = \epsilon_i + C_i \nu_i(A) \stackrel{(i)}{\leq} \epsilon_i + C_i 2^i \mu_0(A) \leq 2\epsilon_i = \rho(\mu_0(A)).$$

In (i) we used the definition of μ_0 . Finally, if $z = 0$, then similarly as above, Eq. (2) implies that $\mu_{\mathcal{U}}(A) \leq \epsilon_i$ for all $i \geq 1$ and hence $\mu_{\mathcal{U}}(A) = 0 = \rho(\mu_0(A))$. This ends the proof. ■

Combining Lemmas 7 and 8 directly shows that any weakly threshold learnable family of distributions \mathcal{U} is a generalized smoothed distribution class. It is further known that generalized smoothness classes are learnable for all VC classes (Blanchard and Kpotufe, 2025). This yields the following main characterization theorem.

Theorem 9 *Let \mathcal{U} be a family of distributions on \mathcal{X} . The following are equivalent:*

1. \mathcal{U} is weakly threshold learnable.
2. \mathcal{U} is strongly VC learnable.
3. \mathcal{U} is a generalized smoothed distribution class.

Proof We proved (1) \Rightarrow (3) in Lemmas 7 and 8. Specifically, if \mathcal{U} is weakly threshold-learnable, then Lemma 7 shows that $\mu_{\mathcal{U}}$ is a continuous submeasure and hence Lemma 8 exactly constructs a reference measure μ_0 and a tolerance function ρ for which \mathcal{U} is generalized smooth. (2) \Rightarrow (1) is immediate. (3) \Rightarrow (2): The learnability of all VC classes under generalized smoothed distribution classes was already observed in previous works, e.g. Blanchard and Kpotufe (2025, Corollary 10). We will further give concrete algorithms—ERM and R-COVER—that achieves VC-dimension-dependent regret bounds for any generalized smoothed distribution classes in the realizable and agnostic setting respectively within Theorems 13 and 14. ■

Conceptually, Theorem 9 shows that any form of learnability on generalized thresholds—while being very simple VC=1 function classes—already forces strong structural properties on the distribution class \mathcal{U} , as captured by generalized smoothness. Perhaps more surprisingly, this shows that *asymptotic* learnability guarantees on generalized thresholds imply *non-asymptotic* regret guarantees on all VC classes.

As a remark, this property may not hold for simpler classes than generalized thresholds. For instance, note that any countable function class \mathcal{F} is weakly learnable in the sense that there exists an algorithm which ensures sublinear excess regret compared to any fixed function within \mathcal{F} even for unconstrained adversaries, e.g. see Hanneke (2021, Lemma 34). Hence, $\mathcal{U} = \{\text{all distributions on } \mathcal{X}\}$ is weakly learnable for any countable function class, but it is not learnable for any function class with infinite Littlestone dimension—including threshold function classes.

3.1. Uniform Covers

Another perspective on generalized smoothness is through the lens of uniform covers.

Definition 10 (Uniform Cover) *A distribution class \mathcal{U} is said to admit uniform covers for a function class \mathcal{F} if for any $\epsilon > 0$, there exists a finite set of functions $\mathcal{F}_\epsilon \subseteq \mathcal{F}$ such that for any $\mu \in \mathcal{U}$ and $f \in \mathcal{F}$, there exists $\hat{f} \in \mathcal{F}_\epsilon$ such that $\mu(\{x : f(x) \neq \hat{f}(x)\}) \leq \epsilon$.*

It is easy to see that generalized smoothness implies existence of uniform covers for all VC classes. Indeed, if \mathcal{U} is generalized smooth with respect to some reference measure μ_0 and tolerance function ρ , then for any $\epsilon > 0$, using a covering for \mathcal{F} under μ_0 at scale $\delta = \rho^{-1}(\epsilon)$ yields a uniform cover at scale ϵ for all distributions in \mathcal{U} . We record this formally as [Lemma 30](#).

A distribution class admitting uniform covers for a class \mathcal{F} is also sufficient for achieving low regret if the adversary is *oblivious* and the distribution class \mathcal{U} is known. Indeed, as observed in [Haghtalab \(2018\)](#), a standard Hedge algorithm run on the uniform cover \mathcal{F}_ϵ ensures regret at most $\epsilon T + O(\sqrt{T \log |\mathcal{F}_\epsilon|})$ against any oblivious \mathcal{U} -constrained adversary. But, to the best of our knowledge, there is no known direct way to extend this to the adaptive adversaries setting or the case when \mathcal{U} is unknown, without using the machinery of the coupling lemma introduced by [Haghtalab et al. \(2024\)](#) (which requires stronger assumption on the distribution class than just uniform covers).

A modification of our proof of [Theorem 9](#) actually shows that the existence of uniform covers for all VC classes is indeed equivalent to generalized smoothness.

Proposition 11 *Let \mathcal{U} be a distribution class. Then \mathcal{U} has uniform covers for all VC classes (equivalently, all generalized threshold classes) if and only if it is generalized smooth.*

Thus, our result can be seen as showing that if we insist on VC dependent regret bounds then uniform cover is also a necessary and sufficient condition for learnability.

4. Universal Algorithms

In the following two sections, we provide algorithms that achieve low regret under generalized smoothed distribution classes. As a reminder, as per the definitions of learnability in [Definitions 2 and 3](#), a distribution class \mathcal{U} is learnable if there exists a corresponding algorithm which achieves the desired learning guarantee. Importantly, the considered algorithm a priori requires knowledge of the distribution class (and the function class \mathcal{F} , which is unavoidable). In this section, we instead focus on algorithms that do not require this prior knowledge on \mathcal{U} , yet still achieve learning under VC classes (or generalized thresholds) and any distribution class \mathcal{U} for which this is achievable—equivalently, any generalized smoothed distribution class \mathcal{U} , by [Theorem 9](#). We refer to such adaptive algorithms as *optimistically universal*, following the corresponding literature on “learning whenever possible” ([Hanneke, 2021](#); [Blanchard, 2022](#); [Blanchard and Jaillet, 2023](#)). Note that their existence is not a priori guaranteed.⁴

To obtain universal algorithms with low regret under generalized smoothness, we leverage known results for smooth adversaries, which correspond to generalized smoothed classes with linear tolerance function $\rho(\epsilon) = \epsilon/\sigma$. Specifically, for smooth adversaries, Empirical Risk Minimization (ERM) already achieves sublinear regret for VC classes in the realizable setting ([Block et al.](#),

4. For instance, no optimistically universal algorithm exist for inductive supervised learning ([Hanneke, 2021](#)), or online adversarial contextual bandits ([Blanchard et al., 2023](#)).

2024b), while R-COVER (Blanchard, 2025)—an algorithm based on recursive empirical covers—is known to achieve sublinear regret for VC classes in the agnostic setting.

Coupling reduction lemma. We then lift these guarantees to generalized smooth adversaries using a simple coupling reduction to smooth adversaries. Fix a generalized smoothed class \mathcal{U} with base measure μ_0 and tolerance ρ . Recall that by definition, for any ϵ , any distribution places at most mass $\rho(\epsilon)$ on sets of μ_0 -measure ϵ . Hence, distributions in \mathcal{U} are intuitively $\epsilon/\rho(\epsilon)$ -smooth with respect to μ_0 up to a mass $\rho(\epsilon)$. More formally, we can couple the sequence generated by a \mathcal{U} -constrained adversary with a $\mathcal{O}(\epsilon/\rho(\epsilon))$ -smooth sequence which agree except on a fraction $\mathcal{O}(\rho(\epsilon))$ of the samples for which we replace the original sample with a dummy variable x_\emptyset , as detailed below. This argument will allow us to import regret bounds from the smoothed online learning literature to the case of generalized smoothed adversaries.

For convenience, we say that a tolerance function $\rho : [0, 1] \rightarrow \mathbb{R}_+$ is *well-behaved* if it is non-decreasing, $\lim_{\epsilon \rightarrow 0} \rho(\epsilon)$, and $\epsilon \in (0, 1] \mapsto \rho(\epsilon)/\epsilon$ is non-increasing.

Lemma 12 *Let \mathcal{U} be a generalized smoothed distribution class on \mathcal{X} with a well-behaved tolerance function $\rho : [0, 1] \rightarrow \mathbb{R}_+$ and a horizon $T \geq 1$. For convenience, fix a dummy new variable $x_\emptyset \notin \mathcal{X}$. Consider any \mathcal{U} -constrained adaptive adversary to generate samples x_1, \dots, x_T .*

For any $\epsilon > 0$, there exists a coupling of these samples with a sequence $\tilde{x}_1, \dots, \tilde{x}_T \in \mathcal{X} \cup \{x_\emptyset\}$ which is generated from a $\frac{\epsilon}{2\rho(\epsilon)}$ -smooth adversary on $\mathcal{X} \cup \{x_\emptyset\}$ such that for all $t \in [T]$, $\tilde{x}_t \in \{x_t, x_\emptyset\}$ and

$$\mathbb{P}[\tilde{x}_t = x_\emptyset \mid \tilde{x}_{<t}, x_{<t}] \leq \rho(\epsilon).$$

As a remark, Lemma 12 additionally assumes that the tolerance function ρ is such that $\rho(\epsilon)/\epsilon$ is non-increasing, which is a mild technical regularity condition. In fact, this property is automatic if distributions in \mathcal{U} are non-atomic (see Proposition 24 for a formal statement and proof). More generally, note that this is without loss of generality: any generalized smoothed distribution class \mathcal{U} admits a concave tolerance function ρ with $\rho(0) = 0$, up to replacing its original tolerance function with its concave envelope $\tilde{\rho}$ (the smallest concave function $\tilde{\rho}$ with $\tilde{\rho}(x) \geq \rho(x)$ for all $x \geq 0$). In most scenarios, this will not change the achieved rates.

Realizable setting. We start with the realizable setting, where there exists a function $f^* \in \mathcal{F}$ fixed *a priori*, and such that $y_t = f^*(x_t)$ for all $t \geq 1$. In this setting, we show that ERM universally learns VC function classes under any threshold learnable distribution class \mathcal{U} .

Theorem 13 *Let $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ be a function class with finite VC dimension and let \mathcal{U} be a generalized smoothed distribution class on \mathcal{X} with a well-behaved tolerance function $\rho : [0, 1] \rightarrow \mathbb{R}_+$. Then, for any realizable \mathcal{U} -constrained adversary, ERM makes predictions \hat{y}_t such that*

$$\text{Reg}_T(\text{ERM}) = \mathbb{E} \left[\sum_{t=1}^T \ell_t(\hat{y}_t) \right] \leq \inf_{\epsilon > 0} \left\{ C \sqrt{\frac{\rho(\epsilon)}{\epsilon} d T \ln^5 T} + \rho(\epsilon) T \right\}.$$

The proof essentially combines the coupling reduction Lemma 12 with known regret bounds for ERM in the realizable setting. Naturally, for any $\epsilon > 0$, compared the bound for the $\mathcal{O}(\epsilon/\rho(\epsilon))$ -smooth adversaries from Block et al. (2024b, Theorem 8), the regret bound from Theorem 13 includes an additive term $\rho(\epsilon)T$ corresponding to the expected number of disagreements between the original and smoothed sequence constructed in Lemma 12.

Agnostic setting. Next, we turn to the general agnostic setting in which no assumptions are made on the labels y_t . We show that the algorithm R-COVER from Blanchard (2025) also universally learns VC classes under any threshold learnable distribution class \mathcal{U} .

Theorem 14 *Let $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ be a function class with finite VC dimension and let \mathcal{U} be a generalized smoothed distribution class on \mathcal{X} with a well-behaved tolerance function $\rho : [0, 1] \rightarrow [0, 1]$. Then, for any adaptive adversary, R-COVER makes predictions \hat{y}_t such that*

$$\mathbb{E} \left[\sum_{t=1}^T \ell_t(\hat{y}_t) - \inf_{f \in \mathcal{F}} \sum_{t=1}^T \ell_t(f(x_t)) \right] \leq C \ln^{3/2} T \cdot \inf_{\epsilon > 0} \left\{ \sqrt{\frac{\rho(\epsilon)}{\epsilon}} dT \cdot \ln T + \rho(\epsilon) T \sqrt{d} \right\},$$

for some universal constant $C > 0$.

Due to the recursive nature of R-COVER, the proof requires additional care to handle the disagreements between the original sequence x_1, \dots, x_T and the smoothed sequence $\tilde{x}_1, \dots, \tilde{x}_T$ from Lemma 12. Specifically, previous samples are used recursively to construct covers, hence disagreements between the two sequences may amplify with the depth of the construction. This increases the additive disagreement term in Theorem 14 by a factor $\sqrt{d} \cdot \text{polylog } T$, compared to the ERM regret bound in Theorem 13.

5. Distribution-class-dependent regret bounds

The previous sections established that generalized smoothness characterizes VC-dependent learnability and that universal algorithms exist which achieve low regret without knowing the specific distribution family \mathcal{U} . A natural follow-up question is: *can we do better when \mathcal{U} is known?*

When the distribution family \mathcal{U} is known to the learner, one can often obtain sharper regret bounds by exploiting additional quantitative structure of \mathcal{U} . In this section, we develop a simple combinatorial parameterization of distribution families and connect it to upper and lower regret bounds. The key insight is that the “complexity” of a distribution family can be measured by how fragmented mass can be: if \mathcal{U} allows mass to be spread across many disjoint regions, learning is harder. This can be formalized by the ϵ -fragmentation number of the distribution family \mathcal{U} , which intuitively measures how “fragmented” mass can be under distributions in \mathcal{U} at scale $\epsilon \in (0, 1]$.

Definition 15 (Fragmentation number) *For any distribution class \mathcal{U} and scale $\epsilon \in (0, 1]$, the ϵ -fragmentation number of \mathcal{U} is defined via*

$$N_{\mathcal{U}}(\epsilon) := \max\{k \in \mathbb{N} : \exists \text{ disjoint } A_1, \dots, A_k \in \mathcal{B}, \forall i \in [k], \mu_{\mathcal{U}}(A_i) \geq \epsilon\}.$$

As a useful remark, $\mu_{\mathcal{U}}$ is a continuous submeasure if and only if $N_{\mathcal{U}}(\epsilon) < \infty$ for all $\epsilon > 0$ (see the beginning of the proof of Lemma 7 for the proof).

In words, the ϵ -fragmentation number is maximum number of disjoint regions that can each carry at least ϵ mass under some $\mu \in \mathcal{U}$. Equivalently, it counts disjoint regions on which a \mathcal{U} -constrained adversary may place $\epsilon \in (0, 1]$ mass. In particular, a \mathcal{U} -constrained adversary may emulate, for a fraction ϵ of the rounds, an adversary which selects the region on which the next sample lies, among these disjoint regions. This observation leads to the following regret lower bound for \mathcal{U} -constrained adversaries.

Theorem 16 Fix a family of distributions \mathcal{U} on \mathcal{X} and $T \geq 1$. For any $\epsilon > 0$ and $d \geq 1$ such that $3d \leq N_{\mathcal{U}}(\epsilon)$. Then, there exists a function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ of VC dimension at most d such that for any algorithm alg ,

$$\text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U}) \gtrsim \sqrt{\epsilon d T \ln \frac{N_{\mathcal{U}}(\epsilon)}{d}} \wedge (\epsilon T).$$

for some oblivious \mathcal{U} -constrained adversary.

We next turn to regret upper bounds. We first give a quantitative version of [Lemma 8](#) using fragmentation numbers. This essentially bounds the continuity modulus of the envelope functional $\mu_{\mathcal{U}}$ as a function of fragmentation numbers.

Lemma 17 Let \mathcal{U} be a family of distributions and $\epsilon \in (0, 1]$ such that $N_{\mathcal{U}}(\epsilon) < \infty$. Then, there exists a distribution μ_{ϵ} on \mathcal{X} such that for any measurable set $A \subseteq \mathcal{X}$,

$$\mu_{\epsilon}(A) \leq \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)^2} \implies \mu_{\mathcal{U}}(A) \leq 2\epsilon.$$

The proof is constructive and largely similar to that of [Lemma 8](#). Intuitively, [Lemma 17](#) shows that at the scale of events of mass $\Omega(\epsilon)$, the distribution class \mathcal{U} is essentially $\mathcal{O}(N_{\mathcal{U}}(\epsilon)^2)$ -smooth with respect to a fixed base distribution μ_{ϵ} . With this property at hand, we can run a simple Hedge algorithm over an appropriate cover of \mathcal{F} , following standard arguments for smoothed adversaries ([Haghtalab et al., 2024](#); [Block et al., 2022](#)), to achieve the following regret upper bound.

Theorem 18 Let \mathcal{U} be a family of distributions on \mathcal{X} and $T \geq 1$. For any function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ of VC dimension $d \geq 1$, there exists an algorithm alg such that

$$\text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U}) \lesssim \inf_{\epsilon > 0} \{ \sqrt{dT \ln(TN_{\mathcal{U}}(\epsilon))} + \epsilon T \}.$$

As an important remark, both upper and lower bounds depend only on the logarithm of fragmentation numbers $N_{\mathcal{U}}(\epsilon)$ for $\epsilon \in (0, 1]$. In comparison, the bounds from ERM and R-COVER qualitatively depend *polynomially* on the fragmentation numbers. Specifically, if $N_{\mathcal{U}}(\epsilon) < \infty$ for all $\epsilon > 0$, [Lemma 17](#) suggests that \mathcal{U} is generalized smooth with a function ρ such that $\rho(\epsilon/N_{\mathcal{U}}(\epsilon)^2) = 2\epsilon$ for $\epsilon \in (0, 1]$. Provided that ρ is additionally concave, [Theorem 13](#) yields the following regret bound for ERM:

$$\text{Reg}_T(\text{ERM}; \mathcal{F}, \mathcal{U}) \lesssim \inf_{\epsilon > 0} \{ N_{\mathcal{U}}(\epsilon) \sqrt{dT \ln^5 T} + \epsilon T \},$$

for function classes \mathcal{F} of VC dimension d . [Theorem 14](#) yields similar regret bounds for R-COVER. Of course, this regret bound is looser than [Theorem 18](#) but has the advantage of being achieved by a universal algorithm that does not require knowledge of the distribution class \mathcal{U} .

6. Differential Privacy

In this section, we will extend the insights from our characterization of distributionally constrained online learning to the setting of differentially private learning. Recall that differential privacy ([Dwork et al., 2006](#)) is a rigorous framework for ensuring that learning algorithms do not leak sensitive information about individual data points. Though differential privacy can be defined in broad generality, we will focus on the setting of learning binary classifiers and present the definition specialized to this setting.

Definition 19 (Differential Private Learning) A randomized algorithm $\mathcal{A} : (\mathcal{X} \times \{0, 1\})^m \rightarrow \{0, 1\}^{\mathcal{X}}$ is (α, β) -differentially private if for any pair of datasets $S, S' \in \mathcal{X}^m$ differing in a single entry, and for any measurable set $E \subseteq \mathcal{H}$,

$$\mathbb{P}(\mathcal{A}(S) \in E) \leq e^\alpha \mathbb{P}(\mathcal{A}(S') \in E) + \beta. \quad (3)$$

Further, for a distribution class \mathcal{U} on \mathcal{X} , a function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ is said to be (α, β) -privately (ϵ, δ) -accurately learnable under \mathcal{U} with sample complexity m if there exists an (α, β) -differentially private algorithm $\mathcal{A} : \mathcal{X}^m \rightarrow \{0, 1\}^{\mathcal{X}}$ such that for any distribution D over $\mathcal{X} \times \{0, 1\}$ with marginal on \mathcal{X} in \mathcal{U} , we have

$$\mathbb{P}_{S \sim D^m} \left(L_D(\mathcal{A}(S)) \leq \inf_{f \in \mathcal{F}} L_D(f) + \epsilon \right) \geq 1 - \delta \quad (4)$$

where $L_D(f) = \mathbb{P}_{(x,y) \sim D}(f(x) \neq y)$.

With this definition at hand, we can now state our characterization of private learnability under distribution constraints. This characterization mirrors our main result for online learning: generalized smoothness is necessary for private learnability with VC-dependent sample complexity.

Theorem 20 *There is a constant $c_0 > 0$ such that the following holds. Let \mathcal{U} be a convex distribution class on \mathcal{X} such that for any generalized threshold class \mathcal{F} and any $\epsilon > 0$, there exists $m(\epsilon, \mathcal{F}) < \infty$ such that for \mathcal{F} is $(0.1, \frac{c_0}{m(\epsilon, \mathcal{F})^2 \log m(\epsilon, \mathcal{F})})$ -privately $(\epsilon, 1/32)$ -accurately learnable under \mathcal{U} with sample complexity $m(\epsilon, \mathcal{F})$. Then, \mathcal{U} is a generalized smooth distribution class.*

As a remark, we believe that a stronger version of this statement may be possible, by showing that generalized smoothness also characterizes private *asymptotic consistency*, for which accuracy is only required asymptotically for each fixed data distribution.

We complement this result by showing that differentially private learnability with VC-dependent sample complexity is achievable under any generalized smooth distribution class. In fact, this is a simple consequence of the fact that under generalized smoothness VC class has uniform covers (Definition 10) and the fact that for finite classes the exponential mechanism (McSherry and Talwar, 2007) provides private learnability with logarithmic dependence on the size of the class.

Theorem 21 (Private Learnability under Generalized Smoothness) *Let \mathcal{U} be a (ρ, μ_0) generalized smooth distribution class on \mathcal{X} . Then, for any function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$ of VC dimension $d \geq 1$, any $\epsilon, \delta \in (0, 1)$, and any $\alpha \in (0, 1)$, there exists an $(\alpha, 0)$ -differentially private algorithm that (ϵ, δ) -accurately learns \mathcal{F} under \mathcal{U} with sample complexity*

$$m \lesssim \frac{d \log(1/\epsilon) + \log(1/\delta)}{\epsilon^2} + \frac{d \log(\rho^{-1}(\epsilon))}{\alpha \epsilon}.$$

This result can be seen as being related to the line of work on private learning with unlabelled public data (Alon et al., 2019a; Bassily et al., 2020; Block et al., 2024a; Block and Shetty, 2025) since unlabelled public data from the base distribution suffices to get the improved sample complexity guarantees. Further, using techniques from Haghtalab et al. (2020) similar results can be obtained for query release and data release tasks under generalized smoothness assumptions. It is an interesting research direction to understand whether oracle-efficient private learning algorithms (Block et al., 2024a; Block and Shetty, 2025) can be obtained under generalized smoothness assumptions.

7. Discussion

We have established that generalized smoothness exactly characterizes when VC-dimension-dependent regret bounds are achievable against distributionally constrained adversaries. This characterization bridges the gap between the classical σ -smooth model and the fully adversarial setting, providing a complete picture of learnability under distribution shifts. Further this characterization extends to the setting of differentially private learning, showing that generalized smoothness is necessary and sufficient for private learnability with VC-dependent sample complexity.

Our work fits into the broader program of understanding “beyond worst-case” models in on-line learning and private learning. The characterization suggests that generalized smoothness is a fundamental structural property, much like how VC dimension and Littlestone dimension are fundamental for i.i.d. and adversarial learning respectively. We hope this perspective inspires further work on understanding the landscape of learnability under various distributional assumptions.

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References

- Alekh Agarwal, Daniel Hsu, Satyen Kale, John Langford, Lihong Li, and Robert Schapire. Taming the monster: A fast and simple algorithm for contextual bandits. In *International conference on machine learning*, pages 1638–1646. PMLR, 2014.
- Noga Alon, Raef Bassily, and Shay Moran. Limits of private learning with access to public data. In *Advances in Neural Information Processing Systems*, volume 32, 2019a.
- Noga Alon, Roi Livni, Maryanthe Malliaris, and Shay Moran. Private pac learning implies finite littlestone dimension. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 852–860, 2019b.
- Raef Bassily, Albert Cheu, Shay Moran, Aleksandar Nikolov, Jonathan Ullman, and Steven Wu. Private query release assisted by public data. In *International Conference on Machine Learning*, pages 695–705. PMLR, 2020.
- Amos Beimel, Hai Brenner, Shiva Prasad Kasiviswanathan, and Kobbi Nissim. Bounds on the sample complexity for private learning and private data release. In *Machine Learning*, volume 94, pages 401–437. Springer, 2014.
- Shai Ben-David, Dávid Pál, and Shai Shalev-Shwartz. Agnostic online learning. In *COLT*, volume 3, page 1, 2009.
- Alina Beygelzimer, John Langford, Lihong Li, Lev Reyzin, and Robert Schapire. Contextual bandit algorithms with supervised learning guarantees. In *Proceedings of the Fourteenth International Conference on Artificial Intelligence and Statistics*, pages 19–26. JMLR Workshop and Conference Proceedings, 2011.

- Moïse Blanchard. Universal online learning: An optimistically universal learning rule. In *Conference on Learning Theory*, pages 1077–1125. PMLR, 2022.
- Moïse Blanchard. Agnostic smoothed online learning. In *Proceedings of the 57th Annual ACM Symposium on Theory of Computing*, pages 1997–2006, 2025.
- Moïse Blanchard and Patrick Jaillet. Universal regression with adversarial responses. *The Annals of Statistics*, 51(3):1401–1426, 2023.
- Moïse Blanchard and Samory Kpotufe. Distributionally-constrained adversaries in online learning. *arXiv preprint arXiv:2506.10293*, 2025.
- Moïse Blanchard, Steve Hanneke, and Patrick Jaillet. Adversarial rewards in universal learning for contextual bandits. *arXiv preprint arXiv:2302.07186*, 2023.
- Adam Block and Yury Polyanskiy. The sample complexity of approximate rejection sampling with applications to smoothed online learning. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 228–273. PMLR, 2023.
- Adam Block and Abhishek Shetty. Small loss bounds for online learning separated function classes: A gaussian process perspective, 2025. URL <https://arxiv.org/abs/2502.10292>.
- Adam Block, Yuval Dagan, Noah Golowich, and Alexander Rakhlin. Smoothed online learning is as easy as statistical learning. In *Conference on Learning Theory*, pages 1716–1786. PMLR, 2022.
- Adam Block, Mark Bun, Rathin Desai, Abhishek Shetty, and Steven Wu. Oracle-efficient differentially private learning with public data, 2024a. URL <https://arxiv.org/abs/2402.09483>.
- Adam Block, Alexander Rakhlin, and Abhishek Shetty. On the performance of empirical risk minimization with smoothed data. *arXiv preprint arXiv:2402.14987*, 2024b.
- Fan Chen, Audrey Huang, Noah Golowich, Sadhika Malladi, Adam Block, Jordan T Ash, Akshay Krishnamurthy, and Dylan J Foster. The coverage principle: How pre-training enables post-training. *arXiv preprint arXiv:2510.15020*, 2025.
- Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of Cryptography Conference*, pages 265–284. Springer, 2006.
- David A Freedman. On tail probabilities for martingales. *the Annals of Probability*, pages 100–118, 1975.
- Nika Haghtalab. *Foundation of Machine Learning, by the People, for the People*. PhD thesis, Microsoft Research, 2018.
- Nika Haghtalab, Tim Roughgarden, and Abhishek Shetty. Smoothed analysis of online and differentially private learning. *Advances in Neural Information Processing Systems*, 33:9203–9215, 2020.

- Nika Haghtalab, Tim Roughgarden, and Abhishek Shetty. Smoothed analysis with adaptive adversaries. *Journal of the ACM*, 71(3):1–34, 2024.
- Steve Hanneke. Learning whenever learning is possible: Universal learning under general stochastic processes. *Journal of Machine Learning Research*, 22(130):1–116, 2021.
- Steve Hanneke and Samory Kpotufe. Adaptive sample aggregation in transfer learning, 2025. URL <https://arxiv.org/abs/2408.16189>.
- David Haussler. Sphere packing numbers for subsets of the boolean n-cube with bounded vapnik-chervonenkis dimension. *Journal of Combinatorial Theory, Series A*, 69(2):217–232, 1995.
- Nigel J Kalton and James W Roberts. Uniformly exhaustive submeasures and nearly additive set functions. *Transactions of the American Mathematical Society*, 278(2):803–816, 1983.
- Shiva Prasad Kasiviswanathan, Homin K Lee, Kobbi Nissim, Sofya Raskhodnikova, and Adam Smith. What can we learn privately? *SIAM Journal on Computing*, 40(3):793–826, 2011.
- Nick Littlestone. Learning quickly when irrelevant attributes abound: A new linear-threshold algorithm. *Machine learning*, 2(4):285–318, 1988.
- Frank McSherry and Kunal Talwar. Mechanism design via differential privacy. In *48th Annual IEEE Symposium on Foundations of Computer Science (FOCS'07)*, pages 94–103. IEEE, 2007.
- Alexander Rakhlin, Karthik Sridharan, and Ambuj Tewari. Online learning: Stochastic, constrained, and smoothed adversaries. *Advances in neural information processing systems*, 24, 2011.
- Norbert Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.
- Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.
- Abhishek Vasantha Shetty. *Learning in a Changing World: Covariate Shift, Subset Selection and Optimal PAC Bounds*. PhD thesis, University of California, Berkeley, 2024.
- Michel Talagrand. Maharam’s problem. *Annals of Mathematics*, 168(3):981–1009, 2008.
- Leslie G Valiant. A theory of the learnable. *Communications of the ACM*, 27(11):1134–1142, 1984.
- V. N. Vapnik and A. Ya. Chervonenkis. On the uniform convergence of relative frequencies of events to their probabilities. *Theory of Probability & Its Applications*, 16(2):264–280, 1971. doi: 10.1137/1116025.
- Vladimir Vapnik and Alexey Chervonenkis. Theory of pattern recognition, 1974.

Appendix A. Proofs from Section 3

We start by proving that for threshold learnable distribution classes \mathcal{U} , their envelope measure $\mu_{\mathcal{U}}$ is a continuous submeasure.

Proof of Lemma 7 The first two properties of continuous submeasures are immediate to check by definition of $\mu_{\mathcal{U}} := \sup_{\mu \in \mathcal{U}} \mu$. We focus on proving the continuity property.

Suppose by contradiction that $\mu_{\mathcal{U}}$ is not continuous and fix a sequence of measurable sets $B_i \downarrow \emptyset$ and $\epsilon > 0$ such that for all $i \geq 1$, $\mu_{\mathcal{U}}(B_i) \geq 2\epsilon$. From these decreasing sets, we construct by induction a sequence of disjoint measurable sets $(A_i)_{i \geq 1}$ such that for $i \geq 1$ we have $\mu_{\mathcal{U}}(A_i) \geq \epsilon$, as follows. Let $j_1 = 1$. For a given $i \geq 1$, let $\mu \in \mathcal{U}$ with $\mu(B_{j_i}) > \frac{3}{2}\epsilon$, which exists by construction of $(B_i)_{i \geq 1}$. Since $B_j \downarrow \emptyset$, there exists $j_{i+1} > j_i$ such that $\mu(B_{j_i} \setminus B_{j_{i+1}}) \geq \epsilon$. We then pose $A_i := B_{j_i} \setminus B_{j_{i+1}}$; in particular, $\mu_{\mathcal{U}}(A_i) \geq \mu(A_i) \geq \epsilon$. By construction, $(A_i)_{i \geq 1}$ are all disjoint, which ends the construction.

Fix any bijection $\phi : \mathbb{Q} \rightarrow \mathbb{N}$ between positive integers and rationals. For any rational $r \in \mathbb{R}$, we define the function $f_r : \mathcal{X} \rightarrow \{0, 1\}$ to be the indicator of the set $\bigcup_{y \leq x, y \in \mathbb{Q}} A_{\phi(y)}$. We then consider the function class which collects of all such functions: $\mathcal{F} := \{f_r : r \in \mathbb{R}\}$. By construction, we can check that \mathcal{F} is a generalized threshold function class since for any $r \leq s \in \mathbb{R}$ one has $f_r \leq f_s$. For convenience, in the rest of the proof we use the following abuse of notation. For any interval $I \subseteq \mathbb{R}$ and $x \in \mathcal{X}$ we write $x \in I$ if and only if $x \in \bigcup_{r \in I \cap \mathbb{Q}} A_{\phi(r)}$.

We now show that \mathcal{U} is not weakly learnable for \mathcal{F} . The corresponding construction follows classical arguments. Specifically, we fix an online learner alg and construct an instance on which they incur linear regret. To do so, we recursively a sequence of rationals $(a_t, b_t, r_t)_{t \geq 1}$ such that $a_t < b_t$ for all $t \geq 1$, and corresponding distributions $(\mu_t)_{t \geq 1}$. By default, initially we can pose $a_0 = 0$ and $b_0 = 1$). Fix $t \geq 1$ and suppose $a_{t-1} < b_{t-1}$ have been constructed. We first pose

$$c_t := \frac{a_{t-1} + b_{t-1}}{2}.$$

By construction, there exists $\mu_t \in \mathcal{U}$ such that

$$\mu_t(A_{\phi(c_t)}) \geq \frac{\mu_{\mathcal{U}}(A_{\phi(c_t)})}{2} \geq \frac{\epsilon}{2}. \quad (5)$$

Next, we consider the online learning setup for which x_1, \dots, x_t are sampled independently from μ_1, \dots, μ_t respectively and consider the values $y_{t'} = f_{c_t}(x_{t'})$ for all $t' < t$. We now define the value

$$z_t := \arg \min_{z \in \{0, 1\}} \mathbb{P}(\hat{y}_t^{(t)} = z, x_t \in \{c_t\}), \quad (6)$$

where we denoted by $\hat{y}_t^{(t)}$ the prediction of the algorithm at time t for this online learning instance. If $z = 0$ we pose $b_t = c_t$ and fix $a_t \in (a_{t-1}, c_t)$ such that

$$\mathbb{P}_{x_{t'} \sim \mu_{t'}} (\exists t' \in [t] : x_{t'} \in (a_t, b_t)) \leq \epsilon/8. \quad (7)$$

Note that this is indeed possible since as $a_t \rightarrow c_t$, the subset of \mathcal{X} corresponding to interval (a_t, c_t) decreases to the empty set. Conversely, if $z_t = 1$, we pose $a_t = c_t$ and fix any $b_t \in (c_t, b_{t-1})$ such that Eq. (7) holds. This ends the recursive construction of the sequences. By construction, note that $(a_t)_{t \geq 1}$ and $(b_t)_{t \geq 1}$ are convergent to the same real value which we denote $r^* \in \mathbb{R}$.

We now check that for the sequence of instances $(\mu_t)_{t \geq 1}$ and function $f^* := f_{r^*} \in \mathcal{F}$, the learner induces linear regret. By construction, note that f^* has value $1 - z_t$ on the set $\{c_t\} = A_{\phi(c_t)}$ for all $t \geq 1$. Therefore, for any $T \geq 1$,

$$\mathbb{E} \left[\sum_{t=1}^T \mathbb{1}[\hat{y}_t \neq f^*(x_t)] \right] \geq \sum_{t=1}^T \mathbb{P}(\hat{y}_t = z_t, x_t \in \{c_t\}).$$

Note that by construction we have $c_t, r^* \in (a_{t-1}, b_{t-1})$. In particular, f^* and f_{c_t} yield the same values $y_{t'}$ for $t' < t$ whenever for all previous samples $t' < t$ one has $x_{t'} \notin (a_{t-1}, b_{t-1})$. Hence, for any $t \geq 1$,

$$\begin{aligned} \mathbb{P}(\hat{y}_t = z_t, x_t \in \{c_t\}) &\geq \mathbb{P}(\hat{y}_t^{(t)} = z_t, x_t \in \{c_t\}) - \mathbb{P}(\exists t' \in [t-1] : x_{t'} \in (a_{t-1}, b_{t-1})) \\ &\stackrel{(i)}{\geq} \frac{\mathbb{P}(x_t \in \{c_t\})}{2} - \frac{\epsilon}{8} \stackrel{(ii)}{\geq} \frac{\epsilon}{8}. \end{aligned}$$

In (i) we used Eqs. (6) and (7) and in (ii) we used Eq. (5). Plugging these estimates within the previous equation shows that for all $T \geq 1$,

$$\mathbb{E} \left[\sum_{t=1}^T \mathbb{1}[\hat{y}_t \neq f^*(x_t)] \right] \geq \frac{\epsilon T}{8}.$$

This ends the proof that \mathcal{U} is not weakly learnable for \mathcal{F} , reaching a contradiction. Hence $\mu_{\mathcal{U}}$ is a continuous submeasure. \blacksquare

Appendix B. Further preliminaries

B.1. Additional definitions

We give the definition of the Littlestone dimension below, which is known to characterize learnable function classes for fully unconstrained adversaries Littlestone (1988); Ben-David et al. (2009).

Definition 22 (Littlestone Dimension) *The Littlestone dimension of a function class $\mathcal{F} : \mathcal{X} \rightarrow \{0, 1\}$, denoted by $\text{Ldim}(\mathcal{F})$, is the largest integer d such that there exists a complete binary tree of depth d with internal nodes labeled by elements of \mathcal{X} and edges labeled by $\{0, 1\}$, such that for every root-to-leaf path $(x_1, b_1), \dots, (x_d, b_d)$, there exists a function $f \in \mathcal{F}$ satisfying $f(x_i) = b_i$ for all $i \in [d]$. If such trees exist for arbitrarily large d , then the Littlestone dimension is infinite.*

B.2. Concentration inequalities

Freedman's inequality Freedman (1975) gives tail probability bounds for martingales. The statement below is for instance taken from (Beygelzimer et al., 2011, Theorem 1) or (Agarwal et al., 2014, Lemma 9).

Theorem 23 (Freedman's inequality) *Let $(Z_t)_{t \in T}$ be a real-valued martingale difference sequence adapted to filtration $(\mathcal{F}_t)_t$. If $|Z_t| \leq R$ almost surely, then for any $\eta \in (0, 1/R)$, with probability at least $1 - \delta$,*

$$\sum_{t=1}^T Z_t \leq \eta \sum_{t=1}^T \mathbb{E}[Z_t^2 \mid \mathcal{F}_{t-1}] + \frac{\ln 1/\delta}{\eta}.$$

B.3. Properties of Generalized Smoothness

In this section, we give further properties of generalized smoothness, starting with short proof of the equivalence between generalized smoothness and having uniform covers for all VC classes.

Proof of Proposition 11 We first check that a (μ_0, ρ) -generalized smooth distribution class \mathcal{U} has uniform covers for any VC class \mathcal{F} . Fix $\epsilon > 0$. The distribution μ_0 admits a finite cover \mathcal{F}_ϵ at scale $\delta := \rho^{-1}(\epsilon) > 0$ (Haussler, 1995), that is, for any $f \in \mathcal{F}$ there exists $\hat{f} \in \mathcal{F}_\epsilon$ such that $\mu_0(\{x : f(x) \neq \hat{f}(x)\}) \leq \delta$. By the generalized smoothness property, \mathcal{F}_ϵ is therefore a uniform cover for \mathcal{U} for scale $\epsilon > 0$.

To complete the proof, it suffices to check that if \mathcal{U} has uniform covers for all generalized thresholds, then it is generalized smooth. Indeed, if it doesn't hold, Lemma 8 shows that $\mu_{\mathcal{U}}$ is not a continuous submeasure. Following the standard argument in the beginning of the proof of Lemma 7, this shows that there exists $\epsilon > 0$ and a sequence of disjoint measurable sets $(A_i)_{i \geq 1}$ such that $\mu_{\mathcal{U}}(A_i) \geq \epsilon$ for all $i \geq 1$. We then construct the generalized threshold class $\mathcal{F} := \{x \mapsto \mathbb{1}[x \in \bigcup_{j \geq i} A_j] : i \geq 1\}$ which by construction is an infinite ϵ -packing for $\mu_{\mathcal{U}}$. That is, for any $f \neq f' \in \mathcal{F}$, there exists $\mu \in \mathcal{U}$ such that $\mu(\{x : f(x) \neq f'(x)\}) \geq \epsilon$. In particular, \mathcal{F} does not admit finite uniform covers for scale $\epsilon/2$, ending the proof. \blacksquare

Next, we show that the tolerance function ρ for a (μ_0, ρ) -generalized smooth distribution class \mathcal{U} is necessarily well-behaved if all distribution in \mathcal{U} are non-atomic.

Proposition 24 *Let \mathcal{U} be a (μ_0, ρ) -generalized smooth distribution class where $\rho : [0, 1] \rightarrow \mathbb{R}_+$ is non-decreasing and $\lim_{\epsilon \rightarrow 0} \rho(\epsilon) = 0$, such that all $\mu \in \mathcal{U}$ are non-atomic (i.e., $\mu(A) = 0$ for all atoms A). Then, the function $\tilde{\rho}$ defined via*

$$\tilde{\rho}(\epsilon) = \sup\{\mu(A) : \mu \in \mathcal{U}, A \in \mathcal{B}, \mu_0(A) \leq \epsilon\}, \quad \epsilon \geq 0,$$

is well-behaved and \mathcal{U} is $(\mu_0, \tilde{\rho})$ -generalized smooth. Also, $\tilde{\rho}(\epsilon) \leq \rho(\epsilon)$ for all $\epsilon \geq 0$.

Proof First, since \mathcal{U} is (μ_0, ρ) -generalized smooth, for any $\epsilon \geq 0$, $\mu \in \mathcal{U}$ and $A \in \mathcal{B}$ with $\mu_0(A) \leq \epsilon$ we have $\mu(A) \leq \rho(\epsilon)$. Hence, by construction of $\tilde{\rho}$, we have $\tilde{\rho}(\epsilon) \leq \rho(\epsilon)$. Also, by construction we directly have $\tilde{\rho}(0) = 0$.

It remains to check that $\tilde{\rho}$ is well-behaved. To do so, we first show that for any non-atomic distribution $\mu \ll \mu_0$ on \mathcal{X} , $A \in \mathcal{B}$ and $\delta \in [0, \mu_0(A)]$, there exists $B \subseteq A$ such that $\mu_0(B) \leq \delta$ and $\mu(B) \geq \mu(A) \frac{\delta}{\mu_0(A)}$. To do so, for any $z \geq 0$ we define the set $B_z := A \cap \{x : \frac{d\mu}{d\mu_0}(x) \geq z\}$. Since $\mu \ll \mu_0$ we have $B_z \downarrow 0$ as $z \rightarrow \infty$, while $B_0 = A$. Hence, there exists $z_\delta \geq 0$ such that $\mu_0(B_{z_\delta}) \geq \delta$ but $\mu_0(A \cap \{x : \frac{d\mu}{d\mu_0}(x) > z_\delta\}) \leq \delta$. As a result, there exists a set B such that

$$A \cap \left\{x : \frac{d\mu}{d\mu_0}(x) > z_\delta\right\} \subseteq B \subseteq A \cap \left\{x : \frac{d\mu}{d\mu_0}(x) \geq z_\delta\right\} = B_{z_\delta}$$

and an atom $C \in \mathcal{B}$ such that $B \cup C \subseteq B_{z_\delta}$, $\mu_0(B) \leq \delta$ but $\mu_0(B \cup C) \geq \delta$. Now we compute

$$\begin{aligned} \mu(A) &= \mu(B \cup C) + \mu(A \setminus (B \cup C)) \stackrel{(i)}{\leq} \mu(B \cup C) + z_\delta \mu_0(A \setminus (B \cup C)) \\ &\stackrel{(ii)}{\leq} \mu(B \cup C) + \frac{\mu(B \cup C)}{\mu_0(B \cup C)} \mu_0(A \setminus (B \cup C)) \stackrel{(iii)}{=} \mu(B) \frac{\mu_0(A)}{\mu_0(B \cup C)}. \end{aligned}$$

In (i) we used the fact that $A \setminus (B \cup C) \subseteq \{x : \frac{d\mu}{d\mu_0}(x) \leq z_2\}$ and in (ii) we used $B \cup C \subseteq B_{z_\delta}$. In (iii) we used $\mu(B) = \mu(B \cup C)$ since μ is non-atomic. This exactly gives the desired bound $\mu(B) \geq \mu(A) \frac{\delta}{\mu_0(A)}$ while $\mu_0(B) \leq \delta$.

With this property at hand, we can check that $\tilde{\rho}(\epsilon)/\epsilon$ is non-increasing in ϵ . First, note that all $\mu \in \mathcal{U}$ satisfy $\mu \ll \mu_0$ otherwise we would have $\lim_{\epsilon \rightarrow 0} \rho(\epsilon) > 0$. Also, by assumption, all distributions $\mu \in \mathcal{U}$ are non-atomic. Now fix any $0 < \epsilon_1 \leq \epsilon_2$. For any $\eta > 0$ there is $\mu \in \mathcal{U}$, $A \in \mathcal{B}$ with $\mu_0(A) \leq \epsilon_2$ and $\mu(A) \geq \tilde{\rho}(\epsilon_2) - \eta$. From the previous paragraph, there is $B \subseteq A$ with $\mu_0(B) \leq \epsilon_1$ and $\mu(B) \geq (\tilde{\rho}(\epsilon_2) - \eta) \frac{\epsilon_1}{\epsilon_2}$. This holds for any $\eta > 0$ and hence $\tilde{\rho}(\epsilon_1) \geq \tilde{\rho}(\epsilon_2) \frac{\epsilon_1}{\epsilon_2}$. ■

Appendix C. Proofs from Section 4

We start by proving the reduction coupling Lemma 12 from generalized smooth adversaries to (standard) smooth adversaries.

Proof of Lemma 12 Denote by μ_0 the base measure for the generalized smoothed class \mathcal{U} . We also denote by $\tilde{\mu}_0 := \frac{1}{2}\mu_0 + \frac{1}{2}\delta_{x_\emptyset}$ the mixture of the base measure and a Dirac at the dummy variable x_\emptyset . For convenience, we denote $\eta := \rho(\epsilon)/\epsilon$. For any $t \in [T]$, denote by μ_t the distribution of x_t conditional on the past history $x_{<t}$. We then construct the sample \tilde{x}_t via

$$\tilde{x}_t := \begin{cases} x_t & \text{if } \frac{d\mu_t}{d\mu_0}(x_t) \leq \eta \\ x_\emptyset & \text{otherwise.} \end{cases}$$

Denote by $\tilde{\mu}_t$ the distribution of \tilde{x}_t conditional to $x_{<t}$. We can immediately check that for any $x \in \mathcal{X}$,

$$\frac{d\tilde{\mu}_t}{d\tilde{\mu}_0}(x) \leq 2 \frac{d\tilde{\mu}_t}{d\mu_0}(x) = 2 \frac{d\mu_t}{d\mu_0}(x) \mathbb{1} \left[\frac{d\mu_t}{d\mu_0}(x) \leq \eta \right] \leq 2\eta.$$

On the other hand, we clearly have $\frac{d\tilde{\mu}_t}{d\tilde{\mu}_0}(x_\emptyset) \leq 2$ since $\tilde{\mu}_0$ places 1/2 mass onto x_\emptyset . This ends the proof that $\tilde{x}_1, \dots, \tilde{x}_T$ is $\frac{1}{2\eta}$ -smooth with respect to $\tilde{\mu}_0$. Next, for any $t \in [T]$,

$$\mathbb{P}[\tilde{x}_t = x_\emptyset \mid \tilde{x}_{<t}, x_{<t}] = \mathbb{P} \left[\frac{d\mu_t}{d\mu_0}(x_t) > \eta \mid x_{<t} \right] = \mu_t \left(\left\{ \frac{d\mu_t}{d\mu_0}(x) > \eta \right\} \right).$$

For convenience, denote $A_t := \left\{ \frac{d\mu_t}{d\mu_0}(x) > \eta \right\}$. Then, by construction, $\eta\mu_0(A_t) < \mu_t(A_t) \leq \rho(\mu_0(A_t))$, where in the last inequality we used $\mu_t \in \mathcal{U}$ and the generalized smoothed property. Recalling that $\eta = \frac{\rho(\epsilon)}{\epsilon}$ and $\alpha \mapsto \frac{\rho(\alpha)}{\alpha}$ is non-increasing, we obtain $\mu_0(A_t) \leq \epsilon$ and in turn, $\mu_t(A_t) \leq \rho(\epsilon)$. This ends the proof. ■

Next, we prove Theorem 13 which gives regret bounds for ERM under any VC class and generalized smooth distribution class \mathcal{U} .

Proof of Theorem 13 We will use Lemma 12 to reduce the problem to smooth adversaries, for which Block et al. (2024b) gives regret lower bounds. To do so, we fix $\epsilon > 0$ and letting $\sigma := \frac{\epsilon}{2\rho(\epsilon)}$, we use Lemma 12 to construct the σ -smooth sequence $\tilde{x}_1, \dots, \tilde{x}_T$ as guaranteed by this result. We then extend the function class \mathcal{F} to $\mathcal{X} \cup \{x_\emptyset\}$ by posing $f(x_\emptyset) = 0$ for all $f \in \mathcal{F}$. Throughout, we use tildes to distinguish between quantities computed for the smooth sequence \tilde{x} or the original sequence x .

Denote by f_t the prediction function chosen by the ERM algorithm at time t (which was trained on the available data $(x_{t'}, f^*(x_{t'}))_{t' < t}$). The main observation is that f_t would still be an ERM prediction function on the training data $(\tilde{x}_{t'}, f^*(\tilde{x}_{t'}))_{t' < t}$ by construction of $\tilde{x}_{<t}$ and since all functions coincide on x_\emptyset . For convenience, denote $\tilde{y}_t := f_t(\tilde{x}_t)$. Since the sequence \tilde{x} is σ -smooth, the regret bound from (Block et al., 2024b, Theorem 8) applies:

$$\mathbb{E} \left[\sum_{t=1}^T \mathbb{1}[\tilde{y}_t \neq f^*(\tilde{x}_t)] \right] \leq C \sqrt{\frac{d}{\sigma} T \ln^5 T} = C \sqrt{\frac{2\rho(\epsilon)}{\epsilon} d T \ln^5 T}$$

for some constant $C > 0$. As a slight note, the (Block et al., 2024b, Theorem 8) gives a dependency of the regret in $1/\sigma$, but this can be improved to $1/\sqrt{\sigma}$ with the same proof as noted in Blanchard (2025). As a result, we obtain

$$\mathbb{E} \left[\sum_{t=1}^T \ell_t(\hat{y}_t) \right] \leq \mathbb{E} \left[\sum_{t=1}^T \mathbb{1}[\tilde{y}_t \neq f^*(\tilde{x}_t)] + \mathbb{1}[\tilde{x}_t = x_\emptyset] \right] \leq C \sqrt{\frac{2\rho(\epsilon)}{\epsilon} d T \ln^5 T} + \rho(\epsilon) T,$$

where in the last inequality we used the last property from Lemma 12. Taking the infimum over $\epsilon > 0$ ends the proof. \blacksquare

Finally, we prove the regret bound on R-COVER from Theorem 14 for agnostic distributionally constrained adversaries.

Proof of Theorem 14 Again, we use Lemma 12 to reduce the problem to smooth adversaries, for which Blanchard (2025) gives regret lower bounds. We use the same notations for the $\sigma := \frac{\epsilon}{2\rho(\epsilon)}$ -smooth sequence $\tilde{x}_1, \dots, \tilde{x}_T$ as in the proof of Theorem 13.

Importantly, the only part of the original proof which uses smoothness is in (Blanchard, 2025, Lemma 11) which bounds some discrepancy terms denoted $\Gamma_k^{(p,r)}$ (for both classification and regression), and when going from oblivious regret bounds to adaptive regret bounds (only for classification). In particular, their regret decomposition still holds even for the regression setup. We rewrite these bounds here for completeness. For context, the R-COVER algorithm proceeds by epochs on several layers $p \in \{0, \dots, P = \lfloor \log_2 T \rfloor\}$: these layer- p epochs are denoted $E_1^{(p)}, \dots, E_{N_p}^{(p)}$ and form a roughly-balanced partition of $[T]$ and $N_p = 2^{P-p}$. Their end points are denoted $E_k^{(p)} := (T_{k-1}^{(p)}, T_k^{(p)})$ for $k \in [N_p]$. As a first step of the original proof (see (Blanchard, 2025, Equation (14)), the regret of the learning-with-experts algorithm at layer p during epoch $E_k^{(p)}$ for $k \in [N_p]$ are bounded using the following quantity (with simplified notation)

$$\Delta_k^{(p)} := \sum_{t \in E_k^{(p)}} \mathbb{E}_{r \sim \hat{p}_t} [(\ell_t(\hat{y}_t) - \ell_t(\mathcal{A}_r(t)))^2].$$

Here, \hat{p}_t is the distribution among experts used by the algorithm at time t , \hat{y}_t denotes the prediction of the expert, and $\mathcal{A}_{r,t}(t)$ denotes the prediction of expert r at time t . The main point is that we can bound

$$\Delta_k^{(p)} \leq \bar{\Delta}_k^{(p)} + \sum_{t \in E_k^{(p)}} \mathbb{1}[\tilde{x}_t = x_\emptyset] \quad \text{where} \quad \bar{\Delta}_k^{(p)} := \sum_{t \in E_k^{(p)}} \mathbb{E}_{r \sim \hat{p}_t} [(\ell_t(\hat{y}_t) - \ell_t(\mathcal{A}_r(t)))^2] \mathbb{1}[\tilde{x}_t = x_t]. \quad (8)$$

As a note, $\bar{\Delta}_k^{(p)}$ is distinct from the quantity $\tilde{\Delta}_k^{(p)}$ which would correspond to using the smoothed sequence \tilde{x}_t throughout the game instead of x_t , since R-COVER constructs covers (which affect \hat{p}_t) using the available data x_t . Similarly, we can bound the following quantity which is also used within the original proof:

$$\Lambda_k^{(p)} := \sum_{t \in E_k^{(p)}} \ell_t \left(f_{k,S}^{(p)}(x_t) \right) - \ell_t(f^*(x_t)) \leq \tilde{\Lambda}_k^{(p)} + \sum_{t \in E_k^{(p)}} \mathbb{1}[\tilde{x}_t = x_\emptyset], \quad (9)$$

where we recall that $\tilde{\Lambda}_k^{(p)}$ denotes the quantity to $\Lambda_k^{(p)}$ but for the smoothed sequence \tilde{x}_t .

We further the bounds as in the original proof, using (Blanchard, 2025, Lemma 10). Formally, we first define

$$\mathcal{P}_k^{(p)} := \{f \in \mathcal{F} : \max_{t \in [T_{k-1}^{(p)}]} |f(x_t) - f^*(x_t)| \leq 2\alpha\},$$

which corresponds to the set of functions approximately consistent with f^* on instances from previous layer- p epochs. Then, for any $r \geq 1$, we define

$$\bar{\Gamma}_k^{(p,r)} := \sum_{t \in E_k^{(p)}} \bar{\gamma}^{(p,r)}(t) \quad \text{where} \quad \bar{\gamma}^{(p,r)}(t) := \sup_{f \in \mathcal{P}_k^{(p)}} \mathbb{E}[|f(x_t) - f^*(x_t)|^r \mathbb{1}[\tilde{x}_t = x_t] \mid \mathcal{H}_{t-1}], \quad t \in E_k^{(p)}$$

which intuitively quantifies the ℓ_r discrepancy between the queries on epoch $E_k^{(p)}$ and queries prior to this epoch. Since within $\bar{\Delta}_k^{(p)}$, we only focus on times when \tilde{x}_t and x_t coincide, the same proof as for (Blanchard, 2025, Lemma 10) shows that with probability at least $1 - \delta$, we have $\bar{\Delta}_k^{(p)} \leq 5\bar{\Gamma}_k^{(p,2)} + 16 \ln \frac{T}{\delta}$ for all $p \in [P]$, $k \in [N_p]$. A main observation is that since all functions agree on x_\emptyset , observing the value of f^* on this instance does not bring any new information. Formally, for all layers p and epochs k , $\mathcal{P}_k^{(p)} \subseteq \tilde{\mathcal{P}}_k^{(p)}$. Hence, for any $t \in R_k^{(p)}$, one has

$$\bar{\gamma}^{(p,r)}(t) \leq \tilde{\gamma}^{(p,r)}(t) := \sup_{f \in \tilde{\mathcal{P}}_k^{(p)}} \mathbb{E}[|f(\tilde{x}_t) - f^*(\tilde{x}_t)|^r \mid \mathcal{H}_{t-1}].$$

In particular, we obtained $\bar{\Gamma}_k^{(p,r)} \leq \tilde{\Gamma}_k^{(p,r)} := \sum_{t \in E_k^{(p)}} \tilde{\gamma}^{(p,r)}(t)$ for all $r \geq 1$, which corresponds to the quantity used within the original proof if we were working with the smoothed data \tilde{x}_t .

Altogether, the regret decomposition bound from (Blanchard, 2025, Equation (14)) shows that for any $p_0 \in \{0, \dots, P\}$, with probability at least $1 - 4\delta$,

$$\begin{aligned}
 & \sum_{t=1}^T \ell_t(\hat{y}_t) - \ell_t(f^*(x_t)) \\
 & \leq 12 \sum_{p=\max(p_0,1)}^P \sum_{k \in [N_p]} \sqrt{2 \max(\Delta_k^{(p)}, 2) \ln(\mathcal{N}(\mathcal{F}; \alpha, T) + 1)} + 8N_{p_0} \ln \frac{2T}{\delta} + \sum_{k \in [N_{p_0}]} \Lambda_k^{(p)} \\
 & \stackrel{(i)}{\leq} 12 \sum_{p=\max(p_0,1)}^P \sum_{k \in [N_p]} \sqrt{2 \max(\tilde{\Delta}_k^{(p)}, 2) \ln(\mathcal{N}(\mathcal{F}; \alpha, T) + 1)} + 8N_{p_0} \ln \frac{2T}{\delta} + \sum_{k \in [N_{p_0}]} \tilde{\Lambda}_k^{(p)} \\
 & + (12P\sqrt{2 \ln(\mathcal{N}(\mathcal{F}; \alpha, T) + 1)} + 1) \sum_{t \in [T]} \mathbb{1}[\tilde{x}_t = x_\emptyset] \\
 & \leq 12 \sum_{p=\max(p_0,1)}^P \sum_{k \in [N_p]} \sqrt{2 \left(5\tilde{\Gamma}_k^{(p,2)} + 16 \ln \frac{T}{\delta} \right) \ln(\mathcal{N}(\mathcal{F}; \alpha, T) + 1)} + 8N_{p_0} \ln \frac{2T}{\delta} \\
 & + \sum_{k \in [N_{p_0}]} \left(2\tilde{\Gamma}_k^{(p_0,1)} + 3 \ln \frac{T}{\delta} \right) + (12P\sqrt{2 \ln(\mathcal{N}(\mathcal{F}; \alpha, T) + 1)} + 1) \sum_{t \in [T]} \mathbb{1}[\tilde{x}_t = x_\emptyset].
 \end{aligned}$$

where $\mathcal{N}(\mathcal{F}; \alpha, T)$ is the α -covering number of \mathcal{F} and α is the parameter used within the algorithm to construct coverings. In (i) we used Eqs. (8) and (9) together with the inequalities $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$ and $\sqrt{a} \leq a$ for any non-negative integer. Importantly, except for the term proportional to $\sum_{t \in [T]} \mathbb{1}[\tilde{x}_t = x_\emptyset]$ the rest of the last term exactly corresponds to the regret bound from the original proof for the smoothed data \tilde{x}_t . In particular, the rest of the proof directly applies. Also, in the binary classification setting, we can take the parameter $\alpha = 0$ and by Sauer-Shelah lemma (Sauer, 1972; Shelah, 1972) we have $\ln \mathcal{N}(\mathcal{F}; \alpha, T) \lesssim d \ln T$. In particular, in the binary classification setting, the rest of the proof of (Blanchard, 2025, Proposition 13) implies that for any $f^* \in \mathcal{F}$, with probability at least $1 - \delta$,

$$\sum_{t=1}^T \ell_t(\hat{y}_t) - \sum_{t=1}^T \ell_t(f^*(x_t)) \leq C \sqrt{\frac{(d \ln^2 T + d \ln \ln \frac{1}{\delta} + \ln \frac{1}{\delta}) \ln^3 T}{\sigma}} \cdot T + C \sqrt{d \ln^3 T} \sum_{t=1}^T \mathbb{1}[\tilde{x}_t = x_\emptyset],$$

for some universal constant $C > 0$. Further, from the last claim of Lemma 12, we can apply Freedman's inequality Theorem 23 to the martingale increments $\mathbb{1}[\tilde{x}_t = x_\emptyset] - \mathbb{P}[\tilde{x}_t = x_\emptyset \mid \tilde{x}_{<t}, x_{<t}]$ to show that with probability at least $1 - \delta$,

$$\sum_{t=1}^T \mathbb{1}[\tilde{x}_t = x_\emptyset] \leq 2\rho(\epsilon)T + 2 \ln \frac{1}{\delta}. \tag{10}$$

The last step of the proof is to generalize this result for oblivious benchmark f^* to adaptive benchmarks (where f^* may be data-dependent). The same arguments from (Blanchard, 2025, Section 5.2) apply here with minor adjustments. As in the original proof, we construct an α -cover \mathcal{H} of \mathcal{F} for the base measure $\tilde{\mu}_0$ with $\ln |\mathcal{H}| \leq 2d \ln(e^2/\alpha)$ for $\alpha = \frac{\sigma}{T} \ln \frac{T}{\delta}$ and bound the adaptive regret

using the regret of the algorithm compared to functions within \mathcal{H} (see (Blanchard, 2025, Eq (30))):

$$\sum_{t=1}^T \ell_t(\hat{y}_t) - \inf_{f \in \mathcal{F}} \sum_{t=1}^T \ell_t(f(x_t)) \leq \sum_{t=1}^T \ell_t(\hat{y}_t) - \inf_{f \in \mathcal{H}} \sum_{t=1}^T \ell_t(f(x_t)) + \sup_{g \in \mathcal{G}} \sum_{t=1}^T g(x_t),$$

where $\mathcal{G} := \{\mathbb{1}[f \neq h_f], f \in \mathcal{F}\}$. The regret compared to \mathcal{H} is bounded using the previous fixed-benchmark-high-probability regret bound and the union bound, as in the original proof. We then note that

$$\sup_{g \in \mathcal{G}} \sum_{t=1}^T g(x_t) \leq \sum_{t=1}^T \mathbb{1}[\tilde{x}_t = x_\emptyset] + \sup_{g \in \mathcal{G}} \sum_{t=1}^T g(\tilde{x}_t).$$

Hence, from there we can use the same arguments as in the original proof to bound the last term. Putting everything together as in the original proof, together with Eq. (10), we obtained with probability at least $1 - 3\delta$,

$$\sum_{t=1}^T \ell_t(\hat{y}_t) - \inf_{f \in \mathcal{F}} \sum_{t=1}^T \ell_t(f(x_t)) \lesssim \sqrt{\frac{(d \ln^2 T + d \ln \ln \frac{1}{\delta} + \ln \frac{1}{\delta}) \ln^3 T}{\sigma}} \cdot T + \rho(\epsilon) T \sqrt{d \ln^3 T}.$$

This ends the proof. ■

Appendix D. Proofs from Section 5

We first prove Lemma 17 which bounds the modulus of continuity of the envelope functional $\mu_{\mathcal{U}}$ whenever \mathcal{U} has finite fragmentation numbers. This gives a quantitative version of Lemma 8.

Proof of Lemma 17 The proof follows the same iterative construction as Lemma 8, but with parameters tuned to track the dependence on $N_{\mathcal{U}}(\epsilon)$. Fix \mathcal{U} and $\epsilon \in (0, 1]$ satisfying the assumptions. We iteratively construct a sequence of distributions $(\mu_i)_{i \geq 1}$ together with measurable sets $(A_i)_{i \geq 1}$ as follows. Suppose we have constructed μ_j, A_j for all $j < i$ for some $i \geq 1$. If there exists a measurable set A such that

$$\sum_{j < i} \mu_j(A) \leq \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)} \quad \text{and} \quad \exists \mu \in \mathcal{U} \text{ with } \mu(A) \geq 2\epsilon,$$

we fix A_i satisfying this property and $\mu_i \in \mathcal{U}$ with $\mu_i(A_i) \geq 2\epsilon$. Otherwise, we end the recursive construction. Next, if A_i is defined, we introduce $B_i := A_i \setminus \bigcup_{j > i} A_j$. By construction, all sets B_i are disjoint and if $i \leq N_{\mathcal{U}}(\epsilon) + 1$,

$$\mu_i(B_i) \geq \mu_i(A_i) - \sum_{j > i} \mu_i(A_j) \geq 2\epsilon - (i - 1) \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)} \geq \epsilon.$$

By definition of $N_{\mathcal{U}}(\epsilon)$, this implies that the construction must have ended at some index $i_{\max} \leq N_{\mathcal{U}}(\epsilon)$. We then pose $\mu_\epsilon := \frac{1}{i_{\max}} \sum_{j=1}^{i_{\max}} \mu_j$. Then, for any measurable set $A \subseteq \mathcal{X}$ with $\mu_\epsilon(A) \leq \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)^2}$ we have

$$\sum_{j=1}^{i_{\max}} \mu_j(A) \leq i_{\max} \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)^2} \leq \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)}.$$

Since the construction ended at i_{\max} , this precisely implies that $\mu_{\mathcal{U}}(A) \leq 2\epsilon$, ending the proof. \blacksquare

We next prove the regret lower bound from [Theorem 16](#) based on fragmentation numbers.

Proof of [Theorem 16](#) The proof uses classical arguments from [Ben-David et al. \(2009\)](#) which gives a lower bound $\Omega(\sqrt{\text{LD}(\mathcal{F})T} \wedge T)$ on the regret for adversarial online learning for a function class \mathcal{F} . We recall the definition of the Littlestone dimension $\text{LD}(\mathcal{F})$ in [Definition 22](#) in the appendix for completeness. Fix $\epsilon > 0$ and for convenience, denote $N = N_{\mathcal{U}}(\epsilon)$. Let A_1, \dots, A_N be disjoint measurable sets such that for all $i \in [N]$, $\mu_{\mathcal{U}}(A_i) \geq \epsilon$. Let $I_j := \{i \in [N] : i \in [(j-1)N/d, jN/d]\}$ for $j \in [d]$. For convenience, we enumerate $I_j := \{i_l^{(j)}, l \in [N_j]\}$. For each $l \in \{0, \dots, N_1\} \times \dots \times \{0, \dots, N_d\}$, we define the function $f_l : \mathcal{X} \rightarrow \{0, 1\}$ via

$$f_l(x) := \begin{cases} 1 & \text{if } x \in \bigcup_{l > l_j} A_{i_l^{(j)}}, j \in [d] \\ 0 & \text{otherwise.} \end{cases}$$

We then consider the function class \mathcal{F} which contains all such functions. For convenience, we also consider a simplified version of this function class $\mathcal{F}' : I_1 \sqcup \dots \sqcup I_d \rightarrow \{0, 1\}$ where we identify all sets A_i to a single point i : it is the product of thresholds on each set I_j for $j \in [d]$. The Littlestone dimension of both function classes is the sum of the Littlestone dimension of each underlying threshold function class:

$$\text{LD}(\mathcal{F}) = \text{LD}(\mathcal{F}') \geq \sum_{j \in [d]} \lceil \log_2(|I_j| + 1) \rceil \gtrsim (d \wedge N) \log_2(\lceil N/d \rceil + 1). \quad (11)$$

Fix a learning algorithm alg . We then emulate the proof of the oblivious regret lower bound for online learning on this function class for a horizon $T' := \lceil \epsilon T/2 \rceil$ as follows. For each new iteration t , let $i_t \in I_1 \sqcup \dots \sqcup I_d$ be the next element which would be chosen by the online learning lower bound adversary. At time t , we then define μ_t to be a distribution such that $\mu_t(A_{i_t}) \geq \epsilon$, which exists by construction. If the sample $x_t \sim \mu_t$ satisfies $x_t \in A_{i_t}$, we use the same labeling y_t as the online learning lower bound adversary. Otherwise, we randomly label $y_t \sim \mathcal{B}(1/2)$ and ignore this iteration for the online learning adversary. Conditionally on the event \mathcal{E} that we were able to emulate T' iterations of the online learning adversary, the regret lower bound for online learning ([Ben-David et al., 2009](#)) implies the desired lower bound

$$\text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U} \mid \mathcal{E}) \gtrsim \sqrt{\text{LD}(\mathcal{F})T'} \wedge T',$$

since the ignored iterations can only increase the regret in expectation. Here, $\text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U} \mid \mathcal{E})$ denotes the conditional regret on \mathcal{E} . By construction, each iteration $t \in [T]$ emulates a step of the online learning adversary with probability ϵ . Therefore, by Azuma-Hoeffding's inequality,

$$\mathbb{P}[\mathcal{E}^c] = \mathbb{P} \left[\sum_{t \in [T]} \mathbb{1}[x_t \in A_{i_t}] < \frac{\epsilon T}{2} \right] \stackrel{(i)}{\leq} \mathbb{P}_{Y \sim \mathcal{B}(T, \epsilon)} \left[Y < \frac{\epsilon T}{2} \right] \stackrel{(ii)}{\leq} e^{-\epsilon T/8}.$$

In (i) we stochastically dominated the quantity of interest by a binomial of parameters T and ϵ , and in (ii) we used Chernoff's inequality. Therefore, the final regret lower bound becomes

$$\begin{aligned} \text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U}) &\geq \text{Reg}_T(\text{alg}; \mathcal{F}, \mathcal{U} \mid \mathcal{E})(1 - \mathbb{P}[\mathcal{E}^c]) \\ &\gtrsim \sqrt{\text{LD}(\mathcal{F})\epsilon T} \wedge (\epsilon T) \\ &\gtrsim \sqrt{\epsilon(d \wedge N_{\mathcal{U}}(\epsilon))T \ln\left(\frac{N_{\mathcal{U}}(\epsilon)}{d} + e\right)} \wedge (\epsilon T), \end{aligned}$$

where in the last inequality we used [Eq. \(11\)](#). This implies the desired result when $3d \leq N_{\mathcal{U}}(\epsilon)$. ■

Finally, we prove the regret upper bound from [Theorem 18](#) for \mathcal{U} -dependent algorithms. This uses a quantitative variant of the coupling [Lemma 12](#) whose proof is deferred to [Appendix E](#).

Proof of [Theorem 18](#) Fix a distribution class \mathcal{U} and a function class \mathcal{F} of VC dimension d . Fix $\epsilon > 0$. From [Lemma 17](#), we can construct a distribution μ_ϵ such that for any measurable set $A \subseteq \mathcal{X}$ with $\mu_\epsilon(A) \leq \frac{\epsilon}{N_{\mathcal{U}}(\epsilon)^2}$, one has $\mu_{\mathcal{U}}(A) \leq 2\epsilon$.

Using [Lemma 25](#), we then couple the original adversary with a sequence $\tilde{x}_1, \dots, \tilde{x}_T$ generated by an $(\epsilon/(4N_{\mathcal{U}}(\epsilon)^2))$ -smooth adversary (with respect to a base measure $\tilde{\mu}_\epsilon$) such that for all $t \in [T]$,

$$\mathbb{P}[\tilde{x}_t \neq x_t \mid \tilde{x}_{<t}, x_{<t}] \leq 2\epsilon.$$

This reduces the analysis to the smoothed model, plus an additive term accounting for coupling failures.

Specifically, consider the algorithm alg_α that runs Hedge over an α -cover of \mathcal{F} with respect to the base measure $\tilde{\mu}_\epsilon$; denote such a cover by $\mathcal{F}(\alpha)$. Here, $\alpha > 0$ is a tuning parameter. Following ([Haghtalab et al., 2020](#), Theorem 3.1), we can bound the expected regret of alg_α by

$$\begin{aligned} &\mathcal{O}\left(\sqrt{T \ln |\mathcal{H}(\alpha)|}\right) + \mathbb{E}\left[\max_{f \in \mathcal{F}} \min_{g \in \mathcal{F}(\alpha)} \sum_{t=1}^T \mathbb{1}[f(x_t) \neq g(x_t)]\right] \\ &\leq \mathcal{O}\left(\sqrt{T \ln |\mathcal{H}(\alpha)|}\right) + \mathbb{E}\left[\max_{f \in \mathcal{F}} \min_{g \in \mathcal{F}(\alpha)} \sum_{t=1}^T \mathbb{1}[f(\tilde{x}_t) \neq g(\tilde{x}_t)]\right] + \mathbb{E}\left[\sum_{t=1}^T \mathbb{1}[\tilde{x}_t \neq x_t]\right]. \end{aligned}$$

By construction, the last term is bounded by $2\epsilon T$. The first two terms exactly correspond to the regret analysis for the smoothed data $\tilde{x}_{\leq T}$. In turn, the arguments from ([Haghtalab et al., 2020](#), Theorem 3.1) directly imply that with an appropriate value of α ,

$$\text{Reg}_T(alg_\alpha; \mathcal{F}, \mathcal{U}) \lesssim \sqrt{Td \ln\left(\frac{4TN_{\mathcal{U}}(\epsilon)^2}{d\epsilon}\right)} + d \ln\left(\frac{4TN_{\mathcal{U}}(\epsilon)^2}{d\epsilon}\right) + \epsilon T.$$

We then minimize this value over $\epsilon \in (0, 1]$. First, note that without loss of generality we may focus on $\epsilon \geq 1/T$: smaller values cannot improve the right-hand side by more than a constant factor, since the first term is already $\Omega(1)$ in that regime.

Next, we can ignore the second term on the right-hand side: if it dominates the first term, then necessarily $d \ln\left(\frac{4TN_{\mathcal{U}}(\epsilon)^2}{d\epsilon}\right) \gtrsim T$, in which case the bound is already trivial (it exceeds T). Tuning the parameter $\epsilon \in (1/T, 1]$ then gives the desired result. ■

Appendix E. A quantitative coupling lemma

The next lemma is a variant of [Lemma 12](#) that is convenient when one only has a guarantee of the form $\mu_0(A) \leq \epsilon \Rightarrow \mu_{\mathcal{U}}(A) \leq \eta$ for some base measure μ_0 .

Lemma 25 *Let \mathcal{U} be a generalized smoothed distribution class on \mathcal{X} , let $\epsilon, \eta \in (0, 1]$, and let μ_0 be a probability measure such that for any measurable set $A \subseteq \mathcal{X}$, if $\mu_0(A) \leq \epsilon$ then $\mu_{\mathcal{U}}(A) \leq \eta$. Consider any \mathcal{U} -constrained adaptive adversary generating samples x_1, \dots, x_T . Fix a dummy symbol $x_\emptyset \notin \mathcal{X}$.*

Then there exists a coupling with a sequence $\tilde{x}_1, \dots, \tilde{x}_T \in \mathcal{X} \cup \{x_\emptyset\}$ which is generated by an $(\epsilon/4)$ -smooth adversary on $\mathcal{X} \cup \{x_\emptyset\}$ such that for all $t \in [T]$,

1. $\tilde{x}_t \in \{x_t, x_\emptyset\}$, and
2. $\mathbb{P}[\tilde{x}_t = x_\emptyset \mid \tilde{x}_{<t}, x_{<t}] \leq \eta$.

Proof The proof is similar to that of [Lemma 12](#). Denote by μ_t the distribution of x_t conditional on the past history $x_{<t}$. Define the mixture base measure

$$\tilde{\mu}_0 := \frac{1}{2}\mu_0 + \frac{1}{2}\delta_{x_\emptyset}.$$

For convenience, set $\sigma := \eta/\epsilon$.

For each $t \in [T]$, construct \tilde{x}_t from x_t as

$$\tilde{x}_t := \begin{cases} x_t & \text{if } \frac{d\mu_t}{d\mu_0}(x_t) \leq 2\sigma \text{ or } \mu_0(\{x_t\}) > \epsilon, \\ x_\emptyset & \text{otherwise.} \end{cases}$$

Let $\tilde{\mu}_t$ be the distribution of \tilde{x}_t conditional on $x_{<t}$. As in the proof of [Lemma 12](#), one checks that the density ratio is bounded by $4/\epsilon$ with respect to $\tilde{\mu}_0$, hence $\tilde{x}_1, \dots, \tilde{x}_T$ is $(\epsilon/4)$ -smooth.

Next, for any $t \in [T]$, define

$$A_t := \left\{ x \in \mathcal{X} : \frac{d\mu_t}{d\mu_0}(x) > 2\sigma \right\} \cap \left\{ x \in \mathcal{X} : \mu_0(\{x\}) \leq \epsilon \right\}.$$

By construction,

$$\mathbb{P}[\tilde{x}_t = x_\emptyset \mid \tilde{x}_{<t}, x_{<t}] = \mu_t(A_t).$$

If $\mu_0(A_t) \leq \epsilon$, then by assumption $\mu_t(A_t) \leq \mu_{\mathcal{U}}(A_t) \leq \eta$. Otherwise, suppose $\mu_0(A_t) > \epsilon$. By [Lemma 26](#), there exists a measurable $B \subseteq A_t$ with $\mu_0(B) \in (\epsilon/2, \epsilon]$. On the one hand, by the defining property of A_t we have $\mu_t(B) \geq 2\sigma \mu_0(B) > \sigma\epsilon = \eta$. On the other hand, $\mu_0(B) \leq \epsilon$ implies $\mu_t(B) \leq \mu_{\mathcal{U}}(B) \leq \eta$, a contradiction. Therefore $\mu_t(A_t) \leq \eta$ in all cases, completing the proof. \blacksquare

Lemma 26 *Let $\epsilon > 0$, let μ be a probability measure on \mathcal{X} , and let $A \subseteq \mathcal{X}$ be measurable such that $\mu(A) > \epsilon$ and for all $x \in A$ we have $\mu(\{x\}) \leq \epsilon$. Then there exists a measurable set $B \subseteq A$ such that $\mu(B) \in (\epsilon/2, \epsilon]$.*

Proof If there exists an atom $x \in A$ with $\mu(\{x\}) \in (\epsilon/2, \epsilon]$, then take $B = \{x\}$.

Otherwise, every atom in A has mass at most $\epsilon/2$. Let $A_{\text{at}} \subseteq A$ be the (at most countable) set of atoms and write $m := \mu(A_{\text{at}}) \in [0, 1]$. If $m \geq \epsilon/2$, pick a finite subset of atoms $\{x_1, \dots, x_k\} \subseteq A_{\text{at}}$ by adding atoms greedily until the partial sum first exceeds $\epsilon/2$. Since each atom has mass at most $\epsilon/2$, the resulting set $B = \{x_1, \dots, x_k\}$ satisfies $\mu(B) \in (\epsilon/2, \epsilon]$.

If $m < \epsilon/2$, then the non-atomic part $A \setminus A_{\text{at}}$ has measure $\mu(A \setminus A_{\text{at}}) > \epsilon - m > \epsilon/2$. Since μ restricted to $A \setminus A_{\text{at}}$ is non-atomic, there exists a measurable $C \subseteq A \setminus A_{\text{at}}$ with $\mu(C) = \epsilon/2$. Taking $B := A_{\text{at}} \cup C$ yields $\mu(B) = m + \epsilon/2 \in (\epsilon/2, \epsilon]$. \blacksquare

Appendix F. Proofs from Section 6

F.1. Proof of Theorem 20

We start by showing the following lemma which is a consequence of Turán's theorem which lower bounds the independence number of graphs with few edges.

Lemma 27 Fix $\epsilon, \delta \in (0, 1]$, $N \geq 1$, and \mathcal{U} be a distribution class such that $N_{\mathcal{U}}(\epsilon) \geq N$. Then, there exist $k \geq N\delta/2$ disjoint measurable sets A_1, \dots, A_k together with distributions $\mu_1, \dots, \mu_k \in \mathcal{U}$ such that

1. for all $i \in [k]$, $\mu_i(A_i) \geq \epsilon$,
2. and for all $i \neq j \in [k]$, $\mu_i(A_j) \leq \delta$.

Additionally, if $N_{\mathcal{U}}(\epsilon) = \infty$, then there exist disjoint measurable sets $(A_i)_{i \geq 1}$, distributions $(\mu_i)_{i \geq 1}$ in \mathcal{U} such that for all $i \geq 1$, $\mu_i(A_i) \geq \epsilon$, and for all $n \geq 1$,

$$\sup_{2^n \leq i \neq j < 2^{n+1}} \mu_i(A_j) \leq 2^{-n}.$$

Proof Fix \mathcal{U} such that $N_{\mathcal{U}}(\epsilon) \geq N$. Then there exist B_1, \dots, B_N disjoint measurable sets together with distributions $\nu_1, \dots, \nu_N \in \mathcal{U}$ with $\nu_i(B_i) \geq \epsilon$ for all $i \in [N]$. Construct the graph $\mathcal{G} = ([N], E)$ on $[N]$ such that $(i, j) \in E$ if and only if $\nu_i(B_j) \geq \delta$ or $\nu_j(B_i) \geq \delta$ for some $\delta \in (0, 1]$. Note that \mathcal{G} has at most N/δ edges since for each $i \in [N]$ there can be at most $1/\delta$ indices $j \in [N]$ for which $\nu_i(B_j) \geq \delta$ given that all sets B_j are disjoint. Therefore, \mathcal{G} has average node degree at most $1/\delta$. Hence, Turán's theorem shows that there is an independent set S of size at least $N/(1 + 1/\delta) \geq N\delta/2$. Focusing only on the indices $i \in S$ with the corresponding sets B_i and distribution ν_i gives the first desired result.

We now prove the second claim when $N_{\mathcal{U}}(\epsilon) = \infty$. Fix a sequence $(B_j)_{j \geq 1}$ of disjoint measurable sets and corresponding distributions $(\nu_j)_{j \geq 1}$ in \mathcal{U} such that $\nu_j(B_j) \geq \epsilon$ for all $j \geq 1$. We then essentially stitch together the above construction for the parameters $\delta_n = 2^{-n}$ for $n \geq 1$. Specifically, having constructed indices $j_i \in \mathbb{N}$ for all $i < 2^n$, we focus on indices that have not been used yet $\mathcal{I}_n := \mathbb{N} \setminus \{j_l : l < 2^n\}$. Among the first 2^{2n} elements of \mathcal{I}_n , the previous arguments show that we can construct distinct indices $i_j \in \mathcal{I}_n$ for $j \in [2^n, 2^{n+1})$ such that letting $A_j = B_{i_j}$ and $\mu_j = \nu_{i_j}$ we have $\nu_i(A_j) \leq 2^{-n}$ for all $i \neq j \in [2^n, 2^{n+1})$. This proves the desired properties and ends the proof. \blacksquare

With this tool at hand we now show that algorithms that privately learn generalized thresholds under \mathcal{U} distributions at scale ϵ with sample complexity m , induce private algorithms for learning thresholds of size $\Omega(\sqrt{N_{\mathcal{U}}(\epsilon)}/m)$.

Lemma 28 *Let \mathcal{U} be a convex distribution class. Fix $m \geq 1$ and $\eta \in (0, 1]$. If for any generalized threshold function class \mathcal{F} , there exists an (ϵ, δ) -differentially private (α, β) -accurate algorithm on distributions in \mathcal{U} with sample complexity m , then there exists an (ϵ, δ) -differentially private $(2\alpha/\eta, 2\beta)$ -accurate PAC learner for thresholds in $[K]$ with sample complexity m , where $K = \Omega\left(\sqrt{\frac{\beta N_{\mathcal{U}}(\eta)}{m}}\right)$.*

Next, suppose that $N_{\mathcal{U}}(\eta) = \infty$ for some $\eta \in (0, 1]$. Then, if for any generalized threshold function class \mathcal{F} , there exists an (ϵ, δ) -differentially private (α, β) -accurate algorithm on distributions in \mathcal{U} with some sample complexity $m(\mathcal{F}) < \infty$, then there exists $m < \infty$ and an (ϵ, δ) -differentially private $(2\alpha/\eta, 2\beta)$ -accurate PAC learner for thresholds in $[K]$ with sample complexity m for any $K \geq 1$.

Proof Let $\gamma \in (0, 1]$ be a parameter to be fixed later. We first use [Lemma 27](#) to construct $K = \lfloor N_{\mathcal{U}}(\eta)\gamma/2 \rfloor$ disjoint measurable sets A_i and distributions $\mu_i \in \mathcal{U}$ for $i \in [K]$ satisfying $\mu_i(A_i) \geq \eta$ and $\mu_i(A_j) \leq \gamma$ for all $i \neq j \in [K]$. We now consider the generalized threshold class on these sets $\mathcal{F} := \{x \in \mathcal{X} \mapsto \mathbb{1}[x \in \bigcup_{j \geq i} A_j] : i \in [K+1]\}$.

Given an (ϵ, δ) -differentially private (α, β) -accurate algorithm for \mathcal{F} on distributions in \mathcal{U} with sample complexity m , we construct a new algorithm \mathcal{A}' as follows. For any dataset $\mathcal{D}' = (a_i, y_i)_{i \in [m]} \in ([K] \times \{0, 1\})^m$, we replace each datapoint (a_i, y_i) with $(x_i, y'_i := y_i \mathbb{1}[x_i \in A_i])$ where $x_i \sim \mu_{a_i}$ are sampled independently for each $i \in [m]$. Denote by \mathcal{D} the corresponding dataset and let h be the output of \mathcal{A} on \mathcal{D} . \mathcal{A}' returns the vector $(\mathbb{1}[\mathbb{E}_{x \sim \mu_i}[h(x) \mid x \in A_i] \geq 1/2])_{i \in [K]}$. Since \mathcal{A} is (ϵ, δ) -private, so is the algorithm that would directly output the hypothesis h (note that \mathcal{A}' can be viewed as a mixture of (ϵ, δ) -private algorithms, one for each potential realization of $x_i \sim \mu_i$ for $i \in [K]$). Hence \mathcal{A}' is also (ϵ, δ) -private.

We next show that \mathcal{A}' is a PAC learner for the class of thresholds on $[K]$ which we denote by $\mathcal{F}' := \{i \in [K] \mapsto \mathbb{1}[i \geq i_0] : i_0 \in [K+1]\}$. Fix any distribution ν on $[K]$ and a threshold $g' \in \mathcal{F}'$. Let $\mathcal{D}' = (a_i, y_i)_{i \in [m]}$ be a dataset corresponding of m i.i.d. samples from ν labeled by g' . Note that for any sample $i \in [m]$, letting $x_i \sim \mu_{a_i}$ be the sampled instance, if $x_i \notin \bigcup_{j \neq i} A_j$, then the value $y_i \mathbb{1}[x_i \in A_i]$ is consistent with the function $g : x \in \mathcal{X} \mapsto \mathbb{1}[x \in \bigcup_{i \in [K]: g'(i)=1} A_i] \in \mathcal{F}$. In particular, under the event

$$\mathcal{E} := \left\{ \forall i \in [m] : x_i \notin \bigcup_{j \neq i} A_j \right\},$$

\mathcal{D}' coincides with a dataset composed of m i.i.d. samples from the mixture $\mu := \sum_{i \in [K]} \nu_i \mu_i$ labeled by g . Note that $\mu \in \mathcal{U}$ since \mathcal{U} is convex and that by construction,

$$\mathbb{P}[\mathcal{E}] \leq mK\gamma.$$

Since \mathcal{A} is (α, β) -accurate, this shows that with probability at least $1 - \beta - mK\gamma$ on the sampled dataset \mathcal{D}' , the output h of \mathcal{A} on the corresponding dataset \mathcal{D} satisfies

$$\begin{aligned} \alpha &\geq \mathbb{P}_{x \sim \mu}[h(x) \neq g(x)] \\ &\geq \mathbb{E}_{i \sim \nu} [\mu_i(A_i) \mathbb{P}_{x \sim \mu_i}[h(x) \neq g'(i) \mid x \in A_i]] \\ &\geq \frac{\eta}{2} \mathbb{E}_{i \sim \nu}[h'(i) \neq g'(i)], \end{aligned}$$

where h' is the final output of \mathcal{A}' . This shows that \mathcal{A}' is an $(2\alpha/\eta, \beta + mK\gamma)$ -accurate PAC learner for \mathcal{F}' . Note that $mK\gamma \leq mN_{\mathcal{U}}(\eta)\gamma^2/2$. Taking $\gamma = \sqrt{\frac{2\beta}{mN_{\mathcal{U}}(\eta)}}$ gives the desired result for the first claim.

The second claim is proved analogously. Suppose that $N_{\mathcal{U}}(\eta) = \infty$. Then, we use [Lemma 27](#) to construct disjoint measurable sets A_i and distributions $\mu_i \in \mathcal{U}$ for $i \geq 1$ satisfying $\mu_i(A_i) \geq \eta$ for all $i \geq 1$, and for all $n \geq 1$ and $i \neq j \in [2^n, 2^{n+1})$, $\mu_i(A_j) \leq 2^{-n}$. As above, we construct the generalized threshold class on these sets $\mathcal{F} := \{x \in \mathcal{X} \mapsto \mathbb{1}[x \in \bigcup_{j \geq i} A_j] : i \geq 1\}$. The main point is that for any $n \geq 1$, this function class essentially encapsulates the previous construction for $\gamma_n = 2^{-n}$, by focusing only on the subclass $\mathcal{F}_n := \{x \in \mathcal{X} \mapsto \mathbb{1}[x \in \bigcup_{j \geq i} A_j] : i \in [2^n, 2^n + K_n]\}$ where $K_n := \lfloor 2^n/m \rfloor$. The only slight modification in the construction of the PAC learner \mathcal{A}' for the thresholds on $[K_n]$ is that we additionally always label $y'_i = 1$ if $x_i \in \bigcup_{j < 2^n} A_j$, to preserve realizability in that case. Following the exact same arguments therefore shows that if there is an (ϵ, δ) -differentially private (α, β) -accurate algorithm for \mathcal{F} on distributions in \mathcal{U} with sample complexity m , then for any $n \geq 1$, there is an $(2\alpha/\eta, \beta + mK_n\gamma_n \leq 2\beta)$ -accurate PAC learner for \mathcal{F}' . This ends the proof. \blacksquare

Combining this result with the sample complexity lower bound from [Alon et al. \(2019b, Theorem 1\)](#) for privately learning thresholds yields the following result.

Theorem 29 *There is a constant $c_0 > 0$ such that the following holds. Let \mathcal{U} be a convex distribution class, $\epsilon \in (0, 1]$, and $m \geq 1$. If for any generalized threshold class \mathcal{F} , there exists a $(0.1, \frac{c_0}{m^2 \log m})$ -differentially private $(\epsilon/32, 1/32)$ -accurate algorithm on distributions from \mathcal{U} , with sample complexity m ; then*

$$m \geq \Omega(\log^* N_{\mathcal{U}}(\epsilon)).$$

Also, if \mathcal{U} is a convex distribution such that $N_{\mathcal{U}}(\epsilon) = \infty$ for some $\epsilon \in (0, 1]$, then there exists a generalized threshold class \mathcal{F} such that for any $m \geq 1$, there does not exist a $(0.1, \frac{c_0}{m^2 \log m})$ -differentially private $(\epsilon/32, 1/32)$ -accurate algorithm on distributions from \mathcal{U} , with sample complexity m .

Proof Given such an algorithm for generalized threshold classes, [Lemma 28](#) constructs a $(0.1, \frac{c_0}{m^2 \log m})$ -differentially private $(1/16, 1/16)$ -accurate PAC learner for thresholds on $[K]$ where $K = \Omega(\sqrt{N_{\mathcal{U}}(\epsilon)/m})$. Therefore, for $c_0 > 0$ sufficiently small, [Alon et al. \(2019b, Theorem 1\)](#) shows that

$$m \geq \Omega(\log^* K) = \Omega(\log^* N_{\mathcal{U}}(\epsilon) - \log^* m).$$

In turn, this shows the desired bound $m \geq \Omega(\log^* N_{\mathcal{U}}(\epsilon))$.

The second claim is proved similarly using the second claim from [Lemma 28](#). \blacksquare

We are now ready to prove [Theorem 20](#) as an immediate consequence of [Theorem 29](#) and the previous characterization of generalized-smooth distribution classes.

Proof of [Theorem 20](#) If \mathcal{U} is not generalized smooth, then [Lemma 8](#) implies that $\mu_{\mathcal{U}}$ is not a continuous submeasure, which in turn implies that there is $\epsilon > 0$ for which $N_{\mathcal{U}}(\epsilon) = \infty$. Then, the desired impossibility result for private learnability is a direct consequence from [Theorem 29](#). ■

E.2. Proof of [Theorem 21](#)

The proof follows the same structure as private learning algorithms for VC classes under smoothness and with unlabelled public data ([Haghtalab et al., 2020](#); [Alon et al., 2019a](#)) but we include it here for completeness. First, recall that under generalized smoothness, that any VC class has finite uniform cover from [Definition 10](#). We present this result below for completeness.

Lemma 30 (Uniform Covers under Generalized Smoothness) *Let \mathcal{U} be a (ρ, μ_0) generalized smooth class of distributions on \mathcal{X} . Then, for any function class \mathcal{F} of VC dimension d , any $\epsilon \in (0, 1]$, there is a class \mathcal{F}_ϵ which is an ϵ -uniform cover of \mathcal{F} with respect to \mathcal{U} and satisfies*

$$\log |\mathcal{F}_\epsilon| \leq d \log(C \cdot \rho^{-1}(\epsilon))$$

for some universal constant $C > 0$.

Proof The idea is to construct an ϵ' -cover of \mathcal{F} with respect to the base measure μ_0 . For any f, f' and $\mu \in \mathcal{U}$ we have that $\mu(\{x : f(x) \neq f'(x)\}) \leq \rho(\mu_0(\{x : f(x) \neq f'(x)\})) \leq \rho(\epsilon')$. Setting $\epsilon' = \rho^{-1}(\epsilon)$, we see that any ϵ' -cover with respect to μ_0 is an ϵ -uniform cover with respect to \mathcal{U} . The result then follows from the classical VC dimension covering number bound. ■

The required private learning algorithm is then a direct application of the private learning algorithm based on the exponential mechanism from [Kasiviswanathan et al. \(2011\)](#) using uniform covers. We recall the main result below for completeness.

Theorem 31 ([Kasiviswanathan et al. \(2011\)](#)) *Let \mathcal{F} be a finite function class. For any $\epsilon, \delta \in (0, 1]$, there exists an (α, β) -differentially private algorithm that is (α, β) -accurate for learning \mathcal{F} with sample complexity*

$$m = \frac{d + \log(1/\delta)}{\epsilon^2} + \frac{\log |\mathcal{F}|}{\alpha \epsilon}.$$

and note that a $(\alpha, 0)$ private algorithm (ϵ, δ) accurate learning algorithm for \mathcal{F}_ϵ is also a $(\alpha, 0)$ private algorithm $(2\epsilon, \delta)$ accurate learning algorithm for \mathcal{F} for any distribution with marginal in \mathcal{U} .