Efficient Swap Multicalibration of Elicitable Properties

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Abstract

Multicalibration (Hébert-Johnson et al., 2018) is an algorithmic fairness perspective which demands that the predictions of a predictor are correct conditional on themselves and membership in a collection of potentially overlapping subgroups of a population. The work of Noarov and Roth (2023) established a surprising connection between multicalibration for an arbitrary property Γ (e.g., mean or median) and property elicitation: a property Γ can be multicalibrated if and only if it is elicitable, where elicitability is the notion that the true property value of a distribution can be obtained by solving a regression problem over the distribution. In the adversarial (online) setting, Noarov and Roth (2023) proposed an inefficient algorithm that achieves $\tilde{O}(\sqrt{T})$ ℓ_2-multicalibration error for a hypothesis class of group membership functions and an elicitable property Γ, after T rounds of interaction between a forecaster and adversary.

In this paper, we generalize multicalibration for an elicitable property Γ from group membership functions to arbitrary bounded hypothesis classes and introduce a stronger notion — swap multicalibration, following Gopalan et al. (2023b). Subsequently, we propose an oracle-efficient algorithm which when given access to an online agnostic learner, achieves $\tilde{O}(T^{\frac{1}{r+1}})$ ℓ_r-swap multicalibration error with high probability ($r \geq 2$) for a hypothesis class with bounded sequential Rademacher complexity and an elicitable property Γ. For the special case of $r = 2$, this implies an oracle-efficient algorithm that achieves $\tilde{O}(T^{\frac{1}{3}})$ ℓ_2-swap multicalibration error, which significantly improves on the previously established bounds for the problem (Noarov and Roth, 2023; Ghuge et al., 2025; Luo et al., 2025a), and completely resolves an open question raised in Garg et al. (2024) on the possibility of an oracle-efficient algorithm that achieves $\tilde{O}(\sqrt{T})$ ℓ_2-mean multicalibration error by answering it in a strongly affirmative sense.

Keywords: Calibration, (Swap) Multicalibration, Elicitable Property, Online Agnostic Learning

1. Introduction

With the rise of machine learning models in several real-world applications such as healthcare, medical support, law, finance, and safety-critical control, quantifying the model's confidence in its predictions is becoming an increasingly serious problem. A widely studied metric to quantify the uncertainty in a model's predictions is *calibration* (Dawid, 1982), which demands that the model's predictions are correct conditional on themselves, e.g., on all days when the model predicts the probability of rain is 60%, does it rain on 60% of those days? To define calibration more formally, we set up some notation. Let $\mathcal{X}$ be the instance space, $\mathcal{Y} = \{0, 1\}$ be the label space, $D \in \Delta(\mathcal{X} \times \mathcal{Y})$ be an unknown distribution over $\mathcal{X} \times \mathcal{Y}$, and $p : \mathcal{X} \to [0, 1]$ be a predictor. The predictor p is (mean) calibrated if $\mathbb{E}_{(x,y)\sim D}[y|p(x) = v] = v$ for all $v \in \text{Range}(p)$. Calibration ensures a certain level of trustworthiness since the predictions are a true representation of the ground truth. However, calibration only ensures unbiasedness at the scale of the level sets of the predictor and cannot address broader reliability and fairness concerns, e.g., when the instance space $\mathcal{X}$ also consists of several refinements corresponding to potentially overlapping demographic groups, and the model's predictions should not lead to any detectable bias among the groups. To incentivize predictors to be unbiased not just marginally, but also for a collection of protected subgroups of a population, Hébert-Johnson et al. (2018) introduced *multicalibration*. Given a set $\mathcal{S}$ of potentially overlapping subsets of $\mathcal{X}$, the predictor p is (mean) multicalibrated if

$$\mathbb{E}_{(x,y)\sim D}[y|p(x) = v, x \in s] = v, \text{for all } v \in \text{Range}(p), s \in \mathcal{S}.$$

Although multicalibration was initially proposed in the distributional (offline) setting by Hébert-Johnson et al. (2018) as a mechanism to incentivize fair predictions, recent years have witnessed surprising connections with several different areas, e.g., complexity theory (Casacuberta et al., 2024, 2025), algorithmic fairness (Obermeyer et al., 2019), learning theory (Gopalan et al., 2022a,b), indistinguishability and pseudo-randomness (Dwork et al., 2021; Gopalan et al., 2023a; Hu and Vadhan, 2025; Dwork and Tankala, 2025), and so on, leading to a surge of interest in the community. Despite its conceptual importance and broad influence, achieving multicalibration in a sample-efficient manner has remained an outstanding problem, with existing constructions (Hébert-Johnson et al., 2018; Gopalan et al., 2022a; Globus-Harris et al., 2023) requiring $\gtrsim \varepsilon^{-10}$ samples to efficiently learn a predictor with multicalibration error at most ε. The above problem also manifests in the more challenging adversarial (online) setting, where an online forecaster and adversary interact sequentially for T rounds, and the goal of the forecaster is to minimize the multicalibration error at the end of their interaction. Notably, existing online multicalibration algorithms suffer from one or more of the following issues: (a) they achieve error rates that scale poorly with T; (b) are inefficient; (c) offer strong theoretical guarantees only under specific settings; and (d) do not readily generalize beyond multicalibration for means (we define multicalibration for an arbitrary property in Subsection 1.2). Although in recent years, online multicalibration has gained significant attention (Gupta et al., 2022; Haghtalab et al., 2023; Garg et al., 2024; Noarov et al., 2025; Luo et al., 2025a; Ghuge et al., 2025), the aforementioned issues remain inherent to the proposed algorithms. The motivating goal of this paper is to devise algorithms that simultaneously circumvent the challenges posed above.

1.1. Online (Mean) Multicalibration

Motivated by the above concerns, in this paper, we primarily consider online multicalibration — a sequential decision-making problem between an online forecaster and an adversary over T rounds of interaction. In online (mean) multicalibration, at each time $t \in [T]$, (a) the adversary reveals a context $x_t \in \mathcal{X}$, where $\mathcal{X}$ is the instance space; (b) the forecaster predicts a distribution $p_t \in [0, 1]$ over binary outcomes; (c) the adversary reveals the true label $y_t \in \mathcal{Y} = \{0, 1\}$. For a bounded hypothesis class $\mathcal{F} \subset [-1, 1]^{\mathcal{X}}$, the set of forecasts $p_1, \ldots, p_T$ is multicalibrated if no hypothesis f in $\mathcal{F}$ is marginally able to detect any correlation $f(x_t)(y_t - p_t)$ with the residual error $y_t - p_t$ when conditioned on the rounds where the prediction made is $p_t = p$. Mathematically, this is quantified by minimizing the ℓ_r-multicalibration error ($r \geq 1$), defined as

$$\mathsf{MCal}_{\text{mean},r}(\mathcal{F}) \coloneqq \sup_{f \in \mathcal{F}} \sum_{p \in [0,1]} n_p \left| \frac{1}{n_p} \sum_{t=1}^{T} \mathbb{I}\big[p_t = p\big] f(x_t)(y_t - p_t) \right|^r, \tag{1}$$

where $n_p \coloneqq \sum_{t=1}^{T} \mathbb{I}\big[p_t = p\big]$ denotes the number of rounds where the prediction made is $p_t = p$. Note that the summation $\sum_{p \in [0,1]}$ is an abuse of notation and that it is really over the (at most T) distinct predictions made by the forecaster. Online calibration ($\mathsf{Cal}_{\text{mean},r}$) is a special case of multicalibration where $\mathcal{F} = \{1\}$ and 1 denotes the constant function that evaluates to 1 for all $x \in \mathcal{X}$:

$$\mathsf{Cal}_{\text{mean},r} \coloneqq \sum_{p \in [0,1]} n_p \left| \frac{1}{n_p} \sum_{t=1}^{T} \mathbb{I}\big[p_t = p\big](y_t - p_t) \right|^r.$$

Throughout the paper, we consider forecasters that make predictions in a finite discretization $\mathcal{Z} = \{z_1, \dots, z_N\}$ of $[0,1]$, where $z_i = \frac{i}{N}$ and N is a parameter to be specified later. We let n_i be a shorthand for n_{z_i}.

In a line of work initiated to understand the implications of multicalibration towards *omniprediction* — a simultaneous loss-minimization paradigm introduced by Gopalan et al. (2022a), Gopalan et al. (2023b) introduced a related notion called *swap multicalibration*. Informally, while multicalibration requires a marginal (over p) guarantee for the conditional correlation errors, swap multicalibration demands that the conditional correlation error for each p be small. Formally, the ℓ_r-swap multicalibration error incurred by the forecaster is defined as

$$\mathsf{SMCal}_{\text{mean},r}(\mathcal{F}) \coloneqq \sum_{p \in [0,1]} n_p \sup_{f \in \mathcal{F}} \left| \frac{1}{n_p} \sum_{t=1}^{T} \mathbb{I}\big[p_t = p\big] f(x_t)(y_t - p_t) \right|^r. \tag{2}$$

Compared to multicalibration, swap multicalibration is a stronger notion which requires that for all $p \in [0,1]$, even when conditioned on the rounds where the prediction made is $p_t = p$, the no correlation guarantee holds for each $f \in \mathcal{F}$. Clearly, we have $\mathsf{SMCal}_{\text{mean},r}(\mathcal{F}) \geq \mathsf{MCal}_{\text{mean},r}(\mathcal{F})$.

1.2. A Connection between Multicalibration and Property Elicitation

Although conventionally stated in the context of means, mutlicalibration can be defined more generally for any distributional property Γ, e.g., median, q-th quantiles, etc. In the distributional setting, let $D \in \Delta(\mathcal{X} \times \mathcal{Y})$ be an unknown distribution, where, as usual, $\mathcal{X}$ denotes the instance space and $\mathcal{Y} \subseteq [0,1]$ represents the label space (not necessarily $\{0,1\}$). Let $\mathcal{S}$ be a collection of potentially overlapping subsets of $\mathcal{X}$ and $\Gamma : \mathcal{P} \to [0,1]$ be a property (without any loss of generality, we assume $\text{Range}(\Gamma) = [0,1]$), where $\mathcal{P}$ is a family of probability distributions over $\mathcal{Y}$. A predictor $\gamma : \mathcal{X} \to [0,1]$ is Γ-*multicalibrated* if $\Gamma(D|\gamma(x) = v, x \in s) = v$ holds for all $v \in [0,1], s \in \mathcal{S}$. In other words, the predictions of γ about the property Γ are correct even when conditioned on the level sets of γ and membership in a collection of potentially overlapping subsets of $\mathcal{X}$. Similarly, Γ-*calibration* is the marginal requirement that $\Gamma(D|\gamma(x) = v) = v$ for all $v \in [0,1]$. Jung et al. (2021) initiated the study of moment multicalibration and showed that it is impossible to multicalibrate variance and other higher moments since there exists a simple distribution D on which even the true distributional predictor for variance $\gamma_{\text{var}}(x) \coloneqq \mathbb{E}[(y - \mathbb{E}[y|x])^2|x]$ is not calibrated. Subsequent work by Noarov and Roth (2023) established a remarkable connection between multicalibration and property elicitation (Lambert et al., 2008; Steinwart et al., 2014). Formally, a property Γ is *elicitable* if there exists a loss function $\ell : [0,1] \times \mathcal{Y} \to \mathbb{R}$ whose minimizer in expectation over P yields the true property value $\Gamma(P)$, i.e., $\Gamma(P) \in \operatorname{argmin}_{\gamma \in [0,1]} \mathbb{E}_{y \sim P}[\ell(\gamma, y)]$ for all $P \in \mathcal{P}$ (we say ℓ elicits Γ). For instance, the squared loss $\ell(\gamma, y) = \frac{1}{2}(\gamma - y)^2$ elicits the mean, where as the ℓ_1-loss $\ell(\gamma, y) = \frac{1}{2}\left|\gamma - y\right|$ elicits the median.

A striking result due to Noarov and Roth (2023) is that a property Γ can be multicalibrated if and only if Γ is elicitable. In both the distributional and online settings, Noarov and Roth (2023) propose canonical algorithms for achieving multicalibration for an elicitable property Γ. The result of Noarov and Roth (2023) builds on the following key characterization of property elicitation established by Steinwart et al. (2014): a property Γ is elicitable if and only if there exists an *identification* function for Γ — a function $V : [0,1] \times \mathcal{Y} \to \mathbb{R}$ such that $\mathbb{E}_{y \sim P}[V(\gamma, y)] = 0 \iff \gamma = \Gamma(P)$ for all $P \in \mathcal{P}$. In other words, the true property value $\Gamma(P)$ is a root of the expected identification function for all $P \in \mathcal{P}$, e.g., for mean $V(\gamma, y) = \gamma - y$, for median $V(\gamma, y) = \mathbb{I}\left[y \le \gamma\right] - \frac{1}{2}$. Based on the concept of an identification function, we generalize the definition of online multicalibration for an elicitable property beyond group membership functions and introduce swap multicalibration. Specifically, for an elicitable property Γ with identification function V (without any loss of generality, we assume $\mathrm{Range}(V) = [-1, 1]$ since the loss that elicits a property can be scaled appropriately), $r \ge 1$, and a hypothesis class $\mathcal{F} \subset [-1,1]^{\mathcal{X}}$, we define the ℓ_r-multicalibration and ℓ_r-swap multicalibration errors incurred by the forecaster respectively as:

$$\mathsf{MCal}_{\Gamma,r}(\mathcal{F}) := \sup_{f \in \mathcal{F}} \sum_{p \in [0,1]} n_p \left| \frac{1}{n_p} \sum_{t=1}^{T} \mathbb{I}\left[p_t = p\right] f(x_t) V(p_t, y_t) \right|^r, \tag{3}$$

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) := \sum_{p \in [0,1]} n_p \sup_{f \in \mathcal{F}} \left| \frac{1}{n_p} \sum_{t=1}^{T} \mathbb{I}\left[p_t = p\right] f(x_t) V(p_t, y_t) \right|^r. \tag{4}$$

Calibration with a property Γ ($\mathsf{Cal}_{\Gamma,r}$) is a special case of multicalibration where $\mathcal{F} = \{1\}$, therefore is non-contextual. When Γ is the mean (and thus $V(\gamma, y) = \gamma - y$), (3) and (4) clearly recover (1) and (2). Since the identification function for an elicitable property is not unique, we assume that there exists a canonical identification function V (e.g., for mean $V(\gamma, y) = \gamma - y$, for median $V(\gamma, y) = \mathbb{I}\left[y \le \gamma\right] - 0.5$) and that the identification function in (3), (4) is the canonical identification function. More generally, our results apply for any identification function satisfying Assumption 1.

1.3. Related Work

Although there are several algorithms that achieve (mean) multicalibration in both the offline (Hébert-Johnson et al., 2018; Jung et al., 2021; Gopalan et al., 2022a,a; Gupta et al., 2022; Globus-Harris et al., 2023; Gopalan et al., 2023b; Deng et al., 2023) and online settings (Gupta et al., 2022; Haghtalab et al., 2023; Garg et al., 2024; Luo et al., 2025a; Ghuge et al., 2025), the resulting offline sample complexities/online multicalibration errors are either quite worse compared to the calibration counterparts, or do not generalize beyond simple classes of functions, e.g., linear functions. For the ease of comparison, in this subsection, we review the relevant results for ℓ_2-(swap) multicalibration and defer discussion for ℓ_1-(swap) multicalibration to Subsection 1.4 (they are related via the Cauchy-

Schwartz inequality as $\mathsf{SMCal}_{\Gamma,2}(\mathcal{F}) \le \mathsf{SMCal}_{\Gamma,1}(\mathcal{F}) \le \sqrt{T \cdot \mathsf{SMCal}_{\Gamma,2}(\mathcal{F})}$ and $\mathsf{MCal}_{\Gamma,2}(\mathcal{F}) \le \mathsf{MCal}_{\Gamma,1}(\mathcal{F}) \le \sqrt{T \cdot \mathsf{MCal}_{\Gamma,2}(\mathcal{F})}$). Particularly, in the online setting:

- Garg et al. (2024) proposed an oracle-efficient algorithm that achieves $\mathsf{SMCal}_{\text{mean},2}(\mathcal{F}) = \tilde{O}(T^{\frac{3}{4}})$ for any hypothesis class $\mathcal{F}$ with bounded online complexity, and raised the open problem of whether it is possible to devise an oracle-efficient algorithm that achieves $\mathsf{MCal}_{\text{mean},2}(\mathcal{F}) = \tilde{O}(\sqrt{T})$. In a restricted setting where the hypothesis class $\mathcal{F}$ is finite, Garg et al. (2024) also developed an inefficient algorithm with running time proportional to $|\mathcal{F}|$ that achieves $\mathsf{MCal}_{\text{mean},2}(\mathcal{F}) = \tilde{O}(\sqrt{T})$.
- Subsequent work by Luo et al. (2025a) partially answered the question raised by Garg et al. (2024) in the affirmative by proposing an efficient algorithm that achieves $\mathsf{SMCal}_{\text{mean},2}(\mathcal{F}) = \tilde{O}(T^{\frac{1}{3}})$ when $\mathcal{F}$ is the class of bounded linear functions.

Beyond mean multicalibration:

- For an elicitable property Γ, Noarov and Roth (2023) proposed an inefficient algorithm that achieves $\mathsf{MCal}_{\Gamma,2}(\mathcal{F}) = \tilde{O}(\sqrt{T})$.[1]
- In the special case that Γ is the q-th quantile of a distribution, Garg et al. (2024) proposed an oracle efficient algorithm that achieves $\mathsf{SMCal}_{\Gamma,2}(\mathcal{F}) = \tilde{O}(T^{\frac{3}{4}})$ for any hypothesis class $\mathcal{F}$ with bounded online complexity, where as it is possible to achieve $\mathsf{MCal}_{\Gamma,2} = \tilde{O}(\sqrt{T})$ via an inefficient algorithm (Bastani et al., 2022; Roth, 2022; Noarov and Roth, 2023; Garg et al., 2024).

We summarize the above results along with a comparison to ours in Table 1.

	Γ	$\mathcal{F}$	$\mathsf{MCal}_{\Gamma,2}(\mathcal{F})$	$\mathsf{SMCal}_{\Gamma,2}(\mathcal{F})$	Efficient
This work	general	general	$\tilde{O}(T^{\frac{1}{3}})$	$\tilde{O}(T^{\frac{1}{3}})$	✓
Luo et al. (2025a)	mean	linear	$\tilde{O}(T^{\frac{1}{3}})$	$\tilde{O}(T^{\frac{1}{3}})$	✓
Roth (2022); Garg et al. (2024)	mean or quantile	finite	$\tilde{O}(\sqrt{T})$	N/A	✗
Garg et al. (2024)	mean or quantile	general	$\tilde{O}(T^{\frac{3}{4}})$	$\tilde{O}(T^{\frac{3}{4}})$	✓
Noarov et al. (2025)	general	group membership functions	$\tilde{O}(\sqrt{T})$	N/A	✗

Table 1: A comparison of our results to the previously established multicalibration error bounds. Blue cells indicate the strongest result for the corresponding columns.

When compared against the achievable rates for mean multicalibration, the above results raise the following immediate question: "*Is it possible to devise an oracle-efficient algorithm which achieves improved (swap) multicalibration guarantees compared to Table 1 for an arbitrary elicitable property Γ and a general hypothesis class $\mathcal{F}$?*"

1. Noarov and Roth (2023) use a slightly different definition of multicalibration compared to the one considered in this paper. However, their analysis extends gracefully to the definition of $\mathsf{MCal}_{\Gamma,2}$ considered in this paper.

1.4. Contributions

In this paper, we answer the question above in a strongly affirmative sense by proposing an oracle-efficient algorithm which meets all the above requirements simultaneously: for an elicitable property Γ with a ρ-Lipschitz identification function,[2] fixed $r \geq 2$,[3] and $\mathcal{F}$ with bounded sequential Rademacher complexity, Algorithm 2 in Appendix B achieves $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}(T^{\frac{1}{r+1}})$ with high probability. Notably, for $r = 2$, this improves the dependence of T significantly over all the rows in Table 1 except for the work of Luo et al. (2025a), while improving on Luo et al. (2025a) by generalizing to $r > 2$, a broader hypothesis class $\mathcal{F}$, and an elicitable property Γ. Towards achieving this result, we first propose an inefficient algorithm (Algorithm 1 in Appendix B) that achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\rho^r T^{\frac{1}{r+1}} + T^{\frac{1}{r+1}}\left(\log\frac{|\mathcal{F}|}{\delta}\right)^{\frac{r}{2}}\right)$$

with probability at least $1-\delta$ for a finite class $\mathcal{F}$. Subsequently, we propose our oracle-efficient algorithm by invoking an online agnostic learner in Algorithm 1.

Definition 1 (Online Agnostic Learning (Ben-David et al., 2009; Beygelzimer et al., 2015)) *Consider the following interaction between an online agnostic learner (*OAL*) and adversary for n rounds: at each time $t \in [n]$, (a) the adversary reveals a context $x_t \in \mathcal{X}$; (b)* OAL *responds with a prediction $q_t(x_t)$, where $q_t : \mathcal{X} \to [-1, 1]$; (c) adversary reveals the outcome $\kappa_t \in [-1, 1]$. In online agnostic learning, the goal of* OAL *is to output a sequence of test functions $q_1, \dots, q_T$ whose cumulative correlation $\sum_{t=1}^{T} q_t(x_t)\kappa_t$ with the outcome sequence $\kappa_1, \dots, \kappa_T$ is comparable with the best hypothesis in a given hypothesis class $\mathcal{F}$, i.e.,*

$$\sup_{f\in\mathcal{F}} \sum_{t=1}^{n} f(x_t)\kappa_t \leq \sum_{t=1}^{n} q_t(x_t)\kappa_t + \mathsf{Reg}(\mathcal{F}, n), \tag{5}$$

where $\mathsf{Reg}(\mathcal{F}, n) > 0$ *denotes the regret incurred by* OAL.

Algorithm 1 is inefficient since it requires enumerating over each $f \in \mathcal{F}$, whereas Algorithm 2 achieves oracle-efficiency by instantiating $2N$ copies of OAL, parameterized by $(i, \sigma) \in [N] \times \{\pm 1\}$, and auditing against the auxiliary test function $q_{t,i,\sigma}$ provided by $\mathsf{OAL}_{i,\sigma}$ for each (i, σ) instead. Next, we provide a brief comparison of the number of instantiated copies of OAL and per-round oracle calls made by Algorithm 2 with the most relevant prior work.

Remark 2 *The recent work of Okoroafor et al. (2025) considered a related problem of omniprediction (Gopalan et al., 2022a) and proposed near optimal algorithms for omniprediction. The OAL abstraction in our paper is motivated by their work. Both Okoroafor et al. (2025) and our work make $O(1)$ many oracle calls per-round, however, since we consider the swap variant of a problem, we instantiate our algorithm with $O(N) = \tilde{O}(T^{\frac{1}{3}})$ oracles. In contrast, Okoroafor et al. (2025) consider*

2. This is a standard assumption invoked in several prior works, e.g., (Noarov and Roth, 2023) invoke it for multicalibrating an elicitable property Γ, where as (Jung et al., 2023; Garg et al., 2024) invoke it for multicalibrating q-th quantiles.
3. Although we also have an explicit bound on $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$ for $r \in [1, 2)$, for the sake of brevity, we refer to the bound for $r \geq 2$ throughout, since a bound on SMCal_r for $r \in [1, 2)$ can be obtained from a bound on SMCal_2 by applying the Hölder's inequality; see Theorem 3.

the non-swap variant of omniprediction, therefore, they require $O(1)$ many oracles. However, both our algorithm and theirs are oracle-efficient. The work of Garg et al. (2024), which our work most notably improves upon, considered the problem of swap multicalibration, however, they use the online squared-error regression oracle. Their proposed algorithm is oracle-efficient and instantiates $O(T^{\frac{1}{4}})$ oracles with $O(1)$ many oracle calls per-round. However, compared to our work, they achieve a suboptimal swap multicalibration error.

In the following result, which is a special case of Theorem 15 (the main result of this paper; refer Appendix B.2), we derive a bound on $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$ by assuming the existence of an OAL with a favorable regret guarantee.

Theorem 3 *Fix a $r \geq 1$ and an elicitable property Γ with a ρ-Lipschitz identification function, and assume that there exists an OAL for which $\mathsf{Reg}(\mathcal{F}, n) = \tilde{O}(\sqrt{n} \cdot \mathsf{Comp}(\mathcal{F}))$, where $\mathsf{Comp}(\mathcal{F})$ is a complexity measure of $\mathcal{F}$ that is independent of n. Then, for $r \geq 2$, Algorithm 2 achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + \mathsf{Comp}(\mathcal{F})^r\right) T^{\frac{1}{r+1}}\right),$$

with probability at least $1-\delta$. Consequently, for $r \in [1, 2)$, Algorithm 2 achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + \mathsf{Comp}(\mathcal{F})^r\right) T^{1-\frac{r}{3}}\right)$$

with probability at least $1-\delta$.[4]

To instantiate the above result:

- When $\mathcal{F}$ is the class of linear functions, the Online Gradient Descent (OGD) algorithm (Zinkevich, 2003) achieves $\mathsf{Reg}(\mathcal{F}, n) = O(\sqrt{n})$, therefore, $\mathsf{SMCal}_{\Gamma,2}(\mathcal{F}) = \tilde{O}(T^{\frac{1}{3}})$. This not only recovers the result of Luo et al. (2025a) for $\mathsf{SMCal}_{\text{mean},2}(\mathcal{F})$ but also generalizes it for any $r > 2$ and an arbitrary elicitable property Γ.

- When $\mathcal{F}$ is a finite class, the Multiplcative Weights Update (MWU) algorithm (Freund and Schapire, 1997) achieves $\mathsf{Reg}(\mathcal{F}, n) = O(\sqrt{n \log|\mathcal{F}|})$, thereby implying that $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}(T^{\frac{1}{r+1}}(\log|\mathcal{F}|)^{\frac{r}{2}})$.

- Similar to Okoroafor et al. (2025), via standard learning-theoretic arguments and results from Rakhlin et al. (2010), we show that there exists an OAL that achieves $\mathsf{Reg}(\mathcal{F}, n) = O(n \cdot \mathfrak{R}^{\text{seq}}(\mathcal{F}, n))$, where $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n)$ is the sequential Rademacher complexity of $\mathcal{F}$ over n rounds of interaction between an algorithm and adversary. Therefore, whenever $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n) = \tilde{O}\left(\frac{\mathsf{Comp}(\mathcal{F})}{\sqrt{n}}\right)$, we obtain $\mathsf{Reg}(\mathcal{F}, n) = \tilde{O}(\sqrt{n}\mathsf{Comp}(\mathcal{F}))$ and $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}(T^{\frac{1}{r+1}})$. Both the linear class and finite $\mathcal{F}$ satisfy $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n) = \tilde{O}\left(\frac{\mathsf{Comp}(\mathcal{F})}{\sqrt{n}}\right)$ with $\mathsf{Comp}(\mathcal{F}) = O(1)$ for the linear class and $\mathsf{Comp}(\mathcal{F}) = \log|\mathcal{F}|$ for the finite class, however, importantly, they admit an explicit algorithm that achieves $\mathsf{Reg}(\mathcal{F}, n) = \tilde{O}(\sqrt{n}\mathsf{Comp}(\mathcal{F}))$. For hypothesis classes (beyond the above two classes) for which $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n) = \tilde{O}\left(\frac{\mathsf{Comp}(\mathcal{F})}{\sqrt{n}}\right)$, we refer to Rakhlin and Sridharan (2014).

4. Our bounds are achieved without the knowledge of ρ, $\mathsf{Reg}(\mathcal{F}, n)$, and δ. We did not make an attempt to optimize our bounds when these parameters are known.

When instantiated for means, the above set of results not only completely resolves the open question raised in Garg et al. (2024) on the possibility of an oracle-efficient algorithm that achieves $\mathsf{MCal}_{\text{mean},2}(\mathcal{F}) = \tilde{O}(\sqrt{T})$, but also shows that it is possible to achieve a much stronger guarantee $\mathsf{SMCal}_{\text{mean},2}(\mathcal{F}) = \tilde{O}(T^{\frac{1}{3}})$. More generally, for an elicitable property Γ, Algorithm 2 achieves $\mathsf{SMCal}_{\Gamma,2} = \tilde{O}(T^{\frac{1}{3}})$, thereby improving upon several previous works (Noarov and Roth, 2023; Garg et al., 2024; Ghuge et al., 2025; Luo et al., 2025a).

Remark 4 *For ℓ_1-multicalibration, Ghuge et al. (2025) proposed an inefficient algorithm that achieves $\mathsf{MCal}_{mean,1}(\mathcal{F}) = \tilde{O}(T^{\frac{2}{3}})$ for a finite class $\mathcal{F}$ (see also Noarov et al. (2025) for a different approach to obtain the same result) and an oracle-efficient algorithm that achieves $\mathsf{MCal}_{mean,1}(\mathcal{F}) = \tilde{O}(T^{\frac{3}{4}})$ for a general hypothesis class $\mathcal{F}$. Since $\mathsf{SMCal}_{\Gamma,1}(\mathcal{F}) \le \sqrt{T \cdot \mathsf{SMCal}_{\Gamma,2}(\mathcal{F})}$, we achieve $\mathsf{SMCal}_{mean,1}(\mathcal{F}) = \tilde{O}(T^{\frac{2}{3}})$ in an oracle-efficient manner, thereby also significantly improving upon Noarov et al. (2025); Ghuge et al. (2025). More recently, Collina et al. (2026) have shown that $\mathsf{MCal}_{mean,1} = \tilde{\Omega}(T^{\frac{2}{3}})$, therefore, $\mathsf{SMCal}_{mean,1} = \tilde{\Omega}(T^{\frac{2}{3}})$ and $\mathsf{SMCal}_{mean,2} = \tilde{\Omega}(T^{\frac{1}{3}})$. Conclusively, Algorithm 2 is the first oracle-efficient algorithm that is also minimax optimal for $\mathsf{SMCal}_{mean,1}$ and $\mathsf{SMCal}_{mean,2}$.*

2. Technical Overview

We provide a technical overview of our work in this section.

2.1. Warm up with $\mathsf{Cal}_{\text{mean},2}$: A new general approach

We begin by considering the simpler problem of online ℓ_2-calibration for means. Prior works for calibrating means achieve $\mathsf{Cal}_{\text{mean},2} = \tilde{O}(T^{\frac{1}{3}})$ by either using the equivalence between ℓ_2-calibration and swap regret of the squared loss (Luo et al., 2025b; Fishelson et al., 2025) or via the concept of calibeating (Foster and Hart, 2023). However, in Lemma 22 in Appendix C, we show that when calibrating q-th quantiles, the ℓ_2-quantile calibration error cannot be bounded by any strictly increasing and invertible function ζ of the swap regret of any loss function, thereby rendering the swap regret-based approach futile. Furthermore, the calibeating-based approach also does not readily generalize beyond means.

To circumvent this, we discover yet another approach to achieve $\mathsf{Cal}_{\text{mean},2} = \tilde{O}(T^{\frac{1}{3}})$, which, unlike the previous approaches, can be directly generalized to properties beyond means. Specifically, we observe that to achieve $\mathsf{Cal}_{\text{mean},2} = \tilde{O}(T^{\frac{1}{3}})$, it suffices to derive the high probability bound

$$\left|\sum_{t=1}^{T} \mathbb{I}\left[p_t = z_i\right](y_t - p_t)\right| = \tilde{O}\left(\sqrt{n_i} + \frac{n_i}{N}\right), \tag{6}$$

where we have ignored dependence on the failure probability δ for simplicity. Before explaining why this is sufficient, we first point out that (6) is a natural goal due to the following interpretation: if the forecaster could witness the conditional distribution of y_t (e.g., if the adversary moves first instead and at every time $t \in [T]$, it reveals the conditional distribution of y_t), then a simple truthful strategy which computes $\tilde{p}_t$ as the conditional expectation of y_t and predicts $p_t = \operatorname{argmin}_{z \in \mathcal{Z}} \left|z - \tilde{p}_t\right|$ by rounding $\tilde{p}_t$ to the nearest point in $\mathcal{Z}$ achieves (6). This is because, such a forecasting strategy has to pay for the rounding error $\frac{1}{N}$ per round for n_i rounds, and a variance term $\tilde{O}(\sqrt{n_i})$ due to not predicting the true label but rather its expectation. Of course, since the forecaster has to decide p_t without any knowledge about y_t, it cannot implement this strategy, however, the discussion points towards the possibility that (6) might be achievable.

Notably, (6) indeed implies that $\mathsf{Cal}_{\text{mean},2} = \tilde{O}(T^{\frac{1}{3}})$ with high probability. This is because, $n_i \left(\frac{1}{n_i}\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right](y_t - p_t)\right)^2 = \tilde{O}\left(1 + \frac{n_i}{N^2}\right)$, which when summed over i yields $\mathsf{Cal}_{\text{mean},2} = \sum_{i\in[N]} n_i \left(\frac{1}{n_i}\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right](y_t - p_t)\right)^2 = \tilde{O}\left(N + \frac{T}{N^2}\right) = \tilde{O}(T^{\frac{1}{3}})$ on choosing the discretization size $N = \Theta(T^{\frac{1}{3}})$. More generally, for $r > 2$, we obtain

$$n_i \left|\frac{1}{n_i}\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right](y_t - p_t)\right|^r = \tilde{O}\left(n_i^{1-r/2} + \frac{n_i}{N^r}\right) = \tilde{O}\left(1 + \frac{n_i}{N^r}\right),$$

and thus $\mathsf{Cal}_{\text{mean},r} = \tilde{O}\left(N + \frac{T}{N^r}\right) = \tilde{\Theta}(T^{\frac{1}{r+1}})$ on choosing $N = \Theta(T^{\frac{1}{r+1}})$. Furthermore, this approach is readily generalizable to an arbitrary elicitable property Γ with an identification function V, by simply replacing $y_t - p_t$ in (6) with $V(p_t, y_t)$.

So how can one achieve (6)? In fact, a recent work by Hu and Wu (2024) almost achieves (6) in their algorithm designed for a different problem of simultaneous swap regret minimization. The only caveat is that their algorithm only ensures an in expectation bound on $\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right](y_t - p_t)\right|$, which, as one can verify, is not sufficient to achieve $\mathsf{Cal}_{\text{mean},2} = \tilde{O}(T^{\frac{1}{3}})$ or even $\mathbb{E}[\mathsf{Cal}_{\text{mean},2}] = \tilde{O}(T^{\frac{1}{3}})$. Since the algorithm of Hu and Wu (2024) uses the special expert algorithm MsMwC of (Chen et al., 2021) as a subroutine, a natural approach to achieve the desired high probability bound would be to obtain a high probability external regret guarantee for MsMwC, which is unfortunately mentioned as an open problem in Chen et al. (2021) and is still unsolved to the best of our knowledge.

Despite this fact, it turns out that (6) can still be achieved in a simple manner that is agnostic to the high probability external regret guarantee of MsMwC, which is quite surprising. Towards achieving this, we propose two simplifications to the algorithm of Hu and Wu (2024), the first of which is critical for achieving (6) and the other is important for subsequent developments in the contextual setting (Subsection 2.2). We start with a brief introduction of the algorithm of Hu and Wu (2024). For each $i \in [N-1]$, let $\mathcal{I}_i = \left[\frac{i-1}{N}, \frac{i}{N}\right)$ and $\mathcal{I}_N = \left[\frac{N-1}{N}, 1\right]$, so that $\mathcal{I}_1, \ldots, \mathcal{I}_N$ represents a partition of $[0,1]$. Informally, their algorithm defines a $2N$-expert problem corresponding to a careful choice of the gain function $\ell_{i,\sigma}(p, y) \coloneqq \sigma\mathbb{I}\left[p \in \mathcal{I}_i\right](p - y)$ for each expert $(i, \sigma) \in [N] \times \{\pm 1\}$. At each time $t \in [T]$, the algorithm obtains a probability distribution $\{w_{t,i,\sigma}\}$ over the experts via an expert problem subroutine ALG, which is instantiated to be the MsMwC algorithm. Subsequently, it hedges against the experts by defining a function $h_t(P)$ that depends on the weights $\{w_{t,i,\sigma}\}$ and the gains $\{\ell_{i,\sigma}\}$ as (with $\mathscr{F}_{t-1}$ being the filtration generated by the random variables $p_1, \ldots, p_{t-1}, y_1, \ldots, y_{t-1}$)

$$h_t(P) \coloneqq \max_{y\in\{0,1\}} \mathbb{E}_{p\sim P}\left[\sum_{(i,\sigma)} w_{t,i,\sigma}\ell_{i,\sigma}(p, y)|\mathscr{F}_{t-1}\right],$$

obtains a distribution P_t satisfying $h_t(P_t) = O\left(\frac{1}{T}\right)$, samples $\tilde{p}_t \sim P_t$, and forecasts $p_t = z_{i_t}$, where $i_t \in [N]$ is such that $\tilde{p}_t \in \mathcal{I}_{i_t}$. Finally, after observing y_t, we feed the gain $\ell_{t,i,\sigma} \coloneqq \ell_{i,\sigma}(\tilde{p}_t, y_t)$ for each (i, σ) to ALG. Our proposed modifications are as follows:

1. Instead of feeding the loss $\ell_{i,\sigma}(\tilde{p}_t, y_t)$, we feed a modified loss $\ell_{t,i,\sigma} = \mathbb{E}_{p\sim P_t}[\ell_{i,\sigma}(p, y_t)|\mathscr{F}_{t-1}]$ for each (i, σ).

2. While Hu and Wu (2024) argue that a distribution P_t satisfying $h_t(P_t) = \mathcal{O}\left(\frac{1}{T}\right)$ can be obtained in poly(T) time by using the FTPL-based approach suggested by Noarov et al. (2025), we provide a simple closed-form construction of P_t that is supported on at most two points, following a similar idea utilized in Gupta et al. (2022); Okoroafor et al. (2025). To state our construction, we define the function $\Phi_t : [0,1] \to [-1,1]$ as $\Phi_t(p) := \sum_{(i,\sigma)} w_{t,i,\sigma} \sigma \mathbb{I}\left[p \in \mathcal{I}_i\right]$. If $\Phi_t(0) > 0$, we choose P_t that is only supported on 0; else if $\Phi_t(1) \le 0$, we choose P_t that is only supported on 1; else we choose $i \in \{0, \ldots, T-1\}$ such that $\Phi_t\left(\frac{i}{T}\right)\Phi_t\left(\frac{i+1}{T}\right) \le 0$ and P_t such that

$$P_t\left(\frac{i}{T}\right) = \frac{\left|\Phi_t\left(\frac{i+1}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|}, \quad P_t\left(\frac{i+1}{T}\right) = \frac{\left|\Phi_t\left(\frac{i}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|}.$$

To see why the first modification leads to (6), we realize the following:

$$\begin{aligned}\left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right](y_t - p_t)\right| &\le \left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right](y_t - \tilde{p}_t)\right| + \left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right](\tilde{p}_t - p_t)\right| \le \max_{\sigma \in \{\pm 1\}} \sum_{t=1}^{T} \ell_{i,\sigma}(\tilde{p}_t, y_t) + \frac{n_i}{N} \\ &\le \max_{\sigma \in \{\pm 1\}} \sum_{t=1}^{T} \ell_{t,i,\sigma} + \max_{\sigma \in \{\pm 1\}} \sum_{t=1}^{T} \ell_{i,\sigma}(\tilde{p}_t, y_t) - \ell_{t,i,\sigma} + \frac{n_i}{N},\end{aligned}$$

where the second inequality follows from the definition of $\ell_{i,\sigma}$ and $\left|\tilde{p}_t - p_t\right| \le \frac{1}{N}$. Since the MsMwC algorithm receives gain $\ell_{t,i,\sigma}$ for each (i, σ), it follows from the regret guarantee of MsMwC that (with $\mathbb{P}_t[.] := \mathbb{P}(.|\mathscr{F}_{t-1})$)

$$\sum_{t=1}^{T} \ell_{t,i,\sigma} \le \sum_{t=1}^{T} \sum_{(i',\sigma')} w_{t,i',\sigma'} \ell_{t,i',\sigma'} + \tilde{\mathcal{O}}\left(\sqrt{\sum_{t=1}^{T} \ell_{t,i,\sigma}^2}\right) = \tilde{\mathcal{O}}\left(\sqrt{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)}\right),$$

where the equality follows since for each t, $\sum_{(i',\sigma')} w_{t,i',\sigma'} \ell_{t,i',\sigma'} \le h_t(P_t) = \mathcal{O}(\frac{1}{T})$ and $\ell_{t,i,\sigma}^2 \le \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$ by the definition of $\ell_{t,i,\sigma}$. Next, observe that the sequence $X_t := \ell_{i,\sigma}(\tilde{p}_t, y_t) - \ell_{t,i,\sigma} = \ell_{i,\sigma}(\tilde{p}_t, y_t) - \mathbb{E}_{p \sim P_t}[\ell_{i,\sigma}(p, y_t)|\mathscr{F}_{t-1}]$ is a martingale difference sequence (conditioned on $\mathscr{F}_{t-1}$, the random variables $\tilde{p}_t$ (or p_t) and y_t are independent) with cumulative variance $\sum_{t=1}^{T} \mathbb{E}_t\left[X_t^2\right] \le \sum_{t=1}^{T} \mathbb{E}_t[\ell_{i,\sigma}(\tilde{p}_t, y_t)^2] \le \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$, where $\mathbb{E}_t[.] := \mathbb{E}[.|\mathscr{F}_{t-1}]$. Therefore, applying Freedman's inequality (Lemma 21 in Appendix C), we obtain $\left|\sum_{t=1}^{T} X_t\right| = \tilde{\mathcal{O}}\left(\sqrt{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)}\right)$ with high probability. Combining everything,

$$\left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right](y_t - p_t)\right| = \tilde{\mathcal{O}}\left(\sqrt{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)} + \frac{n_i}{N}\right) = \tilde{\mathcal{O}}\left(\sqrt{n_i} + \frac{n_i}{N}\right),$$

since $\sqrt{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)} = \tilde{\mathcal{O}}(\sqrt{n_i})$ with high probability by applying Freedman's inequality again to the sequence $Z_t := \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) - \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right]$. This thus achieves (6) without the need of getting a high-probability version of the regret guarantee of MsMwC.

We remark again that while the second modification has not been used yet in the above discussion, it will subsequently be quite helpful in the contextual setting, where we utilize it to obtain tail bounds on the sum of a sequence that is not necessarily a martingale difference sequence. Moreover, we do not flesh out details of the above discussion in the paper since it is a special case of the contextual setting (Appendix B.1) and Lemma 10 is a generalized version of (6).

2.2. An inefficient generalization to the contextual setting

By combining the two modifications above, we generalize to an arbitrary elicitable property and also to the contextual setting where the forecaster receives a context $x_t \in \mathcal{X}$, predicts $p_t \in [0, 1]$, and finally observes $y_t \sim Y_t$, where Y_t is a distribution over $\mathcal{Y} \subseteq [0, 1]$ that is chosen by the adversary. In Appendix B.1, we propose an inefficient algorithm (Algorithm 1), which for a fixed $r \geq 2$, finite $\mathcal{F}$, and an elicitable property Γ with an identification function V so that the marginal $V(., Y) := \mathbb{E}_{y\sim Y}[V(., y)]$ is ρ-Lipschitz (Assumption 1 in Appendix A), achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\rho^r T^{\frac{1}{r+1}} + T^{\frac{1}{r+1}}\left(\log\frac{|\mathcal{F}|}{\delta}\right)^{\frac{r}{2}}\right)$$

with probability at least $1-\delta$ (Theorem 11 in Appendix B.1). Algorithm 1 defines a $2N|\mathcal{F}|$-expert problem corresponding to a choice of the gain function $\ell_{f,i,\sigma}(p, x, y) := \sigma\mathbb{I}\left[p \in \mathcal{I}_i\right] f(x)V(p, y)$ for each $(f, i, \sigma) \in \mathcal{F} \times [N] \times \{\pm 1\}$. As mentioned in Subsection 2.1 above, at each time $t \in [T]$, the algorithm hedges against the experts by defining the auxiliary function $h_t(P) := \sup_{y\in\mathcal{Y}} \mathbb{E}_{p\sim P}\left[\sum_{(f,i,\sigma)} w_{t,f,i,\sigma}\ell_{f,i,\sigma}(p, x_t, y)|\mathscr{F}_{t-1}\right]$, where $\{w_{t,f,i,\sigma}\}$ is the probability distribution over experts as output by the expert problem subroutine ALG. Subsequently, it obtains a distribution P_t satisfying $h_t(P_t) = O\left(\frac{1}{T}\right)$ in a similar manner as Subsection 2.1, samples $\tilde{p}_t \sim P_t$, and forecasts $p_t = \frac{i_t}{N}$, where $i_t \in [N]$ is the unique index such that $\tilde{p}_t \in \mathcal{I}_{i_t}$. Finally, after observing y_t, we feed the gain $\ell_{t,f,i,\sigma} := \mathbb{E}_{p\sim P_t}\left[\ell_{f,i,\sigma}(p, x_t, y_t)|\mathscr{F}_{t-1}\right]$ for each (f, i, σ) to ALG.

Although the proposed modifications and generalization to the algorithm of Hu and Wu (2024) are simple, their combination with a martingale-based analysis using Freedman's inequality (Lemma 21) in the style of Luo et al. (2025a) results in a substantially different and technical proof compared to both Hu and Wu (2024) and Luo et al. (2025a). Moreover, the central idea in Luo et al. (2025a); Garg et al. (2024) is a reduction of swap multicalibration to contextual swap regret, which is quite different from our approach. As a highlight of our proof, we utilize our explicit construction of P_t to obtain tail bounds on $\left|\sum_{t=1}^T W_t\right|$ for a non-martingale sequence $\{W_t\}_{t=1}^T$. To elaborate on this further, we provide a detailed proof sketch of Theorem 11 here. Similar to calibration for means (Subsection 2.1), we begin by deriving a high probability bound on the quantity $\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^T \mathbb{I}\left[p_t = z_i\right] f(x_t)V(p_t, y_t)\right|$ that holds for all $i \in [N]$ (Lemma 10), in the following manner:

(i) Fix $i \in [N], f \in \mathcal{F}$, and failure probability δ. Recall that $n_i = \sum_{t=1}^T \mathbb{I}\left[p_t = z_i\right] = \sum_{t=1}^T \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right]$ is the number of time instants the prediction made is $p_t = z_i$, or equivalently $\tilde{p}_t \in \mathcal{I}_i$. It follows from the triangle inequality that $\left|\sum_{t=1}^T \mathbb{I}\left[p_t = z_i\right] f(x_t)V(p_t, y_t)\right| \leq \left|\sum_{t=1}^T X_t\right| + \left|\sum_{t=1}^T U_t\right| + \left|\sum_{t=1}^T Z_t\right| + \left|\sum_{t=1}^T W_t\right|$, where the sequences $\{X_t\}, \{U_t\}, \{Z_t\}$, and $\{W_t\}$ are defined as

$$\begin{aligned}
X_t &:= \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t) \cdot \left(V(z_i, y_t) - \mathbb{E}_t[V(z_i, y_t)]\right), \\
U_t &:= \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t) \cdot (\mathbb{E}_t[V(z_i, y_t)] - \mathbb{E}_t[V(\tilde{p}_t, y_t)]), \\
Z_t &:= \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t) \cdot V(\tilde{p}_t, y_t),
\end{aligned}$$

$$W_t := \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t) \cdot (\mathbb{E}_t[V(\tilde{p}_t, y_t)] - V(\tilde{p}_t, y_t)).$$

In the next steps, we discuss our approach to bound the sum of each individual sequence. As already mentioned before, conditioned on $\mathscr{F}_{t-1}$, the random variables p_t (or $\tilde{p}_t$) and y_t are independent. Since $V(., Y_t)$ is ρ-Lipschitz and $\left|p_t - \tilde{p}_t\right| \le \frac{1}{N}$, we obtain $|U_t| \le \frac{\rho}{N}$.

(ii) Observe that $\{X_t\}$ is a martingale difference sequence with $|X_t| \le 2$. Applying Freedman's inequality carefully over a dyadic partition of the interval $\left[\tilde{\Theta}\left(\frac{\log \frac{1}{\delta}}{\sqrt{T}}\right), \frac{1}{2}\right]$, similar to (Luo et al., 2025a, Lemma 8), we obtain $\left|\sum_{t=1}^T X_t\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{1}{\delta}} + \log \frac{1}{\delta}\right)$.

(iii) Next, we bound $\left|\sum_{t=1}^T Z_t\right|$. It follows from the definition of the gain function $\ell_{f,i,\sigma}$ that $\left|\sum_{t=1}^T Z_t\right| = \max_{\sigma \in \{\pm 1\}} \sum_{t=1}^T \ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t)$ and

$$\max_{\sigma \in \{\pm 1\}} \sum_{t=1}^T \ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t) \le \max_{\sigma \in \{\pm 1\}} \sum_{t=1}^T \ell_{t,f,i,\sigma} + \max_{\sigma \in \{\pm 1\}} \sum_{t=1}^T \ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t)|\mathscr{F}_{t-1}].$$

Invoking the regret guarantee of ALG (Lemma 9 in Appendix B.1), we obtain

$$\max_{\sigma \in \{\pm 1\}} \sum_{t=1}^T \ell_{t,f,i,\sigma} \le \sum_{t=1}^T \sum_{(f,i,\sigma)} w_{t,f,i,\sigma} \ell_{t,f,i,\sigma} + \tilde{O}\left(\log(|\mathcal{F}|N) + \sqrt{\left(\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log(|\mathcal{F}|N)}\right).$$

Moreover, $\sum_{t=1}^T \sum_{(f,i,\sigma)} w_{t,f,i,\sigma} \ell_{t,f,i,\sigma} \le \sum_{t=1}^T h_t(P_t) = O(1)$. For a fixed $\sigma \in \{\pm 1\}$, the sequence $\ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t)|\mathscr{F}_{t-1}]$ is a martingale difference sequence. Repeating a similar analysis as that done for the sequence $\{X_t\}$, we obtain $\left|\sum_{t=1}^T Z_t\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{|\mathcal{F}|N}{\delta}} + \log \frac{|\mathcal{F}|N}{\delta}\right)$.

(iv) Towards bounding $\left|\sum_{t=1}^T W_t\right|$, we observe that the sequence $\{W_t\}$ is not necessarily a martingale difference sequence, due to the indicator $\mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right]$. However, using the explicit form of P_t (thanks to our second modification to Hu and Wu (2024)) and ρ-Lipschitzness of $V(., Y_t)$, we argue that $|\mathbb{E}_t[W_t]| \le \frac{\rho}{T}$. Thereafter, we consider the sequence $\{W_t - \mathbb{E}_t[W_t]\}$, which is clearly a martingale difference sequence, and bound $\left|\sum_{t=1}^T W_t - \mathbb{E}_t[W_t]\right|$ in an exactly same manner as Step (ii). Since $\left|\sum_{t=1}^T W_t\right| \le \left|\sum_{t=1}^T W_t - \mathbb{E}_t[W_t]\right| + \left|\sum_{t=1}^T \mathbb{E}_t[W_t]\right|$ and $|\mathbb{E}_t[W_t]| \le \frac{\rho}{T}$, we obtain $\left|\sum_{t=1}^T W_t\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{1}{\delta}} + \log \frac{1}{\delta}\right)$.

(v) Finally, we relate $\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$ with n_i by bounding $\left|\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) - \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right]\right|$. To achieve so, we consider the sequence $\left\{\mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) - \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right]\right\}$, which is again a martingale difference sequence. Proceeding similarly as Step (ii), we obtain $\left|\sum_{t=1}^T \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] - \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{1}{\delta}} + \log \frac{1}{\delta}\right)$.

Let $\alpha := \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i), \beta = \log \frac{1}{\delta}$, and $\gamma > 0$ be the constant (including logarithmic terms) hidden in the $\tilde{O}(.)$ notation. Then the equation in the display above can be expressed as $|n_i - \alpha| \le \gamma(\sqrt{\alpha\beta} + \beta)$. Solving for α, we obtain $\alpha \le \gamma\sqrt{\beta} + \sqrt{n_i} + \sqrt{\gamma\beta} = \tilde{O}\left(\sqrt{\log \frac{1}{\delta}} + \sqrt{n_i}\right)$.

Combining everything and taking a union bound over all $i \in [N]$, $f \in \mathcal{F}$, we obtain

$$\sup_{f \in \mathcal{F}} \left| \sum_{t=1}^{T} \mathbb{I}\left[p_t = z_i\right] f(x_t) V(p_t, y_t) \right| = \tilde{O}\left(\frac{\rho n_i}{N} + \sqrt{n_i \log \frac{|\mathcal{F}| N}{\delta}} + \log \frac{|\mathcal{F}| N}{\delta} \right)$$

with probability at least $1 - \delta$. The result of Theorem 11 then follows by directly plugging the above bound in the definition of $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$ (4).

Finally, in Appendix B.2, we propose an oracle-efficient algorithm (Algorithm 2) by reducing to online agnostic learning. Our oracle-efficient algorithm builds on Algorithm 1 and achieves efficiency by using OAL, similar to Okoroafor et al. (2025). We remark that while the idea of making Algorithm 1 oracle-efficient is adopted from Okoroafor et al. (2025), its combination with other techniques proposed in this paper results in an overall different algorithm and a significantly different analysis compared to Okoroafor et al. (2025).

Organization. In Appendix A, we set up some notation and introduce the necessary background for property elicitation, regret minimization, and formally define our problem setting. In Appendix B, we propose our inefficient algorithm (Algorithm 1) and subsequently our oracle-efficient algorithm (Algorithm 2), and derive their swap multicalibration error rates (Theorem 11 and Theorem 15 respectively). Finally, we instantiate our main result (Theorem 15) for specific hypothesis classes.

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Appendix A. Preliminaries

Notation. For any $m \in [N]$, $[m]$ denotes the index set $\{1, \dots, m\}$. Throughout the paper, we reserve calligraphic alphabets, e.g., $\mathcal{P}, \mathcal{Z}$ to denote sets, where as probability distributions over sets are represented by upper-case letters. For a set $\mathcal{A}$, $\Delta(\mathcal{A})$ represents the simplex over $\mathcal{A}$, i.e., the set of all probability distributions over $\mathcal{A}$. We use $\mathbb{I}[.]$ to represent the indicator function, which evaluates to 1 if the predicate is true and 0 otherwise. For a set $\mathcal{S}$, $\bar{\mathcal{S}}$ represents the complement $\Omega \backslash \mathcal{S}$ of $\mathcal{S}$, where the sample set Ω shall be clear from the context. Finally, we use the notations $\tilde{O}, \tilde{\Omega}, \tilde{\Theta}$ to hide lower-order logarithmic terms in T.

A.1. Property Elicitation

Let the label space be $\mathcal{Y} \subseteq [0, 1]$ and $\mathcal{P}$ be a family of probability distributions supported over $\mathcal{Y}$. We view a property $\Gamma : \mathcal{P} \to [0, 1]$ as a mapping from a distribution to a scalar value. A loss function $\ell : [0, 1] \times \mathcal{Y} \to \mathbb{R}$ takes as input a prediction γ about the property value, a label y, and maps it to a real number. We say that a property Γ is *elicitable* if there exists a loss function that is minimized at the true property value. This is formalized in the following definition, which is central to our work.

Definition 5 (Strictly consistent loss function, Property elicitation) *Fix a property Γ, a family of probability distributions $\mathcal{P}$. A loss function ℓ is strictly $\mathcal{P}$-consistent if $\Gamma(P) \in \operatorname{argmin}_{\gamma \in [0,1]} \ell(\gamma, P)$ for all $P \in \mathcal{P}$, where $\ell(\gamma, P) := \mathbb{E}_{y \sim P}[\ell(\gamma, y)]$. We also say that ℓ elicits Γ and that Γ is elicitable if there exists a strictly $\mathcal{P}$-consistent loss function.*

In other words, ℓ is strictly $\mathcal{P}$-consistent if for each $P \in \mathcal{P}$, the true property value $\Gamma(P)$ can be obtained by minimizing the expected loss of ℓ over samples drawn from P. We provide several examples of elicitable properties:

- The squared loss $\ell(\gamma, y) = \frac{1}{2}(\gamma - y)^2$ elicits the mean; more generally, $\ell(\gamma, y) = (\gamma - y^k)^2$ elicits the *raw moment* $\mathbb{E}[y^k]$, where $k \in \mathbb{N}$, and $\ell(\gamma, y) = (\gamma - \phi(y))^2$ elicits $\mathbb{E}[\phi(y)]$ for any given function ϕ.

- The ℓ_1-loss $\ell(\gamma, y) = \frac{1}{2}\left|\gamma - y\right|$ elicits the median; more generally, the pinball loss $\ell(\gamma, y) = (\gamma - y)\left((1-q)\mathbb{I}\left[y \leq \gamma\right] - q\mathbb{I}\left[y > \gamma\right]\right)$ elicits the q-th quantile for $q \in [0, 1]$.

- For a discrete distribution with finite support, the $0 - 1$ loss $\ell(\gamma, y) = \mathbb{I}\left[\gamma \neq y\right]$ elicits the *mode*.

- For a fixed $\tau \in (0, 1)$, the loss $\ell(\gamma, y) = \frac{1}{2}(y - \gamma)^2\left|\tau - \mathbb{I}\left[\gamma - y \geq 0\right]\right|$ elicits the *expectile*, which corresponds to a t that satisfies $\int_{-t}^{\infty}|t - z|\, d\mu(z) = \tau \int_{-\infty}^{\infty}|t - z|\, d\mu(z)$, where μ is the cumulative distribution function of y.

- Let P be a distribution supported over $\{0, 1\}$ and $\Gamma(P) = \mathbb{P}(y = 1)$. A loss $\ell : [0, 1] \times \{0, 1\} \to \mathbb{R}$ is called *proper* if $\mathbb{E}_{y\sim\gamma}[\ell(\gamma, y)] \leq \mathbb{E}_{y\sim\gamma}[\ell(\gamma', y)]$ for all $\gamma, \gamma' \in [0, 1]$, e.g., squared loss, log loss $\ell(\gamma, y) = -y\log\gamma - (1 - y)\log(1 - \gamma)$, spherical loss $\ell(\gamma, y) = -\frac{\gamma y + (1-\gamma)(1-y)}{\sqrt{\gamma^2 + (1-\gamma)^2}}$, Tsallis entropy $\ell(\gamma, y) = (\alpha - 1)\gamma^{\alpha} - \alpha\gamma^{\alpha-1}y$ for $\alpha > 1$, etc. More generally, given any concave function $f(\gamma)$, the loss $\ell(\gamma, y) = f(\gamma) + \left\langle \partial f(\gamma), y - \gamma \right\rangle$ is proper Gneiting and Raftery (2007), where ∂f represents a supergradient of f. Clearly, any proper loss ℓ elicits Γ by definition.

Next, we define the notion of an identification function for a property, which intuitively measures the overestimate/underestimate of the true property value.

Definition 6 (Identification function) *Fix a property* Γ *and a space of probability distributions* $\mathcal{P}$. *A function* $V : [0, 1] \times \mathcal{Y} \to \mathbb{R}$ *is called a* $\mathcal{P}$*-identification function for* Γ *if* $V(\gamma, P) = 0 \iff \gamma = \Gamma(P)$ *for all* $P \in \mathcal{P}$, *where* $V(\gamma, P) := \mathbb{E}_{y\sim P}[V(\gamma, y)]$. *We say that* Γ *is identifiable if there exists a* $\mathcal{P}$*-identification function for* Γ.

As can be concluded from the definition, for any distribution $P \in \mathcal{P}$, the true property value $\Gamma(P)$ is a root of the expected identification function over samples drawn from P. For example, $V(\gamma, y) = \gamma - y$ is an identification function for mean, whereas $V(\gamma, y) = \mathbb{I}\left[y \leq \gamma\right] - 0.5$ is an identification function for median; more generally, $V(\gamma, y) = \mathbb{I}\left[y \leq \gamma\right] - q$ is an identification function for the q-th quantile.

A seminal result due to Steinwart et al. (2014) is that under mild technical assumptions on the family $\mathcal{P}$ and the property Γ, the following conditions are equivalent: (a) Γ is elicitable; (b) there exists a strictly consistent loss function for Γ; (c) there exists an identification function for Γ; (d) the level sets $\Gamma_\alpha := \{P \in \mathcal{P}; \Gamma(P) = \alpha\}$ of Γ are convex. An immediate consequence of the characterization is that it allows us to identify several properties that are not elicitable. For example, consider the conditional value at risk (also known as expected shortfall), which for a target quantile $q \in [0, 1]$ is defined as $\mathrm{CVaR}_q(P) := \mathbb{E}_{y\sim P}[y | y > f_q(P)]$, where $f_q(P)$ represents the q-th quantile of P. CVaR is of central significance in financial risk assesment, however, is not elicitable since its level sets are not convex Gneiting (2011). Similarly, when y is a binary random variable and $\mathcal{P}$ is the set of all Bernoulli distributions with mean p for $p \in [0, 1]$, the variance has non-convex level sets, therefore is not elicitable.

A.2. Problem Setting

Let Γ be an elicitable property. Following Noarov and Roth (2023), we consider the following online learning protocol: at each time $t = 1, \ldots, T$, (a) the adversary selects a feature vector $x_t \in \mathcal{X}$ and a distribution $Y_t \in \Delta(\mathcal{Y})$ over the label space $\mathcal{Y} \subseteq \{0, 1\}$, and reveals x_t; (b) the forecaster randomly predicts $p_t \in [0, 1]$; (c) the adversary reveals $y_t \sim Y_t$. Throughout, (a) we assume that the adversary is adaptive, i.e., x_t, Y_t are chosen with complete knowledge about the forecaster's algorithm and $p_1, \ldots, p_{t-1}$, and (b) the forecaster predicts $p_t \in \mathcal{Z} \subset [0, 1]$ for all $t \in [T]$, where $\mathcal{Z} = \{z_1, \ldots, z_N\}$ is a finite discretization of $[0, 1]$ and $z_i = \frac{i}{N}$ for all $i \in [N]$. We let $\mathbb{E}_t[.], \mathbb{P}_t[.]$ represent the conditional expectation, probability respectively, where the conditioning is over all randomness till time $t-1$ (inclusive). Formally, letting $\mathscr{F}_{t-1}$ be the filtration generated by the random variables $p_1, \ldots, p_{t-1}, y_1, \ldots, y_{t-1}$, we have $\mathbb{E}_t[.] := \mathbb{E}[.|\mathscr{F}_{t-1}], \mathbb{P}_t[.] := \mathbb{P}[.|\mathscr{F}_{t-1}]$ respectively.

For a property Γ, an identification function $V : [0, 1] \times \mathcal{Y} \to [-1, 1]$ takes as input the forecaster's prediction p, the adversary's chosen outcome y and maps it to a real number in $[-1, 1]$ (assumed without any loss of generality). Similar to Noarov and Roth (2023), we assume that the marginal $V(p, Y_t) := \mathbb{E}_{y \sim Y_t}[V(p, y)|\mathscr{F}_{t-1}]$ is ρ-Lipschitz in p.

Assumption 1 *The function $V(p, Y_t)$ satisfies*

$$\left|V(p_2, Y_t) - V(p_1, Y_t)\right| \leq \rho \left|p_2 - p_1\right| \text{ for all } 0 \leq p_1, p_2 \leq 1. \tag{7}$$

Furthermore, $V(0, Y) \leq 0$ and $V(1, Y) \geq 0$ for all $Y \in \Delta(\mathcal{Y})$.

For specific instantiations of (7), observe that for means $V(p, Y_t) = p - \mathbb{E}_t[y]$ is trivially 1-Lipschitz in p, whereas for quantiles $V(p, Y_t) = \mathbb{P}_t(y \leq p) - q$ and (7) corresponds to the assumption that the CDF of Y_t is ρ-Lipschitz, which is a standard assumption invoked in the relevant literature on quantile calibration (Jung et al., 2023; Garg et al., 2024). The second condition in Assumption 1 is without any loss of generality; it is clearly true for means and quantiles and more generally can be interpreted as an underestimate/overestimate of the true property value. For instance, for expectiles, $V(p, y) = (p - y)(1 - \tau)$ if $p \geq y$ and $V(p, y) = (p - y)\tau$ if $p \leq y$. Clearly, $V(p, Y)$ is Lipschitz in p and satisfies $V(0, Y) \leq 0$ and $V(1, Y) \geq 0$ for all $Y \in \Delta(\mathcal{Y})$. Similarly, Assumption 1 can be verified to be true for raw moments, $\Gamma(P) = \mathbb{P}(y = 1)$ under specific proper losses such as Tsallis entropy with $\alpha \geq 3$, spherical loss, etc.

For a bounded hypothesis class $\mathcal{F} \subset [-1, 1]^{\mathcal{X}}$, ℓ_r-multicalibration ($\mathsf{MCal}_{\Gamma,r}(\mathcal{F})$) and ℓ_r-swap multicalibration ($\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$) errors ($r \geq 1$) are given by (3) and (4) respectively. The goal of the forecaster is to make predictions $p_1, \ldots, p_T$ such that $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$ is minimized.

A.3. Regret Minimization

As mentioned, regret minimization algorithms are important subroutines of our final algorithm. Consider the following interaction between an algorithm and adversary over T rounds: at each time $t \in [T]$, the algorithm takes an action $a_t \in \mathcal{A}$ and simultaneously the adversary reveals an outcome $y_t \in \mathcal{Y}$; the algorithm observes y_t and incurs loss $\ell(a_t, y_t)$ for a known loss function $\ell : \mathcal{A} \times \mathcal{Y} \to \mathbb{R}$. The *external regret* $\mathsf{Reg} \coloneqq \sum_{t=1}^{T} \ell(a_t, y_t) - \inf_{a \in \mathcal{A}} \sum_{t=1}^{T} \ell(a, y_t)$ compares the cumulative loss incurred by the algorithm with that incurred by the best fixed action chosen in hindsight. Compared to external regret, *swap regret* is a stronger benchmark that compares the total loss incurred by the algorithm with that of the best swap (or modification) function $\nu : \mathcal{A} \to \mathcal{A}$ chosen in hindsight, i.e., $\mathsf{SReg} \coloneqq \sup_{\nu:\mathcal{A}\to\mathcal{A}} \sum_{t=1}^{T} \ell(a_t, y_t) - \ell(\nu(a_t), y_t)$.

Expert problem. The K-expert problem is a canonical problem in online learning, where at each time $t \in [T]$, an online algorithm outputs a probability distribution $w_t \in \Delta([K])$ over the experts and subsequently the adversary reveals a gain (or reward) vector $\ell_t \in [-1, 1]^K$, where $\ell_t(i)$ is the gain incurred by expert i. The regret $\mathsf{Reg} \coloneqq \max_{i^\star \in [K]} \sum_{t=1}^{T} \ell_t(i^\star) - \sum_{t=1}^{T} \langle w_t, \ell_t \rangle$ measures the difference between the gain of the best fixed expert chosen in hindsight and the expected gain of the algorithm.

Sequential Rademacher Complexity. A $\mathcal{X}$-valued binary tree is one in which all nodes have values that lie in $\mathcal{X}$. For a $\mathcal{X}$-valued tree $\boldsymbol{x}$ with depth n, a path from the root to a leaf can be represented as $\boldsymbol{\epsilon} = (\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i \in \{-1, 1\}$ and $-1, +1$ correspond to traversing left, right respectively (ϵ_n is clearly irrelevant). Moreover, $\boldsymbol{x}$ can be represented by a sequence of mappings $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$, where $\boldsymbol{x}_i : \{-1, 1\}^{i-1} \to \mathcal{X}$ represents the value of a node at the i-th level of $\boldsymbol{x}$. For notational convenience, we represent $\boldsymbol{x}_i(\epsilon_1, \ldots, \epsilon_{i-1})$ with $\boldsymbol{x}_i(\boldsymbol{\epsilon})$, with the understanding that $\epsilon_i, \ldots, \epsilon_n$ are irrelevant when describing a node at the i-th level.

Definition 7 (Sequential Rademacher Complexity) *For a hypothesis class $\mathcal{F}$ and a $\mathcal{X}$-valued tree $\boldsymbol{x}$ with depth n, the conditional sequential Rademacher complexity $\mathfrak{R}^{seq}(\mathcal{F}, n; \boldsymbol{x})$ is defined as*

$$\mathfrak{R}^{seq}(\mathcal{F}, n; \boldsymbol{x}) \coloneqq \frac{1}{n} \sup_{f \in \mathcal{F}} \mathbb{E}_{\boldsymbol{\epsilon}} \left[\sum_{i=1}^{n} \epsilon_i f(\boldsymbol{x}_i(\boldsymbol{\epsilon})) \right],$$

where the expectation is taken over the uniform distribution over $\{-1, 1\} \times \cdots \times \{-1, 1\}$. The (unconditional) sequential Rademacher complexity $\mathfrak{R}^{seq}(\mathcal{F}, n)$ is then defined as

$$\mathfrak{R}^{seq}(\mathcal{F}, n) \coloneqq \sup_{\boldsymbol{x}} \mathfrak{R}^{seq}(\mathcal{F}, n; \boldsymbol{x}),$$

where the supremum is taken over all $\mathcal{X}$-valued trees of depth n.

Similar to the role of Rademacher complexity in describing the learnability of a hypothesis class $\mathcal{F}$ in the distributional setting, the sequential Rademacher complexity characterizes learnability in the online setting.

Appendix B. Algorithm

In this section, we propose an algorithm for multicalibrating an elicitable property Γ with an identification function V that satisfies Assumption 1. We first propose an inefficient algorithm and subsequently propose an efficient algorithm by reducing to online agnostic learning.

B.1. Achieving Multicalibration Inefficiently

As mentioned in Section 1, we build upon the algorithm proposed by Hu and Wu (2024) in a different context. For each $i \in [N-1]$, let $\mathcal{I}_i = \left[\frac{i-1}{N}, \frac{i}{N}\right)$ and $\mathcal{I}_N = \left[\frac{N-1}{N}, 1\right]$ so that $\mathcal{I}_1, \ldots, \mathcal{I}_N$ represents a partition of $[0,1]$. We consider a $2N|\mathcal{F}|$-expert problem where for each $f \in \mathcal{F}, i \in [N]$, and $\sigma \in \{\pm 1\}$, we define the gain function $\ell_{f,i,\sigma} : [0,1] \times \mathcal{X} \times \mathcal{Y} \to [-1,1]$ corresponding to the expert characterized by the tuple (f, i, σ) as $\ell_{f,i,\sigma}(p, x, y) := \sigma \mathbb{I}\left[p \in \mathcal{I}_i\right] f(x) V(p, y)$. At each time t, Algorithm 1 obtains a probability distribution $\{w_{t,f,i,\sigma}\}$ over the $2|\mathcal{F}|N$ experts via an expert problem subroutine ALG, where $w_{t,f,i,\sigma}$ is the probability allotted to the expert corresponding to (f, i, σ). As shall be shown in Proposition 8, in Line 4 we obtain a distribution P_t satisfying $h_t(P_t) \leq \frac{\rho}{T}$, where the function $h_t : \Delta([0,1]) \to [-1,1]$ is defined as

$$h_t(P) := \sup_{y \in \mathcal{Y}} \mathbb{E}_{p \sim P}\left[\sum_{(f,i,\sigma)} w_{t,f,i,\sigma} \ell_{f,i,\sigma}(p, x_t, y) | \mathscr{F}_{t-1}\right].$$

Subsequently, Algorithm 1 samples $\tilde{p}_t \sim P_t$ and predicts $p_t = \frac{i_t}{N}$, where $i_t \in [N]$ is such that $\tilde{p}_t \in \mathcal{I}_{i_t}$. Finally, on observing y_t, we feed the gain $\ell_{t,f,i,\sigma} := \mathbb{E}_{p \sim P_t}\left[\ell_{f,i,\sigma}(p, y_t, x_t) | \mathscr{F}_{t-1}\right]$ for each (f, i, σ) to ALG. A subtle difference between $\ell_{f,i,\sigma}$ and $\ell_{t,f,i,\sigma}$ is that the latter represents the actual gain fed to ALG, while the former is an auxiliary function used to define the latter.

Algorithm 1 Multicalibration for an Elicitable Property (Computationally Inefficient Version)

Initialize: An expert problem subroutine ALG;

1: **for** $t = 1, \ldots, T$,
2: Receive context x_t;
3: Obtain weights $\{w_{t,f,i,\sigma}\}$ from ALG;
4: Define the function $\Phi_t : [0,1] \to [-1,1]$ as

$$\Phi_t(p) := \sum_{(f,i,\sigma)} w_{t,f,i,\sigma} \sigma \mathbb{I}\left[p \in \mathcal{I}_i\right] f(x_t).$$

If $\Phi_t(0) > 0$, choose P_t that is only supported on 0; else if $\Phi_t(1) \leq 0$, choose P_t that is only supported on 1; else choose $i \in \{0, \ldots, T-1\}$ such that $\Phi_t\left(\frac{i}{T}\right) \Phi_t\left(\frac{i+1}{T}\right) \leq 0$ and P_t such that

$$P_t\left(\frac{i}{T}\right) = \frac{\left|\Phi_t\left(\frac{i+1}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|}, \quad P_t\left(\frac{i+1}{T}\right) = \frac{\left|\Phi_t\left(\frac{i}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|}.$$

5: Sample $\tilde{p}_t \sim P_t$ and predict $p_t = z_{i_t} = \frac{i_t}{N}$, where $i_t \in [N]$ is such that $\tilde{p}_t \in \mathcal{I}_{i_t}$;
6: Observe $y_t \in \mathcal{Y}$;
7: For each (f, i, σ), feed $\ell_{t,f,i,\sigma} = \mathbb{E}_{p \sim P_t}\left[\ell_{f,i,\sigma}(p, x_t, y_t) | \mathscr{F}_{t-1}\right]$ to ALG;

Proposition 8 *Algorithm 1 satisfies $h_t(P_t) \leq \frac{\rho}{T}$ for all $t \in [T]$.*

Proof When $\Phi_t(0) > 0$, we have

$$h_t(P_t) = \sup_{Y \in \Delta(\mathcal{Y})} \mathbb{E}_{y \sim Y}\left[\Phi_t(0) V(0, y)\right] = \sup_{Y \in \Delta(\mathcal{Y})} \Phi_t(0) V(0, Y) \leq 0,$$

where the inequality follows from Assumption 1. Similarly, when $\Phi_t(1) \leq 0$, we have

$$h_t(P_t) = \sup_{Y \in \Delta(\mathcal{Y})} \mathbb{E}_{y \sim Y}[\Phi_t(1)V(1,y)] = \sup_{Y \in \Delta(\mathcal{Y})} \Phi_t(1)V(1,Y) \leq 0.$$

For the last case, we have

$$\begin{aligned}
h_t(P_t) &= \sup_{Y \in \Delta(\mathcal{Y})} \mathbb{E}_{p \sim P_t, y \sim Y}\left[\Phi_t(p)V(p,y)\right] \\
&= \sup_{Y \in \Delta(\mathcal{Y})} \mathbb{E}_{y \sim Y}\left[\frac{\Phi_t\left(\frac{i}{T}\right)\left|\Phi_t\left(\frac{i+1}{T}\right)\right| V\left(\frac{i}{T}, y\right)}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|} + \frac{\Phi_t\left(\frac{i+1}{T}\right)\left|\Phi_t\left(\frac{i}{T}\right)\right| V\left(\frac{i+1}{T}, y\right)}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|}\right] \\
&\leq \sup_{Y \in \Delta(\mathcal{Y})} \left[\frac{\left|\Phi_t\left(\frac{i}{T}\right)\right|\left|\Phi_t\left(\frac{i+1}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right| + \left|\Phi_t\left(\frac{i+1}{T}\right)\right|} \cdot \left|V\left(\frac{i}{T}, Y\right) - V\left(\frac{i+1}{T}, Y\right)\right|\right] \leq \frac{\rho}{T},
\end{aligned}$$

where the last inequality follows since due to the Lipschitzness of $V(., Y)$ (7) and that Φ_t takes value in $[-1, 1]$. This completes the proof. ■

Instantiating ALG with the MsMwC algorithm (Chen et al., 2021), we obtain the following lemma.

Lemma 9 *The MsMwC algorithm (Chen et al., 2021) ensures the following regret bound*

$$\sum_{t=1}^{T} \ell_{t,f,i,\sigma} - \sum_{t=1}^{T} \sum_{(f',i',\sigma')} w_{t,f',i',\sigma'} \ell_{t,f',i',\sigma'} = \mathcal{O}\left(\log(|\mathcal{F}|NT) + \sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log(|\mathcal{F}|NT)}\right)$$

against each expert (f, i, σ) simultaneously.

Note that a canonical expert minimization algorithm such as Multiplicative Weights Update would incur $\tilde{O}(\sqrt{T})$ regret against each expert, which is not sufficient to achieve the sub-$\sqrt{T}$ rates that we seek for in this paper. Instead, the MsMwC algorithm ensures a regret bound that depends on $\sqrt{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)}$, which can be potentially much smaller than $\sqrt{T}$. Moreover, compared to (Hu and Wu, 2024, Lemma 5.4) which incurs a factor proportional to $\sqrt{\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right]}$, we incur a factor of $\sqrt{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)}$ instead, since MsMwC actually ensures a regret bound $\tilde{O}\left(\sqrt{\sum_{t=1}^{T} \ell_{t,f,i,\sigma}^2}\right)$ (ignoring additive constants) against the expert (f, i, σ). For our choice of $\ell_{t,f,i,\sigma}$, it follows that $\ell_{t,f,i,\sigma}^2 \leq \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$, thereby introducing a different factor.

Lemma 10 *Algorithm 1 ensures that*

$$\sup_{f \in \mathcal{F}} \left|\sum_{t=1}^{T} \mathbb{I}\left[p_t = z_i\right] f(x_t) V(p_t, y_t)\right| = \tilde{O}\left(\frac{\rho n_i}{N} + \sqrt{n_i \log \frac{|\mathcal{F}|N}{\delta}} + \log \frac{|\mathcal{F}|N}{\delta}\right)$$

for all $i \in [N]$ with probability at least $1 - \delta$.

Proof Recall that $n_i = \sum_{t=1}^T \mathbb{I}[p_t = z_i] = \sum_{t=1}^T \mathbb{I}[\tilde{p}_t \in \mathcal{I}_i]$ is the number of time instants the prediction made is $p_t = z_i = \frac{i}{N}$. Fixing a $i \in [N]$, $f \in \mathcal{F}$, it follows from the triangle inequality that

$$\left|\sum_{t=1}^T \mathbb{I}[p_t = z_i] f(x_t) V(p_t, y_t)\right| \leq \left|\sum_{t=1}^T X_t\right| + \left|\sum_{t=1}^T U_t\right| + \left|\sum_{t=1}^T Z_t\right| + \left|\sum_{t=1}^T W_t\right|,$$

where the sequences $\{X_t\}, \{U_t\}, \{Z_t\}$, and $\{W_t\}$ are defined as

$$\begin{aligned}
X_t &:= \mathbb{I}[\tilde{p}_t \in \mathcal{I}_i] f(x_t) \cdot \left(V(z_i, y_t) - \mathbb{E}_t[V(z_i, y_t)]\right), \\
U_t &:= \mathbb{I}[\tilde{p}_t \in \mathcal{I}_i] f(x_t) \cdot (\mathbb{E}_t[V(z_i, y_t)] - \mathbb{E}_t[V(\tilde{p}_t, y_t)]), \\
Z_t &:= \mathbb{I}[\tilde{p}_t \in \mathcal{I}_i] f(x_t) \cdot V(\tilde{p}_t, y_t), \\
W_t &:= \mathbb{I}[\tilde{p}_t \in \mathcal{I}_i] f(x_t) \cdot (\mathbb{E}_t[V(\tilde{p}_t, y_t)] - V(\tilde{p}_t, y_t)).
\end{aligned}$$

Note that conditioned on $\mathscr{F}_{t-1}$, the random variables p_t (or $\tilde{p}_t$) and y_t are independent. In the next steps, we bound the sum of each sequence individually.

Bounding $\left|\sum_{t=1}^T U_t\right|$. Since $V(., Y_t)$ is ρ-Lipschitz and $\left|p_t - \tilde{p}_t\right| \leq \frac{1}{N}$, we immediately obtain $|U_t| \leq \frac{\rho}{N}$.

Bounding $\left|\sum_{t=1}^T X_t\right|$. Clearly, $|X_t| \leq 2$ and $\mathbb{E}_t[X_t] = 0$; thus, $\{X_t\}$ is a bounded martingale difference sequence. Furthermore, the cumulative variance $V_X := \sum_{t=1}^T \mathbb{E}_t\left[X_t^2\right]$ can be bounded as $V_X \leq 4\sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$. Therefore, for a fixed $\mu \in [0, \frac{1}{2}]$ and $\delta \in [0, 1]$, it follows from Lemma 21 that

$$\left|\sum_{t=1}^T X_t\right| \leq 4\mu \sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) + \frac{1}{\mu} \log \frac{2}{\delta} \tag{8}$$

with probability at least $1 - \delta$. For the subsequent steps, we assume that $m \in \mathbb{N}$ is such that

$$\frac{1}{2^m} \leq \sqrt{\frac{\log \frac{2(m+1)}{\delta}}{T}} < \frac{1}{2^{m-1}}, \tag{9}$$

and partition the interval $\mathcal{J} = \left[\frac{1}{2}\sqrt{\frac{\log \frac{2(m+1)}{\delta}}{T}}, \frac{1}{2}\right]$ into dyadic intervals as $\mathcal{J} = \mathcal{J}_m \cup \cdots \cup \mathcal{J}_1 \cup \mathcal{J}_0$, where $\mathcal{J}_k = \left[\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$ for all $k \in [m-1]$, $\mathcal{J}_0 = \left\{\frac{1}{2}\right\}$, and $\mathcal{J}_m = \left[\frac{1}{2}\sqrt{\frac{\log \frac{2(m+1)}{\delta}}{T}}, \frac{1}{2^m}\right)$. For each interval $\mathcal{J}_k$, we associate a parameter $\mu_k = \frac{1}{2^{k+1}}$. Taking a union bound over all $k \in \{0, \ldots, m\}$, we obtain

$$\left|\sum_{t=1}^T X_t\right| \leq 4\mu_k \sum_{t=1}^T \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) + \frac{1}{\mu_k} \log \frac{2(m+1)}{\delta}$$

with probability at least $1 - \delta$ (simultaneously for all k). Each interval corresponds to a condition on $\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$ for which the optimal

$$\mu = \frac{1}{2} \min\left(1, \sqrt{\frac{\log \frac{2(m+1)}{\delta}}{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)}}\right)$$

falls within that interval. In particular, for each $k \in [m-1]$, the interval $\mathcal{J}_k$ shall correspond to the condition that

$$\frac{1}{2^k} \le \sqrt{\frac{\log \frac{2(m+1)}{\delta}}{\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)}} < \frac{1}{2^{k-1}} \equiv 4^{k-1} \log \frac{2(m+1)}{\delta} < \sum_{t=1}^{T} \mathbb{P}(\tilde{p}_t \in \mathcal{I}_i) \le 4^k \log \frac{2(m+1)}{\delta}. \quad (10)$$

Additionally, $\mathcal{J}_0$ shall represent the condition that $0 \le \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) \le \log \frac{2(m+1)}{\delta}$. Finally, $\mathcal{J}_m$ shall correspond to $4^{m-1} \log \frac{2(m+1)}{\delta} < \sum_{t=1}^{T} \mathbb{P}(\tilde{p}_t \in \mathcal{I}_i) \le T$. Next, we prove an uniform upper bound on $\left|\sum_{t=1}^{T} X_t\right|$ by analyzing the bound for each interval. Towards this end, let $k \in [m-1]$. Then,

$$\left|\sum_{t=1}^{T} X_t\right| \le \frac{1}{2^{k-1}} \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) + 2^{k+1} \log \frac{2(m+1)}{\delta} \le 6\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{2(m+1)}{\delta}},$$

where the second inequality follows from (10). Similarly, for $k = 0$ we have

$$\left|\sum_{t=1}^{T} X_t\right| \le 2 \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) + 2 \log \frac{2(m+1)}{\delta} \le 4 \log \frac{2(m+1)}{\delta}.$$

Finally, for $k = m$ we have

$$\begin{aligned}
\left|\sum_{t=1}^{T} X_t\right| &\le \frac{1}{2^{m-1}} \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) + 2^{m+1} \log \frac{2(m+1)}{\delta} \\
&\le 2\sqrt{\frac{\log \frac{2(m+1)}{\delta}}{T}} \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i) + 2^{m+1} \log \frac{2(m+1)}{\delta} \\
&\le 6\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{2(m+1)}{\delta}},
\end{aligned}$$

where the first inequality follows from (9), while the second inequality follows from the condition characterizing $\mathcal{J}_m$. Combining the three bounds above, we have shown that with probability at least $1 - \delta$,

$$\begin{aligned}
\left|\sum_{t=1}^{T} X_t\right| &\le 6\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{2(m+1)}{\delta}} + 4 \log \frac{2(m+1)}{\delta} \\
&= \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{1}{\delta}} + \log \frac{1}{\delta}\right),
\end{aligned}$$

where the equality follows since $m = \tilde{\Theta}(\log T)$. Taking a union bound over all $f \in \mathcal{F}, i \in [N]$, we obtain

$$\sup_{f\in\mathcal{F}} \left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t) \left(V(z_i, y_t) - \mathbb{E}_t[V(z_i, y_t)]\right)\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{N|\mathcal{F}|}{\delta}} + \log \frac{N|\mathcal{F}|}{\delta}\right)$$

for all $i \in [N]$ simultaneously with probability at least $1 - \delta$.

Bounding $\left|\sum_{t=1}^{T} Z_t\right|$. It follows from the definition of the gain function $\ell_{f,i,\sigma}$ that

$$\begin{aligned}\left|\sum_{t=1}^{T} Z_t\right| &= \max_{\sigma\in\{\pm 1\}} \sum_{t=1}^{T} \ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t) \\ &\le \max_{\sigma\in\{\pm 1\}} \sum_{t=1}^{T} \ell_{t,f,i,\sigma} + \max_{\sigma\in\{\pm 1\}} \sum_{t=1}^{T} \ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t)|\mathscr{F}_{t-1}].\end{aligned}$$

Invoking the regret guarantee of ALG (Lemma 9), we obtain

$$\max_{\sigma\in\{\pm 1\}} \sum_{t=1}^{T} \ell_{t,f,i,\sigma} \le \sum_{t=1}^{T} \sum_{(f,i,\sigma)} w_{t,f,i,\sigma} \ell_{t,f,i,\sigma} + \tilde{O}\left(\log(|\mathcal{F}|N) + \sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log(|\mathcal{F}|N)}\right).$$

Moreover, $\sum_{t=1}^{T} \sum_{(f,i,\sigma)} w_{t,f,i,\sigma} \ell_{t,f,i,\sigma} \le \sum_{t=1}^{T} h_t(P_t) = O(1)$. For a fixed $\sigma \in \{\pm 1\}$, the sequence $\ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{f,i,\sigma}(\tilde{p}_t, x_t, y_t)|\mathscr{F}_{t-1}]$ is a martingale difference sequence. Repeating a similar analysis as that done for the sequence $\{X_t\}$, we obtain

$$\left|\sum_{t=1}^{T} Z_t\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{|\mathcal{F}|N}{\delta}} + \log \frac{|\mathcal{F}|N}{\delta}\right).$$

Taking a union bound over all $f \in \mathcal{F}, i \in [N]$, we obtain that

$$\sup_{f\in\mathcal{F}} \left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t) V(\tilde{p}_t, y_t)\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log \frac{|\mathcal{F}|N}{\delta}} + \log \frac{|\mathcal{F}|N}{\delta}\right).$$

holds for all $i \in [N]$ with probability at least $1 - \delta$.

Bounding $\left|\sum_{t=1}^{T} W_t\right|$. Observe that $\{W_t\}$ is not necessarily a martingale difference sequence. However, since the distribution P_t randomizes between at most 2 points and $V(., Y)$ is Lipschitz, $|\mathbb{E}_t[W_t]| \le \frac{\rho}{T}$. This is because,

$$\mathbb{E}_t[W_t] = f(x_t)\mathbb{E}_{\tilde{p}_t}\left[\mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right](\mathbb{E}_{\tilde{p}_t}[V(\tilde{p}_t, Y_t)] - V(\tilde{p}_t, Y_t))\right],$$

where we have simplified the expression by not mentioning the filtration $\mathscr{F}_{t-1}$, which is still implicit in $\mathbb{E}_{\tilde{p}_t}[.]$. Subsequently, if P_t picks 0 or 1 deterministically, $\mathbb{E}_t[W_t] = 0$. In the remaining case, let

P_t randomize between 2 points $\frac{j}{T}$ and $\frac{j+1}{T}$ at time t. Then, if the sampled $\tilde{p}_t$ is $\frac{j}{T}$, we have

$$\left|\mathbb{E}_{\tilde{p}_t}[V(\tilde{p}_t, Y_t)] - V(\tilde{p}_t, Y_t)\right| = \left|P_t\left(\frac{j}{T}\right)V\left(\frac{j}{T}, Y_t\right) + P_t\left(\frac{j+1}{T}\right)V\left(\frac{j+1}{T}, Y_t\right) - V\left(\frac{j}{T}, Y_t\right)\right|$$
$$= \left|P_t\left(\frac{j+1}{T}\right)\left(V\left(\frac{j+1}{T}, Y_t\right) - V\left(\frac{j}{T}, Y_t\right)\right)\right| \leq \frac{\rho}{T},$$

where the inequality follows due to the Lipschitzness of $V(., Y)$ (7). The above bound also holds if the sampled $\tilde{p}_t = \frac{j+1}{T}$. Therefore, $|\mathbb{E}_t[W_t]| \leq \frac{\rho}{T}$. Equipped with this, we can bound $\sum_{t=1}^{T} W_t$ by considering the martingale difference sequence $W_t - \mathbb{E}_t[W_t]$ instead and repeating a similar analysis using Freedman's inequality. Particularly, $|W_t - \mathbb{E}_t[W_t]| \leq 4$ since $|W_t| \leq 2$ and the conditional variance can be bounded as $\sum_{t=1}^{T} \mathbb{E}_t[(W_t - \mathbb{E}_t[W_t])^2] = \sum_{t=1}^{T} \mathbb{E}_t[W_t^2] - (\mathbb{E}_t[W_t])^2 \leq \sum_{t=1}^{T} \mathbb{E}_t[W_t^2] \leq 4 \cdot \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$. Repeating a similar analysis as that done for the sequence $\{X_t\}$, we obtain

$$\left|\sum_{t=1}^{T} W_t - \mathbb{E}_t[W_t]\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log\frac{1}{\delta}} + \log\frac{1}{\delta}\right).$$

Since $\left|\sum_{t=1}^{T} W_t\right| \leq \left|\sum_{t=1}^{T} W_t - \mathbb{E}_t[W_t]\right| + \left|\sum_{t=1}^{T} \mathbb{E}_t[W_t]\right|$ and $|\mathbb{E}_t[W_t]| \leq \frac{\rho}{T}$, we obtain

$$\left|\sum_{t=1}^{T} W_t\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log\frac{1}{\delta}} + \log\frac{1}{\delta}\right).$$

Taking a union bound over all $f \in \mathcal{F}, i \in [N]$, we obtain that

$$\sup_{f \in \mathcal{F}} \left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t)(\mathbb{E}_t[V(\tilde{p}_t, y_t)] - V(\tilde{p}_t, y_t))\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log\frac{|\mathcal{F}|N}{\delta}} + \log\frac{|\mathcal{F}|N}{\delta}\right)$$

holds for all $i \in [N]$ with probability at least $1 - \delta$.

Relating $\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$ with n_i. Next, we obtain a high probability bound that relates $\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$ with n_i. Consider the sequence $A_t := \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] - \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)$. Again, $\{A_t\}$ is a martingale difference sequence, therefore, repeating the exact same analysis as that for the sequence $\{X_t\}$, we obtain that

$$\left|\sum_{t=1}^{T} \mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] - \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right| = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right) \log\frac{N}{\delta}} + \log\frac{N}{\delta}\right). \tag{11}$$

holds for all $i \in [N]$ with probability at least $1 - \delta$. Let $\alpha := \sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i), \beta = \log\frac{N}{\delta}$, and $\gamma > 0$ be the constant (including logarithmic terms) hidden in the $\tilde{O}(.)$ notation. Then, (11) can be expressed as $|n_i - \alpha| \leq \gamma(\sqrt{\alpha\beta} + \beta)$, therefore α satisfies $\alpha - \gamma\sqrt{\alpha\beta} - (n_i + \gamma\beta) \leq 0$, which implies that $\sqrt{\alpha} \leq \frac{\gamma\sqrt{\beta} + \sqrt{\gamma^2\beta + 4n_i + 4\gamma\beta}}{2} \leq \gamma\sqrt{\beta} + \sqrt{n_i} + \sqrt{\gamma\beta} = \tilde{O}\left(\sqrt{\log\frac{N}{\delta}} + \sqrt{n_i}\right)$.

Combining everything. For all $i \in [N]$, with probability at least $1-\delta$, we have

$$\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t)\left(V(z_i,y_t)-\mathbb{E}_t[V(z_i,y_t)]\right)\right| = \tilde{O}\left(\sqrt{n_i\log\frac{|\mathcal{F}|N}{\delta}}+\log\frac{|\mathcal{F}|N}{\delta}\right),$$

$$\max_{\sigma\in\{\pm1\}}\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\ell_{f,i,\sigma}(\tilde{p}_t,x_t,y_t)-\mathbb{E}_{\tilde{p}_t}[\ell_{f,i,\sigma}(\tilde{p}_t,x_t,y_t)|\mathscr{F}_{t-1}]\right| = \tilde{O}\left(\sqrt{n_i\log\frac{|\mathcal{F}|N}{\delta}}+\log\frac{|\mathcal{F}|N}{\delta}\right),$$

$$\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[\tilde{p}_t \in \mathcal{I}_i\right] f(x_t)(\mathbb{E}_t[V(\tilde{p}_t,y_t)]-V(\tilde{p}_t,y_t))\right| = \tilde{O}\left(\sqrt{n_i\log\frac{|\mathcal{F}|N}{\delta}}+\log\frac{|\mathcal{F}|N}{\delta}\right).$$

Combining the three bounds above with the bound on $\left|\sum_{t=1}^{T}U_t\right|$, we obtain

$$\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right] f(x_t)V(p_t,y_t)\right| = \tilde{O}\left(\frac{\rho n_i}{N}+\sqrt{n_i\log\frac{|\mathcal{F}|N}{\delta}}+\log\frac{|\mathcal{F}|N}{\delta}\right).$$

This completes the proof. ■

Equipped with Lemma 10, we prove the main result of this subsection.

Theorem 11 *For a fixed $r \geq 2$, Algorithm 1 achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\rho^r T^{\frac{1}{r+1}} + T^{\frac{1}{r+1}}\left(\log\frac{|\mathcal{F}|}{\delta}\right)^{\frac{r}{2}}\right)$$

with probability at least $1-\delta$. Consequently, for $r \in [1,2)$, Algorithm 1 achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\rho^r T^{1-\frac{r}{3}} + T^{1-\frac{r}{3}}\left(\log\frac{|\mathcal{F}|}{\delta}\right)^{\frac{r}{2}}\right)$$

with probability at least $1-\delta$.

Proof Fix a $r \geq 2$. We bound $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$ using the high probability bound in Lemma 10. From Lemma 10:

$$\frac{1}{n_i}\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right] f(x_t)V(p_t,y_t)\right| \leq \begin{cases}\tilde{O}\left(\sqrt{\frac{\log\frac{|\mathcal{F}|N}{\delta}}{n_i}}+\frac{\rho}{N}\right) & i\in\mathcal{S},\\ 1 & i\in\bar{\mathcal{S}},\end{cases} \tag{12}$$

where $\mathcal{S} := \left\{i\in[N]; n_i = \Omega\left(\log\frac{|\mathcal{F}|N}{\delta}\right)\right\}$. Therefore,

$$\begin{aligned}\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) &= \sum_{i\in[N]} n_i\left(\frac{1}{n_i}\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right] f(x_t)V(p_t,y_t)\right|\right)^r\\ &= \sum_{i\in\mathcal{S}} n_i\cdot\tilde{O}\left(\frac{\rho^r}{N^r}+\left(\frac{\log\frac{|\mathcal{F}|N}{\delta}}{n_i}\right)^{\frac{r}{2}}\right)+O\left(\left|\bar{\mathcal{S}}\right|\log\frac{|\mathcal{F}|N}{\delta}\right)\\ &= \tilde{O}\left(\rho^r\frac{T}{N^r}+N\left(\log\frac{|\mathcal{F}|N}{\delta}\right)^{\frac{r}{2}}\right)\end{aligned}$$

where the first equality follows since $(u+v)^r \le 2^{r-1}(u^r+v^r)$, which holds for all u, v via Jensen's inequality applied to the function $x \to x^r$, which is convex for $r \ge 1$; we incur the $\mathcal{O}\left(\left|\bar{\mathcal{S}}\right| \log \frac{|\mathcal{F}|N}{\delta}\right)$ term due to (12), and since $n_i = \mathcal{O}\left(\log \frac{|\mathcal{F}|N}{\delta}\right)$ for $i \in \bar{\mathcal{S}}$. To obtain the second equality, we bound $n_i^{1-\frac{r}{2}} < 1$. Setting $N = \Theta(T^{\frac{1}{r+1}})$ to balance the terms $\frac{T}{N^r}$ and N, we obtain the desired bound on $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$. For $r \in [1,2)$, since $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) \le T^{1-\frac{r}{2}}(\mathsf{SMCal}_{\Gamma,2}(\mathcal{F}))^{\frac{r}{2}}$ by Hölder's inequality, we obtain

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{\mathcal{O}}\left(T^{1-\frac{r}{2}} \cdot \left(\rho^2 T^{\frac{1}{3}} + T^{\frac{1}{3}} \log \frac{|\mathcal{F}|}{\delta}\right)^{\frac{r}{2}}\right) = \tilde{\mathcal{O}}\left(\rho^r T^{1-\frac{r}{3}} + T^{1-\frac{r}{3}}\left(\log \frac{|\mathcal{F}|}{\delta}\right)^{\frac{r}{2}}\right),$$

where the second equality follows since $(u+v)^{\frac{r}{2}} \le \sqrt{2^{r-1}(u^r+v^r)} \le 2^{\frac{r-1}{2}}(u^{\frac{r}{2}}+v^{\frac{r}{2}})$. This completes the proof. ■

B.2. Achieving Oracle-Efficient Multicalibration

Clearly, Algorithm 1 is inefficient since it requires enumerating over $\mathcal{F}$. In this section, we propose an oracle-efficient algorithm by reducing to online agnostic learning.

Definition 12 (Online Agnostic Learning (Ben-David et al., 2009; Beygelzimer et al., 2015)) *Consider the following interaction between an online agnostic learner (*OAL*) and adversary for n rounds: at each time $t \in [n]$, (a) the adversary reveals a context $x_t \in \mathcal{X}$; (b)* OAL *responds with a prediction $q_t(x_t)$, where $q_t : \mathcal{X} \to [-1,1]$; (c) adversary reveals the outcome $\kappa_t \in [-1,1]$. In online agnostic learning, the goal of* OAL *is to output a sequence of test functions $q_1, \dots, q_T$ whose cumulative correlation $\sum_{t=1}^T q_t(x_t)\kappa_t$ with the outcome sequence $\kappa_1, \dots, \kappa_T$ is comparable with the best hypothesis in a given hypothesis class $\mathcal{F}$, i.e.,*

$$\sup_{f \in \mathcal{F}} \sum_{t=1}^n f(x_t)\kappa_t \le \sum_{t=1}^n q_t(x_t)\kappa_t + \mathsf{Reg}(\mathcal{F}, n), \tag{5}$$

where $\mathsf{Reg}(\mathcal{F}, n) > 0$ *denotes the regret incurred by* OAL.

For a randomized online agnostic learner that achieves (5) in expectation, due to linearity of (5) in the prediction $q_t(x_t)$, it is sufficient to predict the expectation of $q_t(x_t)$ at each time t to achieve the same guarantee, therefore, without any loss of generality, we may assume that OAL is deterministic.

Remark 13 (Oracle comparison) *Note that compared to Garg et al. (2024), which assumes an online squared error regression oracle for $\mathcal{F}$ and Ghuge et al. (2025), which assumes an offline optimization (multicalibration evaluation) oracle, we assume an online agnostic learner for $\mathcal{F}$, which is incomparable to both oracles. Since multicalibration is generally considered a stronger notion compared to omniprediction (Gopalan et al., 2022a) and recent breakthrough by Okoroafor et al. (2025) assumes an online agnostic learner for $\mathcal{F}$ for achieving oracle-efficient online omniprediction, it is reasonable to assume the same for multicalibration. Even in the distributional setting, multicalibration auditing using a weak agnostic learner is quite standard (see, for example (Hébert-Johnson et al., 2018)).*

Towards making Algorithm 1 efficient, we instantiate $2N$ copies of $\mathsf{OAL}\{\mathsf{OAL}_{i,\sigma}\}_{i\in[N],\sigma\in\{\pm 1\}}$ and only maintain a probability distribution over the experts parametrized by (i,σ). At each time t, each $\mathsf{OAL}_{i,\sigma}$ provides an auditing function $q_{t,i,\sigma}$, which we utilize along with the probability distribution $\{w_{t,i,\sigma}\}$ over the experts to define the distribution

$$h_t(P) := \sup_{y\in\mathcal{Y}} \mathbb{E}_{p\sim P}\left[\sum_{(i,\sigma)} w_{t,i,\sigma}\sigma\mathbb{I}\left[p\in\mathcal{I}_i\right]q_{t,i,\sigma}(x_t)V(p,y)|\mathscr{F}_{t-1}\right].$$

Next, we obtain P_t satisfying $h_t(P_t)\le\frac{\rho}{T}$ similar to Line 4 in Algorithm 1. Subsequently, after predicting p_t and observing y_t, we feed the gain $\ell_{t,i,\sigma} := \mathbb{E}_{p\sim P_t}\left[\ell_{t,i,\sigma}(p,y_t)|\mathscr{F}_{t-1}\right]$ to ALG, where the gain function $\ell_{t,i,\sigma}(p,y)$ is defined as $\ell_{t,i,\sigma}(p,y) := \sigma\mathbb{I}\left[p\in\mathcal{I}_i\right]q_{t,i,\sigma}(x_t)V(p,y)$, and the outcome $\sigma V(p_t,y_t)$ to $\mathsf{OAL}_{i_t,\sigma}$ for each $\sigma\in\{-1,+1\}$ (no feedback to $\mathsf{OAL}_{i,\sigma}$ with $i\ne i_t$). The full algorithm is summarized in Algorithm 2.

Algorithm 2 Multicalibration for an Elicitable Property (Oracle-Efficient Version)

Initialize: An expert problem subroutine ALG, online agnostic learner ($\mathsf{OAL}_{i,\sigma}$) for each (i,σ);

for $t = 1,\dots,T$,

  Receive context $x_t\in\mathcal{X}$;

  Obtain weights $\{w_{t,i,\sigma}\}$ from ALG;

  For each (i,σ), if no outcome was feed to $\mathsf{OAL}_{i,\sigma}$ in the last round, let $q_{t,i,\sigma}$ be the most recent output of $\mathsf{OAL}_{i,\sigma}$; else obtain a new test function from $\mathsf{OAL}_{i,\sigma}$ as $q_{t,i,\sigma}$;

  Define the function $\Phi_t : [0,1]\to[-1,1]$ as

$$\Phi_t(p) := \sum_{(i,\sigma)} w_{t,i,\sigma}\sigma\mathbb{I}\left[p\in\mathcal{I}_i\right]q_{t,i,\sigma}(x_t).$$

  If $\Phi_t(0)>0$, choose a distribution P_t that is only supported on 0; else if $\Phi_t(1)\le 0$, choose P_t that is only supported on 1; else choose $i\in\{0,\dots,T-1\}$ such that $\Phi_t\left(\frac{i}{T}\right)\Phi_t\left(\frac{i+1}{T}\right)\le 0$ and P_t such that

$$P_t\left(\frac{i}{T}\right)=\frac{\left|\Phi_t\left(\frac{i+1}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right|+\left|\Phi_t\left(\frac{i+1}{T}\right)\right|},\quad P_t\left(\frac{i+1}{T}\right)=\frac{\left|\Phi_t\left(\frac{i}{T}\right)\right|}{\left|\Phi_t\left(\frac{i}{T}\right)\right|+\left|\Phi_t\left(\frac{i+1}{T}\right)\right|}.$$

  Sample $\tilde{p}_t\sim P_t$ and predict $p_t=\frac{i_t}{N}$, where $i_t\in[N]$ is such that $\tilde{p}_t\in\mathcal{I}_{i_t}$;

  Observe y_t;

  For each (i,σ), feed the gain $\ell_{t,i,\sigma}=\mathbb{E}_{p\sim P_t}\left[\ell_{t,i,\sigma}(p,y_t)|\mathscr{F}_{t-1}\right]$ to ALG, where $\ell_{t,i,\sigma}(p,y)=\sigma\mathbb{I}\left[p\in\mathcal{I}_i\right]q_{t,i,\sigma}(x_t)V(p,y)$;

  Feed the outcome $\sigma V(p_t,y_t)$ to $\mathsf{OAL}_{i_t,\sigma}$ for each $\sigma\in\{-1,+1\}$;

The following lemma is in a similar spirit as Lemma 10 in Appendix B.1.

Lemma 14 *Algorithm 2 ensures that*

$$\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t=z_i\right]f(x_t)V(p_t,y_t)\right| = \tilde{O}\left(\frac{\rho n_i}{N}+\sqrt{n_i\log\frac{N}{\delta}}+\log\frac{N}{\delta}+\mathsf{Reg}(\mathcal{F},n_i)\right)$$

for all $i \in [N]$ with probability at least $1-\delta$.

Proof For convenience, let $\kappa_{t,i,\sigma} := \sigma\mathbb{I}\big[p_t = z_i\big]V(p_t, y_t)$. Fix a $i \in [N]$ and failure probability δ. We begin by realizing that

$$\begin{aligned}
\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\big[p_t = z_i\big]f(x_t)V(p_t,y_t)\right| &= \sup_{\sigma\in\{\pm1\},f\in\mathcal{F}}\sum_{t=1}^{T}\sigma\mathbb{I}\big[p_t = z_i\big]f(x_t)V(p_t,y_t) \\
&= \sup_{\sigma\in\{\pm1\},f\in\mathcal{F}}\sum_{t=1}^{T}\kappa_{t,i,\sigma}f(x_t) \\
&\le \sup_{\sigma\in\{\pm1\}}\sum_{t=1}^{T}\kappa_{t,i,\sigma}q_{t,i,\sigma}(x_t) + \mathsf{Reg}(\mathcal{F}, n_i),
\end{aligned}$$

where the inequality follows from (5), since $\mathsf{OAL}_{i,\sigma}$ is updated exactly on the rounds where the prediction made is $p_t = z_i$. Next, we obtain a high probability bound on the quantity $\sum_{t=1}^{T}\kappa_{t,i,\sigma}q_{t,i,\sigma}(x_t)$ (for a fixed σ) by proceeding in a similar manner as Lemma 10. We begin with the decomposition $\sum_{t=1}^{T}\kappa_{t,i,\sigma}q_{t,i,\sigma}(x_t) = \sum_{t=1}^{T}X_t + U_t + Z_t + W_t$, where the sequences $\{X_t\},\{U_t\},\{V_t\},\{W_t\}$ are defined as

$$\begin{aligned}
X_t &:= \sigma\mathbb{I}\big[\tilde{p}_t \in \mathcal{I}_i\big]q_{t,i,\sigma}(x_t)\cdot\big(V(z_i,y_t) - \mathbb{E}_t[V(z_i,y_t)]\big), \\
U_t &:= \sigma\mathbb{I}\big[\tilde{p}_t \in \mathcal{I}_i\big]q_{t,i,\sigma}(x_t)\cdot(\mathbb{E}_t[V(z_i,y_t)] - \mathbb{E}_t[V(\tilde{p}_t,y_t)]), \\
Z_t &:= \sigma\mathbb{I}\big[\tilde{p}_t \in \mathcal{I}_i\big]q_{t,i,\sigma}(x_t)\cdot V(\tilde{p}_t,y_t), \\
W_t &:= \sigma\mathbb{I}\big[\tilde{p}_t \in \mathcal{I}_i\big]q_{t,i,\sigma}(x_t)\cdot(\mathbb{E}_t[V(\tilde{p}_t,y_t)] - V(\tilde{p}_t,y_t)).
\end{aligned}$$

Clearly, $|U_t| \le \frac{\rho}{N}$. Since $q_{t,i,\sigma}$ is $\mathscr{F}_{t-1}$ measurable, $\{X_t\}$ is a martingale difference sequence, where as $\{W_t\}$ is not necessarily so. Following the same reasoning as in the proof of Lemma 10, we obtain

$$\begin{aligned}
\sum_{t=1}^{T}X_t &= \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T}\mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right)\log\frac{1}{\delta}} + \log\frac{1}{\delta}\right), \\
\sum_{t=1}^{T}W_t &= \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T}\mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right)\log\frac{1}{\delta}} + \log\frac{1}{\delta}\right)
\end{aligned}$$

with probability at least $1-\delta$. To bound $\sum_{t=1}^{T}Z_t$, we first decompose $\sum_{t=1}^{T}Z_t = \sum_{t=1}^{T}\ell_{t,i,\sigma}(\tilde{p}_t,y_t) = \sum_{t=1}^{T}\ell_{t,i,\sigma} + \sum_{t=1}^{T}\ell_{t,i,\sigma}(\tilde{p}_t,y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{t,i,\sigma}(\tilde{p}_t,y_t)|\mathscr{F}_{t-1}]$. It follows from the regret guarantee of ALG (Lemma 9) that

$$\begin{aligned}
\sum_{t=1}^{T}\ell_{t,i,\sigma} &\le \sum_{t=1}^{T}\sum_{(i,\sigma)}w_{t,i,\sigma}\ell_{t,i,\sigma} + O\left(\log(NT) + \sqrt{\left(\sum_{t=1}^{T}\mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right)\log(NT)}\right) \\
&= \tilde{O}\left(\log N + \sqrt{\left(\sum_{t=1}^{T}\mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right)\log N}\right),
\end{aligned}$$

since for each t, $\sum_{(i,\sigma)} w_{t,i,\sigma}\ell_{t,i,\sigma} \leq h_t(P_t) \leq \frac{\rho}{T}$ by the choice of P_t (exact same reasoning as Proposition 8). The deviation $\sum_{t=1}^{T} \ell_{t,i,\sigma}(\tilde{p}_t, y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{t,i,\sigma}(\tilde{p}_t, y_t)|\mathscr{F}_{t-1}]$ can be bounded similar to Lemma 10; we obtain

$$\sum_{t=1}^{T} \ell_{t,i,\sigma}(\tilde{p}_t, y_t) - \mathbb{E}_{\tilde{p}_t}[\ell_{t,i,\sigma}(\tilde{p}_t, y_t)|\mathscr{F}_{t-1}] = \tilde{O}\left(\sqrt{\left(\sum_{t=1}^{T} \mathbb{P}_t(\tilde{p}_t \in \mathcal{I}_i)\right)\log\frac{1}{\delta}} + \log\frac{1}{\delta}\right).$$

Combining everything and taking a union bound over $i \in [N]$, we obtain

$$\sum_{t=1}^{T} \kappa_{t,i,\sigma} q_{t,i,\sigma}(x_t) = \tilde{O}\left(\frac{\rho n_i}{N} + \sqrt{n_i \log\frac{N}{\delta}} + \log\frac{N}{\delta}\right)$$

with probability at least $1-\delta$. This completes the proof. ■

Theorem 15 *Fix a $r \geq 1$ and assume that there exists an* OAL *for which* $\mathsf{Reg}(\mathcal{F}, n) = \tilde{O}(n^{\alpha}\mathsf{Comp}(\mathcal{F}))$, *where $\alpha \in [0,1)$ and* $\mathsf{Comp}(\mathcal{F})$ *is a complexity measure of $\mathcal{F}$ that is independent of n. Then, for $r \geq 2$, Algorithm 2 achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\rho^r T^{\frac{1}{r+1}} + T^{\frac{1}{r+1}}\left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + T^{1-r+\frac{r}{r+1}+\frac{\alpha r^2}{r+1}}\mathsf{Comp}(\mathcal{F})^r + T^{\frac{1}{r+1}}\mathsf{Comp}(\mathcal{F})^r\right)$$

with probability at least $1-\delta$. Consequently, for $r \in [1,2)$, Algorithm 2 achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\rho^r T^{1-\frac{r}{3}} + T^{1-\frac{r}{3}}\left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + T^{1+\frac{2r(\alpha-1)}{3}}\mathsf{Comp}(\mathcal{F})^r + T^{1-\frac{r}{3}}\mathsf{Comp}(\mathcal{F})^r\right)$$

with probability at least $1-\delta$.

Proof Fix a $r \geq 2$. It follows from Lemma 14 that

$$\frac{1}{n_i}\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right]f(x_t)V(p_t, y_t)\right| = \begin{cases}\tilde{O}\left(\frac{\rho}{N} + \sqrt{\frac{\log\frac{N}{\delta}}{n_i}} + \frac{\mathsf{Reg}(\mathcal{F},n_i)}{n_i}\right) & i \in \mathcal{S}, \\ 1 & \text{otherwise,}\end{cases} \tag{13}$$

where $\mathcal{S} := \{i \in [N]; n_i = \Omega(\log\frac{N}{\delta})\}$. Therefore,

$$\begin{aligned}\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) &= \sum_{i\in[N]} n_i\left(\frac{1}{n_i}\sup_{f\in\mathcal{F}}\left|\sum_{t=1}^{T}\mathbb{I}\left[p_t = z_i\right]f(x_t)V(p_t, y_t)\right|\right)^r \\ &= \sum_{i\in\mathcal{S}} n_i \cdot \tilde{O}\left(\frac{\rho^r}{N^r} + \left(\frac{\log\frac{N}{\delta}}{n_i}\right)^{\frac{r}{2}} + \left(\frac{\mathsf{Reg}(\mathcal{F}, n_i)}{n_i}\right)^r\right) + O\left(\left|\bar{\mathcal{S}}\right|\log\frac{N}{\delta}\right) \\ &= \tilde{O}\left(\rho^r\frac{T}{N^r} + N\left(\log\frac{N}{\delta}\right)^{\frac{r}{2}} + \sum_{i\in\mathcal{S}} n_i\left(\frac{\mathsf{Reg}(\mathcal{F}, n_i)}{n_i}\right)^r\right),\end{aligned}$$

where the first equality follows since $(u+v+w)^r \le 3^{r-1}(u^r+v^r+w^r)$ via Jensen's inequality applied to the function $x \to x^r$, which is convex for $r \ge 1$; the $O\left(\left|\bar{\mathcal{S}}\right| \log \frac{N}{\delta}\right)$ term is because of (13), and since $n_i = O(\log \frac{N}{\delta})$ for $i \in \bar{\mathcal{S}}$. To obtain the second equality, we bound $n_i^{1-\frac{r}{2}} < 1$. Next, observe that as a function of n_i, $\psi(n_i) := n_i \left(\frac{\mathsf{Reg}(\mathcal{F}, n_i)}{n_i}\right)^r$ is either concave or convex in n_i. Particularly, since the dependence of n_i in $\mathsf{Reg}(\mathcal{F}, n_i)$ is n_i^{α} for some $\alpha \in [0,1)$, the dependence of n_i in $\psi(n_i)$ is n_i^{β}, where $\beta = 1 + r(\alpha - 1) < 1$. If $\psi(n_i)$ is concave in n_i (corresponds to $\beta \in [0,1)$), we have

$$\sum_{i \in \mathcal{S}} \psi(n_i) \le \sum_{i \in [N]} \psi(n_i) \le N\psi\left(\frac{1}{N}\sum_{i=1}^{N} n_i\right) = N\psi\left(\frac{T}{N}\right) = T\left(\frac{N}{T}\mathsf{Reg}\left(\mathcal{F}, \frac{T}{N}\right)\right)^r,$$

where the second inequality follows from Jensen's inequality. On the contrary, if $\psi(n_i)$ is convex in n_i ($\beta < 0$), bounding $n_i^{\beta} \le 1$, we obtain

$$\sum_{i \in \mathcal{S}} \psi(n_i) = \sum_{i \in \mathcal{S}} n_i^{\beta} \mathsf{Comp}(\mathcal{F})^r \le |\mathcal{S}| \, \mathsf{Comp}(\mathcal{F})^r \le N \mathsf{Comp}(\mathcal{F})^r.$$

Choosing $N = \Theta(T^{\frac{1}{r+1}})$ and combining both cases yields the desired bound on $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F})$. For $r \in [1,2)$, since $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) \le T^{1-\frac{r}{2}}(\mathsf{SMCal}_{\Gamma,2}(\mathcal{F}))^{\frac{r}{2}}$ via Hölder's inequality, we obtain

$$\begin{aligned}\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) &= \tilde{O}\left(T^{1-\frac{r}{2}} \cdot \left\{\rho^2 T^{\frac{1}{3}} + T^{\frac{1}{3}} \log\frac{1}{\delta} + T^{\frac{4\alpha-1}{3}} \mathsf{Comp}(\mathcal{F})^2 + T^{\frac{1}{3}} \mathsf{Comp}(\mathcal{F})^2\right\}^{\frac{r}{2}}\right) \\ &= \tilde{O}\left(\rho^r T^{1-\frac{r}{3}} + T^{1-\frac{r}{3}}\left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + T^{1+\frac{2r(\alpha-1)}{3}} \mathsf{Comp}(\mathcal{F})^r + T^{1-\frac{r}{3}} \mathsf{Comp}(\mathcal{F})^r\right),\end{aligned}$$

where the last equality follows since $(u+v+w+x)^{\frac{r}{2}} \le \sqrt{4^{r-1}(u^r+v^r+w^r+x^r)} \le 2^{r-1}(u^{\frac{r}{2}} + v^{\frac{r}{2}} + w^{\frac{r}{2}} + x^{\frac{r}{2}})$. This completes the proof. ■

Observe that whenever the dependence of n_i in $\mathsf{Reg}(\mathcal{F}, n_i)$ is $\sqrt{n_i}$ or better ($\alpha \le \frac{1}{2}$), for $r \ge 2$, the dependence of T in the third term in Theorem 15 is at most $T^{1-r+\frac{r}{r+1}+\frac{r^2}{2(r+1)}} = T^{\frac{2+2r-r^2}{2(r+1)}} \le T^{\frac{1}{r+1}}$, thereby resulting in $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(T^{\frac{1}{r+1}}\right)$ with high probability. Similarly, for $r \in [1,2)$, the dependence of T in the third term is at most $T^{1-\frac{r}{3}}$, thereby resulting in $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(T^{1-\frac{r}{3}}\right)$ with high probability. However, when the dependence of n_i is worse than $\sqrt{n_i}$, the dependence of T can be quite worse. In the next section, we consider several hypothesis classes for which either we can derive a concrete algorithm or prove an existential result that $\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(T^{\frac{1}{r+1}}\right)$.

B.3. Bounds for Specific Hypothesis Classes

We begin by implementing the online agnostic learner in Definition 1. As already mentioned, in order to achieve the $T^{\frac{1}{r+1}}$ dependence, we want the online agnostic learner $\mathsf{OAL}_{i,\sigma}$ to satisfy $\mathsf{Reg}(\mathcal{F}, n) = \tilde{O}(\sqrt{n}\mathsf{Comp}(\mathcal{F}))$. We discuss two specific classes for which one can derive an explicit algorithm satisfying the desired guarantee. For a general hypothesis class $\mathcal{F}$, we show that under the assumption that the sequential Rademacher complexity of $\mathcal{F}$ satisfies $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n) = \tilde{O}\left(\frac{\mathsf{Comp}(\mathcal{F})}{\sqrt{n}}\right)$, there indeed exists such an algorithm.

Finite $\mathcal{F}$. When $\mathcal{F}$ is finite, such an algorithm is possible by invoking the Multiplicative Weights Update (MWU) algorithm and predicting $q_t(x_t) = \sum_{f\in\mathcal{F}} w_{t,f} f(x_t)$, where each $f \in \mathcal{F}$ corresponds to an expert and $\{w_{t,f}\}$ represents a probability distribution over the experts output by MWU.

Algorithm 3 Online agnostic learner for finite $\mathcal{F}$

Initialize: Step size $\eta_t = \sqrt{\frac{\log|\mathcal{F}|}{t}}$, weights $w_{1,f} = \frac{1}{|\mathcal{F}|}$ for all $f \in \mathcal{F}$;
1: **for** $t = 1, \ldots, n,$
2: Receive context x_t;
3: Predict $q_t(x_t) = \sum_{f\in\mathcal{F}} w_{t,f} \cdot f(x_t)$ and observe κ_t;
4: Update $w_{t+1,f} = \frac{\exp\left(\eta_t \sum_{\tau=1}^{t} f(x_\tau)\kappa_\tau\right)}{\sum_{f\in\mathcal{F}} \exp\left(\eta_t \sum_{\tau=1}^{t} f(x_\tau)\kappa_\tau\right)}$ for each $f \in \mathcal{F}$;

The following result is immediate regret guarantee of MWU (Arora et al., 2012).

Lemma 16 *Algorithm 3 ensures that* $\mathsf{Reg}(\mathcal{F}, n) = O\left(\sqrt{n \log|\mathcal{F}|}\right)$.

Therefore, when $\mathcal{F}$ is finite, instantiating each $\mathsf{OAL}_{i,\sigma}$ with Algorithm 3, we obtain the following corollary of Theorem 15.

Corollary 17 *Fix a* $r \geq 1$ *and let* $\mathcal{F}$ *be a finite class. Then, Algorithm 2 with Algorithm 3 as* OAL *achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + (\log|\mathcal{F}|)^{\frac{r}{2}}\right) T^{\frac{1}{r+1}}\right)$$

for $r \geq 2$ *with probability at least* $1-\delta$. *Consequently, for* $r \in [1,2)$ *it achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + (\log|\mathcal{F}|)^{\frac{r}{2}}\right) T^{1-\frac{r}{3}}\right)$$

with probability at least $1-\delta$.

Linear class $\mathcal{F}$. Let $\mathcal{F} = \{f_\theta(x) := \langle \theta, x\rangle ; \theta \in \mathbb{B}_2^d\}$ be the set of bounded linear functions, where $\mathbb{B}_2^d := \{\theta; \|\theta\|_2 \leq 1\}$ is the unit norm ball. Then the Online Gradient Descent (OGD) algorithm (Algorithm 4) can be used to instantiate OAL. In Algorithm 4, $\Pi_{\mathbb{B}_2^d}(x) := \operatorname{argmin}_{y\in\mathbb{B}_2^d} \|x-y\|$ represents the projection operator. Clearly, $\Pi_{\mathbb{B}_2^d}(x) = x$ if $x \in \mathbb{B}_2^d$, and $\frac{x}{\|x\|}$ otherwise.

The following result is immediate from the regret guarantee of OGD (Hazan et al., 2016).

Lemma 18 *Algorithm 4 ensures that* $\mathsf{Reg}(\mathcal{F}, n) = O(\sqrt{n})$.

An immediate corollary of Theorem 15 for the linear class is the following.

Algorithm 4 Online agnostic learner for the linear class

Initialize: $\theta_1 \in \mathbb{B}_2^d$;

for $t = 1, \ldots, n,$

Receive context x_t;

Predict $q_t(x) = \langle \theta_t, x \rangle$;

Observe κ_t and update $\theta_{t+1} = \Pi_{\mathbb{B}_2^d}(\theta_t + \eta_t \kappa_t x_t)$, where $\eta_t = \frac{2}{\sqrt{t}}$;

Corollary 19 *Fix a $r \geq 1$ and let $\mathcal{F}$ be the linear class. Then, Algorithm 2 with Algorithm 4 as* OAL *achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + 1\right) T^{\frac{1}{r+1}}\right)$$

for $r \geq 2$ with probability at least $1 - \delta$. Consequently, for $r \in [1, 2)$ it achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + 1\right) T^{1-\frac{r}{3}}\right)$$

with probability at least $1 - \delta$.

$\mathcal{F}$ with bounded Rademacher complexity. Similar to Okoroafor et al. (2025), via standard learning-theoretic arguments, for a general hypothesis class $\mathcal{F}$ we can derive a bound on $\mathsf{Reg}(\mathcal{F}, n)$ that depends on $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n)$ by analyzing the minimax value of the online learning game between the oracle and the adversary. Recall that as per Definition 1, at each time $t \in [n]$ in this game, the adversary reveals a context $x_t \in \mathcal{X}$, the oracle responds with a prediction $q_t(x_t) \in [-1, 1]$, and subsequently the adversary reveals the true outcome $\kappa_t \in [-1, 1]$. Since the adversary is adaptive, the value of the game $\mathcal{V}^{\text{seq}}(\mathcal{F}, n)$ can be expressed as

$$\mathcal{V}^{\text{seq}}(\mathcal{F}, n) = \inf_{Q_1} \sup_{(x_1,\kappa_1)} \mathbb{E}_{q_1 \sim Q_1} \ldots \inf_{Q_n} \sup_{(x_n,\kappa_n)} \mathbb{E}_{q_n \sim Q_n} \left[\sup_{f \in \mathcal{F}} \sum_{t=1}^{n} f(x_t)\kappa_t - q_t(x_t)\kappa_t \right], \tag{14}$$

where $Q_t \in \Delta(\mathcal{X} \to [-1, 1])$ represents a distribution over all predictors mapping from $\mathcal{X}$ to $[-1, 1]$. Since the associated loss in (14) is $\ell(q, \kappa) := -q \cdot \kappa$, which is linear and 1-Lipschitz in q (for a fixed κ), it follows from (Rakhlin et al., 2015, Theorem 8) that $\mathcal{V}^{\text{seq}}(\mathcal{F}, n) = O(n\mathfrak{R}^{\text{seq}}(\mathcal{F}, n))$. When the sequential Rademacher complexity of $\mathcal{F}$ satisfies $\mathfrak{R}^{\text{seq}}(\mathcal{F}, n) = \tilde{O}\left(\frac{\mathsf{Comp}(\mathcal{F})}{\sqrt{n}}\right)$, there exists an OAL that achieves $\mathrm{Reg}(\mathcal{F}, n) = \tilde{O}(\sqrt{n}\mathsf{Comp}(\mathcal{F}))$. Let $\mathsf{ALG}_{\mathsf{exist}}$ be the existential algorithm in this case. Theorem 15 then implies the following corollary.

Corollary 20 *Fix a $r \geq 1$ and let $\mathcal{F}$ be such that $\mathfrak{R}^{seq}(\mathcal{F}, n) = \tilde{O}\left(\frac{\mathsf{Comp}(\mathcal{F})}{\sqrt{n}}\right)$. Then, Algorithm 2 with ALG_{exist} as* OAL *achieves*

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + \mathsf{Comp}(\mathcal{F})^r\right) T^{\frac{1}{r+1}}\right)$$

for $r \geq 2$ with probability at least $1-\delta$. Consequently, for $r \in [1,2)$ it achieves

$$\mathsf{SMCal}_{\Gamma,r}(\mathcal{F}) = \tilde{O}\left(\left(\rho^r + \left(\log\frac{1}{\delta}\right)^{\frac{r}{2}} + \mathsf{Comp}(\mathcal{F})^r\right) T^{1-\frac{r}{3}}\right)$$

with probability at least $1-\delta$.

Appendix C. Miscellaneous

Lemma 21 (Freedman's Inequality (Beygelzimer et al., 2011)) *Let $X_1, \ldots, X_n$ be a martingale difference sequence where $|X_i| \leq B$ for all $i = 1, \ldots, n$, and B is a fixed constant. Define $V := \sum_{i=1}^n \mathbb{E}_i[X_i^2]$. Then, for any fixed $\mu \in \left[0, \frac{1}{B}\right]$, $\delta \in [0,1]$, with probability at least $1-\delta$ we have*

$$\left|\sum_{i=1}^n X_i\right| \leq \mu V + \frac{\log\frac{2}{\delta}}{\mu}.$$

Lemma 22 *There does not exist a function $f_q : [0,1] \times [0,1] \to \mathbb{R}$ such that*

$$\sum_{p\in[0,1]} n_p \left(q - \frac{1}{n_p}\sum_{t=1}^T \mathbb{I}\left[p_t = p, y_t \leq p\right]\right)^2 \leq \zeta\left(\sup_{\nu:[0,1]\to[0,1]} \sum_{t=1}^T f_q(p_t, y_t) - f_q(\nu(p_t), y_t)\right) + C \tag{15}$$

holds for every $T \in \mathbb{N}$, $q \in [0,1]$, and sequences $\{y_t\}, \{p_t\}$ for some strictly increasing invertible function ζ with $\zeta(0) = 0$, and an absolute constant $C > 0$.

Proof Assume on the contrary, that there exists a function f_q such that (15) holds for all $q \in [0,1]$ and sequences $\{y_t\}_{t=1}^T, \{p_t\}_{t=1}^T$. Fix a $q \in (0,1)$, $p \in [0,1]$, $y \in (0,1)$, and let $p_t = p$, $y_t = y$ for all $t \in [T]$. Then, (15) simplifies to

$$T\left(q - \mathbb{I}\left[y \leq p\right]\right)^2 \leq \zeta(T(f_q(p,y) - \inf_{p^\star\in[0,1]} f_q(p^\star, y))) + C.$$

Since ζ is strictly increasing and invertible, this is equivalent to

$$f_q(p,y) - \inf_{p^\star\in[0,1]} f_q(p^\star, y) \geq \frac{1}{T}\zeta^{-1}\left(T\left(q - \mathbb{I}\left[y \leq p\right]^2\right) - C\right).$$

Define the function $\psi(y) := \inf_{p^\star\in[0,1]} f_q(p^\star, y)$. Then, the inequality above implies that

$$\begin{aligned}\psi(y) &\geq \psi(y) + \frac{1}{T}\inf_{p^\star\in[0,1]} \zeta^{-1}\left(T\left(q - \mathbb{I}\left[y \leq p^\star\right]^2\right) - C\right) \\ &= \psi(y) + \frac{1}{T}\min\left(\zeta^{-1}\left(Tq^2 - C\right), \zeta^{-1}\left(T(1-q)^2 - C\right)\right).\end{aligned}$$

Let $T \in \mathbb{N}$ be such that $Tq^2 \geq C$ and $T(1-q)^2 \geq C$, so that both terms $\zeta^{-1}\left(Tq^2 - C\right)$ and $\zeta^{-1}\left(T(1-q)^2 - C\right)$ are non-negative. It follows from the inequality above that either $\zeta^{-1}(Tq^2 - C) = 0$ or $\zeta^{-1}(T(1-q)^2 - C) = 0$. However, since the only solution to $\zeta^{-1}(x) = 0$ is $x = 0$, either $C = Tq^2$ or $C = T(1-q)^2$. Clearly, this is a contradiction since C is an absolute constant, therefore cannot depend on T. This completes the proof. ■