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# Recovery of Planted Subgraphs

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## Abstract

Understanding the fundamental limits of recovering planted subgraphs in random graphs is a central challenge in high-dimensional statistics and theoretical computer science. While existing work has largely focused on special subgraph families such as cliques, bicliques, or dense blocks, the exact recovery of a general planted subgraph in Erdős–Rényi random graphs remains poorly understood. In this paper, we study the exact recovery of an arbitrary planted subgraph $\Gamma = \Gamma_n$ embedded in a dense Erdős–Rényi random graph $\mathcal{G}(n, q_n)$, where edges within $\Gamma$ are present independently with probability $p_n > q_n$.

Our main results identify sharp conditions under which exact recovery is possible with high probability, and we establish matching lower bounds showing the necessity of these conditions. The resulting statistical threshold is characterized by a new graph-theoretic quantity, which we term the *minimal maximum subgraph density*. This quantity is defined as the maximum subgraph density of the smallest induced balanced subgraph of $\Gamma$.

We then turn to the problem of recovery under polynomial-time constraints. We propose a computationally efficient recovery algorithm that applies to arbitrary planted subgraphs and analyze its performance in terms of certain spectral properties of the adjacency matrix. In addition, we derive computational lower bounds for recovery using the low-degree polynomial framework, establishing regimes where recovery is statistically possible but computationally hard. Finally, we consider several extensions of our setting, including recovery in semi-random models and weaker notions of recovery.

**Keywords:** Planted subgraph, recovery, statistical-computational gaps, random graphs.

## 1. Introduction

The study of structured signals in networks lies at the intersection of graph theory, computer science, and statistics, with applications ranging from social networks to computational biology. A central question in this area is whether one can reliably reconstruct hidden or anomalous structures embedded in otherwise random graphs. Much of the existing literature has focused on identifying communities or clusters of vertices with unusually high internal connectivity. While early work emphasized *detection*, namely determining whether such a structure is present, an equally fundamental and arguably more challenging task is *recovery*: given an observed network, can one pinpoint the precise location of the hidden structure? This recovery perspective is crucial in applications such as anomaly localization, motif discovery in biological networks, and targeted monitoring in social networks, and it raises new theoretical questions regarding the statistical and computational limits of inference.

As in many inference problems, recovery exhibits both statistical and computational facets. The statistical aspect concerns the feasibility of recovery given unlimited computational power, whereas the computational aspect asks whether recovery can be achieved efficiently. Traditionally, these aspects were studied separately, with information-theoretic tools providing fundamental limits. However, recent work has highlighted the critical role of computational constraints in high-dimensional inference. A growing body of research (see, e.g., Berthet and Rigollet (2013); Ma and Wu (2015); Cai et al. (2017); Chen and Xu (2016); Hopkins and Steurer (2017); Hopkins B (2018); Gamarnik et al. (2020); Barak et al. (2016); Zdeborová and Krzakala (2016); Lesieur et al. (2015); Hajek et al. (2015b); Bandeira et al. (2018); Brennan et al. (2018, 2019); Brennan and Bresler (2020), among many others) has identified striking statistical–computational gaps in planted combinatorial problems, that is, regimes where recovery is information-theoretically possible but no known polynomial-time algorithm succeeds.

In this paper, we investigate the problem of recovering an *arbitrary* subgraph planted in an Erdős–Rényi random graph. Formally, let $n \in \mathbb{N}$, let $p_n \in (0,1)$ and $q_n \in (0,1)$ satisfy $q_n < p_n \leq 1$, and let $\Gamma = \Gamma_n$ denote an arbitrary sequence of undirected graphs, referred to as the planted subgraph. We observe a graph $\mathsf{G}$ generated as follows: first, a copy $\Gamma_n^\star$ of $\Gamma_n$ is chosen uniformly at random among all embeddings in the complete graph on $n$ vertices; edges within $\Gamma_n^\star$ are then included independently with probability $p_n$, while all remaining edges are included independently with probability $q_n$. The inferential task is to recover the precise location of $\Gamma_n^\star$, namely to achieve *exact recovery*, with high probability.

Over the past several decades, numerous special cases of planted subgraph recovery have been studied. The canonical example is the planted clique problem Jerrum (1992), where the goal is to recover a hidden $k$-clique embedded in $\mathcal{G}(n, 1/2)$. Other notable examples include the planted dense subgraph problem Arias-Castro and Verzelen (2014); Arias-Castro et al. (2015); Hajek et al. (2015b), the recovery of planted trees Massoulié et al. (2019), Hamiltonian cycles Bagaria et al. (2020), matchings Moharrami et al. (2021), and bipartite structures Rotenberg et al. (2024). Despite shared methodological features, the statistical and computational behaviors vary significantly across different planted structures. Some subgraphs exhibit sharp recovery thresholds and conjectured computationally hard regimes, as in the planted clique problem, whereas others, such as paths or stars, appear not to exhibit any computational barrier Massoulié et al. (2019).

There has been recent progress toward unified frameworks for planted subgraph inference. For the *detection problem*, where the goal is to distinguish between a pure Erdős–Rényi random graph and the planted model described above, Addario-Berry et al. (2010); Huleihel (2022); Yu et al. (2024); Elimelech and Huleihel (2025b,a) studied general detection models for arbitrary planted subgraphs in various settings and established both statistical and computational thresholds, primarily in the dense regime. Collectively, these results provide a nearly complete understanding of detection: when it is statistically possible and when it can be achieved efficiently.

By contrast, the landscape for the *recovery problem* remains far less understood. Important progress was made in Mossel et al. (2023); Lee et al. (2025), which studied weak recovery for arbitrary planted subgraphs and derived a general variational formula for the limiting minimum mean squared error (MMSE) under mild structural density assumptions. These results yield a powerful statistical characterization of recovery in the weak sense. Nevertheless, several key questions remain open. First, the focus there is on approximate or fractional recovery, as captured by small MMSE, rather than on exact recovery or other natural notions of recovery. Different recovery criteria can lead to qualitatively different thresholds, governed by fundamentally different graph-theoretic measures. Second, while these works characterize statistical limits, they do not address computational tractability: general computational lower and upper bounds for recovery are still missing. As a result, it remains unclear when a statistical–computational gap arises and how efficient recovery can be achieved.

In this paper, we take a step toward addressing these challenges by developing a general framework for recovery in the arbitrary planted subgraph setting. Our approach simultaneously addresses statistical and computational aspects and accommodates multiple notions of recovery. Motivated by the gaps described above, we pose the following guiding questions:

*What graph-theoretic properties of $\Gamma$ govern the statistical and computational limits of recovery? For which planted structures does a statistical–computational gap emerge?*

**Main contributions (informal).** In this paper, we aim to answer the questions posed above. We begin with what is arguably the most canonical setting of the problem: the dense regime, where the edge probabilities $(p_n, q_n)$ are fixed constants. Our first objective is to characterize the statistical (information-theoretic) limits of exact recovery of the planted subgraph $\Gamma = \Gamma_n$, as defined above (see Section 2 for formal details).

A key ingredient in our analysis is the *onion decomposition* of $\Gamma = \Gamma_n$, a concept recently introduced in Lee et al. (2025). Informally, this decomposition iteratively peels off subgraphs achieving the maximum subgraph density of $\Gamma$, layer by layer, until the entire graph is decomposed into the union of such layers. A precise definition is given in Definition 4. Our results are expressed in terms of a graph-theoretic quantity that we call the *minimal maximum subgraph density*, defined as

$$\mu_{\mathsf{min}}(\Gamma_n) \triangleq \min_{\mathsf{S} \subseteq \Gamma_n} \max_{\mathsf{S} \subsetneq \mathsf{F} \subseteq \Gamma_n} \eta(\mathsf{F}|\mathsf{S}). \tag{1}$$

Here, $\eta(\cdot|\cdot)$ denotes the relative density, as defined in Definition 3. This quantity admits a natural operational interpretation: it coincides with the maximum subgraph density of the final, and hence minimal, layer in the onion decomposition of $\Gamma$. Our first main result shows that the statistical threshold for exact recovery is governed (up to a constant factor) by

$$\boxed{\mu_{\mathsf{min}}(\Gamma_n) \underset{\text{Impossible}}{\overset{\text{Possible}}{\gtrless}} \mathsf{C}\frac{\log n}{d_{\mathsf{KL}}(p_n||q_n)}\,,}$$

where $d_{\mathsf{KL}}(p_n||q_n)$ denotes the Kullback–Leibler divergence between $\mathsf{Bern}(p_n)$ and $\mathsf{Bern}(q_n)$, and $\mathsf{C} \geq 1$ is a universal constant. In particular, if the minimal maximum subgraph density of $\Gamma$ exceeds the right-hand side, then exact recovery is achievable with high probability; otherwise, exact recovery is information-theoretically impossible. The achievability result is obtained by applying a maximum-likelihood estimator (MLE) layer by layer along the onion decomposition, while the converse follows from a genie-aided argument combined with Fano's inequality. Together, these results characterize the statistical limits of exact planted subgraph recovery in the dense regime.

We next turn to the computational aspect of the problem. We propose a general, computationally efficient recovery algorithm and analyze its performance guarantees. This method can be viewed as a semidefinite relaxation of the optimal MLE. We show that it succeeds whenever certain spectral properties of an appropriate transformation of the adjacency matrix of $\Gamma$, specifically its rank and coherence number, satisfy suitable conditions. Notably, the resulting algorithm and its guarantees recover, as special cases, the best-known polynomial-time procedures for classical planted subgraph problems, including cliques, bipartite graphs, and related models. For certain choices of $\Gamma$, however, there remains a gap between the information-theoretic limits established above and the performance of this efficient algorithm. We conjecture that this gap is intrinsic, in the sense that below the corresponding computational threshold, no polynomial-time algorithm can achieve recovery.

To provide evidence for this conjecture, we adopt the low-degree polynomial framework of Schramm and Wein (2022) and establish computational lower bounds for recovery. Roughly speaking, we show that all polynomials of degree $O(\log n)$ fail to recover $\Gamma$ (even in a weak sense) whenever its density satisfies

$$\boxed{\eta(\Gamma_n) \triangleq \frac{|e(\Gamma_n)|}{|v(\Gamma_n)|} \ll \sqrt{n}\,.}$$

For instance, when $\Gamma_n$ is a clique, this condition predicts computational hardness whenever $|v(\Gamma_n)| \ll \sqrt{n}$. Similarly, for a complete bipartite graph with $k_{\mathsf{L}}$ left and $k_{\mathsf{R}}$ right vertices, recovery is conjectured to be computationally hard when $\min\{k_{\mathsf{L}}, k_{\mathsf{R}}\} \ll \sqrt{n}$. These predictions align with well-known folklore conjectures. We also establish complementary upper bounds, showing that low-degree polynomials succeed whenever $\eta(\Gamma_n) \gg \sqrt{n}$ for a broad class of nearly regular subgraphs, including cliques, balanced bipartite graphs, and sparse expander graphs. In these cases, the low-degree predictions match the performance of the best known polynomial-time algorithms.

Finally, we consider several extensions of the baseline planted subgraph model. The standard formulation assumes a purely random generative process, which may be fragile under adversarial perturbations. To address this issue, we study semi-random models in which an adversary may delete edges outside the planted subgraph and add edges inside it prior to observation. Importantly, the statistician does not know which edges have been modified. For this setting, known as the monotone adversary model Feige and Krauthgamer (2000); Feige and Kilian (2001), we show that both the optimal MLE and the efficient algorithm proposed above remain robust and achieve the same recovery guarantees as in the non-adversarial model.

Motivated by the observation that exact recovery is fundamentally impossible for planted subgraphs consisting of a dense core with a very sparse appendage, such as kite graphs, since their minimal maximum subgraph density $\mu_{\mathsf{min}}$ is sub-logarithmic in the number of vertices, we also study weaker notions of recovery. Specifically, we consider exact layer recovery, where the goal is to recover a fixed number of initial layers in the onion decomposition, as well as almost-exact recovery Hajek et al. (2017); Wu and Xu (2020), where the estimator is required to be close to the planted subgraph in Hamming distance. For both recovery criteria, we establish asymptotically tight thresholds.

The rest of this paper is organized as follows. In Section 2, we introduce the problem setup and provide some necessary preliminaries. Section 3 presents our main results, discussions, and examples. Due to space constraints, the main body presents a streamlined version of the paper, focusing on the formulation and main results. The appendix contains a full version of the paper, including complete proofs, additional results, examples, and extended discussions.

## 2. Problem Setup and Preliminaries

In this section, we describe the setting we study, along with several important preliminaries. Let $\Gamma = (\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of graphs such that, for each $n \in \mathbb{N}$, $\Gamma_n = (v(\Gamma_n), e(\Gamma_n))$ is an undirected graph without isolated vertices and with $|v(\Gamma_n)| \leq n$. Let $\mathcal{S}_{\Gamma_n}$ denote the set of all isomorphic copies of $\Gamma_n$ in the complete graph on $n$ vertices. We refer to $\Gamma_n$ as the *planted* (or *hidden*) structure. Fix parameters $p_n, q_n$ satisfying $0 < q_n < p_n \leq 1$. The *planted subgraph model* $\mathcal{G}_{\Gamma_n}(n, p_n, q_n)$ is defined as the distribution of a random graph $\mathsf{G}$ on $n$ vertices generated as follows: first draw an arbitrary but fixed copy $\Gamma_n^\star \in \mathcal{S}_{\Gamma_n}$; then include each edge $e \in e(\Gamma_n^\star)$

independently with probability $p_n$, and include each edge $e \notin e(\Gamma_n^\star)$ independently with probability $q_n$. Equivalently, $\mathsf{G}$ can be viewed as the union of a noisy copy of $\Gamma_n^\star$ and an Erdős–Rényi random graph $\mathcal{G}(n, q_n)$.

A learner observes a single sample $\mathsf{G} \sim \mathcal{G}_{\Gamma_n}(n, p_n, q_n)$, and the goal is to recover the hidden copy $\Gamma_n^\star$. We study this framework in the asymptotic regime where $n \to \infty$. Given $\mathsf{G}$, an estimator $\widehat{\Gamma} : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_{\Gamma_n}$ aims to output $\Gamma_n^\star$. Define the worst-case error probability associated with an estimator $\widehat{\Gamma}$ as

$$\mathsf{E}_n(\hat{\Gamma}) \triangleq \sup_{\Gamma^\star \in \mathcal{S}_{\Gamma_n}} \mathbb{P}_{\mathcal{G}_{\Gamma_n^\star}(n,p_n,q_n)}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^\star], \tag{2}$$

and the optimal error probability as $\mathsf{E}_n^\star \triangleq \inf_{\hat{\Gamma}:\{0,1\}^{n(2)}} \mathsf{E}_n(\hat{\Gamma})$. A sequence of estimators $(\widehat{\Gamma}_n)_{n\in\mathbb{N}}$, where $\widehat{\Gamma}_n : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_{\Gamma_n}$, achieves *exact recovery* if $\limsup_{n\to\infty} \mathsf{E}_n(\widehat{\Gamma}_n) = 0$; conversely, *exact recovery is impossible* if $\liminf_{n\to\infty} \mathsf{E}_n^\star > 0$. Our results will be expressed in terms of the following graph-theoretic measures. We let $\eta(\Gamma_n) \triangleq |e(\Gamma_n)|/|v(\Gamma_n)|$ denote the density of $\Gamma_n$.

**Definition 1 (Maximum subgraph density Bollobás (2001))** *Let $\mathsf{G}$ be an undirected graph. The maximum subgraph density of $\mathsf{G}$ is $\mu(\mathsf{G}) \triangleq \max\{\eta(\mathsf{H}) : \mathsf{H} \subseteq \mathsf{G}, \mathsf{H} \neq \emptyset\}$.*

**Definition 2 (Graph-cut)** *Let $n$ be a positive integer. A graph-cut on $n$ vertices is a triplet $\mathsf{H} = (V, S, E)$, where $S \subseteq V \subseteq [n]$, and*

$$E \subseteq \mathcal{K}_V \setminus \mathcal{K}_S \triangleq \{(u,v) : u, v \in V \text{ and at most one of } u, v \text{ belongs to } S\}. \tag{3}$$

*We define the number of edges of the graph-cut as $|\mathsf{H}| \triangleq |E|$, and the number of non-selected vertices as $|v(\mathsf{H})| \triangleq |V \setminus S|$.*

We note that throughout the paper, we use $\mathsf{H}$ to denote either a graph (identified with its edge set) or a graph-cut, depending on the context. When $\mathsf{H}$ is a graph, $v(\mathsf{H})$ denotes its vertex set; when $\mathsf{H} = (V, S, E)$ is a graph-cut, $v(\mathsf{H})$ denotes the set of non-selected vertices $V \setminus S$. This distinction will be clear from the context. An important graph-theoretic quantity that will play a central role is the *relative density* and the *maximum subgraph relative density*, defined as follows.

**Definition 3 (Relative densities)** *Given graphs $\mathsf{H}' \subseteq \mathsf{H}$ (viewed as edge sets), we define the induced graph-cut by $\mathsf{H}|\mathsf{H}' \triangleq (v(\mathsf{H}), v(\mathsf{H}'), \mathsf{H} \setminus \mathsf{H}')$. The relative density is defined as*

$$\eta(\mathsf{H}|\mathsf{H}') \triangleq \frac{|\mathsf{H}| - |\mathsf{H}'|}{|v(\mathsf{H}) \setminus v(\mathsf{H}')|}. \tag{4}$$

*If $\mathsf{H} \setminus \mathsf{H}' = \emptyset$, we define $\eta(\mathsf{H}|\mathsf{H}') \triangleq \infty$. The maximum subgraph relative density is*

$$\mu(\mathsf{H}|\mathsf{H}') \triangleq \max\left\{\eta(\mathsf{J}|\mathsf{H}') : \mathsf{H}' \subsetneq \mathsf{J} \subseteq \mathsf{H}\right\}. \tag{5}$$

Note that, given $\mathsf{H}'$, we choose $\mathsf{J} \supsetneq \mathsf{H}'$ in (5) to maximize the "cut density", namely the number of new edges per new vertex, counting also edges from the new vertices back into $\mathsf{H}'$. Our statistical lower and upper bounds will rely, both in the statements and in the proofs, on a canonical decomposition of the planted subgraph $\Gamma$, introduced in (Lee et al., 2025, Definition 3.3).

**Definition 4 (Onion decomposition)** *Let $\Gamma = \Gamma_n$ be an arbitrary graph. The* onion decomposition *of $\Gamma$ is an increasing sequence of subgraphs $\Gamma^{(0)}, \Gamma^{(1)}, \ldots$ constructed as follows: [(i)] Initialize with $\Gamma^{(0)} \triangleq \emptyset$; [(ii)] For each $\ell > 0$, let $\Gamma^{(\ell)}$ be a maximal subgraph that maximizes $\eta(\mathsf{H}|\Gamma^{(\ell-1)})$ among all $\mathsf{H}$ s.t. $\Gamma^{(\ell-1)} \subsetneq \mathsf{H}$;[1]; [(iii)] Stop if $\Gamma^{(\ell)} = \Gamma$. Let $M = M(\Gamma) \leq |\Gamma|$ denote the total number of steps until termination. The sequence $\{\Gamma^{(\ell)}\}_{\ell=0}^{M}$ is referred to as the onion decomposition of $\Gamma$. Finally, define the remainder subgraphs $\mathcal{D}^{(\ell)} \triangleq \Gamma^{(\ell)} \setminus \Gamma^{(\ell-1)}$ for $\ell = 1, 2, \ldots, M$.*

Intuitively, this process iteratively selects the densest remaining subgraph, removes it, and continues on the remainder. The onion decomposition in Definition 4 is uniquely determined for any graph $\Gamma$; see Lemma 17 and its proof in Appendix G.2, as well as (Lee et al., 2025, Thm. 3.6). Moreover, the relative density of the final layer satisfies $\eta(\Gamma^{(M)}|\Gamma^{(M-1)}) = \mu_{\mathsf{min}}(\Gamma)$, where $\mu_{\mathsf{min}}(\Gamma)$ is defined in (1); see Lemma 18 and Appendix G.3 for a proof. Finally, throughout the paper, we occasionally suppress the explicit dependence of various quantities on the index $n$. For instance, we write the sequence of planted graphs as $\Gamma = (\Gamma_n)_n$, and similarly denote the sequences of edge probabilities by $p = (p_n)_n$ and $q = (q_n)_n$.

## 3. Main Results

In this section, we present our main results on the statistical and computational limits of planted subgraph recovery, along with several extensions.

### 3.1. Statistical limits

We first characterize the fundamental statistical limits of exact recovery, ignoring computational constraints. Our results identify a sharp threshold, governed by the minimal maximum subgraph density $\mu_{\mathsf{min}}(\Gamma)$, that separates the regimes in which exact recovery is information-theoretically impossible and possible. The following lower and upper bounds are proved in Appendices D and E.1.

**Theorem 5 (Statistical threshold for exact recovery)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, and assume $p_n, q_n = \Theta(1)$. Then:*

- *(Impossibility) Exact recovery is statistically impossible if $\mu_{\mathsf{min}}(\Gamma_n) \leq \frac{(1-\varepsilon)\log n}{d_{\mathsf{KL}}(p||q)}$, for any $\varepsilon > 0$.*
- *(Achievability) Exact recovery is statistically possible if $\mu_{\mathsf{min}}(\Gamma_n) \geq C\,\frac{(1+\varepsilon)\log n}{d_{\mathsf{KL}}(p||q)}$, for any $\varepsilon > 0$ and a universal constant $C > 0$.*

We briefly illustrate the threshold in two cases. If $\Gamma$ consists of $L$ disjoint cliques with sizes $k_1, \ldots, k_L$ and $k_{\min} = \min_i k_i$, then $\mu_{\mathsf{min}}(\Gamma) = \frac{k_{\min}-1}{2}$. Exact recovery is possible if and only if $k_{\min} \geq C \log n$. If $\Gamma$ is a "kite", namely, $k$-clique with an additional pendant edge, then $\mu_{\mathsf{min}}(\Gamma) = \frac{1}{2}$, and exact recovery is statistically impossible for all $n$. This reflects the fact that a sparse appendage can destroy exact recovery even in the presence of a dense core. In such cases, it is reasonable to consider alternative recovery criteria (such as weak recovery), which allow for a nonzero fraction of errors. We discuss such criteria in Subsection 3.4.

The lower bound follows from a genie-aided Fano-type argument. For balanced graphs Bollobás (1981), we additionally provide a tight alternative lecture-style proof based on Bayes risk analysis;

1. Maximality implies that there does not exist a subset $\Gamma^{(\ell)} \subsetneq \Gamma'$ for which $\eta(\Gamma'|\Gamma^{(\ell-1)})$ is also maximized.

see Appendix I.[2] The achievability result is attained by the maximum-likelihood estimator (MLE), implemented via a recursive peeling procedure aligned with the onion decomposition of $\Gamma$. A precise description of the algorithm, is deferred to Appendix C.1, along with several additional examples.

Finally, it is worth emphasizing that the proof of the achievability relies on an important property of the layers obtained in the onion decomposition of $\Gamma$, namely a uniqueness property of the planted copy on its full vertex set. Specifically, one of the steps in the proof analyzes $|v(\mathcal{D}' \cap \mathcal{D}^{(\ell),\star})|$, the intersection between an isomorphic copy $\mathcal{D}' \neq \mathcal{D}^{(\ell),\star}$ of the actual planted $\ell$th layer $\mathcal{D}^{(\ell),\star}$. In principle, it could happen that $|v(\mathcal{D}' \cap \mathcal{D}^{(\ell),\star})| = |v(\mathcal{D}^{(\ell),\star})|$ yet $\mathcal{D}' \neq \mathcal{D}^{(\ell),\star}$; for example, this can occur in the case of a kite. When this happens, exact recovery becomes impossible. Fortunately, we show that this cannot occur for the layers selected by the onion procedure when the parameters lie in the achievability regime.

It is instructive to contrast these results with the corresponding *detection* problem, where the goal is to distinguish $\mathcal{G}(n,q)$ from $\mathcal{G}_{\Gamma_n}(n,p,q)$. Prior work Elimelech and Huleihel (2025b) shows that detection can be statistically possible for subgraphs with sub-logarithmic density, i.e., $\mu(\Gamma_n) = o(\log |v(\Gamma_n)|)$. By contrast, Theorem 5 implies that exact recovery is impossible whenever $\mu_{\mathsf{min}}(\Gamma_n) = o(\log n)$. This reveals a fundamental *detection–recovery gap*: in many regimes, including paths, trees, and stars, detection is feasible while exact recovery is information-theoretically impossible. Even in super-logarithmic density regimes, recovery may fail despite successful detection, as illustrated by the kite example.

### 3.2. Computationally efficient algorithm

We now turn to recovery under polynomial-time constraints. Our approach is based on a convex relaxation of the maximum-likelihood estimator that promotes low-rank structure in the planted subgraph via a nuclear-norm constraint. This yields a general semidefinite programming (SDP)–based recovery algorithm that applies to arbitrary planted subgraphs and recovers, as special cases, the best-known polynomial-time algorithms for classical models such as planted cliques and planted bipartite graphs.

We represent the planted copy $\Gamma^\star$ by its embedded adjacency matrix $\mathsf{X}^\star \in \{0,1\}^{n\times n}$, where $\mathsf{X}^\star_{ii} = 0$ and $\mathsf{X}^\star_{ij} = 1$ if and only if $(i,j) \in e(\Gamma^\star)$. Let $\mathsf{A}$ denote the adjacency matrix of the observed graph, and define the centered observation matrix $\mathsf{W}$ by

$$\mathsf{W}_{ij} \triangleq \begin{cases} q^{-1}\mathsf{A}_{ij} - 1, & i \neq j, \\ 0, & i = j. \end{cases} \tag{6}$$

To control the rank of the planted structure, we introduce a diagonal shift. Fix a hyper-parameter $\alpha \in \mathbb{R}_+$. For any symmetric matrix $\mathsf{X}$ with zero diagonal and any vector $\mathbf{s} \in \mathbb{R}^n$, define

$$\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha) \triangleq \mathsf{X} + \alpha \cdot \mathsf{Diag}(\mathbf{s}). \tag{7}$$

At the planted solution, $\mathbf{s}^\star = \mathbb{1}\{i \in v(\Gamma^\star)\}$ and $\mathsf{Ext}(\mathsf{X}^\star, \mathbf{s}^\star; \alpha)$ is a diagonally shifted adjacency matrix of $\Gamma^\star$. We denote this matrix by

$$\mathsf{S}_\Gamma^{(\alpha)} \triangleq \mathsf{X}^\star + \alpha \cdot \mathsf{Diag}(v(\Gamma^\star)). \tag{8}$$

2. As noted in (Wein, 2025, pg. 7), even in the special case of a planted clique, the "$2\log_2 n$" information-theoretic threshold had been established for detection but not for exact recovery, although the latter was expected to hold. We confirm this expectation and in fact generalize much beyond.

The key spectral quantity governing the performance of our algorithm is the *coherence*. For a real symmetric matrix $\mathsf{Q}$ of rank $r$ with singular-vector matrix $\mathsf{U}_\mathsf{Q} \in \mathbb{R}^{n\times r}$, we define

$$\mathsf{coh}(\mathsf{Q}) \triangleq \frac{n}{r} \cdot \max_{i\in[n]} \|[\mathsf{U}_\mathsf{Q}]_{i,:}\|_2^2 . \tag{9}$$

Intuitively, coherence measures how localized the singular vectors are; low coherence corresponds to well-spread eigenvectors and is favorable for recovery. Our recovery algorithm is the solution to the following convex optimization problem:

$$\begin{aligned} \hat{\mathsf{X}}_{\mathrm{con}}^{(\alpha)} = \quad & \arg\max_{\mathsf{X}\in\mathbb{R}^{n\times n}, \mathbf{s}\in[0,1]^n} && \langle \mathsf{W}, \mathsf{X}\rangle \\ & \text{s.t.} && \mathbf{s}^\top \mathbf{1} = |v(\Gamma)|, \quad \mathsf{X}_{ij} \le \min(\mathbf{s}_i, \mathbf{s}_j),\ \forall i \ne j, \\ & && \|\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha)\|_\star \le \|\Gamma + \alpha\cdot \mathsf{Diag}(v(\Gamma))\|_\star , \\ & && \mathbf{0} \le \mathsf{X} \le \mathbf{J}, \quad \mathsf{X} = \mathsf{X}^\top, \quad \mathsf{X}_{ii} = 0,\ \forall i, \\ & && \langle \mathbf{J}, \mathsf{X}\rangle = 2|e(\Gamma)|. \end{aligned} \tag{10}$$

where the inequality $\mathbf{0} \le \mathsf{X} \le \mathbf{J}$ is to be interpreted entrywise, and $\mathbf{J}$ is the all-ones matrix. Here, the vector $\mathbf{s} \in [0,1]^n$ serves as a relaxed selector for the vertex set of the planted subgraph: each coordinate $\mathbf{s}_i$ indicates whether vertex $i$ belongs to $v(\Gamma^\star)$. At the planted solution, $\mathbf{s}_i^\star = \mathbb{1}\{i \in v(\Gamma^\star)\}$. Similarly, $\mathsf{X} \in \mathbb{R}^{n\times n}$ is a convex relaxation of the planted adjacency matrix $\mathsf{X}^\star$. One might ask why the auxiliary variable $\mathbf{s}$ is needed, rather than encoding the vertex set directly via self-loops in $\mathsf{X}$. The issue is that such an encoding can significantly weaken the discriminative power of the nuclear norm. For instance, certain subgraphs, such as complete bipartite graphs, become full-rank under this representation, rendering the nuclear-norm prior ineffective. Introducing $\mathbf{s}$ allows us to control the number of selected vertices through the linear constraint $\sum_{i\in[n]} \mathbf{s}_i = |v(\Gamma)|$, while applying the low-rank prior to the shifted matrix $\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha)$.

Furthermore, the constraint $\mathsf{X}_{ij} \le \min(\mathbf{s}_i, \mathbf{s}_j)$ enforces consistency between selected vertices and edges, and the nuclear-norm constraint promotes low-rank structure in the diagonally shifted matrix $\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha)$. The diagonal shift controlled by $\alpha$ prevents rank inflation due to forced self-loops; for instance, choosing $\alpha = 1$ for cliques yields a rank-one shifted matrix, whereas $\alpha = 0$ for complete bipartite graphs yields rank two. Further extended discussions that motivate the program above can be found in Appendix C.2. Our main guarantee for this algorithm is as follows. The proof appears in Appendix E.2.

**Theorem 6 (Efficient algorithm)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, $\alpha \in \mathbb{R}_+$, and assume $p_n, q_n = \Theta(1)$. Exact recovery of $\Gamma^\star$ via the convex program (10) is possible if*

$$\mathsf{coh}(\mathsf{S}_\Gamma^{(\alpha)}) \cdot \mathsf{rank}(\mathsf{S}_\Gamma^{(\alpha)}) \le \min\left\{ c_1\sqrt{n},\ c_2 \frac{n}{\sqrt{|v(\Gamma)|\log n}} \right\}, \tag{11}$$

*for some constants $c_1, c_2 > 0$.*

Theorem 6 recovers, as special cases, the best-known polynomial-time guarantees for several classical planted subgraph models. For the planted clique problem, taking $\alpha = 1$ yields $\mathsf{rank}(\mathsf{S}_{\mathcal{K}_k}^{(1)}) = 1$ and $\mathsf{coh}(\mathsf{S}_{\mathcal{K}_k}^{(1)}) = n/k$, leading to exact recovery when $k \gtrsim \sqrt{n}$, in agreement with classical spectral and SDP-based algorithms Alon et al. (1998); Dekel et al. (2014); Montanari (2015); Hajek et al. (2016); Chen and Xu (2016). For the complete bipartite graph $\mathcal{K}_{k_{\mathsf{L}},k_{\mathsf{R}}}$, taking $\alpha = 0$ yields a rank-two shifted matrix and recovery once $\min\{k_{\mathsf{L}}, k_{\mathsf{R}}\} \gtrsim \sqrt{n}$ up to logarithmic factors, consistent with known results for planted bipartite subgraphs Levanzov (2018); Kumar et al. (2022a). More generally, the same framework applies to a broad class of nearly regular planted subgraphs, including balanced Turán graphs, triangular graphs, and unions of cliques; see Appendix C.2 for further examples and analysis.

### 3.3. Computational lower bounds

In this subsection, we derive computational lower bounds for recovery using the low-degree polynomial (LDP) framework. This framework quantifies the best possible performance among estimators that are polynomials of bounded degree in the observations, and has proved predictive for computational thresholds across a broad range of high-dimensional inference problems Schramm and Wein (2022); Bandeira et al. (2022). Our presentation follows Schramm and Wein (2022) and adapts it to the planted-graph recovery model in Section 2.

We start by describing the low-degree framework. Fix $n$, $0 < q_n < p_n \leq 1$, and a planted structure $\Gamma_n = (v(\Gamma_n), e(\Gamma_n))$ with $|v(\Gamma_n)| \leq n$ and no isolated vertices. A planted copy $\Gamma_n^\star \in \mathcal{S}_{\Gamma_n}$ is selected uniformly at random and observed through the binary edge–channel

$$\mathsf{Y}_e | \, \Gamma_n^\star \sim \mathsf{Bern}(\mathsf{X}_e), \qquad \mathsf{X}_e = \begin{cases} p_n, & e \in e(\Gamma_n^\star), \\ q_n, & e \notin e(\Gamma_n^\star), \end{cases} \tag{12}$$

independently over $e \in \binom{[n]}{2}$. We write $N \triangleq \binom{n}{2}$ and view $\mathsf{Y} \in \{0,1\}^N$ as the input to any estimator. To study recovery via low-degree polynomials, we focus on the one-bit anchor

$$x \triangleq \mathbb{1}\{1 \in v(\Gamma_n^\star)\} \in \{0,1\}. \tag{13}$$

Since $\Gamma_n^\star$ is uniform over isomorphic copies, the joint law of $(x, \mathsf{Y})$ does not depend on the choice of the ambient vertex. For a degree budget $D \in \mathbb{N}$, let $\mathbb{R}[\mathsf{Y}]_{\leq D}$ denote the space of real polynomials in the entries of $\mathsf{Y}$ of total degree at most $D$. The degree-$D$ minimum mean-squared error is

$$\mathsf{MMSE}_{\leq D} \triangleq \inf_{f \in \mathbb{R}[\mathsf{Y}]_{\leq D}} \mathbb{E}\left[(f(\mathsf{Y}) - x)^2\right], \tag{14}$$

and the associated degree-$D$ maximum correlation is

$$\mathsf{Corr}_{\leq D} \triangleq \sup_{f \in \mathbb{R}[\mathsf{Y}]_{\leq D}} \frac{\mathbb{E}\left[f(\mathsf{Y}) \cdot x\right]}{\sqrt{\mathbb{E}\left[f(\mathsf{Y})^2\right]}}. \tag{15}$$

These quantities satisfy the identity $\mathsf{MMSE}_{\leq D} = \mathbb{E}[x^2] - \mathsf{Corr}_{\leq D}^2$. Thus, upper bounds on $\mathsf{Corr}_{\leq D}$ yield lower bounds on $\mathsf{MMSE}_{\leq D}$, ruling out nontrivial recovery by low-degree polynomials.

The results in this subsection differ in nature from those in the preceding subsections. While Subsections 3.1–3.2 focus on *exact recovery* of the planted subgraph, the LDP framework addresses a weaker notion of recovery, formalized via MMSE. In particular, the LDP *lower bounds* rule out any polynomial-time algorithm from achieving nontrivial weak recovery; since exact recovery implies weak recovery, these results *also* yield computational barriers for exact recovery. By contrast, the LDP *upper bounds* establish achievability of weak recovery only, and do not imply exact recovery.

The following theorem shows that degree-$D$ polynomials with $D = (\log n)^{o(1)}$ cannot achieve nontrivial recovery when the planted graph has $\eta(\Gamma_n) \ll \sqrt{n}$. Note that the trivial estimator $f(\mathsf{Y}) \equiv \mathbb{E}[x]$ achieves $\mathbb{E}[(f(\mathsf{Y}) - x)^2] = \mathsf{Var}(x)$. The proof is given in Appendix F.

**Theorem 7 (Computational lower bound)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, and assume $p_n, q_n = \Theta(1)$. If $\eta(\Gamma_n) \leq n^{\frac{1}{2}-\varepsilon}$, for any fixed $\varepsilon > 0$, and $D = D_n$ scales as $D \leq (\log n)^\alpha$ for some fixed $\alpha < 1$, then $\mathsf{MMSE}_{\leq D} \geq (1 - o(1)) \cdot \mathsf{Var}(x)$.*

Theorems 6 and 7 are stated in terms of different quantities, but coincide in several canonical examples such as planted cliques and bipartite graphs. Any discrepancy may reflect the weak-versus-exact recovery distinction, or looseness in either the algorithmic upper bound or the computational lower bound.

We next present explicit low-degree polynomial estimators that achieve nontrivial recovery beyond the barrier in Theorem 7. Following (Schramm and Wein, 2022, Sec. 4.2), these estimators correspond to one or multiple rounds of power iteration from the all-ones vector, followed by polynomial thresholding.

Fix $L \in \mathbb{N}$ and define $\mathsf{Z}_{ij} \triangleq \mathsf{Y}_{ij} - q$ for $i, j \in [n]$. Let $\mathcal{P}_L$ denote the set of all simple undirected paths of length $L$ in the complete graph starting at vertex 1 and visiting pairwise distinct vertices, i.e., $P = (u_0, u_1, \ldots, u_L)$ with $u_0 = 1$. For each $P \in \mathcal{P}_L$, define $\mathsf{Z}(P) \triangleq \prod_{t=0}^{L-1} \mathsf{Z}_{u_t u_{t+1}}$, and the degree-$L$ walk polynomial

$$\mathsf{W}_L \triangleq \sum_{P \in \mathcal{P}_L} \mathsf{Z}(P). \tag{16}$$

For $u \in v(\Gamma)$ let $W_L(\Gamma; u)$ be the number of simple paths of length $L$ in $\Gamma$ and starting at vertex $u$, and $W_L^{\min}(\Gamma) \triangleq \min_{u \in v(\Gamma)} W_L(\Gamma; u)$. Our degree-$D$ estimator is

$$f_{L,m}(\mathsf{Y}) = \tau_m(\mathscr{X}_L), \quad \text{where} \quad \mathscr{X}_L \triangleq \frac{2}{(p-q)^L} \frac{\mathsf{W}_L}{W_L^{\min}(\Gamma)}, \tag{17}$$

where $\tau_m$ is a polynomial threshold of degree $D = 2m + 1$ approximating a step function (see Lemma 43). In the special case $L = 1$, this reduces to

$$f_{1,m}(\mathsf{Y}) = \tau_m \left( \frac{1}{(p-q)\eta(\Gamma)} \sum_{i=2}^{n} (\mathsf{Y}_{1i} - q) \right). \tag{18}$$

**Theorem 8 (LDP upper bound)** *Assume $p_n, q_n = \Theta(1)$.*

1. *Single iteration: consider (18). Fix $\epsilon > 0$, and let $\Gamma_n$ be any sequence of subgraphs with*

$$\mathsf{Dis}(\Gamma) \triangleq \max_{v \in v(\Gamma)} \frac{|d_\Gamma(v) - \eta(\Gamma)|}{\eta(\Gamma)} \leq \frac{r}{12} \quad \text{and} \quad \eta(\Gamma_n) \geq n^{\frac{1}{2}+\epsilon}, \tag{19}$$

   *for some fixed $0 < r < 1$ and all sufficiently large $n$. If $D = D(n) \leq (\log n)^\alpha$ for any fixed $\alpha > 0$, then $\mathbb{E}\left[(f_{1,m}(\mathsf{Y}) - x)^2\right] \leq C D^2 r^{D-1}$, for some $C > 0$.*

2. *Multiple iteration: consider (17). Fix $L \in \mathbb{N}$ and let $r \in (0, 1/4]$. Let $\Gamma_n$ be any sequence of subgraphs with*

$$W_L^{\min}(\Gamma) \geq C^\star(L, p, q) \left[ n^{L/2} + k^{L-1/2} \right] \sqrt{\log n}, \tag{20}$$

   *for some $C^\star(L, p, q) > 0$. Assume that $m = \omega(1)$ and $D \leq C \log\log n$, for some constant $C > 0$. Then $\mathbb{E}\left[(f_{L,m}(\mathsf{Y}) - x)^2\right] \leq (\log n)^{-\Omega(1)}$.*

We see that for "almost-regular" structures satisfying (19), the single-iteration bound complements the computational lower bound in Theorem 23. For example, consider the case where $\Gamma_n = \mathcal{K}_k$. Here, $\eta(\Gamma_n) = \frac{k-1}{2}$, so Theorem 7 predicts computational hardness when $k \leq n^{\frac{1}{2}-\varepsilon}$. The single-iteration estimator in Theorem 8 achieves recovery once $k \geq n^{\frac{1}{2}+\varepsilon}$, matching the classical algorithmic barrier for planted clique Alon et al. (1998); Dekel et al. (2014); Montanari (2015); Hajek et al. (2016); Chen and Xu (2016). For $\Gamma_n = \mathcal{K}_{k_{\mathsf{L}},k_{\mathsf{R}}}$, the same conclusions hold with $k$ replaced by $\min\{k_{\mathsf{L}}, k_{\mathsf{R}}\}$, consistent with known thresholds for planted bi-cliques Levanzov (2018); Kumar et al. (2022a). Finally, when $\Gamma$ violates the almost-regularity condition (19), the multi-iteration guarantee in Theorem 8 can still apply (see Appendix F for additional examples).

### 3.4. Extensions

We present two extensions of the vanilla planted model. We first note that our main algorithms are robust to a monotone semi-random adversary, and then develop guarantees for the weaker notion of almost-exact recovery. Complete details can be found in Appendix C.4.

**Semi-random model.** We consider the semi-random recovery setting of Feige and Krauthgamer (2000). As before, a planted copy $\Gamma^\star \in \mathcal{S}_\Gamma$ is sampled and then observed through $\mathsf{G} \sim \mathcal{G}_{\Gamma_n}(n,p,q)$. An adversary then outputs $\mathsf{G}_{\mathsf{Adv}} = \mathsf{Adv}(\mathsf{G},\Gamma^\star)$ by *deleting* edges outside $\Gamma^\star$ and *adding* edges inside $\Gamma^\star$ (equivalently, it applies a monotone modification with respect to $\Gamma^\star$). Let $\mathcal{A}$ denote the family of (possibly randomized) adversaries obeying this monotonicity constraint, and let $\mathsf{Adv}(\mathcal{G}_{\Gamma_n}(n,p,q))$ be the induced semi-random family of graph distributions.

A recovery algorithm is any $\hat{\Gamma} : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_\Gamma$. Its worst-case semi-random error is

$$\mathsf{E}_{\mathsf{adv}}(\hat{\Gamma}) \triangleq \sup_{\Gamma^\star \in \mathcal{S}_\Gamma} \sup_{\mathsf{Adv} \in \mathcal{A}} \mathbb{P}[\hat{\Gamma}(\mathsf{G}_{\mathsf{Adv}}) \neq \Gamma^\star], \qquad \mathsf{E}^\star_{\mathsf{adv}} \triangleq \inf_{\hat{\Gamma}} \mathsf{E}_{\mathsf{adv}}(\hat{\Gamma}). \tag{21}$$

Exact recovery is possible if $\limsup_{n\to\infty} \mathsf{E}^\star_{\mathsf{adv}} = 0$. The next result shows that whenever the MLE or the convex program in (10) succeeds uniquely in the vanilla planted model, it remains uniquely optimal under monotone adversarial perturbations.

**Theorem 9 (Robustness to monotone adversaries)** *Both the maximum-likelihood estimator $\hat{\Gamma}_{\mathsf{MLE}}$ and the convex program $\hat{\mathsf{X}}^{(\alpha)}_{\mathsf{con}}$ in (10) remain uniquely optimal under $\mathsf{Adv}(\mathcal{G}_{\Gamma_n}(n,p,q))$ whenever they are uniquely optimal under $\mathcal{G}_{\Gamma_n}(n,p,q)$.*

**Almost-exact recovery.** The strict notion of exact recovery precludes recovering subgraphs with very sparse layers (e.g., as in the kite example). One way to bypass this inherent limitation is to consider weaker notions of recovery. For instance, one might be interested in exactly recovering only the subgraph $\mathsf{H} \subseteq \Gamma$ that achieves the maximum subgraph density of $\Gamma$, or more generally, exactly recover the first $\ell$ layers of $\Gamma$, namely $\Gamma^{(\kappa)} = \bigcup_{\ell=1}^{\kappa} \mathcal{D}^{(\ell)}$; this is studied in Appendix C.4.2. Here, we focus on almost-exact recovery (see, e.g., Hajek et al. (2017); Wu and Xu (2020)).

**Definition 10 (Almost-exact recovery)** *An estimator $\hat{\Gamma}$ almost-exactly recovers $\Gamma^\star$ if, as $n \to \infty$, $\mathsf{d}_{\mathsf{H}}(\hat{\Gamma},\Gamma^\star)/|e(\Gamma)| \to 0$ in probability, where $\mathsf{d}_{\mathsf{H}}$ denotes the Hamming distance between the adjacency matrices of $\hat{\Gamma}$ and $\Gamma^\star$.*

Exact recovery lower bounds do not generally imply almost-exact recovery lower bounds (though any exact-recovery algorithm is automatically almost-exact). To state our results, consider the onion decomposition of $\Gamma_n$ from Definition 4, and define the $\ell^{\text{th}}$ leftover-edge fraction $\mathsf{Res}_\ell^{(n)} \triangleq |e(\Gamma_n \setminus \Gamma_n^{(\ell)})|/|e(\Gamma_n)|$, for $\ell \in 0 \cup [M(\Gamma)]$. Fix any null sequence $\varepsilon_n \downarrow 0$ and define

$$\ell_{\mathsf{LB}}(n) \triangleq \max\{\ell \in 0 \cup [M(\Gamma)] : \ \mathsf{Res}_\ell^{(n)} > \varepsilon_n\}. \tag{22}$$

Heuristically, $\ell_{\mathsf{LB}}$ is the last index for which the leftover-edge fraction is $\Omega(1)$. For almost-exact recovery, a dichotomy emerges depending on the last non-negligible leftover component $\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})}$. Roughly, if its relative density is sub-logarithmic in an appropriate sense, then almost-exact recovery is impossible; if it is super-logarithmic, impossibility persists throughout a logarithmic regime. The precise statement is as follows (proved in Appendix J).

**Theorem 11** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, assume $p_n, q_n = \Theta(1)$, consider the onion decomposition of $\Gamma$ in Definition 4, and denote $k_{\mathsf{R}} \triangleq |v(\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})})|$.*

1. *If $k_{\mathsf{R}} = o(n)$ and*

$$\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) = o\left(\frac{\log k_{\mathsf{R}}}{\log\log k_{\mathsf{R}}}\right), \tag{23}$$

   *then almost-exact recovery is impossible.*

2. *If $\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) \geq \alpha_n \cdot \log k_{\mathsf{R}}$, for some $\alpha_n = \Omega(1)$, then there exists a constant $\underline{C} > 0$ such that almost-exact recovery is impossible if*

$$\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) \leq \underline{C} \cdot \log n. \tag{24}$$

3. *Almost-exact recovery of is possible if $\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) \geq \mathsf{C}\frac{(1+\varepsilon)\cdot\log n}{d_{\mathsf{KL}}(p||q)}$, for any $\varepsilon > 0$ and some constant $\mathsf{C} > 0$.*

We briefly comment on the proof ideas (details in Appendix J). In both (23) and (24), the first step reduces almost-exact recovery for $\Gamma$ to recovering the last non-negligible leftover component $\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})}$. To prove (23), one of the main ingredients is a generalization of the subgraph expectation threshold Kahn and Kalai (2007), and more specifically, of the modified subgraph expectation threshold studied in Mossel et al. (2022), which analyzes the threshold for the appearance of *any* isomorphic copy of $\Gamma$ in $\mathsf{G} \sim \mathcal{G}(n, q)$. For our purposes, however, not all copies are admissible. Indeed, recall that the recovery problem here is supplied with $\Gamma^{(\ell_{\mathsf{LB}})}$ and is tasked with finding $\Gamma$. Thus, the admissible copies are precisely those extending $\Gamma^{(\ell_{\mathsf{LB}})}$, i.e., the copies contained in $\mathcal{M}(\Gamma^{(\ell_{\mathsf{LB}})}, \Gamma)$. To handle this, we derive a generalization of the modified subgraph expectation threshold that accounts for the appearance of such constrained copies.

To prove (24), we establish a connection between almost-exact recovery and the hypothesis-testing variant of the problem, and then leverage the impossibility result (Elimelech and Huleihel, 2025b, Thm. 1) for the latter. Most notably, in detection, there exists a region (i.e., the sub-logarithmic regime where $\mu(\Gamma_n) = o(\log|v(\Gamma_n)|)$) where detection is statistically possible while, as we show above, almost-exact recovery is always impossible. This highlights why hypothesis-testing–based bounds are insufficient in this regime.

We comment on the scope of Theorem 11. When $\Gamma$ is balanced and has super-logarithmic maximum density, i.e., $\mu(\Gamma_n) = \Omega(\log |v(\Gamma_n)|)$, the lower bounds for almost-exact recovery coincide (up to constants) with those for exact recovery in Theorem 5. In general, however, the two notions differ: while exact recovery can fail due to the presence of very sparse appendages, almost-exact recovery may still be possible since such components can be ignored when their contribution is $o(|e(\Gamma_n)|)$, such as in the kite example. Further discussions and illustrative examples are provided in Appendix J.

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## Appendix A. Preliminaries

### A.1. Related work

This paper lies within a broad literature on planted combinatorial structure in random graphs and matrices, studied from both statistical and computational perspectives. We highlight work most pertinent to recovery; for a fuller survey, see Elimelech and Huleihel (2025b).

**Planted subgraphs and matrices.** One of the earliest and most influential recovery problems is the planted clique. Alon, Krivelevich, and Sudakov Alon et al. (1998) showed that spectral methods can identify a clique of size $k = \Omega(\sqrt{n})$. Since then, a variety of algorithmic approaches have been studied—including combinatorial heuristics Feige and Krauthgamer (2000); McSherry (2001); Feige and Ron (2010), convex relaxations such as semidefinite programming and nuclear-norm minimization Ames and Vavasis (2011); Deshpande and Montanari (2015a), and approximate message passing Chen and Xu (2016). Despite these advances, all known polynomial-time algorithms require $k = \Omega(\sqrt{n})$, giving rise to the widely accepted planted clique conjecture, which asserts that recovery is computationally intractable when $k = o(\sqrt{n})$. Beyond cliques, researchers have explored other planted subgraph models:

- *Independent sets.* Feige and Krauthgamer Feige and Ofek (2005) proposed a spectral method for recovering planted independent sets, while Coja-Oghlan Coja-Oghlan (2003) established polynomial-time recovery guarantees in sparse regimes where $q = \Theta(n^{-\alpha})$, provided the independence number scales appropriately. Additional work has analyzed the limits of greedy and local algorithms in these settings Coja-Oghlan and Efthymiou (2015); Gamarnik and Sudan (2014); Rahman et al. (2017).

- *Dense subgraphs and communities.* The problem of recovering a dense community embedded in an Erdős–Rényi background has attracted extensive attention Arias-Castro and Verzelen (2014); Butucea and Ingster (2013); Verzelen and Arias-Castro (2015); Hajek et al. (2015a); Montanari (2015); Candogan and Chandrasekaran (2018); Hajek et al. (2017); Chen and Xu (2016). A key milestone was the reduction from planted clique to planted dense subgraph, due to Hajek, Wu, and Xu Hajek et al. (2015b), which established hardness in regimes where $p = cq$ and $q = \Theta(n^{-\alpha})$. Brennan et al. Brennan et al. (2018) later extended this reduction to nearly the full range of $p > q$, with transitions to denser regimes analyzed in Bhaskara et al. (2010).

- *Other planted structures.* A variety of recovery problems have been studied for specific combinatorial templates, including: planted trees Massoulié et al. (2019), Hamiltonian cycles Bagaria et al. (2020), perfect matchings Moharrami et al. (2021), bipartite subgraphs Rotenberg et al. (2024), and cycles Cheng Mao (2024); Mao et al. (2023). These studies demonstrate that the algorithmic and statistical behavior can differ drastically depending on the underlying subgraph: some exhibit sharp thresholds and conjectured computational barriers, while others allow efficient algorithms down to information-theoretic limits.

- *Matrices and Gaussian models.* Beyond graphs, analogous recovery problems appear in high-dimensional statistics. A prime example is Gaussian biclustering, where one seeks to identify a planted submatrix with elevated mean. Detection aspects were analyzed in Butucea and Ingster (2013); Ma and Wu (2015); Montanari et al. (2015), while recovery guarantees were developed in Shabalin et al. (2009); Kolar et al. (2011); Balakrishnan et al. (2011); Cai et al. (2017); Chen and Xu (2016); Hajek et al. (2017); Brennan et al. (2019); Dadon et al. (2024b,a). Closely related are spectral analyses of the spiked Wigner model, beginning with Péché (2006); Féral and Péché (2007); Capitaine et al. (2009), and later work on spectral thresholds and Bayesian algorithms Montanari et al. (2015); Perry et al. (2020, 2018); Banks et al. (2018); Hopkins et al. (2017).

Together, this body of research illustrates the diversity of recovery phenomena: in some cases (e.g., planted cliques and dense subgraphs), computational-statistical gaps appear central, while in others (e.g., certain trees or paths), efficient recovery is achievable down to statistical thresholds.

**General planting models.** Research has long studied planting arbitrary substructures in random graphs, exploring both detection and recovery under varied generative models. Early works like Addario-Berry et al. (2010) developed general probabilistic bounds for testing in the planted-vector setting—though their results were broad and not always tight across all cases. More recent lines of inquiry have introduced two foundational paradigms:

1. *Detection.* Several works have developed frameworks for the detection of arbitrary planted subgraphs. The work Huleihel (2022) introduced two models for planting arbitrary subgraphs in random graphs: the union model (where the planted edges are superimposed on an Erdős–Rényi background) and the induced model (where the planted subgraph appears as an induced copy). They analyzed detection thresholds in the dense regime, providing both information-theoretic bounds and computational insights. Recently, Yu et al. (2024) studied the computational limits of detection in the dense regime and showed that optimal constant-degree polynomial tests are always given by counting stars. This result highlights the limitations of low-degree polynomials for detection of arbitrary planted subgraphs. Recently, Elimelech and Huleihel (2025b) extended this line of work by analyzing detection in the union model more generally. Their results provide sharp characterizations across both sparse and dense regimes, demonstrating how the feasibility of detection depends delicately on graph density and subgraph structure. Together, these works clarify the statistical and algorithmic landscape for detection in general planted subgraph models, identifying both tractable and hard regimes. Finally, Elimelech and Huleihel (2025a) established fundamental statistical limits for detecting arbitrary planted subgraph under a semi-random model where an adversary is allowed to remove edges outside the planted subgraph before the graph is provided to the statistician; the goal is to derive robust detection algorithms.

2. *Recovery.* The problem of recovery, where the goal is to reconstruct the precise location of the planted subgraph, has also been addressed in several recent studies. In Huleihel (2022) considered recovery in the induced model, deriving bounds in the dense regime. While their focus was primarily on detection, they established statistical thresholds for recovery as well. In Mossel et al. (2023) investigated recovery for arbitrary planted subgraphs in the dense regime. They provided tight upper and lower bounds for recovery in this setting, focusing on specific families of subgraphs, and highlighted where statistical–computational gaps arise. Finally, Lee et al. (2025) advanced this direction by deriving an exact formula for the asymptotic MMSE curve for recovering arbitrary planted subgraphs. They also proposed an efficient algorithm for approximating this threshold in dense graphs, based on a novel decomposition technique, extending "all-or-nothing" phenomena to general subgraphs. Taken together, these works establish the first general frameworks for recovery in arbitrary planted subgraph models, with Mossel et al. (2023) and Lee et al. (2025) providing sharp results in the dense setting, and Huleihel (2022) bridging induced subgraphs and recovery.

**Computational hardness.** Over the past decade, major progress has been made toward a rigorous understanding of the fundamental limits of efficient algorithms for high-dimensional inference problems with planted structure. A recurring theme in this line of work is the emergence of a statistical–computational gap: the number of samples required by any known polynomial-time algorithm is strictly larger than the information-theoretic minimum Berthet and Rigollet (2013); Ma and Wu (2015); Cai et al. (2017); Krauthgamer et al. (2015); Hajek et al. (2015b); Chen and Xu (2016); Wang et al. (2016a,b); Gao et al. (2017); Brennan et al. (2018, 2019); Wu and Xu (2020);

Brennan and Bresler (2020); Hopkins and Steurer (2017); Hopkins B (2018); Bandeira et al. (2020); Cherapanamjeri et al. (2020); Gamarnik et al. (2020); Barak et al. (2016); Deshpande and Montanari (2015b); Meka et al. (2015); Ma and Wigderson (2015); Kothari et al. (2017); Hopkins et al. (2018); Raghavendra et al. (2019); Hopkins et al. (2017); Mohanty et al. (2020); Feldman et al. (2017, 2015); Diakonikolas et al. (2017, 2019); Zdeborová and Krzakala (2016); Lesieur et al. (2015, 2016); Krzakala et al. (2007); Ricci-Tersenghi et al. (2019); Bandeira et al. (2018); Schramm and Wein (2022); Brennan et al. (2021); Wein (2021); Abhishek Dhawan (2023); Elimelech and Huleihel (2025b). The evidence for these gaps typically falls into two broad categories:

1. *Failure of classes of algorithms.* One line of evidence comes from showing that broad algorithmic paradigms fail in the conjectured hard regime. For instance, low-degree polynomials provide a unifying lens for analyzing high-dimensional inference, and their failure below certain thresholds suggests sharp computational barriers Hopkins and Steurer (2017); Hopkins B (2018); Bandeira et al. (2020); Cherapanamjeri et al. (2020); Gamarnik et al. (2020). Similarly, the sum-of-squares hierarchy, despite its power as a family of semidefinite relaxations, has been shown to fall short in planted clique, planted dense subgraph, and related problems Barak et al. (2016); Deshpande and Montanari (2015b); Meka et al. (2015); Ma and Wigderson (2015); Kothari et al. (2017); Hopkins et al. (2018); Raghavendra et al. (2019); Hopkins et al. (2017); Mohanty et al. (2020). The statistical query model, which captures a wide class of algorithms accessing only expectations of data-dependent functions, has also yielded strong lower bounds in this context Feldman et al. (2017, 2015); Diakonikolas et al. (2017, 2019); Brennan et al. (2021). Finally, message-passing algorithms such as belief propagation and approximate message passing often achieve optimal performance in "easy" regimes but exhibit provable failures in the hard phase across multiple planted models Zdeborová and Krzakala (2016); Lesieur et al. (2015, 2016); Krzakala et al. (2007); Ricci-Tersenghi et al. (2019); Bandeira et al. (2018).

2. *Average-case reductions.* Another powerful approach is to establish hardness by reducing one planted problem to another conjectured-to-be-hard task, most prominently planted clique. This technique has been used to show that recovery (or even detection) in models such as planted dense subgraph or biclustering is at least as hard as planted clique in certain parameter regimes Berthet and Rigollet (2013); Ma and Wu (2015); Cai et al. (2017); Chen and Xu (2016); Hajek et al. (2015b); Wang et al. (2016a,b); Gao et al. (2017); Brennan et al. (2018, 2019); Wu and Xu (2020); Brennan and Bresler (2020).

**Semi-random models.** The notion of robustness via semi-random perturbations dates back to Blum and Spencer Blum and Spencer (1995), with stronger adversarial variants introduced in the semi-random planted clique model Feige and Krauthgamer (2000); Feige and Kilian (2001). These ideas have since been generalized to community detection in the SBM Charikar et al. (2017); Moitra et al. (2016); Banks et al. (2021); Bhaskara et al. (2024), and to recovery of cliques Steinhardt (2017); McKenzie et al. (2020); Buhai et al. (2023); Błasiok et al. (2024); Guruswami and Wang (2025), bipartite graphs Kumar et al. (2022b), and $r$-colorable graphs Louis et al. (2025). While convex relaxations such as SDP often remain robust, spectral and other classical methods can fail under semi-random noise. In the planted dense subgraph model, Brennan et al. Brennan and Bresler (2020) conjectured that below certain density thresholds even weak recovery is computationally hard with adversarial edge deletions, supporting their claim via reductions from the planted clique with side information.

### A.2. Notation

For an integer $n \in \mathbb{N}$, we write $[n] \triangleq \{1, \ldots, n\}$ and define $n^{(2)} \triangleq \binom{n}{2}$. For $i \in \mathbb{N}$, the collection of all subsets of $[n]$ of size $i$ is denoted by $\binom{[n]}{i}$. For real numbers $a, b \in \mathbb{R}$, we write $a \vee b$ for their maximum. We use the standard asymptotic notations $O(\cdot)$, $o(\cdot)$, $\Omega(\cdot)$, and $\omega(\cdot)$ to describe growth rates of sequences, and write $a_n \ll b_n$ to indicate that $a_n$ is polynomially smaller than $b_n$, i.e., $\limsup_{n\to\infty} \log_n a_n < \liminf_{n\to\infty} \log_n b_n$.

For matrices and vectors, we denote by $\mathbf{J}_{m\times n}$ the all-ones $m \times n$ matrix, and by $\mathbf{1}_m$ and $\mathbf{0}_m$ the all-ones and all-zeros vectors in $\mathbb{R}^m$, respectively; subscripts are omitted when dimensions are clear from the context. For square matrices $\mathsf{A}$ and $\mathsf{B}$ of the same size, $\langle \mathsf{A}, \mathsf{B} \rangle$ denotes the Hilbert–Schmidt inner product, defined as $\mathsf{Tr}(\mathsf{A}^\top \mathsf{B})$. We write $\|\mathsf{A}\|_\star$ for the nuclear (trace, Schatten-1) norm of $\mathsf{A}$, given by the sum of its singular values, and $\|\mathsf{A}\|_F$ for its Frobenius norm. Additional matrix norms used throughout include the spectral norm $\|\mathsf{X}\|_{\mathsf{op}}$, the entrywise $\ell_1$ norm $\|\mathsf{X}\|_{\ell_1} = \sum_{i,j} |\mathsf{X}_{ij}|$, the entrywise $\ell_\infty$ norm $\|\mathsf{X}\|_{\ell_\infty} = \max_{i,j} |\mathsf{X}_{ij}|$, the maximum absolute row sum $\|\mathsf{X}\|_{\ell_\infty\to\ell_\infty} \triangleq \max_i \sum_j |\mathsf{X}_{ij}|$, and the maximum row $\ell_2$ norm $\|\mathsf{X}\|_{2,\infty} \triangleq \max_{i\in[n]} \|\mathsf{X}_{i,:}\|_2$.

For probability measures $\mathbb{P}_0$ and $\mathbb{P}_1$ on the same measurable space, we use the total variation distance $d_{\mathsf{TV}}(\mathbb{P}_0, \mathbb{P}_1) = \frac{1}{2}\int |\mathrm{d}\mathbb{P}_0 - \mathrm{d}\mathbb{P}_1|$, the $\chi^2$-divergence $\chi^2(\mathbb{P}_0||\mathbb{P}_1) = \int \frac{(\mathrm{d}\mathbb{P}_0 - \mathrm{d}\mathbb{P}_1)^2}{\mathrm{d}\mathbb{P}_1}$, and the Kullback–Leibler (KL) divergence $d_{\mathsf{KL}}(\mathbb{P}_0||\mathbb{P}_1) = \mathbb{E}_{\mathbb{P}_0} \log \frac{\mathrm{d}\mathbb{P}_0}{\mathrm{d}\mathbb{P}_1}$. When $\mathbb{P}_0 = \mathsf{Bern}(p)$ and $\mathbb{P}_1 = \mathsf{Bern}(q)$, we write $\chi^2(p||q) = \frac{(p-q)^2}{q(1-q)}$ and $d_{\mathsf{KL}}(p||q) = p \log \frac{p}{q} + (1-p)\log\frac{1-p}{1-q}$. For a finite set $S$, $\mathsf{Unif}(S)$ denotes the uniform distribution over $S$.

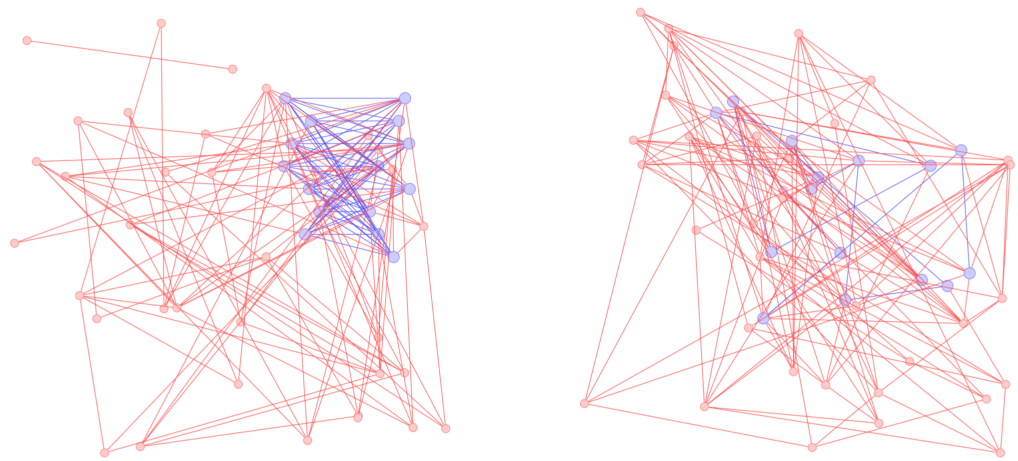
We use standard graph-theoretic notation. An undirected graph is denoted by $\mathsf{G} = (v(\mathsf{G}), e(\mathsf{G}))$, with vertex set $v(\mathsf{G})$ and edge set $e(\mathsf{G})$. Since we primarily consider graphs without isolated vertices, for a subgraph $\mathsf{H} \subseteq \mathsf{G}$ we write $|\mathsf{H}|$ to denote the number of edges in $e(\mathsf{H})$. If $|v(\mathsf{G})| \leq n$, we let $\mathcal{S}_\mathsf{G}$ denote the set of all isomorphic copies of $\mathsf{G}$ in the complete graph $\mathcal{K}_n$. That is, $\mathsf{H} \subseteq \mathcal{K}_n$ belongs to $\mathcal{S}_\mathsf{G}$ if there exists a bijection $f : v(\mathsf{G}) \to v(\mathsf{H})$ such that $(u, v) \in e(\mathsf{G})$ if and only if $(f(u), f(v)) \in e(\mathsf{H})$. Given $\Gamma \subseteq \mathcal{K}_n$ and a subgraph $\mathsf{H} \subseteq \Gamma$, we define $\mathcal{N}(\mathsf{H}, \Gamma)$ as the set of copies of $\mathsf{H}$ in $\Gamma$, and $\mathcal{M}(\mathsf{H}, \Gamma)$ as the set of copies of $\Gamma$ in $\mathcal{K}_n$ that contain $\mathsf{H}$. For any graph $\mathsf{G}$, $\mathsf{A}_\mathsf{G}$ denotes its adjacency matrix, indexed by $v(\mathsf{G})$, where $\mathsf{A}_\mathsf{G}(i, j) = \mathbb{1}\{\{i, j\} \in e(\mathsf{G})\}$.

### A.3. Organization

The rest of this appendix is organized as follows. In Appendix B, we introduce the problem setup and provide some necessary preliminaries. Appendix C presents our main results, discussions, and examples. Appendices D–J are devoted to the the derivation of our lower and upper bounds. Finally, in Appendix K we conclude our paper, and discuss a few directions for future research.

## Appendix B. Problem Setup and Preliminaries

In this section, we describe the setting we study, along with several important preliminaries. Let $\Gamma = (\Gamma_n)_{n\in\mathbb{N}}$ be a sequence of graphs such that, for each $n \in \mathbb{N}$, $\Gamma_n = (v(\Gamma_n), e(\Gamma_n))$ is an undirected graph without isolated vertices and with $|v(\Gamma_n)| \leq n$. Let $\mathcal{S}_{\Gamma_n}$ denote the set of all isomorphic copies of $\Gamma_n$ in the complete graph on $n$ vertices. We refer to $\Gamma_n$ as the *planted* (or *hidden*) structure. Fix parameters $p_n, q_n$ satisfying $0 < q_n < p_n \leq 1$. The *planted subgraph model* $\mathcal{G}_{\Gamma_n}(n, p_n, q_n)$ is defined as the distribution of a random graph $\mathsf{G}$ on $n$ vertices generated as follows: first draw an arbitrary but fixed copy $\Gamma_n^\star \in \mathcal{S}_{\Gamma_n}$; then include each edge $e \in e(\Gamma_n^\star)$



(a) Planted $\Gamma_n^\star = \mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}$. (b) Planted $\Gamma_n^\star = \mathcal{P}_k$

**Figure 1:** The observed graph $\mathsf{G}$ is the union of an Erdős–Rényi graph and a planted subgraph $\Gamma_n^\star$. In (a) $\Gamma_n^\star$ is a bipartite subgraph, and in (b) $\Gamma_n^\star$ is a path.

independently with probability $p_n$, and include each edge $e \notin e(\Gamma_n^\star)$ independently with probability $q_n$. Equivalently, $\mathsf{G}$ can be viewed as the union of a noisy copy of $\Gamma_n^\star$ and an Erdős–Rényi random graph $\mathcal{G}(n, q_n)$.

A learner observes a single sample $\mathsf{G} \sim \mathcal{G}_{\Gamma_n}(n, p_n, q_n)$, and the goal is to recover the hidden copy $\Gamma_n^\star$. We study this framework in the asymptotic regime where $n \to \infty$. Figure 1 illustrates typical graph observations. Given $\mathsf{G}$, an estimator $\widehat{\Gamma} : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_{\Gamma_n}$ aims to output $\Gamma_n^\star$. Define the worst-case error probability associated with an estimator $\widehat{\Gamma}$ as

$$\mathsf{E}_n(\hat{\Gamma}) \triangleq \sup_{\Gamma^\star \in \mathcal{S}_{\Gamma_n}} \mathbb{P}_{\mathcal{G}_{\Gamma_n^\star}(n,p_n,q_n)}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^\star], \tag{25}$$

and the optimal error probability as

$$\mathsf{E}_n^\star \triangleq \inf_{\hat{\Gamma}:\{0,1\}^{n^{(2)}} \to \mathcal{S}_{\Gamma_n}} \sup_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}_{\mathcal{G}_{\Gamma_n^\star}(n,p_n,q_n)}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^\star]. \tag{26}$$

We say that a sequence of estimators $(\widehat{\Gamma}_n)_{n\in\mathbb{N}}$, where $\widehat{\Gamma}_n : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_{\Gamma_n}$, achieves *exact recovery* if $\limsup_{n\to\infty} \mathsf{E}_n(\widehat{\Gamma}_n) = 0$. Conversely, we say that *exact recovery is impossible* if $\liminf_{n\to\infty} \mathsf{E}_n^\star > 0$.

**Remark 12** *In the above, we focused on a worst-case definition of error probability. However, as we show in Appendix G.1, the Bayesian (average-case) definition, where an expectation over* $\Gamma^\star \sim \mathsf{Unif}(\mathcal{S}_{\Gamma_n})$ *is taken instead of a supremum, is equivalent. Indeed, the uniform measure over* $\mathcal{S}_{\Gamma_n}$ *induces a permutation-invariant statistical model for which the least favorable prior is also uniform.*

Our results will be expressed in terms of the following graph-theoretic measures. We let $\eta(\Gamma_n) \triangleq |e(\Gamma_n)|/|v(\Gamma_n)|$ denote the density of $\Gamma_n$, and we recall the following definition of the *maximum subgraph density*.

**Definition 13 (Maximum subgraph density Bollobás (2001))** *Let* $\mathsf{G}$ *be an undirected graph. The maximum subgraph density of* $\mathsf{G}$ *is*

$$\mu(\mathsf{G}) \triangleq \max \{\eta(\mathsf{H}) : \mathsf{H} \subseteq \mathsf{G}, \mathsf{H} \neq \emptyset\} . \tag{27}$$

Next, we introduce the notion of a graph-cut.

**Definition 14 (Graph-cut)** *Let* $n$ *be a positive integer. A graph-cut on* $n$ *vertices is a triplet* $\mathsf{H} = (V, S, E)$*, where* $S \subseteq V \subseteq [n]$*, and*

$$E \subseteq \mathcal{K}_V \setminus \mathcal{K}_S \triangleq \{(u,v) : u,v \in V \text{ and at most one of } u,v \text{ belongs to } S\} . \tag{28}$$

*We define the number of edges of the graph-cut as* $|\mathsf{H}| \triangleq |E|$*, and the number of non-selected vertices as* $|v(\mathsf{H})| \triangleq |V \setminus S|$.

An important graph-theoretic quantity that will play a central role is the *relative density* and the *maximum subgraph relative density*, defined as follows.

**Definition 15 (Relative densities)** *Given graphs* $\mathsf{H}' \subseteq \mathsf{H}$*, we define the induced graph-cut by* $\mathsf{H}|\mathsf{H}' \triangleq (v(\mathsf{H}), v(\mathsf{H}'), \mathsf{H} \setminus \mathsf{H}')$*. The relative density is defined as*

$$\eta(\mathsf{H}|\mathsf{H}') \triangleq \frac{|\mathsf{H}| - |\mathsf{H}'|}{|v(\mathsf{H}) \setminus v(\mathsf{H}')|}. \tag{29}$$

*If* $\mathsf{H} \setminus \mathsf{H}' = \emptyset$*, we define* $\eta(\mathsf{H}|\mathsf{H}') \triangleq \infty$*. The maximum subgraph relative density is*

$$\mu(\mathsf{H}|\mathsf{H}') \triangleq \max \left\{\eta(\mathsf{J}|\mathsf{H}') : \mathsf{H}' \subsetneq \mathsf{J} \subseteq \mathsf{H}\right\} . \tag{30}$$

Note that, given $\mathsf{H}'$, we choose $\mathsf{J} \supsetneq \mathsf{H}'$ in (30) to maximize the "cut density", namely the number of new edges per new vertex, counting also edges from the new vertices back into $\mathsf{H}'$. This corresponds to the densest subgraph in the cut, rather than in the induced remainder. For example, let $\mathsf{H}' = \mathcal{K}_4$ be a clique on 4 vertices (with 6 edges). Add many leaf vertices, each joined to $\mathcal{K}_4$ by exactly one edge and with no edges among the leaves, and denote the resulting graph by $\mathsf{H}$. Then every added vertex contributes exactly one cross-edge to $\mathsf{H}'$, and hence $\mu(\mathsf{H}|\mathsf{H}') = 1$.

Next, our statistical lower and upper bounds will rely, both in the statements and in the proofs, on a canonical decomposition of the planted subgraph $\Gamma$, introduced in (Lee et al., 2025, Definition 3.3).

**Definition 16 (Onion decomposition)** *Let* $\Gamma = \Gamma_n$ *be an arbitrary graph. The* onion decomposition *of* $\Gamma$ *is an increasing sequence of subgraphs* $\Gamma^{(0)}, \Gamma^{(1)}, \ldots$ *constructed as follows:*

*(i) Initialize with* $\Gamma^{(0)} \triangleq \emptyset$.

*(ii) For each* $\ell > 0$*, let* $\Gamma^{(\ell)}$ *be a maximal subgraph that maximizes* $\eta(\mathsf{H}|\Gamma^{(\ell-1)})$ *among all subgraphs* $\Gamma^{(\ell-1)} \subsetneq \mathsf{H}$.[3]

*(iii) Stop if* $\Gamma^{(\ell)} = \Gamma$.

*Let* $M = M(\Gamma) \leq |\Gamma|$ *denote the total number of steps until termination. The sequence* $\{\Gamma^{(\ell)}\}_{\ell=0}^{M}$ *is referred to as the onion decomposition of* $\Gamma$*. Finally, define the remainder subgraphs* $\mathcal{D}^{(\ell)} \triangleq \Gamma^{(\ell)} \setminus \Gamma^{(\ell-1)}$ *for* $\ell = 1, 2, \ldots, M$.

3. Maximality implies that there does not exist a subset $\Gamma^{(\ell)} \subsetneq \Gamma'$ for which $\eta(\Gamma'|\Gamma^{(\ell-1)})$ is also maximized.

Intuitively, this process iteratively selects the densest remaining subgraph, removes it, and continues on the remainder. In the case of balanced graphs Bollobás (1981), where $\max_{\mathsf{H}\subseteq\Gamma}\frac{|\mathsf{H}|}{|v(\mathsf{H})|}=\frac{|\Gamma|}{|v(\Gamma)|}$, the process terminates in a single step. Note that the decomposition elements $\{\Gamma^{(\ell)}\}_{\ell\geq 0}$ are, by definition, edge-disjoint; however, they may share vertices. The following result, proved in Appendix G.2, shows that the decomposition is unique. An alternative proof appears in (Lee et al., 2025, Thm. 3.6) as well.

**Lemma 17 (Uniqueness)** *Let $\Gamma=\Gamma_n$ be an arbitrary graph, and let $\{\Gamma^{(\ell)}\}_{\ell=0}^{M}$ (and $\{\mathcal{D}^{(\ell)}\}_{\ell=0}^{M}$) denote its onion decomposition in Definition 16. Then, for every $\ell=0,1,\ldots,M-1$, the choice of $\Gamma^{(\ell+1)}$ in step (ii) is unique. Thus, the onion decomposition of any graph is uniquely determined.*

We note that computing $\mu(\Gamma|\Gamma^{(\ell-1)})=\eta(\Gamma^{(\ell)}|\Gamma^{(\ell-1)})$ for $\ell\in[M(\Gamma)]$ requires, at least naively, constructing the entire sequence $\{\Gamma^{(\ell)}\}_{\ell=0}^{M}$. Nonetheless, the following result, proved in Appendix G.3, provides a non-sequential equivalent characterization. In particular, a key quantity in our results and analysis is the *minimal maximum subgraph density*, given by $\eta(\Gamma^{(M)}|\Gamma^{(M-1)})$, corresponding to the relative density of the final layer in the onion decomposition of $\Gamma$.

**Lemma 18 (Equivalent characterization)** *Let $\Gamma=\Gamma_n$ be an arbitrary graph, and denote by $\{\Gamma^{(\ell)}\}_{\ell=0}^{M}$ its onion decomposition in Definition 16. Let*

$$\Lambda(\Gamma)\triangleq\{\mu(\Gamma|\mathsf{J}):\mathsf{J}\subseteq\Gamma\},\tag{31}$$

*and list the distinct values of $\Lambda(\Gamma)$ in strictly decreasing order*

$$\lambda_1>\lambda_2>\cdots>\lambda_T.\tag{32}$$

*Then $T=M(\Gamma)$ and, for every $\ell=1,\ldots,M$,*

$$\eta\left(\Gamma^{(\ell)}|\Gamma^{(\ell-1)}\right)=\lambda_\ell,\tag{33}$$

*i.e., the $\ell^{\mathsf{th}}$ layer value is the $\ell^{\mathsf{th}}$ largest distinct value taken by $\mathsf{J}\mapsto\mu(\Gamma|\mathsf{J})$. In particular, define*

$$\mu_{\mathsf{min}}(\Gamma)\triangleq\min_{\mathsf{S}\subseteq\Gamma}\max_{\mathsf{S}\subsetneq\mathsf{F}\subseteq\Gamma}\eta(\mathsf{F}|\mathsf{S}).\tag{34}$$

*Then, $\eta(\Gamma^{(M)}|\Gamma^{(M-1)})=\mu_{\mathsf{min}}(\Gamma)$.*

Thus, for any fixed layer index $\ell\in[M(\Gamma)]$, we have

$$\eta\left(\Gamma^{(\ell)}|\Gamma^{(\ell-1)}\right)=\text{the }\ell^{\mathsf{th}}\text{ largest distinct value of }\{\mu(\Gamma|\mathsf{J}):\mathsf{J}\subseteq\Gamma\},\tag{35}$$

and this value can be defined without explicit reference to the onion construction. At the extremes:

- For $\ell=1$ we have $\lambda_1=\mu(\Gamma|\emptyset)=\max_{\mathsf{F}\subseteq\Gamma}\eta(\mathsf{F}|\emptyset)$, which is the classical maximum subgraph density $\mu(\Gamma)$.
- For $\ell=M$ we have $\lambda_M=\min_{\mathsf{J}\subseteq\Gamma}\mu(\Gamma|\mathsf{J})$, which is the minimal maximum subgraph density $\mu_{\mathsf{min}}(\Gamma)$.

Throughout this paper, we sometimes suppress the explicit dependence of various parameters on the index $n$. For example, we write the sequence of planted graphs as $\Gamma=(\Gamma_n)_n$, the sequences of edge probabilities as $p=(p_n)_n$ and $q=(q_n)_n$, and so on.

## Appendix C. Main Results

In this section, we present our main results. We begin by analyzing the statistical limits of the recovery problem, setting computational considerations aside, and identify a sharp threshold at which the optimal recovery error probability undergoes a phase transition from 0 to 1 as $n \to \infty$. We then address recovery under polynomial-time constraints by proposing a general, computationally efficient procedure and providing statistical guarantees on its performance. Next, we establish several computational lower bounds, showing that polynomial-time algorithms can incur an inherent gap relative to the optimal (information-theoretic) solution. Finally, we consider extensions of the vanilla planted model, including recovery under semi-random (monotone-adversary) perturbations as well as other weaker notions of recovery.

### C.1. Statistical limits

In this subsection, we establish the statistical limits of the problem, beginning with information-theoretic lower bounds and continuing with matching upper bounds.

**Lower bound.** We start with the following general lower bound, which holds for an arbitrary planted subgraph $\Gamma$.

**Theorem 19 (Statistical lower bound)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, and assume $p_n, q_n = \Theta(1)$. Exact recovery is statistically impossible if*

$$\mu_{\mathsf{min}}(\Gamma_n) \leq \frac{(1-\varepsilon)\cdot\log n}{d_{\mathsf{KL}}(p||q)}, \tag{36}$$

*for any $\varepsilon > 0$.*

To better understand Theorem 19, we now consider a few examples.

**Example 1 (Planted clique)** *Consider the case where $\Gamma = \mathcal{K}_k$ is a clique with $k = |v(\mathcal{K}_k)|$ vertices. A clique is balanced (i.e., $\mu(\mathcal{K}_k) = \eta(\mathcal{K}_k) = \frac{k-1}{2}$), and thus $\mu_{\mathsf{min}}(\mathcal{K}_k) = \frac{k-1}{2}$. Therefore, Theorem 19 tells us that exact recovery is statistically impossible if $k \leq (1-\varepsilon)\frac{2\log n}{d_{\mathsf{KL}}(p||q)}$, which is consistent with folklore results (e.g., Jerrum (1992)).*

**Example 2 (Union of disjoint cliques)** *Consider the case where $\Gamma$ is a union of $L$ disjoint cliques with sizes $k_1, \ldots, k_L$. Define $k_{\min} \triangleq \min_{i\in[L]} k_i$. A straightforward calculation reveals that $\mu_{\mathsf{min}}(\Gamma) = \frac{k_{\min}-1}{2}$, and so Theorem 19 implies that exact recovery is statistically impossible if $k_{\min} \leq (1-\varepsilon)\frac{2\log n}{d_{\mathsf{KL}}(p||q)}$.*

**Example 3 (A kite)** *Consider the case where $\Gamma$ is a kite on $k+1$ vertices, namely, a clique with $k = |v(\mathcal{K}_k)|$ vertices, with one of its vertices connected by an edge to an additional vertex. It is not difficult to see that, in this case, the onion decomposition layers are $\mathcal{D}^{(1)} = \mathcal{K}_k$ and $\mathcal{D}^{(2)} = \{\cdot, k+1\}$, which is a single edge. Thus, $\mu_{\mathsf{min}}(\mathcal{K}_k) = \frac{1}{2}$. Therefore, Theorem 19 tells us that exact recovery of a kite is statistically impossible. This aligns with intuition: the noisy random background can alter (or "tweak") the single edge with positive probability.*

Example 3 illustrates that if $\Gamma$ contains a dense core together with a very sparse appendage, then the latter can preclude exact recovery. In such cases, it is reasonable to consider alternative recovery criteria (such as weak recovery), which allow for a nonzero fraction of errors. We discuss such criteria in Subsection C.4. Next, it is instructive to compare and contrast the above results with the detection (hypothesis-testing) variant, given by

$$\mathcal{H}_0 : \mathsf{G} \sim \mathcal{G}(n, q) \quad \text{vs.} \quad \mathcal{H}_1 : \mathsf{G} \sim \mathcal{G}_{\Gamma_n}(n, p, q). \tag{37}$$

Here, the task is to decide whether or not a copy of $\Gamma$ was planted in $\mathsf{G}$. Specifically, (Elimelech and Huleihel, 2025b, Thm. 1) shows that

- If $\Gamma_n$ has sub-logarithmic density, i.e., $\mu(\Gamma_n) = o(\log |v(\Gamma_n)|)$, then detection is statistically impossible provided that

$$|e(\Gamma_n)| \vee d^2_{\max}(\Gamma_n) \ll n. \tag{38}$$

- If $\Gamma_n$ has super-logarithmic density, i.e., $\mu(\Gamma_n) = \Omega(\log |v(\Gamma_n)|)$, then detection is statistically impossible provided that

$$\mu(\Gamma_n) \leq \underline{C} \cdot \log n, \tag{39}$$

  for some $\underline{C} > 0$.

It is further shown in the same paper that these bounds are asymptotically tight, namely, there exist algorithms that achieve them. Comparing the above with Theorem 19, we see that the statistical barriers for detection and recovery hinge on different graph-theoretic quantities, revealing a distinct *detection–recovery gap*. Specifically, in the sub-logarithmic density regime, which includes graphs such as paths, stars, trees, and related structures, detection can be statistically possible, in stark contrast to exact recovery, which is always statistically impossible in this regime. In fact, this impossibility extends beyond subgraphs with sub-logarithmic density to (arguably) all subgraphs with sub-logarithmic minimal maximum subgraph density $\mu_{\mathsf{min}}$. Thus, a clear *detection–recovery gap* emerges in this setting. By contrast, the super-logarithmic density regime includes sufficiently dense subgraphs, such as cliques and bipartite graphs, where detection and recovery thresholds may coincide when the planted structure is "sufficiently nice" (e.g., when $\Gamma$ is a clique). However, recovery may still be statistically impossible even when detection is feasible, mirroring the behavior in the sub-logarithmic regime, as illustrated by the kite example.

Finally, for the special case where $\Gamma$ is balanced Bollobás (1981) (e.g., a clique), we can establish tight information-theoretic lower bounds using a different technique, namely Bayes risk analysis, which relates exact recovery to a variant of the detection problem. The details are provided in Appendix I.[4] We now move to our statistical upper bounds.

4. As noted in (Wein, 2025, pg. 7), even in the special case of a planted clique, the "$2\log_2 n$" information-theoretic threshold had been established for detection but not for exact recovery, although the latter was expected to hold. In this work, we confirm this expectation by providing a lecture-style proof that establishes $2\log_2 n$ as the sharp threshold for exact recovery as well, and in fact we generalize this result to arbitrary planted subgraphs.

**Upper bound.** It is relatively straightforward to show that the optimal maximum-likelihood estimator (MLE) of $\Gamma^\star$ is given by (see Appendix H for details)

$$\hat{\Gamma}_{\mathsf{MLE}}(\mathsf{G}) = \arg\max_{\Gamma' \in \mathcal{S}_\Gamma} \left| e(\Gamma') \cap \mathsf{G} \right|. \tag{40}$$

Interestingly, it turns out to be both simpler and more intuitive to analyze the following recursive peeling MLE procedure. Specifically, let $\{\Gamma^{(\ell)}\}_{\ell \geq 1}$ denote the onion decomposition of $\Gamma$ in Definition 16, and recall that $\mathcal{M}(\mathsf{H}, \Gamma)$ denotes the set of copies of $\Gamma$ in $\mathcal{K}_n$ which contain $\mathsf{H}$. At step $\ell \in [M]$, the recovery algorithm produces the MLE estimate for $\Gamma^{(\ell)}$ as follows:

$$\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}} = \arg \max_{\mathcal{D} \in \mathcal{M}(\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}, \Gamma^{(\ell)})} \left| e(\mathcal{D}) \cap \left( \mathsf{G} \setminus \hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} \right) \right|. \tag{41}$$

That is, the estimator searches for a copy of $\Gamma^{(\ell)}$, restricted to those copies that extend the previously estimated layer $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}$, within the observed graph $\mathsf{G}$, after removing the previously estimated layer $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}$. We define $\hat{\Gamma}^{(0)}_{\mathsf{MLE}} = \emptyset$. At the final step, the algorithm outputs the complete estimate

$$\hat{\Gamma}_{\mathsf{pMLE}} = \hat{\Gamma}^{(M)}_{\mathsf{MLE}}. \tag{42}$$

The following result complements the information-theoretic lower bound in Theorem 19.

**Theorem 20 (Optimal algorithm)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, and assume $p_n, q_n = \Theta(1)$. Exact recovery of $\Gamma^\star$, via the likelihood peeling algorithm in (42), is possible if*

$$\mu_{\mathsf{min}}(\Gamma_n) \geq \mathsf{C}\frac{(1+\varepsilon)\cdot \log n}{d_{\mathsf{KL}}(p||q)}, \tag{43}$$

*for any $\varepsilon > 0$ and some constant $\mathsf{C} > 0$.*

Thus, Theorem 20 matches the lower bound in Theorem 19 up to a constant factor. For $p = 1$, the proof of Theorem 20 yields $\mathsf{C} = 4$, and for any $p < 1$ we obtain $\mathsf{C} = 16$. We have not made a serious effort to optimize this constant, but we conjecture that $\mathsf{C} = 1$ for any $p \in [0, 1]$.[5]

It is worth emphasizing that the proof of Theorem 20 relies on an important property of the layers obtained in the onion decomposition of $\Gamma$, namely a uniqueness property of the planted copy on its full vertex set. Specifically, one of the steps in the proof analyzes $|v(\mathcal{D}' \cap \mathcal{D}^{(\ell),\star})|$, the intersection between an isomorphic copy $\mathcal{D}' \neq \mathcal{D}^{(\ell),\star}$ of the actual planted $\ell$th layer $\mathcal{D}^{(\ell),\star}$. In principle, it could happen that $|v(\mathcal{D}' \cap \mathcal{D}^{(\ell),\star})| = |v(\mathcal{D}^{(\ell),\star})|$ yet $\mathcal{D}' \neq \mathcal{D}^{(\ell),\star}$; for example, this can occur in the case of a kite. When this happens, exact recovery becomes impossible. Fortunately, we show that this cannot occur for the layers selected by the onion procedure.

### C.2. Computationally efficient algorithm

We now introduce a polynomial-time recovery algorithm. To describe the procedure, we first establish notation and record several definitions. The Hilbert–Schmidt inner product (also referred to as the Frobenius inner product) between two matrices $\mathsf{A}$ and $\mathsf{B}$ of identical dimensions is

$$\langle \mathsf{A}, \mathsf{B} \rangle \triangleq \mathsf{Tr}(\mathsf{A}\mathsf{B}^\top) = \sum_{i,j} \mathsf{A}_{i,j}\mathsf{B}_{i,j}. \tag{44}$$

5. As remarked in the proof, improving the constant would require splitting the various sums involved into different asymptotic regimes and analyzing each case separately.

This inner product will be used throughout our analysis.

The formulation in (40) often serves as a starting point for constructing computationally tractable recovery algorithms, typically via relaxation. In our work, we employ semidefinite programming (SDP), a matrix-based generalization of linear programming, as one such relaxation. A standard form of an SDP is

$$\begin{aligned} \max_{\mathsf{Z}\in\mathbb{R}^{n\times n}} \quad & \langle \mathsf{C}, \mathsf{Z}\rangle \\ \text{s.t.} \quad & \mathsf{Z} \succeq 0 \\ & \langle \mathsf{B}_i, \mathsf{Z}\rangle \leq \beta_i \quad \forall i \in [m], \end{aligned} \tag{45}$$

for a collection of matrices $\mathsf{C}, \{\mathsf{B}_i\}_{i=1}^m$, constants $\{\beta_i\}_{i=1}^m$, and $m \in \mathbb{N}$. Here, $\mathsf{Z} \succeq 0$ means that $\mathsf{Z}$ is symmetric and positive semidefinite. SDPs are convex optimization problems and therefore admit efficient solution techniques, including interior-point methods and first-order algorithms; see, for instance, Nesterov and Nemirovskii (1994); Boyd and Vandenberghe (2004). A typical use case is to approximate solutions to optimization problems with nonconvex constraints, such as integer programs, by relaxing them into a semidefinite form.

Recall that the nuclear norm $\|\mathsf{B}\|_\star$ of a matrix $\mathsf{B}$ is defined as the $\ell_1$-norm of its singular value vector. When $\mathsf{B}$ is symmetric, this equals the sum of the absolute values of its eigenvalues. Slightly abusing notation, we denote by $\|\mathsf{G}\|_\star$ the nuclear norm of the adjacency matrix associated with a graph $\mathsf{G}$, interpreted as the adjacency matrix of the subgraph induced on $|v(\mathsf{G})|$ vertices. Importantly, if $|v(\mathsf{G})| \leq n$, then padding the adjacency matrix with zeros to embed $\mathsf{G}$ into an $n$-vertex graph does not change its nuclear norm.

To facilitate the discussion, we associate the planted subgraph $\Gamma^\star$ with a clustered adjacency matrix $\mathsf{X}^\star \in \{0,1\}^{n\times n}$, defined such that $\mathsf{X}^\star_{ii} = 0$ for all $i \in [n]$, $\mathsf{X}^\star_{ij} = 1$ if $(i,j) \in e(\Gamma)$, and $\mathsf{X}^\star_{ij} = 0$ otherwise. Thus, $\mathsf{X}^\star$ can be interpreted as the adjacency matrix of $\Gamma^\star$ embedded in a graph on $n$ vertices, where the additional $n - |v(\Gamma^\star)|$ vertices are isolated. Note that there is a one-to-one correspondence between each $\Gamma^\star$ and its matrix representation $\mathsf{X}^\star$.

Our main result depends on spectral and structural properties of real-valued symmetric matrices. Let $\mathsf{Q}$ be a generic real-valued symmetric matrix. Denote its singular value decomposition (SVD) by $\mathsf{Q} = \mathsf{U}_\mathsf{Q}\Sigma\mathsf{U}_\mathsf{Q}^T$, where $\Sigma \triangleq \mathsf{Diag}(\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^{r\times r}$ contains the (non-negative) singular values $\{\sigma_i\}_{i=1}^r$. The columns of $\mathsf{U}_\mathsf{Q} = [u_1, \ldots, u_r] \in \mathbb{R}^{n\times r}$ are orthonormal and are referred to as the singular vectors. The quantity $r \triangleq \mathsf{rank}(\mathsf{Q})$ denotes the rank of $\mathsf{Q}$.

In our setting, $\mathsf{Q}$ will exhibit a block structure; for example,

$$\mathsf{Q} = \begin{bmatrix} \mathsf{Q}_\mathcal{S} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \tag{46}$$

where $\mathsf{Q}_\mathcal{S}$ denotes the restriction of $\mathsf{Q}$ to $\mathcal{S} \subseteq [n]$. Accordingly, $\mathsf{U}_\mathsf{Q}$ is supported on $\mathcal{S}$, in the sense that $[\mathsf{U}_\mathsf{Q}]_{i,:} = 0$ for all $i \notin \mathcal{S}$. Finally, we define the *coherence parameter* of $\mathsf{U}_\mathsf{Q}$, which measures how "spread out" (or "localized") its rows are:

$$\mathsf{coh}(\mathsf{U}_\mathsf{Q}) \triangleq \frac{n}{\mathsf{rank}(\mathsf{Q})} \cdot \max_{i\in[n]} \|[\mathsf{U}_\mathsf{Q}]_{i,:}\|_2^2 = \frac{n}{\mathsf{rank}(\mathsf{Q})} \cdot \|[\mathsf{U}_\mathsf{Q}]_{i,:}\|_{2,\infty}^2. \tag{47}$$

We prove in Appendix G.4 that $\frac{n}{|\mathcal{S}|} \leq \mathsf{coh}(\mathsf{U}_\mathsf{Q}) \leq \frac{n}{\mathsf{rank}(\mathsf{Q})}$.[6] With a slight abuse of notation, we also write $\mathsf{coh}(\mathsf{Q}) = \mathsf{coh}(\mathsf{U}_\mathsf{Q})$. In our presentation and analysis, it is convenient to work with the

6. Note that the lower bound is achieved when $\Gamma^\star$ is a clique; in this case $\mathsf{X}^\star$ has rank one with $\mathsf{U} = \frac{1}{\sqrt{|v(\Gamma^\star)|}}\mathbf{1}_{v(\Gamma^\star)}$, and hence $\mathsf{coh}(\mathsf{U}) = \frac{n}{|v(\Gamma^\star)|}$.

following transformation of $\mathsf{A}$:

$$\mathsf{W}_{ij} \triangleq \begin{cases} q^{-1}\mathsf{A}_{ij} - 1 & i \neq j \\ 0 & i = j. \end{cases} \tag{48}$$

To present the proposed algorithm and its performance guarantees, we introduce a few additional pieces of notation. Fix a hyper-parameter $\alpha \in \mathbb{R}_+$. For any symmetric matrix $\mathsf{X} \in \mathbb{R}^{n\times n}$ with $\mathsf{Diag}(\mathsf{X}) = \mathbf{0}$, and any vector $\mathbf{s} \in \mathbb{R}^n$, define

$$\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha) \triangleq \mathsf{X} + \alpha \cdot \mathsf{Diag}(\mathbf{s}). \tag{49}$$

Let $\mathbf{s}^\star = \mathsf{supp}(v(\Gamma^\star))$ denote the support of the vertex set of the planted subgraph. At $(\mathsf{X}, \mathbf{s}) = (\mathsf{X}^\star, \mathbf{s}^\star)$, the matrix $\mathsf{Ext}(\mathsf{X}^\star, \mathbf{s}^\star; \alpha)$ is therefore a diagonally shifted version of the adjacency matrix of $\Gamma^\star$. We denote the nuclear norm of the diagonally shifted version of an arbitrary planted subgraph $\Gamma \in \mathcal{S}_\Gamma$ by

$$\|\mathsf{Ext}(\mathsf{X}^\star, \mathbf{s}^\star; \alpha)\|_\star = \|\Gamma + \alpha \cdot \mathsf{Diag}(v(\Gamma))\|_\star . \tag{50}$$

This quantity is an a priori known constant, taking the same value for all $\Gamma \in \mathcal{S}_\Gamma$, and is independent of the particular latent planted subgraph.

We now consider the following convex optimization problem:

$$\begin{aligned} \hat{\mathsf{X}}^{(\alpha)}_{\text{con}} = \underset{\mathsf{X}\in\mathbb{R}^{n\times n}, \mathbf{s}\in[0,1]^n}{\arg\max} \quad & \langle \mathsf{W}, \mathsf{X}\rangle \\ \text{s.t.} \quad & \mathbf{s}^\top \mathbf{1} = |v(\Gamma)|, \ \mathsf{X}_{ij} \leq \min(\mathbf{s}_i, \mathbf{s}_j), \ \forall i \neq j \\ & \|\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha)\|_\star \leq \|\Gamma + \alpha \cdot \mathsf{Diag}(v(\Gamma))\|_\star \\ & \mathbf{0} \leq \mathsf{X} \leq \mathbf{J}, \ \mathsf{X} = \mathsf{X}^\top, \ \mathsf{X}_{ii} = 0, \ \forall i \in [n] \\ & \langle \mathbf{J}, \mathsf{X}\rangle = 2|e(\Gamma)|, \end{aligned} \tag{51}$$

where the inequality $\mathbf{0} \leq \mathsf{X} \leq \mathbf{J}$ is to be interpreted entrywise. In several special cases, such as when $\Gamma$ is a clique or a bipartite graph, the program in (51) can be viewed as a direct relaxation of the maximum-likelihood estimator.

Before proceeding, we briefly interpret the variables and constraints in (51). The vector $\mathbf{s} \in [0,1]^n$ serves as a relaxed selector for the vertex set of the planted subgraph: each coordinate $\mathbf{s}_i$ indicates whether vertex $i$ belongs to $v(\Gamma^\star)$. At the planted solution, $\mathbf{s}^\star_i = \mathbb{1}\{i \in v(\Gamma^\star)\}$. Similarly, $\mathsf{X} \in \mathbb{R}^{n\times n}$ is a convex relaxation of the planted adjacency matrix $\mathsf{X}^\star$. One might ask why the auxiliary variable $\mathbf{s}$ is needed, rather than encoding the vertex set directly via self-loops in $\mathsf{X}$. The issue is that such an encoding can significantly weaken the discriminative power of the nuclear norm. For instance, certain subgraphs, such as complete bipartite graphs, become full-rank under this representation, rendering the nuclear-norm prior ineffective. Introducing $\mathbf{s}$ allows us to control the number of selected vertices through the linear constraint $\sum_{i\in[n]} \mathbf{s}_i = |v(\Gamma)|$, while applying the low-rank prior to the shifted matrix $\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha)$.

In addition, we impose the edge–vertex coupling constraints $\mathsf{X}_{ij} \leq \min(\mathbf{s}_i, \mathbf{s}_j)$ for all $i \neq j$, ensuring that an edge can exist only if both endpoints are selected. These are convex (McCormick envelope-type) relaxations of the nonconvex constraint $\mathsf{X}_{ij} \leq \mathbf{s}_i\mathbf{s}_j$, enforcing that $\mathsf{X}$ is supported on the selected vertex set. The remaining constraints ensure that $\mathsf{X}$ encodes a relaxed simple-graph adjacency matrix: it is symmetric, has zero diagonal, and has fixed off-diagonal mass equal to $2|e(\Gamma)|$, corresponding to the number of edges in the planted subgraph. Finally, we constrain the nuclear norm of the shifted matrix $\mathsf{Ext}(\mathsf{X}, \mathbf{s}; \alpha)$ to be no larger than that of the ground-truth planted subgraph. This promotes low-rank structure in the solution. By choosing $\alpha$ appropriately, we can avoid rank inflation caused by forced self-loops. For example, setting $\alpha = 1$ for cliques ensures that $\Gamma + \alpha \cdot \mathsf{Diag}(v(\Gamma))$ has rank one, while setting $\alpha = 0$ for complete bipartite graphs ensures it has rank two.

The program (51) is convex. The objective $\langle \mathsf{W}, \mathsf{X}\rangle$ is linear in $\mathsf{X}$, and all constraints on $\mathsf{X}$ and $\mathbf{s}$ are linear, except for the nuclear-norm constraint. The latter defines a convex set because $\|\cdot\|_\star$ is a convex norm and $\mathsf{Ext}(\cdot, \cdot; \alpha)$ is affine. Moreover, the nuclear-norm constraint admits a standard semidefinite representation, allowing (51) to be formulated as a semidefinite program (SDP). In particular, the following classical result applies.

**Lemma 21 ((Fazel, 2002, Lemma 2))** *Fix $t \geq 0$. Let $\mathsf{X} \in \mathbb{R}^{m\times n}$. Then, $\|\mathsf{X}\|_\star \leq t$ if and only if there exist $\mathsf{P}_1 \in \mathbb{R}^{m\times m}$ and $\mathsf{P}_2 \in \mathbb{R}^{n\times n}$ such that,*

$$\begin{bmatrix} \mathsf{P}_1 & \mathsf{X} \\ \mathsf{X}^\top & \mathsf{P}_2 \end{bmatrix} \succeq 0 \quad \text{and} \quad \mathsf{Tr}(\mathsf{P}_1) + \mathsf{Tr}(\mathsf{P}_2) \leq 2t. \tag{52}$$

As a result, since all constraints in (51) consist of linear equalities, linear inequalities, and linear matrix inequalities, the program is a semidefinite program (SDP), and therefore admits a polynomial-time solution via standard convex optimization methods. We now state the following result.

**Theorem 22 (Efficient algorithm)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, $\alpha \in \mathbb{R}_+$, and assume $p_n, q_n = \Theta(1)$. Define the diagonally shifted adjacency matrix $\mathsf{S}_\Gamma^{(\alpha)} \triangleq \mathsf{X}^\star + \alpha \cdot \mathsf{Diag}(v(\Gamma^\star))$. Exact recovery of $\mathsf{X}^\star$, using the convex program $\hat{\mathsf{X}}_{\mathrm{con}}$ in (51), is possible if,*

$$\mathsf{coh}(\mathsf{S}_\Gamma^{(\alpha)}) \cdot \mathsf{rank}(\mathsf{S}_\Gamma^{(\alpha)}) \leq \min\left\{c_1\sqrt{n}, c_2\frac{n}{\sqrt{|v(\Gamma)|\log n}}\right\}, \tag{53}$$

*for some constants $c_1, c_2 > 0$.*

To appreciate Theorem 22 let us consider a few examples.

**Example 4 (Planted clique)** *Consider the case where $\Gamma = \mathcal{K}_k$ is a clique with $k \triangleq |v(\mathcal{K}_k)|$ vertices. The adjacency $\mathsf{X}^\star$ in this case has eigenvalues $k-1$ and $-1$ with multiplicity $k-1$. Hence, if we take $\alpha = 1$, we get that the shifted matrix*

$$\mathsf{S}_{\mathcal{K}_k}^{(1)} = \mathsf{X}^\star + \mathsf{Diag}(v(\mathcal{K}_k)), \tag{54}$$

*has rank one, namely, $\mathsf{rank}(\mathsf{S}_{\mathcal{K}_k}^{(1)}) = 1$. Furthermore, the corresponding singular vector $\mathsf{U}$ associated with $\mathsf{S}_{\mathcal{K}_k}^{(1)}$ has a single column equal to $k^{-1/2}\mathbf{1}_{v(\mathcal{K}_k)}$. This implies that $\max_i \|\mathsf{U}_{i,:}\|_2^2 = \frac{1}{k}$, and thus,*

$$\mathsf{coh}(\mathsf{S}_{\mathcal{K}_k}^{(1)}) = \mathsf{coh}(\mathsf{U}) = \frac{n}{\mathsf{rank}(\mathsf{S}_{\mathcal{K}_k}^{(1)})} \cdot \max_{i\in[n]} \|\mathsf{U}_{i,:}\|_2^2 = \frac{n}{k}. \tag{55}$$

*Accordingly, it can be seen that Theorem 22 implies that strong recovery using (51) is possible provided that $k \geq C\sqrt{n}$, for some $C > 0$. This is consistent with how state-of-the-art recovery algorithms perform on the planted clique problem, e.g., Alon et al. (1998); Dekel et al. (2014); Montanari (2015); Hajek et al. (2016); Chen and Xu (2016).*

**Example 5 (Complete bipartite)** *Consider the case where* $\Gamma = \mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}$ *is a complete bipartite graph with partitions of size* $k_\mathsf{L}$ *and* $k_\mathsf{R}$*, and define* $k \triangleq k_\mathsf{L} + k_\mathsf{R}$*. Let us choose* $\alpha = 0$*, and then* $\mathsf{S}^{(0)}_{\mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}} = \mathsf{X}^\star$ *is of the form*

$$\mathsf{S}^{(0)}_{\mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}} = \begin{bmatrix} \mathbf{0} & \mathbf{J}_{k_\mathsf{L},k_\mathsf{R}} \\ \mathbf{J}_{k_\mathsf{R},k_\mathsf{L}} & \mathbf{0} \end{bmatrix}. \tag{56}$$

*Thus, we see that* $\mathsf{S}^{(0)}_{\mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}}$ *has rank 2, i.e.,* $\mathsf{rank}(\mathsf{S}^{(0)}_{\mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}}) = 2$*, with non-zero eigenvalues* $\{\sqrt{k_\mathsf{L}k_\mathsf{R}}, -\sqrt{k_\mathsf{L}k_\mathsf{R}}\}$*. The projector* $\mathsf{P} \triangleq \mathsf{U}\mathsf{U}^\top$ *onto this 2-dimensional subspace satisfies*

$$[\mathsf{P}]_{ii} = \begin{cases} \frac{1}{k_\mathsf{L}}, & i \in \textsf{left side}, \\ \frac{1}{k_\mathsf{R}}, & i \in \textsf{right side}, \end{cases} \tag{57}$$

*which implies that* $\max_i [\mathsf{P}]_{ii} = \frac{1}{\min\{k_\mathsf{L},k_\mathsf{R}\}}$*. Therefore*

$$\mathsf{coh}(\mathsf{S}^{(0)}_{\mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}}) \cdot \mathsf{rank}(\mathsf{S}^{(0)}_{\mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}}) = \frac{n}{\min\{k_\mathsf{L},k_\mathsf{R}\}}. \tag{58}$$

*Accordingly, it can be seen that Theorem 22 implies that strong recovery using (51) is possible provided that*

$$\min\{k_\mathsf{L},k_\mathsf{R}\} \geq C_1\sqrt{n} \quad \textsf{and} \quad \frac{\min\{k_\mathsf{L},k_\mathsf{R}\}}{\sqrt{\max\{k_\mathsf{L},k_\mathsf{R}\}}} \geq C_2\sqrt{\log n}, \tag{59}$$

*for some* $C_1, C_2 > 0$*. In the balanced case, where* $k_\mathsf{L} \approx k_\mathsf{R}$*, the condition at the left-hand side of (59) dominates, and exact recovery is possible once* $\min\{k_\mathsf{L},k_\mathsf{R}\} \geq C_1\sqrt{n}$*. This is consistent with how state-of-the-art recovery algorithms perform on the planted balanced bipartite (or bi-clique) problem, e.g., Levanzov (2018); Kumar et al. (2022a).*

**Example 6 (Balanced Turán graph)** *Consider the case where* $\Gamma = \mathcal{T}(k,r)$ *is the balanced Turán graph, namely, a complete* $r$*-partite graph on* $k = |v(\Gamma)|$ *vertices that avoids a* $(r+1)$*-clique. Denote the size of each partition by* $m \triangleq k/r$*. Let us choose* $\alpha = 0$*. As is well-known, the adjacency matrix* $\mathsf{S}^{(0)}_{\mathcal{T}(k,r)} = \mathsf{X}^\star$ *has the following eigenvalues (see, e.g., Nikiforov (2017)):* $(r-1)m$ *with multiplicity 1,* $-m$ *with multiplicity* $r-1$*, and* $0$ *with multiplicity* $k-r$*. Thus,* $\mathsf{rank}(\mathsf{S}^{(0)}_{\mathcal{T}(k,r)}) = r$*. The range is spanned by the* $r$ *part-indicator vectors. As in the bipartite case, for any vertex* $i \in \Gamma$*, we have* $[\mathsf{P}]_{ii} = \frac{1}{m}$*, and thus,* $\max_i [\mathsf{P}]_{ii} = \frac{1}{m}$*. Hence*

$$\mathsf{coh}(\mathsf{S}^{(0)}_{\mathcal{T}(k,r)}) \cdot \mathsf{rank}(\mathsf{S}^{(0)}_{\mathcal{T}(k,r)}) = \frac{n}{m} = \frac{nr}{k}. \tag{60}$$

*Accordingly, it can be seen that Theorem 22 implies that strong recovery using (51) is possible provided that* $k \geq \max(C_1 r\sqrt{n}, C_2 r^2 \log n)$*.*

**Example 7 (Union of disjoint cliques)** *Consider the case where* $\Gamma$ *is a union of* $L$ *disjoint cliques with sizes sizes* $k_1, \ldots, k_L$*. Define* $k \triangleq \sum_{i=1}^L k_i$ *and* $k_{\min} \triangleq \min_{i\in[L]} k_i$*. Take* $\alpha = 1$*. Similarly to Example 4, it can be shown that* $\mathsf{rank}(\mathsf{S}^{(1)}_\Gamma) = L$*, and that*

$$\mathsf{coh}(\mathsf{S}^{(1)}_\Gamma) \cdot \mathsf{rank}(\mathsf{S}^{(1)}_\Gamma) = \frac{n}{k_{\min}}. \tag{61}$$

*Accordingly, Theorem 22 implies that strong recovery using (51) is possible provided that* $k_{\min} \geq \max(C_1\sqrt{n}, C_1\sqrt{k\log n})$*.*

**Example 8 (Triangular graph)** *Consider the case where* $\Gamma = T(m)$ *is the triangular graph, namely, the line graph of* $\mathcal{K}_m$. *The triangular graph* $T(m)$ *has vertex set* $\binom{[m]}{2}$, *and thus* $k = |v(\Gamma)| = \binom{m}{2}$, *and edges between pairs of* 2*-subsets that intersect in exactly one element. It is the Johnson graph* $J(m,2)$ *(a strongly regular, vertex-transitive graph). It is well-known that the eigenvalues of its adjacency matrix* $\mathsf{X}^\star$ *are given by (see, e.g., Nikiforov (2017)):*

$$\mathsf{Spectrum}(\mathsf{X}^\star) = \Bigg\{\underbrace{2(m-2)}_{\text{mult. } 1}, \underbrace{m-4}_{\text{mult. } m-1}, \underbrace{-2}_{\text{mult. } (m-1)(m-2)/2}\Bigg\}. \tag{62}$$

*Taking* $\alpha = 2$ *we obtain*

$$\mathsf{Spectrum}(\mathsf{S}^{(2)}_{T(m)}) = \Bigg\{\underbrace{2m-2}_{\text{mult. } 1}, \underbrace{m-2}_{\text{mult. } m-1}, \underbrace{0}_{\text{mult. } (m-1)(m-2)/2}\Bigg\}. \tag{63}$$

*Hence* $\mathsf{rank}(\mathsf{S}^{(2)}_{T(m)}) = m$. *Because* $T(m)$ *is vertex-transitive and the range of* $\mathsf{S}^{(2)}_{T(m)}$ *is a direct sum of irreducible representations invariant under* $\mathrm{Aut}(T(m)) \cong \mathbb{S}_m$, *the projector* $\mathsf{P} = \mathsf{U}\mathsf{U}^\top$ *has constant diagonal:*

$$[\mathsf{P}]_{ii} \equiv \frac{\mathsf{rank}(\mathsf{S}^{(2)}_{T(m)})}{k} = \frac{m}{\binom{m}{2}} = \frac{2}{m-1} \ \forall i \in [k]. \tag{64}$$

*Indeed, conjugating* $\mathsf{P}$ *by any permutation matrix induced by an automorphism leaves* $\mathsf{P}$ *invariant; by vertex-transitivity, all diagonal entries must match, and* $\mathsf{trace}(\mathsf{P}) = \mathsf{rank}(\mathsf{S}^{(2)}_{T(m)})$ *gives (64). Therefore*

$$\mathsf{coh}(\mathsf{S}^{(2)}_{T(m)}) \cdot \mathsf{rank}(\mathsf{S}^{(2)}_{T(m)}) = \frac{2n}{m-1}. \tag{65}$$

*Thus, it is not difficult to check that Theorem 22 implies that strong recovery using (51) is possible provided that* $k \geq C\sqrt{n}$, *for some* $C > 0$.

## C.3. Computational lower bounds

In this subsection, we derive computational lower bounds for recovery. We begin by introducing the low-degree polynomial (LDP) framework for *recovery*, setting up the notation used throughout, and recording several basic identities that will be invoked later. Our presentation adapts the general LDP methodology to the planted-graph model from Section B, and follows Schramm and Wein (2022) (see also Bandeira et al. (2022)).

### C.3.1. Background and preliminaries

**Problem setup.** Fix $n$, $0 < q_n < p_n \leq 1$, and a planted structure $\Gamma_n = (v(\Gamma_n), e(\Gamma_n))$ with $|v(\Gamma_n)| \leq n$ and no isolated vertices. A planted copy $\Gamma_n^\star \in \mathcal{S}_{\Gamma_n}$ is selected uniformly at random and then observed through the binary edge–channel

$$\mathsf{Y}_e | \Gamma_n^\star \sim \mathsf{Bern}(\mathsf{X}_e), \qquad \mathsf{X}_e = \begin{cases} p_n, & e \in e(\Gamma_n^\star) \\ q_n, & e \notin e(\Gamma_n^\star), \end{cases} \tag{66}$$

independently over $e \in \binom{[n]}{2}$. We write $N \triangleq \binom{n}{2}$ and view $\mathsf{Y} \in \{0,1\}^N$ as the input to any estimator $\hat{\Gamma} : \{0,1\}^N \to \mathcal{S}_{\Gamma_n}$.

**LDP and performance metrics.** We observe a random vector $\mathsf{Y} \in \mathcal{Y}^N$ and seek to estimate an *anchor* scalar parameter $x \in \mathbb{R}$, defined as a function of the latent signal generating $\mathsf{Y}$ (e.g., one coordinate of the signal). For a degree budget $D \in \mathbb{N}$, let

$$\mathbb{R}[\mathsf{Y}]_{\leq D} \triangleq \left\{ f : \mathcal{Y}^N \to \mathbb{R} \text{ polynomial of total degree at most } D \right\}, \tag{67}$$

namely, the space of polynomials of degree at most $D$. The *degree-$D$ minimum mean-squared error* is defined as

$$\mathsf{MMSE}_{\leq D} \triangleq \inf_{f \in \mathbb{R}[\mathsf{Y}]_{\leq D}} \mathbb{E}\left[ (f(\mathsf{Y}) - x)^2 \right], \tag{68}$$

where the expectation is taken with respect to the joint law of $(x, \mathsf{Y})$. Furthermore, define the associated *degree-$D$ maximum correlation* as

$$\mathsf{Corr}_{\leq D} \triangleq \sup_{f \in \mathbb{R}[\mathsf{Y}]_{\leq D}} \frac{\mathbb{E}\left[ f(\mathsf{Y}) \cdot x \right]}{\sqrt{\mathbb{E}\left[ f(\mathsf{Y})^2 \right]}}. \tag{69}$$

A basic identity links these two quantities:

$$\mathsf{MMSE}_{\leq D} = \mathbb{E}[x^2] - \mathsf{Corr}_{\leq D}^2. \tag{70}$$

This identity allows us to lower bound $\mathsf{MMSE}_{\leq D}$ (and hence derive computational lower bounds) by upper bounding $\mathsf{Corr}_{\leq D}$. Polynomials of degree $D = \mathsf{polylog}(n)$ capture a broad class of efficient estimators, including many spectral and AMP-type procedures via polynomial approximation, and the LDP framework has proved predictive for computational thresholds across a range of high-dimensional problems Schramm and Wein (2022). In particular, if the right-hand side of (69) is $o(1)$ for some $D = \mathsf{polylog}(n)$, then $\mathsf{Corr}_{\leq D} = o(1)$ and hence $\mathsf{MMSE}_{\leq D} \geq \mathbb{E}[x^2] - o(1)$, ruling out any degree-$D$ polynomial from achieving nontrivial recovery of $x$. This yields concrete, model-specific computational lower bounds that often match (up to polylog factors) the best known algorithmic thresholds in classical inference problems.

To study recovery via low-degree polynomials, we focus on a one-bit anchor that is symmetric across the vertices of the planted copy. Fix an *ambient anchor vertex* $v^\star = \{1\}$ and define

$$x \triangleq \mathbb{1}\{v^\star \in v(\Gamma_n^\star)\} \in \{0, 1\}. \tag{71}$$

Because $\Gamma_n^\star$ is drawn uniformly among all of its isomorphic copies, the joint law of $(x, \mathsf{Y})$ does not depend on which ambient vertex is chosen, and the anchor is without loss of generality.

**Binary observation model.** The planted graph problem in this paper fits the general binary model considered in (Schramm and Wein, 2022, Sec. 2.3), where coordinates of $\mathsf{Y}$ are conditionally independent Bernoulli variables with means $\mathsf{X} = (\mathsf{X}_i)_{i \in [N]}$ taking values in $\{\tau_0, \tau_1\}$ with $0 < \tau_0 < \tau_1 < 1$. In this setting, (Schramm and Wein, 2022, Thm. 2.7) derived an upper bound on $\mathsf{Corr}_{\leq D}$ expressed in terms of *joint cumulants*:

$$\mathsf{Corr}_{\leq D}^2 \leq \sum_{\substack{\alpha \subseteq \binom{[n]}{2} \\ |\alpha| \leq D}} \frac{\kappa_\alpha^2}{\left( q_n(1 - p_n) \right)^{|\alpha|}}, \tag{72}$$

where $\kappa_\alpha = \kappa(x, \{\mathsf{X}_e\}_{e\in\alpha})$ is the joint cumulant of $x$ and $\{\mathsf{X}_e\}_{e\in\alpha}$. It is shown in (Schramm and Wein, 2022, Thm. 2.7) that the cumulants $\{\kappa_\alpha\}$ are defined recursively as

$$\kappa_\alpha = \mathbb{E}[x \cdot \mathsf{X}^\alpha] - \sum_{0\leq\beta\prec\alpha} \kappa_\beta \mathbb{E}[\mathsf{X}^{\alpha-\beta}], \tag{73}$$

$$\mathsf{X}^\alpha \triangleq \prod_{e:\ \alpha_e=1} \mathsf{X}_e, \tag{74}$$

with $\beta \prec \alpha$ meaning coordinate–wise inequality and $\beta \neq \alpha$.

### C.3.2. MAIN RESULT

The following result shows that $O(\log n)$-degree polynomials fail at recovery whenever the graph-density satisfies $\eta(\Gamma_n) = \frac{|e(\Gamma_n)|}{|v(\Gamma_n)|} \ll \sqrt{n}$. For reference, the trivial estimator $f(\mathsf{Y}) = \mathbb{E}[x]$ attains mean-squared error $\mathbb{E}[f(\mathsf{Y}) - x]^2 = \mathsf{Var}(x)$. The proof is given in Section F.

**Theorem 23 (Computational lower bound)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, and assume $p_n, q_n = \Theta(1)$. If*

$$\eta(\Gamma_n) \leq n^{\frac{1}{2}-\varepsilon}, \tag{75}$$

*for any fixed $\varepsilon > 0$, and $D = D_n$ scales as $D \leq (\log n)^\alpha$ for some fixed $\alpha < 1$, then*

$$\mathsf{MMSE}_{\leq D} \geq (1 - o(1)) \cdot \mathsf{Var}(x). \tag{76}$$

Next, we prove upper bounds on $\mathsf{MMSE}_{\leq D}$. To that end, following (Schramm and Wein, 2022, Sec. 4.2), we analyze one and multiple rounds of the power iteration method starting from the all-ones vector, followed by thresholding. We now describe this estimator in more detail. Fix the number of iterations $L \in \mathbb{N}$. For simplicity of notation, let $\mathsf{Z}_{ij} \triangleq \mathsf{Y}_{ij} - q$, for any $i, j \in [n]$. Let $\mathcal{P}_L$ denote the set of all simple undirected paths of length $L$ in the ambient complete graph starting at vertex 1 and visiting pairwise distinct vertices, namely, $P = (u_0, u_1, \ldots, u_\ell)$ with $u_0 = 1$. For each $P \in \mathcal{P}_L$, define

$$\mathsf{Z}(P) \triangleq \prod_{\ell=0}^{L-1} \mathsf{Z}_{u_t u_{t+1}}, \tag{77}$$

and the degree-$L$ *walk polynomial*

$$\mathsf{W}_L \triangleq \sum_{P\in\mathcal{P}_L} \mathsf{Z}(P). \tag{78}$$

For $u \in v(\Gamma)$ let

$$W_L(\Gamma; u) \triangleq \left|\{\text{simple paths of length } L \text{ in } \Gamma \text{ starting at } u\}\right|, \tag{79}$$

$$W_L^{\min}(\Gamma) \triangleq \min_{u\in v(\Gamma)} W_L(\Gamma; u). \tag{80}$$

Define the rescaled statistic

$$\mathscr{X}_L \triangleq \frac{2}{(p-q)^L} \frac{\mathsf{W}_L}{W_L^{\min}(\Gamma)}. \tag{81}$$

Our estimator is defined as

$$f_{L,m}(\mathsf{Y}) = \tau_m(\mathscr{Z}_L), \tag{82}$$

where $\tau_m$ of degree $D = 2m+1$ is a *polynomial threshold* that approximates the threshold function using a polynomial of degree $2m+1$ (see Lemma 43). In the special case of a single-iteration power method, the estimator above reduces to

$$f_{1,m}(\mathsf{Y}) = \tau_k\left(\frac{1}{(p-q)\eta(\Gamma)}\sum_{i=2}^{n}(\mathsf{Y}_{1i}-q)\right). \tag{83}$$

We have the following result.

**Theorem 24 (LDP upper bound)** *Assume $p_n, q_n = \Theta(1)$.*

1. *Single iteration: consider the estimator in (83). Fix $\epsilon > 0$, and let $\Gamma_n$ be any sequence of subgraphs with*

$$\mathsf{Dis}(\Gamma) \triangleq \max_{v\in v(\Gamma)} \frac{|d_\Gamma(v)-\eta(\Gamma)|}{\eta(\Gamma)} \le \frac{r}{12}, \tag{84}$$

$$\eta(\Gamma_n) \ge n^{\frac{1}{2}+\epsilon}, \tag{85}$$

*for some fixed $0 < r < 1$ and all sufficiently large $n$. If $D = D(n) \le (\log n)^\alpha$ for any fixed $\alpha > 0$, then*

$$\mathbb{E}\left[(f_{1,m}(\mathsf{Y})-x)^2\right] \le CD^2 r^{D-1}, \tag{86}$$

*for some $C > 0$.*

2. *Multiple iteration: consider the estimator in (82). Fix $L \in \mathbb{N}$ and let $r \in (0, 1/4]$. Let $\Gamma_n$ be any sequence of subgraphs with*

$$W_L^{\min}(\Gamma) \ge C^\star(L,p,q)\left[n^{L/2}+k^{L-1/2}\right]\sqrt{\log n}, \tag{87}$$

*for some $C^\star(L,p,q) > 0$. Assume that $m = \omega(1)$ and $D \le C\log\log n$, for some constant $C > 0$. Then*

$$\mathbb{E}\left[(f_{L,m}(\mathsf{Y})-x)^2\right] \le (\log n)^{-\Omega(1)}. \tag{88}$$

We see that for "almost-regular" structures satisfying (84), the single-iteration bound complements the computational lower bound in Theorem 23. However, there are natural examples for which (84) fails. We next discuss a few such examples to further illustrate the LDP-based lower and upper bounds.

**Example 9 (Planted clique)** *Consider the case where $\Gamma_n = \mathcal{K}_k$. Since $\frac{|e(\Gamma_n)|}{|v(\Gamma_n)|} = \frac{|v(\Gamma_n)|-1}{2}$, Theorem 23 implies that recovery is computationally hard whenever $|v(\Gamma_n)| \le n^{\frac{1}{2}-\varepsilon}$, matching the state-of-the-art algorithmic threshold (see Example 4) and consistent with the folklore planted clique conjecture. Moreover, the first item of Theorem 24 shows that a single iteration of the power method, followed by thresholding, achieves this computational barrier: the regularity condition in (84) is clearly satisfied, and (85) reduces to $|v(\Gamma_n)| \ge n^{\frac{1}{2}+\varepsilon}$. Finally, we note that the multi-iteration bound is also applicable. In this case, $W_L(\Gamma; u) = (k-1)_L \sim k^L$, and hence Theorem 24 implies that recovery is possible whenever $k \gtrsim \sqrt{n}(\log n)^{1/(2L)}$.*

**Example 10 (Complete bipartite clique)** *Consider the case where $\Gamma_n = \mathcal{K}_{k_\mathsf{L},k_\mathsf{R}}$ is a complete bipartite graph with partitions of size $k_\mathsf{L}$ and $k_\mathsf{R}$, and define $k = k_\mathsf{L} + k_\mathsf{R}$. Here $\frac{|e(\Gamma_n)|}{|v(\Gamma_n)|} = \frac{k_\mathsf{L}\cdot k_\mathsf{R}}{k_\mathsf{L}+k_\mathsf{R}} \in [\min\{k_\mathsf{L}, k_\mathsf{R}\}/2, \min\{k_\mathsf{L}, k_\mathsf{R}\}]$. Accordingly, Theorem 23 implies that recovery is computationally hard whenever $\min\{k_\mathsf{L}, k_\mathsf{R}\} \leq n^{\frac{1}{2}-\varepsilon}$, matching the state-of-the-art algorithmic threshold (see Example 5). As in the previous example, the first item of Theorem 24 shows that a single iteration of the power method, followed by thresholding, achieves this computational barrier: the regularity condition in (84) is clearly satisfied, and (85) reduces to $\min\{k_\mathsf{L}, k_\mathsf{R}\} \leq n^{\frac{1}{2}-\varepsilon}$. Finally, we note that the multi-iteration bound is also applicable. In this case, $W_1(\Gamma; u) = \min\{k_\mathsf{L}, k_\mathsf{R}\}$, and hence Theorem 24 implies that recovery is possible under the same condition.*

**Example 11 (Half clique half independent set)** *Split $v(\Gamma)$ into two sets $\mathcal{A}$ and $\mathcal{B}$ with $|\mathcal{A}| = |\mathcal{B}| = k/2$: $\mathcal{A}$ forms a clique, $\mathcal{B}$ is an independent set, and each $b \in \mathcal{B}$ is adjacent to exactly $\sqrt{k}$ vertices in $\mathcal{A}$. Accordingly, Theorem 23 implies that recovery is computationally hard whenever $k \leq n^{\frac{1}{2}-\varepsilon}$. On the other hand, the first item of Theorem 24 does not apply in this example because (84) is not satisfied. Nonetheless, for $u \in \mathcal{B}$ we have $W_L(\Gamma; u) \asymp \sqrt{k}\,(k/2)^{L-1} \asymp k^{L-\frac{1}{2}}$, and therefore the second item of Theorem 24 implies that recovery is possible whenever $k \gtrsim n^{\frac{L}{2L-1}}(\log n)^{\frac{1}{2L-1}}$, which complements our recovery lower bound for any $L = \omega(1)$.*

### C.4. Extensions

In this subsection, we present several direct extensions of the results above. We begin by showing that both the optimal and the efficient algorithms we introduce are robust to a simple monotone adversary, which is allowed to remove arbitrary edges outside the planted subgraph and add edges within it. We then develop guarantees for weaker notions of recovery.

#### C.4.1. Semi-random model

We now consider a framework for the *recovery* problem in a semi-random graph setting known as the *monotone adversary* or "Sandwich Model" introduced in Feige and Krauthgamer (2000). As before, let $\Gamma = (\Gamma_n)_n$ be a sequence of graphs with $|v(\Gamma_n)| \leq n$. To exclude trivial settings, we assume that $\Gamma_n$ contains no isolated vertices. The recovery task is to exactly identify a hidden subgraph embedded in a noisy background graph under semi-random perturbations.

We assume the data is generated according to the following semi-random process. First, we pick a subgraph $\Gamma^\star \in \mathcal{S}_\Gamma$. A random graph $\mathsf{G}$ is then formed by retaining each edge of $\Gamma^\star$ independently with probability $p$, and adding each edge outside $\Gamma^\star$ independently with probability $q$. The resulting distribution over graphs is denoted $\mathcal{G}_{\Gamma_n}(n, p, q)$. An adversary may then modify $\mathsf{G}$ by removing edges that do not belong to the planted structure $\Gamma^\star$, and by adding edges within $\Gamma^\star$. The final observed graph is denoted by $\mathsf{G}_{\mathsf{Adv}}$. We model this perturbation via a (possibly randomized) function $\mathsf{Adv}$ acting on both $\mathsf{G}$ and $\Gamma^\star$, such that $\mathsf{G}_{\mathsf{Adv}} = \mathsf{Adv}(\mathsf{G}, \Gamma^\star) \in \{0,1\}^{\binom{n}{2}}$. The adversary satisfies

$$\mathbb{P}\left[\bigcap_{(i,j)\in\binom{[n]}{2}\setminus e(\Gamma^\star)} \mathsf{A}_{\mathsf{G}_{\mathsf{Adv}}}(i,j) \leq \mathsf{A}_{\mathsf{G}}(i,j), \bigcap_{(i,j)\in e(\Gamma^\star)} \mathsf{A}_{\mathsf{G}_{\mathsf{Adv}}}(i,j) \geq \mathsf{A}_{\mathsf{G}}(i,j)\right] = 1. \tag{89}$$

We denote by $\mathcal{A}$ the collection of all such valid (possibly randomized) functions $\mathsf{Adv}$ satisfying the condition above. When randomized, these functions define conditional distributions over graphs.

Thus, the observed graph is drawn from a distribution in the family $\mathsf{Adv}(\mathcal{G}_{\Gamma_n}(n,p,q))$, induced by applying some $\mathsf{Adv} \in \mathcal{A}$ to a sample from the planted model. Our goal is to design a recovery algorithm that identifies the underlying subgraph $\Gamma^\star$ exactly.

A *recovery algorithm* is a function $\hat{\Gamma}_n : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_\Gamma$ that maps an observed graph to a candidate subgraph. The (worst-case) error of such an algorithm is given by

$$\mathsf{E}_{\mathsf{adv}}(\hat{\Gamma}) \triangleq \sup_{\Gamma^\star \in \mathcal{S}_\Gamma} \sup_{\mathsf{Adv}_1 \in \mathcal{A}_1} \mathbb{P}[\hat{\Gamma}(\mathsf{G}_{\mathsf{Adv}_1}) \neq \Gamma^\star]. \tag{90}$$

We define the worst-case optimal error as

$$\mathsf{E}^\star_{\mathsf{adv}} \triangleq \inf_{\hat{\Gamma}:\{0,1\}^{n(2)} \to \mathcal{S}_\Gamma} \mathsf{E}_{\mathsf{adv}}(\phi). \tag{91}$$

As before, we say that *exact recovery* is possible if $\limsup_{n\to\infty} \mathsf{E}^\star_{\mathsf{adv}} = 0$, and impossible otherwise.

It is straightforward to show that the optimal MLE in the vanilla setting (i.e., without an adversary) is robust to the adversarial model described above. This is summarized in the following result.

**Theorem 25 (MLE is robust)** *Let $\hat{\Gamma}_{\mathsf{MLE}}$ denote the MLE for the recovery problem in Section B. Then, whenever $\hat{\Gamma}_{\mathsf{MLE}} = \Gamma^\star$ is the unique solution to the recovery problem under $\mathcal{G}_{\Gamma_n}(n,p,q)$ (i.e., in the absence of an adversary), it remains the unique solution under the adversarial mode* $\mathsf{Adv}(\mathcal{G}_{\Gamma_n}(n,p,q))$.

In other words, $\mathsf{E}_{\mathsf{adv}}(\hat{\Gamma}_{\mathsf{MLE}}) \to 0$ whenever $\mathsf{E}(\hat{\Gamma}_{\mathsf{MLE}}) \to 0$. In light of the fact that $\mathsf{E}^\star \leq \mathsf{E}^\star_{\mathsf{adv}}$, we see that, at least statistically, the monotone adversary does not alter the performance.
**Proof** [Proof of Theorem 25] Recall that the MLE is given by

$$\hat{\Gamma}_{\mathsf{MLE}}(\mathsf{G}) = \arg\max_{\Gamma' \in \mathcal{S}_\Gamma} \left| e(\Gamma') \cap \mathsf{G} \right|. \tag{92}$$

Denote the corresponding $\mathsf{G}$ after modification as $\mathsf{G}_{\mathsf{adv}}$. Then, for all feasible $\Gamma \neq \Gamma^\star$, we have the following chain of inequalities:

$$\begin{aligned}
|e(\Gamma) \cap \mathsf{G}_{\mathsf{adv}}| &= |e(\Gamma \cap (\Gamma^\star)^c) \cap \mathsf{G}_{\mathsf{adv}}| + |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| && (93)\\
&\leq |e(\Gamma \cap (\Gamma^\star)^c) \cap \mathsf{G}| + |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| && (94)\\
&= |e(\Gamma) \cap \mathsf{G}| + |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| - |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}| && (95)\\
&< |e(\Gamma^\star) \cap \mathsf{G}| + |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| - |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}| && (96)\\
&= |e(\Gamma^c \cap \Gamma^\star) \cap \mathsf{G}| + |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| && (97)\\
&\leq |e(\Gamma^c \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| + |e(\Gamma \cap \Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}| && (98)\\
&= |e(\Gamma^\star) \cap \mathsf{G}_{\mathsf{adv}}|, && (99)
\end{aligned}$$

where the second inequality follows from the fact that outside the planted subgraph (i.e., in $\Gamma \cap (\Gamma^\star)^c$), the monotone adversary can only remove edges. Therefore, replacing $\mathsf{G}_{\mathsf{adv}}$ with $\mathsf{G}$ can only increase the intersection. The second *strict* inequality holds due to the assumption that $\Gamma^\star$ is the unique global maximizer of the original problem in the absence of an adversary. The third inequality follows from the fact that within the planted subgraph (i.e., in $\Gamma^c \cap \Gamma^\star$), the adversary can only add edges, so replacing $\mathsf{G}$ with $\mathsf{G}_{\mathsf{adv}}$ can only increase the intersection. Hence, $\Gamma^\star$ stays optimal even when the graph is modified by the adversary. ■

Next, we show that the computationally efficient algorithm proposed in (51) is also robust to a monotone adversary.

**Theorem 26 (Convex program is robust)** *Let $\hat{\mathsf{X}}^{(\alpha)}_{\mathsf{con}}$ denote the convex program in (51). Then, whenever $\Gamma^\star$ is the unique solution of (51) to the recovery problem under $\mathcal{G}_{\Gamma_n}(n,p,q)$ (i.e., in the absence of an adversary), it remains the unique solution under the adversarial mode $\mathsf{Adv}(\mathcal{G}_{\Gamma_n}(n,p,q))$.*

**Proof** [Proof of Theorem 26] The proof follows the same ideas as in the proof of Theorem 25. Denote the corresponding $\mathsf{W}$ after modification as $\mathsf{W}_{\mathsf{adv}}$. Also, recall that $\mathsf{X}^\star$ denotes the adjacency matrix of the planted subgraph $\Gamma^\star$. Let $\mathsf{X}^\star_{\mathsf{comp}}$ denote the adjacency matrix of $(\Gamma^\star)^c$, and for any feasible $\mathsf{X}$ let $\mathsf{X}_{\mathsf{comp}} \triangleq \mathbf{J} - \mathsf{X}$. Finally, recall that $\odot$ denotes the element-wise Hadamard product. Then, for all feasible $\mathsf{X} \neq \mathsf{X}^\star$, we have the following chain of inequalities:

$$
\begin{aligned}
\langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}\rangle &= \left\langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}\odot \mathsf{X}^\star_{\mathsf{comp}}\right\rangle + \langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}\odot \mathsf{X}^\star\rangle && (100)\\
&\le \left\langle \mathsf{W}, \mathsf{X}\odot \mathsf{X}^\star_{\mathsf{comp}}\right\rangle + \langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}\odot \mathsf{X}^\star\rangle && (101)\\
&= \langle \mathsf{W}, \mathsf{X}\rangle + \langle \mathsf{W}_{\mathsf{adv}} - \mathsf{W}, \mathsf{X}\odot \mathsf{X}^\star\rangle && (102)\\
&< \langle \mathsf{W}, \mathsf{X}^\star\rangle + \langle \mathsf{W}_{\mathsf{adv}} - \mathsf{W}, \mathsf{X}\odot \mathsf{X}^\star\rangle && (103)\\
&= \langle \mathsf{W}, \mathsf{X}^\star\odot \mathsf{X}_{\mathsf{comp}}\rangle + \langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}\odot \mathsf{X}^\star\rangle && (104)\\
&\le \langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}^\star\odot \mathsf{X}_{\mathsf{comp}}\rangle + \langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}\odot \mathsf{X}^\star\rangle && (105)\\
&= \langle \mathsf{W}_{\mathsf{adv}}, \mathsf{X}^\star\rangle, && (106)
\end{aligned}
$$

where the second inequality arises because outside the planted subgraph (that is, over the region $\mathsf{X}\odot\mathsf{X}^\star_{\mathsf{comp}}$), the monotone adversary is limited to deleting edges. As a result, substituting $\mathsf{W}_{\mathsf{adv}}$ with $\mathsf{W}$ can only lead to a larger objective value over this region. The second strict inequality follows from the premise that $\mathsf{X}^\star$ is the unique global maximizer of the original problem when no adversary is present. The third inequality holds because within the planted subgraph (specifically, over $\mathsf{X}^\star\odot\mathsf{X}_{\mathsf{comp}}$), the adversary is only allowed to insert edges. Consequently, replacing $\mathsf{W}$ with $\mathsf{W}_{\mathsf{adv}}$ can only increase the objective contribution over this region, implying that $\mathsf{X}^\star$ remains the optimal solution even under adversarial modifications to the graph. ■

### C.4.2. INDIVIDUAL LAYERS RECOVERY

The strict notion of exact recovery precludes recovering subgraphs with very sparse layers (e.g., as in Example 3). One way to bypass this inherent limitation is to consider weaker notions of recovery. For instance, one might be interested in exactly recovering only the subgraph $\mathsf{H}\subseteq\Gamma$ that achieves the maximum subgraph density of $\Gamma$. More generally, for any $\kappa\in[M(\Gamma)]$—where $M(\Gamma)$ denotes the number of components in the onion decomposition of $\Gamma$, i.e., $\Gamma = \bigcup_{\ell=1}^{M}\mathcal{D}^{(\ell)}$ (see Definition 16)—we could aim to exactly recover the first $\kappa$ layers of $\Gamma$, namely $\Gamma^{(\kappa)} = \bigcup_{\ell=1}^{\kappa}\mathcal{D}^{(\ell)}$. Accordingly, we define the worst-case error probability associated with an estimator $\hat{\Gamma}$ as follows:

$$
\mathsf{E}^{(\kappa)}_n(\hat{\Gamma}) \triangleq \sup_{\Gamma^\star\in\mathcal{S}_\Gamma} \mathbb{P}_{\mathcal{G}_{\Gamma^\star_n}(n,p_n,q_n)}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^{(\kappa),\star}], \tag{107}
$$

and the optimal error probability as $\mathsf{E}_n^{(\kappa),\star} = \inf_{\hat{\Gamma}} \mathsf{E}_n^{(\kappa)}(\hat{\Gamma})$. We say that a sequence of estimators $(\hat{\Gamma}_n)_{n\in\mathbb{N}}$ achieves *$\kappa$-exact recovery* if $\limsup_{n\to\infty} \mathsf{E}_n^{(\kappa)}(\hat{\Gamma}_n) = 0$. Conversely, we say that *$\kappa$-exact recovery is impossible* if $\liminf_{n\to\infty} \mathsf{E}_n^{(\kappa),\star} > 0$. Finally, we define the *truncated likelihood peeling algorithm* as

$$\hat{\Gamma}^{(\kappa)}_{\mathsf{pMLE}} = \hat{\Gamma}^{(\kappa)}_{\mathsf{MLE}}, \tag{108}$$

where $\{\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}}\}_{\ell\geq 1}$ is defined in (41). Recall that $\mu(\Gamma|\Gamma^{(\kappa-1)}) = \eta(\Gamma^{(\kappa)}|\Gamma^{(\kappa-1)})$. We have the following result.

**Theorem 27 ($\kappa$-exact recovery)** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, $\kappa \in [M(\Gamma)]$, and assume $p_n, q_n = \Theta(1)$. Consider the onion decomposition of $\Gamma$ in Definition 16.*

1. *$\kappa$-exact recovery is statistically impossible if*

$$\mu(\Gamma|\Gamma^{(\kappa-1)}) \leq \frac{(1-\varepsilon)\cdot\log n}{d_{\mathsf{KL}}(p||q)}, \tag{109}$$

   *for any $\varepsilon > 0$.*

2. *$\kappa$-exact recovery of $\Gamma^\star$, via the truncated likelihood peeling algorithm in (108), is possible if*

$$\mu(\Gamma|\Gamma^{(\kappa-1)}) \geq \mathsf{C}\frac{(1+\varepsilon)\cdot\log n}{d_{\mathsf{KL}}(p||q)}, \tag{110}$$

   *for any $\varepsilon > 0$ and some constant $\mathsf{C} > 0$.*

Going back to Example 3, if we take $\kappa = 1$, this corresponds to recovering the clique while ignoring the single edge attached to it (which precluded strong recovery). Since $\mathcal{D}^{(1)} = K_k$, Theorem 27 tells us that 1-exact recovery is possible (or impossible) when $k \gtrless C\log n$ for some $C > 0$, just as in the planted clique recovery problem. Finally, the proof of Theorem 27 follows verbatim from the arguments used in the proofs of Theorems 19 and 20, with the recovery restricted to $\Gamma^{(\kappa)}$. The details are therefore omitted.

### C.4.3. ALMOST-EXACT RECOVERY

Next, we turn to the notion of *almost-exact recovery*, defined as follows (see, e.g., Hajek et al. (2017); Wu and Xu (2020)).

**Definition 28 (Almost-exact recovery)** *An estimator $\hat{\Gamma}$ almost-exactly recovers $\Gamma^\star$ if, as $n \to \infty$, $\mathsf{d_H}(\hat{\Gamma}, \Gamma^\star)/|e(\Gamma)| \to 0$ in probability, where $\mathsf{d_H}$ denotes the Hamming distance between the adjacency matrices of $\hat{\Gamma}$ and $\Gamma^\star$.*

In general, lower bounds for exact recovery do not directly imply lower bounds for almost-exact recovery, but any estimator that achieves exact recovery also achieves almost-exact recovery. To state our main results, we first introduce some notation. Consider the onion decomposition of $\Gamma_n$ in Definition 16, and define for $\ell \in 0 \cup [M(\Gamma)]$ and $n \in \mathbb{N}$ the $\ell^{\text{th}}$ *leftover-edge fraction* as

$$\mathsf{Res}_\ell^{(n)} \triangleq \frac{|e(\Gamma_n \setminus \Gamma_n^{(\ell)})|}{|e(\Gamma_n)|}. \tag{111}$$

Fix any null sequence $\varepsilon_n \downarrow 0$ and define

$$\ell_{\mathsf{LB}}(n) \triangleq \max\{\ell \in 0 \cup [M(\Gamma)] : \ \mathsf{Res}_{\ell}^{(n)} > \varepsilon_n\}, \tag{112}$$

$$\ell_{\mathsf{UB}}(n) \triangleq \min\{\ell \in 0 \cup [M(\Gamma)] : \ \mathsf{Res}_{\ell}^{(n)} \leq \varepsilon_n\}. \tag{113}$$

Heuristically, $\ell_{\mathsf{LB}}$ is the last index for which the leftover-edge fraction is $\Omega(1)$, whereas $\ell_{\mathsf{UB}}$ is the first index for which the leftover-edge fraction is $o(1)$. Since $\mathsf{Res}_{\ell}^{(n)}$ is nonincreasing, it follows that $\ell_{\mathsf{UB}}(n) = \ell_{\mathsf{LB}}(n) + 1$. We nevertheless introduce both notations for clarity.

**Lower bounds.** We begin with our impossibility results. For almost-exact recovery, an interesting dichotomy emerges. Roughly speaking, if the last non-negligible leftover component has sub-logarithmic maximum subgraph relative density, then almost-exact recovery is impossible. On the other hand, when this relative density is super-logarithmic, almost-exact recovery remains impossible whenever it is at most logarithmic in $n$. This dichotomy is summarized in the following result, proved in Appendix J.

**Theorem 29** *Fix a sequence of subgraphs $\Gamma = (\Gamma_n)_n$, assume $p_n, q_n = \Theta(1)$, and consider the onion decomposition of $\Gamma$ in Definition 16.*

1. *If $|v(\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})})| = o(n)$ and*

$$\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) = o\left(\frac{\log|v(\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})})|}{\log\log|v(\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})})|}\right), \tag{114}$$

*then almost-exact recovery is impossible.*

2. *If $\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) \geq \alpha_n \cdot \log|v(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})})|$, for some $\alpha_n = \Omega(1)$, then there exists a constant $\underline{C} > 0$ such that almost-exact recovery is impossible if*

$$\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) \leq \underline{C} \cdot \log n. \tag{115}$$

A few remarks are in order. First, the proofs of (114) and (115) rely on different techniques, each tailored to its own regime of relative density and typically suboptimal in the other. In both cases, the first step is to reduce almost-exact recovery for $\Gamma$ to recovering the last non-negligible $\Omega(1)$ leftover-edge fraction of $\Gamma$, namely, $\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})}$.

To prove (114), one of the main ingredients is a generalization of the subgraph expectation threshold Kahn and Kalai (2007), and more specifically, of the modified subgraph expectation threshold studied in Mossel et al. (2022), which analyzes the threshold for the appearance of *any* isomorphic copy of $\Gamma$ in $\mathsf{G} \sim \mathcal{G}(n,q)$. For our purposes, however, not all copies are admissible. Indeed, recall that the recovery problem here is supplied with $\Gamma^{(\ell_{\mathsf{LB}})}$ and is tasked with finding $\Gamma$. Thus, the admissible copies are precisely those extending $\Gamma^{(\ell_{\mathsf{LB}})}$, i.e., the copies contained in $\mathcal{M}(\Gamma^{(\ell_{\mathsf{LB}})}, \Gamma)$. To handle this, we derive a generalization of the modified subgraph expectation threshold that accounts for the appearance of such constrained copies.

To prove (115), we establish a connection between almost-exact recovery and a hypothesis-testing variant of the problem (see (37)), and then leverage the known impossibility result (39) for the latter. Most notably, as discussed right after (38)–(39), in the sub-logarithmic regime there exists a region where detection is statistically possible while, as we show above, almost-exact recovery is always impossible. This highlights why hypothesis-testing–based bounds are insufficient in this regime.

Let us now explain the scope of Theorem 29. Comparing Theorems 19 and 29, we see that if $\Gamma$ is balanced and has super-logarithmic maximum density, i.e., $\mu(\Gamma_n) = \Omega(\log |v(\Gamma_n)|)$, then the two bounds coincide (up to constant factors). In this case, almost-exact and exact recovery are statistically equivalent. In general, however, the bounds need not coincide. As noted after Theorem 19 (see, e.g., Example 3), the strict exact-recovery criterion rules out recovering graphs with a dense core and a very sparse appendage, whereas almost-exact recovery may remain possible since one may effectively "discard" the problematic appendage, provided its contribution is $o(|e(\Gamma_n)|)$. We illustrate this phenomenon with two examples.

**Example 12 (A kite)** *Recall Example 3, where $\Gamma$ is a kite on $k+1$ vertices—namely, a clique with $k = |v(\mathcal{K}_k)|$ vertices, with one of its vertices connected by an edge to an additional vertex. In this case, $\ell_{\mathsf{LB}} = 0$, so $\Gamma^{(\ell_{\mathsf{LB}})} = \emptyset$ and the core is a clique on $k$ vertices. Thus, $\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})} = \Gamma$ and $\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) = \mu(\Gamma) = \mu(\mathcal{K}_k) = \frac{k-1}{2}$. Hence, almost-exact recovery is impossible precisely when recovery of a $k$-clique is impossible, namely for $k \leq C \log n$ for some constant $C > 0$, in agreement with folklore results. Therefore, while the dangling edge precludes exact recovery, it has no effect on almost-exact recovery.*

**Example 13 (A kite graph with a long tail)** *Consider the case where $\Gamma$ is a clique $\mathcal{K}_k$ on $k = |v(\mathcal{K}_k)|$ vertices, with one of its vertices attached to a path of length $k^2$ (i.e., $k^2 - 1$ edges and $k^2$ vertices). In this case, $\Gamma^{(1)} = \mathcal{K}_k$ and $\Gamma^{(2)} = \mathcal{P}_{k^2}$ (a path on $k^2$ vertices). Accordingly, we have $\ell_{\mathsf{LB}} = 1$, so $\Gamma^{(\ell_{\mathsf{LB}})} = \mathcal{K}_k$. Thus, $\Gamma \setminus \Gamma^{(\ell_{\mathsf{LB}})} = \mathcal{P}_{k^2}$ and $\mu(\Gamma|\Gamma^{(\ell_{\mathsf{LB}})}) = \frac{|e(\mathcal{P}_{k^2})|}{k+k^2-k-1} = \frac{k^2-1}{k^2-1} = 1$. Hence, we are in the sub-logarithmic regime of (114), which implies that almost-exact recovery is impossible, in agreement with the exact-recovery criterion. We see that a sparse appendage may still deteriorate almost-exact recovery when its "size" is comparable to that of the dense core. Notably, if the path above had length $o(k^2)$, then almost-exact recovery would be impossible only when recovery of the clique itself is impossible (whereas exact recovery remains impossible).*

We now turn to achievability results and show that the lower bounds above are tight.

**Upper bounds.** Recall the truncated likelihood peeling algorithm in (108). For almost-exact recovery, we propose

$$\hat{\Gamma}^{\mathsf{almost}}_{\mathsf{pMLE}} \triangleq \hat{\Gamma}^{(\ell_{\mathsf{UB}})}_{\mathsf{MLE}}, \tag{116}$$

where $\{\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}}\}_{\ell \geq 1}$ is defined in (41), and $\ell_{\mathsf{UB}}$ is defined in (113). Indeed, (116) aims to recover all but the last layers whose leftover-edge fraction is $o(1)$, since these contribute negligibly to the Hamming error. We have the following result.

**Theorem 30** *Almost-exact recovery of $\Gamma^\star$, via the truncated likelihood peeling algorithm in (116), is possible if*

$$\mu(\Gamma|\Gamma^{(\ell_{\mathsf{UB}}-1)}) \geq \mathsf{C}\frac{(1+\varepsilon)\cdot \log n}{d_{\mathsf{KL}}(p||q)}, \tag{117}$$

*for any $\varepsilon > 0$ and some constant $\mathsf{C} > 0$.*

Since $\ell_{\mathsf{UB}} = \ell_{\mathsf{LB}} + 1$, we see that Theorem 30 matches the lower bound in (115) up to a constant factor. Finally, the proof of Theorem 30 follows verbatim from the arguments used in the proof of Theorem 20, with the recovery restricted to $\Gamma^{(\ell_{\mathsf{UB}})}$. The details are therefore omitted.

## Appendix D. Statistical Lower Bound

In this subsection, we prove Theorem 19. As it turns out, this result can be established using two different approaches. The first relies on information-theoretic tools, while the second is based on a second-moment Bayes risk analysis. Here, we present the first approach and relegate the second to Appendix I.

We adopt an information-theoretic strategy based on Fano's inequality combined with a genie-aided argument. Specifically, we lower bound the optimal recovery error probability by the error probability of an estimator that is granted partial access to the latent planted subgraph $\Gamma^\star$. Let $\{\Gamma^{(\ell),\star}\}_{\ell=1}^M$ denote the (unique) onion decomposition of $\Gamma^\star$ as defined in Definition 16. Suppose that a genie reveals $\Gamma^{(M-1),\star}$ to the recovery algorithm. Under this additional information, recovering $\Gamma^\star$ reduces to recovering only the final layer

$$\mathcal{D}^{(M),\star} \triangleq \Gamma^{(M),\star} \setminus \Gamma^{(M-1),\star} = \Gamma^\star \setminus \Gamma^{(M-1),\star}. \tag{118}$$

Accordingly, we consider the recovery problem of $\Gamma^{(M),\star}$ *given* the observation

$$\mathcal{O} \triangleq (\mathsf{G}, \Gamma^{(M-1),\star}), \tag{119}$$

and let $\hat{\Gamma}^{(M)}(\mathcal{O})$ denote any recovery algorithm that is allowed to depend on this augmented information. Then, by definition,

$$\inf_{\hat{\Gamma}} \max_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^\star] \geq \inf_{\hat{\Gamma}^{(M)}} \max_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}[\hat{\Gamma}^{(M)}(\mathcal{O}) \neq \Gamma^\star], \tag{120}$$

where the infimum on the left-hand side of (120) ranges over all measurable functions of $\mathsf{G}$.

Recall that $\mathcal{M}(\Gamma^{(M-1),\star}, \Gamma)$ denotes the number of ways in which $\Gamma^{(M-1),\star}$ can be extended to a copy $\Gamma$ of $\Gamma^\star$ in the complete graph $\mathsf{K}_n$. Equivalently, it is the number of copies of $\Gamma^\star$ in $\mathsf{K}_n$ that contain $\Gamma^{(M-1),\star}$. In words, $\mathcal{M}(\Gamma^{(M-1),\star}, \Gamma)$ characterizes the set of all *valid* embeddings of $\Gamma^\star$ consistent with the revealed partial structure, and thus captures the sole remaining uncertainty in recovering $\Gamma^\star$ given $\Gamma^{(M-1),\star}$. Accordingly, let $n' \triangleq n - |v(\Gamma^{(M-1),\star})|$ and $k' \triangleq |v(\Gamma^{(M),\star}) \setminus v(\Gamma^{(M-1),\star})|$. Crucially, note that $k'$ represents the number of vertices not yet known to the recovery algorithm. Conversely, recall that although recall that although $\Gamma^{(M),\star} \setminus \Gamma^{(M-1),\star}$ and $\Gamma^{(M-1),\star}$ are edge-disjoint, they may share common vertices. These shared vertices are available to the informed recovery algorithm, since it is provided with $\Gamma^{(M-1),\star}$. Next, we observe that

$$\inf_{\hat{\Gamma}^{(M)}} \max_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}[\hat{\Gamma}^{(M)}(\mathcal{O}) \neq \Gamma^\star] = \inf_{\hat{\Gamma}^{(M)}} \max_{\Gamma^{(M-1),\star}} \max_{\Gamma^\star \in \mathcal{M}(\Gamma^{(M-1),\star}, \Gamma^\star)} \mathbb{P}[\hat{\Gamma}^{(M)}(\mathcal{O}) \neq \Gamma^\star] \tag{121}$$

$$\geq \inf_{\hat{\Gamma}^{(M)}} \max_{\Gamma^{(M-1),\star}} \mathbb{E}_{\Gamma^\star \sim \pi} \mathbb{P}[\hat{\Gamma}^{(M)}(\mathcal{O}) \neq \Gamma^\star], \tag{122}$$

where $\pi \triangleq \mathsf{Unif}(\mathcal{M}(\Gamma^{(M-1),\star}, \Gamma))$ denotes the uniform distribution over $\mathcal{M}(\Gamma^{(M-1),\star}, \Gamma)$, and the inequality follows from lower bounding the worst-case risk by the corresponding average risk. Applying conditional Fano's inequality (see, e.g., (Scarlett and Cevher, 2021, Theorem 3)), we obtain

$$\inf_{\hat{\Gamma}^{(M)}} \max_{\Gamma^{(M-1),\star}} \mathbb{E}_{\Gamma^\star \sim \pi} \mathbb{P}[\hat{\Gamma}^{(M)}(\mathcal{O}) \neq \Gamma^\star]$$

$$\geq \max_{\gamma^{(M-1)}} \left[1 - \frac{I(\Gamma^\star; \mathsf{G} | \Gamma^{(M-1),\star} = \gamma^{(M-1)}) + \log 2}{\log |\mathcal{M}(\gamma^{(M-1)}, \Gamma)|}\right], \tag{123}$$

where $I(X;Z \mid V=v)$ denotes the conditional mutual information between the random variables $X$ and $Z$ given $V=v$, namely, $I(X;Z|V=v)=H(X|V=v)-H(X|Z,V=v)$. We begin by lower bounding $|\mathcal{M}(\gamma^{(M-1)},\Gamma)|$ for an arbitrary realization $\gamma^{(M-1)}$. Although sharper bounds on $|\mathcal{M}(\gamma^{(M-1)},\Gamma)|$ can be derived, the following simple bound suffices for our purposes. Specifically, we claim that $|\mathcal{M}(\gamma^{(M-1)},\Gamma)| \geq \binom{n'}{k'}$. Indeed, fixing $\Gamma^{(M-1),\star}=\gamma^{(M-1)}$, we may first select $k'$ vertices from the remaining $n'$ vertices to host the vertices of the final layer $\Gamma^\star=\Gamma^{(M),\star}$. This yields $\binom{n'}{k'}$ distinct choices. In principle, one should also account for the possible labelings of $\Gamma^\star$, whose number is given by $\frac{k'!}{|\mathsf{Aut}(\Gamma^\star\setminus\gamma^{(M-1)})|}$. For our purposes, however, it is sufficient to note that this factor is always at least 1, and thus the crude lower bound $\binom{n'}{k'}$ is adequate.

Next we upper bound $I(\Gamma^\star;\mathsf{G}|\gamma^{(M-1)})$. With a slight abuse of notation, for any $e\in\binom{[n]}{2}$ we let $\mathsf{G}_e$ denote the indicator for the presence of an edge in $\mathsf{G}$, i.e., $\mathsf{G}_e=1$ if $e\in\mathsf{G}$, and zero otherwise. We define $\Gamma^\star_e$ similarly. Then, note that conditioned on the edges $\gamma^{(M-1)}$, the corresponding edges in $\mathsf{G}$ are deterministically fixed, and thus do not contribute to the mutual information. Specifically, recall that

$$I(\Gamma^\star;\mathsf{G}|\Gamma^{(M-1),\star}=\gamma^{(M-1)})=H(\mathsf{G}|\Gamma^{(M-1),\star}=\gamma^{(M-1)})-H(\mathsf{G}|\Gamma^{(M-1),\star}=\gamma^{(M-1)},\Gamma^\star). \tag{124}$$

Then, applying the chain rule for entropy we get

$$H(\mathsf{G}|\Gamma^{(M-1),\star}=\gamma^{(M-1)})=H\left(\mathsf{G}\setminus\gamma^{(M-1)}\middle|\,\Gamma^{(M-1),\star}=\gamma^{(M-1)}\right) \tag{125}$$

$$\leq\left[\binom{n}{2}-|e(\gamma^{(M-1)})|\right]H(\mathsf{G}_e), \tag{126}$$

for any arbitrary edge $e\in\binom{[n]}{2}\setminus\gamma^{(M-1)}$, where the inequality follows from the fact that $\mathsf{G}_e$'s are identically distributed by symmetry. Similarly

$$H(\mathsf{G}|\Gamma^{(M-1),\star}=\gamma^{(M-1)},\Gamma^\star)=\left[\binom{n}{2}-|e(\gamma^{(M-1)})|\right]H(\mathsf{G}_e|\Gamma^\star_e), \tag{127}$$

again, for any arbitrary edge $e\in\binom{[n]}{2}\setminus\gamma^{(M-1)}$, where we have used the fact that $\mathsf{G}_e$'s are independent conditioned on $\Gamma^\star$. Therefore

$$I(\Gamma^\star;\mathsf{G}|\Gamma^{(M-1),\star})\leq\left[\binom{n}{2}-|e(\gamma^{(M-1)})|\right]I(\mathsf{G}_e;\Gamma^\star_e), \tag{128}$$

for an arbitrary edge $e\in\binom{[n]}{2}\setminus\gamma^{(M-1)}$. Note that

$$\mathsf{G}_e|\Gamma^\star_e\sim\begin{cases}\mathsf{Bern}(p_n), & \text{if } \Gamma^\star_e=1\\ \mathsf{Bern}(q_n), & \text{if } \Gamma^\star_e=0.\end{cases} \tag{129}$$

Next, we find $\eta\triangleq\mathbb{P}[\Gamma^\star_e=1]$ for $e\in\binom{[n]}{2}\setminus\gamma^{(M-1)}$. Recall that $\Gamma^\star\sim\mathsf{Unif}(\mathcal{M}(\Gamma^{(M-1),\star},\Gamma))$ given $\Gamma^{(M-1),\star}=\gamma^{(M-1)}$. Thus, we have

$$\eta=\frac{1}{|\mathcal{M}(\gamma^{(M-1)},\Gamma)|}\sum_{\Gamma'\in\mathcal{M}(\gamma^{(M-1)},\Gamma)}\mathbb{1}\{e\in\Gamma'\}. \tag{130}$$

Observe that each embedding $\Gamma' \in \mathcal{M}(\gamma^{(M-1)}, \Gamma)$ contains the same number of edges $|e(\Gamma^\star \setminus \Gamma^{(M-1),\star})|$, and so

$$\sum_{\Gamma' \in \mathcal{M}(\gamma^{(M-1)},\Gamma)} \sum_{\{e\} \subseteq \binom{[n]}{2} \setminus \gamma^{(M-1)}} \mathbb{1}\{e \in \Gamma'\} = |\mathcal{M}(\gamma^{(M-1)}, \Gamma)| \cdot |e(\Gamma^\star \setminus \Gamma^{(M-1),\star})|. \tag{131}$$

Since every embedding $\Gamma' \in \mathcal{M}(\gamma^{(M-1)}, \Gamma)$ is obtained by first choosing an *unordered* $k'$-subset and then relabeling the vertices of $\Gamma'$, every unordered host pair $\{i, j\}$ appears in precisely the *same* number of embeddings as any other pair. Therefore, combining the above, we may conclude that each fixed pair $\{i, j\}$ appears as an edge in $\frac{|\mathcal{M}(\gamma^{(M-1)},\Gamma)| \cdot |e(\Gamma^\star \setminus \Gamma^{(M-1),\star})|}{\binom{n}{2} - |e(\gamma^{(M-1)})|}$ embeddings, and as so

$$\eta = \frac{\frac{|\mathcal{M}(\gamma^{(M-1)},\Gamma)| \cdot |e(\Gamma^\star \setminus \Gamma^{(M-1),\star})|}{\binom{n}{2} - |e(\gamma^{(M-1)})|}}{|\mathcal{M}(\gamma^{(M-1)}, \Gamma)|} \tag{132}$$

$$= \frac{|e(\Gamma^\star \setminus \Gamma^{(M-1),\star})|}{\binom{n}{2} - |e(\gamma^{(M-1)})|} \tag{133}$$

$$= \frac{\eta(\Gamma^{(M),\star} | \Gamma^{(M-1),\star}) k'}{\binom{n}{2} - |e(\gamma^{(M-1)})|} \tag{134}$$

$$= \frac{\mu_{\mathsf{min}}(\Gamma) k'}{\binom{n}{2} - |e(\gamma^{(M-1)})|} = o(1), \tag{135}$$

where in the third equality we have used the fact that, the onion decomposition in Definition 16 forms the sequence $\{\Gamma^{(\ell),\star}\}_{\ell \geq 0}$ in a way such that each layer $\Gamma^{(\ell)}$ is the maximal subgraph that maximizes $\eta(\mathsf{H}|\Gamma^{(\ell-1)})$ among all subgraphs $\Gamma^{(\ell-1)} \subsetneq \mathsf{H}$, and therefore, $|e(\Gamma^{(M),\star} \setminus \Gamma^{(M-1),\star})| = \eta(\Gamma^{(M),\star}|\Gamma^{(M-1),\star}) \cdot |v(\Gamma^{(M),\star}) \setminus v(\Gamma^{(M-1),\star})| = \mu_{\mathsf{min}}(\Gamma) k'$. The last equality is because $k' \leq n$ and (36). Next, note that by Bayes theorem, $\zeta \triangleq \mathbb{P}[\mathsf{G}_e = 1] = \eta p_n + (1 - \eta) q_n$. Combining the above, we finally obtain

$$I(\mathsf{G}_e; \Gamma_e^\star) = \eta d_{\mathsf{KL}}(p_n || \zeta) + (1 - \eta) d_{\mathsf{KL}}(q_n || \zeta) \tag{136}$$

$$\leq \eta d_{\mathsf{KL}}(p_n || \zeta) + (1 - \eta) \frac{(q_n - \zeta)^2}{\zeta(1 - \zeta)}. \tag{137}$$

Since $\eta = o(1)$, we note that $\zeta = q_n + O(\eta)$, and so, $\frac{(q_n - \zeta)^2}{\zeta(1-\zeta)} = O(\eta^2)$. Furthermore

$$d_{\mathsf{KL}}(p_n || \zeta) = d_{\mathsf{KL}}(p_n || q_n) + O(\eta). \tag{138}$$

Thus

$$I(\mathsf{G}_e; \Gamma_e^\star) \leq \eta d_{\mathsf{KL}}(p_n || q_n) + O(\eta^2), \tag{139}$$

which in light of (128) implies that

$$I(\Gamma^\star; \mathsf{G} | \Gamma^{(M-1),\star}) \leq \left[\binom{n}{2} - |e(\gamma^{(M-1)})|\right] I(\mathsf{G}_e; \Gamma_e^\star) \tag{140}$$

$$\leq \left[\binom{n}{2} - |e(\gamma^{(M-1)})|\right] \eta d_{\mathsf{KL}}(p_n || q_n) \left[1 + O(\eta)\right] \tag{141}$$

$$= \mu_{\mathsf{min}}(\Gamma) k' d_{\mathsf{KL}}(p_n || q_n) \left[1 + O(\eta)\right], \tag{142}$$

where the last equality follows from (135). Substituting in (123) we obtain

$$\inf_{\hat{\Gamma}^{(M)}} \max_{\Gamma^{(M-1),\star}} \mathbb{E}_{\Gamma^\star\sim\pi}\mathbb{P}[\hat{\Gamma}^{(M)}(\mathcal{O}) \neq \Gamma^\star] \geq 1 - \frac{\mu_{\mathsf{min}}(\Gamma)k' d_{\mathsf{KL}}(p_n||q_n)\,[1+O(\eta)] + \log 2}{\log \binom{n'}{k'}}. \tag{143}$$

For any $\delta > 0$, the term at the right-hand side of (143) is bounded below by $\delta$ provided that

$$\mu_{\mathsf{min}}(\Gamma)k' d_{\mathsf{KL}}(p_n||q_n)\,[1+O(\eta)] \leq (1-\delta)\log\binom{n'}{k'}. \tag{144}$$

Using the facts that $\log\binom{n'}{k'} \geq k'\log(n'/k')$, $k' \leq k = o(n)$, $n' = (1-o(1))n$, and $\eta = o(1)$, it is evident that (143) is implied by (36), for large enough $n$, which concludes the proof.

## Appendix E. Upper Bounds

In this section, we establish the sufficient conditions for the planted graph recovery.

### E.1. Peeling MLE

**Proof** [Proof of Theorem 20] We analyze the MLE peeling algorithm in (42). Let $\{\Gamma^{(\ell),\star}\}_{\ell=0}^M$ denote the onion decomposition of the planted subgraph $\Gamma^\star$ as defined in Definition 16. We have

$$\begin{aligned}
\mathbb{P}[\hat{\Gamma}_{\mathsf{pMLE}} \neq \Gamma^\star] &= \mathbb{P}[\hat{\Gamma}^{(M)}_{\mathsf{MLE}} \neq \Gamma^\star] && (145)\\
&= \mathbb{P}\left[\bigcup_{\ell=1}^M \left\{\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}} \neq \Gamma^{(\ell),\star}\right\}\right] && (146)\\
&\leq \sum_{\ell=1}^M \mathbb{P}\left[\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}} \neq \Gamma^{(\ell),\star}\middle|\,\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} = \Gamma^{(\ell-1),\star}\right], && (147)
\end{aligned}$$

where the inequality follows from the fact that for a set of monotonically increasing nested events $(\mathcal{A}_i)_{i=1}^n$ we have $\mathbb{P}[\bigcup_{i=1}^n \mathcal{A}_i] \leq \sum_{i=1}^n \mathbb{P}[\mathcal{A}_i|\mathcal{A}_{i-1}^c]$.[7] Accordingly, in what follows, we analyze the error probability at the $\ell^{\mathsf{th}}$ iteration of the algorithm, conditioned on the event that at the $(\ell-1)^{\mathsf{th}}$ iteration the estimated subgraph is correct. It should be emphasized that, in the case where $v(\Gamma^{(\ell),\star}\setminus\Gamma^{(\ell-1),\star})\cap v(\Gamma^{(\ell-1),\star}) \neq \emptyset$, the previously estimated layer $\Gamma^{(\ell-1),\star}$ correctly identified those common vertices.

Recall that $\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}}$ in (41) scans over all copies $\mathcal{M} \triangleq \mathcal{M}(\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}, \Gamma^{(\ell)})$, i.e., copies of $\Gamma^{(\ell)}$ in $\mathcal{K}_n$ which contain $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}$. Loosely speaking, we shall refer to this set as the collection of all possible copies of $\mathcal{D}^{(\ell),\star}$, each taken in union with $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}$. Now, the MLE $\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}}$ of $\Gamma^{(\ell,\star)}$, given $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} = \Gamma^{(\ell-1),\star}$, outputs any element (copy) of $\mathcal{M}(\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}, \Gamma^{(\ell)})$ in the observed graph $\mathsf{G}$. Accordingly, this estimator succeeds if, with high probability, the observed graph contains such a *unique* copy. Let $\{\mathcal{W}_j\}_{j=1}^{|\mathcal{M}|}$ denote the $|\mathcal{M}(\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}, \Gamma^{(\ell)})|$ possible copies (note that each copy is an extension of $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}$). Without loss of generality, we may assume that the actual planted copy is $\Gamma^{(\ell,\star)} = \mathcal{W}_1$.

7. Define $\mathcal{B}_1 = \Omega$, $\mathcal{B}_i = \bigcap_{j<i}\mathcal{A}_j^c$ for $i \geq 2$. Then the union splits into disjoint pieces $\bigcup_{i=1}^n \mathcal{A}_i = \bigsqcup_{i=1}^n(\mathcal{A}_i\cap\mathcal{B}_i)$, and so $\mathbb{P}(\bigcup_{i=1}^n \mathcal{A}_i) = \sum_{i=1}^n \mathbb{P}(\mathcal{A}_i\cap\mathcal{B}_i) = \sum_{i=1}^n \mathbb{P}(\mathcal{A}_i|\mathcal{B}_i)\mathbb{P}(\mathcal{B}_i) \leq \sum_{i=1}^n \mathbb{P}(\mathcal{A}_i|\mathcal{B}_i)$, because $\mathbb{P}(\mathcal{B}_i) \leq 1$ for each $i$.

Given $\mathsf{G}$, let $\mathsf{N}_{\mathcal{M}}(\mathsf{G}) \triangleq \sum_{j=1}^{|\mathcal{M}|} \mathbb{1}\{\mathcal{W}_j \in \mathsf{G}\}$ denote the number of copies in $\mathcal{M}(\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}, \Gamma^{(\ell)})$ that appear in $\mathsf{G}$. Define the event $\mathcal{F}_\ell \triangleq \{\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} = \Gamma^{(\ell-1),\star}\}$. Using the above argument and Markov's inequality, we have:

$$\begin{aligned}
\mathbb{P}\left[\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}} \neq \Gamma^{(\ell),\star} \middle| \mathcal{F}_\ell\right] &\leq \mathbb{P}[\text{At least two extensions in } \mathcal{G}_{\Gamma_n}(n,p_n,q_n)|\mathcal{F}_\ell] && (148)\\
&= \mathbb{P}_{\mathcal{G}_{\Gamma_n}(n,p_n,q_n)}\left[\mathsf{N}_{\mathcal{M}}(\mathsf{G}) \geq 2|\mathcal{F}_\ell\right] && (149)\\
&= \mathbb{P}_{\mathcal{G}_{\Gamma_n}(n,p_n,q_n)}\left[\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G}) \geq 1|\mathcal{F}_\ell\right] && (150)\\
&\leq \mathbb{E}\left[\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G})|\mathcal{F}_\ell\right], && (151)
\end{aligned}$$

where the second equality follows because under $\mathcal{G}_{\Gamma_n}(n,p_n,q_n)$ the copy $\mathcal{W}_1$ is planted—and thus appear—in $\mathsf{G}$, and $\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G})$ counts the number of copies in $\mathsf{G}$ except the planted one $\mathcal{W}_1$, i.e., $\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G}) \triangleq \sum_{j=2}^{|\mathcal{M}|} \mathbb{1}\{\mathcal{W}_j \in \mathsf{G}\}$.

We first consider the case where $p = 1$, and show that if $\mu_{\mathsf{min}}(\Gamma_n) \geq \frac{(4+\varepsilon)\cdot\log n}{d_{\mathsf{KL}}(1||q)}$, then the error probability is small; this differs by a factor of 4 from the lower bound stated in Theorem 19. To that end, we first note that each copy $\mathcal{W}_j$ can be written as $\mathcal{W}_j = \hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} \cup (\mathcal{W}_j \setminus \hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}})$, for all $j \in [|\mathcal{M}|]$. Conditioned on $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} = \hat{\Gamma}^{(\ell-1),\star}$, we have $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} \in \mathsf{G}$ with probability one. Thus, for any $j \in [|\mathcal{M}|]$

$$\begin{aligned}
\mathbb{P}\left[\mathcal{W}_j \in \mathsf{G}|\mathcal{F}_\ell\right] &= \mathbb{P}\left[\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} \cup (\mathcal{W}_j \setminus \hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}) \in \mathsf{G} \middle| \mathcal{F}_\ell\right] && (152)\\
&= \mathbb{P}\left[\mathcal{W}_j \setminus \hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} \in \mathsf{G} \middle| \mathcal{F}_\ell\right] && (153)\\
&= \mathbb{P}\left[\mathcal{W}_j \setminus \Gamma^{(\ell-1),\star} \in \mathsf{G} \middle| \mathcal{F}_\ell\right] && (154)\\
&= q^{|e(\Gamma^{(\ell),\star}\setminus\Gamma^{(\ell-1),\star})|-|e((\mathcal{W}_j\setminus\Gamma^{(\ell-1),\star})\cap(\Gamma^{(\ell),\star}\setminus\Gamma^{(\ell-1),\star}))|} && (155)\\
&= q^{|e(\Gamma^{(\ell),\star}\setminus\Gamma^{(\ell-1),\star})|-|e((\mathcal{W}_j\cap\Gamma^{(\ell),\star})\setminus\Gamma^{(\ell-1),\star})|}, && (156)
\end{aligned}$$

where $|e(\Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star})| - |e((\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus \Gamma^{(\ell-1),\star})|$ is the number of edges in $\mathcal{W}_j \setminus \Gamma^{(\ell-1),\star}$ that are not part of the planted subgraph $\Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star}$, whose edges appear in the graph with probability one. Thus

$$\begin{aligned}
\mathbb{E}\left[\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G})|\mathcal{F}_\ell\right] &= \sum_{j\geq 2} \mathbb{P}\left[\mathcal{W}_j \in \mathsf{G}|\mathcal{F}_\ell\right] && (157)\\
&= \sum_{j\geq 2} q^{|e(\Gamma^{(\ell),\star}\setminus\Gamma^{(\ell-1),\star})|-|e((\mathcal{W}_j\cap\Gamma^{(\ell),\star})\setminus\Gamma^{(\ell-1),\star})|}. && (158)
\end{aligned}$$

The onion decomposition in Definition 16 forms the sequence $\{\Gamma^{(\ell),\star}\}_{\ell\geq 0}$ in a way such that each layer $\Gamma^{(\ell)}$ is the maximal subgraph that maximizes $\eta(\mathsf{H}|\Gamma^{(\ell-1)}) = \frac{|e(\mathsf{H})|-|e(\Gamma^{(\ell-1)})|}{|v(\mathsf{H})\setminus v(\Gamma^{(\ell-1)})|}$ among all subgraphs $\Gamma^{(\ell-1)} \subsetneq \mathsf{H}$. Therefore,

$$\eta(\Gamma^{(\ell),\star}|\Gamma^{(\ell-1),\star}) = \sup_{\Gamma^{(\ell-1),\star}\subsetneq \mathsf{H}} \eta(\mathsf{H}|\Gamma^{(\ell-1)}) \tag{159}$$

$$\geq \frac{|e(\Gamma^{(\ell-1),\star} \cup (\mathcal{W}_j \cap \Gamma^{(\ell),\star}))| - |e(\Gamma^{(\ell-1),\star})|}{|v(\Gamma^{(\ell-1),\star} \cup (\mathcal{W}_j \cap \Gamma^{(\ell),\star})) \setminus v(\Gamma^{(\ell-1),\star})|} \tag{160}$$

$$= \frac{|e((\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus \Gamma^{(\ell-1),\star})|}{|v(\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|}. \tag{161}$$

Furthermore, $|e(\Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star})| = \eta(\Gamma^{(\ell),\star}|\Gamma^{(\ell-1),\star}) \cdot |v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$. Thus, we get

$$\mathbb{E}\left[\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G})|\mathcal{F}_\ell\right] \leq \sum_{j\geq 2} q^{\eta(\Gamma^{(\ell),\star}|\Gamma^{(\ell-1),\star})\cdot\left[|v(\Gamma^{(\ell),\star})\setminus v(\Gamma^{(\ell-1),\star})|-|v(\mathcal{W}_j\cap\Gamma^{(\ell),\star})\setminus v(\Gamma^{(\ell-1),\star})|\right]}. \tag{162}$$

Denote $\mathcal{D}^{(\ell),\star} \triangleq \Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star}$. It can be seen that (162) depends on each possible embedding $\{\mathcal{W}_j\}_{j\geq 2}$ only through the number of vertices it shares with $\Gamma^{(\ell),\star}$ but with the vertices of $\Gamma^{(\ell-1),\star}$ removed. Accordingly, let us count the number of embeddings $\{\mathcal{W}_j\}_{j\geq 2}$ with overlap size $i = |v(\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$, and then sum over all possible values of $i$. This count can be upper bounded quite easily. Indeed, we first consider the number of ways to choose $i$ vertices from $v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})$ to overlap with $v(\mathcal{W}_j) \setminus v(\Gamma^{(\ell-1),\star})$—that is, we are selecting which $i$ vertices of $v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})$ will align with the vertex set of $v(\mathcal{W}_j) \setminus v(\Gamma^{(\ell-1),\star})$. This count is given by $\binom{|v(\Gamma^{(\ell),\star})\setminus v(\Gamma^{(\ell-1),\star})|}{i}$. Next, we count the number of ways to select the remaining $|v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})| - i$ fresh vertices from the $n - |v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$ vertices not in $v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})$. This is given by $\binom{n-|v(\Gamma^{(\ell),\star})\setminus v(\Gamma^{(\ell-1),\star})|}{|v(\Gamma^{(\ell),\star})\setminus v(\Gamma^{(\ell-1),\star})|-i}$. Therefore, the total number of injective vertex mappings of $v(\mathcal{W}_j \cap \Gamma^{(\ell),\star})$ into the vertex set $[n]$ such that exactly $i$ of the vertices are shared with $v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})$, and the remaining $|v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})| - i$ vertices are disjoint from it, is bounded above by $\binom{k_\ell}{i}\binom{n-k_\ell}{k_\ell-i}$, where $k_\ell = triangleq|v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$.

Before we plug in the above and sum over the index $i$, there is a subtle point that must be taken into account—namely, the uniqueness property of the planted copy on its full vertex set. Specifically, in theory, it could be the case that $|v(\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})| = |v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$, yet $\mathcal{W}_j \neq \Gamma^{(\ell),\star}$. For example, consider the case where $\Gamma^{(\ell),\star}$ is a clique plus a single edge attached to one of its vertices. This scenario implies that the range of $i$ should be $0 \leq i \leq |v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$. However, this possibility is precluded by the property that, at each step of the onion decomposition (see Definition 16), the selected layer maximizes the relative subgraph density. We now prove that this is indeed the case. The following result is general and holds for any step in the onion decomposition of $\Gamma$ as defined in Definition 16. Accordingly, consider the $\ell$th step in the onion decomposition, and recall that $\mathcal{D}^{(\ell),\star} \triangleq \Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star}$.

**Lemma 31 (Uniqueness at full overlap)** *Consider the onion decomposition of $\Gamma^\star$ in Definition 16. Let $\mathcal{W}' \in \mathcal{M}(\Gamma^{(\ell-1),\star}, \Gamma^{(\ell)})$ and $\mathcal{D}' = \mathcal{W}' \setminus \Gamma^{(\ell-1),\star}$. Assume that $\mathcal{D}'$ satisfies $v(\mathcal{D}') = v(\mathcal{D}^{(\ell),\star})$. Then $e(\mathcal{D}') = e(\mathcal{D}^{(\ell),\star})$; hence $\mathcal{D}' = \mathcal{D}^{(\ell),\star}$. In particular, there exists no copy distinct from the planted one whose vertex overlap with $\mathcal{D}^{(\ell),\star}$ equals $|v(\mathcal{D}^{(\ell),\star})|$.*

**Proof** [Proof of Lemma 31] By definition of $\mathcal{M}(\Gamma^{(\ell-1),\star}, \Gamma^{(\ell)})$ there is an isomorphism $\phi_1 : v(\mathcal{D}^{(\ell)}) \to v(\mathcal{D}^{(\ell),\star})$ with $\phi_1(e(\mathcal{D}^{(\ell)})) = e(\mathcal{D}^{(\ell),\star})$. Furthermore, because we assume that there exist $\mathcal{W}' \in \mathcal{M}(\Gamma^{(\ell-1),\star}, \Gamma^{(\ell)})$ such that $\mathcal{D}' = \mathcal{W}' \setminus \Gamma^{(\ell-1),\star}$ with $v(\mathcal{D}') = v(\mathcal{D}^{(\ell),\star})$, there is an isomorphism $\phi_2 : v(\mathcal{D}^{(\ell)}) \to v(\mathcal{D}^{(\ell),\star})$ with $\phi_2(e(\mathcal{D}^{(\ell)})) = e(\mathcal{D}')$. Compose them to obtain an automorphism $\psi = \phi_1^{-1} \circ \phi_2 : v(\mathcal{D}^{(\ell)}) \to v(\mathcal{D}^{(\ell)})$. Now, let us show that $\psi$ fixes $v(\Gamma^{(\ell-1)})$. Indeed, since $v(\Gamma^{(\ell-1)}) \subsetneq v(\Gamma^{(\ell)}) = v(\mathcal{D}^{(\ell),\star})$, both $\phi_1$ and $\phi_2$ coincide with the identity on $v(\Gamma^{(\ell-1)})$, hence so does $\psi$. Next, we claim that $\psi$ is the identity on $v(\mathcal{D}^{(\ell)})$. Otherwise set $\widetilde{\Gamma}^{(\ell)} = \Gamma^{(\ell-1)} \cup \psi(\mathcal{D}^{(\ell)})$. The vertex set of $\widetilde{\Gamma}^{(\ell)}$ equals that of $\Gamma^{(\ell)}$ and $e(\psi(\mathcal{D}^{(\ell)})) = e(\mathcal{D}') \neq e(\mathcal{D}^{(\ell),\star}) = e(\mathcal{D}^{(\ell)})$. Consequently, $|e(\widetilde{\Gamma}^{(\ell)})| > |e(\Gamma^{(\ell)})|$. This contradicts the *maximality* of $\Gamma^{(\ell)}$ in step (ii) of Definition 16. Hence $\psi$ is the identity, and $e(\mathcal{D}') = e(\mathcal{D}^{(\ell),\star})$. ■

Returning to (162) and its notation, we observe that, by Lemma 31, the set

$$\left\{\mathcal{W}' \in \mathcal{M} : \mathcal{W}' \neq \Gamma^{(\ell),\star},\ |v(\mathcal{W}' \cap \Gamma^{(\ell),\star})| = |v(\Gamma^{(\ell),\star})|\right\} \tag{163}$$

is empty. Recall that $k_\ell \triangleq |v(\Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|$, and further denote $\eta_\ell \triangleq \eta(\Gamma^{(\ell),\star}|\Gamma^{(\ell-1),\star})$. Therefore

$$\begin{aligned}
\mathbb{E}\left[\mathsf{N}_{\mathcal{M}\setminus\mathcal{W}_1}(\mathsf{G})|\mathcal{F}_\ell\right] &\leq \sum_{j\geq 2} q^{\eta_\ell \cdot \left[k_\ell - |v(\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus v(\Gamma^{(\ell-1),\star})|\right]} && (164)\\
&\leq \sum_{i=0}^{k_\ell - 1} q^{\eta_\ell \cdot (k_\ell - i)} \binom{k_\ell}{i}\binom{n-k_\ell}{k_\ell - i} && (165)\\
&\leq \sum_{i=0}^{k_\ell - 1} k_\ell^{k_\ell - i} n^{k_\ell - i} q^{\eta_\ell \cdot (k_\ell - i)} && (166)\\
&= \sum_{i=0}^{k_\ell - 1} \left[k_\ell n q^{\eta_\ell}\right]^{k_\ell - i} && (167)\\
&\leq \frac{\delta}{M(\Gamma)}, && (168)
\end{aligned}$$

where the second inequality is because $\binom{k_\ell}{i} = \binom{k_\ell}{k_\ell - i} \leq k_\ell^{k_\ell - i}$ and $\binom{n-|v(\mathcal{D})|}{k_\ell - i} \leq n^{k_\ell - i}$, the last inequality holds provided that $k_\ell n q^{\eta_\ell} \leq \frac{\delta/2}{M(\Gamma)}$, for any $\delta > 0$, and $M(\Gamma)$ is the number of subgraphs in the onion decomposition of $\Gamma^\star$. This concludes the analysis of the $\ell^{\mathsf{th}}$ step of the MLE peeling algorithm. Applying the same argument for each step, by using (147) we get

$$\mathbb{P}[\hat{\Gamma}_{\mathsf{pMLE}} \neq \Gamma^\star] \leq \sum_{\ell=1}^{M} \mathbb{P}\left[\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}} \neq \Gamma^{(\ell),\star} \middle| \hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} = \Gamma^{(\ell-1),\star}\right] \leq \delta, \tag{169}$$

provided that $k_\ell n q^{\eta_\ell} \leq \frac{\delta/2}{M(\Gamma)}$, for all $\ell \geq 1$. Since $M(\Gamma) \leq |e(\Gamma)| \leq n^2$ and $k_\ell \leq n$, this condition is clearly satisfied when $\eta_\ell \geq \frac{(4+\varepsilon)\cdot \log n}{d_{\mathsf{KL}}(1||q)}$, for all $\ell \geq 1$, namely, $\mu_{\mathsf{min}}(\Gamma_n) \geq \frac{(4+\varepsilon)\cdot \log n}{d_{\mathsf{KL}}(1||q)}$, for any $\varepsilon > 0$, which concludes the proof for $p = 1$.

**Remark 32 (Why the argument fails for the whole graph $\Gamma$)** *The conclusion of Lemma 31 relies on the* edge–maximality *property (27) of each layer $\Gamma^{(\ell)}$. The full graph $\Gamma$ need not be edge–maximal on its vertex set. If $\Gamma$ contains a dense core and a sparse appendage (e.g., a clique $K_k$ plus one extra edge), one can keep the entire vertex set fixed and relocate only the sparse part,* *producing $\Theta(n^2)$ distinct copies $\Gamma' \neq \Gamma^\star$ with full vertex overlap $i = |v(\Gamma^\star)|$. These copies must be included in the global first–moment sum, and their aggregate contribution is already unbounded, so the counting/Markov argument used for a peeling layer cannot be transferred verbatim to the global MLE.*

Finally, we adapt the above proof to the case where planted edges appear with probability $q < p \leq 1$. Recall (147), and as above, let us analyze the $\ell^{\mathsf{th}}$ step of the MLE peeling algorithm. To that end, for any $j \in [|\mathcal{M}(\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}}, \Gamma^{(\ell)})|]$, define

$$\mathcal{A}(\mathcal{W}_j) \triangleq \sum_{(i,j)\in\mathcal{W}_j} \mathsf{A}_{ij}, \tag{170}$$

and $\triangle(\mathcal{W}_j) \triangleq \mathcal{A}(\mathcal{W}_1) - \mathcal{A}(\mathcal{W}_j)$, where $\mathcal{W}_1$ denotes the actual planted subgraph, namely, $\mathcal{W}_1 = \Gamma^{(\ell),\star}$. We next find conditions under which $\triangle(\mathcal{W}_j) > 0$, for any $j \neq 1$. By definition, note that $\hat{\Gamma}^{(\ell-1)}_{\mathsf{MLE}} \subsetneq \mathcal{W}_j$ and since we condition on $\mathcal{F}_\ell$ (see (147)) we have $\Gamma^{(\ell-1),\star} \subsetneq \mathcal{W}_j$, for any $\mathcal{W} \in \mathcal{M}$. For simplicity of notations, we define $\mathcal{D}^{(\ell),\star} \triangleq \mathcal{W}_1 \setminus \Gamma^{(\ell-1),\star} = \Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star}$ and $\mathcal{D}_j \triangleq \mathcal{W}_j \setminus \Gamma^{(\ell-1),\star}$, for all $j \geq 2$. Thus,

$$\triangle(\mathcal{W}_j) = \sum_{(i,j)\in\mathcal{W}_1} \mathsf{A}_{ij} - \sum_{(i,j)\in\mathcal{W}_j} \mathsf{A}_{ij} \tag{171}$$

$$= \sum_{(i,j)\in\mathcal{D}^{(\ell),\star}} [\mathsf{A}_{ij} - \mathbb{E}\mathsf{A}_{ij}] - \sum_{(i,j)\in\mathcal{D}_j} [\mathsf{A}_{ij} - \mathbb{E}\mathsf{A}_{ij}] + \sum_{(i,j)\in\mathcal{D}^{(\ell),\star}} \mathbb{E}\mathsf{A}_{ij} - \sum_{(i,j)\in\mathcal{D}_j} \mathbb{E}\mathsf{A}_{ij} \tag{172}$$

$$= (p-q)\cdot|\mathcal{D}_j \setminus \mathcal{D}^{(\ell),\star}| + 2\sum_{(i,j)\in\mathcal{D}^{(\ell),\star}\setminus\mathcal{D}_j, i<j} [\mathsf{A}_{ij} - p] - 2\sum_{(i,j)\in\mathcal{D}_j\setminus\mathcal{D}^{(\ell),\star}, i<j} [\mathsf{A}_{ij} - q] \tag{173}$$

$$\triangleq (p-q)\cdot|\mathcal{D}_j \setminus \mathcal{D}^{(\ell),\star}| + 2\mathsf{B}_1 - 2\mathsf{B}_2, \tag{174}$$

where $\mathsf{B}_1$ and $\mathsf{B}_2$ are independent random variables, each composed of a sum of $\frac{1}{2}|\mathcal{D}^{(\ell),\star} \setminus \mathcal{D}_j| = \frac{1}{2}|\mathcal{D}_j \setminus \mathcal{D}^{(\ell),\star}| = \frac{1}{2}\left(|e(\mathcal{D}^{(\ell),\star})| - |\mathcal{D}_j \cap \mathcal{D}^{(\ell),\star}|\right)$ i.i.d. centered Bernoulli random variables with parameters $p$ and $q$, respectively. Let $\mathsf{J}(\mathcal{D}_j) \triangleq |e(\mathcal{D}^{(\ell),\star})| - |\mathcal{D}_j \cap \mathcal{D}^{(\ell),\star}|$. Chernoff's bound implies that,

$$\mathbb{P}_{\mathsf{Bern}(p)^{\otimes \mathsf{J}(\mathcal{D}_j)}}\left[\mathsf{B}_1 \leq -\frac{p-q}{4}\mathsf{J}(\mathcal{D}_j)\right] \leq \exp\left[-\frac{\mathsf{J}(\mathcal{D}_j)}{2} d_{\mathsf{KL}}\left(\frac{p+q}{2}\middle\| p\right)\right], \tag{175}$$

and

$$\mathbb{P}_{\mathsf{Bern}(q)^{\otimes \mathsf{J}(\mathcal{D}_j)}}\left[\mathsf{B}_2 \geq \frac{p-q}{4}\mathsf{J}(\mathcal{D}_j)\right] \leq \exp\left[-\frac{\mathsf{J}(\mathcal{D}_j)}{2} d_{\mathsf{KL}}\left(\frac{p+q}{2}\middle\| q\right)\right]. \tag{176}$$

For simplicity of notation define $\kappa \triangleq \frac{p+q}{2}$. Then

$$\mathbb{P}\left[\hat{\Gamma}^{(\ell)}_{\mathsf{MLE}} \neq \Gamma^{(\ell),\star} \middle| \mathcal{F}_\ell\right] \leq \mathbb{P}\left[\bigcup_{j\geq 2} \triangle(\mathcal{W}_j) < 0\right] \tag{177}$$

$$= \mathbb{P}\left[\bigcup_{j\geq 2}\left\{\mathsf{B}_2 - \mathsf{B}_1 < \frac{(p-q)}{2}\cdot|\mathcal{D}_j \setminus \mathcal{D}^{(\ell),\star}|\right\}\right] \tag{178}$$

$$\leq \mathbb{P}\left[\bigcup_{j\geq 2}\left\{\mathsf{B}_1 \leq -\frac{p-q}{4}\mathsf{J}(\mathcal{D}_j)\right\}\cup\left\{\mathsf{B}_2 \geq \frac{p-q}{4}\mathsf{J}(\mathcal{D}_j)\right\}\right] \tag{179}$$

$$\leq \sum_{j\geq 2}\left[\mathbb{P}\left(\mathsf{B}_1 \leq -\frac{p-q}{4}\mathsf{J}(\mathcal{D}_j)\right) + \mathbb{P}\left(\mathsf{B}_2 \geq \frac{p-q}{4}\mathsf{J}(\mathcal{D}_j)\right)\right] \tag{180}$$

$$\leq \sum_{j\geq 2}\left[e^{-\frac{\mathsf{J}(\mathcal{D}_j)}{2}d_{\mathsf{KL}}(\kappa\|q)} + e^{-\frac{\mathsf{J}(\mathcal{D}_j)}{2}d_{\mathsf{KL}}(\kappa\|p)}\right] \tag{181}$$

$$= \sum_{j\geq 2}\left[e^{-\frac{\mathsf{J}(\mathcal{D}_j)}{2}d_{\mathsf{KL}}(\kappa\|q)} + e^{-\frac{\mathsf{J}(\mathcal{D}_j)}{2}d_{\mathsf{KL}}(\kappa\|p)}\right], \tag{182}$$

where the first inequality follows from the fact that for any two random variables $X, Y$ and any $\zeta \in \mathbb{R}$, we have $\mathbb{P}[X \geq Y] \leq \mathbb{P}[X \geq \zeta \cup Y \leq \zeta]$. Since $\mathsf{J}(\mathcal{D}_j) = |e(\Gamma^{(\ell),\star} \setminus \Gamma^{(\ell-1),\star})| - |e((\mathcal{W}_j \cap \Gamma^{(\ell),\star}) \setminus \Gamma^{(\ell-1),\star})|$, we notice the resemblance between (182) and (158). Accordingly, by following the same arguments as in (158)–(169), we obtain that $\mathbb{P}[\hat{\mathcal{D}}_{\mathsf{MLE}} \neq \mathcal{D}^\star] \leq \frac{\delta}{|M(\Gamma)|}$, provided that

$$k_\ell \cdot n \cdot \exp\left(-\frac{d_{\mathsf{KL}}(\kappa||q) \wedge d_{\mathsf{KL}}(\kappa||p)}{2}\eta_\ell\right) \leq \frac{\delta/2}{M(\Gamma)}. \tag{183}$$

Accordingly, $\mathbb{P}[\hat{\Gamma}_{\mathsf{pMLE}} \neq \Gamma^\star] \to 0$, provided by (183), for all $\ell \geq 1$. This is equivalent to $\mu_{\mathsf{min}}(\Gamma_n) \geq \frac{(8+\varepsilon)\cdot\log n}{d_{\mathsf{KL}}(\kappa||q)\wedge d_{\mathsf{KL}}(\kappa||p)}$, for any $\varepsilon > 0$. ■

### E.2. Convex relaxation

**Proof** [Proof of Theorem 22] We will show that, with high probability, the objective function of any feasible $\mathsf{X} \neq \mathsf{X}^\star$ is inferior, i.e.,

$$\langle \mathsf{X}, \mathsf{W}\rangle < \langle \mathsf{X}^\star, \mathsf{W}\rangle. \tag{184}$$

To that end, we start by establishing some notations. Denote the singular value decomposition (SVD) of the underlying *diagonally shifted* adjacency matrix $\mathsf{S}^\star \triangleq \mathsf{Ext}(\mathsf{X}^\star, \mathsf{s}^\star; \alpha) = \mathsf{X}^\star + \alpha\mathsf{Diag}(\mathbf{s}^\star)$ associated with the planted subgraph $\Gamma^\star$ by

$$\mathsf{S}^\star = \mathsf{U}\Sigma\mathsf{U}^\top, \tag{185}$$

and define the projections projections onto the tangent space $\mathcal{T}$ of the manifold of matrices as $\mathcal{P}_{\mathcal{T}}(M) \triangleq \mathsf{U}\mathsf{U}^T M + M\mathsf{U}\mathsf{U}^T - \mathsf{U}\mathsf{U}^T M\mathsf{U}\mathsf{U}^T$ and $\mathcal{P}_{\mathcal{T}^\perp}(M) \triangleq M - \mathcal{P}_{\mathcal{T}}(M)$. We note that $\mathbb{E}\mathsf{W} = c\mathsf{X}^\star$

with $c \triangleq \frac{p}{q} - 1 > 0$. With these notations, we can write,

$$\langle \mathsf{X}^\star - \mathsf{X}, \mathsf{W}\rangle = \underbrace{\langle \mathsf{X}^\star - \mathsf{X}, c\mathsf{X}^\star\rangle}_{\text{(a)}} + \underbrace{\langle \mathsf{X}^\star - \mathsf{X}, \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\rangle}_{\text{(b)}} + \underbrace{\langle \mathsf{X}^\star - \mathsf{X}, \mathcal{P}_{\mathcal{T}}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\rangle}_{\text{(c)}}, \tag{186}$$

and we next bound each one of the above terms (a)–(c). As for (a), using the feasibility condition $\langle \mathbf{J}, \mathsf{X}\rangle = 2|e(\Gamma)|$ in (51), we get

$$\langle \mathsf{X}^\star - \mathsf{X}, c\mathsf{X}^\star\rangle = \frac{c}{2}\left\|\mathsf{X}^\star - \mathsf{X}\right\|_{\ell_1}. \tag{187}$$

Next, we analyze term (b). To that end, we follow a standard approach and analyze the subgradient of $\|\cdot\|_\star$ at $\mathsf{S}^\star$. Using (Chen and Xu, 2016, Corollary 6.1), we have $\partial\|\cdot\|_\star(\mathsf{S}^\star) = \{\mathsf{U}\mathsf{U}^\top + \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{Y}) : \|\mathsf{Y}\|_{\mathsf{op}} \leq 1\}$. Thus, using the above fact and the nuclear-norm feasibility condition $\|\mathsf{X} + \alpha\mathsf{Diag}(\mathsf{s})\|_\star \leq \|\mathsf{S}^\star\|_\star$, we obtain

$$\begin{aligned} 0 &\geq \|\mathsf{X} + \alpha\mathsf{Diag}(\mathsf{s})\|_\star - \|\mathsf{S}^\star\|_\star && (188)\\ &\geq \langle \mathsf{X} - \mathsf{X}^\star, \mathsf{U}\mathsf{U}^\top\rangle + \langle \mathsf{X} - \mathsf{X}^\star, \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{Y})\rangle + \alpha\langle \mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star), \mathsf{U}\mathsf{U}^\top + \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{Y})\rangle, && (189)\end{aligned}$$

and thus, by taking $\mathsf{Y} = \frac{\mathsf{W} - \mathbb{E}[\mathsf{W}]}{\|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}}$, and using Hölder's inequality, we have

$$\begin{aligned} |\langle \mathsf{X} - \mathsf{X}^\star, \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\rangle| &\leq \|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}\left|\langle \mathsf{X} - \mathsf{X}^\star, \mathsf{U}\mathsf{U}^\top\rangle\right| \\ &\quad + \alpha\|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}\left(\left\|\mathsf{U}\mathsf{U}^\top\right\|_\infty \|\mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star)\|_{\ell_1} + |\langle \mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star), \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{Y})\rangle|\right). \end{aligned} \tag{190}$$

Since $(\mathsf{W} - \mathbb{E}[\mathsf{W}])$ has zero diagonal, so does $\mathsf{Y}$. Hence

$$\langle \mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star), \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{Y})\rangle = -\langle \mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star), \mathcal{P}_{\mathcal{T}}(\mathsf{Y})\rangle, \tag{191}$$

and therefore, by duality,

$$|\langle \mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star), \mathcal{P}_{\mathcal{T}^\perp}(\mathsf{Y})\rangle| \leq \|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty}\|\mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star)\|_{\ell_1}. \tag{192}$$

Finally, as for (c), using Hölder's inequality once again

$$\langle \mathsf{X}^\star - \mathsf{X}, \mathcal{P}_{\mathcal{T}}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\rangle \leq \|\mathcal{P}_{\mathcal{T}}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\|_{\ell_\infty}\left\|\mathsf{X} - \mathsf{X}^\star\right\|_{\ell_1}. \tag{193}$$

Combining the above we obtain

$$\begin{aligned} \langle \mathsf{X}^\star - \mathsf{X}, \mathsf{W}\rangle &\geq \left[\frac{c}{2} - \left\|\mathsf{U}\mathsf{U}^\top\right\|_{\ell_\infty}\|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}} - \|\mathcal{P}_{\mathcal{T}}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\|_{\ell_\infty}\right]\|\mathsf{X} - \mathsf{X}^\star\|_{\ell_1} \\ &\quad - \alpha\|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}\left(\left\|\mathsf{U}\mathsf{U}^\top\right\|_\infty + \|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty}\right)\|\mathsf{Diag}(\mathsf{s} - \mathsf{s}^\star)\|_{\ell_1}, \end{aligned} \tag{194}$$

where $\mathsf{Y} = \frac{\mathsf{W} - \mathbb{E}[\mathsf{W}]}{\|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}}$. Let us bound each one of the terms at the right-hand side of (194). Since $\mathsf{W} - \mathbb{E}\mathsf{W}$ is a symmetric and i.i.d. matrix, with zero mean, it is well-known that (see, e.g., (Vershynin, 2018, Corollary 4.4.7)) with probability at least $1 - \delta$,

$$\|\mathsf{W} - \mathbb{E}\mathsf{W}\|_{\mathsf{op}} \leq \frac{C}{q}\sqrt{n} + \frac{C}{q}\sqrt{\log\frac{4}{\delta}}. \tag{195}$$

Thus,

$$\|\mathsf{W} - \mathbb{E}[\mathsf{W}]\|_{\mathsf{op}} \leq C_q' \sqrt{n}, \tag{196}$$

for some large enough constant $C_q' > 0$. Furthermore, we have the following lemma, which we will prove later on.

**Lemma 33** *Let* $\mathsf{U} \in \mathbb{R}^{n\times r}$ *have orthonormal columns and set* $\mathsf{P} \triangleq \mathsf{U}\mathsf{U}^\top$. *For each* $i$, *let* $\ell_i \triangleq \|\mathsf{U}_{i,:}\|_2^2$ *and thus* $\mathsf{coh}(\mathsf{U}) = \frac{n}{r}\max_{1\leq i\leq n}\ell_i$. *Then*

$$\frac{\sqrt{r}}{n} \leq \|\mathsf{P}\|_{\ell_\infty} \leq \frac{r}{n}\mathsf{coh}(\mathsf{U}). \tag{197}$$

Lemma 33 implies that

$$\left\|\mathsf{U}\mathsf{U}^\top\right\|_{\ell_\infty} \leq \frac{\mathsf{rank}(\mathsf{S}^\star)}{n}\mathsf{coh}(\mathsf{U}). \tag{198}$$

We finally upper bound $\|\mathcal{P}_{\mathcal{T}}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\|_{\ell_\infty}$ and $\|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty}$ where $\mathsf{Y} = \frac{\mathsf{W}-\mathbb{E}[\mathsf{W}]}{\|\mathsf{W}-\mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}}$. To that end, we let, for simplicity of notation, $\mathsf{P} \triangleq \mathsf{U}\mathsf{U}^\top$, namely, orthogonal projector onto $\mathsf{range}(\mathsf{S}^\star)$, with $\mathsf{U} \in \mathbb{R}^{n\times r}$ having orthonormal columns. Denote $r \triangleq \mathsf{rank}(\mathsf{S}^\star)$, and recall that

$$\|\mathsf{U}\|_{2,\infty} \triangleq \max_{i\in[n]} \|\mathsf{U}_{i,:}\|_2, \qquad \mathsf{coh}(\mathsf{U}) = \frac{n}{r}\cdot\|\mathsf{U}\|_{2,\infty}^2 \tag{199}$$

Accordingly, we note that $\mathsf{P}_{ii} = \|\mathsf{U}^\top e_i\|_2^2 \leq \|\mathsf{U}\|_{2,\infty}^2 = \mathsf{coh}(\mathsf{U})r/n$, for any $i \in [n]$, and furthermore,

$$|\mathsf{P}_{ij}| \leq \|\mathsf{U}\|_{2,\infty}^2 = \frac{r}{n}\mathsf{coh}(\mathsf{U}), \tag{200}$$

for all $i, j \in [n]$. We have the following lemma.

**Lemma 34** *Let* $\mathsf{Z}$ *be a real-valued symmetric matrix, and* $\mathcal{P}_{\mathcal{T}}$ *be the projection matrix onto the tangent space, i.e.,* $\mathcal{P}_{\mathcal{T}}(\mathsf{Z}) = \mathsf{PZ} + \mathsf{ZP} - \mathsf{PZP}$. *Then*

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{Z})\|_{\ell_\infty} \leq (2 + \|\mathsf{P}\|_{\ell_\infty\to\ell_\infty})\cdot\|\mathsf{PZ}\|_{\ell_\infty} \tag{201}$$

$$\leq \left(2 + \sqrt{\frac{|v(\Gamma)|r\mathsf{coh}(\mathsf{U})}{n}}\right)\cdot\|\mathsf{PZ}\|_{\ell_\infty} \tag{202}$$

*Furthermore, for a diagonal matrix* $D$

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{D})\|_{\ell_\infty} \leq 3\frac{\mathsf{coh}(\mathsf{U})r}{n}\cdot\|\mathsf{D}\|_{\ell_\infty}. \tag{203}$$

Apply Lemma 34 with $\mathsf{Z} = \mathsf{W} - \mathbb{E}[\mathsf{W}]$ we have

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\|_{\ell_\infty} \leq \left(2 + \sqrt{\frac{|v(\Gamma)|\mathsf{coh}(\mathsf{U})r}{n}}\right)\cdot\|\mathsf{P}(\mathsf{W} - \mathbb{E}[\mathsf{W}])\|_{\ell_\infty}. \tag{204}$$

Now, note that for any $(i,j)$

$$[\mathsf{P}(\mathsf{W}-\mathbb{E}[\mathsf{W}])]_{ij}=\sum_{s=1}^{n}\mathsf{P}_{is}[\mathsf{W}-\mathbb{E}[\mathsf{W}]]_{sj}. \tag{205}$$

Furthermore $([\mathsf{W}-\mathbb{E}[\mathsf{W}]]_{sj})_{s=1}^{n}$ has independent, mean-zero, bounded entries with $\psi_2$-norm $\le K_q$). Then a weighted Bernstein's bound gives with probability at least $1-n^{-3}$ that,

$$[\mathsf{P}(\mathsf{W}-\mathbb{E}[\mathsf{W}])]_{ij}\le K_q\sqrt{\sum_{s=1}^{n}\mathsf{P}_{is}^2\log n}, \tag{206}$$

for a large enough constant $K_q$. Because $\mathsf{P}$ is a projector, $\sum_{s=1}^{n}\mathsf{P}_{is}^2=[\mathsf{P}^2]_{ii}=[\mathsf{P}]_{ii}\le\|\mathsf{U}\|_{2,\infty}^2=\mathsf{coh}(\mathsf{U})r/n$. Hence, by a union bound over all pairs $(i,j)\in[n]\times[n]$, with probability at least $1-n^{-1}$,

$$\|\mathsf{P}(\mathsf{W}-\mathbb{E}[\mathsf{W}])\|_{\ell_\infty}\le K_q\sqrt{\frac{\mathsf{coh}(\mathsf{U})r}{n}\log n}, \tag{207}$$

Likewise, with $\mathsf{Z}=\mathsf{Y}=\frac{\mathsf{W}-\mathbb{E}[\mathsf{W}]}{\|\mathsf{W}-\mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}}$ we get

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty}\le\left(2+\sqrt{\frac{|v(\Gamma)|\mathsf{coh}(\mathsf{U})r}{n}}\right)\cdot\frac{K_q}{\|\mathsf{W}-\mathbb{E}[\mathsf{W}]\|_{\mathsf{op}}}\sqrt{\frac{\mathsf{coh}(\mathsf{U})r}{n}\log n}. \tag{208}$$

Looking at (194), it is only left to deal $\|\mathsf{Diag}(\mathbf{s}-\mathbf{s}^\star)\|_{\ell_1}$. Let $\delta(\Gamma^\star)$ denote the minimum degree of $\Gamma^\star$. The edge–vertex coupling constraints $\mathsf{X}_{ij}\le\min(\mathbf{s}_i,\mathbf{s}_j)$ together with $\langle\bar{\mathsf{J}},\mathsf{X}\rangle=2|e(\Gamma)|$ imply

$$\|\mathsf{Diag}(\mathbf{s}-\mathbf{s}^\star)\|_{\ell_1}\le\frac{1}{\delta(\Gamma^\star)}\|\mathsf{X}-\mathsf{X}^\star\|_{\ell_1}. \tag{209}$$

Indeed, let $\mathcal{N}_i$ denote the set of neighbors of $i$ in $\Gamma^\star$. For any $i\in v(\Gamma^\star)$,

$$\sum_{j\in\mathcal{N}_i}\mathsf{X}_{ij}\le\sum_{j\in\mathcal{N}_i}\min\{\mathbf{s}_i,\mathbf{s}_j\}\le\mathsf{deg}_{\Gamma^\star}(i)\mathbf{s}_i. \tag{210}$$

Since $\sum_{j\in\mathcal{N}_i}\mathsf{X}_{ij}^\star=\mathsf{deg}_{\Gamma^\star}(i)$, we get

$$\sum_{j\in\mathcal{N}_i}\left(\mathsf{X}_{ij}^\star-\mathsf{X}_{ij}\right)\ge\mathsf{deg}_{\Gamma^\star}(i)(1-\mathbf{s}_i). \tag{211}$$

Taking positive parts only makes the left-hand side larger, so

$$\sum_{j\in\mathcal{N}_i}\left(\mathsf{X}_{ij}^\star-\mathsf{X}_{ij}\right)_+\ge\mathsf{deg}_{\Gamma^\star}(i)(1-\mathbf{s}_i)\ge\delta(\Gamma^\star)(1-\mathbf{s}_i), \tag{212}$$

where $\delta(\Gamma^\star)\triangleq\min_{k\in v(\Gamma^\star)}\mathsf{deg}_{\Gamma^\star}(k)$. Now, for any scalars $a,b$, it holds that $|a-b|=(a-b)_++(b-a)_+\ge(b-a)_+$. Summing (212) over ordered pairs $(i,j)$ with $j\in\mathcal{N}_i$ and using symmetry,

$$\|\mathsf{X}-\mathsf{X}^\star\|_{\ell_1}=\sum_{i,j}|\mathsf{X}_{ij}-\mathsf{X}_{ij}^\star|\ge2\sum_{i\in v(\Gamma^\star)}\sum_{j\in\mathcal{N}_i}\left(\mathsf{X}_{ij}^\star-\mathsf{X}_{ij}\right)_+\ge2\delta(\Gamma^\star)\sum_{i\in v(\Gamma^\star)}(1-\mathbf{s}_i). \tag{213}$$

By feasibility $\sum_i \mathbf{s}_i = \sum_i \mathbf{s}_i^\star = |v(\Gamma^\star)|$, and thus

$$\sum_{i\in v(\Gamma^\star)} (1-\mathbf{s}_i) = \sum_{i\notin v(\Gamma^\star)} \mathbf{s}_i = \frac{1}{2}\sum_{i\in[n]} |\mathbf{s}_i - \mathbf{s}_i^\star| = \frac{1}{2}\|\mathsf{Diag}(s-s^\star)\|_{\ell_1}. \tag{214}$$

Combining together reveals that,

$$\|\mathsf{X}-\mathsf{X}^\star\|_{\ell_1} \geq 2\delta(\Gamma^\star)\cdot\frac{1}{2}\|\mathsf{Diag}(\mathbf{s}-\mathbf{s}^\star)\|_{\ell_1} = \delta(\Gamma^\star)\|\mathsf{Diag}(\mathbf{s}-\mathbf{s}^\star)\|_{\ell_1}, \tag{215}$$

which is equivalent to

$$\|\mathsf{Diag}(\mathbf{s}-\mathbf{s}^\star)\|_{\ell_1} \leq \frac{1}{\delta(\Gamma^\star)}\|\mathsf{X}-\mathsf{X}^\star\|_{\ell_1}, \tag{216}$$

which yields (209). Thus, plugging (196), (198), (204), (208), and (209) in (194), we get with probability at least $1-n^{-1}$,

$$\begin{aligned}\langle \mathsf{X}^\star-\mathsf{X},\mathsf{W}\rangle \geq \Bigg[&\frac{c}{2} - \frac{C_q' r}{\sqrt{n}}\mathsf{coh}(\mathsf{U}) - \left(2+\sqrt{\frac{|v(\Gamma)|\mathsf{coh}(\mathsf{U})r}{n}}\right)K_q\sqrt{\frac{\mathsf{coh}(\mathsf{U})r}{n}\log n}\\ &-\frac{\alpha}{\delta(\Gamma)}\left(\frac{C_q' r}{\sqrt{n}}\mathsf{coh}(\mathsf{U}) + \left(2+\sqrt{\frac{|v(\Gamma)|\mathsf{coh}(\mathsf{U})r}{n}}\right)K_q\sqrt{\frac{\mathsf{coh}(\mathsf{U})r}{n}\log n}\right)\Bigg]\|\mathsf{X}-\mathsf{X}^\star\|_{\ell_1}. \end{aligned}\tag{217}$$

Accordingly, since $\alpha$ is a fixed constant, and $\delta(\Gamma)\geq 1$, there exist constants $c_1, c_2 > 0$ such that if

$$\mathsf{coh}(\mathsf{U})r \leq c_1\sqrt{n} \tag{218}$$

$$\mathsf{coh}(\mathsf{U})r\sqrt{|v(\Gamma)|} \leq c_2\frac{n}{\sqrt{\log n}}, \tag{219}$$

then, for any $\mathsf{X}\neq\mathsf{X}^\star$, we have $\langle \mathsf{X}^\star-\mathsf{X},\mathsf{W}\rangle > 0$. Finally, let us check when (218) dominates (219). Assuming that (218) holds, it can be seen that (219) is satisfied if

$$|v(\Gamma)| \leq \frac{c_2^2}{c_1^2}\frac{n}{\log n}, \tag{220}$$

otherwise, (219) dominates, which concludes the proof. ■

We finally prove Lemmata 33 and 34.

**Proof** [Proof of Lemma 33] Since $\mathsf{U}$ has orthonormal columns, $\mathsf{P}=\mathsf{U}\mathsf{U}^\top$ is the orthogonal projector onto $\mathsf{range}(\mathsf{U})$. Its entries satisfy

$$\mathsf{P}_{ij} = \mathsf{U}_{i,:}\mathsf{U}_{j,:}^\top = u_i^\top u_j, \tag{221}$$

where $u_i \triangleq \mathsf{U}_{i,:}\in\mathbb{R}^r$. Also $\mathsf{P}_{ii} = \|u_i\|_2^2 = \ell_i$ and $\sum_{i=1}^n \ell_i = \mathsf{trace}(\mathsf{P}) = r$. By Cauchy–Schwarz inequality,

$$|\mathsf{P}_{ij}| = |u_i^\top u_j| \leq \|u_i\|_2\|u_j\|_2 \leq \max_k \|\mathsf{U}_{k,:}\|_2^2 = \max_k \ell_k = \frac{r}{n}\mathsf{coh}(\mathsf{U}). \tag{222}$$

Taking the maximum over $i, j$ gives $\|\mathsf{P}\|_{\ell_\infty} \leq \frac{r}{n}\mathsf{coh}(\mathsf{U})$. For the lower bound, note that the Frobenius norm satisfies

$$\|\mathsf{P}\|_F^2 = \mathsf{trace}(\mathsf{P}^2) = \mathsf{trace}(\mathsf{P}) = r. \tag{223}$$

Since $\|\mathsf{P}\|_F^2 = \sum_{i,j} \mathsf{P}_{ij}^2 \leq n^2 \|\mathsf{P}\|_{\ell_\infty}^2$, we obtain

$$n^2 \|\mathsf{P}\|_{\ell_\infty}^2 \geq r, \tag{224}$$

and thus $\|\mathsf{P}\|_{\ell_\infty} \geq \frac{\sqrt{r}}{n}$. This proves the claim. ∎

**Proof** [Proof of Lemma 34] By triangle inequality, note that,

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty} \leq \|\mathsf{PY}\|_{\ell_\infty} + \|\mathsf{YP}\|_{\ell_\infty} + \|\mathsf{PYP}\|_{\ell_\infty} \tag{225}$$

Since $\mathsf{Y}$ is symmetric, $\|\mathsf{YP}\|_{\ell_\infty} = \|\mathsf{PY}\|_{\ell_\infty}$. Moreover,

$$\|\mathsf{PYP}\|_{\ell_\infty} \leq \|\mathsf{P}\|_{\ell_\infty \to \ell_\infty} \|\mathsf{PY}\|_{\ell_\infty}. \tag{226}$$

Therefore,

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty} \leq (2 + \|\mathsf{P}\|_{\ell_\infty \to \ell_\infty}) \cdot \|\mathsf{PY}\|_{\ell_\infty}. \tag{227}$$

Let us bound $\|\mathsf{P}\|_{\ell_\infty \to \ell_\infty}$. Note that $\mathsf{U}$ is supported on a set $S \subset [n]$ of size $|v(\Gamma)|$ (i.e., $\mathsf{U}_{i,:} = 0$ for $i \notin S$). Thus, for any $i \notin S$, we have $\mathsf{P}_{i,:} = 0$. For $i \in S$,

$$\|\mathsf{P}_{i,:}\|_1 = \sum_{j \in S} |\langle \mathsf{U}_{i,:}, \mathsf{U}_{j,:} \rangle| \leq \sum_{j \in S} \|\mathsf{U}_{i,:}\|_2 \cdot \|\mathsf{U}_{j,:}\|_2 = \|\mathsf{U}_{i,:}\|_2 \cdot \sum_{j \in S} \|\mathsf{U}_{j,:}\|_2, \tag{228}$$

by Cauchy–Schwarz on each inner product. Apply Cauchy–Schwarz to the sum over $j \in S$:

$$\sum_{j \in S} \|\mathsf{U}_{j,:}\|_2 \leq \sqrt{|v(\Gamma)|} \left( \sum_{j \in S} \|\mathsf{U}_{j,:}\|_2^2 \right)^{1/2} = \sqrt{|v(\Gamma)|} \cdot \|\mathsf{U}\|_F = \sqrt{|v(\Gamma)| r}, \tag{229}$$

since $\|\mathsf{U}\|_F^2 = \mathrm{tr}(\mathsf{U}^\top \mathsf{U}) = r$, and rows outside $S$ are zero. From the definition of coherence,

$$\|\mathsf{U}_{i,:}\|_2 \leq \max_t \|\mathsf{U}_{t,:}\|_2 = \sqrt{\frac{\mathsf{coh}(\mathsf{U}) r}{n}}. \tag{230}$$

Combining the above,

$$\|\mathsf{P}_{i,:}\|_1 \leq \sqrt{\frac{\mathsf{coh}(\mathsf{U}) r}{n}} \cdot \sqrt{kr} = \sqrt{\frac{|v(\Gamma)| \mathsf{coh}(\mathsf{U}) r}{n}}. \tag{231}$$

Taking the maximum over $i$ gives

$$\|\mathsf{P}\|_{\ell_\infty \to \ell_\infty} = \max_i \|\mathsf{P}_{i,:}\|_1 \leq \sqrt{\frac{|v(\Gamma)| \mathsf{coh}(\mathsf{U}) r}{n}}. \tag{232}$$

Thus, substituting (232) in (227) we obtain

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{Y})\|_{\ell_\infty} \le \left(2 + \sqrt{\frac{|v(\Gamma)|\mathsf{coh}(\mathsf{U})r}{n}}\right) \cdot \|\mathsf{PY}\|_{\ell_\infty}. \tag{233}$$

Finally, for a diagonal matrix $\mathsf{D}$, we have $[\mathsf{PD}]_{ij} = \mathsf{P}_{ij}\mathsf{D}_{jj}$, for any $i, j \in [n]$, and thus

$$\|\mathsf{PD}\|_{\ell_\infty} \le \|\mathsf{P}\|_{\ell_\infty} \cdot \|\mathsf{D}\|_{\ell_\infty} \le \|\mathsf{U}\|_{2,\infty}^2 \cdot \|\mathsf{D}\|_{\ell_\infty}, \tag{234}$$

so that

$$\|\mathcal{P}_{\mathcal{T}}(\mathsf{D})\|_{\ell_\infty} \le 3\,\|\mathsf{PD}\|_{\ell_\infty} \le 3\|\mathsf{U}\|_{2,\infty}^2 \cdot \|\mathsf{D}\|_{\ell_\infty} = 3\frac{\mathsf{coh}(\mathsf{U})r}{n} \cdot \|\mathsf{D}\|_{\ell_\infty}. \tag{235}$$

■

## Appendix F. Computational Bounds

### F.1. Lower bound

We follow the same notations and definitions established in Section C. Recall that our goal is to upper bound

$$\mathsf{Corr}_{\le D}^2 \le \sum_{\alpha\in\{0,1\}^N:|\alpha|\le D} \frac{\kappa_\alpha^2}{(q_n(1-p_n))^{|\alpha|}}, \tag{236}$$

where $\kappa_\alpha = \mathbb{E}[x \cdot \mathsf{X}^\alpha] - \sum_{0\le\beta\lneq\alpha} \kappa_\beta \mathbb{E}[\mathsf{X}^{\alpha-\beta}]$. Equivalently, $\kappa_\alpha$ is the joint cumulant of one copy of $x$ and the set of coordinates $\{\mathsf{X}_e : \alpha_e = 1\}$. Therefore, we see that the task of upper bounding the correlation reduces to the problem of upper bounding the joint cumulants. To that end, we introduce a few important notations.

**Definition 35 (Rooted pattern)** *A rooted pattern is a finite simple graph $\mathsf{H} = (v(\mathsf{H}), e(\mathsf{H}))$ together with a distinguished vertex $r^\star \in v(\mathsf{H})$, called the root. We say that $\mathsf{H}$ is* connected *if the underlying graph is connected.*

We recall the definitions of injective homomorphism and embedding

**Definition 36 (Injective homomorphism and embedding)** *Let $\mathsf{G} = (v(\mathsf{G}), e(\mathsf{G}))$ be a finite graph. An* injective homomorphism *(or* embedding*) $\psi : \mathsf{H} \hookrightarrow \mathsf{G}$ is an injective map $\psi : v(\mathsf{H}) \to v(\mathsf{G})$ such that*

$$\{u, v\} \in e(\mathsf{H}) \implies \{\psi(u), \psi(v)\} \in e(\mathsf{G}). \tag{237}$$

*Thus, an embedding preserves adjacency and does not identify distinct vertices of $\mathsf{H}$.*

**Definition 37 (Root-preserving embedding)** *Let $\mathsf{H} = (v(\mathsf{H}), e(\mathsf{H}), r^\star)$ be a rooted pattern and let $v \in v(\mathsf{G})$ be a vertex of a graph $v(\mathsf{G})$. A* root-preserving embedding *of $\mathsf{H}$ into $\mathsf{G}$ with root at $v$ is an injective homomorphism $\psi : \mathsf{H} \hookrightarrow \mathsf{G}$ such that $\psi(r^\star) = v$.*

**Definition 38 (Rooted embedding counts)** *Fix integers $t \geq 2$ and $d \geq 1$. For a template graph $\Gamma_n$ and a vertex $v \in v(\Gamma_n)$, we define*

$$\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n, v) \triangleq |\{\psi : \mathsf{H} \hookrightarrow \Gamma_n : \text{ H is a connected } r^\star\text{-rooted pattern with } |v(\mathsf{H})| = t,\ |e(\mathsf{H})| = d,\ \psi(r^\star) = v\}| . \tag{238}$$

*That is, $\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n; v)$ is the number of root-preserving embeddings of all connected rooted patterns with $t$ vertices and $d$ edges, where the root of the pattern is mapped to the specified vertex $v$ of $\Gamma_n$. We also define the total count*

$$\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n) \triangleq \sum_{v \in v(\Gamma_n)} \mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n; v), \tag{239}$$

*which sums over all possible root locations in $\Gamma_n$. Finally, we define*

$$\mathsf{Emb}_{t,d}(\Gamma_n; v) \triangleq \sum_{\substack{\mathsf{H}:\, |v(\mathsf{H})|=t,\, |e(\mathsf{H})|=d \\ \text{H is connected root } r^\star}} \mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n; v), \tag{240}$$

$$\mathsf{Emb}_{t,d}(\Gamma_n) \triangleq \sum_{v \in v(\Gamma_n)} \mathsf{Emb}_{t,d}(\Gamma_n; v). \tag{241}$$

The role of the root in the above definition is to enforce alignment with a fixed *anchor* vertex in the ambient graph (say vertex $1 \in [n]$). The distinction between $\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n, v)$ and $\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n)$ is that the former counts embeddings anchored at a specific template vertex, while the latter sums over all template roots. Next, let $\mathsf{H}$ be a rooted pattern with $t$ vertices and $d$ edges. Let $\phi : v(\Gamma_n) \hookrightarrow [n]$ be a uniformly random injection, and set $\Gamma_n^\star = \phi(\Gamma_n)$. Consider the event

$$\mathcal{E}(\mathsf{H}) \triangleq \{\exists\, v \in v(\Gamma_n) \text{ and } \psi : \mathsf{H} \hookrightarrow \Gamma_n \text{ injective with } \psi(r^\star) = v, \text{ and } (\phi \circ \psi)(r^\star) = v^\star\} . \tag{242}$$

Intuitively, $\mathcal{E}(\mathsf{H})$ is the event that the rooted pattern $\mathsf{H}$ appears inside the planted copy $\Gamma_n^\star$ with the root landing at the ambient anchor $v^\star$. The following lemma is a key ingredient in the proof of our upper bound on $\mathsf{Corr}_{\leq D}$.

**Lemma 39** *For any connected rooted pattern $\mathsf{H}$ as above,*

$$\mathbb{P}[\mathcal{E}(\mathsf{H})] = \frac{1}{|v(\Gamma_n)|} \cdot \frac{\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n)}{(n-1)_{t-1}}, \tag{243}$$

*where $(n-1)_{t-1} \triangleq (n-1)(n-2)\cdots(n-t+1)$.*

**Proof** [Proof of Lemma 39] Let $\mathcal{R} \triangleq \phi^{-1}(v^\star)$ be the (random) template vertex mapped by $\phi$ to the ambient anchor $v^\star$. Because $\phi$ is a uniform injection, $\mathcal{R}$ is uniform on $v(\Gamma_n)$, hence

$$\mathbb{P}[\mathcal{E}(\mathsf{H})] = \frac{1}{|v(\Gamma_n)|} \sum_{v \in v(\Gamma_n)} \mathbb{P}[\mathcal{E}(\mathsf{H}) | \mathcal{R} = v] . \tag{244}$$

Fix $v \in v(\Gamma_n)$. Conditional on $\mathcal{R} = v$, we have $\phi(v) = v^\star$, and the remaining $k-1$ template vertices are mapped to $[n] \setminus \{v^\star\}$ uniformly without replacement. For a *fixed* root-preserving template embedding $\psi : \mathsf{H} \hookrightarrow \Gamma_n$ with $\psi(r^\star) = v$, define the event

$$\mathcal{E}_\psi \triangleq \{\phi(\psi(w)) = a_w \text{ for all } w \in v(\mathsf{H}) \setminus \{r^\star\}\}, \tag{245}$$

where $\{a_w : w \neq r\} \subset [n] \setminus \{v^\star\}$ is any *fixed* set of $t-1$ distinct ambient vertices that we want the non-root pattern vertices to land on. Conditional on $\mathcal{R} = v$, the probability that $\phi$ realizes $\mathcal{E}_\psi$ is exactly

$$\mathbb{P}\left[\mathcal{E}(\mathsf{H}) | \mathcal{R} = v\right] = \frac{1}{(n-1)_{t-1}}, \tag{246}$$

because once $\phi(v) = v^\star$ is fixed, the remaining $t-1$ pattern vertices must occupy the $t-1$ distinct labels $\{a_w\}$ in a one-to-one fashion, and the remaining $|v(\Gamma_n)| - 1$ labels are uniform without replacement.

Crucially, for a fixed $v$, the events $\{\mathcal{E}_\psi\}_\psi$ over *distinct* root-preserving embeddings $\psi$ are disjoint: two different $\psi$'s require $\phi$ to send (at least) one different template vertex to a *specific* ambient label; since $\phi$ is injective and the target $\{a_w\}$ has size $t-1$, no single $\phi$ can satisfy two distinct $\psi$'s simultaneously. Therefore

$$\mathbb{P}\left[\mathcal{E}(\mathsf{H}) | \mathcal{R} = v\right] = \sum_{\psi:\, \psi(r^\star) = v} \mathbb{P}(\mathcal{E}_\psi | \mathcal{R} = v) \tag{247}$$

$$= \mathsf{Emb}^{\text{vert}}_{t,d}(\mathsf{H} \to \Gamma_n; v) \cdot \frac{1}{(n-1)_{t-1}}. \tag{248}$$

Averaging over $v$ gives

$$\mathbb{P}\left[\mathcal{E}(\mathsf{H})\right] = \frac{1}{|v(\Gamma_n)|} \sum_{v \in v(\Gamma_n)} \mathsf{Emb}^{\text{vert}}_{t,d}(\mathsf{H} \to \Gamma_n; v) \cdot \frac{1}{(n-1)_{t-1}} \tag{249}$$

$$= \frac{1}{|v(\Gamma_n)|} \cdot \frac{\mathsf{Emb}_{t,d}(\mathsf{H} \to \Gamma_n)}{(n-1)_{t-1}}, \tag{250}$$

which comcludes the proof. ∎

We are now ready to state and prove our upper bound on $\mathsf{Corr}_{\leq D}$.

**Lemma 40** *Let $D \geq 1$ and $0 < q_n < p_n < 1$. With the notation above, let $\lambda_n = (p_n - q_n)/(\sqrt{q_n(1-p_n)})$. Then*

$$\mathsf{Corr}^2_{\leq D} \leq \mathbb{E}^2[x] + \sum_{d=1}^{D} \sum_{t=2}^{(d+1)\wedge|v(\Gamma_n)|} [(d+1)t\lambda_n]^{2d} \frac{\mathsf{Emb}_{t,d}(\Gamma_n)}{|v(\Gamma_n)|^2 (n-1)_{t-1}}, \tag{251}$$

*where $(n-2)^{t-2} \triangleq (n-2)(n-3)\cdots(n-t)$. Consequently,*

$$\mathsf{MMSE}_{\leq D} \geq \mathsf{Var}(x) - \sum_{d=1}^{D} \sum_{t=2}^{(d+1)\wedge|v(\Gamma_n)|} [(d+1)t\lambda_n]^{2d} \frac{\mathsf{Emb}_{t,d}(\Gamma_n)}{|v(\Gamma_n)|^2 (n-1)_{t-1}}. \tag{252}$$

**Proof** [Proof of Lemma 40] Recall that $\{\mathsf{X}_e\}_e$ are the mean parameters of the Bernoulli observations, i.e., $\mathsf{X}_e = \mathbb{E}[\mathsf{Y}_e|\Gamma_n] \in \{q_n, p_n\}$, and $x = \mathbb{1}\{v^\star \in \Gamma_n\}$. Using (Schramm and Wein, 2022, Claim 2.14) and (Schramm and Wein, 2022, Prop. 2.13), with $\mathsf{I}_e \triangleq \mathbb{1}\{e \in \Gamma_n\}$ we have

$$\kappa_\alpha = \kappa(x, \{\mathsf{X}_e\}_{e\in\alpha}) = (p_n - q_n)^{|\alpha|}\kappa(x, \{\mathsf{I}_e\}_{e\in\alpha}), \tag{253}$$

and thus

$$\mathsf{Corr}^2_{\leq D} \leq \sum_{\alpha\in\{0,1\}^N:|\alpha|\leq D} \lambda_n^{2|\alpha|}(\kappa(x, \{\mathsf{I}_e\}_{e\in\alpha}))^2, \tag{254}$$

where $\lambda_n \triangleq \frac{p_n - q_n}{\sqrt{q_n(1-p_n)}}$. Next, fix any edge-set $\alpha \subset \binom{[n]}{2}$ of size $d = |\alpha|$, whose union with a fixed anchor $v^\star$ spans $t$ vertices; let $\mathsf{H}_\alpha$ be the induced ambient rooted pattern (root at $v^\star$). Note that $t \leq d + 1$. Define

$$\mathsf{Z}_0 \triangleq x = \mathbb{1}\{v^\star \in \Gamma_n\}, \tag{255}$$

$$\mathsf{Z}_j \triangleq \mathbb{1}\{e_j \in \Gamma_n\}, \tag{256}$$

for $1 \leq j \leq d$, where $\{e_1, \ldots, e_d\} = \alpha$. By the combinatorial formula for joint cumulants (see, (Schramm and Wein, 2022, Def. 2.10)),

$$\kappa(\mathsf{Z}_0, \ldots, \mathsf{Z}_d) = \sum_{\pi\in\mathcal{P}_{d+1}} (|\pi| - 1)!(-1)^{|\pi|-1} \prod_{B\in b(\pi)} \mathbb{E}\left[\prod_{j\in B} \mathsf{Z}_j\right], \tag{257}$$

where $\mathcal{P}_{d+1}$ denotes the set of all partitions of $[d + 1]$ (that is, partitions of $d + 1$ labeled elements into nonempty, unlabeled blocks). For a partition $\pi \in \mathcal{P}_{d+1}$, we write $b(\pi)$ for the collection of its blocks and $|\pi|$ for the number of blocks. Taking absolute values and bounding $\mathbb{E}\left[\prod_{j\in B} \mathsf{Z}_j\right] \leq 1$, and using

$$\sum_{\pi\in\mathcal{P}_{d+1}} (|\pi| - 1)! = \sum_{k=1}^{d+1} S(d + 1, k)(k - 1)! \leq \sum_{k=1}^{d+1} k^d \leq (d + 1)^{d+1} \triangleq C_d, \tag{258}$$

we obtain

$$|\kappa(\mathsf{Z}_0, \ldots, \mathsf{Z}_d)| \leq C_d \cdot \max_{\pi\in\mathcal{P}_{d+1}} \prod_{B\in b(\pi)} \mathbb{E}\left[\prod_{j\in B} \mathsf{Z}_j\right]. \tag{259}$$

Among all blocks $B$, the block containing $\mathsf{Z}_0 = x$ yields the smallest event, hence for every partition $\pi$,

$$\prod_{B\in b(\pi)} \mathbb{E}\left[\prod_{j\in B} \mathsf{Z}_j\right] \leq \mathbb{E}\left[x \prod_{e\in\alpha} \mathsf{I}_e\right] \tag{260}$$

$$= \mathbb{P}\left\{\{v^\star\} \cup \alpha \subseteq \Gamma_n\right\}. \tag{261}$$

Therefore

$$|\kappa(x, \{\mathsf{I}_e\}_{e\in\alpha})| \leq C_d \cdot \mathbb{P}\{\{v^\star\} \cup \alpha \subseteq \Gamma_n\}. \tag{262}$$

But the event $\{v^\star\} \cup \alpha \subseteq \Gamma_n$ is exactly $\mathcal{E}(\mathsf{H}_\alpha)$ in (242). Thus, by the Lemma 39

$$\mathbb{P}\left\{\{v^\star\} \cup \alpha \subseteq \Gamma_n\right\} = \frac{1}{|v(\Gamma_n)|} \cdot \frac{\mathsf{Emb}_{t,d}(\mathsf{H}_\alpha \to \Gamma_n)}{(n-1)_{t-1}} \tag{263}$$

$$\leq \frac{1}{|v(\Gamma_n)|} \cdot \frac{\mathsf{Emb}_{t,d}(\Gamma_n)}{(n-1)_{t-1}}, \tag{264}$$

where the inequality follows from the definition in (241). Combining that above gives

$$|\kappa(x, \{\mathsf{I}_e\}_{e\in\alpha})| \leq \frac{C_d}{|v(\Gamma_n)|} \cdot \frac{\mathsf{Emb}_{t,d}(\Gamma_n)}{(n-1)_{t-1}}. \tag{265}$$

Now, for fixed $d, t$, the number of ambient edge-sets $\alpha$ with $|\alpha| = d$ whose union with $v^\star$ spans exactly $t$ vertices is at most

$$\binom{n-1}{t-1}\binom{\binom{t}{2}}{d} \leq (n-1)_{t-1} t^{2d} \leq n^{t-1} t^{2d}. \tag{266}$$

Indeed, choose the additional $t-1$ vertices among $n-1$ options, then choose $d$ edges among the $\binom{t}{2}$ possible on those $t$ vertices. The crude bounds $\binom{n-1}{t-1} \leq (n-1)_{t-1} \leq n^{t-1}$ and $\binom{\binom{t}{2}}{d} \leq \binom{t^2/2}{d} \leq t^{2d}$ yield the claim. Grouping the sum (254) by $(t, d)$, and applying (265)–(266), we obtain

$$\mathsf{Corr}^2_{\leq D} \leq \mathbb{E}^2[x] + \sum_{d=1}^{D} \sum_{t=2}^{(d+1)\wedge|v(\Gamma_n)|} (n-1)_{t-1} t^{2d} \left((d+1)^d \lambda_n^d \frac{\mathsf{Emb}_{t,d}(\Gamma_n)}{|v(\Gamma_n)|(n-1)_{t-1}}\right)^2 \tag{267}$$

$$= \mathbb{E}^2[x] + \sum_{d=1}^{D} \sum_{t=2}^{(d+1)\wedge|v(\Gamma_n)|} [(d+1)t\lambda_n]^{2d} \frac{\mathsf{Emb}^2_{t,d}(\Gamma_n)}{|v(\Gamma_n)|^2 (n-1)_{t-1}}. \tag{268}$$

Finally, the identity $\mathsf{MMSE}_{\leq D} = \mathbb{E}[x] - \mathsf{Corr}^2_{\leq D}$ yields the MMSE bound. ■

Next, to prove Theorem 23, we show that, roughly speaking, the slice corresponding to $(t, d) = (2, 1)$ in the sum in (251) dominates the entire expression. The slice $(t, d) = (2, 1)$ corresponds to a single planted edge incident to the anchor. In this case, the root-averaged count equals the *average degree* of the planted template:

$$\mathsf{Emb}_{2,1}(\Gamma_n) = \sum_{v \in v(\Gamma_n)} \mathsf{deg}_{\Gamma_n}(v) = 2 \cdot |e(\Gamma_n)|. \tag{269}$$

This in turn implies that $\mathsf{Corr}^2_{\leq D} = o(1)$ (and hence recovery is computationally hard) whenever

$$\frac{\mathsf{Emb}^2_{2,1}(\Gamma_n)}{|v(\Gamma_n)|^2 n} \ll 1 \implies \eta(\Gamma_n) = \frac{|e(\Gamma_n)|}{|v(\Gamma_n)|} \ll \sqrt{n}. \tag{270}$$

Let us now establish this rigorously.

**Lemma 41** *Let $\Gamma_n$ be a simple graph on $k$ vertices with average degree*

$$\bar{d}(\Gamma_n) = \frac{1}{k} \sum_{u \in v(\Gamma_n)} \mathsf{deg}_{\Gamma_n}(u) = \frac{2|e(\Gamma_n)|}{k}. \tag{271}$$

*Fix integers $t \geq 2$ and $d \geq t-1$. There exists an absolute constant $C \geq 1$ such that*

$$\frac{1}{k}\mathsf{Emb}_{t,d}(\Gamma_n) \leq \big(Ct\big)^{C(d+t)} \bar{d}(\Gamma_n)^{t-1}. \tag{272}$$

We begin with an intuitive sketch of the lemma before giving the formal proof. Our goal is to bound the anchor-averaged number of rooted embeddings of any $t$-vertex, $d$-edge connected pattern into $\Gamma_n$. The proof reinterprets every embedding as a *growth process*: starting from the root, we reveal the pattern one vertex at a time according to a chosen *exploration scheme* (a spanning tree together with an exposure order). At each step the number of valid extensions is controlled by the *edge boundary* of the already-embedded set in $\Gamma$. Averaging over relabelings makes every vertex of $\Gamma$ "typical", so that on average an $s$-set has boundary size about $s \cdot \bar{d}(\Gamma)$. This exchangeability turns the complicated boundary terms into a clean multiplicative factor, leading to the bound $\prod_{s=1}^{t-1}[s \cdot \bar{d}(\Gamma)]$. The only remaining combinatorial work is to account for the number of possible exploration schemes, which contributes an additional factor $(Ct)^{C(d+t)}$.

**Proof** [Proof of Lemma 41] Let $\mathfrak{H}_{t,d}$ denote the family of connected rooted patterns on $t$ vertices and $d$ edges (root distinguished but otherwise labeled). The proof proceeds in five steps.

*1) Exploration schemes.* For a connected rooted pattern $H \in \mathfrak{H}_{t,d}$, fix a rooted spanning tree $T \subseteq H$ of size $t-1$, and fix an *exploration order* $r = v_1, v_2, \ldots, v_t$ in which each $v_{s+1}$ is adjacent in $T$ to some earlier vertex among $v_1, \ldots, v_s$. An *exploration scheme* $\mathsf{S}$ comprises:

(i) the rooted spanning tree $T$;

(ii) the exploration order;

(iii) the choice of the extra (non-tree) edges of $H$ (there are $d-(t-1)$ of them).

Denote by $\mathcal{S}_{t,d}$ the set of all exploration schemes over all $H \in \mathfrak{H}_{t,d}$. Let us bound its cardinality. A crude combinatorial bound suffices: choose $T$ (at most $t^{t-2}$ rooted labeled trees by Cayley), choose an order (at most $t!$), and choose the extra edges (at most $\binom{\binom{t}{2}}{d-(t-1)} \leq (Ct)^{2(d-(t-1))}$). Thus

$$|\mathcal{S}_{t,d}| \leq t^{t-2} \cdot t! \cdot (Ct)^{2(d-(t-1))} \leq (Ct)^{C(d+t)}. \tag{273}$$

*2) Partial histories and the boundary recursion.* Fix a relabeling $\sigma$ of $v(\Gamma_n)$, and write $\Gamma_n^\sigma$ for the relabeled graph. For an exploration scheme $\mathsf{S} \in \mathcal{S}_{t,d}$ and an anchor $v \in v(\Gamma_n^\sigma)$, a *partial history of length $s$* ($1 \leq s \leq t$) is a choice of an injective map sending $v_1, \ldots, v_s$ to distinct vertices of $v(\Gamma_n^\sigma)$ that respects the (tree) adjacencies required by $\mathsf{S}$ among $v_1, \ldots, v_s$. Let $\mathcal{F}_s(\sigma)$ be the multiset of all partial histories of length $s$, formed by ranging over all anchors $v \in v(\Gamma_n^\sigma)$ and all schemes $\mathsf{S} \in \mathcal{S}_{t,d}$.

For $f \in \mathcal{F}_s(\sigma)$ write $U(f) \subseteq v(\Gamma_n^\sigma)$ for the current image (so $|U(f)| = s$), and

$$\partial_{\Gamma_n^\sigma}(U) \triangleq \big|\{\{x,y\} \in e(\Gamma_n^\sigma) : x \in U,\ y \notin U\}\big| \tag{274}$$

for the (undirected) edge boundary size of $U$. Each extension from $s$ to $s+1$ chooses the next image vertex $v_{s+1}$ outside $U(f)$ that is adjacent, in $\Gamma_n^\sigma$, to the specified predecessor of $v_{s+1}$ in the tree $T$; consequently, for every $f \in \mathcal{F}_s(\sigma)$, the number of admissible choices is at most $\partial_{\Gamma_n^\sigma}(U(f))$. Summing over all $f$ yields the recursion

$$|\mathcal{F}_{s+1}(\sigma)| \leq \sum_{f \in \mathcal{F}_s(\sigma)} \partial_{\Gamma_n^\sigma}\big(U(f)\big). \tag{275}$$

Indeed, non-tree edges in $\mathsf{S}$ impose additional adjacency constraints and thus only *decrease* the number of admissible choices, so (275) remains valid.

*3) Averaging over relabelings.* By construction, $\mathsf{Emb}_{t,d}(\Gamma_n)$ is invariant under relabelings of $v(\Gamma_n)$, so

$$\mathsf{Emb}_{t,d}(\Gamma_n) = \mathbb{E}_\sigma\left[\mathsf{Emb}_{t,d}(\Gamma_n^\sigma)\right]. \tag{276}$$

Furthermore, every root-preserving embedding counted by $\mathsf{Emb}_{t,d}(\Gamma_n^\sigma)$ corresponds to at least one full history of length $t$ (for some $\mathsf{S}$), so $\mathsf{Emb}_{t,d}(\Gamma_n^\sigma) \leq |\mathcal{F}_t(\sigma)|$. Therefore

$$\mathsf{Emb}_{t,d}(\Gamma_n) \leq \mathbb{E}_\sigma\left[|\mathcal{F}_t(\sigma)|\right]. \tag{277}$$

Taking expectations of (275), we are reduced to bounding

$$\mathbb{E}_\sigma\left[\sum_{f \in \mathcal{F}_s(\sigma)} \partial_{\Gamma_n^\sigma}\big(U(f)\big)\right] \tag{278}$$

in terms of $\mathbb{E}_\sigma\left[|\mathcal{F}_s(\sigma)|\right]$. We have the following lemma.

**Lemma 42 (Average boundary at size $s$)** *For each $1 \leq s \leq t-1$,*

$$\mathbb{E}_\sigma\left[\sum_{f \in \mathcal{F}_s(\sigma)} \partial_{\Gamma_n^\sigma}\big(U(f)\big)\right] \leq s\bar{d}(\Gamma_n) \cdot \mathbb{E}_\sigma\left[|\mathcal{F}_s(\sigma)|\right]. \tag{279}$$

**Proof** [Proof of Lemma 42] For any $U \subseteq v(\Gamma_n^\sigma)$ we have $\partial_{\Gamma_n^\sigma}(U) \leq \sum_{u \in U} \mathsf{deg}_{\Gamma_n^\sigma}(u)$. Thus

$$\sum_{f \in \mathcal{F}_s(\sigma)} \partial_{\Gamma_n^\sigma}(U(f)) \leq \sum_{f \in \mathcal{F}_s(\sigma)} \sum_{u \in U(f)} \mathsf{deg}_{\Gamma_n^\sigma}(u) \tag{280}$$

$$= \sum_{u \in v(\Gamma_n^\sigma)} \mathsf{deg}_{\Gamma_n^\sigma}(u) \cdot N_s(u;\sigma), \tag{281}$$

where $N_s(u;\sigma) \triangleq |\{f \in \mathcal{F}_s(\sigma) : u \in U(f)\}|$ is the multiplicity with which $u$ appears among the size-$s$ images. By vertex-exchangeability under the uniform relabeling $\sigma$, $\mathbb{E}_\sigma[N_s(u;\sigma)]$ is the *same* for all $u$. Moreover,

$$\sum_{u \in v(\Gamma_n^\sigma)} N_s(u;\sigma) = \sum_{f \in \mathcal{F}_s(\sigma)} |U(f)| \tag{282}$$

$$= s|\mathcal{F}_s(\sigma)|, \tag{283}$$

hence $\mathbb{E}_\sigma[N_s(u;\sigma)] = (s/k)\mathbb{E}_\sigma[|\mathcal{F}_s(\sigma)|]$ for every $u$. Taking expectations and using the fact that $\sum_u \mathsf{deg}_{\Gamma_n^\sigma}(u) = 2|e(\Gamma_n)| = k\bar{d}(\Gamma_n)$ gives

$$\mathbb{E}_\sigma\left[\sum_{f\in\mathcal{F}_s(\sigma)} \partial_{\Gamma_n^\sigma}(U(f))\right] \le \sum_u \mathsf{deg}_{\Gamma_n}(u)\cdot\frac{s}{k}\cdot\mathbb{E}_\sigma[|\mathcal{F}_s(\sigma)|] \tag{284}$$

$$= s\bar{d}(\Gamma_n)\cdot\mathbb{E}_\sigma[|\mathcal{F}_s(\sigma)|], \tag{285}$$

proving (279). ■

*4) Induction and base size.* Combining (275) and Lemma 42, and iterating for $s = 1, 2, \ldots, t-1$, we obtain

$$\mathbb{E}_\sigma\big[|\mathcal{F}_t(\sigma)|\big] \le \left(\prod_{s=1}^{t-1} s\bar{d}(\Gamma_n)\right)\cdot\mathbb{E}_\sigma\big[|\mathcal{F}_1(\sigma)|\big] \tag{286}$$

$$\le t!\bar{d}(\Gamma_n)^{t-1}\cdot\mathbb{E}_\sigma\big[|\mathcal{F}_1(\sigma)|\big]. \tag{287}$$

A partial history of length 1 consists of choosing an anchor $v \in v(\Gamma_n^\sigma)$ and a scheme $\mathsf{S} \in \mathcal{S}_{t,d}$; hence $|\mathcal{F}_1(\sigma)| = k|\mathcal{S}_{t,d}|$. Using (273) and $t! \le (Ct)^t$, (286) yields

$$\mathbb{E}_\sigma\big[|\mathcal{F}_t(\sigma)|\big] \le k\cdot(Ct)^{C(d+t)}\cdot\bar{d}(\Gamma_n)^{t-1}. \tag{288}$$

*5) Combining.* Combining the above steps together we obtain

$$\mathsf{Emb}_{t,d}(\Gamma_n) = \mathbb{E}_\sigma\big[\mathsf{Emb}_{t,d}(\Gamma_n^\sigma)\big] \tag{289}$$

$$\le \mathbb{E}_\sigma\big[|\mathcal{F}_t(\sigma)|\big] \tag{290}$$

$$\le k\cdot(Ct)^{C(d+t)}\cdot\bar{d}(\Gamma_n)^{t-1}. \tag{291}$$

Dividing both sides by $k$ gives (272). ■

We are now in a position to finish the proof of Theorem 23. We first recall our upper bound in Lemma 40:

$$\mathsf{Corr}^2_{\le D} \le \mathbb{E}^2[x] + \sum_{d=1}^{D}\sum_{t=2}^{(d+1)\wedge k_n}[(d+1)t\lambda_n]^{2d}\,\frac{\mathsf{Emb}_{t,d}(\Gamma_n)^2}{k_n^2\,(n-1)_{t-1}}. \tag{292}$$

Lemma 41 state that

$$\frac{1}{k_n}\mathsf{Emb}_{t,d}(\Gamma_n) \le (Ct)^{C(d+t)}\bar{d}(\Gamma_n)^{t-1}, \tag{293}$$

for a universal $C \ge 1$. Since $t \le d+1 \le D+1 = o(n)$, we have the falling-factorial lower bound

$$(n-1)_{t-1} \ge \left(\frac{n}{2}\right)^{t-1}, \tag{294}$$

for all large $n$. Define

$$r_n \triangleq \frac{\bar{d}(\Gamma_n)^2}{n} \le n^{-2\varepsilon}, \tag{295}$$

where the inequality follows from the assumptions of Theorem 23, for any $\varepsilon > 0$. Plugging (293) and (294) into (292), and noting the connectivity constraint $d \geq t-1$, we obtain

$$\mathsf{Corr}_{\leq D}^2 \leq \mathbb{E}^2[x] + \sum_{t=2}^{D+1} \sum_{d=t-1}^{\min\{D,\binom{t}{2}\}} [(d+1)t\lambda_n]^{2d}(Ct)^{2C(d+t)} \cdot (2r_n)^{t-1} \tag{296}$$

$$= \mathbb{E}^2[x] + \sum_{t=2}^{D+1} \Phi_t \cdot (2r_n)^{t-1}, \tag{297}$$

where for any $t \geq 2$,

$$\Phi_t \triangleq \sum_{d=t-1}^{\min\{D,\binom{t}{2}\}} [(d+1)t\lambda_n]^{2d}(Ct)^{2C(d+t)}. \tag{298}$$

Because we only sum over simple connected patterns, we always have $d \leq \binom{t}{2} \leq t^2/2$. For such $d$,

$$(d+1)t\lambda_n \leq ct^3, \tag{299}$$

for some $c \geq 1$, and hence

$$[(d+1)t\lambda_n]^{2d} \leq (ct^3)^{2d} \leq (ct^3)^{t^2} = \exp\left(t^2 \log(ct^3)\right) \leq \exp\left(C_1\, t^2 \log t\right), \tag{300}$$

for a universal $C_1$. Also,

$$(Ct)^{2C(d+t)} \leq (Ct)^{2C(t^2/2+t)} \leq \exp\left(C_2 t^2 \log t\right). \tag{301}$$

Multiplying the two results above and summing over at most $\binom{t}{2} - (t-1) + 1 = O(t^2)$ values of $d$, we can absorb the polynomial factor into the exponential envelope and get

$$\Phi_t \leq \exp\left(C'' t^2 \log t\right), \tag{302}$$

for for some universal $C'' \geq 1$. Now, for $t \geq 2$ define

$$\alpha_t \triangleq \exp\left(C'' t^2 \log t\right) \cdot (2r_n)^{t-1}. \tag{303}$$

Then (297) and (302) give

$$\mathsf{Corr}_{\leq D}^2 \leq \mathbb{E}^2[x] + \sum_{t=2}^{D+1} \alpha_t. \tag{304}$$

We next show that $\sum_{t=2}^{D+1} \alpha_t = o(1)$.

**Case I.** Consider the regime where $D = o\left(\frac{\log n}{\log\log n}\right)$, and let $t_\star \triangleq D+1$. Since $r_n = n^{-2\varepsilon}$,

$$\log \alpha_{t_\star} \leq C'' D^2 \log D - 2\varepsilon D \log n + D \log 2. \tag{305}$$

Since $D \log D = o(\log n)$ in this regime, the negative term dominates, so $\log \alpha_{t_\star} = -\omega(1)$ and thus $\alpha_{t_\star} = n^{-\omega(1)}$. Because the sum in (304) has at most $D = o(\frac{\log n}{\log\log n})$ terms and the sequence $\{a_t\}$ is positive,

$$\sum_{t=2}^{D+1} \alpha_t \leq D \cdot \alpha_{t_\star} = o(1). \tag{306}$$

**Case II.** Consider the case where $D \leq (\log n)^\alpha$, with a fixed $\alpha < 1$. Again take $t_\star \triangleq D + 1 \leq 2(\log n)^\alpha$ and note that

$$\log \alpha_{t_\star} \leq C'' t_\star^2 \log t_\star - 2\varepsilon(t_\star - 1)\log n + (t_\star - 1)\log 2. \tag{307}$$

Using $t_\star \leq 2(\log n)^\alpha$ and $\log t_\star \leq \log[2(\log n)^\alpha] = O(\log\log n)$, we get

$$\log \alpha_{t_\star} \leq C_3 (\log n)^{2\alpha} \log\log n - 2\varepsilon(\log n)^{\alpha+1} + O\left((\log n)^\alpha\right). \tag{308}$$

Because $\alpha + 1 > 2\alpha$ for every $\alpha < 1$, the negative term dominates, hence $\log \alpha_{t_\star} = -\omega(1)$ and $\alpha_{t_\star} = n^{-\omega(1)}$. Since the sum in (304) has at most $D \leq (\log n)^\alpha$ terms,

$$\sum_{t=2}^{D+1} \alpha_t \leq D \cdot \alpha_{t_\star} = o(1). \tag{309}$$

Combining Case I or II with (304) yields $\mathsf{Corr}^2_{\leq D} \leq \mathbb{E}^2[x] + o(1)$. This concludes the proof of Theorem 23.

## F.2. Upper bounds

**Single-iteration power method.** Following the approach of (Schramm and Wein, 2022, Sec. 4.2), to derive an upper bound on the truncated MMSE we analyze the performance of a simple algorithm: a single round of the power iteration method initialized with the all-ones vector, followed by thresholding. We begin by recalling two key results from Schramm and Wein (2022). Throught this section, we let $\nu_{p,q} \triangleq \min\{p, 1-p, q, 1-q\}$ and we recall that $\lambda \triangleq (p-q)/\sqrt{q(1-p)}$.

**Lemma 43** *(Schramm and Wein, 2022, Prop. 4.1) There is a universal constant $c_0 \in (0,\infty)$ such that for each $k \in \mathbb{N}$ there exists a degree-$(2k+1)$ polynomial $\tau_k : \mathbb{R} \to \mathbb{R}$ with:*

$$\text{for } \ell \in \{0,1\}, \quad |y - \ell| \leq \Delta \leq \tfrac{1}{2} \implies |\tau_k(y) - \ell| \leq (k + \tfrac{1}{2})(c_0\Delta)^k. \tag{310}$$

**Lemma 44** *(Schramm and Wein, 2022, Thm. 4.2) Let $f$ be a degree-$D$ polynomial in the independent edge-variables $\{Y_{ij}\}_{1\leq i<j\leq n}$, where each $Y_{ij} \in \{0,1\}$ with parameter $p$ or $q$. Then*

$$\mathbb{E}\left[f^4(Y)\right] \leq \left(\frac{9}{\nu_{p,q}}\right)^D \mathbb{E}^2\left[f^2(Y)\right]. \tag{311}$$

The first lemma provides a subroutine for constructing a polynomial approximation to the threshold function. The second lemma is a hypercontractivity result for mixed Bernoulli random variables, which, roughly speaking, asserts that the moments of low-degree polynomials are well behaved.

Next, let us define the proposed estimator. Fix a normalization $d_\star > 0$ to be specified. Define the centered/rescaled row-sum at vertex 1:

$$g(\mathsf{Y}) \triangleq \frac{1}{(p-q)d_\star} \sum_{i=2}^{n} (\mathsf{Y}_{1i} - q). \tag{312}$$

We feed $g$ into the *polynomial threshold* $\tau_k$ of degree $2k+1$ in Lemma 43 and set

$$f(\mathsf{Y}) \triangleq \tau_k\left(g(\mathsf{Y})\right), \qquad \deg f = 2k + 1 \triangleq D. \tag{313}$$

Recall that $\phi : v(\Gamma) \hookrightarrow [n]$ is the uniform injection, we set $e^\star = \{\{\phi(u), \phi(v)\} : \{u,v\} \in e(\Gamma)\}$, and $x \triangleq \mathbb{1}\{1 \in \phi(v(\Gamma))\}$. We have the following lemma.

**Lemma 45 (Moments and tails of the row-sum)** *Condition on $\phi$. If $x=0$, then the $\{Y_{1i}-q\}_{i=2}^n$ are i.i.d. centered Bernoullis with variance $q(1-q)$, and*

$$\mathbb{E}[g(\mathsf{Y})|\phi, x=0] = 0, \tag{314}$$

$$\mathsf{Var}(g(\mathsf{Y})|\phi, x=0) = \frac{q(1-q)}{(p-q)^2}\cdot\frac{n-1}{d_\star^2}. \tag{315}$$

*If $x=1$ and $u\in v(\Gamma)$ is the (random) root with $\phi(u)=1$, then exactly $d_\Gamma(u)$ summands have mean $(p-q)$ and the rest have mean 0, hence*

$$\mathbb{E}[g(\mathsf{Y})|\phi, x=1] = \frac{d_\Gamma(u)}{d_\star}, \tag{316}$$

$$\mathsf{Var}(g(\mathsf{Y})|\phi, x=1) \le \frac{p(1-p)+(n-2)q(1-q)}{(p-q)^2}\cdot\frac{1}{d_\star^2} \le \frac{(p+q)(n-1)}{(p-q)^2}\cdot\frac{1}{d_\star^2}. \tag{317}$$

*Moreover, for any $t>0$,*

$$\mathbb{P}\left(|g(\mathsf{Y})|\ge t|\phi, x=0\right) \le 2\exp\left[-\frac{t^2(p-q)^2d_\star^2}{2q(n-1)+\frac{2}{3}t(p-q)d_\star}\right], \tag{318}$$

$$\mathbb{P}\left[\left|g(\mathsf{Y})-\tfrac{d_\Gamma(u)}{d_\star}\right|\ge t\,\middle|\,\phi, x=1\right] \le 2\exp\left(-\frac{t^2(p-q)^2d_\star^2}{2(p+q)(n-1)+\frac{2}{3}t(p-q)d_\star}\right). \tag{319}$$

**Proof** [Proof of Lemma 45] All results follow from independence, the variance computations above, and Bernstein's inequality for sums of bounded mean-zero variables (each summand has range $\le 1$ after centering by $q$ and rescaling). ■

**Lemma 46 (From high probability to mean-square via hypercontractivity)** *Let $Z=f(\mathsf{Y})-x$ for a degree-$D$ polynomial $f$. Suppose for some $\varepsilon>0$ and $\delta\in(0,1)$ we have*

$$\mathbb{P}\{Z^2\le\varepsilon\}\ge 1-\delta. \tag{320}$$

*Then, using (311),*

$$\mathbb{E}[Z^2] \le \varepsilon + \left(\frac{9}{\nu_{p,q}}\right)^{D/2}\mathbb{E}[Z^2]\sqrt{\delta}. \tag{321}$$

*Consequently, if $\delta\le\frac{1}{4}(\nu_{p,q}/9)^D$ then $\mathbb{E}[Z^2]\le 2\varepsilon$; if $\delta\le\frac{1}{16}(\nu_{p,q}/9)^D$ then $\mathbb{E}[Z^2]\le\frac{4}{3}\varepsilon$.*

**Proof** [Proof of Lemma 46] Decompose $\mathbb{E}[Z^2]=\mathbb{E}[Z^2\mathbb{1}_{\mathcal{G}}]+\mathbb{E}[Z^2\mathbb{1}_{\mathcal{B}}]\le\varepsilon+\sqrt{\mathbb{E}[Z^4]\mathbb{P}(\mathcal{B})}$ with $\mathcal{G}=\{Z^2\le\varepsilon\}$ and $\mathcal{B}=\mathcal{G}^c$. Apply (311) to bound $\mathbb{E}[Z^4]\le(9/\nu_{p,q})^D\mathbb{E}[Z^2]^2$ and rearrange. ■

We are now ready to state and prove our main upper bound on the $\mathsf{MMSE}_{\le D}$.

**Theorem 47 (General upper bound)** *Fix $0<r<1$, $k\in\mathbb{N}$, and set $D=2k+1$. Assume there exists $d_\star>0$ such that*

$$\max_{u\in V(\Gamma)}\frac{|d_\Gamma(u)-d_\star|}{d_\star}\le\frac{r}{12}. \tag{322}$$

*Assume further that*

$$(p-q)^2 d_\star^2 \geq \frac{C_1}{r^2}(p+q)(n-1)\left[\log 4 + D\log\frac{9}{\nu_{p,q}}\right], \tag{323}$$

*for an absolute constant $C_1 \geq 432$. Let $g$ be as in (312), let $\tau_k$ be as in Lemma 43, and set $f = \tau_k \circ g$. Then*

$$\mathsf{MMSE}_{\leq D} \leq \mathbb{E}\left[(f(\mathsf{Y})-x)^2\right] \leq C_2 D^2 r^{D-1}, \tag{324}$$

*for an absolute constant $C_2 \geq 1$.*

**Proof** [Proof of Theorem 47] Define

$$\Delta_{\text{noise}} \triangleq \sqrt{\frac{3(p+q)\left[\log 4 + D\,\log(9/\nu_{p,q})\right]}{(p-q)^2 d_\star^2 (n-1)}}. \tag{325}$$

By (323) (with $C_1 \geq 432$), we have $\Delta_{\text{noise}} \leq r/12$. By Lemma 45, using $\Delta_{\text{noise}} \leq r/12$ and a union bound over the two regimes $x \in \{0,1\}$,

$$\mathbb{P}\left[\,|g(\mathsf{Y})-x| \leq r/6|\,\phi\right] \geq 1-\delta_D, \tag{326}$$

where $\delta_D \triangleq \frac{1}{4}\left(\frac{\nu_{p,q}}{9}\right)^D$; the choice of $\delta_D$ follows by setting the exponents in (318)—(319) equal to $\log 4 + D\log(9/\nu_{p,q})$ so that each tail at most $\frac{1}{2}(\nu_{p,q}/9)^D$, and then union bound. Now, on the event $\{|g(\mathsf{Y})-x| \leq r/6\}$, Lemma 43 with $\Delta = r/6$ gives

$$|f(\mathsf{Y})-x| \leq (k+\tfrac{1}{2})\left(c_0\tfrac{r}{6}\right)^k \leq Ckr^k, \tag{327}$$

for $C$ absorbing $c_0^k$ and $6^{-k}$; we may fix $c_0 = 6$ to get $|f-x| \leq (k+\frac{1}{2})r^k$. Hence $(f-x)^2 \leq (k+\frac{1}{2})^2 r^{2k}$ on the good event. Next, we apply Lemma 46 with $\varepsilon = (k+\frac{1}{2})^2 r^{2k}$ and $\delta = \delta_D \leq \frac{1}{4}(\nu_{p,q}/9)^D$. We obtain

$$\mathsf{MMSE}_{\leq D} \leq \mathbb{E}[(f(\mathsf{Y})-x)^2] \leq 2(k+\tfrac{1}{2})^2 r^{2k} \leq C_2\, D^2\, r^{D-1}, \tag{328}$$

since $D = 2k+1$ and $r \in (0,1)$ imply $r^{2k} \leq r^{D-1}$ and $(k+\frac{1}{2})^2 \leq D^2$ up to a fixed constant factor. This completes the proof. ■

Theorem 47 holds true for any choice of $d_\star$. In many cases, the choice $d_\star = \eta(\Gamma_n) = \frac{|e(\Gamma_n)|}{|v(\Gamma_n)|}$ is optimal.

**Corollary 48** *Fix $\epsilon > 0$ and $0 < q < p < 1$. Let $\Gamma_n$ be any template sequence with*

$$\mathsf{Dis}(\Gamma) \triangleq \max_{v\in v(\Gamma)} \frac{|d_\Gamma(v)-\eta(\Gamma)|}{\eta(\Gamma)} \leq \frac{r}{12}, \tag{329}$$

$$\eta(\Gamma_n) \geq n^{\frac{1}{2}+\epsilon}, \tag{330}$$

*for some fixed $0 < r < 1$ and all large $n$. If $D = D(n) \leq (\log n)^\alpha$ for any fixed $\alpha > 0$, then*

$$\mathbb{E}\left[(f(\mathsf{Y})-x)^2\right] \leq C_2\, D^2 r^{D-1} \to 0, \tag{331}$$

*as $n \to \infty$.*

**Proof** [Proof of Corollary 48] The proof follows from Theorem 47. Indeed, (322) is satisfied under assumption (329), while (330) implies that the term on the left-hand side of (323) is $\asymp n^{1+2\epsilon}$, while left-hand side of (323) is $O(n(\log n)^{\alpha})$ for any fixed $\alpha$. Thus (323) holds for large enough $n$. ■

Thus, for any sequence of subgraphs $\Gamma = \Gamma_n$ such that $\mathsf{Dis}(\Gamma)$ is bounded away from one, the MSE of the algorithm proposed above matches and complements the computational lower bound in Theorem 23. While the result above is certainly nontrivial, it does not cover the many cases in which $\mathsf{Dis}(\Gamma)$ fails to satisfy the condition in (329). Consider the following example.

**Example 14** *Split $v(\Gamma)$ into two sets $A$ and $B$ with $|A| = |B| = k/2$: $A$ forms a clique, $B$ is an independent set, and each $b \in B$ is adjacent to exactly $\sqrt{k}$ vertices in $A$. In this construction, $\eta(\Gamma) = \Theta(k)$ and $\mathsf{Dis}(\Gamma) \asymp 1 \not\lesssim \frac{1}{12}$. Hence, Corollary 48 does not apply, since condition (329) fails, whereas our lower bound yields $k \ll \sqrt{n}$. This simple example therefore reveals a gap.*

**Multi-iteration power method.** We now propose a stronger bound obtained by applying $L$ iterations of the power method (rather than the single iteration considered above). Let us define the proposed estimator. For simplicity of notations, we let $\mathsf{Z}_{ij} \triangleq \mathsf{Y}_{ij} - q$, for any $i, j \in [n]$. Fix $L \in \mathbb{N}$, and let $\mathcal{P}_L$ denote the set of all simple undirected paths of length $L$ in the ambient complete graph starting at vertex 1 and pairwise distinct vertices, namely, $P = (u_0, u_1, \ldots, u_\ell)$ with $u_0 = 1$. For each $P \in \mathcal{P}_L$, define

$$\mathsf{Z}(P) \triangleq \prod_{\ell=0}^{L-1} \mathsf{Z}_{u_t u_{t+1}}, \tag{332}$$

and the degree-$L$ *walk polynomial*

$$\mathsf{W}_L \triangleq \sum_{P \in \mathcal{P}_L} \mathsf{Z}(P). \tag{333}$$

Let $B_L \triangleq c_L n^{L/2}$ where $c_L > 0$ is a constant depending only on $(L, p, q)$ to be chosen in the sequel, and set

$$\mathscr{X}_L \triangleq \frac{\mathsf{W}_L}{B_L} = \frac{1}{c_L n^{L/2}} \mathsf{W}_L. \tag{334}$$

By Lemma 43, for an integer $m \geq 1$, let $\tau_m : \mathbb{R} \to [0, 1]$ be a degree-$(2m+1)$ polynomial with the following uniform approximation property: for any $r \in (0, 1/4]$,

$$|y| \leq \tfrac{r}{2} \implies |\tau_m(y) - 0| \leq (6r)^m, \quad y \geq r \implies |\tau_m(y) - 1| \leq (6r)^m. \tag{335}$$

For $u \in v(\Gamma)$ let

$$W_L(\Gamma; u) \triangleq |\{\text{simple paths of length } L \text{ in } \Gamma \text{ starting at } u\}|, \tag{336}$$

$$W_L^{\min}(\Gamma) \triangleq \min_{u \in v(\Gamma)} W_L(\Gamma; u). \tag{337}$$

Recall that $\phi : v(\Gamma) \hookrightarrow [n]$ is the uniform injection, we set $e^\star = \{\{\phi(u), \phi(v)\} : \{u, v\} \in e(\Gamma)\}$, and $x \triangleq \mathbb{1}\{1 \in \phi(v(\Gamma))\}$. We have the following lemma.

**Lemma 49** *Let $C_L \triangleq 2[q(1-q)]^L$. Then,*

$$\mathbb{E}[\mathsf{W}_L||\phi, x=0] = 0, \tag{338}$$

$$\mathsf{Var}(\mathsf{W}_L|\phi, x=0) \le C_L n^L. \tag{339}$$

*Consequently, with $B_L = c_L n^{L/2}$ and $c_L = \sqrt{C_L}$ we have* $\mathsf{Var}(\mathscr{X}_L|\phi, x=0) \le 1$.

**Proof** [Proof of Lemma 49] Conditioned on $\{\phi, x=0\}$ all edges are i.i.d. $\mathsf{Bern}(q)$, hence $\mathbb{E}[\mathsf{Z}_{ij}] = 0$ and $\mathbb{E}[\mathsf{Z}_{ij}^2] = \mathsf{Var}(\mathsf{Y}_{ij}) = q(1-q) \le 1/4$, for any $(i,j) \in \binom{[n]}{2}$. Thus, $\mathbb{E}[\mathsf{W}_L|\phi, x=0] = 0$ follows since every $\mathsf{Z}(P)$ is a product of mean-zero independent factors. For the variance, write

$$\mathsf{Var}(\mathsf{W}_L|\phi, x=0) = \sum_{P\in\mathcal{P}_L} \mathbb{E}[\mathsf{Z}^2(P)|\phi, x=0] + \sum_{\substack{P,P'\in\mathcal{P}_L\\ P\neq P'}} \mathbb{E}[\mathsf{Z}(P)\mathsf{Z}(P')|\phi, x=0], \tag{340}$$

where we have used the fact that all $\mathsf{Z}(P)$ are mean zero. If $P$ and $P'$ do *not* use exactly the same set of edges, then there exists an edge $(i,j)$ that appears in exactly one of $P, P'$, hence independence and $\mathbb{E}[\mathsf{Z}_{ij}] = 0$ imply $\mathbb{E}[\mathsf{Z}(P)\mathsf{Z}(P')] = 0$. When $P \neq P'$ but have the same undirected edge-set (i.e., the reversed path), we have

$$\mathbb{E}[\mathsf{Z}(P)\mathsf{Z}(P')|\phi, x=0] = \prod_{e\in P} \mathbb{E}[\mathsf{Z}_e^2] \tag{341}$$

$$\le [q(1-q)]^L \le 4^{-L}. \tag{342}$$

Note that the number of such pairs is at most $|\mathcal{P}_L|$. Similarly, for each $P$,

$$\mathbb{E}[\mathsf{Z}^2(P)|\phi, x=0] = \prod_{e\in P} \mathbb{E}[\mathsf{Z}_e^2|\phi, x=0] \le 4^{-L}. \tag{343}$$

Since $|\mathcal{P}_L| = (n-1)_L = \Theta(n^L)$, we obtain $\mathsf{Var}(\mathsf{W}_L|\phi, x=0) \le C_L n^L$ where $C_L = 2[q(1-q)]^L$. Scaling by $B_L = \sqrt{C_L} n^{L/2}$ yields $\mathsf{Var}(\mathscr{X}_L|\phi, x=0) \le 1$. ■

**Lemma 50** *Let $u \in v(\Gamma)$ be such that $\phi(u) = 1$. Then*

$$\mathbb{E}\left[\mathsf{W}_L|\phi, x=1\right] = (p-q)^L W_L(\Gamma; u). \tag{344}$$

**Proof** [Proof of Lemma 50] If a path $P \in \mathcal{P}_L$ is *not* fully contained in the planted edge-set, then some factor $\mathsf{Z}_{ij}$ on that path has mean zero (conditional on $\phi$), hence $\mathbb{E}[\mathsf{Z}(P)|\phi] = 0$. If $P$ is fully planted, then the $\mathsf{Z}_{ij}$ along $P$ are i.i.d. centered with mean $(p-q)$, so $\mathbb{E}[\mathsf{Z}(P)|\phi, x=1] = (p-q)^L$. The number of fully planted simple ambient paths equals the number of simple paths of length $L$ in $\Gamma$ starting at $u$, namely $W_L(\Gamma; u)$. Summing over $P \in \mathcal{P}_L$ yields the claim. ■

**Lemma 51** *Let $u \in v(\Gamma)$ be such that $\phi(u) = 1$. Then there exists a constant $C_L = C_L(L, p, q)$ such that*

$$\mathsf{Var}\left(\mathsf{W}_L|\phi, x=1\right) \le C_L \left(n^L + k^{2L-1}\right), \tag{345}$$

*where $k \triangleq |v(\Gamma)|$.*

**Proof** We define $\mathcal{P}_L^{\mathsf{pl}}$ as the set of simple ambient $L$-paths fully contained in the planted vertex set $\phi(v(\Gamma))$, and $\mathcal{P}_L^{\mathsf{mix}} \triangleq \mathcal{P}_L \setminus \mathcal{P}_L^{\mathsf{pl}}$. Decompose $\mathsf{W}_L = \mathsf{W}_L^{\mathsf{pl}} + \mathsf{W}_L^{\mathsf{mix}}$ with the corresponding meanings. We have

$$\begin{aligned}\mathsf{Var}\left(\mathsf{W}_L|\phi, x=1\right) &\leq \mathsf{Var}\left(\mathsf{W}_L^{\mathsf{pl}}|\phi, x=1\right) + \mathsf{Var}\left(\mathsf{W}_L^{\mathsf{mix}}|\phi, x=1\right) \\ &\quad + 2\sqrt{\mathsf{Var}\left(\mathsf{W}_L^{\mathsf{pl}}|\phi, x=1\right)\mathsf{Var}\left(\mathsf{W}_L^{\mathsf{mix}}|\phi, x=1\right)},\end{aligned} \tag{346}$$

and we next bound the variances $\mathsf{W}_L^{\mathsf{pl}}$ and $\mathsf{W}_L^{\mathsf{mix}}$.

**Mixed part.** If $P \in \mathcal{P}_L^{\mathsf{mix}}$, then $\mathsf{Z}(P)$ contains at least one non-planted edge; since $\mathbb{E}[\mathsf{Z}_{ij}|\phi, x = 1] = 0$ on non-planted edges and the edges remain independent conditional on $\phi$, we have $\mathbb{E}[\mathsf{Z}(P)|\phi, x = 1] = 0$. Hence expanding the variance,

$$\mathsf{Var}(\mathsf{W}_L^{\mathsf{mix}}|\phi, x=1) = \sum_{P\in\mathcal{P}_L^{\mathsf{mix}}} \mathbb{E}[\mathsf{Z}(P)^2|\phi, x=1] + \sum_{\substack{P\neq P' \\ P,P'\in\mathcal{P}_L^{\mathsf{mix}}}} \mathbb{E}[\mathsf{Z}(P)\mathsf{Z}(P')|\phi, x=1]. \tag{347}$$

Each diagonal term is $\mathbb{E}[\mathsf{Z}(P)^2|\phi, x = 1] = \prod_{e\in P} \mathbb{E}[\mathsf{Z}_e^2|\phi, x = 1] \leq \max\{p(1-p), q(1-q)\}^L \leq (1/4)^L$. For off-diagonals, independence implies $\mathbb{E}[\mathsf{Z}(P)\mathsf{Z}(P')|\phi, x = 1] = 0$ unless *every* edge that appears with multiplicity one across $P \cup P'$ is planted. In particular, any non-planted edge that appears in exactly one of $P, P'$ nullifies the expectation. Therefore, for a given $P \in \mathcal{P}_L^{\mathsf{mix}}$ the number of $P' \in \mathcal{P}_L^{\mathsf{mix}}$ with $\mathbb{E}[\mathsf{Z}(P)\mathsf{Z}(P')|\phi, x = 1] \neq 0$ is bounded by a constant depending only on $L$ (one must pick exactly the same set of non-planted edges, and there are $O_L(1)$ ways to complete to a simple path once those are fixed). Consequently,

$$\mathsf{Var}(\mathsf{W}_L^{\mathsf{mix}}|\phi, x=1) \leq C_L^{(1)}|\mathcal{P}_L| \leq C_L^{(1)} n^L. \tag{348}$$

**Planted-only part.** Write $M \triangleq |\mathcal{P}_L^{\mathsf{pl}}| \leq (k-1)_L = O(k^L)$. For $P \in \mathcal{P}_L^{\mathsf{pl}}$, all edges of $P$ are planted, hence $\mathbb{E}[\mathsf{Z}(P)|\phi, x = 1] = (p-q)^L$, and $\mathsf{Var}(\mathsf{Z}(P)|\phi, x = 1) = \prod_{e\in P} \mathbb{E}[\mathsf{Z}_e^2|\phi, x = 1] - (p-q)^{2L} \leq (1/4)^L$. Moreover, for $P \neq P'$, the covariance vanishes unless $P$ and $P'$ share at least one edge: if $e(P) \cap e(P') = \emptyset$, then $\mathsf{Z}(P)$ and $\mathsf{Z}(P')$ are independent (disjoint edge sets), so $\mathsf{Cov}(\mathsf{Z}(P), \mathsf{Z}(P')|\phi, x = 1) = 0$. For a fixed $P$, the number of $P'$ sharing at least one edge with $P$ is $O_L(k^{L-1})$ (a shared edge reduces one free choice). Therefore,

$$\begin{aligned}\mathsf{Var}(\mathsf{W}_L^{\mathsf{pl}}|\phi, x=1) &= \sum_P \mathsf{Var}(\mathsf{Z}(P)|\phi, x=1) + \sum_{\substack{P\neq P' \\ e(P)\cap e(P')\neq\emptyset}} \mathsf{Cov}(\mathsf{Z}(P), \mathsf{Z}(P')|\phi, x=1) && (349)\\ &\leq C_L^{(2)}\left(k^L + k^{2L-1}\right), && (350)\end{aligned}$$

since each covariance is $O(1)$ (uniform in $k$) and there are $O(k^{2L-1})$ overlapping pairs.

Finally, by Cauchy–Schwarz inequality,

$$\begin{aligned}|\mathsf{Cov}(\mathsf{W}_L^{\mathsf{mix}}, \mathsf{W}_L^{\mathsf{pl}}|\phi, x=1)| &\leq \sqrt{\mathsf{Var}(\mathsf{W}_L^{\mathsf{mix}}|\phi, x=1)}\sqrt{\mathsf{Var}(\mathsf{W}_L^{\mathsf{pl}}|\phi, x=1)} && (351)\\ &\leq \sqrt{C_L^{(1)}C_L^{(2)}}n^{L/2}\,k^{L-1/2}. && (352)\end{aligned}$$

This is absorbed by $C_L(n^L + k^{2L-1})$. Combining the above three bounds yields (345). ■

We are now in a position to define the proposed estimator. Let

$$\mu_L^{\min}(\Gamma) \triangleq \frac{(p-q)^L}{B_L} W_L^{\min}(\Gamma), \tag{353}$$

$$t_n \triangleq \frac{\mu_L^{\min}(\Gamma)}{2}. \tag{354}$$

Define the rescaled statistic $\mathscr{Z}_L \triangleq \mathscr{X}_L/t_n$, and let $\tau_m$ be as in (335). Define the estimator

$$f_{L,m}(\mathsf{Y}) = \tau_m(\mathscr{Z}_L) \tag{355}$$

of total degree $D = L + 2m + 1$. We have the following result.

**Lemma 52** *Fix $r \in (0, 1/4]$. Assume*

$$W_L^{\min}(\Gamma) \geq C^\star(L,p,q)\left[n^{L/2} + k^{L-1/2}\right]\sqrt{\log n}, \tag{356}$$

*where*

$$C^\star(L,p,q) \geq \max\left\{\frac{4c_L}{(p-q)^L}, \frac{2\sqrt{C_L}}{c_L(p-q)^L}\right\}. \tag{357}$$

*Then for all large $n$,*

$$\mathbb{P}\left[|\mathscr{Z}_L| \leq \tfrac{r}{2} | x = 0\right] \geq 1 - \frac{1}{\log n}, \tag{358}$$

$$\mathbb{P}\left[\mathscr{Z}_L \geq r | x = 1\right] \geq 1 - \frac{1}{\log n}. \tag{359}$$

**Proof** [Proof of Lemma 52] By Lemma 49, we have $\mathbb{E}[\mathscr{X}_L | x = 0] = 0$ and $\mathsf{Var}(\mathscr{X}_L | x = 0) \leq 1$. Hence $\mathsf{Var}(\mathscr{Z}_L | x = 0) = \mathsf{Var}(\mathscr{X}_L | x = 0)/t_n^2 \leq 1/t_n^2$. Chebyshev's inequality then gives

$$\mathbb{P}\left[|\mathscr{Z}_L| \geq r/2 | x = 0\right] \leq \frac{4\mathsf{Var}(\mathscr{Z}_L)}{r^2} \tag{360}$$

$$\leq \frac{4}{r^2 t_n^2} \tag{361}$$

$$= \frac{16 B_L^2}{r^2 (p-q)^{2L} (W_L^{\min}(\Gamma))^2} \tag{362}$$

$$\leq \frac{16 B_L^2}{r^2 (p-q)^{2L} [C^\star(L,p,q)]^2 n^L \log n} \tag{363}$$

$$= \frac{16 c_L^2}{r^2 (p-q)^{2L} [C^\star(L,p,q)]^2 \log n} \tag{364}$$

$$\leq \frac{1}{\log n}, \tag{365}$$

where in the second inequality we have used (356), and in the last inequality we used the fact that $C^\star(L,p,q) \geq \frac{4c_L}{(p-q)^L}$. Under $x = 1$ and conditioning on $\phi$ with $\phi(u) = 1$, Lemma 50 gives $\mathbb{E}[\mathscr{X}_L|\phi] = \mu_L(u) \triangleq \frac{(p-q)^L}{B_L} W_L(\Gamma; u) \geq \mu_L^{\min}(\Gamma) = 2t_n$, and thus $\mathbb{E}[\mathscr{Z}_L|\phi] \geq 2$. Lemma 51 gives $\mathsf{Var}\left(\mathsf{W}_L|\phi, x=1\right) \leq C_L\left(n^L + k^{2L-1}\right)$ and so $\mathsf{Var}(\mathscr{Z}_L|\phi, x=1) \leq (C_L\left(n^L + k^{2L-1}\right))/(c_L^2 B_L^2 t_n^2)$. Thus, Chebyshev's inequality yields

$$\begin{aligned}
\mathbb{P}\left[\mathscr{Z}_L < 1|\phi, x=1\right] &\leq \frac{\mathsf{Var}(\mathscr{Z}_L|\phi, x=1)}{(2-1)^2} && (366)\\
&\leq \frac{C_L\left(n^L + k^{2L-1}\right)}{c_L^2 B_L^2 t_n^2} && (367)\\
&= \frac{4C_L\left(n^L + k^{2L-1}\right)}{c_L^2 B_L^2 (\mu_L^{\min}(\Gamma))^2} && (368)\\
&= \frac{4B_L^2 C_L\left(n^L + k^{2L-1}\right)}{c_L^2 B_L^2 (p-q)^{2L} (W_L^{\min}(\Gamma))^2} && (369)\\
&\leq \frac{4C_L\left(n^L + k^{2L-1}\right)}{c_L^2 (p-q)^{2L}\left[C^\star(L,p,q)\left[n^{L/2} + k^{L-1/2}\right]\sqrt{\log n}\right]^2} && (370)\\
&\leq \frac{4C_L}{c_L^2 (p-q)^{2L}[C^\star(L,p,q)]^2 \log n} && (371)\\
&\leq \frac{1}{\log n}, && (372)
\end{aligned}$$

where in the last inequality we have used the fact that $C^\star(L,p,q) \geq \sqrt{\frac{4C_L}{c_L^2(p-q)^{2L}}}$. Since $r \leq 1/4$, the event $\{\mathscr{Z}_L \geq 1\}$ implies $\{\mathscr{Z}_L \geq r\}$. Averaging over $\phi$ proves the claim. ■

**Theorem 53 ($L$-step low-degree upper bound for general $\Gamma$)** *Fix $L \in \mathbb{N}$ and $0 < q < p < 1$. Let $r \in (0, 1/4]$. Assume the rooted simple-path mass satisfies*

$$W_L^{\min}(\Gamma) \geq C^\star(L,p,q)\left[n^{L/2} + k^{L-1/2}\right]\sqrt{\log n}, \tag{373}$$

*Consider the estimator $f_{L,m}(\mathsf{Y})$ in (355). Assume that $m = \omega(1)$ and $D \leq C \log\log n$, for some constants $C > 0$. Then*

$$\mathbb{E}\left[(f_{L,m}(\mathsf{Y}) - x)^2\right] \leq (\log n)^{-\Omega(1)} \xrightarrow[n\to\infty]{} 0. \tag{374}$$

**Proof** [Proof of Theorem 53] By Lemma 52, with $\delta_n \triangleq 1/\log n$ we have

$$\mathbb{P}\left[|\mathscr{Z}_L| \leq \tfrac{r}{2}|x=0\right] \geq 1 - \delta_n, \tag{375}$$
$$\mathbb{P}\left[\mathscr{Z}_L \geq r|x=1\right] \geq 1 - \delta_n. \tag{376}$$

By (335), on $\{|\mathscr{Z}_L| \leq r/2\}$ we have $|\tau_m(\mathscr{Z}_L) - 0| \leq (6r)^m$, and on $\{\mathscr{Z}_L \geq r\}$ we have $|\tau_m(\mathscr{Z}_L) - 1| \leq (6r)^m$. Thus the pointwise squared error $(\tau_m(\mathscr{Z}_L) - x)^2$ is at most $\varepsilon_n^2 \triangleq (6r)^{2m}$ on the respective "good" events, whose complements have probability at most $\delta_n$. Now, apply Lemma 46 with $\bar{f} = \tau_m(\mathscr{Z}_L) - x$, noting that $\bar{f}$ is a polynomial of total degree $D = L + 2m + 1$. We obtain

$$\mathbb{E}[(\tau_m(\mathscr{Z}_L) - x)^2] \leq \varepsilon_n^2 + \left(\frac{9}{\nu}\right)^{D/2} \mathbb{E}[(\tau_m(\mathscr{Z}_L) - x)^2]\sqrt{\delta_n}. \tag{377}$$

Choose $D \leq C \log\log n$ with $C$ small enough so that $\sqrt{\delta_n} = (\log n)^{-1/2} \leq \frac{1}{4}(\nu/9)^{D/2}$ for large $n$, e.g., any $C < \frac{1}{\log(9/\nu)}$ works. Then the second term can be absorbed to the left, yielding $\mathbb{E}[(\tau_m(\mathscr{Z}_L) - x)^2] \leq 2\varepsilon_n^2$. Finally, $\varepsilon_n^2 = (6r)^{2m} = o(1)$ since $m = \omega(1)$, which proves the claim. ■

## Appendix G. Auxiliary Lemmata

### G.1. Equivalence of worst-case and Bayes risks

This appendix establishes the equivalence between the worst-case and Bayesian error probabilities when the prior over $\Gamma^\star$ is uniform on $\mathcal{S}_\Gamma$.

**Lemma 54** *Fix $n \in \mathbb{N}$ and let $\Gamma_n$ be an arbitrary graph with no isolated vertices and at most $n$ vertices. Denote by $\mathcal{S}_{\Gamma_n}$ the set of (labelled) copies of $\Gamma_n$ inside the complete graph $\mathcal{K}_n$. For $\Gamma^\star \in \mathcal{S}_{\Gamma_n}$ write $\mathbb{P}_{\Gamma^\star}$ for the distribution $\mathcal{G}_{\Gamma^\star}(n, p_n, q_n)$ of the observed graph $\mathsf{G}$. Given an estimator $\hat{\Gamma}_n : \{0,1\}^{\binom{n}{2}} \to \mathcal{S}_{\Gamma_n}$, define the* risk function

$$\mathsf{R}_n(\hat{\Gamma}_n; \Gamma^\star) \triangleq \mathbb{P}_{\Gamma^\star}\left[\hat{\Gamma}_n(\mathsf{G}) \neq \Gamma^\star\right], \quad \Gamma^\star \in \mathcal{S}_{\Gamma_n}. \tag{378}$$

*Let*

$$\mathsf{E}_n(\hat{\Gamma}_n) \triangleq \sup_{\Gamma^\star \in \mathcal{S}_{\Gamma_n}} \mathsf{R}_n(\hat{\Gamma}_n; \Gamma^\star), \tag{379}$$

$$\overline{\mathsf{E}}_n(\hat{\Gamma}_n) \triangleq \mathbb{E}_{\Gamma^\star \sim \mathrm{Unif}(\mathcal{S}_{\Gamma_n})}\left[\mathsf{R}_n(\hat{\Gamma}_n; \Gamma^\star)\right]. \tag{380}$$

*Then for every estimator $\hat{\Gamma}_n$ there exists an equivariant estimator $\hat{\Gamma}_n^{\mathrm{eq}}$ such that*

$$\mathsf{R}_n(\hat{\Gamma}_n^{\mathrm{eq}}; \Gamma^\star) \equiv \overline{\mathsf{E}}_n(\hat{\Gamma}_n), , \tag{381}$$

*for all $\Gamma^\star \in \mathcal{S}_{\Gamma_n}$, and*

$$\mathsf{E}_n(\hat{\Gamma}_n^{\mathrm{eq}}) = \overline{\mathsf{E}}_n(\hat{\Gamma}_n^{\mathrm{eq}}) = \overline{\mathsf{E}}_n(\hat{\Gamma}_n) \leq \mathsf{E}_n(\hat{\Gamma}_n). \tag{382}$$

*Consequently*

$$\inf_{\hat{\Gamma}_n : \{0,1\}^{n(2)} \to \mathcal{S}_{\Gamma_n}} \mathsf{E}_n(\hat{\Gamma}_n) = \inf_{\hat{\Gamma}_n : \{0,1\}^{n(2)} \to \mathcal{S}_{\Gamma_n}} \overline{\mathsf{E}}_n(\hat{\Gamma}_n), \tag{383}$$

*for every $n \in \mathbb{N}$. That is, the uniform prior on $\mathcal{S}_{\Gamma_n}$ is least–favorable, and the minimax error probability coincides with the Bayes error probability under that prior.*

**Proof** Let $\mathbb{S}_n$ be the permutation group on the vertex set $[n] = \{1, \ldots, n\}$. For $\pi \in \mathbb{S}_n$ and a graph $H$ on $[n]$, define $\pi \circ H$ to be the graph whose adjacency indicator satisfies

$$(\pi \circ H)_{ij} = H_{\pi^{-1}(i), \pi^{-1}(j)}, \qquad 1 \leq i < j \leq n. \tag{384}$$

The action extends to subgraphs: $\pi \circ \Gamma^\star \triangleq \pi(\Gamma^\star)$ for $\Gamma^\star \in \mathcal{S}_{\Gamma_n}$. Because the generative rule depends *only* on (i) which edges belong to $\Gamma^\star$ and (ii) the probabilities $(p_n, q_n)$, for every $\pi \in \mathbb{S}_n$ and every measurable $A \subseteq \{0,1\}^{\binom{n}{2}}$

$$\mathbb{P}_{\Gamma^\star}[\mathsf{G} \in A] = \mathbb{P}_{\pi\circ\Gamma^\star}[\mathsf{G} \in \pi \circ A], \tag{385}$$

where $\pi \circ A = \{\pi \circ H : H \in A\}$. Also, note that the indicator loss $\mathbb{1}\{\hat{\Gamma}_n(\mathsf{G}) \neq \Gamma^\star\}$ is invariant in the sense that

$$\mathbb{1}\{\hat{\Gamma}_n(\mathsf{G}) \neq \Gamma^\star\} = \mathbb{1}\left\{\pi \circ \hat{\Gamma}_n(\pi \circ \mathsf{G}) \neq \pi \circ \Gamma^\star\right\}. \tag{386}$$

Now, given an arbitrary estimator $\hat{\Gamma}_n$, define a *randomized* rule

$$\hat{\Gamma}_n^{\mathrm{eq}}(\mathsf{G}) \triangleq \Pi^{-1} \circ \hat{\Gamma}_n(\Pi \circ \mathsf{G}), \tag{387}$$

where $\Pi \sim \mathrm{Unif}(\mathbb{S}_n)$ is an independent, auxiliary random permutation. For any fixed $\pi \in \mathbb{S}_n$,

$$\hat{\Gamma}_n^{\mathrm{eq}}(\pi \circ \mathsf{G}) = \Pi^{-1} \circ \hat{\Gamma}_n(\Pi\pi \circ \mathsf{G}) \stackrel{d}{=} \pi \circ \left[\Pi^{-1} \circ \hat{\Gamma}_n(\Pi \circ \mathsf{G})\right] = \pi \circ \hat{\Gamma}_n^{\mathrm{eq}}(\mathsf{G}), \tag{388}$$

hence the rule is *equivariant*. Furthermore, for any $\Gamma^\star \in \mathcal{S}_{\Gamma_n}$,

$$\begin{aligned}
\mathsf{R}_n(\hat{\Gamma}_n^{\mathrm{eq}}; \Gamma^\star) &= \mathbb{E}_{\Gamma^\star}\left[\mathbb{1}\left\{\Pi^{-1} \circ \hat{\Gamma}_n(\Pi \circ \mathsf{G}) \neq \Gamma^\star\right\}\right] &(389)\\
&= \frac{1}{|\mathbb{S}_n|}\sum_{\pi\in\mathbb{S}_n} \mathbb{E}_{\Gamma^\star}\left[\mathbb{1}\left\{\hat{\Gamma}_n(\pi \circ \mathsf{G}) \neq \pi \circ \Gamma^\star\right\}\right] &(390)\\
&= \frac{1}{|\mathbb{S}_n|}\sum_{\pi\in\mathbb{S}_n} \mathbb{E}_{\pi\circ\Gamma^\star}\left[\mathbb{1}\left\{\hat{\Gamma}_n(\mathsf{G}) \neq \pi \circ \Gamma^\star\right\}\right] &(391)\\
&= \mathbb{E}_{\widetilde{\Gamma}\sim\mathrm{Unif}(\mathcal{S}_{\Gamma_n})}\mathsf{R}_n(\hat{\Gamma}_n; \widetilde{\Gamma}) = \overline{\mathsf{E}}_n(\hat{\Gamma}_n), &(392)
\end{aligned}$$

which is independent of $\Gamma^\star$, and the third equality follows from (385)–(386). This proves (381). Next, we compare the worst-case and Bayes risks. Because the equivariant rule has equal risk *everywhere*, its worst–case and average risks coincide: $\mathsf{E}_n(\hat{\Gamma}_n^{\mathrm{eq}}) = \overline{\mathsf{E}}_n(\hat{\Gamma}_n^{\mathrm{eq}})$. Moreover, Jensen's inequality applied to the averaging in Step 3 gives $\overline{\mathsf{E}}_n(\hat{\Gamma}_n^{\mathrm{eq}}) \leq \overline{\mathsf{E}}_n(\hat{\Gamma}_n)$, while by definition $\mathsf{E}_n(\hat{\Gamma}_n^{\mathrm{eq}}) \leq \mathsf{E}_n(\hat{\Gamma}_n)$. This yields the chain of equalities and inequalities in (382). Finally, taking the infimum over *all* estimators on both sides of (381) gives

$$\inf_{\hat{\Gamma}_n:\{0,1\}^{n(2)}\to\mathcal{S}_{\Gamma_n}} \mathsf{E}_n(\hat{\Gamma}_n) = \inf_{\hat{\Gamma}_n:\{0,1\}^{n(2)}\to\mathcal{S}_{\Gamma_n}} \overline{\mathsf{E}}_n(\hat{\Gamma}_n), \tag{393}$$

so the uniform prior on $\mathcal{S}_{\Gamma_n}$ achieves the minimax value. ■

### G.2. Proof of Lemma 17

Fix a step $\ell$ and let $\Gamma^{(\ell-1)}$ denote the graph remaining at the beginning of that step. Recall that for any subgraph $\Gamma^{(\ell-1)} \subsetneq \mathcal{K}$, we define

$$\eta(\mathcal{K}|\Gamma^{(\ell-1)}) \triangleq \frac{|\mathcal{K} \setminus \Gamma^{(\ell-1)}|}{|v(\mathcal{K}) \setminus v(\Gamma^{(\ell-1)})|}. \tag{394}$$

By construction, $\Gamma^{(\ell)}$ is a *maximal* subgraph that attains the maximum value

$$\eta^\star = \max_{\Gamma^{(\ell-1)} \subsetneq \mathcal{K}} \eta(\mathcal{K}|\Gamma^{(\ell-1)}). \tag{395}$$

Assume, toward a contradiction, that two distinct maximisers $\mathcal{A}, \mathcal{B} \subsetneq \Gamma^{(\ell-1)}$ exist, i.e.,

$$\eta(\mathcal{A}|\Gamma^{(\ell-1)}) = \eta(\mathcal{B}|\Gamma^{(\ell-1)}) = \eta^\star, \tag{396}$$

with $\mathcal{A} \neq \mathcal{B}$. Let $\mathcal{C} \triangleq \mathcal{A} \cup \mathcal{B}$ and define

$$e_{\mathcal{X}} \triangleq |\mathcal{X} \setminus \Gamma^{(\ell-1)}|, \tag{397}$$

$$v_{\mathcal{X}} \triangleq |v(\mathcal{X}) \setminus v(\Gamma^{(\ell-1)})|, \tag{398}$$

for $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}, \mathcal{C}\}$. Then $e_{\mathcal{C}} = e_{\mathcal{A}} + e_{\mathcal{B}} - \delta$ and $v_{\mathcal{C}} = v_{\mathcal{A}} + v_{\mathcal{B}} - \gamma$, where $\delta, \gamma \geq 0$ count the edges and vertices already shared by $\mathcal{A}$ and $\mathcal{B}$ outside $\Gamma^{(\ell-1)}$. Because $\eta(\mathcal{A}|\Gamma^{(\ell-1)}) = \eta(\mathcal{B}|\Gamma^{(\ell-1)}) = \eta^\star$, we have $e_{\mathcal{A}}/v_{\mathcal{A}} = e_{\mathcal{B}}/v_{\mathcal{B}} = \eta^\star$. Notice that by the maximiality of $\eta^\star$, we have $\frac{\delta}{\gamma} \leq \eta^\star$, and consequently

$$\eta(\mathcal{C}|\Gamma^{(\ell-1)}) = \frac{e_{\mathcal{A}} + e_{\mathcal{B}} - \delta}{v_{\mathcal{A}} + v_{\mathcal{B}} - \gamma} \geq \frac{e_{\mathcal{A}} + e_{\mathcal{B}}}{v_{\mathcal{A}} + v_{\mathcal{B}}} = \eta^\star, \tag{399}$$

with equality only if $\frac{\delta}{\gamma} = \eta^\star$. Now, if the inequality is strict, namely, $\eta(\mathcal{C}|\Gamma^{(\ell-1)}) > \eta^\star$, then this contradicts the optimality of $\eta^\star$. If equality holds, then $\mathcal{C} \supsetneq \mathcal{A}, \mathcal{B}$ attains the same maximal density, contradicting the *maximality* of $\Gamma^(\ell)$ as $\mathcal{A} \neq \mathcal{B}$. Thus, no two distinct maximizers coexist.

### G.3. Proof of Lemma 18

Recall that for any $\mathsf{J} \subseteq \Gamma$

$$\mu(\Gamma|\mathsf{J}) \triangleq \max_{\mathsf{J} \subsetneq \mathsf{F} \subseteq \Gamma} \eta(\mathsf{F}|\mathsf{J}), \qquad S \subseteq \Gamma. \tag{400}$$

Denote the onion decomposition of $\Gamma$ by

$$\emptyset = \Gamma^{(0)} \subsetneq \Gamma^{(1)} \subsetneq \cdots \subsetneq \Gamma^{(M)} = \Gamma, \tag{401}$$

and set

$$d_\ell \triangleq \eta\left(\Gamma^{(\ell)}|\Gamma^{(\ell-1)}\right) = \mu(\Gamma|\Gamma^{(\ell-1)}), \tag{402}$$

for $\ell = 1, \ldots, M$. We proceed through four simple lemmas. All subgraphs are understood to be subgraphs of $\Gamma$.

**Lemma 55** *If* $\mathsf{J} \subseteq \mathsf{J}'$ *then* $\mu(\Gamma|\mathsf{J}) \geq \mu(\mu(\Gamma|\mathsf{J}')$.

**Proof** [Proof of Lemma 55] The feasible set $\{\mathsf{F} : \mathsf{J} \subsetneq \mathsf{F} \subseteq \Gamma\}$ contains $\{\mathsf{F} : \mathsf{J}' \subsetneq \mathsf{F} \subseteq \Gamma\}$. Maximizing over a larger set cannot give a smaller value. ■

**Lemma 56** *Fix* $\mathsf{J}$*. Let* $\mathcal{F}_{\mathsf{J}} \triangleq \{\mathsf{F} : \mathsf{J} \subsetneq \mathsf{F} \subseteq \Gamma,\ \eta(\mathsf{F}|\mathsf{J}) = \mu(\Gamma|\mathsf{J})\}$*. If* $\mathsf{F}_1, \mathsf{F}_2 \in \mathcal{F}_{\mathsf{J}}$ *then* $\mathsf{F}_1 \cup \mathsf{F}_2 \in \mathcal{F}_{\mathsf{J}}$*. Consequently,* $\mathcal{F}_{\mathsf{J}}$ *has a unique inclusion-wise maximal element (its union).*

**Proof** [Proof of Lemma 56] With $d \triangleq \mu(\Gamma|\mathsf{J})$, set $\mathsf{U} \triangleq \mathsf{F}_1 \cup \mathsf{F}_2$, $\mathsf{I} \triangleq \mathsf{F}_1 \cap \mathsf{F}_2$. Then

$$\begin{aligned} |e(\mathsf{U}|\mathsf{J})| - d \cdot |v(\mathsf{U}|\mathsf{J})| = \left[|e(\mathsf{F}_1|\mathsf{J})| - d \cdot |v(\mathsf{F}_1|\mathsf{J})|\right] + \left[|e(\mathsf{F}_2|\mathsf{J})| - d \cdot |v(\mathsf{F}_2|\mathsf{J})|\right] \\ - \left[|e(\mathsf{I}|\mathsf{J})| - d \cdot |v(\mathsf{I}|\mathsf{J})|\right] \geq 0, \end{aligned} \tag{403}$$

because $\mathsf{I}$ is feasible and hence $\eta(\mathsf{I}|\mathsf{J}) \leq d$. Thus $\eta(\mathsf{U}|\mathsf{J}) \geq d$, and since $d$ is maximal, equality holds. ■

**Lemma 57** *Recall that* $d_\ell \triangleq \eta\left(\Gamma^{(\ell)}|\Gamma^{(\ell-1)}\right)$*, for* $\ell = 1, \ldots, M$*. Then,* $d_1 > d_2 > \cdots > d_M$*.*

**Proof** [Proof of Lemma 57] Suppose $d_{\ell+1} \geq d_\ell$. Let $\mathsf{F}$ maximize $\mu(\Gamma|\Gamma^{(\ell)})$, so $\eta(\mathsf{F}|\Gamma^{(\ell)}) = d_{\ell+1} \geq d_\ell$. Then

$$\eta(\mathsf{F}|\Gamma^{(\ell-1)}) = \frac{|e(\Gamma^{(\ell)}|\Gamma^{(\ell-1)})| + |e(\mathsf{F}|\Gamma^{(\ell)})|}{|v(\Gamma^{(\ell)}|\Gamma^{(\ell-1)})| + |v(\mathsf{F}|\Gamma^{(\ell)})|} \ \geq\ d_\ell. \tag{404}$$

If the inequality is strict, this contradicts maximality of $d_\ell = \mu(\Gamma|\Gamma^{(\ell-1)})$. If equality holds, then $\mathsf{F}$ is also a maximizer for seed $\Gamma^{(\ell-1)}$; by Lemma 56 and the onion step's maximality, we cannot have a strict superset $\mathsf{F} \supsetneq \Gamma^{(\ell)}$. This leads to a contradiction. ■

**Lemma 58** *Fix* $\ell$ *and set* $\mathsf{A} \triangleq \Gamma^{(\ell-1)}$, $\mathsf{B} \triangleq \Gamma^{(\ell)}$*,* $d \triangleq d_\ell = \eta(\mathsf{B}|\mathsf{A}) = \mu(\Gamma|\mathsf{A})$*. Then for every* $\mathsf{J}$ *with* $\mathsf{A} \subseteq \mathsf{J} \subseteq \mathsf{B}$*, we have* $\mu(\Gamma|\mathsf{J}) = d$*.*

**Proof** [Proof of Lemma 58] On the one hand, by the inequality in the proof of Lemma 57 with $\mathsf{F} = \mathsf{B}$,

$$\eta(\mathsf{B}|\mathsf{J}) \geq \eta(\mathsf{B}|\mathsf{A}) = d \quad \Rightarrow \quad \mu(\Gamma|\mathsf{J}) \geq d. \tag{405}$$

On the other hand, by Lemma 55, since $\mathsf{A} \subseteq \mathsf{J}$,

$$\mu(\Gamma|\mathsf{J}) \leq \mu(\Gamma|\mathsf{A}) = d. \tag{406}$$

Together, $\mu(\Gamma|\mathsf{J}) = d$. ■

We are now ready to prove Lemma 18, and we start by proving that $T = M$. Indeed, for each $\ell$, $d_\ell = \mu(\Gamma|\Gamma^{(\ell-1)}) \in \Lambda(\Gamma)$. Hence $\{d_1, \ldots, d_M\} \subseteq \Lambda(\Gamma)$. In particular, the number $T$ of distinct values satisfies $T \geq M$. Now, let $\mathsf{J} \subseteq \Gamma$ be arbitrary, and let $\ell$ be the minimal index with $\mathsf{J} \subseteq \Gamma^{(\ell)}$. Then $\Gamma^{(\ell-1)} \subseteq \mathsf{J} \subseteq \Gamma^{(\ell)}$. Lemma 58 gives $\mu(\Gamma|\mathsf{J}) = d_\ell$. Thus $\Lambda(\Gamma) \subseteq \{d_1, \ldots, d_M\}$, and so $T \leq M$. Combining the above, we conclude $T = M$ and the sets of distinct values coincide:

$$\{\lambda_1 > \cdots > \lambda_M\} = \{d_1 > \cdots > d_M\}. \tag{407}$$

By Lemma 57, both sides are strictly decreasing, and thus their $\ell$-th entries match:

$$\lambda_\ell = d_\ell = \eta(\Gamma^{(\ell)}|\Gamma^{(\ell-1)}), \qquad \ell = 1, \ldots, M, \tag{408}$$

which concludes the proof.

**Minimal maximum subgraph density.** We provide here an alternative proof for the fact that $\eta(\Gamma^{(M)}|\Gamma^{(M-1)}) = \mu_{\mathsf{min}}(\Gamma)$, where $\mu_{\mathsf{min}}(\Gamma)$ is defined in (34). The proof follows from several facts established in Lee et al. (2025). Specifically, for any $\mathsf{S} \subseteq \Gamma$, define $G(\mathsf{S}) \triangleq \max_{\mathsf{S}\subsetneq\mathsf{F}\subseteq\Gamma} \eta(\mathsf{F}|\mathsf{S})$. Then, we claim that

$$\min_{\mathsf{S}\subseteq\Gamma} G(\mathsf{S}) = \min_{\alpha\in[0,1]} \min_{\substack{\mathsf{S}\subseteq\Gamma \\ |\mathsf{S}|\leq\alpha|\Gamma|}} G(\mathsf{S}). \tag{409}$$

Indeed, for every $\alpha \in [0,1]$, the inner minimum is over a subset of all $\mathsf{S}$, so

$$\min_{\substack{\mathsf{S}\subseteq\Gamma \\ |\mathsf{S}|\leq\alpha|\Gamma|}} G(\mathsf{S}) \geq \min_{\mathsf{S}\subseteq\Gamma} G(\mathsf{S}). \tag{410}$$

Taking $\min_{\alpha\in[0,1]}$ over both sides we get

$$\min_{\alpha\in[0,1]} \min_{\substack{\mathsf{S}\subseteq\Gamma \\ |\mathsf{S}|\leq\alpha|\Gamma|}} G(\mathsf{S}) \geq \min_{\mathsf{S}\subseteq\Gamma} G(\mathsf{S}). \tag{411}$$

Conversely, let $\mathsf{S}^\star$ attain $\min_{\mathsf{S}\subseteq\Gamma} G(\mathsf{S})$ and take $\alpha^\star \triangleq |\mathsf{S}^\star|/|\Gamma|$; then $\mathsf{S}^\star$ is feasible for $\alpha^\star$, so

$$\min_{\substack{\mathsf{S}\subseteq\Gamma \\ |\mathsf{S}|\leq\alpha|\Gamma|}} G(\mathsf{S}) \leq G(\mathsf{S}^\star) = \min_{\mathsf{S}\subseteq\Gamma} G(\mathsf{S}). \tag{412}$$

Thus equality holds. If for $\alpha \in [0,1]$ we define

$$\phi(\alpha) \triangleq \min_{\substack{\mathsf{S}\subseteq\Gamma \\ |\mathsf{S}|\leq\alpha|\Gamma|}} \max_{\mathsf{S}\subsetneq\mathsf{F}\subseteq\Gamma} \eta(\mathsf{F}|\mathsf{S}), \tag{413}$$

then the above implies that

$$\min_{\mathsf{S}\subseteq\Gamma} \max_{\mathsf{S}\subsetneq\mathsf{F}\subseteq\Gamma} \eta(\mathsf{F}|\mathsf{S}) = \min_{\alpha\in[0,1]} \phi(\alpha). \tag{414}$$

Now, by (Lee et al., 2025, Thm. 3.6(b)), $\phi(\alpha)$ is piecewise constant in $\alpha$ with breakpoints $\alpha_i \triangleq |e(\Gamma^{(i)})|/|e(\Gamma)|$, and on the plateau $\alpha \in [\alpha_i, \alpha_{i+1})$,

$$\phi(\alpha) = \eta(\Gamma^{(i+1)}|\Gamma^{(i)}). \tag{415}$$

Furthermore, by (Lee et al., 2025, Lemma 5.4) these plateau values are *non-increasing* in $i$, so the minimum over all $\alpha \in [0,1]$ is attained on the last plateau, i.e., for $\alpha \in (\alpha_{M-1}, 1]$, and equals

$$\min_{\alpha\in[0,1]} \phi(\alpha) = \rho(\Gamma^{(M)}|\Gamma^{(M-1)}). \tag{416}$$

Combining (414) and (416) completes the proof.

## G.4. Bounds on coherence

**Lemma 59** *Let* $\mathsf{U} \in \mathbb{R}^{n\times r}$ *be a matrix with orthonormal columns, and supported on the set* $\mathcal{S}$ *of size* $k = |\mathcal{S}|$*; that is, the* $i$*th row of* $\mathsf{U}$ *satisfies* $\mathsf{U}_{i,:} = 0$ *for all* $i \notin \mathcal{S}$*. Then*

$$\frac{n}{k} \le \mathsf{coh}(\mathsf{U}) \le \frac{n}{r}. \tag{417}$$

**Proof** [Proof of Lemma 59] Since the columns of $\mathsf{U}$ are orthonormal, we have

$$\sum_{i=1}^{n} \|\mathsf{U}_{i,:}\|_2^2 = \mathsf{trace}(\mathsf{U}\mathsf{U}^\top) \tag{418}$$

$$= \mathsf{trace}(\mathsf{U}^\top\mathsf{U}) = \mathsf{rank}(\mathsf{X}^\star). \tag{419}$$

To establish the upper bound, note that for any row $i$, the squared norm satisfies:

$$\|\mathsf{U}_{i,:}\|_2^2 \le 1, \tag{420}$$

because $\mathsf{U}_{i,:} \in \mathbb{R}^r$ and the total squared norm across all rows sums to $r$. Hence

$$\mathsf{coh}(\mathsf{U}) = \frac{n}{r} \cdot \max_i \|\mathsf{U}_{i,:}\|_2^2 \tag{421}$$

$$\le \frac{n}{r} \cdot 1 = \frac{n}{r}. \tag{422}$$

This proves the upper bound. For the lower bound, due to (419), the average row norm squared over the $k$ non-zero rows is $\frac{r}{k}$. Thus

$$\mathsf{coh}(\mathsf{U}) \ge \frac{n}{r} \cdot \frac{r}{k} = \frac{n}{k}. \tag{423}$$

■

## G.5. Spectral-degree bound on coherence

Consider the following result.

**Theorem 60 (Spectral/degree bound on coherence)** *Let* $\mathsf{X} \in \{0,1\}^{n\times n}$ *be symmetric (e.g., the adjacency matrix of an undirected graph), let* $r \triangleq \mathsf{rank}(\mathsf{X}) \ge 1$*, and let* $\mathsf{U} \in \mathbb{R}^{n\times r}$ *have orthonormal columns spanning* $\mathrm{range}(\mathsf{X})$*. Define the coherence*

$$\mathsf{coh}(\mathsf{U}) \triangleq \frac{n}{r} \max_{1\le i\le n} \|\mathsf{U}_{i,:}\|_2^2. \tag{424}$$

*Let* $\sigma_{\min} > 0$ *be the smallest nonzero singular value of* $\mathsf{X}$*, and let* $d_i \triangleq \|x_i\|_2^2$ *where* $x_i^\top$ *is the* $i$*-th row of* $\mathsf{X}$*, with* $d_{\max} \triangleq \max_i d_i$*. Then*

$$\mathsf{coh}(\mathsf{U}) \le \frac{n}{r} \frac{d_{\max}}{\sigma_{\min}^2}. \tag{425}$$

*In particular, if* $\mathsf{X}$ *is a* $\{0,1\}$ *adjacency matrix (with or without self-loops), then* $d_i$ *equals the (loop–inclusive) degree of vertex* $i$*.*

**Proof** Let $\mathsf{P} \triangleq \mathsf{U}\mathsf{U}^\top$ be the orthogonal projector onto $\mathsf{range}(\mathsf{X})$. The *leverage scores* are $\ell_i \triangleq \|\mathsf{U}_{i,:}\|_2^2 = \mathsf{P}_{ii}$, and

$$\mathsf{coh}(\mathsf{U}) = \frac{n}{r} \max_i \ell_i. \tag{426}$$

We express $\mathsf{P}$ using only $\mathsf{X}$. Since $\mathsf{X}$ is symmetric, its singular values are the absolute values of its (nonzero) eigenvalues. Writing the spectral decomposition as $\mathsf{X} = Q\Lambda Q^\top$ with $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$, where $\lambda_k \neq 0$ for $k \leq r$), one checks that

$$\begin{aligned} \mathsf{P} &= \mathsf{U}\mathsf{U}^\top && (427)\\ &= Q_r Q_r^\top && (428)\\ &= \mathsf{X}\mathsf{X}^\dagger && (429)\\ &= \mathsf{X}(\mathsf{X}^2)^\dagger \mathsf{X}, && (430) \end{aligned}$$

where $Q_r \triangleq [q_1 \; \cdots \; q_r]$ collects the eigenvectors associated with the nonzero eigenvalues, and $\mathsf{X}^\dagger$ denotes the Moore–Penrose pseudoinverse. Therefore, for each $i$,

$$\begin{aligned} \ell_i &= e_i^\top P e_i && (431)\\ &= e_i^\top \mathsf{X}(\mathsf{X}^2)^\dagger \mathsf{X} e_i && (432)\\ &= x_i^\top (\mathsf{X}^2)^\dagger x_i. && (433) \end{aligned}$$

Let $\sigma_{\min} > 0$ be the smallest nonzero singular value of $\mathsf{X}$. Then the nonzero spectrum of $\mathsf{X}^2$ is $\{\sigma_k^2\}_{k=1}^r$, so the operator norm of $(\mathsf{X}^2)^\dagger$ equals $1/\sigma_{\min}^2$. Using (431) and the Cauchy–Schwarz/operator-norm bound,

$$\begin{aligned} \ell_i &= x_i^\top (\mathsf{X}^2)^\dagger x_i && (434)\\ &\leq \|(\mathsf{X}^2)^\dagger\|_2 \|x_i\|_2^2 && (435)\\ &= \frac{\|x_i\|_2^2}{\sigma_{\min}^2} && (436)\\ &= \frac{d_i}{\sigma_{\min}^2}. && (437) \end{aligned}$$

Taking the maximum over $i$ yields

$$\max_i \ell_i \leq \frac{d_{\max}}{\sigma_{\min}^2}, \tag{438}$$

and hence

$$\begin{aligned} \mathsf{coh}(\mathsf{U}) &= \frac{n}{r} \max_i \ell_i && (439)\\ &\leq \frac{n}{r} \frac{d_{\max}}{\sigma_{\min}^2}. && (440) \end{aligned}$$

This completes the proof. ■

## Appendix H. Derivation of the Maximum Likelihood Estimator

Consider the following recovery task. Pick a copy $\Gamma^\star \in \mathcal{S}_\Gamma$. A random graph $\mathsf{G}$ with $n$ vertices is formed as follows: keep the edges of $\Gamma$ with probability $p$, and the edges outside $\Gamma$ with probability $q$. We denote the ensemble of such planted graphs by $\mathcal{G}_{\Gamma^\star}(n,p,q)$. Given $\mathsf{G}$, the goal is to recover the graph $\Gamma^\star$. Then, we have

$$\mathbb{P}_{\mathcal{G}_{\Gamma^\star}(n,p,q)}(\mathsf{G};\Gamma) = \prod_{(i,j)\in e(\Gamma)} p^{\mathsf{A}_{ij}}(1-p)^{1-\mathsf{A}_{ij}} \prod_{(i,j)\in\binom{[n]}{2}\setminus e(\Gamma)} q^{\mathsf{A}_{ij}}(1-q)^{1-\mathsf{A}_{ij}} \tag{441}$$

$$\propto \prod_{(i,j)\in e(\Gamma)} \left[\frac{p(1-q)}{q(1-p)}\right]^{\mathsf{A}_{ij}} \tag{442}$$

$$= \left[\frac{p(1-q)}{q(1-p)}\right]^{\sum_{(i,j)\in e(\Gamma)} \mathsf{A}_{ij}} \tag{443}$$

Thus, we see that in the regime $p > q$, the MLE is given by

$$\hat{\Gamma}_{\mathsf{MLE}} = \arg\max_{\Gamma\in\mathcal{S}_\Gamma} \sum_{(i,j)\in e(\Gamma)} \mathsf{A}_{ij}. \tag{444}$$

## Appendix I. Lower Bound Via Bayes Risk Analysis

In this appendix, we provide an alternative proof of the information-theoretic lower bounds on recovery, derived from hypothesis testing risk analysis. The resulting bound is tight for balanced graphs, i.e., graphs $\Gamma$ with $\mu(\Gamma) = \eta(\Gamma)$, and for which $\mu_{\mathsf{min}}(\Gamma) = \mu(\Gamma)$, and with super-logarithmic maximum density, namely, $\mu(\Gamma) \geq \alpha_n \log|v(\Gamma)|$, for some $\alpha_n = \Omega(1)$ (which is, in fact, the interesting region). Therefore, we focus on such graphs, although the method applies to arbitrary graphs as well. To that end, we begin by showing that it suffices to consider the canonical case where $p_n = 1$. We then provide brief preliminaries on the detection problem of an arbitrary planted subgraph in random graphs. Next, we propose and prove a generalized notion of the subgraph expectation threshold, which plays an important role in the proof of Theorem 19, specialized for balanced graphs, presented in Appendix I.3.

We first observe that we can safely focus on the case where $p = 1$. This follows from the fact that for any given instance of $\mathcal{G}_{\Gamma_n}(n, p_n, q_n)$, there exist a fixed $\tilde{q}$ such that recovery is statistically easier over $\mathcal{G}_{\Gamma_n}(n, 1, \tilde{q}_n)$. Indeed, let $\hat{\Gamma}$ be any successful estimation algorithm for the recovery of $\Gamma$ over $\mathcal{G}_{\Gamma_n}(n, p_n, q_n)$, i.e., $\mathbb{P}_{\mathcal{G}_{\Gamma_n}(n,p_n,q_n)}(\hat{\Gamma} \neq \Gamma) \leq \varepsilon$, for any $\varepsilon > 0$. Now, let $\mathsf{G} \sim \mathcal{G}_{\Gamma_n}(n, 1, q_n/p_n)$, and consider the random map $\phi : \{0,1\}^{\binom{n}{2}} \to \{0,1\}^{\binom{n}{2}}$, which receives the graph $\mathsf{G}$ as an input, and if $(i,j) \in \mathsf{G}$, then we keep it with probability $p_n$, otherwise, if $(i,j) \notin \mathsf{G}$, then it remains the same. It should be clear that the random graph $\phi(\mathsf{G})$ is distributed as follows: if $(i,j) \in \Gamma$, then $\mathbb{P}\left[[\phi(\mathsf{G})]_{ij} = 1\right] = p_n \cdot 1$, otherwise, if $(i,j) \notin \Gamma$, then $\mathbb{P}\left[[\phi(\mathsf{G})]_{ij} = 1\right] = p_n \cdot (q_n/p_n) = q_n$, and thus, $\phi(\mathsf{G}) \sim \mathcal{G}_{\Gamma_n}(n, p_n, q_n)$. Furthermore, the estimator $\hat{\Gamma}(\phi(\mathsf{G}))$ is successful by construction. Therefore, proving the impossibility of recovery over $\mathcal{G}_{\Gamma_n}(n, 1, q_n/p_n)$ implies immediately the impossibility of recovery over $\mathcal{G}_{\Gamma_n}(n, p_n, q_n)$. Below, with abuse of notation, we focus on $\mathcal{G}_{\Gamma_n}(n, 1, q_n)$.

### I.1. Preliminaries on detection

In statistical analysis of detection problems, one of the goals is to establish a lower bound on the optimal risk from below, thereby ruling out the possibility of successful detection. A general recipe for this is as follows. Recall the likelihood ratio functional,

$$\mathsf{L}(\mathsf{G}) \triangleq \frac{\mathrm{d}\mathbb{P}_{\mathcal{H}_1}}{\mathrm{d}\mathbb{P}_{\mathcal{H}_0}}(\mathsf{G}), \tag{445}$$

which is the Radon-Nikodym derivative of $\mathbb{P}_{\mathcal{H}_1}$ w.r.t. the measure $\mathbb{P}_{\mathcal{H}_0}$. It is well known (see, e.g., (Tsybakov, 2004, Theorem 2.2)) that the optimal test $\phi^\star$ that minimizes the risk $\mathsf{R}_n$ is the likelihood ratio test defined as,

$$\phi^\star(\mathsf{G}) \triangleq \begin{cases} 1, & \text{if } \mathsf{L}(\mathsf{G}) \geq 1 \\ 0, & \text{otherwise,} \end{cases} \tag{446}$$

and the associated optimal risk is $\mathsf{R}^\star \triangleq \mathsf{R}(\phi^\star(\mathsf{G})) = 1 - d_{\mathsf{TV}}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1})$. Recalling that $\chi^2(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}) = \mathbb{E}_{\mathcal{H}_0}[\mathsf{L}(\mathsf{G})^2] - 1$, it can be shown that (see, e.g., (Tsybakov, 2004, Sec. 2) and (Sason, 2014, Prop. 3)),

$$\chi^2(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}) \geq \max\left(\frac{1}{2\left(1 - d_{\mathsf{TV}}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1})\right)} - 1, \left(2 d_{\mathsf{TV}}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1})\right)^2\right), \tag{447}$$

and thus,

$$\mathsf{R}^\star = 1 - d_{\mathsf{TV}}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}) \geq \max\left(1 - \frac{1}{2}\sqrt{\chi^2(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1})}, \frac{1}{2(1 + \chi^2(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}))}\right). \tag{448}$$

In particular, we see that $\mathsf{R}^\star$ is bounded away from zero, namely, strong detection is impossible, if $\mathbb{E}[\mathsf{L}(\mathsf{G})^2]$ is bounded. Similarly, $\mathsf{R}^\star$ converge to unity, i.e., weak detection is impossible if $\mathbb{E}_{\mathcal{H}_0}[\mathsf{L}(\mathsf{G})^2] = 1 + o(1)$. Accordingly, to rule out the possibility of detection (either strong or weak) it suffices to upper bound the second moment of the likelihood function.

Let us derive the likelihood function in our case, where the null distribution reflects the distribution of $\mathcal{G}(n, q_n)$, and the alternative distribution is exactly the one we consider in the recovery problem. Then,

$$\mathsf{L}(\mathsf{G}) = \mathbb{E}_\Gamma\left[\frac{\mathbb{P}_{\mathcal{H}_1|\Gamma}(\mathsf{G}|\Gamma)}{\mathbb{P}_{\mathcal{H}_0}(\mathsf{G})}\right], \tag{449}$$

and

$$\frac{\mathbb{P}_{\mathcal{H}_1|\Gamma}(\mathsf{G}|\Gamma)}{\mathbb{P}_{\mathcal{H}_0}(\mathsf{G})} = q^{-|\Gamma|} \cdot \mathbb{1}\left\{\Gamma \subseteq \mathsf{G}\right\}. \tag{450}$$

Thus,

$$\mathsf{L}(\mathsf{G}) = \frac{q^{-e(\Gamma)}}{|\mathcal{S}_\Gamma|} \sum_{\ell=1}^{|\mathcal{S}_\Gamma|} \mathbb{1}\left\{\Gamma_\ell \subseteq \mathsf{G}\right\}, \tag{451}$$

where $\{\Gamma_\ell\}_{\ell=1}^{|\mathcal{S}_\Gamma|}$ are the $|\mathcal{S}_\Gamma|$ possible copies of $\Gamma$ in $\mathcal{K}_n$. Let $\mathsf{N}_\Gamma(\mathsf{G}) \triangleq \sum_{\ell=1}^{|\mathcal{S}_\Gamma|} \mathbb{1}\{\Gamma_\ell \subseteq \mathsf{G}\}$ denote the graph copies enumerator. Then, we note that

$$\mathbb{E}_{\mathcal{H}_0}[\mathsf{N}_\Gamma(\mathsf{G})] = |\mathcal{S}_\Gamma| \cdot q^{|\Gamma|}, \tag{452}$$

and thus,

$$\mathsf{L}(\mathsf{G}) = \frac{\mathsf{N}_\Gamma(\mathsf{G})}{\mathbb{E}_{\mathcal{H}_0}[\mathsf{N}_\Gamma(\mathsf{G})]}. \tag{453}$$

Therefore, we obtain that

$$\mathbb{E}_{\mathcal{H}_0}[\mathsf{L}^2(\mathsf{G})] = \frac{\mathbb{E}_{\mathcal{H}_0}[\mathsf{N}_\Gamma^2(\mathsf{G})]}{\mathbb{E}_{\mathcal{H}_0}^2[\mathsf{N}_\Gamma(\mathsf{G})]}. \tag{454}$$

In particular, if, under some conditions we have $\mathbb{E}_{\mathcal{H}_0}[\mathsf{L}^2(\mathsf{G})] = 1 + o(1)$, implying that detection is statistically impossible, then, under the same conditions, we have,

$$\begin{aligned}\mathbb{E}_{\mathcal{H}_0}[\mathsf{N}_\Gamma^2(\mathsf{G})] &= (1+o(1)) \cdot \mathbb{E}_{\mathcal{H}_0}^2[\mathsf{N}_\Gamma(\mathsf{G})] &(455)\\ &= (1+o(1)) \cdot |\mathcal{S}_\Gamma|^2 \cdot q^{2|\Gamma|}. &(456)\end{aligned}$$

It was recently shown in Elimelech and Huleihel (2025b) that these conditions are: If $\mu(\Gamma_n) \geq \alpha_n \cdot \log|v(\Gamma_n)|$, for some $\alpha_n = \Omega(1)$, then there exists a constant $\underline{C} > 0$ such that weak detection is impossible if,

$$\mu(\Gamma_n) \leq \underline{C} \cdot \log n. \tag{457}$$

If, on the other hand, $\mu(\Gamma_n) = o(\log|v(\Gamma_n)|)$, then for every $\varepsilon > 0$, weak detection is impossible if,

$$|e(\Gamma_n)| \vee d_{\max}^2(\Gamma_n) \leq n^{1-\varepsilon}. \tag{458}$$

### I.2. Generalized subgraph expectation threshold

A central problem in probabilistic combinatorics is to understand, for a fixed graph $\Gamma$, the minimum edge probability $p$ such that the random graph $G(n,p)$ contains $\Gamma$ as a subgraph with probability at least $1/2$. This threshold is commonly known as the *critical threshold* for the appearance of $\Gamma$, denoted by $p_c(\Gamma)$. Formally, for graphs $\Gamma$ and $\mathsf{G}$ let us denote the number of copies of $\Gamma$ in $\mathsf{G}$ by $\mathcal{N}(\Gamma,\mathsf{G})$. The critical probability of $\Gamma$ is defined as,

$$q_c(\Gamma) \triangleq \min\left\{q \in [0,1] \,\middle|\, \mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}[\mathcal{N}(\Gamma,\mathsf{G}) \geq 1] \geq \frac{1}{2}\right\}. \tag{459}$$

A long-standing conjecture by Kahn and Kalai Kahn and Kalai (2007) suggests that this critical threshold is closely approximated—up to a logarithmic factor—by a more tractable quantity called the *subgraph expectation threshold*. In Mossel et al. (2022), the critical probability was bounded by a modified subgraph expectation threshold, defined as follows,

$$\tilde{q}_E(\Gamma) \triangleq \min\left\{q \in [0,1] \,\middle|\, \mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,q)}[\mathcal{N}(\mathsf{H},\mathsf{G})] \geq \frac{\mathcal{N}(\mathsf{H},\Gamma)}{2} \text{ for all } \mathsf{H} \subseteq \Gamma\right\}, \tag{460}$$

where only subgraphs $\mathsf{H} \subseteq \Gamma$ with no isolated vertices are considered.

**Theorem 61** *(Mossel et al., 2022, Theorem 1) There exists a universal constant $C$ such that for any graph $\Gamma$,*

$$\tilde{q}_E(\Gamma) \leq q_c(\Gamma) \leq C \cdot \tilde{q}_E \cdot \log|e(\Gamma)|. \tag{461}$$

We would like to generalize the above notions to the scenario of the appearance of $L \in \mathbb{N}$ distinct copies. Specifically, define the critical probability as,

$$q_c^{(L)}(\Gamma) \triangleq \min\left\{ q \in [0,1] \,\middle|\, \mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\Gamma,\mathsf{G}) \geq L\right] \geq \frac{1}{2} \right\}, \tag{462}$$

and accordingly,

$$\tilde{q}_E^{(L)}(\Gamma) \triangleq \min\left\{ q \in [0,1] \,\middle|\, \mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{H},\mathsf{G})\right] \geq \frac{L\cdot\mathcal{N}(\mathsf{H},\Gamma)}{2} \text{ for all } \mathsf{H} \subseteq \Gamma \right\}. \tag{463}$$

Note that for $q \geq q_c^{(L)}(\Gamma)$, by Markov's inequality, for any $\mathsf{H} \subseteq \Gamma$,

$$\frac{1}{2} \leq \mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\Gamma,\mathsf{G}) \geq L\right] \tag{464}$$

$$\leq \mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{H},\mathsf{G}) \geq L\cdot\mathcal{N}(\mathsf{H},\Gamma)\right] \tag{465}$$

$$\leq \frac{\mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{H},\mathsf{G})\right]}{L\cdot\mathcal{N}(\mathsf{H},\Gamma)}, \tag{466}$$

and thus, $\tilde{q}_E^{(L)}(\Gamma) \leq q_c^{(L)}(\Gamma)$. Finally, note that we can rewrite,

$$\tilde{q}_E^{(L)}(\Gamma) \triangleq \max\left\{ \left(\frac{L\cdot\mathcal{N}(\mathsf{H},\Gamma)}{2|\mathcal{S}_{\mathsf{H}}|}\right)^{1/|e(\mathsf{H})|} :\ \mathsf{H} \subseteq \Gamma \right\}. \tag{467}$$

We have the following result.

**Theorem 62** *There exists a universal constant $C$ such that for any graph $\Gamma$,*

$$\tilde{q}_E^{(L)}(\Gamma) \leq q_c^{(L)}(\Gamma) \leq C \cdot L \cdot \tilde{q}_E^{(L)} \cdot \log(L|e(\Gamma)|). \tag{468}$$

The proof of Theorem 62 relies on a powerful probabilistic tool known as the *spread lemma*, which has been instrumental in several recent breakthroughs. To describe the lemma in our context, consider a probability distribution $\pi$ supported on subgraphs of the complete graph $\mathcal{K}_n$. Let $\alpha > 1$. We say that the distribution $\pi$ is $\alpha$-*spread* if, for every (non-empty) subgraph $\mathsf{H} \subseteq \mathcal{K}_n$,

$$\pi(\mathsf{H} \subseteq \Gamma) \leq \alpha^{-|e(\mathsf{H})|}, \tag{469}$$

where $\Gamma \sim \pi$. The following result—adapted from Theorem 1.6 in Frankl et al. (2021)—gives a threshold condition under which a random graph drawn from $G(n,p)$ is likely to intersect a family of such subgraphs.

**Lemma 63 (Spread lemma (Mossel et al., 2022, Lemma 2))** *Fix integers $k \geq 1$ and $M \geq 1$. Let $\mathcal{G}_M \triangleq \{\mathsf{G}_1,\ldots,\mathsf{G}_M\} \subseteq K_n$ satisfy $|e(\mathsf{G}_i)| \leq k$ for every $i$. Let $\pi$ be the uniform measure on $\mathcal{G}_M$, and assume the $\alpha$-*spread *condition in (469). Then, there is an absolute constant $C > 0$ such that if $q > C\frac{\log k}{\alpha}$, then a sample from $\mathcal{G}(n,q)$ contains one of the $\mathsf{G}_i$'s with probability at least $1/2$.*

We now propose the following generalization of the above spread lemma.

**Lemma 64 (Generalized spread lemma)** *Fix integers $k \geq 1$ and $M \geq L \geq 1$. Let $\mathcal{G}_M \triangleq \{\mathsf{G}_1, \ldots, \mathsf{G}_M\} \subseteq K_n$ satisfy $|e(\mathsf{G}_i)| \leq k$ for every $i$. Let $\pi$ be the uniform measure on $\mathcal{G}_M$, and assume the $\alpha$-*spread *condition in (469). Then, there is an absolute constant $C > 0$ such that if,*

$$q > C\frac{L\log(kL)}{\alpha} \tag{470}$$

*then,*

$$\mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathsf{G} \text{ contains } L \text{ distinct graphs among } \mathcal{G}_M\right] \geq \frac{1}{2}. \tag{471}$$

**Proof** [Proof of Lemma 64] Define,

$$\Sigma \triangleq \left\{(i_1, \ldots, i_L) \in [M]^L : i_1, \ldots, i_L \text{ distinct}\right\}, \tag{472}$$

and thus, $|\Sigma| = \frac{M!}{(M-L)!}$. For $\sigma = (i_1, \ldots, i_L) \in \Sigma$, set,

$$\mathsf{G}_\sigma \triangleq \bigcup_{\ell=1}^{L} \mathsf{G}_{i_\ell}. \tag{473}$$

Let

$$\pi^{\langle L\rangle} = \mathsf{Unif}(\Sigma), \tag{474}$$

be the *uniform* law on the index set $\Sigma$. Sampling $\sigma \sim \pi^{\langle L\rangle}$ and putting $\mathsf{G}^* = \mathsf{G}_\sigma$ chooses an ordered $L$-tuple without replacement and bundles its $L$ distinct graphs into one target. Now, fix a non-empty subgraph $\mathsf{H} \subseteq \mathcal{K}_n$. For $j \in [L]$ let,

$$q_j \triangleq \mathbb{P}_{\mathsf{G}\sim\pi}\left[\mathsf{H} \subseteq \mathsf{G} | \mathsf{G} \notin \{\mathsf{G}_{i_1}, \ldots, \mathsf{G}_{i_{j-1}}\}\right]. \tag{475}$$

Since $\pi$ is assumed $\alpha$-spread, and because conditioning can only decrease the chance that $\mathsf{H} \subseteq \mathsf{G}$, we have,

$$q_j \leq \mathbb{P}_{\mathsf{G}\sim\pi}\left[\mathsf{H} \subseteq \mathsf{G}\right] \leq \alpha^{-|e(\mathsf{H})|}. \tag{476}$$

Therefore,

$$\pi^{\langle L\rangle}(\mathsf{H} \subseteq \mathsf{G}^*) = 1 - \prod_{j=1}^{L}(1 - q_j) \tag{477}$$

$$\leq L\alpha^{-|e(\mathsf{H})|} \tag{478}$$

$$\leq (L/\alpha)^{|e(\mathsf{H})|}. \tag{479}$$

Thus $\pi^{\langle L\rangle}$ is $(\alpha/L)$-spread. Now, we invoke the single spread lemma in Lemma 63. Note that each element $\mathsf{G}^*$ has at most $kL$ edges, and there are at most $|\Sigma| \leq M^L$ such unions. Applying Lemma 63 with parameters $k' = kL$ and $\alpha' = \alpha/L$, we see that (470) guarantees $\mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}[\mathsf{G} \supseteq \mathsf{G}^*] \geq \frac{1}{2}$, which concludes the proof. ∎

We are now in a position to prove Theorem 62.

**Proof** [Proof of Theorem 62] Let $\pi = \pi_\Gamma$ denote the uniform distribution over all copies of a fixed graph $\Gamma$ within the complete graph $\mathcal{K}_n$. Let $\Gamma' \sim \pi_\Gamma$ be a random sample from this distribution. Now, consider a subgraph $\mathsf{H} \subset \Gamma$. Let $\pi_{\mathsf{H}}$ denote the uniform distribution over all copies of $\mathsf{H}$ in $\mathcal{K}_n$, and let $\mathsf{H}' \sim \pi_{\mathsf{H}}$. For any fixed instances $\mathsf{H}_0 \subseteq \mathcal{K}_n$ and $\Gamma_0 \subseteq \mathcal{K}_n$, copies of $\mathsf{H}$ and $\Gamma$, respectively, we can compute the inclusion probability in two equivalent ways,

$$\pi_\Gamma(\mathsf{H}_0 \subseteq \Gamma') = \pi_{\mathsf{H}}(\mathsf{H}' \subseteq \Gamma_0) = \frac{\mathcal{N}(\mathsf{H},\Gamma)}{|\mathcal{S}_{\mathsf{H}}|}. \tag{480}$$

Now, combining (480) with (467), we obtain,

$$\pi_\Gamma(\mathsf{H}_0 \subseteq \Gamma') = \frac{\mathcal{N}(\mathsf{H},\Gamma)}{|\mathcal{S}_{\mathsf{H}}|} \tag{481}$$

$$\leq \frac{2}{L} \cdot \left[\widetilde{q}_E^{(L)}(\Gamma)\right]^{|e(\mathsf{H})|} \tag{482}$$

$$\leq \left(\frac{1}{2\widetilde{q}_E^{(L)}(\Gamma)}\right)^{-|e(\mathsf{H})|}. \tag{483}$$

Since the bound holds uniformly over all subgraphs $\mathsf{H}_0$, we conclude that $\pi_\Gamma$ is $\alpha$-spread with $\alpha = \frac{1}{2\widetilde{q}_E^{(L)}(\Gamma)}$. An application of Lemma 64 with $k = |e(\Gamma)|$ then completes the proof of Theorem 62. ∎

Next, we prove the following result.

**Theorem 65** *Fix $L \geq 1$. Let $\Gamma = (\Gamma_n)$ be a sequence of graphs such that $\omega(1) \leq |v(\Gamma)| \leq n$ and*

$$(1+\varepsilon)\mu(\Gamma_n) \cdot [\log\log(L|e(\Gamma_n)|) + \log(CL)] + \log|v(\Gamma_n)| \leq \log n, \tag{484}$$

*for some $\varepsilon > 0$ and $C > 0$. Then, for any fixed $q$, a sample from $\mathcal{G}(n,q)$ contains at least $L$ isomorphic copies of $\Gamma$, with probability at least $1/2$.*

In order to prove Theorem 65 we need to bound the probability that a uniform random copy of $\Gamma$ in $\mathcal{K}_n$ contains an arbitrary isomorphic copy of a subgraph $\mathsf{H} \subseteq \Gamma$ in $\mathcal{K}_n$, namely, $\mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma]$.

**Lemma 66** *For any $\mathsf{H} \subseteq \Gamma$,*

$$\mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma] \leq \left(\frac{|v(\Gamma)|}{n}\right)^{|v(\mathsf{H})|}. \tag{485}$$

**Proof** [Proof of Lemma 66] Let $m = |v(\Gamma)|$ and $k = |v(\mathsf{H})|$. Fix a particular copy $\mathsf{H}_0$ of $\mathsf{H}$ in $\mathcal{K}_n$ with vertex set $U = \{u_1, \ldots, u_k\}$. Generate a uniformly random copy of $\Gamma$ by first picking its vertex set $S \subset [n]$ uniformly among all $m$-subsets (the internal labeling of $\Gamma$ can only lower the probability of containing $\mathsf{H}_0$, so it suffices to control this step). If the sampled copy contains $\mathsf{H}_0$, then necessarily $U \subseteq S$. Thus

$$\mathbb{P}_\Gamma[\mathsf{H}_0 \subseteq \Gamma] \leq \mathbb{P}[U \subseteq S] \tag{486}$$

$$= \prod_{i=1}^{k} \mathbb{P}[u_i \in S | u_1, \ldots, u_{i-1} \in S]. \tag{487}$$

Conditioned on $u_1, \ldots, u_{i-1} \in S$, there are $n-(i-1)$ remaining vertices and $m-(i-1)$ remaining slots in $S$, so

$$\mathbb{P}[u_i \in S | u_1, \ldots, u_{i-1} \in S] = \frac{m-(i-1)}{n-(i-1)} \tag{488}$$

$$\leq \frac{m}{n}. \tag{489}$$

Multiplying these $k$ bounds gives

$$\mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma] \leq \mathbb{P}[U \subseteq S] \tag{490}$$

$$\leq \left(\frac{m}{n}\right)^k = \left(\frac{|v(\Gamma)|}{n}\right)^{|v(\mathsf{H})|}. \tag{491}$$

■

We are now ready to prove Theorem 65.

**Proof** [Proof of Theorem 65] From the definition of $q_c^{(L)}(\Gamma)$ in (462), our goal is to understand for which $\Gamma$'s we have $q_c^{(L)}(\Gamma) \leq q$, for any fixed $q \in (0,1]$. By Theorem 62, it is sufficient to show that $\tilde{q}_E^{(L)} \leq \frac{q}{CL\log(L|e(\Gamma)|)}$ for any fixed $q$, which holds if and only if,

$$\inf_{\mathsf{H} \subseteq \Gamma} \frac{\mathbb{E}_{\mathsf{G} \sim \mathcal{G}(n,\tilde{q})}[\mathcal{N}(\mathsf{H},\mathsf{G})]}{\mathcal{N}(\mathsf{H},\Gamma)} \geq \frac{L}{2}, \tag{492}$$

where $\tilde{q} \triangleq \frac{q}{CL\log(L|e(\Gamma)|)}$. We note that,

$$\mathbb{E}_{\mathsf{G} \sim \mathcal{G}(n,\tilde{q})}[\mathcal{N}(\mathsf{H},\mathsf{G})] = |\mathcal{S}_\mathsf{H}| \cdot \tilde{q}^{|\mathsf{H}|}, \tag{493}$$

where we recall that $|\mathcal{S}_\mathsf{H}| = \mathcal{N}(\mathsf{H}, \mathcal{K}_n)$ denotes the number of copies of $\mathsf{H}$ in the complete graph, and $|\mathsf{H}|$ denotes the number of edges in $\mathsf{H}$. Thus, (492) holds if and only if for any $\mathsf{H} \subseteq \Gamma$ we have,

$$\frac{\mathcal{N}(\mathsf{H},\Gamma)}{|\mathcal{S}_\mathsf{H}|} \leq \frac{2}{L}\tilde{q}^{|\mathsf{H}|} = \frac{2}{L}\left(\frac{q}{CL\log(L|e(\Gamma)|)}\right)^{|\mathsf{H}|}. \tag{494}$$

An easy combinatorial argument shows that the expression on the left-hand side of (494) equals $\mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma]$ (see, (Elimelech and Huleihel, 2025b, Lemma 2)). Thus, (494) holds if,

$$\log \mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma] - \log\frac{2}{L} - |\mathsf{H}|\left(\log q - \log(CL) - \log\log(L|e(\Gamma)|)\right) \leq 0. \tag{495}$$

By the assumptions that $|v(\Gamma)| = \omega(1)$, and that $\Gamma$ has no isolated vertices, for any $\varepsilon$ and for sufficiently large $n$, the above holds if,

$$\log \mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma] + |\mathsf{H}| \log(CL) + (1+\varepsilon)|\mathsf{H}| \log\log(L|e(\Gamma)|) \leq 0. \tag{496}$$

Finally, using Lemma 66, the left-hand side of (496) can be upper bounded by,

$$\log \mathbb{P}_\Gamma[\mathsf{H} \subseteq \Gamma] + |\mathsf{H}| \log(CL) + (1+\varepsilon)|\mathsf{H}| \log\log(L|e(\Gamma)| \tag{497}$$

$$\leq |v(\mathsf{H})| \left(\log |v(\Gamma)| - \log n\right) + |\mathsf{H}| \log(CL) + (1+\varepsilon)|\mathsf{H}| \log\log(L|e(\Gamma)|) \tag{498}$$

$$= |v(\mathsf{H})| \cdot \left(\log |v(\Gamma)| + \frac{|\mathsf{H}|}{|v(\mathsf{H})|} \log(CL) + (1+\varepsilon)\frac{|\mathsf{H}|}{|v(\mathsf{H})|} \log\log(L|e(\Gamma)|) - \log n\right) \tag{499}$$

$$\overset{(a)}{\leq} |v(\mathsf{H})| \cdot (\log |v(\Gamma)| + (1+\varepsilon)\mu(\Gamma)[\log\log |e(\Gamma)| + \log(CL)] - \log n) \leq 0, \tag{500}$$

where $(a)$ follows from the definition of the maximal subgraph density, and the last inequality follows from (484). This concludes the proof. ■

### I.3. Proof of Theorem 19

**Recovery through detection.** As mentioned before we focus on the case where $\Gamma$ is on balanced. First note that the worst-case error probability can be lower bounded by the average risk as follows, Accordingly,

$$\inf_{\hat{\Gamma}^M} \max_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}[\hat{\Gamma}^M(\mathsf{G}; \Gamma^{(M-1)}) \neq \Gamma^{(M)}] = \inf_{\hat{\Gamma}^M} \max_{\Gamma^{(M)} \in \mathcal{M}(\Gamma^{(M-1)}, \Gamma)} \mathbb{P}[\hat{\Gamma}^M(\mathsf{G}; \Gamma^{(M-1)}) \neq \Gamma^{(M)}] \tag{501}$$

$$\max_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^\star] \geq \mathbb{E}_{\Gamma \sim \mathsf{Unif}(\mathcal{S}_\Gamma)} \mathbb{P}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma], \tag{502}$$

and thus we next focus on the case where $\Gamma$ is drawn uniformly at random. It proves more convenient to analyze the probability of correct recovery, i.e.,

$$\mathbb{P}_{\mathcal{G}_{\Gamma_n}}\left[\hat{\Gamma}(\mathsf{G}) = \Gamma\right] = \mathbb{E}\left[\mathbb{P}\left[\hat{\Gamma}(\mathsf{G}) = \Gamma \middle| \mathsf{G}\right]\right], \tag{503}$$

where we have used the law of total expectation. Let us analyze the distribution of $\Gamma$ *given* $\mathsf{G}$. First, we note that for any $\Gamma' \in \mathcal{S}_\Gamma$,

$$\mathbb{P}_{\mathcal{G}_{\Gamma_n}}(\mathsf{G}|\Gamma = \Gamma') = q^{\binom{n}{2} - |\Gamma|} \mathbb{1}\left\{\Gamma' \subseteq \mathsf{G}\right\}, \tag{504}$$

and thus,

$$\mathbb{P}_{\mathcal{G}_{\Gamma_n}}(\Gamma = \Gamma'|\mathsf{G}) = \frac{\mathbb{P}_{\mathcal{G}_{\Gamma_n}}(\mathsf{G}, \Gamma = \Gamma')}{\mathbb{P}_{\mathcal{G}_{\Gamma_n}}(\mathsf{G})} \tag{505}$$

$$= \frac{\mathbb{P}_{\mathcal{G}_{\Gamma_n}}(\mathsf{G}|\Gamma = \Gamma')\mathbb{P}(\Gamma = \Gamma')}{\sum_{\Gamma'' \in \mathcal{S}_\Gamma} \mathbb{P}_{\mathcal{G}_{\Gamma_n}}(\mathsf{G}|\Gamma = \Gamma'')\mathbb{P}(\Gamma = \Gamma'')} \tag{506}$$

$$= \frac{q^{\binom{n}{2} - |\Gamma|} \mathbb{1}\left\{\Gamma' \subseteq \mathsf{G}\right\}}{\sum_{\Gamma'' \in \mathcal{S}_\Gamma} q^{\binom{n}{2} - |\Gamma''|} \mathbb{1}\left\{\Gamma'' \subseteq \mathsf{G}\right\}} \tag{507}$$

$$= \frac{\mathbb{1}\left\{\Gamma' \subseteq \mathsf{G}\right\}}{\mathsf{N}_\Gamma(\mathsf{G})}, \tag{508}$$

where the second equality follows from the fact that $\Gamma$ is drawn uniformly over $\mathcal{S}_\Gamma$, the last inequality is because $|\Gamma'| = |\Gamma''|$, for any $\Gamma', \Gamma'' \in \mathcal{S}_\Gamma$, and we recall that $\mathsf{N}_\Gamma(\mathsf{G})$ counts the number of copies of $\Gamma$ in $\mathsf{G}$. Thus, using the above we see that,

$$\mathbb{P}_{\mathcal{G}_{\Gamma_n}}\left[\hat{\Gamma}(\mathsf{G}) = \Gamma\right] = \mathbb{E}\left[\mathbb{P}\left[\hat{\Gamma}(\mathsf{G}) = \Gamma \middle| \mathsf{G}\right]\right] \tag{509}$$

$$= \mathbb{E}\left[\frac{\mathbb{1}\left\{\hat{\Gamma}(\mathsf{G}) \subseteq \mathsf{G}\right\}}{\mathsf{N}_\Gamma(\mathsf{G})}\right] \tag{510}$$

$$\leq \mathbb{E}\left[\mathsf{N}_\Gamma^{-1}(\mathsf{G})\right]. \tag{511}$$

Luckily, we understand very well the distribution of $\mathsf{N}_\Gamma(\mathsf{G})$ when $\mathsf{G} \sim \mathcal{G}(n, q_n)$; here, on the other hand, the expectation is taken w.r.t. $\mathcal{G}_{\Gamma_n}$. Nonetheless, we note to the following simple observation. Let $\mathsf{G} \setminus \Gamma^\star$ be defined as the graph obtained by removing from $\mathsf{G}$, the planted subgraph $\Gamma^\star$, all the edges between the vertices of $\Gamma^\star$ in $\mathsf{G}$, and the edges from $\Gamma^\star$ to the vertices in $\mathsf{G}$. Then, it is clear that $\mathsf{G} \setminus \Gamma^\star \sim \mathcal{G}(n - |v(\Gamma)|, q_n)$. Furthermore, if we let $\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}}) \triangleq \mathsf{N}_\Gamma(\mathsf{G} \setminus \Gamma^\star)$, then clearly, $\mathsf{N}_\Gamma(\mathsf{G} \setminus \Gamma^\star) \leq \mathsf{N}_\Gamma(\mathsf{G})$. Thus, if we fix $\ell \in \mathbb{N}$, then,

$$\mathbb{E}\left[\frac{1}{\mathsf{N}_\Gamma(\mathsf{G})}\right] = \mathbb{E}\left[\frac{1}{\mathsf{N}_\Gamma(\mathsf{G})}\mathbb{1}\left\{\mathsf{N}_\Gamma(\mathsf{G}) > \ell\right\}\right] + \mathbb{E}\left[\frac{1}{\mathsf{N}_\Gamma(\mathsf{G})}\mathbb{1}\left\{\mathsf{N}_\Gamma(\mathsf{G}) \leq \ell\right\}\right] \tag{512}$$

$$\leq \ell^{-1} + \mathbb{P}\left[\mathsf{N}_\Gamma(\mathsf{G}) \leq \ell\right] \tag{513}$$

$$\leq \ell^{-1} + \mathbb{P}\left[\mathsf{N}_\Gamma(\mathsf{G} \setminus \Gamma^\star) \leq \ell\right], \tag{514}$$

where in the second inequality we used the fact that $\mathsf{N}_\Gamma(\mathsf{G}) \geq 1$ with probability one.

Finally, we prove Theorem 19, again, assuming that $\Gamma$ is balanced. Recalling (454), let $\bar{\mathsf{L}}(\mathsf{G}) \triangleq \frac{\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})}{\mathbb{E}\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}}}$ denote the respective likelihood function. By Chebyshev's inequality, for $0 < \alpha < 1$,

$$\mathbb{P}\left[\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}}) \leq \alpha \cdot \mathbb{E}\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})\right] \leq \mathbb{P}\left[\left|\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}}) - \mathbb{E}\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})\right| \geq (1-\alpha) \cdot \mathbb{E}\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})\right] \tag{515}$$

$$\leq \frac{\mathsf{Var}\left(\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})\right)}{(1-\alpha)^2\mathbb{E}^2\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})}. \tag{516}$$

Also,

$$\frac{\mathsf{Var}\left(\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})\right)}{\mathbb{E}^2\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})} = \frac{\mathbb{E}\bar{\mathsf{N}}^2_\Gamma(\bar{\mathsf{G}})}{\mathbb{E}^2\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})} - 1 \tag{517}$$

$$= \mathbb{E}[\bar{\mathsf{L}}^2(\bar{\mathsf{G}})] - 1, \tag{518}$$

where the expectation is w.r.t. $\bar{\mathsf{G}} \sim \mathcal{G}(n - |v(\Gamma)|, q_n)$. The above second moment is, almost, the quantity we bound when lower bounding the risk of the corresponding *detection problem*; it is almost, because here the underlying graphs are over $n - |v(\Gamma)|$ rather than $n$ vertices. However, this can be accounted for by replacing $n$ with $n - |v(\Gamma)|$ in (457). Specifically, for sufficiently large $n$, it should be clear that there exist constants $\underline{C}$ and $\varepsilon > 0$, such that (457) hold with $n$ replaced by $n - |v(\Gamma)|$, and accordingly, under these conditions, using (456),

$$\frac{\mathsf{Var}\left(\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})\right)}{\mathbb{E}^2\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})} = \mathbb{E}[\bar{\mathsf{L}}^2(\bar{\mathsf{G}})] - 1 = o(1). \tag{519}$$

Thus, with $\ell = \alpha \cdot \mathbb{E}\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}})$,

$$\mathbb{P}\left[\mathsf{N}_\Gamma(\mathsf{G}) \leq \ell\right] = o(1). \tag{520}$$

Finally, notice that, if we let $k = |v(\Gamma)|$, then,

$$\begin{aligned}
\ell &= \alpha \cdot \mathbb{E}\bar{\mathsf{N}}_\Gamma(\bar{\mathsf{G}}) && (521)\\
&= \binom{n-k}{k}\frac{k!}{|\mathsf{Aut}(\Gamma)|}\cdot q^{|\Gamma|} && (522)\\
&\geq \left(\frac{n-k}{k}\right)^k q^{|\Gamma|} && (523)\\
&= 2^{k\log\frac{n-k}{k} - |\Gamma|\log\frac{1}{q}} && (524)\\
&= 2^{k\left(\log\frac{n-k}{k} - \frac{|\Gamma|}{k}\log\frac{1}{q}\right)} && (525)\\
&\geq 2^{k\left(\log\frac{n-k}{k} - \mu(\Gamma_n)\log\frac{1}{q}\right)} \to \infty, && (526)
\end{aligned}$$

for some $\underline{C}$ under (36) for graphs with super-logarithmic density. Thus, using the above fact we get from (502), (511), and (514), that,

$$\max_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}[\hat{\Gamma}(\mathsf{G}) \neq \Gamma^\star] \geq 1 - o(1), \tag{527}$$

under the same conditions as in (457) albeit with a different constant $\underline{C}$.

## Appendix J. Almost-exact recovery

In this appendix, we prove Theorems 29. To that end, we start with some preliminaries.

### J.1. Preliminaries and auxiliary results

**Equivalent notions of recovery.** Recall that an estimator $\hat{\Gamma}$ almost-exactly recover $\Gamma^\star$ if, as $n \to \infty$, $\mathsf{d_H}(\hat{\Gamma}, \Gamma^\star)/|e(\Gamma)| \to 0$ in probability. We have the following result.

**Lemma 67** *An estimator $\hat{\Gamma}$ almost-exactly recover $\Gamma^\star$ if and only if $\mathbb{E}\mathsf{d_H}(\hat{\Gamma}, \Gamma^\star)/|e(\Gamma)| \to 0$.*

**Proof** [Proof of Lemma 67] The forward implication follows immediately, since $L_1$-convergence entails convergence in probability (Folland, 1999, Proposition 6.14). For the other direction, assume there is $\hat{\Gamma}$ with

$$\frac{d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma})}{m} \xrightarrow{\mathsf{P}} 0, \tag{528}$$

where $m \triangleq |e(\Gamma)|$. Let $\Pi_m(\hat{\Gamma})$ be any $m$-sparse vector minimizing Hamming distance to $\hat{\Gamma}$, i.e., delete or add ones to reach size $m$). Then, by triangle inequality,

$$\begin{aligned}
d_H(\Gamma^\star, \Pi_m(\hat{\Gamma})) &\leq d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma}) + d_{\mathsf{H}}(\hat{\Gamma}, \Pi_m(\hat{\Gamma})) && (529)\\
&= d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma}) + \left||\hat{\Gamma}| - m\right| && (530)\\
&= d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma}) + \left||\mathsf{supp}(\hat{\Gamma})| - |\mathsf{supp}(\Gamma^\star)|\right| && (531)\\
&\leq 2 \cdot d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma}). && (532)
\end{aligned}$$

Furthermore, $|\Gamma^\star| = |\Pi_m(\hat{\Gamma})| = m$ gives $d_H(\Gamma^\star, \Pi_m(\hat{\Gamma})) \le 2m$. Hence

$$\frac{d_H(\Gamma^\star, \Pi_m(\hat{\Gamma}))}{m} \le 2\left(\frac{d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma})}{m} \wedge 1\right), \tag{533}$$

with probability one. Now, for any nonnegative $X$, we have $X \wedge 1 = \int_0^1 \mathbb{1}\{X > t\}\,\mathrm{d}t$. Therefore

$$\mathbb{E}\left[\frac{d_H(\Gamma^\star, \Pi_m(\hat{\Gamma}))}{m}\right] \le 2\mathbb{E}\left[\left(\frac{d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma})}{m} \wedge 1\right)\right] \tag{534}$$

$$= 2\int_0^1 \mathbb{P}\left(\frac{d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma})}{m} > t\right)\mathrm{d}t. \tag{535}$$

Because $\frac{d_{\mathsf{H}}(\Gamma^\star,\hat{\Gamma})}{m} \xrightarrow{\mathsf{P}} 0$, for every fixed $t > 0$ we have $\mathbb{P}(d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma}) > mt) \to 0$; the integrand is bounded by $1$. By dominated convergence,

$$\int_0^1 \mathbb{P}\left(\frac{d_{\mathsf{H}}(\Gamma^\star, \hat{\Gamma})}{m} > t\right)\mathrm{d}t \to 0, \tag{536}$$

which implies that $\mathbb{E}[d_H(\Gamma, \Pi_m(\hat{\Gamma}))]/m \to 0$. Thus, the estimator $\tilde{\Gamma} \triangleq \Pi_m(\hat{\Gamma})$ satisfies $\mathbb{E}[d_H(\Gamma^\star, \Pi_m(\hat{\Gamma}))/m] \to 0$, as required. ■

**Genie argument for recovery.** To prove our lower bounds, we rely on the following genie argument. Consider the onion decomposition in Definition 16, and for each $\Gamma^{(\ell)}$ define its onion tail as $T_\ell(\Gamma) \triangleq \Gamma \setminus \Gamma^{(\ell)}$. In light of Lemma 67, the (global) minimax risk is

$$\mathsf{E}^\star_{\mathsf{almost}} \triangleq \inf_{\hat{\Gamma}} \sup_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{E}\left[\frac{d_{\mathsf{H}}(\hat{\Gamma}, \Gamma^\star)}{|e(\Gamma^\star)|}\right]. \tag{537}$$

Define the *genie minimax risk* at level $\ell$ as follows. The oracle reveals the *true* $\Gamma^{(\ell)}$ and we only need to recover the tail $T_\ell(\Gamma^\star)$; the loss is the *normalized* Hamming error on the tail

$$\mathsf{e}^\star_{\mathsf{almost}}(\ell) \triangleq \inf_{\hat{T}} \sup_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{T}(\mathsf{G}, \Gamma^{(\ell)}),\, T_\ell(\Gamma^\star)\right)}{|e(T_\ell(\Gamma^\star))|}\right]. \tag{538}$$

We have the following result.

**Lemma 68** *For any fixed index $\ell \in [M(\Gamma)]$,*

$$\mathsf{E}^\star_{\mathsf{almost}} \ge \frac{|e(T_\ell(\Gamma^\star))|}{|e(\Gamma^\star)|} \cdot \mathsf{e}^\star_{\mathsf{almost}}(\ell). \tag{539}$$

**Proof** [Proof of Lemma 68] Fix any estimator $\hat{\Gamma}$. For each $\Gamma^\star$, let $\Pi_{T_\ell(\Gamma^\star)}(\hat{\Gamma})$ denote the projection of $\hat{\Gamma}$ into $T_\ell(\Gamma^\star)$. Then, disagreements on the tail are a subset of global disagreements, and so

$$d_{\mathsf{H}}(\hat{\Gamma}, \Gamma^\star) \ge d_{\mathsf{H}}\left(\hat{\Gamma}|_{T_\ell(\Gamma^\star)}, T_\ell(\Gamma^\star)\right). \tag{540}$$

Divide by $|e(\Gamma^\star)|$ and take expectation:

$$\mathbb{E}\left[\frac{d_{\mathsf{H}}(\hat{\Gamma},\Gamma^\star)}{|e(\Gamma^\star)|}\right] \geq \frac{|e(T_\ell(\Gamma^\star))|}{|e(\Gamma^\star)|}\mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{\Gamma}|_{T_\ell(\Gamma^\star)},T_\ell(\Gamma^\star)\right)}{|e(T_i(\Gamma^\star))|}\right]. \tag{541}$$

Consequently, by the definition of (538), we clearly have

$$\sup_{\Gamma^\star\in\mathcal{S}_\Gamma}\mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{\Gamma}|_{T_\ell(\Gamma^\star)},T_\ell(\Gamma^\star)\right)}{|e(T_i(\Gamma^\star))|}\right] \geq \mathsf{e}^\star_{\mathsf{almost}}(\ell). \tag{542}$$

Taking $\sup_{\Gamma^\star}$ in (541) and then $\inf_{\hat{\Gamma}}$,

$$\begin{aligned}
\mathsf{E}^\star_{\mathsf{almost}} &= \inf_{\hat{\Gamma}}\sup_{\Gamma^\star\in\mathcal{S}_\Gamma}\mathbb{E}\left[\frac{d_{\mathsf{H}}(\hat{\Gamma},\Gamma^\star)}{|e(\Gamma^\star)|}\right]\\
&\geq \inf_{\hat{\Gamma}}\sup_{\Gamma^\star\in\mathcal{S}_\Gamma}\frac{|e(T_\ell(\Gamma^\star))|}{|e(\Gamma^\star)|}\mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{\Gamma}|_{T_\ell(\Gamma^\star)},T_\ell(\Gamma^\star)\right)}{|e(T_\ell(\Gamma^\star))|}\right]\\
&= \frac{|e(T_\ell(\Gamma^\star))|}{|e(\Gamma^\star)|}\inf_{\hat{\Gamma}}\sup_{\Gamma^\star\in\mathcal{S}_\Gamma}\mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{\Gamma}|_{T_\ell(\Gamma^\star)},T_\ell(\Gamma^\star)\right)}{|e(T_\ell(\Gamma^\star))|}\right]\\
&\geq \frac{|e(T_\ell(\Gamma^\star))|}{|e(\Gamma^\star)|}\cdot\mathsf{e}^\star_{\mathsf{almost}}(\ell),
\end{aligned} \tag{543}$$

where the second equality follows from the fact that $\frac{|e(T_\ell(\Gamma^\star))|}{|e(\Gamma^\star)|}$ is the same for all $\Gamma^\star\in\mathcal{S}_\Gamma$. ■

Define

$$r_\ell^{(n)} \triangleq \frac{|e(\Gamma_n^\star\setminus\Gamma_n^{(\ell)})|}{|e(\Gamma_n^\star)|}. \tag{544}$$

Fix any sequence $\varepsilon_n\downarrow 0$ and define

$$\ell_{\mathsf{LB}}(n) \triangleq \max\{\ell : r_\ell^{(n)} > \varepsilon_n\}. \tag{545}$$

Heuristically, $\ell_{\mathsf{LB}}$ is the last index where the ratio $r_\ell^{(n)}$ is $\Omega(1)$. Then, using Lemma 68 we have

$$\mathsf{E}^\star_{\mathsf{almost}} \geq \frac{|e(T_{\ell_{\mathsf{LB}}}(\Gamma^\star))|}{|e(\Gamma^\star)|}\cdot\mathsf{e}^\star_{\mathsf{almost}}(\ell_{\mathsf{LB}}), \tag{546}$$

and by the definition of $\ell_{\mathsf{LB}}$, we know that the multiplicative factor $\frac{|e(T_{\ell_{\mathsf{LB}}}(\Gamma^\star))|}{|e(\Gamma^\star)|}$ is strictly positive for all $n$. Hence, to rule out the possibility of almost-exact recovery, it suffices to prove that $\mathsf{e}^\star_{\mathsf{almost}}(\ell_{\mathsf{LB}}) = \Omega(1)$. In fact, we first lower bound the worst-case error probability $\mathsf{e}_{\mathsf{almost}}$ by its average-case counterpart as follows:

$$\mathsf{e}^{\star}_{\mathsf{almost}}(\ell) = \inf_{\hat{T}} \sup_{\Gamma^{\ell}} \sup_{T_\ell \in \mathcal{M}(\Gamma^{(\ell)},\Gamma^{\star})} \mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{T}(\mathsf{G},\Gamma^{(\ell)}),\, T_\ell\right)}{|e(T_\ell)|}\right] \tag{547}$$

$$\geq \inf_{\hat{T}} \sup_{\Gamma^{\ell}} \mathbb{E}_{T_\ell\sim\pi} \mathbb{E}\left[\frac{d_{\mathsf{H}}\left(\hat{T}(\mathsf{G},\Gamma^{(\ell)}),\, T_\ell\right)}{|e(T_\ell)|}\right] \tag{548}$$

$$\triangleq \bar{\mathsf{e}}^{\star}_{\mathsf{almost}}(\ell), \tag{549}$$

where $\pi \triangleq \mathsf{Unif}(\mathcal{M}(\Gamma^{(\ell)},\Gamma^{\star}))$ is the uniform measure over $\mathcal{M}(\Gamma^{(\ell)},\Gamma^{\star})$, and the inequality follows from the fact that the worst-case risk is lower bounded by the average-case risk.

### J.2. Lower bound through detection

By the results of the previous subsection, it suffices to establish lower bounds for recovering $\bar{\Gamma}_n \triangleq \Gamma^{\star}_n \setminus \Gamma^{\ell_{\mathsf{LB}}}_n$. Let $\mathsf{G}' \triangleq \mathsf{G} \setminus \Gamma^{\ell_{\mathsf{LB}}}_n$ and $n' \triangleq n - v(|\Gamma^{\ell_{\mathsf{LB}}}_n|)$. We have the following result.

**Theorem 69** *Assume that $p_n, q_n = \Theta(1)$.*

1. *If $\mu(\bar{\Gamma}_n) \geq \alpha_n \cdot \log|v(\bar{\Gamma}_n)|$, for some $\alpha_n = \Omega(1)$, then there exists a constant $\underline{C} > 0$ such that almost-exact recovery of $\bar{\Gamma}$ is impossible if*

$$\mu(\bar{\Gamma}_n) \leq \underline{C} \cdot \log n'. \tag{550}$$

2. *If $\mu(\bar{\Gamma}_n) = o(\log|v(\bar{\Gamma}_n)|)$, then for every $\varepsilon > 0$, almost-exact recovery of $\bar{\Gamma}$ is impossible if*

$$|e(\bar{\Gamma}_n)| \vee d^2_{\max}(\bar{\Gamma}_n) \leq n^{1-\varepsilon}. \tag{551}$$

**Proof** [Proof of Theorem 69] Let $k \triangleq |v(\bar{\Gamma})|$, and $L_n \triangleq |\mathcal{M}(\Gamma^{\ell_{\mathsf{LB}}}_n, \Gamma^{\star})|$ where $\mathcal{M}(\Gamma^{\ell_{\mathsf{LB}}}_n, \Gamma^{\star})$ is the number of ways $\Gamma^{\ell_{\mathsf{LB}}}_n$ can be extended to a copy of $\Gamma^{\star}$ in $\mathsf{K}_n$, or the number of copies of $\Gamma^{\star}$ in $\mathsf{K}_n$ that contain $\Gamma^{\ell_{\mathsf{LB}}}_n$. Define the following quantity:

$$\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma}) \triangleq \frac{1}{\mathcal{N}^2_{\bar{\Gamma}}} \sum_{\ell_1=1}^{L_n} \sum_{\ell_1=2}^{L_n} \mathbb{1}\left[\bar{\Gamma}_{\ell_1} \cap \bar{\Gamma}_{\ell_2} \neq \emptyset,\ \bar{\Gamma}_{\ell_1}, \bar{\Gamma}_{\ell_2} \in \mathsf{G}'\right], \tag{552}$$

where $\bar{\Gamma}_1, \bar{\Gamma}_2, \ldots, \bar{\Gamma}_{L_n}$ are all possible subgraph copies, and $\mathcal{N}_{\bar{\Gamma}} \triangleq \sum_{\ell=1}^{L_n} \mathbb{1}\left[\bar{\Gamma}_\ell \in \mathsf{G}\right]$. Namely, $\mathsf{Int}_{\mathsf{G}}(\bar{\Gamma})$ is the proportion of pairs copies of $\bar{\Gamma}$ in $\mathsf{G}$ whose intersection is nonempty. Let $\mathsf{N}_\ell \triangleq \mathbb{1}\left[\bar{\Gamma}_\ell \in \mathsf{G}\right]$. Then

$$\mathbb{E}_{\mathcal{H}_0}[\mathcal{N}^2_{\bar{\Gamma}}] = \sum_{\ell_1,\ell_2} \mathbb{P}_{\mathcal{H}_0}[\mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}] \tag{553}$$

$$= \sum_{\ell_1,\ell_2:\bar{\Gamma}_{\ell_1}\cap\bar{\Gamma}_{\ell_2}=\emptyset} \mathbb{P}_{\mathcal{H}_0}[\mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}] + \sum_{\ell_1,\ell_2:\bar{\Gamma}_{\ell_1}\cap\bar{\Gamma}_{\ell_2}\neq\emptyset} \mathbb{P}_{\mathcal{H}_0}[\mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}] \tag{554}$$

$$\triangleq \mathsf{A} + \mathsf{B}. \tag{555}$$

We can easily compute $\mathsf{A}$. Indeed

$$\mathsf{A} = L_n \cdot L_{n-k} \cdot q^{2e(\bar{\Gamma})} \sim [\mathbb{E}_{\mathcal{H}_0}(\mathcal{N}_{\bar{\Gamma}})]^2. \tag{556}$$

Now, recall that the likelihood function is defined as $\mathsf{L}(\mathsf{G}') = \mathcal{N}_{\bar{\Gamma}}/\mathbb{E}_{\mathcal{H}_0}(\mathcal{N}_{\bar{\Gamma}})$, and that under conditions (457)–(458), we have $\mathbb{E}_{\mathcal{H}_0}[\mathsf{L}(\mathsf{G}')]^2 = 1 + o(1)$. Therefore, it follows that

$$\frac{\mathsf{B}}{[\mathbb{E}_{\mathcal{H}_0}(\mathcal{N}_{\bar{\Gamma}})]^2} = o(1). \tag{557}$$

Next, we note that

$$\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma}) = \frac{1}{\mathcal{N}_{\bar{\Gamma}}^2} \sum_{\ell_1,\ell_2:\bar{\Gamma}_{\ell_1}\cap\bar{\Gamma}_{\ell_2}\neq\emptyset} \mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}. \tag{558}$$

Therefore, we can decompose $\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})$ as follows

$$\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma}) = \mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') \geq 1/2\right] + \mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') < 1/2\right] \tag{559}$$

$$= \frac{\sum_{\ell_1,\ell_2:\bar{\Gamma}_{\ell_1}\cap\bar{\Gamma}_{\ell_2}\neq\emptyset} \mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}}{\mathbb{E}[\mathcal{N}_{\bar{\Gamma}}^2]} \frac{1}{\mathsf{L}^2(\mathsf{G}')}\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') \geq 1/2\right] + \mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') < 1/2\right], \tag{560}$$

and thus

$$\mathbb{E}_{\mathcal{H}_0}[\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})] \leq \mathbb{E}_{\mathcal{H}_0}\left[\frac{\sum_{\ell_1,\ell_2:\bar{\Gamma}_{\ell_1}\cap\bar{\Gamma}_{\ell_2}\neq\emptyset} \mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}}{\mathbb{E}[\mathcal{N}_{\bar{\Gamma}}^2]} \frac{1}{\mathsf{L}^2(\mathsf{G}')}\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') \geq 1/2\right]\right]$$
$$+ \mathbb{E}_{\mathcal{H}_0}\left[\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') < 1/2\right]\right] \tag{561}$$

$$\leq 2 \cdot \mathbb{E}_{\mathcal{H}_0}\left[\frac{\sum_{\ell_1,\ell_2:\bar{\Gamma}_{\ell_1}\cap\bar{\Gamma}_{\ell_2}\neq\emptyset} \mathsf{N}_{\ell_1}\mathsf{N}_{\ell_2}}{\mathbb{E}[\mathcal{N}_{\bar{\Gamma}}^2]}\right] + \mathbb{E}_{\mathcal{H}_0}\left[\mathbb{1}\left[\mathsf{L}^2(\mathsf{G}') < 1/2\right]\right] \tag{562}$$

$$= 2 \cdot \frac{\mathsf{B}}{[\mathbb{E}_{\mathcal{H}_0}(\mathcal{N}_{\bar{\Gamma}})]^2} + \mathbb{P}_{\mathcal{H}_0}\left[\mathsf{L}^2(\mathsf{G}') < 1/2\right] \tag{563}$$

$$\leq o(1), \tag{564}$$

where in the inequality we have used the fact that $\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma}) \leq 1$ with probability one, and the last equality is due to (557) and Chebyshev's inequality. In order to prove that almost-exact recovery is impossible we will look the following overlap measure

$$\mathsf{over}(\hat{\Gamma}) \triangleq \sum_{(i,j)\in\binom{[n]}{2}} \mathbb{P}_{\mathcal{H}_1}[(i,j) \in \bar{\Gamma} \cap \hat{\bar{\Gamma}}], \tag{565}$$

where $\hat{\bar{\Gamma}}$ is any possible estimator of $\bar{\Gamma}$. Note that $\mathbb{E}d_{\mathsf{H}}(\bar{\Gamma}, \hat{\bar{\Gamma}}) = 2|e(\bar{\Gamma})| - 2\mathsf{over}(\hat{\bar{\Gamma}})$. Thus to rule out almost-exact recovery, it suffices to prove that $\mathsf{over}(\hat{\bar{\Gamma}}) = o(|e(\bar{\Gamma})|)$. To that end, we note that

$\mathsf{over}(\hat{\hat{\Gamma}})$ can be rewritten as follows

$$\mathsf{over}(\hat{\hat{\Gamma}}) = \sum_{\mathsf{G}'} \mathbb{P}_{\mathcal{H}_1}(\mathsf{G}') \sum_{\ell=1}^{L_n} \mathbb{P}_{\mathcal{H}_1}(\bar{\Gamma}_\ell|\mathsf{G}')|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}| \tag{566}$$

$$= \sum_{\mathsf{G}'} \mathbb{P}_{\mathcal{H}_1}(\mathsf{G}') \sum_{\ell=1}^{L_n} \frac{|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}|}{\mathcal{N}_{\bar{\Gamma}}} \tag{567}$$

$$= \sum_{\mathsf{G}'} \mathbb{P}_{\mathcal{H}_0}(\mathsf{G}') \sum_{\ell=1}^{L_n} \frac{|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}|}{\mathcal{N}_{\bar{\Gamma}}} + \sum_{\mathsf{G}'}[\mathbb{P}_{\mathcal{H}_1}(\mathsf{G}') - \mathbb{P}_{\mathcal{H}_0}(\mathsf{G}')] \sum_{\ell=1}^{L_n} \frac{|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}|}{\mathcal{N}_{\bar{\Gamma}}} \tag{568}$$

$$\leq \sum_{\mathsf{G}'} \mathbb{P}_{\mathcal{H}_0}(\mathsf{G}') \sum_{\ell=1}^{L_n} \frac{|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}|}{\mathcal{N}_{\bar{\Gamma}}} + |e(\bar{\Gamma})| \cdot \mathsf{TV}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}), \tag{569}$$

where in the last inequality we have used the definition of the total-variation distance, and the fact that $|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}| \leq |e(\bar{\Gamma})|$, for any $\bar{\Gamma}_\ell$ and $\hat{\hat{\Gamma}}$. Since $\mathsf{TV}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}) \leq \sqrt{\mathbb{E}_{\mathcal{H}_0}(\mathsf{L}^2(\mathsf{G}')) - 1}$, conditions (457)–(458) (with $n$ replaced by $n'$) imply that $\mathsf{TV}(\mathbb{P}_{\mathcal{H}_0}, \mathbb{P}_{\mathcal{H}_1}) = o(1)$, and therefore

$$\mathsf{over}(\hat{\hat{\Gamma}}) \leq \sum_{\mathsf{G}'} \mathbb{P}_{\mathcal{H}_0}(\mathsf{G}') \sum_{\ell=1}^{L_n} \frac{|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}|}{\mathcal{N}_{\bar{\Gamma}}} + o(|e(\bar{\Gamma})|). \tag{570}$$

Next, we can write

$$\mathsf{over}(\hat{\hat{\Gamma}}) \leq \mathbb{E}_{\mathcal{H}_0}\left[\sum_{\ell=1}^{L_n} \frac{|\bar{\Gamma}_\ell \cap \hat{\hat{\Gamma}}|}{\mathcal{N}_{\bar{\Gamma}}}\right] + o(|e(\bar{\Gamma})|) \tag{571}$$

$$= \sum_{(i,j)\in\binom{[n]}{2}}^{n} \mathbb{E}_{\mathcal{H}_0}\left[\mathbb{1}\left[(i,j) \in \hat{\hat{\Gamma}}\right] \sum_{\ell=1}^{L_n} \frac{\mathbb{1}\left[(i,j) \in \bar{\Gamma}_\ell\right]}{\mathcal{N}_{\bar{\Gamma}}}\right] + o(|e(\bar{\Gamma})|), \tag{572}$$

and we note that

$$\left[\sum_{\ell=1}^{L_n} \frac{\mathbb{1}\left[i \in \bar{\Gamma}_\ell\right]}{\mathcal{N}_{\bar{\Gamma}}}\right]^2 = \sum_{\ell_1=1}^{L_n} \sum_{\ell_2=1}^{L_n} \frac{\mathbb{1}\left[(i,j) \in \bar{\Gamma}_{\ell_1}\right] \mathbb{1}\left[(i,j) \in \bar{\Gamma}_{\ell_2}\right]}{\mathcal{N}_{\bar{\Gamma}}^2} \tag{573}$$

$$\leq \mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma}). \tag{574}$$

Thus

$$\mathsf{over}(\hat{\hat{\Gamma}}) \leq \sum_{(i,j)\in\binom{[n]}{2}} \mathbb{E}_{\mathcal{H}_0}\left[\mathbb{1}\left[(i,j) \in \hat{\hat{\Gamma}}\right] \sqrt{\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})}\right] + o(|e(\bar{\Gamma})|) \tag{575}$$

$$\leq |e(\bar{\Gamma})| \cdot \mathbb{E}_{\mathcal{H}_0}\left[\sqrt{\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})}\right] + o(|e(\bar{\Gamma})|) \tag{576}$$

$$\leq |e(\bar{\Gamma})| \cdot \sqrt{\mathbb{E}_{\mathcal{H}_0}\left[\mathsf{Int}_{\mathsf{G}'}(\bar{\Gamma})\right]} + o(|e(\bar{\Gamma})|) \tag{577}$$

$$\leq o(|e(\bar{\Gamma})|), \tag{578}$$

where the third inequality follows from Jensen's inequality, and the last inequality is due to (564). This concludes the proof.

■

### J.3. Lower bound in sub-logarithmic density regime

As it turns out, the proof technique in the previous subsection is not strong enough to capture the correct behavior in the sub-logarithmic density regime. Specifically, Theorem 69 shows that, in this regime, recovery is impossible if (551) holds. However, as we show next, recovery is in fact always impossible in this regime.

**Theorem 70** *Assume that $q \in (0,1)$ is fixed and that $|v(\Gamma^\star \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star})| = o(n)$. If*

$$\mu(\Gamma^\star|\Gamma^{(\ell_{\mathsf{LB}}),\star}) = o\left(\frac{\log|v(\Gamma^\star \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star})|}{\log\log|v(\Gamma^\star \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star})|}\right), \tag{579}$$

*then almost-exact recovery is impossible.*

To prove Theorem 70, we need a generalization of the subgraph expectation threshold that was already discussed in Appendix J.3.1. Specifically, the original subgraph expectation threshold in Theorem 61 analyzes the threshold for the appearance of *any possible* copy of the planted subgraph $\Gamma$ (in the complete graph $\mathcal{K}_n$) within $\mathsf{G} \sim \mathcal{G}(n,q)$. For our purposes, however, some copies are precluded. Let us explain this in detail.

In a nutshell, recall that we are in the scenario where the recovery problem is supplied with $\Gamma^{(\ell_{\mathsf{LB}}),\star}$ and tasked with finding $\Gamma^\star$. Therefore, the admissible $\Gamma^\star$'s are only those that extend $\Gamma^{(\ell_{\mathsf{LB}}),\star}$, namely, the copies contained in $\mathcal{M}(\Gamma^{(\ell_{\mathsf{LB}}),\star},\Gamma^\star)$. We resolve this by deriving a generalization of the subgraph expectation threshold that accounts for the appearance of a constrained set of subgraph copies. The details are provided in Appendix J.3.1, where we also establish the following key result.

**Lemma 71** *Let $\Gamma = (\Gamma_n)$ be a sequence of graphs such that $\omega(1) \leq |v(\Gamma)| \leq n$, and (579) holds. Then, for any fixed $q$, a sample from $\mathcal{G}(n,q)$ contains an isomorphic copy of $\Gamma \in \mathcal{M}(\Gamma^{(\ell_{\mathsf{LB}}),\star},\Gamma^\star)$, with probability at least $1/2$.*

Let us now prove Theorem 70, and then move forward to the proof of Lemma 71.
**Proof** [Proof of Theorem 70] We prove that

$$\inf_{\hat{T}} \sup_{\Gamma^\star \in \mathcal{S}_\Gamma} \mathbb{P}_{\Gamma^\star}\left[d_{\mathsf{H}}\big(\hat{T}(\mathsf{G},\Gamma^{(\ell)}),\, T_\ell(\Gamma^\star)\big) \geq |e(T_\ell(\Gamma^\star))|\right] \geq \frac{1}{4}, \tag{580}$$

where we use $\mathbb{P}_{\Gamma^\star}$ to emphasize that $\mathsf{G} \sim \mathcal{G}_{\Gamma^\star}(n,1,q)$, with $\Gamma^\star$ being the underlying planted subgraph. Proving (580) then readily implies that almost-exact recovery is impossible. Recall that by Lemma 71, for all sufficiently large $n$, and since $|v(\Gamma^{(\ell_{\mathsf{LB}}),\star})| \leq |v(\Gamma)| = o(n)$, under the condition in (579), a typical sample from $\mathcal{G}(n - |v(\Gamma^{(\ell_{\mathsf{LB}}),\star})|, q)$ contains an isomorphic copy of $\Gamma^\star \in \mathcal{M}(\Gamma^{(\ell_{\mathsf{LB}}),\star},\Gamma^\star)$, with probability at least $1/2$. Denote this copy by $\Gamma'$, and note that by construction, both $\Gamma^\star$ and $\Gamma'$ are extensions of the same subgraph $\Gamma^{(\ell_{\mathsf{LB}}),\star}$, and furthermore, $v(\Gamma^\star \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star}) \cap v(\Gamma' \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star}) = \emptyset$, namely, their tails are vertex-disjoint. Accordingly, define the event

$$\mathcal{F}_{\Gamma^\star} \triangleq \left\{\exists\, \Gamma' \in \mathcal{M}(\Gamma^{(\ell_{\mathsf{LB}}),\star},\Gamma^\star) \text{ with } v(\Gamma^\star \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star}) \cap v(\Gamma' \setminus \Gamma^{(\ell_{\mathsf{LB}}),\star}) = \emptyset, e(\Gamma') \subseteq e(\mathsf{G})\right\}. \tag{581}$$

Then $\mathbb{P}_{\Gamma^\star}[\mathcal{F}_{\Gamma^\star}] \geq 1/2$ provided that (579) holds. We will need the following simple observation, which we prove at the end.

**Lemma 72** *Let $\Gamma', \Gamma^\star$ be two vertex-disjoint copies of $\Gamma$. For any graph $\mathsf{g} \in \{0,1\}^{\binom{n}{2}}$, such that all edges in both $\Gamma'$ and $\Gamma^\star$ appear in $\mathsf{g}$, we have*

$$\mathbb{P}_{\Gamma^\star}(\mathsf{G} = \mathsf{g}) = \mathbb{P}_{\Gamma'}(\mathsf{G} = \mathsf{g}). \tag{582}$$

We are now in a position to prove Theorem 70. Fix any estimator $\hat{T}$, and a subgraph $\Gamma^\star$. On the event $\mathcal{F}_{\Gamma^\star}$, pick one disjoint extra copy $\Gamma'$ whose edges are all 1 in $\mathsf{G}$, (existence guaranteed by $\mathcal{F}_{\Gamma^\star}$). Consider the two distributions $\mathbb{P}_{\Gamma^\star}$ and $\mathbb{P}_{\Gamma'}$ and their equal mixture

$$\mathbb{M} \triangleq \frac{1}{2}\mathbb{P}_{\Gamma^\star} + \frac{1}{2}\mathbb{P}_{\Gamma'}. \tag{583}$$

Define the error indicator under a randomly and uniformly chosen planted location $\widetilde{\Gamma} \in \{\Gamma^\star, \Gamma'\}$:

$$\mathcal{E}(\mathsf{G}, \widetilde{\Gamma}) \triangleq \mathbb{1}\left\{d_{\mathsf{H}}\big(\hat{T}(\mathsf{G}), T_\ell(\widetilde{\Gamma})\big) \geq |e(T_\ell(\Gamma^\star))|\right\}. \tag{584}$$

By Lemma 72, for any realized graph $\mathsf{g} \in \{0,1\}^{\binom{n}{2}}$ such that both $e(\Gamma^\star)$ and $e(\Gamma')$ are 1 in $\mathsf{g}$,

$$\mathbb{P}_{\Gamma^\star}(\mathsf{G} = \mathsf{g}) = \mathbb{P}_{\Gamma'}(\mathsf{G} = \mathsf{g}) \quad \Longrightarrow \quad \mathbb{M}(\widetilde{\Gamma} = \Gamma^\star | \mathsf{G} = \mathsf{g}) = \mathbb{M}(\widetilde{\Gamma} = \Gamma' | \mathsf{G} = \mathsf{g}) = \frac{1}{2}. \tag{585}$$

Triangle inequality implies that

$$d_{\mathsf{H}}(\hat{T}(\mathsf{G}), T_\ell(\Gamma^\star)) + d_{\mathsf{H}}(\hat{T}(\mathsf{G}), T_\ell(\Gamma')) \geq d_{\mathsf{H}}(T_\ell(\Gamma'), T_\ell(\Gamma^\star)) = 2|e(T_\ell(\Gamma^\star))|, \tag{586}$$

where we have used the fact that $v(T_\ell(\Gamma^\star)) \cap v(T_\ell(\Gamma')) = \emptyset$. Therefore

$$\max\{d_{\mathsf{H}}(\hat{T}(\mathsf{G}), T_\ell(\Gamma^\star)), d_{\mathsf{H}}(\hat{T}(\mathsf{G}), T_\ell(\Gamma'))\} \geq |e(T_\ell(\Gamma^\star))|. \tag{587}$$

Thus, whenever both copies $\Gamma^\star$ and $\Gamma'$ are present in $\mathsf{g}$, we obtain

$$\mathcal{E}(\mathsf{g}, \Gamma^\star) + \mathcal{E}(\mathsf{g}, \Gamma') \geq 1. \tag{588}$$

Next, let $\mathcal{C}$ be the event "both $\Gamma^\star$ and $\Gamma'$ appear in $\mathsf{G}$". Notice that $\mathcal{C}$ coincides with $\mathcal{F}_{\Gamma^\star}$ when $\mathsf{G} \sim \mathbb{P}_{\Gamma^\star}$, and with $\mathcal{F}_{\Gamma'}$ when $\mathsf{G} \sim \mathbb{P}_{\Gamma'}$. Hence

$$\mathbb{M}(\mathcal{C}) = \frac{1}{2}\mathbb{P}_{\Gamma^\star}(\mathcal{F}_{\Gamma^\star}) + \frac{1}{2}\mathbb{P}_{\Gamma'}(\mathcal{F}_{\Gamma'}) \geq \frac{1}{2}. \tag{589}$$

Conditioning on $\mathsf{G}$ and then averaging over $\widetilde{\Gamma}$ under $\mathbb{M}$,

$$\mathbb{E}_{\mathbb{M}}\big[\mathcal{E}(\mathsf{G}, \widetilde{\Gamma})\mathbb{1}\{\mathcal{C}\}\big] = \mathbb{E}_{\mathbb{M}}\Big[\mathbb{E}_{\mathbb{M}}\big[\mathcal{E}(\mathsf{G}, \widetilde{\Gamma})|\mathsf{G}\big]\,\mathbb{1}\{\mathcal{C}\}\Big] \tag{590}$$

$$\geq \mathbb{E}_{\mathbb{M}}\left[\frac{1}{2}\mathbb{1}\{\mathcal{C}\}\right] \tag{591}$$

$$= \frac{1}{2}\mathbb{M}(\mathcal{C}) \geq \frac{1}{4}, \tag{592}$$

where the inequality follows from the facts that on $\mathcal{C}$ the posterior on $\{\Gamma^\star, \Gamma'\}$ is uniform, and (588) holds. Since $\mathcal{C} \geq 0$, we can drop $\mathbb{1}\{\mathcal{C}\}$ on the left to get

$$\mathbb{E}_{\mathbb{M}}\big[\mathcal{E}(\mathsf{G}, \widetilde{\Gamma})\big] \geq \frac{1}{4}. \tag{593}$$

But, we note that

$$\mathbb{E}_{\mathbb{M}}[\mathcal{E}(\mathsf{G},\widetilde{\Gamma})] = \frac{1}{2}\mathbb{P}_{\Gamma^\star}\left[d_{\mathsf{H}}\big(\hat{T}(\mathsf{G}), T_\ell(\Gamma^\star)\big) \geq |e(T_\ell(\Gamma^\star))|\right] + \frac{1}{2}\mathbb{P}_{\Gamma'}\left[d_{\mathsf{H}}\big(\hat{T}(\mathsf{G}), T_\ell(\Gamma')\big) \geq |e(T_\ell(\Gamma^\star))|\right]. \tag{594}$$

Therefore, at least one of those two probabilities at the right-hand side of (594) is at least $1/4$. This, in turn, implies that

$$\sup_{\bar{\Gamma}\in\mathcal{S}_\Gamma} \mathbb{P}_{\bar{\Gamma}}\left[d_{\mathsf{H}}\big(\hat{T}(\mathsf{G},\Gamma^{(\ell)}),\, T_\ell(\bar{\Gamma})\big) \geq |e(T_\ell(\bar{\Gamma}))|\right] \geq \sup_{\bar{\Gamma}\in\{\Gamma^\star,\Gamma'\}} \mathbb{P}_{\bar{\Gamma}}\left[d_{\mathsf{H}}\big(\hat{T}(\mathsf{G},\Gamma^{(\ell)}),\, T_\ell(\bar{\Gamma})\big) \geq |e(T_\ell(\bar{\Gamma}))|\right] \geq \frac{1}{4}. \tag{595}$$

Since $\hat{T}$ was arbitrary, the infimum over estimators of the left-hand side in (595) is also at least $1/4$, which concludes that proof. ■

Finally, we prove Lemma 72.

**Proof** [Proof of Lemma 72] Under $\mathbb{P}_{\Gamma^\star}$, edges in $e(\Gamma^\star)$ are fixed to $1$, and edges in $\binom{[n]}{2}\setminus e(\Gamma^\star)$ are independent $\mathsf{Bern}(q)$. Hence, for any $\mathsf{g}\in\{0,1\}^{\binom{n}{2}}$,

$$\mathbb{P}_{\Gamma^\star}(\mathsf{G}=\mathsf{g}) = \begin{cases} \prod_{e\in\binom{[n]}{2}\setminus e(\Gamma^\star)} q^{\mathsf{g}_e}(1-q)^{1-\mathsf{g}_e}, & \text{if } \mathsf{g}_e = 1\ \forall e\in e(\Gamma^\star),\\ 0, & \text{otherwise.}\end{cases} \tag{596}$$

An analogous formula holds for $\mathbb{P}_{\Gamma'}$. Assume $\mathsf{g}$ has $\mathsf{g}_e = 1$ for every $e\in e(\Gamma^\star)\cup e(\Gamma')$. Let

$$\mathcal{R} \triangleq \binom{[n]}{2} \setminus \big(e(\Gamma^\star)\cup e(\Gamma')\big). \tag{597}$$

Then

$$\mathbb{P}_{\Gamma^\star}(\mathsf{G}=\mathsf{g}) = \prod_{e\in e(\Gamma')} q^{\mathsf{g}_e}(1-q)^{1-\mathsf{g}_e} \prod_{e\in\mathcal{R}} q^{\mathsf{g}_e}(1-q)^{1-\mathsf{g}_e} \tag{598}$$

$$= q^{|e(\Gamma')|}\prod_{e\in\mathcal{R}} q^{\mathsf{g}_e}(1-q)^{1-\mathsf{g}_e}, \tag{599}$$

because $\mathsf{g}_e = 1$ for all $e\in e(\Gamma')$. Similarly,

$$\mathbb{P}_{\Gamma'}(\mathsf{G}=\mathsf{g}) = q^{|e(\Gamma^\star)|}\prod_{e\in\mathcal{R}} q^{\mathsf{g}_e}(1-q)^{1-\mathsf{g}_e}. \tag{600}$$

Since $|e(\Gamma^\star)| = |e(\Gamma')|$, these expressions are equal:

$$\mathbb{P}_{\Gamma^\star}(\mathsf{G}=\mathsf{g}) = q^{|e(\Gamma^\star)|}\prod_{e\in\mathcal{R}} q^{\mathsf{g}_e}(1-q)^{1-\mathsf{g}_e} = \mathbb{P}_{\Gamma'}(\mathsf{G}=\mathsf{g}). \tag{601}$$

■

### J.3.1. Proof of Lemma 71

In this subsection we prove Lemma 71. To that end, let us introduce a few definitions and notations. Fix a graph $\Gamma$ and a subgraph $\mathsf{J} \subsetneq \Gamma$. Recall that $\mathcal{M}(\mathsf{J},\Gamma)$ is the set of copies of $\Gamma$ in $\mathcal{K}_n$ that contain J. Let

$$\mathcal{N}(\mathsf{J},\Gamma,\mathsf{G}) \triangleq \sum_{\Gamma'\in\mathcal{M}(\mathsf{J},\Gamma)} \mathbb{1}\left\{\Gamma'\in\mathsf{G}\right\}, \tag{602}$$

which counts the number of copies of $\Gamma'\in\mathcal{M}(\mathsf{J},\Gamma)$ which appear in G. The critical probability of $\Gamma$ w.r.t. to $\mathcal{M}$ is defined as

$$q_c(\mathsf{J};\Gamma) \triangleq \min\left\{q\in[0,1]\,\middle|\,\mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{J},\Gamma,\mathsf{G})\geq 1|\mathsf{J}\in\mathsf{G}\right]\geq\frac{1}{2}\right\}. \tag{603}$$

Define the modified subgraph expectation threshold w.r.t. to $\mathcal{M}$ as

$$\tilde{q}_E(\mathsf{J};\Gamma) \triangleq \min\left\{q\in[0,1]\,\middle|\,\mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{J},\mathsf{H},\mathsf{G})|\mathsf{J}\in\mathsf{G}\right]\geq\frac{\mathcal{N}(\mathsf{H},\Gamma)}{2},\ \forall \mathsf{J}\subseteq\mathsf{H}\subseteq\Gamma\right\}, \tag{604}$$

where only subgraphs $\mathsf{J}\subseteq\mathsf{H}\subseteq\Gamma$ with no isolated vertices are considered. We prove the following result.

**Theorem 73** *There exists a universal constant $C$ such that for any graph $\Gamma$,*

$$\tilde{q}_E(\mathsf{J};\Gamma)\leq q_c(\mathsf{J};\Gamma)\leq C\cdot\tilde{q}_E(\mathsf{J};\Gamma)\cdot\log|e(\Gamma\setminus\mathsf{J})|. \tag{605}$$

Note that for $q\geq\tilde{q}_E(\mathsf{J};\Gamma)$, by Markov's inequality, for any $\mathsf{J}\subseteq\mathsf{H}\subseteq\Gamma$,

$$\frac{1}{2}\leq\mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{J},\Gamma,\mathsf{G})\geq 1|\mathsf{J}\in\mathsf{G}\right] \tag{606}$$

$$\leq\mathbb{P}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{J},\mathsf{H},\mathsf{G})\geq\mathcal{N}(\mathsf{H},\Gamma)|\mathsf{J}\in\mathsf{G}\right] \tag{607}$$

$$\leq\frac{\mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,q)}\left[\mathcal{N}(\mathsf{J},\mathsf{H},\mathsf{G})|\mathsf{J}\in\mathsf{G}\right]}{\mathcal{N}(\mathsf{H},\Gamma)}, \tag{608}$$

and thus, $\tilde{q}_E(\mathsf{J};\Gamma)\leq q_c(\mathsf{J};\Gamma)$. Finally, note that we can rewrite

$$\tilde{q}_E(\mathsf{J};\Gamma)\triangleq\max\left\{\left(\frac{\mathcal{N}(\mathsf{H},\Gamma)}{2|\mathcal{M}(\mathsf{J},\mathsf{H})|}\right)^{1/|e(\mathsf{H}\setminus\mathsf{J})|}:\ \mathsf{J}\subseteq\mathsf{H}\subseteq\Gamma\right\}. \tag{609}$$

We are now in a position to prove Theorem 73.

**Proof** [Proof of Theorem 73] Let $\pi_\Gamma=\mathsf{Unif}(\mathcal{M}(\mathsf{J},\Gamma))$ denotes the uniform measure over all copies of a fixed graph $\Gamma$ within $\mathcal{K}_n$ that contain J, and let $\Gamma'\sim\pi_\Gamma$ be a random sample from this distribution. Now, consider a subgraph H such that $\mathsf{J}\subseteq\mathsf{H}\subset\Gamma$. Let $\pi_{\mathsf{H}}=\mathsf{Unif}(\mathcal{M}(\mathsf{J},\mathsf{H}))$ denotes the uniform measure over all copies of H within $\mathcal{K}_n$ that contain J, and let $\mathsf{H}'\sim\pi_{\mathsf{H}}$. For any fixed instances $\mathsf{H}_0,\Gamma_0\subseteq\mathcal{K}_n$, copies of H and $\Gamma$, respectively, we can compute the inclusion probability in two equivalent ways:

$$\pi_\Gamma(\mathsf{H}_0\subseteq\Gamma')=\pi_{\mathsf{H}}(\mathsf{H}'\subseteq\Gamma_0)=\frac{|\mathcal{M}(\mathsf{H},\Gamma)|}{|\mathcal{M}(\mathsf{J},\Gamma)|}=\frac{\mathcal{N}(\mathsf{H},\Gamma)}{|\mathcal{M}(\mathsf{J},\mathsf{H})|}. \tag{610}$$

Now, combining (610) with (609), we obtain

$$\pi_\Gamma(\mathsf{H}_0 \subseteq \Gamma') = \frac{\mathcal{N}(\mathsf{H},\Gamma)}{|\mathcal{M}(\mathsf{J},\mathsf{H})|} \leq 2 \cdot [\tilde{q}_E(\mathsf{J};\Gamma)]^{|e(\mathsf{H}\setminus\mathsf{J})|} \leq \left(\frac{1}{2\tilde{q}_E(\mathsf{J};\Gamma)}\right)^{-|e(\mathsf{H}\setminus\mathsf{J})|}. \tag{611}$$

Since the bound holds uniformly over all subgraphs $\mathsf{H}_0$, we conclude that $\pi_\Gamma$ is $\alpha$-spread with $\alpha = 1/[2\tilde{q}_E(\mathsf{J};\Gamma)]$. An application of Lemma 64 with $k = |e(\Gamma \setminus \mathsf{J})|$ then completes the proof of Theorem 73.

■

Next, we prove the following result.

**Theorem 74** *Let* $\Gamma = (\Gamma_n)$ *and* $\mathsf{J} = (\mathsf{J}_n)$ *be sequences of graphs such that* $\omega(1) \leq |v(\Gamma)| \leq n$, $\mathsf{J} \subseteq \Gamma$, *and*

$$(1+\varepsilon)\mu(\Gamma_n|\mathsf{J}_n) \cdot [\log\log|e(\Gamma_n \setminus \mathsf{J})| + \log C] + \log|v(\Gamma_n) \setminus v(\mathsf{J}_n)| \leq \log n, \tag{612}$$

*for some* $\varepsilon > 0$ *and* $C > 0$. *Then, for any fixed* $q$, *a sample from* $\mathcal{G}(n,q)$ *contains an isomorphic copy of* $\Gamma' \in \mathcal{M}(\mathsf{J},\Gamma)$, *with probability at least* $1/2$.

**Proof** [Proof of Theorem 74] From the definition of $q_c(\mathsf{J};\Gamma)$ in (603), our goal is to understand for which $\Gamma$'s we have $q_c(\mathsf{J};\Gamma) \leq q$, for any fixed $q \in (0,1]$. By Theorem 73, it is sufficient to show that $\tilde{q}_E(\mathsf{J};\Gamma) \leq \frac{q}{C\log(|e(\Gamma\setminus\mathsf{J})|)}$ for any fixed $q$, which holds if and only if

$$\inf_{\mathsf{J}\subseteq\mathsf{H}\subseteq\Gamma} \frac{\mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,\tilde{q})}\left[\mathcal{N}(\mathsf{J},\mathsf{H},\mathsf{G})|\mathsf{J}\in\mathsf{G}\right]}{\mathcal{N}(\mathsf{H},\Gamma)} \geq \frac{1}{2}, \tag{613}$$

where $\tilde{q} \triangleq \frac{q}{C\log(|e(\Gamma\setminus\mathsf{J})|)}$. We note that

$$\mathbb{E}_{\mathsf{G}\sim\mathcal{G}(n,\tilde{q})}\left[\mathcal{N}(\mathsf{J},\mathsf{H},\mathsf{G})|\mathsf{J}\in\mathsf{G}\right] = |\mathcal{M}(\mathsf{J},\mathsf{H})| \cdot \tilde{q}^{|e(\mathsf{H}\setminus\mathsf{J})|}, \tag{614}$$

Thus, (613) holds if and only if for any $\mathsf{J} \subseteq \mathsf{H} \subseteq \Gamma$ we have

$$\frac{\mathcal{N}(\mathsf{H},\Gamma)}{|\mathcal{M}(\mathsf{J},\mathsf{H})|} \leq 2\tilde{q}^{|e(\mathsf{H}\setminus\mathsf{J})|} = 2\left(\frac{q}{C\log(|e(\Gamma\setminus\mathsf{J})|)}\right)^{|e(\mathsf{H}\setminus\mathsf{J})|}. \tag{615}$$

As we have seen in the proof of Theorem 73, the expression on the left-hand side of (615) equals $\pi_\Gamma[\mathsf{H} \subseteq \Gamma]$, where $\pi_\Gamma = \mathsf{Unif}(\mathcal{M}(\mathsf{J},\Gamma))$ denotes the uniform measure over all copies of a fixed graph $\Gamma$ within $\mathcal{K}_n$ that contain $\mathsf{J}$. Thus, (615) holds if,

$$\log\pi_\Gamma[\mathsf{H}\subseteq\Gamma] - \log 2 - |e(\mathsf{H}\setminus\mathsf{J})|\left(\log q - \log C - \log\log|e(\Gamma\setminus\mathsf{J})|\right) \leq 0. \tag{616}$$

By the assumptions that $|v(\Gamma)| = \omega(1)$, and that $\Gamma$ has no isolated vertices, for any $\varepsilon$ and for sufficiently large $n$, the above holds if,

$$\log\pi_\Gamma[\mathsf{H}\subseteq\Gamma] + |e(\mathsf{H}\setminus\mathsf{J})|\log C + (1+\varepsilon)|e(\mathsf{H}\setminus\mathsf{J})|\log\log|e(\Gamma\setminus\mathsf{J})| \leq 0. \tag{617}$$

Finally, using Lemma 75, the left-hand side of (617) can be upper bounded by,

$$
\begin{aligned}
&\log \pi_\Gamma[\mathsf{H} \subseteq \Gamma] + |e(\mathsf{H} \setminus \mathsf{J})| \log C + (1+\varepsilon)|e(\mathsf{H} \setminus \mathsf{J})| \log\log |e(\Gamma \setminus \mathsf{J})| && (618)\\
&\leq |v(\mathsf{H}) \setminus v(\mathsf{J})| \left(\log |v(\Gamma) \setminus v(\mathsf{J})| - \log n\right) + |e(\mathsf{H} \setminus \mathsf{J})| \log C \\
&\qquad + (1+\varepsilon)|e(\mathsf{H} \setminus \mathsf{J})| \log\log |e(\Gamma \setminus \mathsf{J})| && (619)\\
&= |v(\mathsf{H}) \setminus v(\mathsf{J})| \cdot \left(\log |v(\Gamma \setminus v(\mathsf{J}))| + \frac{|e(\mathsf{H} \setminus \mathsf{J})|}{|v(\mathsf{H}) \setminus v(\mathsf{J})|} \log C \right.\\
&\qquad\qquad \left. + (1+\varepsilon)\frac{|e(\mathsf{H} \setminus \mathsf{J})|}{|v(\mathsf{H}) \setminus v(\mathsf{J})|} \log\log |e(\Gamma \setminus \mathsf{J})| - \log n \right) && (620)\\
&\overset{(a)}{\leq} |v(\mathsf{H}) \setminus v(\mathsf{J})| \cdot \left[\log |v(\Gamma) \setminus v(\mathsf{J})| + (1+\varepsilon)\mu(\Gamma|\mathsf{J}) \left[\log\log |e(\Gamma \setminus \mathsf{J})| + \log C\right] - \log n\right] && (621)\\
&\leq 0, && (622)
\end{aligned}
$$

where $(a)$ follows from the definition of the maximum subgraph relative density in (30), and the last inequality follows from (612). This concludes the proof of the statement of Theorem 74, and it is left to prove the following lemma.

**Lemma 75** *For any* $\mathsf{J} \subseteq \mathsf{H} \subseteq \Gamma$*,*

$$
\pi_\Gamma[\mathsf{H} \subseteq \Gamma] \leq \left(\frac{|v(\Gamma) \setminus v(\mathsf{J})|}{n}\right)^{|v(\mathsf{H} \setminus \mathsf{J})|}, \tag{623}
$$

*where* $\pi_\Gamma = \mathsf{Unif}(\mathcal{M}(\mathsf{J}, \Gamma))$.

**Proof** [Proof of Lemma 75] Let $m = |v(\Gamma) \setminus v(\mathsf{J})|$ and $k = |v(\mathsf{H} \setminus v(\mathsf{J})|$. Fix a particular copy $\mathsf{H}_0$ of $\mathsf{H}$ in $\mathcal{K}_n$ with vertex set $U = \{u_1, \ldots, u_k\}$. Generate a uniformly random copy of $\Gamma$ by first picking its vertex set $S \subset [n]$ uniformly among all $m$-subsets (the internal labeling of $\Gamma$ can only lower the probability of containing $\mathsf{H}_0$, so it suffices to control this step). If the sampled copy contains $\mathsf{H}_0$, then necessarily $U \subseteq S$. Thus

$$
\pi_\Gamma[\mathsf{H}_0 \subseteq \Gamma] \leq \mathbb{P}[U \subseteq S] = \prod_{i=1}^{k} \mathbb{P}[u_i \in S | u_1, \ldots, u_{i-1} \in S]. \tag{624}
$$

Conditioned on $u_1, \ldots, u_{i-1} \in S$, there are $n-(i-1)$ remaining vertices and $m-(i-1)$ remaining slots in $S$, so

$$
\mathbb{P}[u_i \in S | u_1, \ldots, u_{i-1} \in S] = \frac{m-(i-1)}{n-(i-1)} \leq \frac{m}{n}. \tag{625}
$$

Multiplying these $k$ bounds gives

$$
\pi_\Gamma[\mathsf{H} \subseteq \Gamma] \leq \mathbb{P}[U \subseteq S] \leq \left(\frac{m}{n}\right)^k = \left(\frac{|v(\Gamma) \setminus v(\mathsf{J})|}{n}\right)^{|v(\mathsf{H} \setminus \mathsf{J})|}. \tag{626}
$$

■

■

We are now in a position to prove Lemma 71. To that end, we set $\mathsf{J} = \Gamma^{(\ell_{\mathsf{LB}})}$ and $\Gamma = \Gamma^\star$ in Theorem 74. Then it is clear that condition (612) is satisfied under (579), which completes the proof.

## Appendix K. Conclusion and Outlook

This work studies the problem of exact recovery of an arbitrary planted subgraph $\Gamma_n$ in the Erdős–Rényi random graph, in the dense regime where both the planting and noise edge probabilities, $p_n$ and $q_n$, are fixed independently of $n$. For this problem, we characterize the statistical limits of recovery: roughly speaking, we show that if a certain graph-theoretic quantity—termed the minimal maximum subgraph density $\mu_{\mathsf{min}}(\Gamma_n)$—is below $\log n$, then recovery is statistically impossible, while it becomes possible (via exhaustive search) when it is above $\log n$. We then turn to the problem of recovery in polynomial-time. We first propose a general algorithm that applies to arbitrary $\Gamma_n$ and provide its statistical guarantees. Next, we derive computational lower bounds based on the low-degree polynomial framework recently developed in Schramm and Wein (2022). Finally, we discuss several extensions, including semi-random models and weaker notions of recovery.

We hope that our work inspires more questions than it answers. We conclude by outlining several avenues for future research:

1. Extending our results to settings in which the edge probabilities $p$ and $q$ depend on $n$, especially when they vanish or approach one (for instance, polynomially in $n$), entails technical and conceptual difficulties. Some of our techniques can be generalized to handle moderately sparse regimes (e.g., when $p$ and $q$ are as small as $n^{-\alpha}$ for certain ranges of $\alpha$), but pushing these arguments further to the general case requires new ideas.

2. While we were able to tightly characterize the statistical limits, a gap remains between our lower and upper computational bounds. At present, the precise computational barrier for recovering an arbitrary planted subgraph is remains a mystery.

3. Our semi-random analysis focused on an adversary limited to edge deletions outside the planted subgraph and edge additions inside it. Considering models in which the adversary is allowed to add a bounded number of edges, or to perform more general perturbations, leads to natural questions concerning robustness and reconstructability.

4. We assume that the statistician observes the entire graph. Extending the analysis to settings where only part of the graph is observable—through adaptive or non-adaptive queries—poses an interesting and challenging problem for arbitrary planted subgraphs.

5. It is of interest, both for the detection variant and for the recovery problem studied in this paper, to provide additional forms of evidence for the statistical–computational gaps that arise. One powerful technique is through average-case reductions. For example, can we show that the hardness of detecting or recovering an arbitrary planted subgraph can be reduced from the detection or recovery of a planted clique—given that the clique is the "easiest" subgraph to infer?