

# Blackwell Approachability and Gradient Equilibrium are Equivalent

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## Abstract

Gradient equilibrium (GEQ) is a recently introduced online optimization framework that generalizes first-order stationarity from offline optimization. Given differentiable losses  $\ell_t : \mathbb{R}^d \rightarrow \mathbb{R}$ , GEQ asks the learner to choose decisions  $\theta_t \in \mathbb{R}^d$  so that the norm of the average gradient vanishes:

$$\left\| \frac{1}{T} \sum_{t=1}^T \nabla \ell_t(\theta_t) \right\|_2 \rightarrow 0 \quad \text{as } T \rightarrow \infty.$$

GEQ offers a natural way to lift statistical problems, such as mean debiasing and conformal prediction, to the online setting. While GEQ has curious similarities with known online learning frameworks, such as regret minimization in online optimization, prior work has shown that GEQ error and regret are incomparable as objectives, leaving open a precise understanding of how GEQ fits into the broader online learning landscape.

In this work, we show that GEQ is equivalent to Blackwell approachability in the algorithmic sense. That is, a Blackwell approachability problem can always be solved using queries to a black-box GEQ oracle, with no asymptotic loss in the oracle’s error rate, and vice versa. Taken together with known equivalences between approachability, regret minimization, and calibration, these results imply an equivalence between GEQ and these frameworks, as well. Hence, while GEQ guarantees are semantically different from known online learning guarantees, GEQ algorithms are equally powerful primitives as classical regret minimization and calibration algorithms.

Our reductions are efficient and can be used to transfer refined guarantees, such as optimism and strong adaptivity, from regret minimization to GEQ. Our techniques can also be used to identify necessary and sufficient conditions for GEQ, and to establish reductions between different notions of GEQ with unconstrained and constrained decision sets, which may be of independent interest.<sup>1</sup>

**Keywords:** Blackwell approachability, gradient equilibrium, regret minimization, calibration

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