The Monotonicity of the Franz–Parisi Potential Is Equivalent to Low-Degree MMSE Lower Bounds

Konstantinos Tsirkas KOSTAS.TSIRKAS@YALE.EDU
Leda Wang LEDA.WANG@YALE.EDU
Ilias Zadik ILIAS.ZADIK@YALE.EDU
Department of Statistics and Data Science, Yale University.

Editors: Steve Hanneke and Tor Lattimore

Abstract

Over the last decades, two distinct approaches have been instrumental to our understanding of the computational complexity of statistical estimation. The statistical physics literature predicts algorithmic hardness through local stability and monotonicity properties of the Franz–Parisi potential (Franz and Parisi, 1995, 1997), while the rigorous average-case complexity literature characterizes hardness via the limitations of restricted algorithmic classes, most notably low-degree polynomial estimators (Hopkins and Steurer, 2017; Schramm and Wein, 2022). For many inference models, these two perspectives yield strikingly consistent predictions, giving rise to the problem of establishing a precise mathematical relationship between them.

Recent works (Bandeira et al., 2022; Chen et al., 2025) addressed this question in the setting of detection, showing that for broad classes of models the success of low-degree polynomials is governed by an area criterion involving the annealed Franz–Parisi potential. In this extended abstract, we show that for estimation problems the relevant low-degree criterion is instead the monotonicity of the annealed Franz–Parisi potential. For a broad family of Gaussian additive models with signal-to-noise ratio λ, letting F_λ denote the annealed Franz–Parisi potential, we prove that the optimal degree-D correlation satisfies, for $D = O(\mathrm{polylog}(N))$,

$$\mathrm{Corr}_{P_0}^{\leq D}\left(\lambda + \frac{d}{dq}F_\lambda(q)\Big|_{q=q(D)}\right)^2 \approx q(D),$$

where $q(D)$ is the e^{-D}-quantile of the overlap between two independent draws from the prior. Consequently, the condition $\frac{d}{dq}F_\lambda(q)\Big|_{q=q(D)} \geq 0$ is equivalent to all degree-D polynomial estimators achieving correlation at most $q(D)$ at signal-to-noise ratio λ. Subject to the low-degree conjecture for these Gaussian additive models, this identifies the polynomial-time estimation threshold with the monotonicity threshold of the annealed Franz–Parisi potential, matching the prediction from the physics literature. Our result can also be viewed as a low-degree analogue of the classical I–MMSE relationship (Guo et al., 2005).

Keywords: Franz–Parisi potential, low-degree MMSE, computational-statistical tradeoffs

1. Extended abstract. Full version appears as [arXiv:2603.20070, v1]

References

Milton Abramowitz and Irene A Stegun. *Handbook of mathematical functions: with formulas, graphs, and mathematical tables*, volume 55. Courier Corporation, 1965.

Gerard Ben Arous, Reza Gheissari, and Aukosh Jagannath. Algorithmic thresholds for tensor pca. *The Annals of Probability*, 48(4):2052–2087, 2020a.

Gérard Ben Arous, Alexander S. Wein, and Ilias Zadik. Free energy wells and overlap gap property in sparse pca. In *Conference on Learning Theory*, pages 479–482. PMLR, 2020b.

Rami Atar and Tsachy Weissman. Mutual information, relative entropy, and estimation in the poisson channel. *IEEE Transactions on Information Theory*, 58(3):1302–1318, 2012.

Afonso S. Bandeira, Ahmed El Alaoui, Samuel Hopkins, Tselil Schramm, Alexander S. Wein, and Ilias Zadik. The franz–parisi criterion and computational trade-offs in high dimensional statistics. *Advances in Neural Information Processing Systems*, 35:33831–33844, 2022.

Jean Barbier, Florent Krzakala, Nicolas Macris, Léo Miolane, and Lenka Zdeborová. Optimal errors and phase transitions in high-dimensional generalized linear models. *Proceedings of the National Academy of Sciences*, 116(12):5451–5460, 2019.

Alexandra Carpentier, Simone Maria Giancola, Christophe Giraud, and Nicolas Verzelen. Low-degree lower bounds via almost orthonormal bases. *arXiv preprint arXiv:2509.09353*, 2025.

Siyu Chen, Theodor Misiakiewicz, Ilias Zadik, and Peiyuan Zhang. An optimized franz–parisi criterion and its equivalence with SQ lower bounds. In *The Thirty-ninth Annual Conference on Neural Information Processing Systems*, 2025. URL https://openreview.net/forum?id=U8BwT6Rmw4.

Zongchen Chen, Conor Sheehan, and Ilias Zadik. On the low-temperature MCMC threshold: the cases of sparse tensor PCA, sparse regression, and a geometric rule. *arXiv preprint arXiv:2408.00746*, 2024.

Bertrand Even, Christophe Giraud, and Nicolas Verzelen. Computational lower bounds in latent models: clustering, sparse-clustering, biclustering. *arXiv preprint arXiv:2506.13647*, 2025a.

Bertrand Even, Christophe Giraud, and Nicolas Verzelen. Computational lower bounds in latent models: clustering, sparse-clustering, biclustering, 2025b. URL https://arxiv.org/abs/2506.13647.

Silvio Franz and Giorgio Parisi. Recipes for metastable states in spin glasses. *Journal de Physique I*, 5(11):1401–1415, 1995.

Silvio Franz and Giorgio Parisi. Phase diagram of coupled glassy systems: A mean-field study. *Physical Review Letters*, 79(13):2486, 1997.

Dongning Guo, Shlomo Shamai, and Sergio Verdú. Mutual information and minimum mean-square error in gaussian channels. *IEEE Transactions on Information Theory*, 51 (4):1261–1282, 2005.

Samuel B. Hopkins and David Steurer. Efficient bayesian estimation from few samples: community detection and related problems. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 379–390. IEEE, 2017.

Samuel B. Hopkins, Pravesh K. Kothari, Aaron Potechin, Prasad Raghavendra, Tselil Schramm, and David Steurer. The power of sum-of-squares for detecting hidden structures. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 720–731, 2017. doi: 10.1109/FOCS.2017.72.

Dmitriy Kunisky, Cristopher Moore, and Alexander S. Wein. Tensor cumulants for statistical inference on invariant distributions, 2024. URL https://arxiv.org/abs/2404.18735.

Marc Lelarge and Léo Miolane. Fundamental limits of symmetric low-rank matrix estimation. In *Conference on Learning Theory*, pages 1297–1301. PMLR, 2017.

Matthias Löffler, Alexander S. Wein, and Afonso S. Bandeira. Computationally efficient sparse clustering, 2021. URL https://arxiv.org/abs/2005.10817.

Yuetian Luo and Chao Gao. Computational lower bounds for graphon estimation via low-degree polynomials. *The Annals of Statistics*, 52(5):2318–2348, 2024.

Yuetian Luo and Anru R. Zhang. Tensor clustering with planted structures: Statistical optimality and computational limits, 2023. URL https://arxiv.org/abs/2005.10743.

Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni. *Formulas and theorems for the special functions of mathematical physics*, volume 52. Springer Science & Business Media, 2013.

Peter McCullagh. Tensor methods in statistics. 1987. URL https://api.semanticscholar.org/CorpusID:120013550.

Andrea Montanari and Emile Richard. A statistical model for tensor PCA, 2014. URL https://arxiv.org/abs/1411.1076.

Andrea Montanari and Alexander S. Wein. Equivalence of approximate message passing and low-degree polynomials in rank-one matrix estimation, 2024. URL https://arxiv.org/abs/2212.06996.

Elchanan Mossel, Jonathan Niles-Weed, Youngtak Sohn, Nike Sun, and Ilias Zadik. Sharp thresholds in inference of planted subgraphs. In *The Thirty Sixth Annual Conference on Learning Theory*, pages 5573–5577. PMLR, 2023.

Jonathan Niles-Weed and Ilias Zadik. The all-or-nothing phenomenon in sparse tensor PCA. *Advances in Neural Information Processing Systems*, 33:17674–17684, 2020.

Galen Reeves and Henry D. Pfister. The replica-symmetric prediction for compressed sensing with gaussian matrices is exact. In *2016 IEEE International Symposium on Information Theory (ISIT)*, pages 665–669. IEEE, 2016.

Galen Reeves, Jiaming Xu, and Ilias Zadik. The all-or-nothing phenomenon in sparse linear regression. In *Conference on Learning Theory*, pages 2652–2663. PMLR, 2019.

Tselil Schramm and Alexander S. Wein. Computational barriers to estimation from low-degree polynomials. *The Annals of Statistics*, 50(3), June 2022. ISSN 0090-5364. doi: 10.1214/22-aos2179. URL http://dx.doi.org/10.1214/22-AOS2179.

Youngtak Sohn and Alexander S. Wein. Sharp phase transitions in estimation with low-degree polynomials, 2025. URL https://arxiv.org/abs/2502.14407.

Aad W. Van der Vaart. *Asymptotic statistics*, volume 3. Cambridge University Press, 2000.

Roman Vershynin. *High-dimensional probability: An introduction with applications in data science*, volume 47. Cambridge University Press, 2018.

Mariia Vladimirova, Stéphane Girard, Hien Nguyen, and Julyan Arbel. Sub-weibull distributions: Generalizing sub-gaussian and sub-exponential properties to heavier tailed distributions. *Stat*, 9(1), January 2020. ISSN 2049-1573. doi: 10.1002/sta4.318. URL http://dx.doi.org/10.1002/sta4.318.

Zhen-Hang Yang and Yu-Ming Chu. On approximating the modified Bessel function of the second kind. *Journal of Inequalities and Applications*, 2017(1):41, February 2017. ISSN 1029-242X. doi: 10.1186/s13660-017-1317-z. URL https://doi.org/10.1186/s13660-017-1317-z.

Lenka Zdeborová and Florent Krzakala. Statistical physics of inference: Thresholds and algorithms. *Advances in Physics*, 65(5):453–552, 2016.