Group-Fair Allocations of Contiguous Blocks of Indivisible Items

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Abstract

We study the problem of allocating contiguous blocks of indivisible items positioned on a line to a set of agents where agents are partitioned into different groups. In the problem, agents must be located on a line, and items allocated to each agent must be contiguous and form a connected block. We study the settings, considered by Suksompong [2019] and Wang et al. [2021], without and with the requirement that items are assigned to their closest agents, respectively. For both settings, we focus on allocations that satisfy intra-group envy fairness (IEF) and inter-group fair share (GFS). In both settings, we show that determining the existence of IEF or GFS allocations is NP-complete. When the number of agents or items is constant, we provide polynomial algorithms to determine the existence of IEF or GFS allocations for both settings. For both settings, we provide upper and lower bounds on the existence of IEF or GFS allocations as well as allocations that satisfy both IEF and GFS simultaneously.

1 INTRODUCTION

In recent decades, fair division of indivisible items has received extensive attention from various communities including AI, theoretical computer science, operations research, and economics [Moulin, 2004, Brams et al., 2003, Amanatidis et al., 2023] due to its theoretical interests and applications to address situations such as divorce settlements [Brams and Taylor, 1996], task allocations [Goldman and Procaccia, 2015], and expense divisions [Goldman and Procaccia, 2015]. In fair division of indivisible items, the typical problems deal with allocating items to agents subject to given fairness notions (e.g., envy-freeness, proportionality, and their variations) based on the agent utility preferences over the items [Amanatidis et al., 2023].

In various scenarios—such as assigning offices to different units in a corridor, designating retail spaces on a street to various retailers—it is preferable to fairly allocate resources contiguously to ensure each agent receives resources forming a contiguous block [Suksompong, 2019, Bouveret et al., 2017, Bilò et al., 2022]. To model these scenarios, Bouveret et al. [2017], Suksompong [2019], Bilò et al. [2022], Truszczynski and Lonc [2020], Greco and Scarcello [2020], Misra et al. [2021] study contiguous fair division of indivisible items located on graphs such that each agent's received items must be connected (or form a contiguous block) under various fairness notions (e.g., envy-freeness and proportionality). Following these studies, subsequent work has explored specific graph structures [Bilò et al., 2022, Igarashi and Peters, 2019, Bei et al., 2022, Truszczynski and Lonc, 2020], chores [Höhne and van Stee, 2021], and closest assignment constraints (i.e., agents need to be located and items are assigned to their closest agents) [Wang et al., 2021, Wang and Zhang, 2021].

All existing studies on contiguous fair division have focused on individual agents in which fairness is considered with respect to a single agent. However, as studied in existing fair division literature (e.g., see a survey [Amanatidis et al., 2023] and our related work section), agents often belong to different fixed groups, items are allocated to agents, and fairness must take agent groups into consideration. For instance, when allocating offices, there are different specialized teams (i.e., agents) within different units (i.e., groups). Each team should be allocated a contiguous block of rooms to facilitate communication (e.g., different departments within different academic units share a space, and each department is often allocated a contiguous space). When allocating retail units, retailers can belong to different groups based on their types or associations (e.g., National Restaurant Association). Each

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retailer should be allocated a contiguous block of space for store extensions and operations.

Naturally, existing group fairness notions for fixed groups [Amanatidis et al., 2023, Segal-Halevi and Suksompong, 2019, Bredereck et al., 2022, Abebe et al., 2017, Aziz et al., 2018, Michorzewski et al., 2020] can be used for contiguous fair division with groups of agents. We examine two classical inter-group and intra-group fairness notions for groups of agents widely considered in the literature and consider allocations that satisfy the two notions.

1.1 OUR CONTRIBUTIONS

We study the problem of allocating contiguous blocks of m indivisible items on a line to a set of n agents. The agents are partitioned into k groups and have additive utilities over the m items. We consider two main settings of contiguous fair division inspired by different situations. The first is the setting of Suksompong [2019] in which each agent's received items (or bundle) must form a contiguous connected block. The second is the setting of Wang et al. [2021] in which we need to locate each agent on the line and assign items to their closest agents. This setting is motivated by the facility location problems in which one needs to locate the facilities (which we refer to as agents) to serve their closest customers (which we refer to as items) fairly. We refer to contiguous allocations as those in which each agent's bundle is contiguous. We use valid allocations to denote contiguous allocations that satisfy the closest assignment requirement.

For these settings, we consider the classical intra-group envy freeness (IEF) and inter-group fair share (GFS) for groups of agents where envy-freeness is only required for agents within each group (e.g., each agent is aware of other agents within their group) and group utility is at least proportional to the group size (e.g., each agent is concerned about the well-being of their group only). See Section 3 for detail.

Below, we summarize our contributions for both settings.

- We show that determining the existence of IEF or GFS allocations is NP-complete.
- We provide algorithms to determine the existence of IEF or GFS allocations. The algorithms are polynomial when the number of agents or the number of items is constant.
- We provide (almost tight or tight) upper and lower bounds on the existence of IEF or GFS allocations as well as allocations that satisfy both IEF and GFS in terms of the commonly studied additive approximation in contiguous fair division (see Table 1).

Fairness		*Upper (U) and Lower (L) Bounds*
IEF	C	U: $\frac{3}{2}u_{max}$ L: u_{max}
	V	U: $(\frac{5}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})u_{max}$ L: u_{max}
GFS	C	U: $\frac{\lvert G_{max}\rvert}{n} - \frac{1}{k} + \frac{k-1}{k}u_{max}$ L: $\frac{\lvert G_{max}\rvert}{n} - \frac{1}{k} + \frac{k-1}{k}u_{max}$
	V	U: $\frac{\lvert G_{max}\rvert}{n} - \frac{1}{2k} + \frac{k-1}{2k}u_{max}$ L: $\frac{\lvert G_{max}\rvert}{n} - \frac{1}{2k} + \frac{k-1}{2k}u_{max}$
IEF	C	U: $(2u_{max}, \frac{1-(m-n)u_{min}+2nu_{max}}{n}\lvert G_{max}\rvert)$ L: $(u_{max}, \frac{1-mu_{min}+2nu_{max}}{n}\lvert G_{max}\rvert)$
+ GFS	V	U: $((\frac{5}{3}\lvert G_{max}\rvert\lceil\frac{m}{n}\rceil + \frac{4}{3})u_{max},$ $\frac{2-(m-n)u_{min}+2nu_{max}}{2n}\lvert G_{max}\rvert)$ L: $(u_{max}, \frac{2-mu_{min}+2nu_{max}}{2n}\lvert G_{max}\rvert)$

Table 1: Our Upper and Lower Bounds for Each Fairness Notion (C = Contiguous, V=Valid, and (α, β) = α-IEF and β-GFS). m is the number of items, n is the number of agents, k is the number of groups, u_{max}/u_{min} is the maximum/minimum utility of any single item of any agent, and $|G_{max}|$ is the maximum size of groups.

When there is a single group, IEF reduces to EF. When the number of groups is equal to the number of agents, GFS is the same as proportionality. It is shown that determining the existence of EF/proportional contiguous or valid allocation is NP-complete [Bouveret et al., 2017, Wang et al., 2021]. We strengthened these hardness results by showing that the hardness results still hold when there are multiple groups (for IEF) and when the number of groups is less than the number of agents (for GFS).

To derive the lower bounds, we carefully construct new specific instances. To derive the upper bounds, our arguments are based on the combination of our new analyses (e.g., allocation rounding and case studies) and existing results in Bouveret et al. [2017], Wang et al. [2021]. For instance, in the case of IEF allocations, we introduce a new rounding technique to ensure that the allocation is contiguous $\frac{3}{2}u_{max}$-IEF, which is better than the previous contiguous $2u_{max}$-IEF in Suksompong [2019]. We present case studies based on different inequalities to derive $(\frac{5}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})$-IEF valid allocations. For GFS allocations, we use a new argument based on analyzing the function f_g (which is the maximum total utility that group G_g can achieve contiguously) to obtain our tight bounds of $(\frac{|G_{max}|}{n} - \frac{1}{k} + \frac{k-1}{k}u_{max})$-GFS contiguous allocation and $(\frac{|G_{max}|}{n} - \frac{1}{2k} + \frac{k-1}{2k}u_{max})$-GFS valid allocation. For IEF+GFS allocations, we pre-allocate a certain block to each group and apply the argument in Suksompong [2019] to obtain the upper bounds for contiguous and valid allocations.

Due to the lack of space, all of the missing proofs are in the supplementary material/appendix.

Outline In Section 2, we present studies most relevant to our settings. In Section 3, we provide the necessary notations and definitions to define our settings formally. In Sections 4 and 5, we present our hardness and upper/lower bound results for IEF and GFS, respectively. In Section 6, we provide upper and lower bounds when considering IEF and GFS together. Finally, we conclude in Section 7.

2 RELATED WORK

In this section, we discuss studies that consider contiguous fair division and fair division with groups of agents.

Contiguous and Valid Allocations. Existing studies have considered the contiguous fair division of indivisible items (see, e.g., [Amanatidis et al., 2023, Biswas et al., 2023, Suksompong, 2021]). We focus on discussing the setting (i.e., lines/paths) and results (i.e., upper and lower bounds) that are most relevant.

When the graph is a line, under the additive fairness relaxation, Suksompong [2019] shows that there is a contiguous $\frac{n-1}{n} \cdot u_{max}$-Prop allocation, and a contiguous $2u_{max}$-EF allocation where u_{max} is the largest value of any player for any item. Bouveret et al. [2017] prove that the problem of determining the existence of a contiguous Prop/EF allocation for an instance is NP-complete. Aziz et al. [2019] study the fair allocation of indivisible goods and chores. They show that a contiguous PROP1 allocation of a path always exists and can be computed in polynomial time, based on a contiguous proportional allocation of a mixed cake. Their PROP1 existence result extends to any graph that admits a Hamiltonian path.

Wang et al. [2021] consider the contiguous fair division of indivisible items subject to the closest assignment constraint. In their setting, the agents (i.e., facilities) have to be located on a line, and items (i.e., customers) must be allocated (or served) by one of the closest agents. They show that the problem of determining the existence of an envy-free or proportional allocation is NP-complete. They also provide upper/lower bounds of $\Omega(u_{max})$ on the existence of envy-free or proportional allocation.

	Envy-free (IEF, $\lvert\mathbf{G}\rvert = 1$)	Proportional (GFS, $\lvert\mathbf{G}\rvert = n$)
Suksompong [2019]	U: $2u_{max}$ L: u_{max}	U: $\frac{n-1}{n}u_{max}$ L: $\frac{n-1}{n}u_{max}$
Wang et al. [2021]	U: $(\frac{3}{5}m + \frac{8}{5})u_{max}$ L: $\frac{m}{4}u_{max}$	U: $\frac{n+m-1}{2n}u_{max}$ L: $\frac{m}{12}u_{max}$

Table 2: Existing Upper and Lower Bound Results.

Group Fairness Notions. While group fairness has not been examined in contiguous fair division, existing studies have considered several notions of group fairness for standard fair division settings. These group fairness notions can be mainly categorized into either when agents do not belong to any fixed groups or belong to predetermined fixed groups. See also a survey [Amanatidis et al., 2023].

In the without fixed group setting, Berliant et al. [1992] define the group envy-freeness (GEF) and Conitzer et al. [2019] extend the notion to group-fairness (GF). Their fairness definitions specify that the allocations to agents need to be fair for any possible group or subset of agents. For the above fainess, Aziz and Rey [2020] and Conitzer et al. [2019] introduce and examine different variants of the "up-to-one" relaxation.

In the fixed group setting, fairness notions are often defined over the fixed groups of agents. For instance, Manurangsi and Suksompong [2022] and Kyropoulou et al. [2020] consider EF1/EFx/EFc for fixed groups of agents, where agents from the same group share the items and group fairness is defined across different groups. In our settings, we do not consider agents within the same group sharing items. Other studies examine fairness notions across different groups. For instance, Suksompong [2017] consider MMS (which focuses on maximizing the minimum utility within each group), Segal-Halevi and Suksompong [2019] consider democratic fairness (which aims to fairly satisfy a certain fraction of agents in each group), and Scarlett et al. [2023] study group weighted envy-freeness (which defines envy-freeness based on group utilities normalized by their group sizes). Instead of comparing groups' utilities, GFS aims to compare each group's utility by the ratio of group size over the total number of agents.

Note that none of the above-mentioned studies considers contiguous or valid allocations.

3 PRELIMINARIES

Let $\mathbf{M} = \{1, 2, \cdots, m\}$ be the set of m items. The items are located on a line segment, represented by the interval $[0, 1]$. Let $\mathbf{F} = \{F_1, F_2, \cdots, F_n\}$ be the set of n agents.

An allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ is a partition of all items into bundles for the agents so that agent F_i receives bundle A_i with $\bigcup_{1 \le i \le n} A_i = \mathbf{M}$ and $A_i \cap A_j = \emptyset$ for $\forall i \neq j$.

Each agent F_i has some nonnegative value $u_i(j) \geq 0$ for item $j \in \mathbf{M}$. Without loss of generality, assume that for every item $j \in \mathbf{M}$, there is some agent F_i with a positive value, i.e., $u_i(j) > 0$. For each agent F_i, define $u_{i,max} = \max_{j \in \mathbf{M}} u_i(j)$ to be the largest value of F_i towards any item. Also, we define $u_{i,min} = \min_{j \in \mathbf{M}} u_i(j)$ to be the smallest value of F_i towards any item. Let $u_{max} =$

$\max_{F_i \in \mathbf{F}} u_{i,max}$ be the largest value of any agent towards any item. Similarly, we have $u_{min} = \min_{F_i \in \mathbf{F}} u_{i,min}$.

We assume that utilities are additive, which means $u_i(\mathbf{M}') = \sum_{j \in \mathbf{M}'} u_i(j)$ for any agent F_i and any subset of items $\mathbf{M}' \subseteq \mathbf{M}$. Assume that for any agent F_i, $u_i(\mathbf{M}) = 1$ (i.e., one can normalize agent utilities in a straightforward manner). All of the above-mentioned assumptions are standard in existing literature [Suksompong, 2019, Wang et al., 2021, Amanatidis et al., 2023].

In our settings, agents are partitioned into $k \geq 2$ disjoint groups. Let $\mathbf{G} = \{G_1, G_2, \cdots, G_k\}$ be the set of disjoint groups of agents over $\mathbf{F}$ such that $\cup_{1 \leq i \leq k} G_g = \mathbf{F}$, and, for any groups G_g and $G_{g'}$, $G_g \cap G_{g'} = \emptyset$. For each group $G_g \in \mathbf{G}$, $|G_g|$ is the number of agents in group G_g. Let $G_{min} = \arg\min_{G_g \in \mathbf{G}} |G_g|$ and $G_{max} = \arg\max_{G_g \in \mathbf{G}} |G_g|$. As for utility, we denote $u_{g,max} = \max_{F_i \in G_g} u_{i,max}$ as the largest value of any agent in G_g towards any item.

We are interested in finding contiguous allocations of items to agents that are fair with respect to the following intra-group and inter-group fairness concepts for groups of agents.

The intra-group fairness notion we consider is intra-group envy-freeness (IEF). A standard (exact and additive approximate) version of IEF is defined below.

Definition 1. *An allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ is intra-group envy-free (IEF) if $u_i(A_i) \geq u_i(A_j)$ for all $G \in \mathbf{G}$ and $F_i, F_j \in G$. For any $\epsilon \geq 0$, the allocation is ϵ-IEF if $u_i(A_i) \geq u_i(A_j) - \epsilon$ for all $G \in \mathbf{G}$ and $F_i, F_j \in G$.*

IEF says that an agent is only required to be envy-free of those agents within their group. This notion has been considered in the fair division literature as a special case of envy-freeness defined on networks [Bredereck et al., 2022, Aziz et al., 2018, Abebe et al., 2017] where each group of agents forms a clique. It is reasonable for various situations mentioned in the introduction in which agents might not know the identity of the other agents.

We also study the inter-group fairness notion of group fair share (GFS). GFS is a generalization of proportionality. An exact and additive approximate GFS is defined below.

Definition 2. *An allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ is inter-group fair share (GFS) if $\sum_{F_i \in G} u_i(A_i) \geq \frac{|G|}{n}$ for all $G \in \mathbf{G}$. For any $\epsilon \geq 0$, the allocation is ϵ-GFS if $\sum_{F_i \in G} u_i(A_i) \geq \frac{|G|}{n} - \epsilon$ for all $G \in \mathbf{G}$.*

GFS says that each group should receive a share that is proportional to their group size. Recall that a proportional allocation ensures that each agent should receive at least $\frac{1}{n}$ [Suksompong, 2019, Wang et al., 2021]. It is easy to see that proportionality implies GFS. However, GFS does not imply proportionality. That is, an agent can still achieve a smaller utility while their group can collectively obtain a combined utility that satisfies GFS. GFS has also been considered in fair resource sharing literature for various domains (see e.g., [Michorzewski et al., 2020, Airiau et al., 2023] when selecting items to be shared by a set of agents). It can be naturally applied to fair division and our setting. For instance, this notion can be useful for ensuring that groups of research teams or groups of retailers (in our introduction examples) receive utilities proportionally fair for all groups with respect to their sizes.

For simplicity, for any instance, we call an allocation "a best-approximate IEF (or GFS) allocation" if and only if the allocation is ϵ-IEF (or ϵ-GFS) with minimum ϵ.

Contiguous and Closest Assignment Requirements. We consider the setting with the contiguous requirement in which each agent should receive a contiguous block. Formally, an allocation $\mathbf{A}$ is contiguous if and only if for each agent F_i, $A_i = \{j \in \mathbf{M} | j \in [l_i, r_i]\}$ or empty. For convenience, we use the interval $[l_i, r_i]$ to represent A_i. Notice that the setting only requires agents to receive contiguous blocks, while the items assigned to agents of the same group are not necessarily contiguous.

We will also consider the setting with the closest assignment requirement [Wang et al., 2021] in which we are required to locate the agents on the interval and assign items to one of their closest agents. This setting is motivated by facility location applications in which facilities have to be located on a line to serve their closest customers to ensure fair customer allocation to facilities. In our context, facilities correspond to the agents, and customers correspond to the items.

Thus, when considering the closest assignment requirement, we want to locate $n \geq 2$ agents $\mathbf{F} = \{F_1, F_2, \cdots, F_n\}$ in different places on the interval $[0, 1]$ and allocate m items to the agents such that the items are assigned contiguously and to one of their closest agents. To distinguish such an allocation from the standard contiguous allocation, we call allocation $\mathbf{A}$ valid if there exists a location profile of the agents such that the locations of agents are pairwise distinct and each item is assigned to an agent that has the smallest distance to the item. Note that a valid allocation is a contiguous allocation. Because the agent locations must be different, a valid allocation implies that the bundle of every agent induces a contiguous block, and thus it is contiguous. However, a contiguous allocation is not necessarily valid. We note that, given an allocation, we can check whether it is valid by locating agents on the line in polynomial time using linear programming [Wang et al., 2021].

4 IEF ALLOCATIONS

In this section, we first show that determining the existence of an IEF contiguous or valid allocation is NP-complete and provide algorithms for determining its existence. We then provide lower and upper bounds to the existence of IEF contiguous or valid allocations.

4.1 COMPUTING IEF ALLOCATIONS

In this subsection, we consider finding IEF contiguous or valid allocations. We begin with providing a simple observation when the minimum group size is one, in which we can assign all items to a group with size one.

Observation 1. *Given any instance with k groups of agents and $|G_{min}| = 1$, there always exists an IEF contiguous or valid allocation.*

Determining the existence of an IEF contiguous or valid allocation is hard in general. Notice that when there is one group with all agents, finding an IEF contiguous or valid allocation reduces to finding an EF contiguous or valid allocation, which is already NP-hard [Bouveret et al., 2017, Wang et al., 2021]. However, as we increase the number of groups, the problem can become easier (e.g., when each agent is independently grouped, where the number of groups k equals the number of agents n). Below, we show that the problem is as hard even if we have multiple groups.

Theorem 2. *Determining the existence of an IEF contiguous or valid allocation is NP-complete, even if there are multiple groups of agents.*

Proof (Sketch). First, we consider the hardness of determining the existence of an IEF valid allocation. The correctness of our reduction is in the appendix, together with a similar proof when considering contiguity.

We show that the problem is in NP. In particular, we can verify whether an allocation is valid by applying the linear programming in Wang et al. [2021]. For the given allocation, we can also verify in polynomial time whether some envy exists between agents from the same group. Therefore, the problem is in NP.

Next, we show the problem is NP-hard. We reduce from the problem of determining the existence of an envy-free valid allocation (EEA), which is NP-complete when there is a single group Wang et al. [2021].

An instance of such a problem is given by $I = (\mathbf{M}, \mathbf{X}, \mathbf{F}, u)$, where $\mathbf{M} = \{1, 2, \cdots, m\}$ is the set of items, $\mathbf{X} = \{x_1, x_2, \cdots, x_m\}$ is the item location profile, $\mathbf{F} = \{F_1, F_2, \cdots, F_n\}$ is the set of agents ($n \geq 2$), and u is the utility matrix whose entry in i-th row and j-th column $u_i(j)$ refers to the utility of agent F_i for item j. Without loss of generality, in the instance, $0 \leq u_i(j) \leq 1$ for i and j.

In the EEA problem, we aim to find valid allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ such that each item is assigned to one of their closest agents and each agent receives utility no less than others in the agent's point of view, i.e., $u_i(A_i) \geq u_i(A_j)$ for any pair of agents F_i, F_j.

Considering k independent instances $I^{(1)}, I^{(2)}, \cdots, I^{(k)}$ of EEA with $I^{(i)} = (\mathbf{M}^{(i)}, \mathbf{X}^{(i)}, \mathbf{F}^{(i)}, u^{(i)})$, we construct the instance of our setting by defining $\mathbf{M} = \cup_{1 \leq i \leq k} \mathbf{M}^{(i)}$, $\mathbf{F} = \cup_{1 \leq i \leq k} \mathbf{F}^{(i)}$, $G = \{G_1, G_2, \cdots, G_k\}$ where $G_i = \mathbf{F}^{(i)}$.

To construct item position profile $\mathbf{X}$, we define $t = \lceil \log k \rceil$ and perform the following. To start with, we have the segment $[0, 1]$. We do recursion t times. In each recursion, we divide a segment $[a, b]$ into two segment $[a, a + \frac{b-a}{10}]$ and $[b - \frac{b-a}{10}, b]$. Eventually, we get 2^t segments. For i-th segment $[a_i, b_i]$, we locate the item $j \in \mathbf{M}^{(i)}$ at $a_i + x_j^{(i)}(b_i - a_i)$ so that the distance between items in $\mathbf{M}^{(i)}$ proportionally shrinks with respect to the length of the segment.

Let utility matrix u be $\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,k} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,k} \end{bmatrix}$, where $u_{i,j}$ is a $n^{(i)} \times m^{(j)}$ submatrix and

$$u_{i,j} = \begin{cases} u^{(i)} & i = j \\ \begin{bmatrix} \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \\ \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \end{bmatrix} & i \neq j \end{cases}$$

Here α_i are extremely small positive numbers and $\alpha_1 \gg \alpha_2 \gg \cdots \gg \alpha_k$. Lastly, we do normalization.

□

In the following, we show that there is a polynomial-time algorithm to compute a (best-approximate) IEF contiguous or valid allocation when the number of agents is constant. The algorithm considers all possible contiguous allocations of items to agents and check, for each contiguous allocation, whether it satisfies IEF and whether it is valid. To determine whether a contiguous allocation is valid, we additionally require linear programming in Wang et al. [2021] and use l to denote the number of bits to represent it.

Theorem 3. *There exists an algorithm that returns either a (best-approximate) IEF contiguous allocation in $O((mn + n^2)\max(m, n)^{n-1})$, or a (best-approximate) IEF valid allocation in $O((mn + n^2 + poly(m, n, l))\max(m, n)^{n-1})$.*

Next, we show that when the number of items is constant, we can obtain a polynomial-time algorithm to compute a (best-approximate) IEF contiguous or valid allocation. Intuitively, we compute the minimum local envy achieved by assigning each partition of the items to each group using matching. Based on minimum local envies, we leverage dynamic programming to obtain a (best-approximate) allocation.

Theorem 4. *There exists an algorithm that returns either a (best-approximate) IEF contiguous allocation in $O(k|G_{max}|^{2+o(1)} \log |G_{max}|4^m)$, or a (best-approximate) IEF valid allocation in $O(k8^m)$.*

4.2 BOUNDS FOR IEF ALLOCATIONS

In this subsection, we provide upper/lower bounds on the existence of IEF contiguous or valid allocations.

For envy-free (EF) contiguous allocations (with only one group), Suksompong [2019] shows there is always a $2u_{max}$-EF contiguous allocation. We show that this approximation can be improved when there are multiple groups for IEF.

Theorem 5. *Given any instance with $k \geq 2$ groups of agents, there always exists a $\frac{3}{2}u_{max}$-IEF contiguous allocation.*

Proof. Consider a cake represented by the interval $[0, m)$ and a group that has the least number of agents. For $F_i \in G_{min}$, agent F_i has utility $u_i(j)$ for the subinterval $[j-1, j)$, which is the same as the utility of item j to agent F_i. According to Stromquist [1980], for any instance, there always exists an EF contiguous allocation for divisible items. Therefore, we use G_{min} and the cake as the instance of the mechanism in Stromquist [1980] to obtain the envy-free division $\mathbf{D} = (D_1, D_2, \cdots, D_{|G_{min}|})$, where D_i is the contiguous piece of cake for $F_i \in G_{min}$. Let $|D|$ be the length of any piece D.

Then we do rounding as follows: for subinterval $[j-1, j)$, if it is covered by some D_i, i.e., $[j-1, j) \subseteq D_i$, we allocate item j to agent F_i. Otherwise, the subinterval is shared by some pieces.

- If there exists a piece D_i such that more than $\frac{1}{2}$ of $[j-1, j)$ is in D_i, i.e., $|D_i \cap [j-1, j)| \geq \frac{1}{2}$, we allocate the item j to agent F_i.
- If there does not exist such a piece, there must exist a piece D_i that is fully contained in $[j-1, j)$, i.e., $D_i \subseteq [j-1, j)$. Then we allocate the item j to agent F_i. If more than one pieces are fully contained, we arbitrarily choose one of them.

For those pieces that are fully contained by some subintervals, they either lose or gain within one item, which is at most u_{max}. Other agents gain or lose utility also at most u_{max} since each boundary contributes at most $\frac{1}{2}u_{max}$.

For those who gain more than $\frac{1}{2}u_{max}$ compared to the initial division $\mathbf{D}$, we let them give up their rightmost item, decreasing their extra utility to be at most $\frac{1}{2}u_{max}$. Notice that the item being given up is unallocated now. Then each agent in G_{min} will gain at most $\frac{1}{2}u_{max}$ extra utility, while it will lose at most u_{max} utility. Hence, the local envy is $\frac{1}{2}u_{max} - (-u_{max}) = \frac{3}{2}u_{max}$.

The case that some agents give up items will occur every two agents at most. Thus, there are at most $\lceil \frac{|G_{min}|-1}{2} \rceil$ items that are given up. We use agents outside G_{min} to hold them one-to-one, i.e., one agent will only keep one item. Since $k \geq 2$ and $|G| \geq |G_{min}| > \lceil \frac{|G_{min}|-1}{2} \rceil$ for any $G \in \mathbf{G}$, there always exist enough agents outside G_{min} for the reallocation. Since the agent outside G_{min} will only receive one item or nothing, the envy inside other groups should be no more than u_{max}.Thus, there always exists a $\frac{3}{2}u_{max}$-IEF contiguous allocation.

□

For envy-free valid allocations (in which there is only one group), Wang et al. [2021] show there is always a $(\frac{3m}{5} + \frac{8}{5})u_{max}$- EF valid allocation. We show that this approximation can be improved considering groups.

Theorem 6. *Given any instance with k groups of agents and $|G_{min}| = 2$, there always exists a u_{max}-IEF valid allocation.*

Theorem 7. *Given any instance with k groups of agents and $|G_{min}| \geq 3$, there always exists a i) $(\frac{5}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})u_{max}$-IEF valid allocation for $k \geq 3$; ii) $(\frac{4}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})u_{max}$-IEF valid allocation for $k = 2$.*

Proof. We first divide items into k blocks as equally as possible, i.e., there are either $\lceil \frac{m}{k} \rceil$ or $\lfloor \frac{m}{k} \rfloor$ items in each block. We assign the blocks arbitrarily to groups and each group will receive exactly one block. For each group G_g, we use the agents in G_g and the corresponding block to generate a $2u_{max}$-EF contiguous allocation using Theorem 5 in Suksompong [2019]. Combining such allocation for each group together, we obtain a $2u_{max}$-IEF contiguous allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$. Applying Proposition 2.4 in Wang et al. [2021], we can obtain a valid allocation $\mathbf{A}' = (A'_1, A'_2, \cdots, A'_n)$, where $u_i(A'_i) \geq \frac{1}{2}u_i(A_i)$ for $F_i \in \mathbf{F}$. For convenience, below we take $G_g \in \mathbf{G}$ as an example.

Since $\mathbf{A}$ is $2u_{max}$-IEF, we have $u_i(A_i) \geq u_i(A_j) - 2u_{max}$, where $F_i, F_j \in G_g$. Notice that the leftmost and the rightmost agent may obtain items from the neighbor blocks, leading to $u_i(A'_j) \leq u_i(A_j) + \lceil \frac{m}{k} \rceil u_{max}$. Then we obtain that for $i \neq j$ and $F_i, F_j \in G_g$,

$$u_i(A'_i) \geq \frac{1}{2}u_i(A_i) \geq \frac{1}{2}u_i(A_j) - u_{max}$$
$$\geq \frac{1}{2}u_i(A'_j) - (\frac{1}{2}\lceil\frac{m}{k}\rceil + 1)u_{max} \quad (1)$$

Since a group can obtain items from two sides, we have

$$u_i(A'_i) + u_i(A'_j) \leq 3\lceil\frac{m}{k}\rceil u_{max}, \quad (2)$$

which gives

$$u_i(A'_i) \geq \frac{1}{2}u_i(A'_j) - (\frac{1}{2}\lceil\frac{m}{k}\rceil + 1)u_{max}$$
$$\geq \frac{3}{4}u_i(A'_j) + \frac{1}{4}u_i(A'_i) - (\frac{5}{4}\lceil\frac{m}{k}\rceil + 1)u_{max}.$$

It implies that

$$u_i(A'_i) \geq u_i(A'_j) - (\frac{5}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})u_{max}.$$

Thus, we obtain a $(\frac{5}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})u_{max}$-IEF contiguous allocation for $k \geq 3$ groups.

As for $k = 2$, inequality (1) still holds, but inequality (2) changes to $u_i(A'_i) + u_i(A'_j) \leq 2\lceil\frac{m}{k}\rceil u_{max}$, and the result changes to $(\frac{4}{3}\lceil\frac{m}{k}\rceil + \frac{4}{3})u_{max}$-IEF valid allocation. □

Recall that in Wang et al. [2021], they show that the existence of $(\frac{m}{4}u_{max} - \delta)$-EF valid allocation is not guaranteed when there is only one group. Even when there are multiple groups, we can still obtain a lower bound of u_{max}.

Theorem 8. *For $\alpha < 1$, the existence of an αu_{max}-IEF contiguous allocation or αu_{max}-IEF valid allocation is not guaranteed.*

5 GFS ALLOCATIONS

In this section, we first show that determining the existence of a GFS contiguous or valid allocation is NP-complete and provide an algorithm for determining its existence. We then provide lower and upper bounds to the existence of GFS contiguous or valid allocations.

5.1 COMPUTING GFS ALLOCATIONS

Notice that when there are n groups for n agents, finding a GFS contiguous or valid allocation reduces to finding a proportional contiguous or valid allocation, which is already NP-hard [Bouveret et al., 2017, Wang et al., 2021].

However, as we decrease the number of groups, the problem can become easier (e.g., when all agents belong to one group). Below, we show that the problem is as hard even if we have a small number of groups.

Theorem 9. *Determining the existence of a GFS contiguous or valid allocation is NP-complete if the number of groups $1 < k < n$.*

Proof (Sketch). First, we consider the hardness of determining the existence of a GFS valid allocation. Inspired by Theorem 3.1 in Wang et al. [2021], we reduce from EXACT-3-COVER (X3C), which is an NP-complete problem Garey and Johnson [1979]. An instance of X3C is given by $I = (X, \mathbf{T})$, where $X = \{x_1, x_2, \cdots, x_{3s}\}$ is a set of elements, and $\mathbf{T} = \{T_1, T_2, \cdots, T_r\}$ is a family of 3-element subsets of X. The answer is "yes" if and only if X can be exactly covered by s sets from $\mathbf{T}$, i.e., each element in X is covered by exactly one of the s sets. This problem remains NP-complete if the frequency $f_x = |\{T \in \mathbf{T} | x \in T\}|$ of each $x \in X$ is at most 3.

Consider an Instance $I = (X, \mathbf{T})$ of X3C, where $f_x \leq 3$ for each $x \in X$ and the elements of T are denoted by x_T^1, x_T^2, x_T^3 for each $T \in \mathbf{T}$. We construct an instance of our problem: there are three items v_T^1, v_T^2, v_T^3 for each set $T \in \mathbf{T}$, a set of s items $B = \{b_1, b_2, \cdots, b_s\}$ and a dummy item w. Let $\epsilon > 0$ be a sufficiently small number. The order of these $m = 3r + s + 1$ items in the line segment $[0, 1]$ is

$$v_{T_1}^1 < v_{T_1}^2 < v_{T_1}^3 < v_{T_2}^1 < \cdots < v_{T_r}^3 < b_1 < \cdots < b_s < w$$

For each $T_i \in \mathbf{T}$, the lengths of both intervals $(v_{T_i}^1, v_{T_i}^2)$ and $(v_{T_i}^2, v_{T_i}^3)$ are ϵ. For $i = 1, \cdots, r-1$, the lengths of both intervals $(v_{T_i}^3, v_{T_{i+1}}^1)$ and $(v_{T_r}^3, b_1)$ are 3ϵ. For $j = 1, \cdots, s-1$, the lengths of intervals (b_j, b_{j+1}) and (b_s, w) are ϵ. Define a T-set to be $V_T = \{v_T^1, v_T^2, v_T^3\}$ for each $T \in \mathbf{T}$.

There is a total of $n' = 3s + r + 1$ real agents: one agent F_T for each $T \in \mathbf{T}$, one agent F_x for each $x \in X$ and one dummy agent F_d. Moreover, we add other sn' dummy agents to construct the group profile $\mathbf{G}$ such that $G_g \in \mathbf{G}$ contains the agent F_g together with s dummy agents $\{F_{g,d_1}, F_{g,d_2}, \cdots, F_{g,d_s}\}$. The values are defined as:

$$u_T(v) = \begin{cases} \frac{1}{3n'} & \text{if } v \in V_T \\ \frac{1}{n'} & \text{if } v \in B \\ \frac{n'-s-1}{n'} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_x(v) = \begin{cases} \frac{1}{n'} & \text{if } v \in V_T \text{ and } x \in T \\ \frac{n'-3f_x}{n'} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_d(v) = \begin{cases} 1 & v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_{g,d}(v) = \begin{cases} 1 & v = w \\ 0 & \text{otherwise} \end{cases}$$

Then each agent has a utility of 1 over the set of all items satisfying normalization.

□

We can use a similar idea in Theorem 3 for identifying approximate GFS contiguous or valid allocations. Enumerating all contiguous allocations, we check whether they satisfy GFS or compute the best approximation ϵ for them.

Theorem 10. *There exists an algorithm that returns either a (best-approximate) GFS contiguous allocation in $O((m+n)\max(m,n)^{n-1})$ or a (best-approximate) GFS valid allocation in $O((m+n+poly(m,n,l))\max(m,n)^{n-1})$.*

The idea for Theorem 4 can also be extended to find a (best-approximate) GFS contiguous allocation.

Theorem 11. *There exists an algorithm that returns either a (best-approximate) GFS contiguous allocation in $O((k|G_g|^{2+o(1)})4^m)$ or a (best-approximate) GFS valid allocation in $O(k8^m)$.*

5.2 BOUNDS FOR GFS ALLOCATIONS

For proportional allocations (in which there are n groups for n agents), Suksompong [2019] shows there is always a $\frac{n-1}{n} \cdot u_{max}$-proportional allocation. Surprisingly, this allocation can provide a $\frac{n-1}{n} \cdot |G_{max}| \cdot u_{max}$-GFS contiguous allocation directly. We show that this bound can be improved and depends on the number of groups.

Theorem 12. *Given any instance, there always exists a contiguous allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ such that for every group G_g, $\sum_{F_i \in G_g} u_i(A_i) \geq \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{g,max})$. In particular, there exists a $(\frac{|G_{max}|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{max})$-GFS contiguous allocation.*

Proof. Define $f_g(\mathbf{M}')$ to be the maximum total utility that group G_g can achieve contiguously for the contiguous bundle of items $\mathbf{M}'$. In Appendix, we show that while computing f_g is NP-hard, there is a dynamic programming algorithm that computes f_g in $O(m^2 n 2^n)$.

Lemma 13. *For each group G_g and each bundle $[l, r]$, the inequality $f_g([l, r]) \leq f_g([l, i]) + f_g([i + 1, r])$ holds for any item i in $[l, r]$.*

We process the items from left to right using the following mechanism, modified from Suksompong [2019]:

1. Set block B to empty.
2. If there exists a group G_g such that $f_g(B) \geq \frac{1}{k} - \frac{k-1}{k} u_{g,max}$, give the block to the group. (If several groups satisfy the condition, choose one arbitrarily)
   - If all groups have received a block as a result of this step, allocate the leftover items to G_g and terminate.
   - Otherwise, remove G_g from consideration and return to step 1.
3. Add the next item to block B and return to step 2.

We show by (backward) induction that when there are t groups who have not been allocated a block, each group G_g among them can achieve a maximum total utility of at least $\frac{t}{k} - \frac{k-t}{k} u_{g,max}$ for the remaining items R. Formally, we prove that $f_g(R) \geq \frac{t}{k} - \frac{k-t}{k} u_{g,max}$ for such group G_g.

The case holds when $t = k$. Assume that the statement holds when there are $t + 1$ groups left, and consider G_g who is not the next group to receive the block B. When there are $t + 1$ groups left, $f_g(R) \geq \frac{t+1}{k} - \frac{k-t-1}{k} u_{g,max}$. Assume that the last item that inserts into the block is j. Since G_g is not the next group to receive B, $f_g(B \setminus j) < \frac{1}{k} - \frac{k-1}{k} u_{g,max}$, meaning that $f_g(B) < \frac{1}{k} + \frac{1}{k} u_{g,max}$. Since $f_g(R \setminus B) + f_g(B) \geq f_g(R)$ by Lemma 13, we have

$$f_g(R \setminus B) \geq f_g(R) - f_g(B) \geq \frac{t}{k} - \frac{k-t}{k} u_{g,max}$$

as desired. Now each group G_g receives total utility at least $\frac{1}{k} - \frac{k-1}{k} u_{g,max}$. In other words, $\sum_{F_i \in G_g} u_i(A_i) \geq \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{g,max})$. □

Recall that in Wang et al. [2021], when every group has a single agent, they show that there always exists a $\frac{n+m-1}{2n} u_{max}$-Prop valid allocation. Their setting is equivalent to ours when there are $k = m$ groups. This allocation can provide a $\frac{n+m-1}{2n} \cdot |G_{max}| \cdot u_{max}$-GFS valid allocation directly. Below, we derive an improved upper bound for instances of our setting with groups by leveraging Theorem 12 and Proposition 2.4 in Wang et al. [2021].

Theorem 14. *Given any instance, there always exists a valid allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ such that for every group G_g, $\sum_{F_i \in G_g} u_i(A_i) \geq \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{max})$. In particular, there exists a $(\frac{|G_{max}|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{max})$-GFS valid allocation.*

Wang et al. [2021] show the existence of a $(\frac{m}{12} u_{max} - \delta)$-GFS valid allocation is not guaranteed when each agent belongs to their own group. Below, we show that even if we have more agents in one group, we can obtain a lower bound.

Theorem 15. *The existence of an $\alpha(\frac{|G_{max}|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{max})$-GFS contiguous allocation or $\alpha(\frac{|G_{max}|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{max})$-GFS valid allocation is not guaranteed for any $\alpha < 1$ when there are $m = k - 1$ items.*

6 IEF AND GFS ALLOCATIONS

When considering an IEF allocation, a group of agents may not receive any item to satisfy envy-freeness (which can be bad for GFS). In a GFS allocation, an agent may receive low utility while other agents in the same group receive high utility (which can be bad for IEF). In this section, we consider allocations that satisfy both IEF and GFS and provide upper/lower bounds on their existence.

When each group has only one agent, any allocation is IEF. In this case, determining the existence of GFS reduces to finding a proportional allocation, which is NP-hard considering contiguous or valid [Bouveret et al., 2017, Wang et al., 2021]. We can use a similar idea as the algorithm in Theorem 3 and Theorem 10 to find a contiguous or valid allocation that approximately satisfies both IEF and GFS.

For the existence of IEF and GFS contiguous allocation, we provide the following upper bound.

Theorem 16. *Given any instance, there always exists a $2u_{max}$-IEF and $\frac{1-(m-n)u_{min}+2nu_{max}}{n}|G_{max}|$-GFS contiguous allocation.*

Proof. Given any contiguous block B_g, a group G_g can always achieve locally $2u_{max}$-EF proved by Suksompong [2019]. Meanwhile, since ϵ-EF implies ϵ-Prop, the allocation is also a $2u_{max}$-Prop allocation. Then for the given B_g, we have the $2u_{max}$-EF and $2u_{max}$-Prop allocation $\mathbf{A}$ such that for $F_i \in G_g$,

$$u_i(A_i) \geq \frac{u_i(B_g)}{|G_g|} - 2u_{max}$$
$$\Rightarrow \sum_{F_i \in G_g} u_i(A_i) \geq \frac{\sum_{F_i \in G_g} u_i(B_g)}{|G_g|} - 2|G_g|u_{max}. \quad (3)$$

We construct the contiguous block B_g by assigning $\lfloor \frac{m}{n} \rfloor |G_g|$ or $\lceil \frac{m}{n} \rceil |G_g|$ items to each group.

$$\sum_{F_i \in G_g} u_i(B_g) \geq \lfloor \frac{m}{n} \rfloor |G_g| \times |G_g| u_{min}$$
$$\geq (\frac{m}{n} - 1)|G_g|^2 u_{min} = \frac{m-n}{n}|G_g|^2 u_{min} \quad (4)$$

After incorporating the inequality (4) into (3), we have

$$\sum_{F_i \in G_g} u_i(A_i) \geq \frac{m-n}{n}|G_g|u_{min} - 2|G_g|u_{max}$$
$$= \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{m-n}{n}|G_g|u_{min} + 2|G_g|u_{max})$$
$$\geq \frac{|G_g|}{n} - \frac{1-(m-n)u_{min}+2nu_{max}}{n}|G_{max}|.$$

□

Next, we provide the following upper bound on the existence of IEF and GFS valid allocation. For GFS, we apply Proposition 2.4 in Wang et al. [2021], which obtains a valid allocation $\mathbf{A}'$ from a contiguous allocation $\mathbf{A}$ such that for each agent F_i, $u_i(A_i') \geq \frac{1}{2}u_i(A_i)$. For IEF, the trick is similar as in Theorem 7, except that some parameters in some inequalities are different.

Theorem 17. *Given any instance with $k \geq 3$ groups of agents, there always exists a $\frac{2-(m-n)u_{min}+2nu_{max}}{2n}|G_{max}|$-GFS and $(\frac{5}{3}\lceil \frac{m}{n} \rceil |G_{max}| + \frac{4}{3})u_{max}$-IEF valid allocation.*

Proof. By applying Proposition 2.4 in Wang et al. [2021], we can construct a valid allocation $\mathbf{A}'$ from $\mathbf{A}$ in Theorem 16 such that for each agent F_i, $u_i(A_i') \geq \frac{1}{2}u_i(A_i)$ by locating the agent at one of the boundaries of their allocation A_i to guarantee that the agent can always get the better half.

Then, we can obtain the following upper bound for GFS:

$$\sum_{F_i \in G_g} u_i(A_i') \geq \frac{1}{2}\sum_{F_i \in G_g} u_i(A_i)$$
$$\geq \frac{m-n}{2n}|G_g|u_{min} - |G_g|u_{max}$$
$$= \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{m-n}{2n}|G_g|u_{min} + |G_g|u_{max})$$
$$\geq \frac{|G_g|}{n} - \frac{2-(m-n)u_{min}+2nu_{max}}{2n}|G_{max}|.$$

For IEF, we can apply the same tricks as in Theorem 7. Notice that the only difference between the $\mathbf{A}'$ in Theorem 7 and the $\mathbf{A}'$ here is the way to distribute items to each group in the construction of $\mathbf{A}$. Instead of $\lceil \frac{m}{k} \rceil$ items, each group can have at most $\lceil \frac{m}{n} \rceil |G_{max}|$ items in the contiguous allocation $\mathbf{A}$. Thus, the approximation of IEF for $\mathbf{A}'$ is $(\frac{5}{3}\lceil \frac{m}{n} \rceil |G_{max}| + \frac{4}{3})u_{max}$. □

Following the above result, as in Section 4, for $k = 2$, the approximation for IEF becomes $(\frac{4}{3}\lceil \frac{m}{n} \rceil |G_{max}| + \frac{4}{3})u_{max}$, while the approximation of GFS remains the same.

Below, we provide a particular instance showing the lower bounds on the existence of IEF or GFS contiguous and valid allocations. The proof is shown in the appendix.

Theorem 18. *The existence of $\alpha\frac{1-mu_{min}+2nu_{max}}{n}|G_{max}|$-GFS or βu_{max}-IEF contiguous allocation is not guaranteed, and the existence of $\alpha\frac{2-mu_{min}+2nu_{max}}{2n}|G_{max}|$-GFS or βu_{max}-IEF valid allocation is not guaranteed for any $\alpha, \beta < 1$ when there are $m = 2n - 1$ items.*

7 CONCLUSION

We consider contiguous fair division of indivisible items to groups, where each agent's bundle must be contiguous, with or without closest assignment. For both settings, we show NP-completeness for IEF and GFS allocation existence, and give polynomial-time approximation algorithms when either the number of agents or items is fixed. We also provide bounds for IEF, GFS, and their combination. For future directions, an immediate direction is to tighten the upper and lower bounds further. Beyond the additive approximation (as often considered in contiguous fair division), it is also interesting to consider relaxations analogous to EF1/EFX/Prop1/PropX. Finally, it would be a good direction to consider other fairness notions for groups of agents within contiguous fair division.

References

Rediet Abebe, Jon Kleinberg, and David C. Parkes. Fair division via social comparison. In *Proceedings of the 16th Conference on Autonomous Agents and MultiAgent Systems*, pages 281–289, 2017.

Stéphane Airiau, Haris Aziz, Ioannis Caragiannis, Justin Kruger, Jérôme Lang, and Dominik Peters. Portioning using ordinal preferences: Fairness and efficiency. *Artificial Intelligence*, 314:103809, 2023. ISSN 0004-3702. doi: https://doi.org/10.1016/j.artint.2022.103809. URL `https://www.sciencedirect.com/science/article/pii/S0004370222001497`.

Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence*, 322:103965, 2023. ISSN 0004-3702. doi: https://doi.org/10.1016/j.artint.2023.103965. URL `https://www.sciencedirect.com/science/article/pii/S000437022300111X`.

Haris Aziz and Simon Rey. Almost group envy-free allocation of indivisible goods and chores. In Christian Bessiere, editor, *Proceedings of the Twenty-Ninth International Joint Conference on Artificial Intelligence, IJCAI-20*, pages 39–45. International Joint Conferences on Artificial Intelligence Organization, 7 2020. doi: 10.24963/ijcai.2020/6. URL `https://doi.org/10.24963/ijcai.2020/6`. Main track.

Haris Aziz, Sylvain Bouveret, Ioannis Caragiannis, Ira Giagkousi, and Jérôme Lang. Knowledge, fairness, and social constraints. *Proceedings of the AAAI Conference on Artificial Intelligence*, 32(1), Apr. 2018.

Haris Aziz, Ioannis Caragiannis, Ayumi Igarashi, and Toby Walsh. Fair allocation of indivisible goods and chores. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI)*, pages 53–59, 2019.

Xiaohui Bei, Ayumi Igarashi, Xinhang Lu, and Warut Suksompong. The price of connectivity in fair division. *SIAM Journal on Discrete Mathematics*, 36(2):1156–1186, 2022.

Marcus Berliant, William Thomson, and Karl Dunz. On the fair division of a heterogeneous commodity. *Journal of Mathematical Economics*, 21(3):201–216, 1992. ISSN 0304-4068. doi: https://doi.org/10.1016/0304-4068(92)90001-N. URL `https://www.sciencedirect.com/science/article/pii/030440689290001N`.

Vittorio Bilò, Ioannis Caragiannis, Michele Flammini, Ayumi Igarashi, Gianpiero Monaco, Dominik Peters, Cosimo Vinci, and William S. Zwicker. Almost envy-free allocations with connected bundles. *Games and Economic Behavior*, 131:197–221, 2022.

Arpita Biswas, Justin Payan, Rik Sengupta, and Vignesh Viswanathan. *The Theory of Fair Allocation Under Structured Set Constraints*, pages 115–129. Studies in Computational Intelligence. Springer Science and Business Media Deutschland GmbH, Germany, 2023. doi: 10.1007/978-981-99-7184-8_7. Publisher Copyright: © The Institution of Engineers (India) 2023.

Sylvain Bouveret, Katarína Cechlárová, Edith Elkind, Ayumi Igarashi, and Dominik Peters. Fair division of a graph. In *Proceedings of the 26th International Joint Conference on Artificial Intelligence*, pages 135–141, 2017.

Steven J. Brams and Alan D. Taylor. A procedure for divorce settlements. *Mediation Quarterly*, 13(3):191–205, 2023/01/12 1996.

Steven J. Brams, Paul H. Edelman, and Peter C. Fishburn. Fair division of indivisible items. *Theory and Decision*, 55(2):147–180, 2003.

Robert Bredereck, Andrzej Kaczmarczyk, and Rolf Niedermeier. Envy-free allocations respecting social networks. *Artificial Intelligence*, 305, 2022.

Li Chen, Rasmus Kyng, Yang P. Liu, Richard Peng, Maximilian Probst Gutenberg, and Sushant Sachdeva. Maximum flow and minimum-cost flow in almost-linear time. In *2022 IEEE 63rd Annual Symposium on Foundations of Computer Science (FOCS)*, pages 612–623, 2022. doi: 10.1109/FOCS54457.2022.00064.

Vincent Conitzer, Rupert Freeman, Nisarg Shah, and Jennifer Wortman Vaughan. Group fairness for the allocation of indivisible goods. *Proceedings of the AAAI Conference on Artificial Intelligence*, 33 (01):1853–1860, Jul. 2019. doi: 10.1609/aaai.v33i01.33011853. URL `https://ojs.aaai.org/index.php/AAAI/article/view/4010`.

M.R. Garey and D.S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-completeness*. Mathematical Sciences Series. Freeman, 1979. ISBN 9780716710448. URL `https://books.google.com.hk/books?id=fjxGAQAAIAAJ`.

Jonathan Goldman and Ariel D. Procaccia. Spliddit: Unleashing fair division algorithms. *SIGecom Exch.*, 13(2): 41–46, jan 2015.

Gianluigi Greco and Francesco Scarcello. The complexity of computing maximin share allocations on graphs. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(02):2006–2013, Apr. 2020.

Felix Höhne and Rob van Stee. Allocating contiguous blocks of indivisible chores fairly. *Information and Computation*, 281, 2021.

Ayumi Igarashi and Dominik Peters. Pareto-optimal allocation of indivisible goods with connectivity constraints. *Proceedings of the AAAI Conference on Artificial Intelligence*, 33(01):2045–2052, 2019.

Shunhua Jiang, Zhao Song, Omri Weinstein, and Hengjie Zhang. Faster dynamic matrix inverse for faster lps. *ArXiv*, abs/2004.07470, 2020. URL `https://api.semanticschozaclar.org/CorpusID:215786164`.

Maria Kyropoulou, Warut Suksompong, and Alexandros Voudouris. Almost envy-freeness in group resource allocation. *Theoretical Computer Science*, 841, 07 2020. doi: 10.1016/j.tcs.2020.07.008.

Pasin Manurangsi and Warut Suksompong. Almost envy-freeness for groups: Improved bounds via discrepancy theory. *Theoretical Computer Science*, 930:179–195, 2022. ISSN 0304-3975. doi: https://doi.org/10.1016/j.tcs.2022.07.022. URL `https://www.sciencedirect.com/science/article/pii/S0304397522004546`.

Marcin Michorzewski, Dominik Peters, and Piotr Skowron. Price of fairness in budget division and probabilistic social choice. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(02):2184–2191, Apr. 2020.

Neeldhara Misra, Chinmay Sonar, PR Vaidyanathan, and Rohit Vaish. Equitable division of a path. *arXiv preprint arXiv:2101.09794*, 2021.

Hervé Moulin. *Fair division and collective welfare*. MIT press, 2004.

Jonathan Scarlett, Nicholas Teh, and Yair Zick. For one and all: Individual and group fairness in the allocation of indivisible goods. In *Proceedings of the 2023 International Conference on Autonomous Agents and Multiagent Systems*, AAMAS '23, page 2466–2468, Richland, SC, 2023. International Foundation for Autonomous Agents and Multiagent Systems. ISBN 9781450394321.

Erel Segal-Halevi and Warut Suksompong. Democratic fair allocation of indivisible goods. *Artificial Intelligence*, 277, 2019.

Walter Stromquist. How to cut a cake fairly. *The American Mathematical Monthly*, 87(8):640–644, 1980. doi: 10.1080/00029890.1980.11995109. URL `https://doi.org/10.1080/00029890.1980.11995109`.

Warut Suksompong. Approximate maximin shares for groups of agents. *CoRR*, abs/1706.09869, 2017. URL `http://arxiv.org/abs/1706.09869`.

Warut Suksompong. Fairly allocating contiguous blocks of indivisible items. *Discrete Applied Mathematics*, 260: 227–236, 2019.

Warut Suksompong. Constraints in fair division. *SIGecom Exch.*, 19(2):46–61, December 2021. doi: 10.1145/3505156.3505162. URL `https://doi.org/10.1145/3505156.3505162`.

Miroslaw Truszczynski and Zbigniew Lonc. Maximin share allocations on cycles. *Journal of Artificial Intelligence Research*, 69:613–655, 2020.

Chenhao Wang and Mengqi Zhang. Fairness and efficiency in facility location problems with continuous demands. In *Proceedings of the 20th International Conference on Autonomous Agents and MultiAgent Systems*, pages 1371–1379, 2021.

Chenhao Wang, Xiaoying Wu, Minming Li, and Hau Chan. Facility's perspective to fair facility location problems. *Proceedings of the AAAI Conference on Artificial Intelligence*, 35(6):5734–5741, May 2021. doi: 10.1609/aaai.v35i6.16719. URL `https://ojs.aaai.org/index.php/AAAI/article/view/16719`.

Appendix

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PROOFS IN SECTION 4

PROOF OF THEOREM 2

Claim. *Determining the existence of an IEF valid allocation is NP-complete, even if there are multiple groups of agents.*

Proof. First, we consider the hardness of determining the existence of an IEF valid allocation. We show that the problem is in NP. In particular, we can verify whether a given allocation to the problem is valid by applying linear programming in Wang et al. [2021]. For the given allocation, we can also verify in polynomial time whether there exists some envy between agents from the same group. Therefore, the problem is in NP.

Next, we show that the problem is NP-hard. We reduce from the problem of determining the existence of an envy-free valid allocation (EEA), which is NP-complete when there is a single group Wang et al. [2021].

An instance of such a problem is given by $I = (\mathbf{N}, \mathbf{X}, \mathbf{F}, u)$, where $\mathbf{N} = \{1, 2, ..., m\}$ is the set of items, $\mathbf{X} = \{x_1, x_2, ..., x_m\}$ is the item location profile, $\mathbf{F} = \{F_1, F_2, ..., F_n\}$ is the set of agents ($n \geq 2$), and u is the utility matrix whose entry in i-th row and j-th column $u_i(j)$ refers to the utility of agent F_i for item j. Without loss of generality, in the instance, $0 \leq u_i(j) \leq 1$ for i and j.

In the EEA problem, we seek to find valid allocation $\mathbf{A} = \{A_1, A_2, ..., A_n\}$ such that each item is assigned to one of their closest agents and each agent receives utility no less than others in the agent's point of view, i.e., $u_i(A_i) \geq u_i(A_j)$ for any pair of agents F_i, F_j.

Considering k independent instances $I^{(1)}, I^{(2)}, ..., I^{(k)}$ of EEA, we construct the instance of our setting by defining $\mathbf{N} = \cup_{1 \leq i \leq k} \mathbf{N}^{(i)}$, $\mathbf{F} = \cup_{1 \leq i \leq k} \mathbf{F}^{(i)}$, $G = \{G_1, G_2, ..., G_k\}$ where $G_i = \mathbf{F}^{(i)}$.

To construct item position profile $\mathbf{X}$, we define $t = \lceil \log k \rceil$ and perform the following. To start with, we have the segment $[0, 1]$. We do recursion t times. In each recursion, we divide a segment $[a, b]$ into two segment $[a, a + \frac{b-a}{10}]$ and $[b - \frac{b-a}{10}, b]$. Eventually, we get 2^t segments. For i-th segment $[a_i, b_i]$, we locate the item $j \in \mathbf{N}^{(i)}$ at $a_i + x_j^{(i)}(b_i - a_i)$ so that the distance between items in $\mathbf{N}^{(i)}$ proportionally shrinks with respect to the length of the segment.

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Let the utility matrix u be $\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,k} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,k} \end{bmatrix}$, where $u_{i,j}$ is a $n^{(i)} \times m^{(j)}$ submatrix and

$$u_{i,j} = \begin{cases} u^{(i)} & i = j \\ \begin{bmatrix} \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \\ \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \end{bmatrix} & i \neq j \end{cases}$$

Here we assume that all α_i are extremely small positive numbers and $\alpha_1 \gg \alpha_2 \gg ... \gg \alpha_k$. Lastly, we normalize the matrix u so that the sum of each row is 1.

Clearly, constructing instance $I = (\mathbf{N}, \mathbf{X}, \mathbf{M}, \mathbf{G}, u)$ for our problem can be done in polynomial time. Next, we prove that the answer to each instance of EEA is "yes" if and only if the answer to our problem is "yes".

Suppose the answer to each instance of EEA is "yes" for all of the instances $I^{(i)}$'s, meaning that the local envy of each group is 0. Then, we can directly combine the envy-free valid allocations of the k instances to obtain an IEF valid allocation.

For the other direction, suppose the answer to our problem is "yes", meaning that there is an IEF valid allocation $\mathbf{A}$. Take $G_i \in \mathbf{G}$ as an example. We introduce a variable x to hold the index of instances for later discussion. If G_i receives some items from $\mathbf{N}^{(i)}$, $x = i$. Otherwise, x is the minimum index of instance that G_i receives items from. There are three cases in G_i's allocation:

Case-1 agent: the agent receives items from instances other than $\mathbf{N}^{(x)}$.

By construction, the utilities of such items are far smaller than the items in $\mathbf{N}^{(x)}$. Therefore, such an agent will definitely envy those who receive items from $\mathbf{N}^{(x)}$, meaning that this case will not happen.

Case-2 agent: the agent receives items from $\mathbf{N}^{(x)}$ and some other instances.

Case-3 agent: the agent receives items only from $\mathbf{N}^{(x)}$.

Because of the closest assignment requirement and the item location profile we construct, there are at most two case-2 agents, and these two agents receive items from $\mathbf{N}^{(x)}$ and some other instances. Since the difference in the utilities of different instances is too large, there must exist some envy between these two agents. Thus, the case where there are two case-2 agents in G_i will never happen. Notice that each group has multiple agents, meaning that there must exist case-3 agents.

Since the utilities of other instances are nonzero, there must exist some difference of utilities that case-2 and case-3 agents receive from $\mathbf{N}^{(x)}$. Meanwhile, the utilities of other instances are too small to remove the difference, meaning that the case-2 agent always envies case-3 agents. Hence, case-2 agents do not exist, meaning that agents in G_i can only receive items from $\mathbf{N}^{(x)}$.

Since $\alpha_x > \sum_y \frac{\alpha_x}{2^y}$, the agent receiving α_x must be envied by other agents when $x \neq i$. Therefore, $x = i$, meaning that agents in G_i can only receive items from $\mathbf{N}^{(i)}$.

Hence, $\mathbf{A}$ is IEF, implying that the allocation for each instance is envy-free. Thus, the answer to each instance is "yes".

□

We show that determining the existence of an IEF contiguous allocation is also NP-complete. In Bouveret et al. [2017], they introduce the problem of fairly allocating a graph such that each agent receives a connected subgraph. They prove that determining the existence of such allocations satisfying envy-freeness is NP-complete, even if the graph is a path. Notice that when the graph is a path, the problem is just the same as our problem when all agents are in a single group. Next, we can give the hardness of finding an IEF contiguous allocation when there are multiple groups of agents.

Claim. *Determining the existence of an IEF contiguous allocation is NP-complete, even if there are multiple groups of agents.*

Proof. First, we show that the problem is in NP. It is easy to verify whether the given allocation is contiguous. We can also verify in polynomial time whether there exists some envy between agents from the same group. Therefore, the problem is in NP.

Next, we show that the problem is NP-hard. We reduce from the problem in Bouveret et al. [2017] that given a path, whether there exists an envy-free allocation such that each agent receives a connected subgraph. We will refer to their envy-free allocation problem as EEAP.

An instance of EEAP is given by $I = ((\mathbf{V}, \mathbf{E}), \mathbf{F}, u)$, where $\mathbf{V} = \{1, 2, ..., m\}$ is the set of vertices, $\mathbf{E}$ is the set of edges, $\mathbf{F} = \{F_1, F_2, ..., F_n\}$ is the set of agents, and u is the utility matrix whose entry in i-th row, j-th column $u_i(j)$ refers to the utility of agent F_i for vertex j. Without loss of generality, in the instance, $\forall i, j, 0 \leq u_i(j) \leq 1$.

Since the graph is a path, vertices can be seen as items ordered from left to right in a line. Then for such a problem, we aim to find valid allocation $A = \{A_1, A_2, ..., A_n\}$ such that each agent receives a contiguous bundle with utility no less than others in the agent point of view, i.e., $u_i(A_i) \geq u_i(A_j)$ for any pair of agents F_i and F_j.

Considering k independent instances $I^{(1)}, I^{(2)}, ..., I^{(k)}$ of EEAP with $I^{(i)} = ((\mathbf{V}^{(i)}, \mathbf{E}^{(i)}), \mathbf{F}^{(i)}, u^{(i)})$, we construct the instance of our setting by defining $\mathbf{N} = \cup_{1 \leq i \leq k} \mathbf{V}^{(i)}$, $\mathbf{F} = \cup_{1 \leq i \leq k} \mathbf{F}^{(i)}$, $\mathbf{G} = \{G_1, G_2, ..., G_k\}$ where $G_i = \mathbf{F}^{(i)}$. Also, the order of items in an instance remains the same, while the instances are also arranged in some order.

Let utility matrix u be $\begin{bmatrix} u_{1,1} & u_{1,2} & \cdots & u_{1,k} \\ u_{2,1} & u_{2,2} & \cdots & u_{2,k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{k,1} & u_{k,2} & \cdots & u_{k,k} \end{bmatrix}$, where $u_{i,j}$ is a $n^{(i)} \times m^{(j)}$ submatrix and

$$u_{i,j} = \begin{cases} u^{(i)} & i = j \\ \begin{bmatrix} \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \\ \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_j & \frac{\alpha_j}{2} & \cdots & \frac{\alpha_j}{2^{n^{(j)}}} \end{bmatrix} & i \neq j \end{cases}$$

Here we assume that all α_i are extremely small positive numbers and $\alpha_1 \gg \alpha_2 \gg ... \gg \alpha_k$. Lastly, we normalize the matrix u so that the sum of each row is 1.

In the proof for valid case above, the allocation is restricted by the constructed utility of the items instead of the distance between items, meaning that the arguments apply not only for valid but also for a contiguous allocation. As such, we can follow the same arguments as in the valid case (i.e., the reduction above for valid allocations) to prove that the answer to each instance of the EEAP is "yes" if and only if the answer to our problem is "yes". Thus, our problem is NP-hard. □

PROOF OF THEOREM 3

Proof. We start with the following lemma to count the number of possible contiguous allocations.

Lemma 19. *There are $O(\max(m, n)^{n-1})$ possible allocations in total.*

Proof. Assume that there are i agents who receive some items, leaving others to receive no items. Then there are $\binom{n}{i}$ ways to pick the i agents. Then we divide the m items into i contiguous and nonempty bundles. There are $\binom{m-1}{i-1}$ divisions (adding breakpoints to the gaps between two adjacent items, $n-1$ breakpoints and $m-1$ gaps; when $i > m$, there is at least one bundle that is empty, meaning that there is no way to divide the m bundles into i contiguous and nonempty bundles).

Moreover, there are $i!$ ways to assign the i bundles to the i agents. Totally, the number of possible allocations is

$$
\begin{aligned}
&\sum_{i=1}^{n} \binom{n}{i} * i! * \binom{m-1}{i-1} \\
\leq &\sum_{i=1}^{n} n! * \binom{m-1}{i-1} \\
= &\sum_{i-1}^{n} \frac{n!}{(i-1)!} * \frac{(m-1)!}{(m-i)!} \\
= &\sum_{i=1}^{n} O(n^{n-i+1}) * O((m-1)^{i-1}) \\
= &\sum_{i=0}^{n-1} O(n^{n-i}) * O(m^{i}) \\
= &O(\sum_{i=0}^{n-1} n^{n-i} m^{i}) \\
= &O(\max(m,n)^{n-1}).
\end{aligned}
$$

□

Thus, we can simply enumerate all possible allocations and check whether each allocation satisfies IEF or compute the best approximation. Since checking whether an allocation is IEF takes $O(mn + n^2)$ (for each agent, computing the utility of all agents under his evaluation takes $O(m)$, while comparing with other agents to check whether it is IEF or calculating the difference with other agents to find the best approximation takes $O(n)$) and there are $O(\max(m,n)^{n-1})$ possible allocations by Lemma 19, the whole procedure takes $O((mn + n^2)\max(m,n)^{n-1})$. Thus, this approach is polynomial when the number of agents n is constant.

To check whether an allocation is valid, we can also apply the linear program in Proposition 2.4 in Wang et al. [2021] to check its validity. Since a linear program can solve in $poly(m,n,l)$ with m items, n agents, l input bits to encode the linear program Jiang et al. [2020], the whole procedure takes $O((mn + n^2 + poly(m,n,l))\max(m,n)^{n-1})$. □

PROOF OF THEOREM 4

Proof. To prove Theorem 4, we first present the following lemma.

Lemma 20. *Given a group of agents G_g and $|G_g|$ contiguous bundles $\{B_1, B_2, \cdots, B_{|G_g|}\}$, we can allocate the bundles to the agents in G_g with the smallest local envy in polynomial time.*

Proof. We first construct a bipartite graph $G = (V_1, V_2, E)$ such that each vertex in V_1 represents an agent in G_g, while each vertex in V_2 represents a contiguous bundle. There is an edge $e_{i,j}$ between each vertex $F_i \in V_1$ and $B_j \in V_2$, whose weight $w_{i,j}$ is the envy caused by assigning B_j to F_i. Thus, we have $w_{i,j} = \max_k(u_i(B_k)) - u_i(B_j)$.

For a given w^* as the maximum local envy, constructing a subgraph G^* of G with the same vertices and a subset of edges $E^* = \{e_{i,j} | w_{i,j} \leq w^*\}$. It is not hard to see that there exists an allocation of the $|G_g|$ bundles with at most w^* local envy if and only if G^* has a perfect matching. Thus, by sorting all the possible weights and applying binary search, we can obtain an allocation with the smallest local envy.

Notice that there are $O(|G_g|^2)$ edges, implying $O(|G_g|^2)$ possible weights. Thus, the binary search takes $O(\log |G_g|)$. Meanwhile, we can apply the algorithm for finding maximum flow to check the existence of a perfect matching, which takes $O(|G_g|^{2+o(1)})$ [Chen et al., 2022]. Therefore, the whole procedure takes $O(|G_g|^{2+o(1)} \log |G_g|)$ and is polynomial. □

We call an interval $[l, r] \subseteq \mathbf{M}'$ for some subset of items $\mathbf{M}'$ a local maximal contiguous bundle if $l - 1 \notin \mathbf{M}'$ and $r + 1 \notin \mathbf{M}'$ and $j \in \mathbf{M}'$ for any $l \leq j \leq r$. Then the following lemma shows that we have $O(2^{m-1})$ different ways to turn at most $|G_g|$ local maximal contiguous bundles into exact $|G_g|$ contiguous bundles.

Lemma 21. *Given a group of agents G_g and a subset of items $\mathbf{M}' \in 2^{\mathbf{M}}$ composed of at most $|G_g|$ local maximal contiguous bundles, there are $O(2^{m-1})$ ways to divide them into exact $|G_g|$ contiguous bundles.*

Proof. Typically, the number of ways to divide the subset of items $\mathbf{M}'$ into G_g bundles is calculated by selecting $G_g - 1$ breakpoints from $|\mathbf{M}'| + |G_g| - 1$ slots. However, this method treats two empty bundles as distinct, leading to many duplicate counts. Thus, we count them by assuming there are i nonempty bundles and summing them up for all possible i. The number of ways to divide the subset of items $\mathbf{M}'$ into i nonempty bundles is calculated by selecting $i-1$ breakpoints from $|\mathbf{M}'| - 1$ gaps between each two adjacent items, which is $\binom{|\mathbf{M}'-1|}{i-1}$. Therefore, the total number is

$$\sum_{i=1}^{|G_g|} \binom{|\mathbf{M}'| - 1}{i-1} \leq \sum_{i=0}^{m-1} \binom{m-1}{i} = 2^{m-1}.$$

□

We define a map $f : 2^{\mathbf{M}} \times \mathbf{G} \to \mathcal{R}$ showing the smallest local envy achieved by assigning a subset of items to a group of agents. For any subset of items $\mathbf{M}'$ and the group G_g, we first check whether $\mathbf{M}'$ is composed of at most $|G_g|$ local maximal contiguous bundles. If no, $f(\mathbf{M}', G_g) = \infty$; if yes, we try all $O(2^{m-1})$ ways to divide them into $|G_g|$ contiguous bundles by Lemma 21, and then apply Lemma 20 for each of the divisions. Eventually, we take the smallest local envy among all divisions as $f(\mathbf{M}', G_g)$. This procedure takes $O(k * 2^m * 2^{m-1} * |G_{max}|^{2+o(1)} \log |G_{max}|) = O(k|G_{max}|^{2+o(1)} \log |G_{max}| 4^m)$.

Next, we leverage dynamic programming. Assume that $dp_{g,s}$ is the smallest local envy achieved by assigning the items in item subset s to the first g groups. Initially, $dp_{0,\emptyset} = 0$. Then we have

$$dp_{g,s} = \min_{s' \subseteq s} \max(dp_{g-1,s-s'}, f(s', G_g)).$$

Therefore, we obtain the best approximation with $dp_{k,\mathbf{M}}$, and we can recover the entire allocation by backtracking. Since enumerating g, s, s' takes $O(k), O(2^m), O(2^m)$, the dynamic programming and backtracking takes $O(k4^m)$ in total.

Overall, the whole algorithm returns a (best-approximated) IEF contiguous allocation and runs in $O(k|G_{max}|^{2+o(1)} \log |G_{max}| 4^m)$.

As for valid allocations, we can enumerate a division of all m items into n contiguous bundles $\mathbf{B} = \{B_1, B_2, \cdots, B_n\}$ and check whether such a division satisfies validity using linear programming in Wang et al. [2021]. Recall we have shown in Lemma 19 that there are $\binom{m-1}{i-1}$ ways to divide the m items into i nonempty contiguous bundles. Hence, there are $\sum \binom{m-1}{i-1} = O(2^m)$ such divisions in total. Given a division $\mathbf{B}$ that satisfies validity, for each group of agents G_g, we enumerate $|G_g|$ bundles for them. Although there are n bundles in $\mathbf{B}$, at most m of them are nonempty. We can only enumerate i nonempty bundles, assuming that the remaining $|G_g| - i$ bundles are empty. Thus, there are

$$\sum_{i=0}^{|G_g|} \binom{m}{i} \leq \sum_{i=0}^{m} \binom{m}{i} = 2^m$$

enumerations for each group G_g. Similarly, we define a map $f : 2^{\mathbf{M}} \times \mathbf{G} \to \mathcal{R}$ showing the smallest local envy achieved by assigning a subset of items to a group of agents. For any subset of items $\mathbf{M}'$ and the group G_g, we first check whether $\mathbf{M}'$ is a union of $|G_g|$ contiguous bundles from $\mathbf{B}$. If no, $f(\mathbf{M}', G_g) = \infty$; if yes, we apply Lemma 20 using the corresponding $|G_g|$ contiguous bundles to get $f(\mathbf{M}', G_g)$. Then we apply the same dynamic programming to find the best approximation and the allocation. Since there are $O(2^m)$ different $\mathbf{B}$, and we do linear programming, calculate f and apply dynamic programming for each $\mathbf{B}$, the overall procedure takes $O(2^m(poly(n, m, l) + k|G_{max}|^{2+o(1)} \log |G_{max}| 2^m + k4^m)) = O(k8^m)$. □

PROOF OF THEOREM 6

Proof. If $|G_{min}| = 2$, assume that G_g achieves $|G_{min}|$. From Theorem 4.2 of Wang et al. [2021], allocating all items to agents $F_i \in G_g$ always guarantees that each agent has at most u_{max} envy over the other one, which indicates a locally u_{max}-EF allocation in group G_g. Since other groups are all locally EF with no items allocated, there always exists a u_{max}-IEF valid allocation for $|G_{min}| = 2$. □

PROOF OF THEOREM 8

Proof. When there does not exist any k natural numbers $a_1, a_2, ..., a_k$ such that the number of items $m = a_1|G_1| + a_2|G_2| + .. + a_k|G_k|$ (One setting satisfying this condition is that each group has even number of agents and the number of items is odd). Suppose each agent has an equal utility of $\frac{1}{n}$ towards each item. We first assign an equal number of items to the agents in the same group. Let $a_1, a_2, ..., a_k$ be the number of items the agents receive in the corresponding groups such that the number of remaining items $r = m - (a_1|G_1| + a_2|G_2| + .. + a_k|G_k|)$ is minimized. Notice that r is less than $|G_{min}|$, otherwise a_{min} can increase by 1 meaning that r is not minimized. Also, r cannot be 0 by the initial assumption. Then there must exist some agents who receive these remaining r items. To minimize envy, each agent that receives additional items will only receive one more item. Thus, the local envy within a group containing such agents is equal to the utility of one item, which is less than u_{max}. Once how many items each agent receives is determined, we can assign the items from left to right constructing the contiguous assignment $\mathbf{A}$, which is u_{max}-IEF. Thus, the existence of αu_{max}-IEF contiguous allocation is not guaranteed.

Considering closest assignment requirement, suppose the segment $[0,1]$ is equally divided by n point $\{x_{1,1}, x_{1,2}, ..., x_{1,|G_1|}, x_{2,1}, ..., x_{k,|G_k|}\}$ from left to right. For each point x_i in $\{x_{g,1}, x_{g,2}, ..., x_{g,|G_g|}\}$, there are a_g items in the neighborhood $(x_i - \delta, x_i + \delta)$, where δ is a very small positive number. Suppose that there is one more item in each of the neighborhoods of the last r points. It is not hard to see that the location profile of agents with the smallest local envy is that each agent in G_g is located separately at the points in $\{x_{g,1}, x_{g,2}, ..., x_{g,|G_g|}\}$ and receives the items in the neighborhood of that point. Since the last r agents will receive one item than other agents in G_k, there is a local envy of u_{max}, and thus the allocation is u_{max}-IEF and valid. □

PROOFS IN SECTION 5

PROOF OF THEOREM 9

Claim. *Determining the existence of a GFS valid allocation is NP-complete, even if the number of groups $1 \leq k \leq n$.*

Proof. First, we consider the hardness of determining the existence of a GFS valid allocation. Inspired by Theorem 3.1 in Wang et al. [2021], we reduce from EXACT-3-COVER (X3C), which is an NP-complete problem Garey and Johnson [1979]. An instance of X3C is given by $I = (X, \mathbf{T})$, where $X = \{x_1, x_2, \cdots, x_{3s}\}$ is a set of elements, and $\mathbf{T} = \{T_1, T_2, \cdots, T_r\}$ is a family of 3-element subsets of X. The answer is "yes" if and only if X can be exactly covered by s sets from $\mathbf{T}$, i.e., each element in X is covered by exactly one of the s sets. This problem remains NP-complete if the frequency $f_x = |\{T \in \mathbf{T} | x \in T\}|$ of each $x \in X$ is at most 3.

Consider an Instance $I = (X, \mathbf{T})$ of X3C, where $f_x \leq 3$ for each $x \in X$ and the elements of T are denoted by x_T^1, x_T^2, x_T^3 for each $T \in \mathbf{T}$. We construct an instance of our problem as follows. There are three items v_T^1, v_T^2, v_T^3 for each set $T \in \mathbf{T}$, a set of s items $B = \{b_1, b_2, \cdots, b_s\}$ and a dummy item w. Let $\epsilon > 0$ be a sufficiently small number. The order of these $m = 3r + s + 1$ items in the line segment $[0,1]$ is

$$v_{T_1}^1 < v_{T_1}^2 < v_{T_1}^3 < v_{T_2}^1 < \cdots < v_{T_r}^3 < b_1 < \cdots < b_s < w$$

For each $T_i \in \mathbf{T}$, the lengths of both intervals $(v_{T_i}^1, v_{T_i}^2)$ and $(v_{T_i}^2, v_{T_i}^3)$ are ϵ. For $i = 1, \cdots, r-1$, the lengths of both intervals $(v_{T_i}^3, v_{T_{i+1}}^1)$ and $(v_{T_r}^3, b_1)$ are 3ϵ. For $j = 1, \cdots, s-1$, the lengths of intervals (b_j, b_{j+1}) and (b_s, w) are ϵ. Define a T-set to be $V_T = \{v_T^1, v_T^2, v_T^3\}$ for each $T \in \mathbf{T}$.

There is a total of $n' = 3s + r + 1$ real agents: one agent F_T for each $T \in \mathbf{T}$, one agent F_x for each $x \in X$ and one dummy agent F_d. Moreover, we add other sn' dummy agents to construct the group profile $\mathbf{G}$ such that $G_g \in \mathbf{G}$ contains the agent F_g together with s dummy agents $\{F_{g,d_1}, F_{g,d_2}, \cdots, F_{g,d_s}\}$. The values are defined as:

$$u_T(v) = \begin{cases} \frac{1}{3n'} & \text{if } v \in V_T \\ \frac{1}{n'} & \text{if } v \in B \\ \frac{n'-s-1}{n'} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_x(v) = \begin{cases} \frac{1}{n'} & \text{if } v \in V_T \text{ and } x \in T \\ \frac{n'-3f_x}{n'} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_d(v) = \begin{cases} 1 & v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_{g,d}(v) = \begin{cases} 1 & v = w \\ 0 & \text{otherwise} \end{cases}$$

Then each agent has a utility of 1 over the set of all items satisfying normalization.

Notice that there are $n = (s+1)n'$ agents in total, and the fair share for each group is $\frac{s+1}{(s+1)n'} = \frac{1}{n'}$. Since all agents in G_d only have utility towards the dummy item w, w is assigned to G_d in any GFS allocation. Thus, in any GFS allocation $\mathbf{A}$, for each group G_g, $\sum_{F_i \in G_g} u_i(A_i) = u_g(A_g)$, which is just equivalent to the cases in Theorem 3.1 in Wang et al. [2021]. Further, we can also conclude that the answer to X3C is "yes" if we can find the GFS allocation for the instance.

Suppose that I admits an exact cover $\mathbf{T}' \subseteq \mathbf{T}$ of size s. Let μ be a perfect matching between $\mathbf{T}'$ and B. Define a location profile of agents and the allocation $\mathbf{A}$ as follows:

- for each $T \in \mathbf{T}'$, agent F_T is located at the position of item $\mu(T) \in B$, and receives the item $\mu(T)$;
- for each $T \notin \mathbf{T}'$, agent F_T is located at the position of item v_T^2, and receives the T-set V_T;
- each agent F_x is located at the position of v_T^k such that $x = v_T^x$ and $T \in \mathbf{T}'$, and receives item v_T^k;
- agent F_d is located at the position of item w, and receives item w.
- agent $F_{g,d}$ is located anywhere that is at the right of item w, and receives nothing.

Then every group receives at least $\frac{1}{n'}$ as its fair share and thus the allocation is valid and GFS. □

Now, we consider a GFS contiguous allocation. Again, in Bouveret et al. [2017], they proved that determining the existence of proportional allocation in a graph such that each agent receives a connected subgraph is NP-complete, even if the graph is a path. Similar to IEF, when the graph is a path, the problem is just the same as determining the existence of a contiguous proportional allocation in our setting. Thus, we can apply a similar method as in Bouveret et al. [2017] and the claim above to show the hardness of determining the existence of a GFS contiguous allocation.

Claim. *Determining the existence of a GFS contiguous allocation is NP-complete, even if the number of groups* $1 \leq k \leq n$.

Proof. We reduce from EXACT-3-COVER (X3C), which is an NP-complete problem Garey and Johnson [1979]. An instance of X3C is given by $I = (X, \mathbf{T})$, where $X = \{x_1, x_2, ..., x_{3s}\}$ is a set of elements, and $\mathbf{T} = \{T_1, T_2, ..., T_r\}$ is a family of 3-element subsets of X. The answer is "yes" if and only if X can be exactly covered by s sets from $\mathbf{T}$, i.e., each element in X is covered by exactly one of the s sets. This problem remains NP-complete even if the frequency $f_x = |\{T \in \mathbf{T} | x \in T\}|$ of each $x \in X$ is at most 3.

Consider an Instance $I = (X, \mathbf{T})$ of X3C, where $f_x \leq 3$ for each $x \in X$ and the elements of T are denoted by x_T^1, x_T^2, x_T^3 for each $T \in \mathbf{T}$. We construct an instance of our problem as follows. There are three items v_T^1, v_T^2, v_T^3 for each set $T \in \mathbf{T}$, a set of s items $B = \{b_1, b_2, ..., b_s\}$ and a dummy item w. The order of these $m = 3r + s + 1$ items in the line segment $[0, 1]$ is

$$v_{T_1}^1 < v_{T_1}^2 < v_{T_1}^3 < v_{T_2}^1 < ... < v_{T_r}^3 < b_1 < ... < b_s < w$$

Define a T-set to be $V_T = \left\{v_T^1, v_T^2, v_T^3\right\}$ for each $T \in \mathbf{T}$.

There is a total of $n' = 3s + r + 1$ real agents: one agent F_T for each $T \in \mathbf{T}$, one agent F_x for each $x \in X$ and one dummy agent F_d. Moreover, we add other sn' dummy agents to construct the group profile $\mathbf{G}$ such that $G_g \in \mathbf{G}$ contains the agent F_g together with s dummy agents $\{F_{g,d_1}, F_{g,d_2}, ..., F_{g,d_s}\}$. The values are defined as:

$$u_T(v) = \begin{cases} \frac{1}{3n'} & \text{if } v \in V_T \\ \frac{1}{n'} & \text{if } v \in B \\ \frac{n'-s-1}{n'} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_x(v) = \begin{cases} \frac{1}{n'} & \text{if } v \in V_T \text{ and } x \in T \\ \frac{n'-3f_x}{n'} & \text{if } v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_d(v) = \begin{cases} 1 & v = w \\ 0 & \text{otherwise} \end{cases}$$

$$u_{g,d}(v) = \begin{cases} 1 & v = w \\ 0 & \text{otherwise} \end{cases}$$

Then each agent has a utility of 1 over the set of all items satisfying normalization.

Notice that there are $n = (s+1)n'$ agents in total, and the fair share for each group is $\frac{s+1}{(s+1)n'} = \frac{1}{n'}$. Since all agents in G_d only have utility towards the dummy item w, w is assigned to G_d in any GFS allocation. Thus, in any GFS allocation $\mathbf{A}$, for each group G_g, $\sum_{F_i \in G_g} u_i(A_i) = u_g(A_g)$, which is just equivalent to the cases in Theorem 3.1 in Wang et al. [2021]. Further, in the hardness proof for valid cases, the allocation is limited by the designed utility of the items instead of the distance between items, meaning that the statements are not only for the valid but also for a contiguous allocation. Then we can conclude that the answer to X3C is "yes" if we can find the GFS allocation for the instance. And thus the problem is NP-complete. □

PROOF OF THEOREM 11

Proof. Given an ϵ-GFS allocation, let ϵ_g be the difference between the total utility of the group and its fair share. Formally, we have $\sum_{F_i \in G_g} u_i(A_i) = \frac{|G_g|}{n} - \epsilon_g$ and $\epsilon = \max_g \epsilon_g$. Based on such a definition, we have the following lemma:

Lemma 22. *Given a group of agents G_g and $|G_g|$ contiguous bundles $\{B_1, B_2, \cdots, B_{|G_g|}\}$, we can allocate the bundles to the agents in G_g with smallest ϵ_g in polynomial time.*

Proof. Since $\epsilon_g = \frac{|G_g|}{n} - \sum_{F_i \in G_g} u_i(A_i)$, to minimize ϵ_g, we need to maximize $\sum_{F_i \in G_g} u_i(A_i)$.

We first construct a bipartite graph $G = (V_1, V_2, E)$ such that each vertex in V_1 represents each agent in G_g, while each vertex in V_2 represents each contiguous bundle. There is an edge $e_{i,j}$ between each vertex $F_i \in V_1$ and $B_j \in V_2$, who has weight $w_{i,j} = u_i(B_j)$.

Notice that since the constructed bipartite graph is a complete bipartite graph with equal vertices on both sides, the matching with maximized total weights must be a perfect matching. This means that each bundle is assigned to an agent, and each agent is allocated a bundle. Meanwhile, the algorithm in Chen et al. [2022] can also be applied here, which takes $O(|G_g|^{2+o(1)})$. □

By replacing Lemma 20 with Lemma 22, we apply the same algorithm in Theorem 4 and find the (best-approximate) GFS contiguous allocation in $O(k|G_{max}|^{2+o(1)}4^m)$ or (best-approximate) GFS valid allocation in $O(k8^m)$.

□

PROOF OF THE NP-HARDNESS OF COMPUTING f_g AND THE DYNAMIC PROGRAMMING ALGORITHM IN THEOREM 12

The problem of finding the maximum total utility when there are n items in a line and m agents with continuous demands subject to the valid condition is proved to be NP-hard by Wang and Zhang [2021] Since they limit the allocation by the utility of items instead of the distance between items, the proof also works for the contiguous setting. Formally, we have the following.

Claim. *Computing f_g is NP-hard.*

Proof. We reduce from EXACT-3-COVER (X3C), which is an NP-complete problem Garey and Johnson [1979]. An instance of X3C is given by $I = (X, \mathbf{T})$, where $X = \{x_1, x_2, ..., x_{3s}\}$ is a set of elements, and $\mathbf{T} = \{T_1, T_2, ..., T_r\}$ is a family of 3-element subsets of X. The answer is "yes" if and only if X can be exactly covered by s sets from $\mathbf{T}$, i.e., each element in X is covered by exactly one of the s sets.

Consider an Instance $I = (X, \mathbf{T})$ of X3C, where the elements of T are denoted by x_T^1, x_T^2, x_T^3 for each $T \in \mathbf{T}$. We construct an instance of our problem as follows. There are three items v_T^1, v_T^2, v_T^3, each of which corresponds to an item in T for each set $T \in \mathbf{T}$, $r-1$ dummy items $D = \{d_1, d_2, ..., d_{r-1}\}$. The order of these $m = 4r - 1$ items in the line segment $[0, 1]$ is

$$v_{T_1}^1 < v_{T_1}^2 < v_{T_1}^3 < d_1 < v_{T_2}^1 < v_{T_2}^2 < v_{T_2}^3 < d_2 < ... < v_{T_r}^3$$

Define a T-set to be $V_T = \{v_T^1, v_T^2, v_T^3\}$ for each $T \in \mathbf{T}$. Denote by $\mathbf{M}$ the set of all items. Define the frequency $f_x = |\{T \in \mathbf{T} | x \in T\}|$ to be the number of subsets that contain x.

There are a total of $n = 2s + 2r - 1$ agents: $r - s$ identical T-type agents, one agent F_x for each $x \in X$, and one agent F_d for each $d \in D$. Denote by $\mathbf{F}$ the set of all agents. For each agent $i \in \mathbf{F}$, we have $u_i(\mathbf{M}) = 1$. For each item v, the valuation is defined as:

$$u_T(v) = \begin{cases} \frac{1}{3r} & \text{if } v \notin D \\ 0 & \text{otherwise} \end{cases}$$

$$u_x(v) = \begin{cases} \frac{1}{f_x} & \text{if } v = v_T^k \text{ and } x = x_T^k \text{ for some } T, k \\ 0 & \text{otherwise} \end{cases}$$

$$u_d(v) = \begin{cases} 1 & v = d \\ 0 & \text{otherwise} \end{cases}$$

Notice that compared to other values, a dummy item has a sufficiently large utility toward the corresponding dummy agent. Then it is easy to see that, in any optimal allocation $\mathbf{A}$ that maximizes the total utility, every d-type agent F_{d_i} must receive a dummy item $d_i \in D$ and the bundle of every x-type or T-type agent cannot contain any dummy items, indicating that $u_x(\mathbf{A}) \leq \frac{1}{f_x}$ and $u_T(\mathbf{A}) \leq \frac{1}{r}$ by the contiguous requirement. So the maximum total utility is at most $(r-1) + \sum_x \frac{1}{f_x} + \frac{1}{r} \times (r - s)$.

We claim that the X3C instance has a solution if and only if the maximum total utility of a contiguous allocation is $(r-1) + \sum_x \frac{1}{f_x} + \frac{1}{r} \times (r-s))$. This gives the reduction.

Suppose there is an exact cover $\mathbf{T}'$. We can construct a contiguous allocation: every x-type agent receives a corresponding item in the cover $\mathbf{T}'$, every T-type agent receives a T-set not in the cover $\mathbf{T}'$, and every d-type agent F_{d_i} receives the corresponding dummy item $d_i \in D$. The total utility of such allocation is $(r-1) + \sum_x \frac{1}{f_x} + \frac{1}{r} \times (r-s))$ and thus it is maximized.

Conversely, suppose the maximum total utility is $(r-1) + \sum_x \frac{1}{f_x} + \frac{1}{r} \times (r-s))$. Consider a contiguous allocation $\mathbf{A}$ that maximizes the total utility. Let $\mathbf{T}' \subseteq \mathbf{T}$ be the family of sets in which at least one element corresponds to an item assigned to an x-type agent, that is,

$$\mathbf{T}' = \{T \in \mathbf{T} | \text{there is a } v_T^k \text{ assigned to some } F_x \text{ with } x = x_T^k\}.$$

The total utility of all x-type agents is at most $\sum_x \frac{1}{f_x}$ and that of all T-type agents is at most $\frac{r-s}{r}$. Then every x-type agent must receive a corresponding item (as otherwise the maximum total utility is less than $(r-1) + \sum_x \frac{1}{f_x} + \frac{1}{r} \times (r-s))$) and thus it must be $|\mathbf{T}'| \geq s$. If $|\mathbf{T}'| > s$, then at least one T-type agent cannot receive a full T-set, and the total utility of all T-type agents is less then $\frac{r-s}{r}$. It indicates that the maximum total utility is less than $(r-1) + \sum_x \frac{1}{f_x} + \frac{1}{r} \times (r-s))$, a contradiction. Therefore, it must be $|\mathbf{T}'| = s$, which is an exact cover. □

Next, we present a dynamic programming algorithm to compute f_g.

Let $dp[S][j]$ be the maximum total utility the agents in $S \subseteq G_g$ receive when considering the first j items. We enumerate $k < j$ and assign the items from $k+1$ to j to an agent F_i in the agent set S. The f_g function can be computed as follows:

$$dp[S][j] = \max_{k<j, F_i \in S} (dp[S - F_i][k] + \sum_{v=k+1}^{j} u_i(v))$$

Finally, we get $f_g(N) = dp[G_g][m]$. Notice that with some initialization, $\sum_{v=k+1}^{j} u_i(v)$ can be calculated in $O(1)$. Since enumerating S is $O(2^n)$, enumerating i is $O(n)$, enumerating j, k is $O(m^2)$, the procedure goes in $O(m^2 n 2^n)$.

PROOF OF LEMMA 13

Proof. Assume that sum_{prefix} is the total utility that G_g gets from $[l, i]$ by the assignment of $f_g([l, r])$, while sum_{suffix} is the total utility that G_g gets from $[i+1, r]$ by the assignment of $f_g([l, r])$. By definition, $f_g([l, i]) \geq sum_{prefix}$ and $f_g([i+1, r]) \geq sum_{suffix}$. Thus, $f_g([l, i]) + f_g([i+1, r]) \geq sum_{prefix} + sum_{suffix} = f_g([l, r])$. □

PROOF OF THEOREM 14

Proof. Applying Theorem 12, we can obtain a $(\frac{|G_{max}|}{n}) - \frac{1}{k} + \frac{k-1}{k} u_{max})$-GFS contiguous allocation $\mathbf{A} = \{A_1, A_2, ..., A_n\}$ such that for group $G_g \in \mathbf{G}$, $\sum_{F_i \in G_g} u_i(A_i) = \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{g,max})$. By Proposition 2.4 in Wang et al. [2021], we can locate each facility at one of its boundaries guaranteeing that it receives more than half of its original utility so that we get a valid allocation $\mathbf{A}' = \{A'_1, A'_2, ..., A'_n\}$, where for each facility, $u_i(A'_i) \geq \frac{1}{2} u_i(A_i)$. Then for each group, we have

$$\begin{aligned}
\sum_{F_i \in G_g} u_i(A'_i) &\geq \frac{1}{2} \sum_{F_i \in G_g} u_i(A_i) \\
&\geq \frac{1}{2} [\frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{g,max})] \\
&= \frac{1}{2k} - \frac{k-1}{2k} u_{max} \\
&= \frac{|G_g|}{n} - (\frac{|G_g|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{max})
\end{aligned}$$

In particular, we get a $(\frac{|G_{max}|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{max})$-GFS valid allocation. □

PROOF OF THEOREM 15

Proof. Suppose that there are $m = k - 1$ items for any of which each agent has an equal utility of $\frac{1}{k-1}$ and each group has the same number of agents, i.e., $|G_g| = |G_{max}|$. The fair share for group G_g is $\frac{|G_g|}{n} = \frac{|G_g|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{max}$. On the other hand, in any allocation, some groups receive nothing and therefore have a utility of 0. For any $\alpha < 1$, the utility of this group is less than her fair share minus $\alpha(\frac{|G_g|}{n} - \frac{n}{k} + \frac{k-1}{k} u_{max})$. We can conclude that the existence of a valid $\alpha(\frac{|G_{max}|}{n} - \frac{1}{k} + \frac{k-1}{k} u_{max})$-GFS contiguous allocation is not guaranteed for $\alpha < 1$.

With the same instance above, we give the lower bound for GFS valid allocation. Notice that $\frac{|G_g|}{n} = \frac{|G_g|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{g,max}$. Then for the same reason, we can conclude that the existence of a valid $\alpha(\frac{|G_{max}|}{n} - \frac{1}{2k} + \frac{k-1}{2k} u_{max})$-GFS contiguous allocation is not guaranteed for $\alpha < 1$.

We notice that the same instance can provide the above lower bounds under different expressions. Both of these lower bounds are mathematically equivalent for this instance. □

PROOFS IN SECTION 6

PROOF OF THEOREM 18

Proof. Suppose that there are $m = 2n - 1$ items where each agent has an equal utility of $\frac{1}{2n-1}$ and k groups with an even number of agents in each group. Notice that the utility of an agent will never become negative. Thus, for any contiguous allocation, we have

$$\begin{aligned}\sum_{F_i \in G_g} u_i(A_i) \geq & 0 \\ = & \frac{|G_g|}{n} - \frac{|G_g|}{n} \\ \geq & \frac{|G_g|}{n} - \frac{|G_{max}|}{n} \\ = & \frac{|G_g|}{n} - \frac{1 - 2nu_{min} + 2nu_{max}}{n}|G_{max}| \\ \geq & \frac{|G_g|}{n} - \frac{1 - (2n-1)u_{min} + 2nu_{max}}{n}|G_{max}| \\ = & \frac{|G_g|}{n} - \frac{1 - mu_{min} + 2nu_{max}}{n}|G_{max}|\end{aligned}$$

Thus, the existence of $\alpha\frac{1-mu_{min}+2nu_{max}}{n}|G_{max}|$-GFS contiguous allocation is not guaranteed for $\alpha < 1$. Similarly, for any valid allocation, we have

$$\begin{aligned}\sum_{F_i \in G_g} u_i(A_i) \geq & 0 \\ = & \frac{|G_g|}{n} - \frac{|G_g|}{n} \\ \geq & \frac{|G_g|}{n} - \frac{2|G_{max}|}{2n} \\ = & \frac{|G_g|}{n} - \frac{2 - 2nu_{min} + 2nu_{max}}{2n}|G_{max}| \\ \geq & \frac{|G_g|}{n} - \frac{2 - (2n-1)u_{min} + 2nu_{max}}{2n}|G_{max}| \\ = & \frac{|G_g|}{n} - \frac{2 - mu_{min} + 2nu_{max}}{2n}|G_{max}|\end{aligned}$$

Thus, the existence of $\alpha\frac{2-mu_{min}+2nu_{max}}{2n}|G_{max}|$-GFS valid allocation is not guaranteed for $\alpha < 1$.

Notice that the instance satisfies the construction of Theorem 8 (an even number of agents in each group and an odd number of items). Thus, the existence of βu_{max}-IEF allocation is still not guaranteed for $\beta < 1$, no matter whether the allocation is contiguous or valid. $\square$