Differentially Private Approval-Based Committee Voting

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Abstract

In this paper, we investigate tradeoffs among differential privacy (DP) and several representative axioms for approval-based committee voting, including justified representation, proportional justified representation, extended justified representation, Pareto efficiency, and Condorcet criterion. Without surprise, we demonstrate that all of these axioms are incompatible with DP, and thus establish both upper and lower bounds for their two-way tradeoffs with DP. Furthermore, we provide upper and lower bounds for three-way tradeoffs among DP and every pairwise combination of such axioms, revealing that although these axioms are compatible without DP, their optimal levels under DP cannot be simultaneously achieved. Our results quantify the effect of DP on the satisfaction and compatibility of the axioms in approval-based committee voting, which can provide insights for designing voting rules that possess both privacy and axiomatic properties.

1 INTRODUCTION

Voting is a fundamental mechanism in elections, where a group of voters express their preferences over a set of alternatives, and a voting rule determines the winner. In this paper, we focus on committee voting rules, where the outcome is a group of alternatives rather than a single winner. Specifically, we study approval-based committee voting rules (*ABC rules*), where each voter's preference is dichotomous, dividing the set of alternatives into approved and disapproved subsets. Similar to single-winner voting, ABC rules are typically evaluated based on axiomatic properties [Plott, 1976], known as voting axioms. A key distinction between committee voting and single-winner voting lies in the emphasis on fairness, often described through proportionality. Proportionality requires ABC rules to fairly consider the preferences of both small and large voter groups [Lackner and Skowron, 2023], and has been extensively studied in social choice theory [Aziz et al., 2017, Peters and Skowron, 2020, Skowron, 2021], participatory budgeting [Brill et al., 2023, Aziz et al., 2024], and machine learning [Chen et al., 2019, Micha and Shah, 2020].

In recent years, privacy concerns in voting have become increasingly prominent. In real-world elections, minimizing privacy leakage is crucial to prevent censorship, coercion, and bribery [Liu et al., 2020]. However, revealing election outcomes can inadvertently compromise privacy. For example, consider a scenario where a minority group of voters, including Alice, cohesively supports a specific candidate c. The privacy risk arises if Alice's single vote is pivotal. If her vote significantly increases the winning probability of c by enabling a representation guarantee like *Justified Representation (JR)*, an adversary could infer her preference from the outcome. In this case, the very mechanism designed to ensure fairness (proportionality) becomes a vector for a privacy breach. From the perspective of differential privacy [Dwork, 2006], which aims to limit such probabilistic inferences, this scenario constitutes a clear privacy violation and highlights a deep-seated tension between representation axioms and privacy.

A growing body of literature has explored the application of differential privacy to single-winner voting rules [Shang et al., 2014, Hay et al., 2017, Yan et al., 2020]. These works primarily apply randomized mechanisms to existing voting rules, evaluating utility loss (often measured by accuracy or mean square error) under privacy guarantees, but largely overlook the satisfaction of voting axioms. While this is a notable omission in single-winner contexts, it becomes a critical, foundational challenge in committee voting. The primary goal of committee voting rules is often to ensure fair representation and proportionality for diverse groups of voters, as captured by axioms like Justified Representation. The introduction of privacy-preserving noise, however,

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Proceedings of the 42nd Conference on Uncertainty in Artificial Intelligence (UAI 2026), PMLR 337:3556–3580.

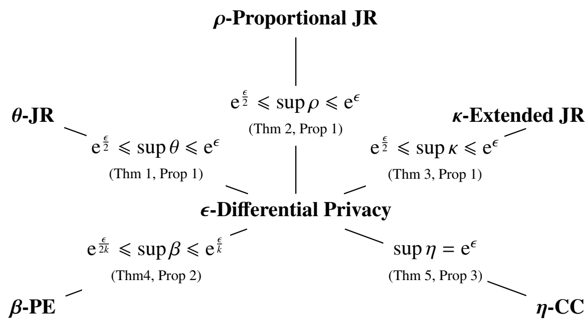

Figure 1: Two-way tradeoffs between DP and axioms. The expressions on the lines illustrate the upper and lower bounds of the achievable level of the corresponding axioms under ϵ-DP.

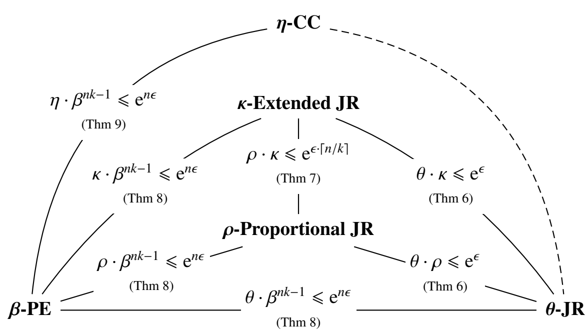

Figure 2: Three-way tradeoffs among DP and axioms (the dash line indicates that standard Condorcet criterion and justified representation are incompatible, so there is no 3-way tradeoff for them).

directly threatens these delicate representation guarantees. An ABC rule that is private but fails to be proportional may defeat the very purpose of holding a committee election. Therefore, understanding the fundamental tradeoff between DP and proportionality is not just an academic exercise, but a crucial step towards designing practical and fair private voting systems. This leads to our central question:

What is the tradeoff between DP and voting axioms for approval-based committee voting rules?

1.1 OUR CONTRIBUTIONS

Our conceptual contributions include defining approximate versions of several axioms for randomized ABC rules, such as justified representation, proportional justified representation, extended justified representation, Pareto efficiency, and Condorcet criterion (Definitions 2–6).

Our theoretical contributions are twofold. First, we investigate the 2-way tradeoff between DP and individual axioms. We establish tradeoff theorems (with both upper and lower bounds) between DP and approximate justified representation, proportional justified representation, extended justified representation, and Pareto efficiency (Theorems 1–4, Propositions 1–2). For the Condorcet criterion, we provide a tight bound for its tradeoff with DP (Theorem 5, Proposition 3). These results are summarized in Figure 1, where the expressions on the lines indicate the bounds of the approximate axioms under ϵ-DP.

Second, we explore the 3-way tradeoffs between DP and pairwise combinations of the axioms. Since justified representation and the Condorcet criterion are inherently incompatible even without DP, we exclude this pair from the 3-way tradeoff analysis. For the remaining pairwise combinations, we derive upper bounds on their tradeoffs under DP (Theorems 6–9). We demonstrate that a tradeoff exists between each pair of axioms, despite their compatibility in the absence of DP. These results are summarized in Figure 2, where the expressions on the lines represent the upper bounds of the 3-way tradeoffs under ϵ-DP.

Due to space limitations, some proofs are omitted from the main text and provided in the Appendix.

1.2 RELATED WORK AND DISCUSSIONS

In the context of social choice, approval-based committee voting has been extensively studied [Faliszewski et al., 2017, Lackner and Skowron, 2023]. Most works focus on standard axiomatic properties, such as proportionality [Aziz et al., 2017, Brill and Peters, 2023], Pareto efficiency [Lackner and Skowron, 2020, Aziz and Monnot, 2020], and the Condorcet criterion [Darmann, 2013]. For approximate axioms, prior work has explored strategyproofness for single-winner rules [Procaccia, 2010, Birrell and Pass, 2011] and proportionality for committee voting [Jiang et al., 2020, Munagala et al., 2022, Cheng et al., 2020].

To the best of our knowledge, the application of DP to voting was pioneered in works that focused on the utility-privacy tradeoff in rank aggregation and single-winner rules [Shang et al., 2014, Hay et al., 2017, Yan et al., 2020]. These early studies designed various randomized mechanisms to preserve properties like accuracy or approximate strategyproofness under DP, but largely overlooked the impact on other core voting axioms [Lee, 2015, Kohli and Laskowski, 2018, Torra, 2019, Wang et al., 2019].

More recently, a line of work has begun to systematically analyze the tradeoff between privacy and voting axioms [Liu et al., 2020, Mohsin et al., 2022]. Notably, Li et al. [2025] investigated both 2-way and 3-way tradeoffs for single-winner rules, showing that DP not only limits the achievable level of individual axioms but also exacerbates their incompatibility. However, these works focus solely on single-winner voting, leaving the analysis of DP-axiom tradeoffs in committee voting largely unexplored.

Beyond social choice, DP has also been applied to other areas of economics, such as mechanism design [Pai and Roth, 2013, Xiao, 2013], auctions [Jamshidi et al., 2024, Jia et al., 2023], and resource allocation [Hsu et al., 2016, Kannan et al., 2018, Angel et al., 2020, Chen et al., 2023].

2 PRELIMINARIES

Let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of $m \geqslant 3$ distinct alternatives. Let $N = \{1, 2, \ldots, n\}$ denote a finite set of n voters. We assume that voters' preferences are approval-based, i.e., voters distinguish between alternatives they approve and those they disapprove. Therefore, each voter i's ballot can be denoted by a (possibly empty) set $P_i \subseteq A$, representing the set of alternatives approved by the i-th voter. Then the *(preference) profile*, i.e., the collection of all voters' preferences is denoted by $P = (P_1, P_2, \ldots, P_n) \in \mathcal{P}(A)^n$, where $\mathcal{P}(A)$ is the power set of A. In addition, we use P_{-i} to denote the profile obtained by removing the voter i's preference from P.

In the paper, we are interested in committees of a specific size $k \in \mathbb{N}_+$. The input for choosing such a committee is called a *voting instance* $E = (P, k)$, consisting of a profile P and a desired committee size k. Under the settings above, a (randomized) *approval-based committee voting rule (ABC rule for short)* is a mapping $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$, where $\mathcal{R}(\mathcal{P}(A))$ denotes the set of all random variables on $\mathcal{P}(A)$. For any $W \in \mathcal{P}(A)$, the winning probability of W is denoted by $\mathbb{P}[f(P, k) = W]$. Then for any voting instance (P, k), we have $\operatorname{Supp} f(P, k) \subseteq \mathcal{P}(A, k)$, where $\operatorname{Supp} f(P, k)$ denotes the support set of $f(P, k)$ and $\mathcal{P}(A, k) = \{W \subseteq A : |W| = k\}$. An ABC rule f is *neutral* if for any voting instance (P, k) and any permutation σ on A, $\sigma \cdot f(P, k) = f(\sigma \cdot P, k)$.

Differential privacy (DP). In a word, DP requires a function to return similar output when receiving similar inputs. Here, the similarity between inputs is ensured by the notion of *neighboring datasets*, i.e., two datasets differing on at most one record. Formally, DP is defined as follows.

Definition 1 (ϵ-Differential privacy, ϵ-DP [Dwork, 2006]). *Given $\epsilon \in \mathbb{R}$, a mechanism $f : \mathcal{D} \to \mathcal{O}$ satisfies ϵ-DP if for all $O \subseteq \mathcal{O}$ and each pair of neighboring databases $D, D' \in \mathcal{D}$,*

$$\mathbb{P}[f(D) \in O] \leqslant \mathrm{e}^{\epsilon} \cdot \mathbb{P}[f(D') \in O].$$

The probability of the above inequality is taken over the randomness of the mechanism f. The smaller ϵ is, the better privacy guarantee can be offered. In the context of multi-winner voting, the mechanism f in Definition 1 is an ABC rule and its domain $\mathcal{D} = \mathcal{P}(A)^n \times \mathbb{N}_+$. Further, the pair of neighboring databases $D = (P, k), D' = (P', k)$ are two voting instances with the same committee size k and their profiles differ on at most one voter's vote, i.e., $P_{-j} = P'_{-j}$ holds for every voter $j \in N$.

Axioms of approval-based committee voting. The specific axioms considered in the paper can be roughly divided into two parts, which are listed below, respectively.

The first part of axioms is usually called proportionality in committee voting, including justified representation, proportional justified representation, and extended justified representation. For the sake of simplicity, we call them *JR-family* throughout the paper. All of the axioms in JR-family rely on the notion of *ℓ-cohesive group*. Given $\ell \geqslant 1$, a group of voters $V \subseteq N$ is ℓ-cohesive if it satisfies $|V| \geqslant \ell \cdot \frac{n}{k}$ and $|\bigcap_{i \in V} P_i| \geqslant \ell$. Based on this definition, the axioms in JR-family are listed as follows.

- **Justified representation (JR, [Aziz et al., 2017])**: A committee $W \in \mathcal{P}(A, k)$ satisfies JR for voting instance $E = (P, k)$ if for any 1-cohesive group V, there exists $i \in V$, such that $|P_i \cap W| \geqslant 1$. The family of committees satisfying JR for voting instance (P, k) is denoted by JR(P, k). An ABC rule f satisfies JR if $\operatorname{Supp} f(P, k) \subseteq$ JR(P, k) holds for any instance $(P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+$.
- **Proportional justified representation (PJR, [Sánchez-Fernández et al., 2017])**: A committee $W \in \mathcal{P}(A, k)$ satisfies PJR for voting instance $E = (P, k)$ if for any ℓ-cohesive group V, $|W \cap (\bigcup_{i \in V} P_i)| \geqslant \ell$. The family of committees satisfying PJR for voting instance (P, k) is denoted by PJR(P, k). An ABC rule f satisfies PJR if $\operatorname{Supp} f(P, k) \subseteq$ PJR(P, k) holds for any voting instance $(P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+$.
- **Extended justified representation (EJR, [Aziz et al., 2017])**: A committee $W \in \mathcal{P}(A, k)$ satisfies EJR for voting instance $E = (P, k)$ if for any ℓ-cohesive group V, there exists $i \in V$, such that $|P_i \cap W| \geqslant \ell$. The family of committees satisfying EJR for voting instance (P, k) is denoted by EJR(P, k). An ABC rule f satisfies EJR if $\operatorname{Supp} f(P, k) \subseteq$ EJR(P, k) holds for any voting instance $(P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+$.

Remark 1. *Empty ballots (i.e., $P_i = \varnothing$) require no separate treatment here. In fact, if $P_i = \varnothing$, then i will not belong to any ℓ-cohesive group for $\ell \geqslant 1$. Thus, empty ballots do not introduce additional JR-family requirements, and all definitions and arguments below apply unchanged to profiles in $\mathcal{P}(A)^n$, including profiles with empty ballots.*

Based on the implication relationship between axioms, there is a hierarchy in JR-family [Lackner and Skowron, 2023], shown in the following diagram, where $a \to b$ means that a implies b, i.e., any rule satisfying a satisfies b.

(*Strongest*) EJR $\to$ PJR $\to$ JR (*Weakest*)

The second part of axioms can be regarded as efficiency notions, including Pareto efficiency and Condorcet criterion, defined as follows. Here, we adopt the definitions in [Lackner and Skowron, 2023].

- **Pareto efficiency (PE)**: An ABC rule f satisfies Pareto efficiency if for any voting instance $E = (P, k)$, $f(E)$ is never Pareto dominated by any other committee, i.e., there does not exist committee $W' \in \mathcal{P}(A, k)$ that satisfies
  1. for all $i \in N$, $|W' \cap P_i| \geqslant |f(E) \cap P_i|$;
  2. for some $i \in N$, $|W' \cap P_i| > |f(E) \cap P_i|$.
- **Condorcet criterion (CC)**: An ABC rule satisfies Condorcet criterion if for any voting instance $E = (P, k)$ where a *Condorcet committee* W exists, $f(E) = W$, where the Condorcet committee is a committee $W \in \mathcal{P}(A, k)$ that satisfies $\big|\{i \in N : |P_i \cap W| > |P_i \cap W'|\}\big| > n/2$ for all $W' \neq W$.

3 TWO-WAY TRADEOFFS BETWEEN DP AND AXIOMS

This section investigates the tradeoffs between privacy and voting axioms for ABC rules (a two-way tradeoff). Specifically, we demonstrate that all axioms defined in Section 2 are incompatible with DP. Consequently, we propose approximate versions of them. Based on these approximations, we establish both upper and lower bounds on their tradeoffs with DP, i.e., the bounds for satisfying approximate axioms under a given privacy budget $\epsilon \in \mathbb{R}_+$. All missing proofs for this section are provided in Appendix B.

The incompatibility. We first demonstrate the incompatibility between DP and voting axioms for ABC rules. Specifically, for any neutral ABC rule satisfying DP, we establish the following lemma.

Lemma 1. *Given $\epsilon \in \mathbb{R}_+$, let $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying neutrality and ϵ-DP. Then, for any $(P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+$ and $W_1, W_2 \in \mathcal{P}(A, k)$,*

$$\mathbb{P}[f(P, k) = W_1] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P, k) = W_2].$$

For any voting instance $E = (P, k)$ and committee $W \in \mathcal{P}(A, k)$, $\mathbb{P}[f(E) = W]$ must be strictly positive; otherwise, Lemma 1 would imply $\mathbb{P}[f(E) = W] = 0$ for all $W \in \mathcal{P}(A, k)$, leading to a contradiction. This yields the following corollary.

Corollary 1. *If a neutral ABC rule $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfies ϵ-DP for some $\epsilon \in \mathbb{R}_+$,*

$$\operatorname{Supp} f(P, k) = \mathcal{P}(A, k), \quad \textit{for all } (P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+.$$

Now, consider a voting instance (P, k) where all voters approve the same k alternatives, i.e., $P_j = W$ for all $j \in N$. It is easy to verify that every $W' \neq W$ is Pareto-dominated by W. Thus, for any ABC rule f satisfying PE, $\operatorname{Supp} f(P, k) = \{W\}$. By Corollary 1, this indicates that PE is incompatible with DP[1]. Similarly, CC and axioms in the JR-family are also incompatible with DP (see Appendix B.1). In the remainder of this section, we propose approximate versions of these axioms and analyze their tradeoffs with DP.

3.1 DP-PROPORTIONALITY TRADEOFF

In this subsection, we explore the tradeoffs between DP and axioms in the JR-family. Intuitively, axioms in the JR-family require ABC rules to distinguish certain committees from others. For instance, any ABC rule f satisfying JR ensures $\operatorname{Supp} f(P, k) \subseteq \mathrm{JR}(P, k)$ for every voting instance $(P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+$, implying that a committee $W \in \mathcal{P}(A, k)$ can win only if $W \in \mathrm{JR}(P, k)$. In other words, such rules aim to distinguish JR committees from non-JR ones. However, Corollary 1 shows that standard JR is not achievable under DP whenever $\mathrm{JR}(P, k) \neq \mathcal{P}(A, k)$. Thus, under DP, we can only distinguish JR committees by assigning them different winning probabilities. The lower bound of the ratio between JR and non-JR committees reflects the strength of this distinction, i.e., the level of JR satisfaction. Similar approaches apply to PJR and EJR. The rest of this subsection discusses the tradeoffs between DP and JR-family axioms individually.

DP against JR. We first explore the tradeoff between DP and JR. To quantify how closely an ABC rule approximates JR, we introduce the following definition, where the parameter $\theta \in \mathbb{R}_+$ indicates the degree of JR satisfaction.

Definition 2 (θ-Justified representation, θ-JR for short). *Given $\theta \in \mathbb{R}_+$, an ABC rule f satisfies θ-JR if for any voting instance $E = (P, k)$, $W_1 \in \mathit{JR}(P, k)$, and $W_2 \notin \mathit{JR}(P, k)$,*

$$\mathbb{P}[f(P, k) = W_1] \geqslant \theta \cdot \mathbb{P}[f(P, k) = W_2].$$

In short, an ABC rule satisfies θ-JR if the winning probability of any JR committee is at least θ times that of any non-JR committee. Thus, a larger θ is more desirable and indicates a higher level of JR. Notably, ∞-JR reduces to its standard form. Further, θ-JR yields a lower bound (which increases with θ) on the probability that $f(P, k)$ satisfies JR, i.e.,

$$\begin{aligned}\mathbb{P}[f(P, k) \in \mathrm{JR}(P, k)] &\geqslant \frac{\theta \cdot |\mathrm{JR}(P, k)|}{\theta \cdot |\mathrm{JR}(P, k)| + \binom{m}{k} - |\mathrm{JR}(P, k)|} \\ &\geqslant \frac{\theta}{\theta + \binom{m}{k} - 1}.\end{aligned} \tag{1}$$

With this definition, we can now discuss the tradeoff between DP and JR. By Lemma 1, we immediately obtain

[1] This incompatibility also holds in non-neutral cases. See Appendix B.1 for details.

a trivial upper bound on θ-JR under ϵ-DP, i.e., $\theta \leqslant \mathrm{e}^{n\epsilon}$. Furthermore, we observe that neighboring voting instances may have distinct JR committees, each unique to its corresponding profile. This leads to the following theorem, which provides a significantly tighter bound than the trivial $\mathrm{e}^{n\epsilon}$.

Theorem 1 (θ-JR, upper bound). *Given any $\epsilon \in \mathbb{R}_+$, there is no ABC rule $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfying ϵ-DP and θ-JR with $\theta > \mathrm{e}^{\epsilon}$.*

Proof. Consider a pair of neighboring voting instances $E = (P, k)$ and $E' = (P', k)$, where $s = \lceil n/k \rceil$:

$$P_j = \begin{cases} \{a_1\}, & 1 \leqslant j \leqslant s, \\ \{a_2\}, & s+1 \leqslant j \leqslant 2s-1, \\ A \backslash \{a_1, a_2\}, & \text{otherwise}. \end{cases}$$

$$P'_j = \begin{cases} \{a_1\}, & 1 \leqslant j \leqslant s-1, \\ \{a_2\}, & s \leqslant j \leqslant 2s-1, \\ A \backslash \{a_1, a_2\}, & \text{otherwise}. \end{cases}$$

Then $W = \{a_1, a_3, a_4, \dots, a_{k+1}\}$ satisfies JR only for E, while $W' = \{a_2, a_3, a_4, \dots, a_{k+1}\}$ satisfies JR only for E'. Letting f be an ABC rule satisfying ϵ-DP and θ-JR, we have

$$\begin{aligned} \mathbb{P}[f(P,k) = W] &\geqslant \theta \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k) = W'] \\ &\geqslant \theta^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k) = W], \end{aligned}$$

i.e., $\theta \leqslant \mathrm{e}^{\epsilon}$, which completes the proof. ☐

In other words, for any ABC rule satisfying ϵ-DP, its JR level is bounded above by e^{ϵ}. Moreover, we show that the randomized response mechanism achieves $\mathrm{e}^{\epsilon/2}$-JR, providing a lower bound on the achievable θ-JR under DP. We will discuss this in the final part of the subsection.

Remark 2 (Another way to define approximate JR-axioms)**.** *The approximate JR-axioms can also be defined in another way, i.e., utilizing the probability that the outcome of the mechanism satisfies a certain axiom. For example, under this setting, the approximate JR requires the probability that the outcome of the mechanism satisfies JR is lower bounded by some constant, i.e., $\mathbb{P}[f(P,k) \in JR(P,k)] \geqslant \theta$. Then the inequality (1) indicates that our θ-JR can guarantee a lower bound of approximate JR under this setting. Please refer to Appendix A for the formal definitions and more detailed analysis.*

DP against PJR. Next, we examine the tradeoff between DP and PJR. Similar to θ-JR, we define approximate PJR by introducing a parameter $\rho \in \mathbb{R}_+$, which quantifies the level of PJR satisfaction.

Definition 3 (ρ-Proportional justified representation, ρ-PJR for short). *Given $\rho \in \mathbb{R}_+$, an ABC rule f satisfies ρ-PJR if for any voting instance $E = (P, k)$, $W_1 \in PJR(P, k)$, and $W_2 \notin PJR(P, k)$,*

$$\mathbb{P}[f(P,k) = W_1] \geqslant \rho \cdot \mathbb{P}[f(P,k) = W_2].$$

In Definition 3, the parameter ρ represents the lower bound on the ratio between the winning probabilities of PJR and non-PJR committees. Thus, a larger ρ is more desirable. Notably, as ρ approaches infinity, ρ-PJR reduces to its standard form. Based on Lemma 1, we derive a trivial upper bound for ρ-PJR, i.e., $\rho \leqslant \mathrm{e}^{n\epsilon}$. A tighter bound is established in the following theorem.

Theorem 2 (ρ-PJR, upper bound). *For any $\epsilon \in \mathbb{R}_+$, no ABC rule satisfies ϵ-DP and ρ-PJR with $\rho > \mathrm{e}^{\epsilon}$.*

Proof. Consider the following voting instances (P, k) and (P', k), where $n = s \cdot k$:

$$P_j = \begin{cases} \{a_1, a_{k+1}\}, & j = 1, \\ \{a_1\}, & 2 \leqslant j \leqslant s, \\ \{a_2\}, & s+1 \leqslant j \leqslant 2s, \\ \dots & \\ \{a_k\}, & m-k+1 \leqslant j \leqslant m. \end{cases}$$

$$P'_j = \begin{cases} \{a_1, a_{k+2}\}, & j = 1, \\ \{a_1\}, & 2 \leqslant j \leqslant s, \\ \{a_2\}, & s+1 \leqslant j \leqslant 2s, \\ \dots & \\ \{a_k\}, & m-k+1 \leqslant j \leqslant m. \end{cases}$$

Let $W = \{a_2, \dots, a_{k+1}\}$ and $W' = \{a_2, a_3, \dots, a_k, a_{k+2}\}$. Then for any $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfying ϵ-DP and ρ-PJR, we can obtain that

$$\begin{aligned} \mathbb{P}[f(P,k) = W] &\geqslant \rho \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k) = W'] \\ &\geqslant \rho^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k) = W], \end{aligned}$$

i.e., $\rho \leqslant \mathrm{e}^{\epsilon}$, which completes the proof. ☐

DP against EJR. Thirdly, we analyze the tradeoff between DP and EJR. Similar to θ-JR and ρ-PJR, we introduce the following definition.

Definition 4 (κ-Extended justified representation, κ-EJR for short). *Given $\kappa \in \mathbb{R}_+$, an ABC rule f satisfies κ-EJR if for any voting instance $E = (P, k)$, $W_1 \in EJR(P, k)$, and $W_2 \notin EJR(P, k)$,*

$$\mathbb{P}[f(P,k) = W_1] \geqslant \kappa \cdot \mathbb{P}[f(P,k) = W_2].$$

In Definition 4, a larger κ is more desirable, as it indicates a higher level of EJR. In particular, ∞-EJR reduces to the standard EJR. Similar to θ-JR and ρ-PJR, Lemma 1 provides a trivial upper bound for κ-EJR, i.e., $\kappa \leqslant \mathrm{e}^{n\epsilon}$. A tighter bound is established in the following theorem.

Theorem 3 (κ-EJR, upper bound). *For any $\epsilon \in \mathbb{R}_+$, no ABC rule satisfies ϵ-DP and κ-EJR with $\kappa > \mathrm{e}^{\epsilon\lceil \frac{n}{k} \rceil}$.*

Mechanism 1: JR-family randomized response

Input: Voting instance (P, k), Noise level ϵ, Axiom $\mathcal{A}$
Output: Winning committee W_{win}

for $W \in \mathcal{P}(A, k)$ **do**
  if W *satisfies* $\mathcal{A}$ **then**
    $\chi(W) \leftarrow 1$;
  else
    $\chi(W) \leftarrow 0$;
Compute the probability distribution p, such that $p(W) \propto \mathrm{e}^{\chi(W)\cdot\epsilon/2}$ for all $W \in \mathcal{P}(A, k)$;
Sample $W_{win} \sim p$;
return W_{win}

The lower bounds. Finally, we establish lower bounds for the achievable level of approximate JR-axioms under ϵ-DP. Specifically, by assigning winning probabilities to committees dichotomously, the following mechanism achieves lower bounds for approximate JR/PJR/EJR under DP.

Intuitively, Mechanism 1 uniformly increases the winning probability of every committee satisfying JR, PJR, or EJR. Formally, we have the following proposition.

Proposition 1. *For any noise level $\epsilon \in \mathbb{R}_+$, voting axiom $\mathcal{A} \in \{JR, PJR, EJR\}$, and voting instance $(P, k) \in \mathcal{P}(A)^n \times \mathbb{N}_+$, Mechanism 1 satisfies ϵ-DP and $\mathrm{e}^{\epsilon/2}$-$\mathcal{A}$.*

3.2 DP-EFFICIENCY TRADEOFF

In this subsection, we discuss the tradeoffs between DP and efficiency axioms, includeing Pareto efficiency (PE) and Condorcet criterion (CC). For their approximate versions, we extend the probabilistic Pareto efficiency and probabilistic Condorcet criterion in [Li et al., 2023].

DP against PE. First, we discuss the tradeoff between DP and PE. As is mentioned previously, PE requires the ABC rule to avoid selecting a Pareto-dominated committee as the winner for any input profile, which cannot be achieved under DP. Therefore, we construct the following definition, where we introduce a parameter β to quantify the level of Pareto efficiency.

Definition 5 (β-Pareto efficiency, β-PE for short). *Given $\beta \in \mathbb{R}_+$, an ABC rule f satisfies Pareto efficiency if for all voting instance $E = (P, k)$, where exist two distinct committee $W_1, W_2 \in \mathcal{P}(A)$ that W_1 Pareto dominates W_2,*

$$\mathbb{P}[f(E) = W_1] \geqslant \beta \cdot \mathbb{P}[f(E) = W_2].$$

In other words, an ABC rule satisfies β-PE if for each pair of Pareto-dominant and dominated committees, the ratio of their winning probability can be lower bounded by β. Therefore, a larger β is more desirable, since it represents a higher level of Pareto efficiency. It is also easy to check that β-PE reduces to the standard PE when β goes to infinity.

Now, we can discuss the tradeoff between DP and Pareto efficiency in the sense of Definition 5. Formally, the following theorem provides an upper bound of this tradeoff.

Theorem 4 (β-PE, upper bound). *For any $\epsilon \in \mathbb{R}_+$, there is no neutral ABC rule satisfying ϵ-DP and β-PE with $\beta > \mathrm{e}^{\frac{\epsilon}{k}}$.*

Proof. Consider the instance $E = (P, k)$, where $m \geqslant n + 2k - 1$ and $\{a_1, \ldots, a_k, b_1, \ldots, b_{n-1}, c_1, \ldots, c_k\} \subseteq A$,

$$P_j = \begin{cases} \{a_1, a_2, \ldots, a_k\}, & j = 1, \\ \{a_1, a_2, \ldots, a_k, b_1\}, & j = 2, \\ \cdots & \\ \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_{n-1}\}, & j = n. \end{cases}$$

Then we construct a series of distinct committees $W_{p,q}$, where $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant n$.

$$W_{p,q} = \begin{cases} \{a_1, \ldots, a_{k-p+1}, c_1, \ldots, c_{p-1}\}, & q = 1, \\ \{a_1, \ldots, a_{k-p}, b_{q-1}, c_1, \ldots, c_{p-1}\}, & q > 1. \end{cases}$$

Further, we can prove that $\{W_{p,q}\}_{p \leqslant k, q \leqslant n}$ forms a chain under the Pareto-dominance relationship. Consider $W_{1,1} = \{a_1, a_2, \ldots, a_k\}$ and $W_{k+1,1} = \{c_1, c_2, \ldots, c_k\}$, we have

$$\mathbb{P}[f(P, k) = W_{1,1}] \geqslant \beta^{nk} \cdot \mathbb{P}[f(P, k) = W_{k+1,1}],$$

for any neutral ABC rule $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfying β-PE. Further, if f satisfies ϵ-DP, we have

$$\mathbb{P}[f(P, k) = W_{1,1}] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P, k) = W_{k+1,1}].$$

In other words, $\beta \leqslant \mathrm{e}^{\frac{\epsilon}{k}}$, which completes the proof. $\square$

Theorem 4 shows an upper bound of the tradeoff between DP and Pareto efficiency. Next, we propose a mechanism to provide a lower bound of the achievable level of β-PE under ϵ-DP, shown in Mechanism 2. Technically, Mechanism 2 is equivalent to the exponential mechanism using the approval voting (AV) score as utility metric, where AV score is defined in the following equation.

$$\mathrm{AV}_P(W) = \sum_{c \in W} |\{j \in N : c \in P_j\}| = \sum_{j \in N} |P_j \cap W|.$$

The following proposition provides the DP and PE level of the mechanism.

Proposition 2 (β-PE, lower bound). *Given any $\epsilon \in \mathbb{R}_+$, Mechanism 2 satisfies ϵ-DP and $\mathrm{e}^{\frac{\epsilon}{2k}}$-PE.*

Mechanism 2: AV score exponential mechanism

Input: Voting instance (P, k), Noise level ϵ
Output: Winning committee W_{win}

for $a \in A$ **do**
  $\chi(a) \leftarrow |\{j \in N : a \in P_j\}|$;
$W_{win} \leftarrow \varnothing$;
while $|W_{win}| < k$ **do**
  Compute the probability distribution p, such that $p(a) \propto \mathrm{e}^{\chi(a)\cdot\epsilon/(2k)}$ for all $a \in A \backslash W_{win}$;
  Sample $a \sim p$;
  $W_{win} \leftarrow W_{win} \cup \{a\}$;
return W_{win}

Mechanism 3: Condorcet randomized response

Input: Voting instance (P, k), Noise level ϵ
Output: Winning committee W_{win}

if *Condorcet committee W_c exists* **then**
  Compute the distribution p on $\mathcal{P}(A, k)$, such that
$$p(W) = \begin{cases} \frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon}+\binom{m}{k}-1}, & W = W_c, \\ \frac{1}{\mathrm{e}^{\epsilon}+\binom{m}{k}-1}, & \text{otherwise;} \end{cases}$$
else
  Let p be the uniform distribution on $\mathcal{P}(A, k)$, i.e., $p(W) = \binom{m}{k}^{-1}$, for all $W \in \mathcal{P}(A, k)$;
Sample $W_{win} \sim p$;
return W_{win}

DP against CC. Secondly, we study the tradeoff between DP and CC. Similar to β-PE, we propose an approximate version of CC by introducing a parameter $\eta \in \mathbb{R}_+$ in it.

Definition 6 (η-Condorcet criterion, η-CC for short). *An ABC rule satisfies η-Condorcet criterion if for any voting instance $E = (P, k)$ where a Condorcet committee W_c exists,*

$$\mathbb{P}[f(E) = W_c] \geqslant \eta \cdot \mathbb{P}[f(E) = W], \quad \textit{for all } W \neq W_c.$$

It is worth noting that in Definition 6, a larger η is more desirable. When η goes to infinity, η-CC approaches the standard form of Condorcet criterion.

With Definition 6, we can quantify the DP-Condorcet tradeoff by investigating the relationship between η and the privacy budget ϵ. In fact, the following theorem illustrates an upper bound of η under ϵ-DP.

Theorem 5 (η-CC, upper bound). *Given any $\epsilon \in \mathbb{R}_+$, there is no ABC rule satisfying ϵ-DP and η-CC with $\eta > \mathrm{e}^{\epsilon}$.*

Proof sketch. Let $n = 2t + 1$ and fix a $(k-1)$-committee W. Consider two neighboring profiles P and P' that differ only on voter $t + 1$: in P, voters $1, \ldots, t + 1$ approve $C_1 = W \cup \{a_1\}$ and the remaining voters approve $C_2 = W \cup \{a_2\}$; in P', voter $t + 1$ switches to approving C_2. Then C_1 is the Condorcet committee for P, while C_2 is the Condorcet committee for P'. Hence,

$$\begin{aligned} \mathbb{P}[f(P, k) = C_1] &\geqslant \eta \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P', k) = C_2] \\ &\geqslant \eta^2 \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P, k) = C_1], \end{aligned}$$

where the two factors η come from η-CC in P and P', and the two factors $\mathrm{e}^{-\epsilon}$ come from ϵ-DP across the neighboring profiles. Therefore, $\eta \leqslant \mathrm{e}^{\epsilon}$. □

Theorem 5 provides an upper bound of the achievable level of CC under ϵ-DP. Further, we show that the bound can be achieved by Mechanism 3.

Technically, we design this mechanism by applying randomized response mechanism to the Condorcet committee. Formally, the following Proposition shows that the mechanism satisfies both DP and approximate CC, which provides a lower bound for achievable level of η-CC under ϵ-DP.

Proposition 3 (η-CC, lower bound). *Given any $\epsilon \in \mathbb{R}_+$, Mechanism 3 satisfies ϵ-DP and e^{ϵ}-CC.*

In fact, Proposition 3 indicates that the upper bound illustrated in Theorem 5 is tight.

3.3 COMPUTATIONAL COMPLEXITIES

In this subsection, we analyze the computational complexities of the proposed mechanisms.

First of all, Mechanism 2 is polynomial-time implementable, with time complexity $O(m(n+k))$. Mechanism 1 is written as an enumeration over $\binom{m}{k}$ committees, but its JR instantiation admits an efficient rejection-sampling implementation: sample a committee uniformly at random, accept it with probability proportional to its JR-RR weight, and otherwise resample. JR membership can be checked in $O(nm)$ time by counting, for voters not represented by the sampled committee, how many such voters approve each alternative outside the committee. Since each rejection-sampling round is accepted with probability at least $\mathrm{e}^{-\epsilon/2}$, the expected number of rounds is at most $\mathrm{e}^{\epsilon/2}$. This argument does not extend to PJR or EJR, because verifying whether a given committee satisfies PJR or EJR is coNP-complete [Aziz et al., 2018]. Similarly, Mechanism 3 requires identifying a Condorcet committee, and even verifying a candidate Condorcet committee is coNP-complete for approval-based committee voting [Darmann, 2013]. Thus, Mechanism 2 and Mechanism 1 instantiated with JR are the practical polynomial-time mechanisms, whereas the PJR/EJR and Condorcet randomized-response mechanisms mainly serve as theoretical lower-bound constructions.

3.4 EMPIRICAL EVALUATION

We also evaluate the practical behavior of the proposed mechanisms on real-world approval data; full experimental details and tables are provided in Appendix D. The code used for the experiments is available at `https://github.com/Categoli/DP-ABC-Voting`. On small PrefLib [Mattei and Walsh, 2013] instances, Mechanism 1 instantiated with JR improves the probability of JR satisfaction over random selection, while Mechanism 2 consistently selects AV-optimal or near-AV-optimal committees and satisfies JR in almost all tested settings. On large Polkadot validator-election instances [Boehmer et al., 2024], Mechanism 2 scales directly, and both JR satisfaction and AV-overlap increase with the privacy budget. We also test permute-and-flip as a practical replacement for the exponential-mechanism step; it slightly improves AV performance at small privacy budgets and behaves nearly identically at larger budgets.

4 THREE-WAY TRADEOFFS AMONG DP AND AXIOMS

This section investigates the 3-way tradeoffs among DP and voting axioms for ABC rules. We focus on the distinction between axiom tradeoffs in classical social choice theory and those under DP. We consider all axioms mentioned in previous sections. Notably, each pairwise combination of axioms (in their standard forms) is compatible, except for pairs involving CC and another axiom in the JR-family. In other words, the incompatibility between CC and the JR-family is not introduced by DP. Indeed, for an ABC rule f satisfying ϵ-DP, η-CC, and θ-JR, and a voting instance (P, k) where the Condorcet committee W_c does not satisfy JR, we have

$$\begin{aligned}\mathbb{P}[f(P,k) = W_c] &\geqslant \eta \cdot \mathbb{P}[f(P,k) = W_0] \\ &\geqslant \eta \cdot \theta \cdot \mathbb{P}[f(P,k) = W_c],\end{aligned}$$

where $W_0 \in \text{JR}(P, k)$. This implies an upper bound independent of ϵ, i.e., $\eta \cdot \theta \leqslant 1$. Moreover, this bound is tight, as it can be achieved by Mechanism 3. Similar statements hold for PJR and EJR. Therefore, we focus on the 3-way tradeoffs among DP and other pairs of axioms in this section. All missing proofs and further discussions on axiom compatibility without DP are provided in Appendix C.

First, we explore the tradeoffs among axioms in the JR-family. Without DP, it is evident that JR, PJR, and EJR are compatible. However, under DP, the objectives of θ-JR, ρ-PJR, and κ-EJR start to diverge. To illustrate this, consider the voting instance (P, k) from Example 4.6 of [Lackner and Skowron, 2023], where voter set is divided into k disjoint groups $V_1, V_2, \ldots, V_k$ with $|V_t| = n/k$ for all t, and

$$P_j = \begin{cases} \{a_1, a_{k+1}, a_{k+2}, \ldots, a_{2k}\}, & j \in V_1, \\ \{a_2, a_{k+1}, a_{k+2}, \ldots, a_{2k}\}, & j \in V_2, \\ \cdots & \\ \{a_k, a_{k+1}, a_{k+2}, \ldots, a_{2k}\}, & j \in V_k. \end{cases}$$

The profile P is visualized in Figure 3, where a block labeled with alternative a above a voter group V indicates that V approves a in P. It is straightforward to verify that $W_1 = \{a_{k+1}, a_{k+2}, \ldots, a_{2k}\}$ is the unique committee satisfying EJR, while $W_2 = \{a_1, a_2, \ldots, a_k\}$ satisfies PJR. Intuitively, for such a voting instance, any ABC rule f achieving optimal ρ-PJR under ϵ-DP should maximize the minimum winning probability of W_1 and W_2 while minimizing the maximum winning probability of other committees. However, κ-EJR focuses solely on maximizing the winning probability of W_1. In other words, κ-EJR requires the ABC rule to distinguish W_1 from W_2, whereas ρ-PJR requires treating them equally. This indicates that DP introduces additional tradeoffs between PJR and EJR. The following theorems formalize these tradeoffs within the JR-family.

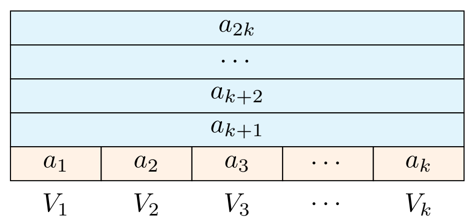

Figure 3: Diagram corresponding to the profile $\boldsymbol{P}$.

Theorem 6 (JR against PJR/EJR). *Given any $\epsilon \in \mathbb{R}_+$, there is no ABC rule satisfying*

(1) *ϵ-DP, θ-JR, and ρ-PJR with $\theta \cdot \rho > \mathrm{e}^{\epsilon}$;*
(2) *ϵ-DP, θ-JR, and κ-EJR with $\theta \cdot \kappa > \mathrm{e}^{\epsilon}$.*

Proof. For both (1) and (2), consider the following neighboring voting instances (P, k) and (P', k), where $s = \lceil 2n/k \rceil$:

$$P_j = \begin{cases} \{a_1, a_2\}, & 1 \leqslant j \leqslant s, \\ \{a_1, a_3\}, & s+1 \leqslant j \leqslant 2s-1, \\ \{a_4, a_5\}, & 2s \leqslant j \leqslant 3s-1, \\ \{a_4\}, & \text{otherwise.} \end{cases}$$

$$P'_j = \begin{cases} \{a_1, a_2\}, & 1 \leqslant j \leqslant s-1, \\ \{a_1, a_3\}, & s \leqslant j \leqslant 2s-1, \\ \{a_4, a_5\}, & 2s \leqslant j \leqslant 3s-1, \\ \{a_4\}, & \text{otherwise.} \end{cases}$$

Then we can construct the following committees.

$$\begin{aligned} W_0 &= \{a_1, a_4, a_6, a_7, \ldots, a_{k+3}\}, \\ W_1 &= \{a_1, a_2, a_4, a_5, a_6, \ldots, a_{k+1}\}, \\ W_1' &= \{a_1, a_3, a_4, a_5, a_6, \ldots, a_{k+1}\}. \end{aligned}$$

With the constructions above, we can finally prove that for any $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfying ϵ-DP, θ-JR, and ρ-PJR,

$$
\begin{aligned}
\mathbb{P}[f(P,k) = W_1] &\geqslant \rho \cdot \mathbb{P}[f(P,k) = W_0] \\
&\geqslant \rho \cdot \theta \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k) = W_1'] \\
&\geqslant \rho^2 \cdot \theta^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k) = W_1],
\end{aligned}
$$

i.e., $\rho \cdot \theta \leqslant \mathrm{e}^{\epsilon}$, which completes the proof. □

Theorem 7 (**PJR against EJR**). *Given any $\epsilon \in \mathbb{R}_+$, there is no ABC rule f satisfying ϵ-DP, ρ-PJR, and κ-EJR with $\rho \cdot \kappa > \mathrm{e}^{\epsilon \lceil \frac{n}{k} \rceil}$.*

Proof sketch. Let $s = \lceil n/k \rceil$. Construct profiles P and P' that differ only on the first s voters. In P, let $W_1 = \{a_1, \ldots, a_k\}$ satisfy EJR and let $W_0 = \{a_{k+1}, \ldots, a_{2k}\}$ satisfy PJR but not EJR. In P', replace the first block of voters so that $W_1' = \{a_1, \ldots, a_{k-1}, a_{2k+1}\}$ satisfies EJR, while W_0 still satisfies PJR. Moreover, W_1' fails PJR for P, and W_1 fails even JR for P'. Therefore,

$$
\begin{aligned}
\mathbb{P}[f(P,k) = W_1] &\geqslant \kappa\rho\mathrm{e}^{-s\epsilon} \cdot \mathbb{P}[f(P',k) = W_1'] \\
&\geqslant \kappa^2\rho^2\mathrm{e}^{-2s\epsilon} \cdot \mathbb{P}[f(P,k) = W_1],
\end{aligned}
$$

where the factors κ, ρ, and $\mathrm{e}^{-s\epsilon}$ come from EJR, PJR, and group privacy over the s changed ballots, respectively. Thus $\kappa\rho \leqslant \mathrm{e}^{s\epsilon} = \mathrm{e}^{\epsilon\lceil n/k \rceil}$. □

Second, we examine the tradeoffs between PE and axioms in the JR-family. In deterministic settings (without DP), the proportional approval voting (PAV) rule satisfies both PE and EJR [Aziz et al., 2017], demonstrating that standard PE is compatible with all axioms in the JR-family. However, under DP, tradeoffs between β-PE and approximate axioms in the JR-family emerge. Intuitively, when a Pareto-dominated committee satisfies JR (or PJR/EJR), β-PE requires the ABC rule to distinguish it from its Pareto-dominator, whereas θ-JR (or ρ-PJR/κ-EJR) treats them equally. Formally, the following theorem establishes upper bounds for the tradeoffs between PE and the JR-family under ϵ-DP.

Theorem 8 (**PE against JR-family**). *Given any $\epsilon \in \mathbb{R}_+$, there is no neutral ABC rule satisfying*

(1) *ϵ-DP, β-PE, and θ-JR with $\beta^{nk-1} \cdot \theta > \mathrm{e}^{n\epsilon}$;*
(2) *ϵ-DP, β-PE, and ρ-PJR with $\beta^{nk-1} \cdot \rho > \mathrm{e}^{n\epsilon}$;*
(3) *ϵ-DP, β-PE, and κ-EJR with $\beta^{nk-1} \cdot \kappa > \mathrm{e}^{n\epsilon}$.*

Finally, we analyze the tradeoff between PE and CC under DP. Similar to the pairwise combinations of axioms discussed earlier, the standard forms of PE and CC are compatible, as the Condorcet committee is never Pareto-dominated by any other committee. However, under DP, η-CC disregards Pareto-dominance relationships among committees other than the Condorcet committee. The following theorem formalizes this tradeoff.

Theorem 9 (**PE against CC**). *Given any $\epsilon \in \mathbb{R}_+$, there is no neutral ABC rule satisfying ϵ-DP, β-PE, and η-CC with $\beta^{nk-1} \cdot \eta > \mathrm{e}^{n\epsilon}$.*

Proof sketch. Again use the Pareto-dominance chain from Theorem 4. For the constructed profile, $W_{1,1}$ is the Condorcet committee. Thus η-CC implies $\mathbb{P}[f(P,k) = W_{1,1}] \geqslant \eta \cdot \mathbb{P}[f(P,k) = W_{1,2}]$, and β-PE along the remaining $nk-1$ edges implies $\mathbb{P}[f(P,k) = W_{1,2}] \geqslant \beta^{nk-1}\mathbb{P}[f(P,k) = W_{k+1,1}]$. Lemma 1 upper-bounds the ratio between $W_{1,1}$ and $W_{k+1,1}$ by $\mathrm{e}^{n\epsilon}$, so $\beta^{nk-1}\eta \leqslant \mathrm{e}^{n\epsilon}$. □

5 CONCLUSION AND FUTURE WORK

In the paper, we investigated the tradeoffs between DP and several axioms in approval-based committee voting, including JR, PJR, EJR, PE, and CC. We demonstrated that DP is fundamentally incompatible with all these axioms and quantified their tradeoffs against DP (2-way tradeoff). Additionally, we explored the tradeoffs between axioms under DP (3-way tradeoff). By establishing upper bounds for 3-way tradeoffs, we revealed that DP introduces additional tradeoffs among axioms. However, some of the bounds presented in this paper are not tight. Therefore, deriving tighter bounds for both 2-way and 3-way tradeoffs remains a promising direction for future research. Our empirical results suggest that the polynomial-time AV exponential mechanism and the JR rejection-sampling implementation are practically usable on real-world data, while designing efficient mechanisms with stronger guarantees for PJR, EJR, and CC remains an interesting avenue for future work.

Acknowledgements

This work was supported by the National Natural Science Foundation of China under Grant 62572007 and Key Laboratory of Interdisciplinary Research of Computation and Economics (Shanghai University of Finance and Economics), Ministry of Education. Lirong Xia acknowledges NSF #2450124, #2517733, and #2518373 for support.

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A ANOTHER DEFINITION OF APPROXIMATE JR-AXIOMS

Except for the Definitions 2–4 in the paper, there is another way to define the approximate JR-axioms, i.e., to utilize the probability that the outcome of the mechanism satisfies the axiom. Based on this concept, the approximate versions of JR-axioms are listed in the following definition. In addition to the axioms discussed in the paper (JR, PJR, and EJR), we also take PJR+ and EJR+ into consideration, of which the definition is given as follows.

- **PJR+**: An ABC rule satisfies *PJR+* if for any voting instance $E = (P, k)$ and $V \subseteq N$ that $|V| \geqslant \ell \cdot \frac{n}{k}$,

$$\bigcap_{i \in V} P_i \subseteq f(E) \text{ or } \left| f(E) \cap \bigcup_{i \in V} P_i \right| \geqslant \ell.$$

- **EJR+**: An ABC rule satisfies *EJR+* if for any voting instance $E = (P, k)$ and $V \subseteq N$ that $|V| \geqslant \ell \cdot \frac{n}{k}$,

$$\bigcap_{i \in V} P_i \subseteq W \text{ or } \exists i \in V, |P_i \cap W| \geqslant \ell.$$

Definition 7 (**Approximate JR-family).** *Let $\mathcal{A} \in \{JR, PJR, EJR, PJR+, EJR+\}$ be any JR-axiom. Given $\theta \in [0, 1]$, an ABC rule f satisfies θ-$\mathcal{A}$ if for any $E = (P, k)$,*

$$\mathbb{P}[f(P, k) \textit{ satisfies } \mathcal{A}] \geqslant \theta.$$

Then it is not hard to see that the implication relationship will be preserved for the approximate JR-axioms. In Brill and Peters [2023], it has been shown that there is a hierarchy in JR-family, described in the following diagram, where $a \to b$ indicates that a implies b, i.e., any ABC rule satisfying a satisfies b.

$$\begin{array}{ccccccccccc} & & & & \text{PJR+} & & & & & & \\ & & & \nearrow & & \searrow & & & & & \\ (\textit{Strongest}) & \text{EJR+} & \to & \text{EJR} & & \to & \text{PJR} & \to & \text{JR} & (\textit{Weakest}) \end{array}$$

Formally, we have the following lemma.

Lemma 2. *Given any $\theta \in [0, 1]$, we have the following diagram, where $a \to b$ indicates that a implies b.*

$$\begin{array}{cccccccc} & & & \theta\text{-PJR+} & & & & \\ & & \nearrow & & \searrow & & & \\ \theta\text{-EJR+} & \to & \theta\text{-EJR} & & \to & \theta\text{-PJR} & \to & \theta\text{-JR} \end{array}$$

Proof. First, we prove θ-EJR $\to$ θ-PJR. Let f be any ABC rule satisfying θ-EJR. Then for any voting instance $E = (P, k)$ and ℓ-cohesive group V, we have

$$\mathbb{P}\left[|f(E) \cap \bigcup\nolimits_{i \in V} P_i| \geqslant \ell\right] \geqslant \mathbb{P}\left[\exists i \in V, |f(E) \cap P_i| \geqslant \ell\right] \geqslant \theta.$$

In other words, θ-EJR $\to$ θ-PJR. As a consequence, θ-EJR+ $\to$ θ-PJR+.

Next, we prove θ-PJR $\to$ θ-JR. Let f be any ABC rule satisfying θ-PJR. Then for any voting instance $E = (P, k)$ and 1-cohesive group V, we have

$$\mathbb{P}\left[\exists i \in V, |f(E) \cap P_i| \geqslant 1\right] = \mathbb{P}\left[|f(E) \cap \bigcup\nolimits_{i \in V} P_i| \geqslant 1\right] \geqslant \theta.$$

In other words, θ-PJR $\to$ θ-JR.

Finally, we prove θ-PJR+ $\to$ θ-PJR. Let f be any ABC rule satisfying θ-PJR+. Then for any voting instance $E = (P, k)$ and 1-cohesive group V, we have

$$\mathbb{P}\left[|f(E) \cap \bigcup\nolimits_{i \in V} P_i| \geqslant 1\right] \geqslant \mathbb{P}\left[\bigcap\nolimits_{i \in V} P_i \subseteq f(E) \text{ or } |f(E) \cap \bigcup\nolimits_{i \in V} P_i| \geqslant \ell;\right] \geqslant \theta.$$

In other words, θ-PJR+ $\to$ θ-PJR. Similarly, we can prove that θ-EJR+ $\to$ θ-EJR, which completes the proof. $\square$

A.1 THE (2-WAY) TRADEOFFS WITH DP

Under the approximate axioms above, we can investigate the upper bounds of the tradeoff between DP and axioms in JR-family. In fact, when the number of voters is n, any pair of profiles P and P' differ on at most n voters' vote. Therefore, the definition of ϵ-DP indicates that for any committee W,

$$\mathbb{P}[f(P,k)=W] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P',k)=W].$$

By neutrality, for any committees W and W',

$$\mathbb{P}[f(P,k)=W] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P,k)=W'].$$

Therefore, there exists an upper bound for the winning probability of any committee W,

$$\mathbb{P}[f(P,k)=W] \leqslant \frac{\mathrm{e}^{n\epsilon}}{\mathrm{e}^{n\epsilon}+\binom{m}{k}-1},$$

which reveals a trivial upper bound for the tradeoffs between DP and any approximate JR-axiom θ-$\mathcal{A}$, i.e.,

$$\theta \leqslant \frac{\mathrm{e}^{n\epsilon}}{\mathrm{e}^{n\epsilon}+\binom{m}{k}-1}. \tag{2}$$

However, this bound is quite loose. Therefore, we construct a special profile, and provide a tighter upper bound for JR-family through this profile, shown in the following theorem.

Theorem 10 (JR-Family, Upper Bound). *Given $\epsilon \in \mathbb{R}_+$, there is no neutral ABC rule satisfying ϵ-DP and θ-JR with*

$$\theta > \frac{\mathrm{e}^{n\epsilon}}{\sum_{s=0}^{k}\binom{k}{s}\binom{m-k}{k-s}\cdot \mathrm{e}^{n\epsilon\cdot s/k}}.$$

PROOF. Let f be any neutral ABC rule satisfying ϵ-DP. Consider the following profile P, where N can be divided into k disjoint 1-cohesive groups (let $p=n/k$).

- **Group 1.** $V_1=\{1,2,\ldots,p\}$, $P_1=\{a_1\}$;
- **Group 2.** $V_2=\{p+1,p+2,\ldots,2p\}$, $P_2=\{a_2\}$;

......

- **Group *k*.** $V_k=\{(k-1)p+1,\ldots,n-1,n\}$, $P_k=\{a_k\}$.

Then $W_0=\{a_1,a_2,\ldots,a_k\}$ is the unique committee satisfying JR for voting instance (P,k). Further, for any committee $W=\{a'_1,a'_2,\ldots,a'_k\}$ that $|W\cap W_0|=s$, consider the following profile P^W (without loss of generality, suppose that $a_j=a'_j$ for all $1\leqslant j\leqslant s$).

- **Group 1.** $V_1=\{1,2,\ldots,p\}$, $P_1=\{a'_1\}$;
- **Group 2.** $V_2=\{p+1,p+2,\ldots,2p\}$, $P_2=\{a'_2\}$;

......

- **Group *k*.** $V_k=\{(k-1)p+1,\ldots,n-1,n\}$, $P_k=\{a'_k\}$.

Then the unique committee satisfying JR for P^W is W. Further, applying the following permutation σ on A, we have $\sigma\cdot P=P^W$ and $\sigma\cdot W_0=W$.

$$\sigma(a)=\begin{cases} a'_i, & a=a_i\\ a_i, & a=a'_i\\ a, & \text{otherwise.}\end{cases}$$

By neutrality, we have

$$\mathbb{P}[f(P,k)=W_0]=\mathbb{P}[f(P^W,k)=W].$$

Further, since P^W can be obtained by changing the preferences of $V_{s+1}, \dots, V_k$ in P, which contains $p(k-s)$ voters in total, the definition of ϵ-DP indicates that

$$\mathbb{P}[f(P,k)=W] \geqslant \mathrm{e}^{(s-k)p\epsilon} \cdot \mathbb{P}[f(P^W,k)=W], \text{ for all } W.$$

Therefore, for any $W \subseteq A$ that $|W|=k$, we have

$$\begin{aligned}\mathbb{P}[f(P,k)=W] &\leqslant \mathrm{e}^{(|W\cap W_0|-k)\cdot p\epsilon}\mathbb{P}[f(P^W,k)=W]\\ &= \mathrm{e}^{(|W\cap W_0|-k)\cdot p\epsilon}\mathbb{P}[f(P,k)=W_0].\end{aligned}$$

Summing both LHS and RHS of the inequality above for all $W \subseteq A$ that $|W|=k$, we have

$$\sum_{W\subseteq A, |W|=k} \mathrm{e}^{(|W\cap W_0|-k)\cdot p\epsilon} \cdot \mathbb{P}[f(P,k)=W_0] \leqslant 1.$$

Replace the summation variable with the size of intersection,

$$\sum_{W\subseteq A, |W|=k} \mathrm{e}^{(|W\cap W_0|-k)\cdot p\epsilon} = \sum_{s=0}^{k} \binom{k}{s}\binom{m-k}{k-s} \cdot \mathrm{e}^{(s-k)p\epsilon}$$

Then we have

$$\theta \leqslant \mathbb{P}[f(P,k)=W_0] \leqslant \frac{\mathrm{e}^{n\epsilon}}{\sum\limits_{s=0}^{k}\binom{k}{s}\binom{m-k}{k-s}\cdot \mathrm{e}^{sp\epsilon}},$$

which completes the proof. □

Theorem 10 illustrates the upper bound on the achievable level of approximate JR, PJR, and EJR when DP is required. In fact, the same bound also works for approximate PJR+ and EJR+ due to the implication relationship. Further, we propose a mechanism (Mechanism 4, proportionality approval voting with randomized response or PAV-RR for short) to show their lower bounds under ϵ-DP. Technically, Mechanism 4 is designed by applying randomized response mechanism to proportional approval voting (PAV) rule, which selects the committee that maximizes the PAV score, defined as follows.

$$\mathrm{PAV}_P(W) = \sum_{i\in N}\sum_{s=1}^{|W\cap P_i|}\frac{1}{s}.$$

Since the original PAV rule satisfies EJR Aziz et al. [2017], it is evident that our Mechanism 4 can guarantee a lower bound of approximate EJR, and the same applies to JR and PJR. Formally, we have the following proposition.

Mechanism 4: PAV-RR$_\epsilon$

Input: Profile P, Committee Size k

Output: Winning Committee

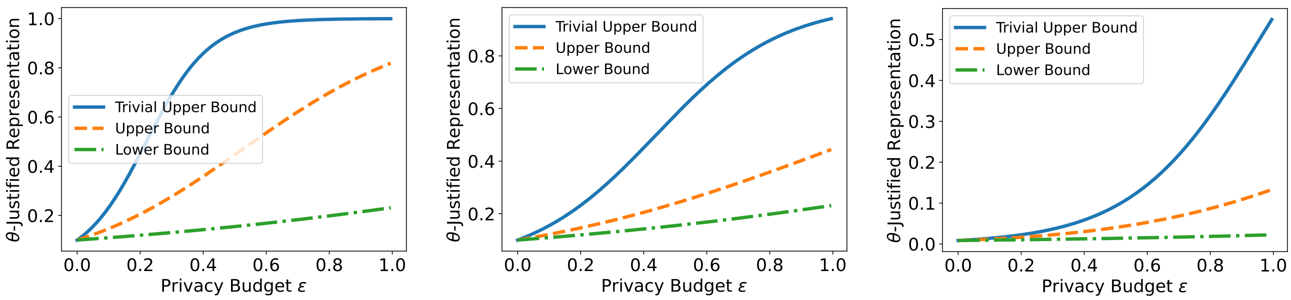
Select a committee W_0 that maximizes the PAV score $\mathrm{PAV}_P(W)$, where $\mathrm{PAV}_P(W) = \sum_{i\in N}\sum_{t=1}^{|P_i\cap W|}\frac{1}{t}$;

Compute the probability distribution $p \in \Delta(\mathcal{P}(A))$, such that $p(W) = \begin{cases} 0, & |W| \neq k \\ \frac{\mathrm{e}^\epsilon}{\mathrm{e}^\epsilon + \binom{m}{k} - 1}, & W = W_0 \\ \frac{1}{\mathrm{e}^\epsilon + \binom{m}{k} - 1}, & \text{otherwise}\end{cases}$;

Sample $W_{win} \sim p$;

return W_{win}

Proposition 4 (JR-Family, Lower Bound). *Given any $\epsilon \in \mathbb{R}_+$, PAV-RR$_\epsilon$ satisfies ϵ-DP and $\frac{\mathrm{e}^\epsilon}{\mathrm{e}^\epsilon+\binom{m}{k}-1}$-EJR+.*



(a) $n = 5, m = 10, k = 3$. (b) $n = 5, m = 5, k = 3$. (c) $n = 10, m = 5, k = 3$.

Figure 4: Tradeoff curves between the probability-based JR-family relaxations and ϵ-DP. The vertical axis is the guaranteed probability that the mechanism output satisfies the corresponding axiom; the settings of n, m, k are shown in the subtitles.

Proof. Let f denote the ABC rule corresponding to the mechanism. First, we prove that f satisfies ϵ-DP. In fact, for any committee W and neighboring profiles P, P' that satisfies $P_i \neq P'_i$, we have

$$\frac{\mathbb{P}[f(P,k) = W]}{\mathbb{P}[f(P',k) = W]} \leqslant \frac{\sup_P \mathbb{P}[f(P,k) = W]}{\inf_P \mathbb{P}[f(P,k) = W]} = \mathrm{e}^{\epsilon},$$

which completes the proof of DP bound.

Next, we prove that f satisfies $\frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon}+\binom{m}{k}-1}$-EJR+. Suppose that W_0 is the committee with highest PAV score selected in the mechanism. Since the original PAV rule satisfies the standard form of EJR+, for any ℓ-cohesive group V, there exists $i \in N$, such that $|P_i \cap W_0| \geqslant \ell$. Therefore,

$$\begin{aligned}\mathbb{P}\left[\exists j \in V, |f(P,k) \cap P_j| \geqslant \ell\right] &\geqslant \mathbb{P}[f(P,k) = W_0] \\ &= \frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon} + \binom{m}{k} - 1},\end{aligned}$$

which indicates that f satisfies $\frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon}+\binom{m}{k}-1}$-EJR. □

The upper and lower bounds of the axioms in JR-family under ϵ-DP are plotted in Figure 4, which visualizes the tradeoff between JR and DP. In the figures, the trivial upper bound refers to the upper bound described in Equation (2).

Remark 3 (Limitations of the mechanism)**.** *It is not hard to see that in Mechanism 4, each k-subset of A is checked. This indicates that the mechanism incurs a time complexity of $\Omega(\binom{m}{k})$, which is exponentially large with respect to k. However, this inefficiency is not introduced by the the mechanism itself, but inherited from the PAV rule. In fact, Skowron et al. [2016] shows that finding the PAV-winner (the committee with the highest PAV score) is NP-hard.*

B MISSING PROOFS IN SECTION 3

B.1 DETAILED DISCUSSIONS ON INCOMPATIBILITY BETWEEN DP AND AXIOMS

First, we prove Lemma 1.

Proof of Lemma 1. Let $W_1 = \{a_1, a_2, \ldots, a_k\}$, $W_2 = \{b_1, \ldots, b_k\}$. Then we can construct a permutation σ on A, such that $\sigma(W_1) = W_2$, $\sigma(W_2) = W_1$, and satisfies $\sigma(a) = a$ for every $a \in A \backslash \{W_1 \cup W_2\}$. By neutrality, we have

$$\mathbb{P}[f(P,k) = W_2] = \mathbb{P}[f(\sigma \cdot P, k) = W_1].$$

Since $W_1 \neq W_2$, there are n different votes in P and $\sigma \cdot P$. Therefore, ϵ-DP indicates that

$$\begin{aligned}\mathbb{P}[f(P,k) = W_1] &\leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(\sigma \cdot P, k) = W_1] \\ &= \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P,k) = W_2],\end{aligned}$$

which completes the proof. □

Further, the following lemma can be regarded as a more general version of Corollary 1, where neutrality is no longer required.

Lemma 3. *If an ABC rule $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfies ϵ-DP for some $\epsilon \in \mathbb{R}_+$, for any voting instances (P, k) and (P', k),*

$$\operatorname{Supp} f(P, k) = \operatorname{Supp} f(P', k).$$

PROOF. Let (P, k) and (P', k) be two voting instances. Since f satisfies ϵ-DP, for any committee $W \in \operatorname{Supp} f(P, k)$, we have

$$\begin{aligned}\mathbb{P}[f(P', k) = W] &\geqslant \mathrm{e}^{-|j \in N : P_j \neq P'_j| \cdot \epsilon} \cdot \mathbb{P}[f(P, k) = W] \\ &\geqslant \mathrm{e}^{-n\epsilon} \cdot \mathbb{P}[f(P, k) = W] \\ &> 0.\end{aligned}$$

In other words, we have $W \in \operatorname{Supp} f(P', k)$, which indicates that $\operatorname{Supp} f(P, k) \subseteq \operatorname{Supp} f(P', k)$. Similarly, we have $\operatorname{Supp} f(P', k) \subseteq \operatorname{Supp} f(P, k)$, which completes the proof. □

In fact, without the assumption of neutrality, we can still obtain the incompatibility between DP and axioms, as shown in the following corollary.

Corollary 2. *If an ABC rule $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ satisfies ϵ-DP for some $\epsilon \in \mathbb{R}_+$, then f does not satisfy JR, PE, or CC.*

PROOF. We only need to prove that any ABC rule satisfying JR, PE, or CC cannot satisfy DP.

Firstly, let f be an ABC rule satisfying JR. Then for any $W, W' \in \mathcal{P}(A, k)$, we can construct two distinct voting instances (P, k) and (P', k), such that W and W' are their unique JR committee, respectively (each consists of k disjoint 1-cohesive groups). Then $\operatorname{Supp} f(P, k) = \{W\} \neq \{W'\} \operatorname{Supp} f(P', k)$, which indicates that f does not satisfy ϵ-DP for any ϵ.

Secondly, let f be an ABC rule satisfying PE. Then for any committee $W \in \mathcal{P}(A, k)$, considering the voting instance (P, k), where $P_j = A \backslash W$, we have $\mathbb{P}[f(P, k) = W] = 0$, since W must be Pareto dominated by another committee. In other words, we have $\mathbb{P}[f(P, k) = W] = 0$ for all $W \in \mathcal{P}(A, k)$, a contradiction.

Finally, for CC, it is easy to construct two voting instance (P, k) and (P', k) with different Condorcet committees W and W'. Then $\operatorname{Supp} f(P, k) \neq \operatorname{Supp} f(P, k)$. By Lemma 3, f does not satisfy ϵ-DP for any ϵ, which completes the proof. □

B.2 MISSING PROOFS IN SUBSECTION 3.1

Lemma 1 has been proved in the previous subsection (Appendix A.1).

PROOF OF THEOREM 1. Suppose $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP and θ-JR. Consider the following voting instances (P, k) and (P', k), where $s = \lceil n/k \rceil$:

$$P_j = \begin{cases} \{a_1\}, & 1 \leqslant j \leqslant s, \\ \{a_2\}, & s+1 \leqslant j \leqslant 2s-1, \\ A \backslash \{a_1, a_2\}, & \text{otherwise.} \end{cases}$$

$$P'_j = \begin{cases} \{a_1\}, & 1 \leqslant j \leqslant s-1, \\ \{a_2\}, & s \leqslant j \leqslant 2s-1, \\ A \backslash \{a_1, a_2\}, & \text{otherwise.} \end{cases}$$

It is not hard to see that $P_s \neq P'_s$ and $P_{-s} = P'_{-s}$, i.e., P and P' are neighboring profiles. Further, voters $\{1, 2, \ldots, s\}$ and $\{s, s+1, \ldots, 2s-1\}$ form 1-cohesive groups for (P, k) and (P', k), respectively. Therefore, committee $W = \{a_1, a_3, a_4, \ldots, a_{k+1}\}$ satisfies JR for (P, k), but does not satisfy JR for (P', k). Similarly, $W' = \{a_2, a_3, a_4, \ldots, a_{k+1}\}$

satisfies JR for (P', k), but does not satisfy JR for (P, k). Then we have

$$\begin{aligned}
\mathbb{P}[f(P,k)=W] &\geqslant \theta \cdot \mathbb{P}[f(P,k)=W'] && (\theta\text{-JR})\\
&\geqslant \theta \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k)=W'] && (\epsilon\text{-DP})\\
&\geqslant \theta^2 \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k)=W] && (\theta\text{-JR})\\
&\geqslant \theta^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k)=W]. && (\epsilon\text{-DP})
\end{aligned}$$

In other words, $\theta^2 \cdot \mathrm{e}^{-2\epsilon} \leqslant 1$, i.e., $\theta \leqslant \mathrm{e}^{\epsilon}$, which completes the proof. □

PROOF OF THEOREM 2. Suppose $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP and ρ-PJR. Consider the following voting instances (P, k) and (P', k), where $n = s \cdot k$:

$$P_j = \begin{cases} \{a_1, a_{k+1}\}, & j = 1,\\ \{a_1\}, & 2 \leqslant j \leqslant s,\\ \{a_2\}, & s+1 \leqslant j \leqslant 2s,\\ \cdots & \\ \{a_k\}, & m-k+1 \leqslant j \leqslant m. \end{cases}$$

$$P'_j = \begin{cases} \{a_1, a_{k+2}\}, & j = 1,\\ \{a_1\}, & 2 \leqslant j \leqslant s,\\ \{a_2\}, & s+1 \leqslant j \leqslant 2s,\\ \cdots & \\ \{a_k\}, & m-k+1 \leqslant j \leqslant m. \end{cases}$$

It is quite evident that P and P' are neighboring profiles, since they only differ on the first voter's vote. Further, we claim that $W = \{a_2, a_3, \ldots, a_k, a_{k+1}\}$ and $W' = \{a_2, a_3, \ldots, a_k, a_{k+2}\}$ satisfies PJR for P and P', respectively. Besides, it is not hard to check that $W \notin \text{PJR}(P', k)$ and $W' \notin \text{PJR}(P, k)$. Therefore,

$$\begin{aligned}
\mathbb{P}[f(P,k)=W] &\geqslant \rho \cdot \mathbb{P}[f(P,k)=W'] && (\rho\text{-PJR})\\
&\geqslant \rho \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k)=W'] && (\epsilon\text{-DP})\\
&\geqslant \rho^2 \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k)=W] && (\rho\text{-PJR})\\
&\geqslant \rho^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k)=W], && (\epsilon\text{-DP})
\end{aligned}$$

which indicates that $\rho^2 \cdot \mathrm{e}^{-2\epsilon} \leqslant 1$, i.e., $\rho \leqslant \mathrm{e}^{\epsilon}$. That completes the proof. □

PROOF OF THEOREM 3. Suppose $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP and κ-EJR. Consider the following voting instances (P, k) and (P', k), where $s = \lceil n/k \rceil$:

$$P_j = \begin{cases} \{a_1\}, & 1 \leqslant j \leqslant s,\\ \{a_2\}, & s+1 \leqslant j \leqslant 2s\\ \cdots & \\ \{a_k\}, & (k-1)s+1 \leqslant j \leqslant m. \end{cases}$$

$$P'_j = \begin{cases} \{a_{k+1}\}, & 1 \leqslant j \leqslant s,\\ \{a_2\}, & s+1 \leqslant j \leqslant 2s\\ \cdots & \\ \{a_k\}, & (k-1)s+1 \leqslant j \leqslant m. \end{cases}$$

Without loss of generality, suppose $n = s \cdot k$. Then for each $t \in \{0, 1, \ldots, m-1\}$, the set of voters $\{ts+1, ts+2, \ldots, (t+1)s\}$ forms a 1-cohesive group in both P and P'. Therefore, $W = \{a_1, a_2, \ldots, a_k\}$ and $W' = \{a_{k+1}, a_2, a_3, \ldots, a_k\}$ are the unique JR committee for (P, k) and (P', k), respectively. Noting that there is no ℓ-cohesive group ($\ell \geqslant 2$) for either P or P', W and W' are the unique EJR committee for (P, k) and (P', k), respectively. Further, since f satisfies κ-EJR, we have

$$\begin{aligned}
\mathbb{P}[f(P,k)=W] &\geqslant \kappa \cdot \mathbb{P}[f(P,k)=W'], \text{ and}\\
\mathbb{P}[f(P',k)=W'] &\geqslant \kappa \cdot \mathbb{P}[f(P',k)=W].
\end{aligned} \tag{3}$$

On the other hand, P and P' differ on s voters' vote. By ϵ-DP, for all $X \in \mathcal{P}(A,k)$

$$\begin{aligned} \mathrm{e}^{-s\epsilon} \cdot \mathbb{P}[f(P',k)=X] &\leqslant \mathbb{P}[f(P,k)=X] \\ &\leqslant \mathrm{e}^{s\epsilon} \cdot \mathbb{P}[f(P',k)=X]. \end{aligned} \tag{4}$$

Further, we have

$$\begin{aligned} \mathbb{P}[f(P,k)=W] &\geqslant \kappa \cdot \mathbb{P}[f(P,k)=W'] && \text{(By Equation (3))} \\ &\geqslant \kappa \cdot \mathrm{e}^{-\lceil \frac{n}{k} \rceil \cdot \epsilon} \cdot \mathbb{P}[f(P',k)=W'] && \text{(By Equation (4))} \\ &\geqslant \kappa^2 \cdot \mathrm{e}^{-\lceil \frac{n}{k} \rceil \cdot \epsilon} \cdot \mathbb{P}[f(P',k)=W] && \text{(By Equation (3))} \\ &\geqslant \kappa^2 \cdot \mathrm{e}^{-2\lceil \frac{n}{k} \rceil \cdot \epsilon} \cdot \mathbb{P}[f(P,k)=W]. && \text{(By Equation (4))} \end{aligned}$$

Therefore, $\kappa^2 \cdot \mathrm{e}^{-2\lceil \frac{n}{k} \rceil \cdot \epsilon} \leqslant 1$, i.e., $\kappa \leqslant \mathrm{e}^{\lceil \frac{n}{k} \rceil \cdot \epsilon}$, which completes the proof. □

PROOF OF PROPOSITION 1. We only prove for JR here, proofs for the cases when $\mathcal{A} = \text{PJR}/\text{EJR}$ can be obtained in the same way. Let $\mathfrak{R}_{\text{JR}} \colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ denote the ABC rule induced by Mechanism 1 with axiom $\mathcal{A} = \text{JR}$ and noise level $\epsilon \in \mathbb{R}_+$. For any voting instance (P,k), supposing $t = |\text{JR}(P,k)|$, the winning probability of each committee $W \in \mathcal{P}(A,k)$ satisfies

$$\mathbb{P}[f(P,k)=W] = \begin{cases} \frac{\mathrm{e}^{\epsilon/2}}{t \cdot \mathrm{e}^{\epsilon/2} + \binom{m}{k} - t}, & W \in \text{JR}(P,k), \\ \frac{1}{t \cdot \mathrm{e}^{\epsilon/2} + \binom{m}{k} - t}, & \text{otherwise.} \end{cases}$$

Then, for any neighboring profile P' of P (let $t' = |\text{JR}(P',k)|$) and committee $W \in \mathcal{P}(A,k)$,

$$\begin{aligned} \frac{\mathbb{P}[f(P,k)=W]}{\mathbb{P}[f(P',k)=W]} &\leqslant \frac{\mathrm{e}^{\epsilon/2}}{t \cdot \mathrm{e}^{\epsilon/2} + \binom{m}{k} - t} \Big/ \frac{1}{t' \cdot \mathrm{e}^{\epsilon/2} + \binom{m}{k} - t'} \\ &= \mathrm{e}^{\epsilon/2} \cdot \frac{t' \cdot \mathrm{e}^{\epsilon/2} + \binom{m}{k} - t'}{t \cdot \mathrm{e}^{\epsilon/2} + \binom{m}{k} - t} \\ &\leqslant \mathrm{e}^{\epsilon/2} \cdot \frac{\mathrm{e}^{\epsilon/2} \cdot \binom{m}{k}}{\binom{m}{k}} \\ &= \mathrm{e}^{\epsilon}, \end{aligned}$$

which indicates that $\mathfrak{R}_{\text{JR}}$ satisfies ϵ-DP. Further, according to the definition of Mechanism 1, it is easy to check that for any voting instance (P,k) and committees $W \in \text{JR}(P,k)$ and $W' \notin \text{JR}(P,k)$,

$$\frac{\mathbb{P}[f(P,k)=W]}{\mathbb{P}[f(P,k)=W']} = \frac{\mathrm{e}^{\chi(W)\cdot\epsilon/2}}{\mathrm{e}^{\chi(W)\cdot\epsilon/2}} = \mathrm{e}^{\epsilon/2},$$

i.e., $\mathfrak{R}_{\text{JR}}$ satisfies $\mathrm{e}^{\epsilon/2}$-JR, which completes the proof. □

B.3 MISSING PROOFS IN SUBSECTION 3.2

PROOF OF THEOREM 4. Let $f \colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP and β-PE. Consider the following voting instance $E = (P,k)$, where $m \geqslant n + 2k - 1$ and $\{a_1, \ldots, a_k, b_1, \ldots, b_{n-1}, c_1, \ldots, c_k\} \subseteq A$.

$$P_j = \begin{cases} \{a_1, a_2, \ldots, a_k\}, & j = 1, \\ \{a_1, a_2, \ldots, a_k, b_1\}, & j = 2, \\ \cdots & \\ \{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_{n-1}\}, & j = n. \end{cases}$$

Then we can construct the following committees. To be more concise, we use $W_{p,q}$ to denote the committee on p-th row and

$$
\begin{array}{llll}
\{a_1,\dots,a_k\} & \{a_1,\dots,a_{k-1},b_1\} & \cdots & \{a_1,\dots,a_{k-1},b_{n-1}\} \\
\{a_1,\dots,a_{k-1},c_1\} & \{a_1,\dots,a_{k-2},b_1,c_1\} & \cdots & \{a_1,\dots,a_{k-2},b_{n-1},c_1\} \\
\vdots & \vdots & & \vdots \\
\{a_1,c_1,\dots,c_{k-1}\} & \{b_1,c_1,\dots,c_{k-1}\} & \cdots & \{b_{n-1},c_1,\dots,c_{k-1}\} \\
\{c_1,\dots,c_k\} & & &
\end{array}
$$

q-th column in the matrix above. Then for all $1 \leqslant p \leqslant k$ and $1 \leqslant q \leqslant n$, we have

$$
W_{p,q} = \begin{cases} \{a_1,\dots,a_{k-p+1},c_1,\dots,c_{p-1}\}, & q=1, \\ \{a_1,\dots,a_{k-p},b_{q-1},c_1,\dots,c_{p-1}\}, & q>1. \end{cases}
$$

For the sake of simplicity, we use $\Omega_P(W)$ to denote the series consisting of $|W \cap P_j|$ for each voter $j \in N$, i.e., $\Omega_P \colon \mathcal{P}(A,k) \to \mathbb{N}^n,\ W \mapsto (|W \cap P_1|, |W \cap P_2|, \dots, |W \cap P_n|)$. Then it is easy to check that $\Omega_P(W_{p,q})$ forms the Table 1. In other words, we have

Table 1: Table of $\boldsymbol{\Omega(W_{p,q})}$

$(k,k,\dots,k)$	$(k-1,k,\dots,k)$	$\cdots$	$(k-1,\dots,k-1,k)$
$(k-1,k-1,\dots,k-1)$	$(k-2,k-1,\dots,k-1)$	$\cdots$	$(k-2,\dots,k-2,k-1)$
$\vdots$	$\vdots$		$\vdots$
$(1,1,\dots,1)$	$(0,1,\dots,1)$	$\cdots$	$(0,\dots,0,1)$
$(0,0,\dots,0)$			

$$
\Omega_P(W_{p,q}) = (\overbrace{k-p,\dots,k-p}^{q-1},\underbrace{k-p+1,\dots,k-p+1}_{n-q+1}).
$$

Then, according to the definition of Pareto-dominance, we have W_{p_1,q_1} Pareto dominates W_{p_2,q_2} if $p_1 < p_2$ or $p_1 = p_2$ and $q_1 < q_2$. The relationship of Pareto-dominance among $W_{p,q}$ can be visualized as the following diagram, where $X \to Y$ indicates that X Pareto-dominates Y.

$$
\begin{array}{ccccccccc}
W_{1,1} & \longrightarrow & W_{1,2} & \longrightarrow & \cdots & \longrightarrow & W_{1,n} & & \\
 & & & & & & \downarrow & & \\
W_{2,n} & \longleftarrow & W_{2,n-1} & \longleftarrow & \cdots & \longleftarrow & W_{2,1} & & \\
\downarrow & & & & & & & & \\
\vdots & & & & & & & & \\
\downarrow & & & & & & & & \\
W_{k,1} & \longrightarrow & W_{k,2} & \longrightarrow & \cdots & \longrightarrow & W_{k,n} & \longrightarrow & W_{k+1,1}
\end{array}
$$

There is totally nk arrows in the diagram. Since f satisfies β-PE, we have

$$
\mathbb{P}[f(P,k) = W_{1,1}] \geqslant \beta^{nk} \cdot \mathbb{P}[f(P,k) = W_{k+1,1}].
$$

However, since f satisfies ϵ-DP, Lemma 1 indicates that

$$
\mathbb{P}[f(P,k) = W_{1,1}] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P,k) = W_{k+1,1}].
$$

In other words, we have $\beta^{nk} \leqslant \mathrm{e}^{n\epsilon}$, i.e., $\beta \leqslant \mathrm{e}^{\epsilon/k}$, which completes the proof. □

Proof of Proposition 2. Let f denote the ABC rule corresponding to Mechanism 2. Then, for any committee $W \in \mathcal{P}(A,k)$, we have

$$
\begin{aligned}
\mathbb{P}[f(P,k) = W] &\propto \prod_{a \in W} \mathrm{e}^{\chi(a)\cdot\epsilon/(2k)} \\
&= \mathrm{e}^{\sum \chi(a) \cdot \epsilon/(2k)} \\
&= \mathrm{e}^{\mathrm{AV}_P(W)\cdot\epsilon/(2k)}.
\end{aligned}
$$

First, we prove f satisfies ϵ-DP. Suppose P and P' are a pair of neighboring profiles, satisfying $P_i \neq P_i'$ and $P_{-i} = P'_{-i}$. Then the scores of any committee W in P and P' satisfy

$$\mathrm{AV}_P(W) - \mathrm{AV}_{P'} \leqslant \mathrm{AV}_{P_i}(W) - \mathrm{AV}_{P_i'}(W) \leqslant k.$$

Therefore, we have

$$\frac{\mathbb{P}[f(P,k) = W]}{\mathbb{P}[f(P',k) = W]} = \frac{\mathrm{e}^{\mathrm{AV}_P(W)\cdot\epsilon/(2k)}}{\mathrm{e}^{\mathrm{AV}_{P'}(W)\cdot\epsilon/(2k)}} \leqslant \mathrm{e}^{\epsilon},$$

which indicates that f satisfies ϵ-DP.

Next, we prove the Pareto bound of f. Suppose that in profile P, committee W Pareto dominates W'. Then we have

- for all $j \in N$, $|P_j \cap W| - |P_j \cap W'| \geqslant 0$;
- for at least one $j \in N$, $|P_j \cap W| - |P_j \cap W'| \geqslant 1$.

Therefore, their scores satisfy

$$\mathrm{AV}_P(W) - \mathrm{AV}_P(W') = \sum_{j \in N} \mathrm{AV}_{P_j}(W) - \mathrm{AV}_{P_j}(W') \geqslant 1.$$

Then we have

$$\frac{\mathbb{P}[f(P,k) = W]}{\mathbb{P}[f(P,k) = W']} \geqslant \mathrm{e}^{\frac{\epsilon}{2k}},$$

which indicates that f satisfies $\mathrm{e}^{\frac{\epsilon}{2k}}$-Pareto efficiency. That completes the proof. □

PROOF OF THEOREM 5. Let $f \colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP and η-CC. Consdier the voting instances (P,k) and (P',k), where $n = 2t+1$ and there exists a $W \in \mathcal{P}(A, k-1)$ that

$$P_j = \begin{cases} W \cup \{a_1\}, & 1 \leqslant j \leqslant t+1, \\ W \cup \{a_2\}, & t+2 \leqslant j \leqslant 2t+1. \end{cases}$$

$$P_j' = \begin{cases} W \cup \{a_1\}, & 1 \leqslant j \leqslant t, \\ W \cup \{a_2\}, & t+1 \leqslant j \leqslant 2t+1. \end{cases}$$

Then P and P' are neighboring profiles, since they only differ on the $(t+1)$-th voter's vote. Further, according to the definition, $W \cup \{a_1\}$ is the Condorcet committee in (P,k), since it is approved by $t+1$ out of $2t+1$ voters (more than a half). Similarly, $W \cup \{a_2\}$ is the Condorcet committee in (P',k). Further, we have

$$\begin{aligned} \mathbb{P}[f(P,k) = W \cup \{a_1\}] &\geqslant \eta \cdot \mathbb{P}[f(P,k) = W \cup \{a_2\}] \\ &\geqslant \mathrm{e}^{-\epsilon} \cdot \eta \cdot \mathbb{P}[f(P',k) = W \cup \{a_2\}] \\ &\geqslant \mathrm{e}^{-\epsilon} \cdot \eta^2 \cdot \mathbb{P}[f(P',k) = W \cup \{a_1\}] \\ &\geqslant \mathrm{e}^{-2\epsilon} \cdot \eta^2 \cdot \mathbb{P}[f(P,k) = W \cup \{a_1\}]. \end{aligned}$$

In other words, $\mathrm{e}^{-2\epsilon} \cdot \eta^2 \leqslant 1$, i.e., $\eta \leqslant \mathrm{e}^{\epsilon}$, which completes the proof. □

PROOF OF PROPOSITION 3. Let f denote the ABC rule corresponding to Mechanism 3. First, we prove that f satisfies ϵ-DP. In fact, given any neighboring voting instances (P,k) and (P',k), for each committee $W \in \mathcal{P}(A,k)$, we have

$$\frac{\mathbb{P}[f(P,k) = W]}{\mathbb{P}[f(P',k) = W]} \leqslant \frac{\sup_W \mathbb{P}[f(P,k) = W]}{\inf_W \mathbb{P}[f(P,k) = W]} = \mathrm{e}^{\epsilon},$$

which indicates that f satisfies ϵ-DP.

Next, we prove that f satisfies e^{ϵ}-Condorcet criterion. Let $P \in \mathcal{P}(A)^n$ be a profile, where exists a Condorcet committee W_c. Then for any $W \in \mathcal{P}(A,k) \backslash \{W_c\}$,

$$\frac{\mathbb{P}[f(P,k) = W_c]}{\mathbb{P}[f(P,k) = W]} = \frac{\mathrm{e}^{\epsilon}}{\mathrm{e}^{\epsilon} + \binom{m}{k} - 1} \Big/ \frac{1}{\mathrm{e}^{\epsilon} + \binom{m}{k} - 1} = \mathrm{e}^{\epsilon}.$$

In other words, f satisfies e^{ϵ}-Condorcet criterion, which completes the proof. □

C MISSING PROOFS IN SECTION 4

C.1 THE COMPATIBILITY BETWEEN AXIOMS WITHOUT DP

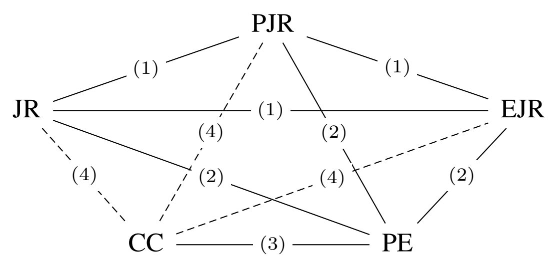


Figure 5: Compatibility between axioms, where a solid line indicates that the axioms are compatible, while a dash line indicates that the axioms are incompatible.

First of all, the compatibility among JR-family (denoted by (1)) is evident, since there are implication relationships among them. The correctness of (2) is ensured by the proportional approval voting (PAV) rule, which satisfies PE and EJR [Aziz et al., 2017]. Further, (3) is correct since the Condorcet committee is never Pareto-dominated. Finally, CC and JR are incompatible (denoted by (4)) since the Condorcet committee does not necessarily satisfy JR. For example, consider the following voting instance (P, k), where $n = 2t + 1$.

$$P_j = \begin{cases} \{a_1, a_2, \ldots, a_k\}, & 1 \leqslant j \leqslant t+1, \\ \{a_{k+1}, a_{k+2}, \ldots, a_{2k}\}, & t+2 \leqslant 2t+1. \end{cases}$$

When $k \geqslant 3$, both $\{1, 2, \ldots, t+1\}$ and $\{t+2, t+3, \ldots, 2t+1\}$ are 1-cohesive for (P, k). However, the Condorcet committee of this voting instance is $\{a_1, a_2, \ldots, a_k\}$, which does not offer any seat to $\{t+2, t+3, \ldots, 2t+1\}$. Therefore, CC is incompatible with JR (and PJR/EJR).

C.2 THE MISSING PROOFS

Proof of Theorem 6. First, we prove (1). Let $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP, θ-JR, and ρ-PJR. Consider the following neighboring voting instances (P, k) and (P', k), where $s = \lceil 2n/k \rceil$:

$$P_j = \begin{cases} \{a_1, a_2\}, & 1 \leqslant j \leqslant s, \\ \{a_1, a_3\}, & s+1 \leqslant j \leqslant 2s-1, \\ \{a_4, a_5\}, & 2s \leqslant j \leqslant 3s-1, \\ \{a_4\}, & \text{otherwise}. \end{cases}$$

$$P'_j = \begin{cases} \{a_1, a_2\}, & 1 \leqslant j \leqslant s-1, \\ \{a_1, a_3\}, & s \leqslant j \leqslant 2s-1, \\ \{a_4, a_5\}, & 2s \leqslant j \leqslant 3s-1, \\ \{a_4\}, & \text{otherwise}. \end{cases}$$

By the definition of cohesive groups, it is easy to check that $V = \{1, 2, \ldots, s\}$ and $V' = \{s, s+1, \ldots, 2s-1\}$ are 2-cohesive for (P, k) and (P', k), respectively. For both P and P', $\{2s, 2s+1, \ldots, 3s-1\}$ forms a 2-cohesive group, while $\{1, 2, \ldots, 2s-1\}$ and $\{2s, 2s+1, \ldots, n\}$ are 1-cohesive. Further, there is no ℓ-cohesive group with $\ell \geqslant 3$ for both voting instances.

Therefore, $W_1 = \{a_1, a_2, a_4, a_5, a_6, \ldots, a_{k+1}\}$ satisfies PJR for (P, k), but does not satisfy even JR for (P', k), since it offers only one seat for the 2-cohesive group V'. Similarly, $W'_1 = \{a_1, a_3, a_4, a_5, a_6, \ldots, a_{k+1}\}$ satisfies PJR for (P', k), but does not satisfy JR for (P, k). Besides, $W_0 = \{a_1, a_4, a_6, a_7, \ldots, a_{k+3}\}$ satisfies JR, but does not satisfy PJR for both voting instances.

Since f satisfies ϵ-DP, θ-JR, and ρ-PJR, we have

$$\begin{aligned}
\mathbb{P}[f(P,k) = W_1] &\geqslant \rho \cdot \mathbb{P}[f(P,k) = W_0] && (\rho\text{-PJR})\\
&\geqslant \rho \cdot \theta \cdot \mathbb{P}[f(P,k) = W_1'] && (\theta\text{-JR})\\
&\geqslant \rho \cdot \theta \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k) = W_1'] && (\epsilon\text{-DP})\\
&\geqslant \rho^2 \cdot \theta \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k) = W_0] && (\rho\text{-PJR})\\
&\geqslant \rho^2 \cdot \theta^2 \cdot \mathrm{e}^{-\epsilon} \cdot \mathbb{P}[f(P',k) = W_1] && (\theta\text{-JR})\\
&\geqslant \rho^2 \cdot \theta^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k) = W_1]. && (\epsilon\text{-DP})
\end{aligned}$$

In other words, we have $\rho^2 \cdot \theta^2 \cdot \mathrm{e}^{-2\epsilon} \leqslant 1$, i.e., $\rho \cdot \theta \leqslant \mathrm{e}^{\epsilon}$, which completes the proof of (1).

For (2), we still consider the two voting instances defined above. In fact, W_1 and W_1' satisfy EJR for (P,k) and (P',k), respectively. Therefore, by comparing the winning probability of W_0, W_1, and W_1', we will also end up with

$$\mathbb{P}[f(P,k) = W_1] \leqslant \kappa^2 \cdot \theta^2 \cdot \mathrm{e}^{-2\epsilon} \cdot \mathbb{P}[f(P,k) = W_1].$$

In other words, we have $\kappa \cdot \theta \leqslant \mathrm{e}^{\epsilon}$, which completes the proof. □

PROOF OF THEOREM 7. Let $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be an ABC rule satisfying ϵ-DP, ρ-PJR, and κ-EJR. Consider the following voting instance (P,k), where $s = \lceil n/k \rceil$:

$$P_j = \begin{cases} \{a_1, a_2, \ldots, a_k, a_{k+1}\}, & 1 \leqslant j \leqslant s,\\ \{a_1, a_2, \ldots, a_k, a_{k+2}\}, & s+1 \leqslant j \leqslant 2s,\\ \cdots & \\ \{a_1, a_2, \ldots, a_k, a_{2k}\}, & (k-1)s+1 \leqslant j \leqslant n. \end{cases}$$

Notice that $\{a_1, a_2, \ldots, a_k\}$ is approved by every voter $j \in N$. As a consequence, for all $\ell \in \{1, 2, \ldots, k\}$, the set of voters $V \subseteq N$ forms an ℓ-cohesive group if and only if $|V| \geqslant \ell \cdot n/k$. Further, it is not hard to check correctness the following statements for (P,k).

1. $W_1 = \{a_1, a_2, \ldots, a_k\}$ satisfies EJR;
2. $W_0 = \{a_{k+1}, a_{k+2}, \ldots, a_{2k}\}$ satisfies PJR, but fails to satisfy EJR.

Now, consider another voting instance (P',k), where

$$P_j' = \begin{cases} \{a_{k+1}, a_{2k+1}, a_{2k+2}, \ldots, a_{3k-1}\}, & 1 \leqslant j \leqslant s,\\ P_j, & \text{otherwise}. \end{cases}$$

Then $\{1, 2, \ldots, s\}$ forms a 1-cohesive group, and any $V \subseteq \{s+1, s+2, \ldots, n\}$ is ℓ-cohesive if and only if $|V| \geqslant \ell \cdot n/k$. Therefore, the following statements are true.

1. $W_1' = \{a_1, a_2, \ldots, a_{k-1}, a_{2k+1}\}$ satisfies EJR for (P',k);
2. W_0 also satisfies PJR for (P',k), but fails to satisfy EJR.

Further, W_1 does not satisfy JR for (P,k), since it does not offer any seat for the 1-cohesive group $\{1, 2, \ldots, s\}$. Similarly, W_1' does not satsify JR for (P,k). Notice that P and P' only differ on the first s voter's vote. Therefore, for any $W \in \mathcal{P}(A,k)$,

$$\mathbb{P}[f(P,k) = W] \leqslant \mathrm{e}^{s\epsilon} \cdot \mathbb{P}[f(P',k) = W].$$

Since f satisfies ϵ-DP, ρ-PJR, and κ-EJR, we have

$$\begin{aligned}
\mathbb{P}[f(P,k) = W_1] &\geqslant \kappa \cdot \mathbb{P}[f(P,k) = W_0] && (\kappa\text{-EJR})\\
&\geqslant \kappa \cdot \rho \cdot \mathbb{P}[f(P,k) = W_1'] && (\rho\text{-PJR})\\
&\geqslant \kappa \cdot \rho \cdot \mathrm{e}^{-s\epsilon} \cdot \mathbb{P}[f(P',k) = W_1'] && (\epsilon\text{-DP})\\
&\geqslant \kappa^2 \cdot \rho \cdot \mathrm{e}^{-s\epsilon} \cdot \mathbb{P}[f(P',k) = W_0] && (\kappa\text{-EJR})\\
&\geqslant \kappa^2 \cdot \rho^2 \cdot \mathrm{e}^{-s\epsilon} \cdot \mathbb{P}[f(P',k) = W_1] && (\rho\text{-PJR})\\
&\geqslant \kappa^2 \cdot \rho^2 \cdot \mathrm{e}^{-2s\epsilon} \cdot \mathbb{P}[f(P,k) = W_1]. && (\epsilon\text{-DP})
\end{aligned}$$

In other words, we have $\kappa \cdot \rho \leqslant \mathrm{e}^{\lceil n/k \rceil \cdot \epsilon}$, which completes the proof. $\square$

Proof of Theorem 8. First, we prove (1). Let $f\colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be a neutral ABC rule satisfying ϵ-DP, β-PE, and θ-JR. Consider the following voting instance $E = (P, k)$, where $m \geqslant n + 2k - 1$ and $\{a_1, \dots, a_k, b_1, \dots, b_{n-1}, c_1, \dots, c_k\} \subseteq A$.

$$P_j = \begin{cases} \{a_1, a_2, \dots, a_k\}, & j = 1, \\ \{a_1, a_2, \dots, a_k, b_1\}, & j = 2, \\ \dots & \\ \{a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_{n-1}\}, & j = n. \end{cases}$$

Then we can construct the following committees, where we use $W_{p,q}$ to denote the committee on p-th row and q-th column in the table. Notice that the voting instance and the table of committees are exactly the same as those in the proof of Theorem

$$\begin{array}{llll}
\{a_1, \dots, a_k\} & \{a_1, \dots, a_{k-1}, b_1\} & \cdots & \{a_1, \dots, a_{k-1}, b_{n-1}\} \\
\{a_1, \dots, a_{k-1}, c_1\} & \{a_1, \dots, a_{k-2}, b_1, c_1\} & \cdots & \{a_1, \dots, a_{k-2}, b_{n-1}, c_1\} \\
\vdots & \vdots & & \vdots \\
\{a_1, c_1, \dots, c_{k-1}\} & \{b_1, c_1, \dots, c_{k-1}\} & \cdots & \{b_{n-1}, c_1, \dots, c_{k-1}\} \\
\{c_1, \dots, c_k\} & & &
\end{array}$$

4. Then the relationship of Pareto-dominance among $W_{p,q}$ can be visualized as the same diagram, where the arrows are labeled from 1 to kn.

$$\begin{tikzcd}
W_{1,1} \arrow[r, "1"] & W_{1,2} \arrow[r, "2"] & \cdots \arrow[r, "n-1"] & W_{1,n} \arrow[d, "n"] \\
W_{2,n} \arrow[d, "2n"] & W_{2,n-1} \arrow[l, "2n-1"'] & \cdots \arrow[l, "2n-2"'] & W_{2,1} \arrow[l, "n+1"'] \\
\vdots \arrow[d, "(k-1)n"] & & & \\
W_{k,1} \arrow[r, "(k-1)n+1"] & W_{k,2} \arrow[r, "(k-1)n+2"] & \cdots \arrow[r, "kn-1"] & W_{k,n} \arrow[d, "kn"] \\
 & & & W_{k+1,1}
\end{tikzcd}$$

Since $a_1, a_2, \dots, a_k$ are approved by all voters, the group of voters $V \subseteq N$ is ℓ-cohesive if and only if $|V| \geqslant \ell \cdot n/k$, for any ℓ. Further, according to the definition of JR, it is easy to check that

$$W_{1,1} \in \mathrm{JR}(P, k) \quad \text{and} \quad W_{k+1,1} \notin \mathrm{JR}(P, k).$$

Therefore, there must exist an arrow $\xrightarrow{t}$ in the diagram, connecting a JR committee and a non-JR committee, i.e., there exists a pair of adjacent committees X, Y in the diagram, satisfying $X \in \mathrm{JR}(P, k)$ and $Y \notin \mathrm{JR}(P, k)$. Then we have

$$\mathbb{P}[f(P, k) = X] \geqslant \theta \cdot \mathbb{P}[f(P, k) = Y].$$

Since there are totally $t - 1$ arrows between $W_{1,1}$ and X, we have

$$\mathbb{P}[f(P, k) = W_{1,1}] \geqslant \beta^{t-1} \cdot \mathbb{P}[f(P, k) = X].$$

Similarly, there are $kn - t$ arrows between Y and $W_{k+1,1}$, which indicates that

$$\mathbb{P}[f(P, k) = Y] \geqslant \beta^{kn-t} \cdot \mathbb{P}[f(P, k) = W_{k+1,1}].$$

According to the three inequalities above, we have

$$\mathbb{P}[f(P, k) = W_{1,1}] \geqslant \beta^{kn-1} \cdot \theta \cdot \mathbb{P}[f(P, k) = W_{k+1,1}]. \tag{5}$$

However, since f satisfies ϵ-DP, Lemma 1 indicates that

$$\mathbb{P}[f(P,k) = W_{1,1}] \leqslant \mathrm{e}^{n\epsilon} \cdot \mathbb{P}[f(P,k) = W_{k+1,1}].$$

In other words, we have $\beta^{nk-1} \cdot \theta \leqslant \mathrm{e}^{n\epsilon}$, which completes the proof of (1). For (2) and (3), we still consider the voting instances and committees defined above. Then it is easy to check that $W_{1,1}$ also satisfies PJR and EJR for (P, k). Therefore, the inequality (5) also holds when θ-JR is replaced by ρ-PJR or κ-EJR. That completes the proof. □

Proof of Theorem 9. Let $f \colon \mathcal{P}(A)^n \times \mathbb{N}_+ \to \mathcal{R}(\mathcal{P}(A))$ be a neutral ABC rule satisfying ϵ-DP, β-PE, and η-CC. We still consider the voting instance (P, k) and the committees $\{W_{p,q}\}$ defined in the proof of Theorem 4. Then it is easy to check that the Condorcet committee for (P, k) is $W_{1,1}$. Therefore,

$$\begin{aligned}\mathbb{P}[f(P,k) = W_{1,1}] &\geqslant \eta \cdot \mathbb{P}[f(P,k) = W_{1,2}] \\ &\geqslant \eta \cdot \beta^{nk-1} \cdot \mathbb{P}[f(P,k) = W_{k+1,1}].\end{aligned}$$

Then, by Lemma 1, we have $\eta \cdot \beta^{nk-1} \leqslant \mathrm{e}^{n\epsilon}$, which completes the proof. □

D ADDITIONAL EXPERIMENTAL DETAILS

The PrefLib [Mattei and Walsh, 2013] experiments in Section 3.4 use 22 approval elections from three datasets: 00026, the 2002 French Presidential Election with 6 districts, $m = 16$, and $n \in [365, 476]$; 00071, the Voter Autrement in situ elections with 14 elections, $m \in [10, 12]$, and $n \in [233, 1321]$; and 00073, the Voter Autrement online elections with 2 elections, $m \in [11, 12]$, and $n \in [1379, 20076]$. Each instance is tested with $k \in \{3, 5\}$ and $\epsilon \in \{0.1, 0.5, 1, 2, 5, 10\}$. For comparison, the Random sampling baseline in Table 2 samples a committee uniformly at random from all size-k committees $\mathcal{P}(A, k)$, independently of the profile. For the Polkadot experiments, we use the validator-election data of Boehmer et al. [2024], whose instances are indexed by election eras. The labels era400, era800, and era1000 refer to three selected Polkadot election instances with $(m, n) = (918, 18102), (1007, 34115), (926, 45627)$, respectively; all three use $k = 297$. Since the original data are weighted, we discard the weights and treat each positive-weight nominator as one unweighted voter. We evaluate Mechanism 2 on these three instances, using 200 trials per privacy budget.

Table 2: Small-scale PrefLib results averaged over 44 configurations and 500 trials per setting.

ϵ	Mechanism	Pr(JR)	Pr(AV)
0.5	Mechanism 1 (JR)	0.752	0.339
	Mechanism 2	1.000	0.925
	Random sampling	0.695	0.330
1	Mechanism 1 (JR)	0.822	0.349
	Mechanism 2	1.000	0.965
	Random sampling	0.696	0.328
5	Mechanism 1 (JR)	0.964	0.394
	Mechanism 2	1.000	0.994
	Random sampling	0.703	0.331

Table 2 summarizes the small-scale PrefLib results and Table 3 reports the large-scale Polkadot results. Table 4 compares Mechanism 2 with a permute-and-flip (PF) implementation on the PrefLib configurations. PF matches or slightly improves the probability of selecting an AV-optimal committee, especially when ϵ is small. On the large Polkadot instances, PF and the exponential mechanism produce nearly identical AV scores and JR probabilities across the tested privacy budgets.

Table 3: Large-scale Polkadot results for Mechanism 2. AV-overlap is the fraction of selected committee members also selected by approval voting.

Era	n	ϵ	$\Pr(\text{JR})$	AV-overlap
era400	18102	1	0.125	0.473
		5	0.445	0.702
		10	0.820	0.801
era800	34115	1	0.540	0.547
		5	1.000	0.831
		10	1.000	0.924
era1000	45627	1	0.805	0.602
		5	0.995	0.877
		10	1.000	0.943

Table 4: Permute-and-flip comparison on PrefLib small-scale configurations.

ϵ	Exp $\Pr(\text{AV})$	PF $\Pr(\text{AV})$	PF wins/ties/losses
0.1	0.706	0.731	43/1/0
0.5	0.925	0.940	30/12/2
1	0.965	0.973	25/19/0
5	0.995	0.997	7/36/1