Online Fair Allocation with Demand-Side Time-Dependent Weight

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Abstract

We study the problem of allocating indivisible items to a group of agents as these items arrive over time. Our focus is on scenarios where the utility of the items is weighted based on the waiting time since the last item was received, which is a demand-side time-dependent utility model. To ensure fairness, we concentrate on two criteria: temporal envy-freeness (TEF) and temporal proportionality (TProp). These criteria require that the allocation remains fair after each item is allocated. When there are two agents, we demonstrate the sufficient and necessary conditions of the weight function that ensure the existence of a TEF1 or TProp1 allocation. Additionally, we provide two polynomial-time algorithms that output a TEF2 or TProp2 allocation, with some restrictions on the weight function. When there are more than two agents, we show the hardness of determining the existence of TEF1 allocations. We also provide polynomial algorithms returning TEF1/TProp1 allocations with some specific valuation profiles. When there are multiple items per round, we also provide a polynomial algorithm returning TEF1/TProp1 allocations.

1 INTRODUCTION

The fair allocation problem, also known as the fair division problem, is a fundamental task in resource allocation where a set of indivisible or divisible goods must be distributed among agents. The goal is to allocate resources under some fairness notions, such as envy-freeness, proportionality, maximin share, and their variations [Amanatidis et al., 2023]. This problem arises in various real-world scenarios, such as divorce settlements [Brams and Taylor, 1996], task allocations [Goldman and Procaccia, 2015], and expense divisions [Goldman and Procaccia, 2015].

Most previous research focused on the problem in an offline setting, where agents and items are known and available at the start of the allocation process. However, many applications involve situations where items or agents arrive over a time horizon. For instance, a food bank [Aleksandrov et al., 2015] may receive food donations and must distribute each donation to one of the parties requesting it. Each donation needs to be allocated immediately upon arrival, and the next donation cannot be predicted. Such applications align with online fair allocation models, where items or agents appear over time and must be allocated as they arrive.

We are particularly interested in the online fair allocation problem with time-dependent utility, where an agent's utility of an item increases or decreases based on the time elapsed since the last item he received. This model is particularly relevant in applications like vaccination scheduling, where the timing of doses critically impacts efficacy. For instance, a short interval between vaccinations may reduce the effectiveness of subsequent doses or cause adverse effects, while an excessively long interval increases infection risks. Additionally, a study [Ghosh et al., 2024] indicates that the costs associated with vaccination, including both the vaccination and treatment expenses, tend to rise as the time gap between doses increases (see Fig. 1) during the COVID-19 pandemic. This underscores the need for allocation strategies that balance fairness and temporal constraints in dynamic, time-sensitive environments.

Unfortunately, existing work on fair resource allocation, such as the study [Mattei et al., 2017, 2018, Banerjee et al., 2023] on perishable resources and the work [Tan and Zhang, 2015] on utility maximization in wireless networks, is inherently incompatible with our scenario, including the demand-side time-dependent utility. They primarily address supply-side perishability where goods spoil stochastically and in-

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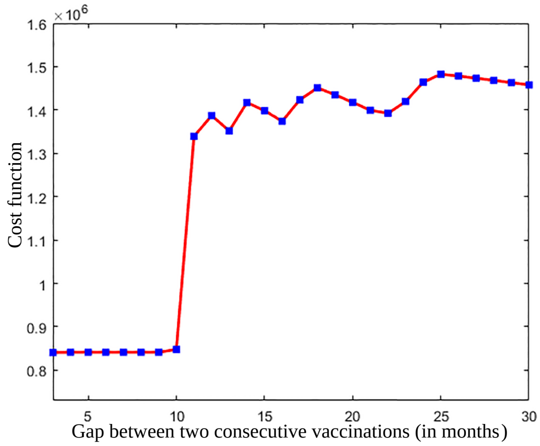

Figure 1: The cost related to the gap between two consecutive vaccinations [Ghosh et al., 2024]

dependently over time according to a known distribution [Mattei et al., 2017, 2018, Banerjee et al., 2023], or static allocations incorporating diminishing marginal utility for varying traffic types with fixed users receiving resources based on constant channel qualities [Tan and Zhang, 2015]. Their assumptions—including transient agents with one-time allocations under linear utilities in Banerjee et al. [2023], and a fixed set of flows with sigmoid or convex utilities dependent solely on allocated amounts without temporal dependencies in Tan and Zhang [2015]—fail to accommodate persistent agents who receive multiple items sequentially, where the utility of each subsequent item decays as a function of the elapsed time since the previous one, as in vaccination scheduling where inter-dose intervals directly impact efficacy and risk. Consequently, their algorithms, which adaptively select guardrail quantities based on pessimistic forecasts of exogenous spoilage under a static allocation schedule in Banerjee et al. [2023], or solve the network utility maximization problem for sufficient/insufficient resource cases assuming all users known upfront in Tan and Zhang [2015], cannot be extended to our scenario, as they do not account for endogenous utility decay tied to personalized timing decisions across rounds, nor do they support redefined envy notions that span multiple interdependent allocations per agent; moreover, their lower bounds on inefficiency and counterfactual envy [Banerjee et al., 2023], derived from unavoidable spoilage due to prediction errors in the static allocation schedule, rely on supply uncertainty that does not translate to demand-side temporal dependencies in our scenario.

1.1 CONTRIBUTION

In this paper, we investigate the allocation of indivisible items to a group of agents as these items arrive over time. We consider a setting where items are allocated over time, and agents' utility for an item depends on the item's original value and how long the agents have waited since they received the previous item, which is a demand-side time-dependent utility model. The model is reflected by a weight function f, the multiplier of items' value relating to the waiting time. To guarantee fairness, we propose two main criteria: Temporal Envy-Freeness (TEF) and Temporal Proportionality (TProp), and their "up-to-d" relaxations (TEFd, TProp$d$). These ensure that, from the perspective of any agent, the allocation is fair after every single item is assigned. We mainly study the cases where there are two agents.

- We show that TEF1 allocations exist if and only if the weight function f is constant.
- We show that TProp1 allocations exist if $f(1)$ is the minimum among $f(t)$ for all t, while TProp1 allocations may not exist if $f(1)$ is the maximum among $f(t)$ for all t.
- We design two polynomial algorithms returning a TEF2 allocation when $f(1) = f(2) = f(3) \geq f(4), f(5)$, or a TProp2 allocation when $f(1) \leq f(2), f(3)$.

We also discuss the cases where there are more than two agents or there are multiple items per round.

- For more than two agents, when agents are identical, we provide a polynomial algorithm returning a TEF1/TProp1 allocation.
- For multi-item per round, we also provide a polynomial algorithm returning a TEF1/TProp1 allocation.

Outline In Section 2, we present studies most relevant to our setting. In Section 3, we provide the necessary notations and definitions to define our setting formally. In Sections 4 and 5, we present our results for TEF and TProp, respectively. In Section 6, we discuss the cases where there are more than two agents. In Section 7, we focus on other cases where multiple items arrive in each round. Finally, we conclude in Section 8.

2 LITERATURE REVIEW

In 1948, Hugo Steinhaus posed the question of how to equitably distribute a "cake" among a group of agents who may have varying preferences for the resource [Steinhaus, 1948]. Later, much work appears assuming the agents and items are all known and fixed at the beginning. However, there exist many real-world situations spanning a long period and items and agents arrive in succession [Walsh, 2014, 2015]. In the following, we explore relevant literature that specifically pertains to our scenario.

Online Fair Division The online fair division problem can be characterized by three aspects: types of resources (divisibility, homogeneity), online features (items are online, agents are online or both), and mechanism features (with or without future information for the mechanism)[Aleksandrov and Walsh, 2020, Amanatidis et al., 2023]. For example, Walsh [2010] considers an online cake-cutting problem where the resource is a single heterogeneous divisible cake and the agents are online. The paper extends several classical cake-cutting methods to the online setting and discusses the fairness. Aleksandrov et al. [2015] study the setting where indivisible items arrive in an online fashion and analyze the strategy-proofness, impact of welfare and envy-freeness of two mechanisms $LIKE$ and $BANLANCED\_LIKE$. Aleksandrov and Walsh [2017a,b,c] also work on the two mechanisms and analyze their other performances, such as the expected outcome, competitive analysis, advice complexity, group strategy-proofness and pure Nash equilibrium. Kash et al. [2013] focus on dividing multiple homogeneous and divisible resources, introducing and analyzing two fairness notions: dynamic envy-freeness (DEF) and dynamic Pareto optimality (DPO). Mattei et al. [2017, 2018] study the online organ matching problem where items (organs) and agents (patients) arrive over time. Since each agent can receive only one item, the allocation is a matching, and they analyze the matching related to several factors in the online organ matching problem. The above works are all in a setting where the mechanism has no information about the future. In addition, Benade et al. [2018] consider the indivisible setting with items arriving online and the uninformed setting where the mechanism has no information of the future. They provide a randomized algorithm that allocates the new item to an agent chosen uniformly at random, which gives a tight bound $\tilde{O}(\sqrt{T/n})$ for cumulative envy. Mertzanidis et al. [2024] model the situation of allocating (indivisible) food donations among food banks, which is located on a graph. The truck with food starts and ends at certain vertices. With the food arriving online supported by known distribution, they provide guarantees of approximated weighted EF allocation together with approximated efficiency for drivers. Vardi et al. [2022] work on dividing a divisible item to agents with arrival and departure. Because of the unknown future, they aim to limit the ratio of the smallest piece to the proportional share by disrupting the historical allocation under some pattern.

Temporal Model Although the works above are on online fair allocation, they consider the fairness after allocating all the items at the end of the time horizon. Recently, temporal models, which require the allocation to be fair over time, have also been studied in social choice literature. He et al. [2019] consider the setting where indivisible items arrive online. For both informed and uninformed settings, they provide bounded times of adjustment on history allocation guaranteeing EF1 in each round. Elkind et al. [2024a] work on the existence of Temporal EF1 in some restricted settings, together with its efficiency. Cookson et al. [2025] study the stochastic dominance extension of EF1 and PROP1 in three online models, per day (the allocation in each day is fair), up to each day (as in our setting), and overall (the final allocation is fair). Amanatidis et al. [2025] work on cases where the valuation profile is personalized 2-value (agents' valuations are of two types) or personalized interval-restricted (agents' valuations are in a certain range). They demonstrate algorithms or impossibility results on the approximation of temporal EF1/MMS allocations with or without foresight of the future. Beyond additive valuation, Wang and Wei [2025] consider submodular binary valuation and provides upper bounds on the approximation of EF1/MMS/USW (optimal utilitarian social welfare) under the temporal model. There are also some other works on temporal models in voting [Chandak et al., 2024, Elkind et al., 2024b,c, Mackenzie and Komornik, 2023, Zech et al., 2024] and scheduling [Elkind et al., 2022, Lackner, 2020].

3 PROBLEM FORMULATION

Define $[x] = \{1, 2, \cdots, x\}$ for any positive integer x. Assume that there is a time horizon of T time slots. At the beginning of each time slot t, an item I_t arrives, and thus we have an item set $\mathbf{I}^t = \{I_1, I_2, \cdots, I_t\}$ after I_t's arrival. Let $\mathbf{N} = \{a_1, a_2, \cdots, a_n\}$ be the set of n agents and $\mathbf{I} = \mathbf{I}^T$ be the set of all T items.

An allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ is a partition of all items into bundles for the agents so that agent a_i receives bundle A_i satisfying $\cup_{1 \le i \le n} A_i = \mathbf{I}$ and $A_i \cap A_j = \emptyset$ for any two agents a_i, a_j.

We consider the allocation problem over the time horizon. The decision maker allocates the stream of items one at a time in an irrevocable manner. At each item's arrival, the decision maker must choose an agent to allocate the item immediately. Notice that the irrevocable manner means that when allocating an item, the historical allocation is fixed, while our algorithms below are the brainstorm of the decision maker before allocating items. Thus, the swaps and reallocations in our algorithms are reasonable. Let $\mathbf{A}^t = (A_1^t, A_2^t, \cdots, A_n^t)$ be the allocation after assigning the item at time slot t and the final allocation $\mathbf{A} = \mathbf{A}^T$.

Each agent a_i has nonnegative valuations $v_{i,t}$ towards each item $I_t \in \mathbf{I}$. In addition, we consider demand-side time-dependent utility. When an agent receives an item, he will gain utility concerning the valuation and the time delay from the last time he received an item. Specifically, agents have a weight function $f : [T] \to [0, \infty]$. Let $\tau_{t,A} = \max_{t' < t, I_{t'} \in A} t'$ be the index of the latest item before time t (I_t excluded) in the set of items A. In case t' does not exist (all items' indices in A are larger than or equal to t), $\tau_{t,A} = 0$. When the agent a_i receives an item at time t with his previous allocation A_i^{t-1}, he will receive utility of $f(t - \tau_{t,A_i^{t-1}}) * v_{i,t}$.

We define utility that the agent a_i receives from any subset of items $\mathbf{I}'$ as $u_i(\mathbf{I}') = \sum_{I_t \in \mathbf{I}'} f(t - \tau_{t,\mathbf{I}'}) * v_{i,t}$, which is the sum of the utility that a_i receives from each item in I'. Notice that the set function u_i is usually a non-additive function (a set function u is additive on $\mathbf{I}$ if $u(S \cup S') = u(S)+u(S')$ for all $S, S' \subseteq \mathbf{I}$ with $S \cap S' = \emptyset$). For instance, assume that there are two items $\mathbf{I} = \{I_1, I_2\}$ with $v_{i,1} = v_{i,2} = 1$ and the weight function is $f(1) = 0.1, f(2) = 1$. Then $u_i(\{I_1, I_2\}) = f(1-0)v_{i,1} + f(2-1)v_{i,2} = 0.2$, while $u_i(\{I_1\}) = f(1-0)v_{i,1} = 0.1$ and $u_i(\{I_2\}) = f(2-0)v_{i,2} = 1$.

Then we have the following proposition:

Proposition 1. *The set function u_i is additive if and only if the weight function f is a constant function.*

Proof. Assume that f is constant and $f(t) = c$ for all $t \in [T]$. Then for any subset S, S' of $\mathbf{I}$ with $S \cap S' = \emptyset$,

$$
\begin{aligned}
u_i(S \cup S') &= \sum_{I_j \in S \cup S'} c \times v_{i,j} \\
&= \sum_{I_j \in S} c \times v_{i,j} + \sum_{I_j \in S'} c \times v_{i,j} \\
&= u_i(S) + u_i(S')
\end{aligned}
$$

On the other hand, assume that u_i is additive. Let S, S' be two arbitrary subset of $\mathbf{I} \setminus \{I_T\}$. Then

$$
\begin{aligned}
&u_i(\{I_T\}) \\
=&u_i(S \cup \{I_T\}) - u_i(S) = f(T - \tau_{T,S}) \times v_{i,T} \\
=&u_i(S' \cup \{I_T\}) - u_i(S') = f(T - \tau_{T,S'}) \times v_{i,T}
\end{aligned}
$$

for any $\tau_{T,S}, \tau_{T,S'} \in [T]$. Therefore, f is a constant function. $\square$

3.1 FAIRNESS NOTIONS

We are interested in finding a fair allocation after each round. The main fairness notions that we consider are envy-freeness (EF) and proportionality (Prop), which are well-studied notions in the problem of allocating indivisible items. Envy-freeness requires that, in each agent's point of view, the other agents' allocation is at most as good as his own, while proportionality requires that each agent's utility is at least $\frac{1}{n}$ fraction of what he can get if he receives the whole set of items.

Definition 1. *An allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ is envy-free (EF) if for all $a_i, a_j \in \mathbf{N}$, $u_i(A_i) \geq u_i(A_j)$.*

Definition 2. *An allocation $\mathbf{A} = (A_1, A_2, \cdots, A_n)$ is Proportional (Prop) if for all $a_i \in \mathbf{N}$, $u_i(A_i) \geq \frac{1}{n} u_i(\mathbf{I})$.*

When considering deterministic mechanisms, we study the "up-to-d" relaxation of the above fairness notion, which is defined as follows:

Definition 3. *An allocation $A = (A_1, A_2, \cdots, A_n)$ is envy-free up to d items (EFd) if for $a_i, a_j \in \mathbf{N}$, there exist d items $\{I_{t_1}, \cdots, I_{t_d}\} \subseteq A_j$ such that $u_i(A_i) \geq u_i(A_j \setminus \{I_{t_1}, \cdots, I_{t_d}\})$.*

Definition 4. *An allocation $A = (A_1, A_2, \cdots, A_n)$ is proportional up to d items (Propd) if for $a_i \in \mathbf{N}$, there exist d items $\{I_{t_1}, \cdots, I_{t_d}\} \in \mathbf{I} \setminus A_i$ such that $u_i(A_i \cup \{I_{t_1}, \cdots, I_{t_d}\}) \geq \frac{1}{n} u_i(\mathbf{I})$.*

In this work, we focus on the problem over a time horizon. Thus, we extend the above fairness notions to temporal envy-freeness (TEF) and temporal proportionality (TProp) and their corresponding "up-to-d" relaxation (TEFd, TProp$d$), which requires that after allocating the item in each round, the allocation of items that have arrived satisfies the fairness.

Definition 5. *For $t \in [T]$ and a fairness notion **X**, the allocation $\mathbf{A}^t$ is temporal **X** (T**X**) if for any $t' \leq t$, $\mathbf{A}^{t'}$ satisfies **X**.*

Notice that when u_i is additive, EFd implies Prop$d$. Thus, it is not hard to obtain the following proposition:

Proposition 2. *When u_i is additive for all $i \in \mathbf{N}$, TEFd implies TProp$d$.*

4 TEMPORAL ENVY-FREENESS

In this section, we show our results considering temporal envy-freeness. We explore the condition of the existence of TEF1 allocations when there are only two agents.

He et al. [2019] show that TEF1 allocations always exist for two agents without time-dependent weight. The key idea is to swap the two agents' partial allocations when they envy each other. However, with time-dependent weight, we find that TEF1 allocations may not exist. Below, we show the sufficient and necessary conditions of f to guarantee the existence of TEF1 allocations.

Theorem 1. *When there are two agents, TEF1 allocations always exist if and only if f is a constant function.*

Proof. For the sufficient condition, when $f(t) = c$ is a constant function, for each instance, we can construct a new valuation profile v' where for each agent a_i and each item I_t, $v'_{i,t} = c \times v_{i,t}$, and the problem becomes finding TEF1 allocations without time-dependent weight. By applying the strategy in He et al. [2019], we can always find a TEF1 allocation.

For the necessary condition, we first show that TEF1 allocations may not exist when f is not a periodic function with period 2 (Lemma 2). Then we show that TEF1 allocations may not exist when f is a periodic function with period 2 but $f(1) \neq f(2)$ (Lemma 3), by two constructed instances, implying f must be a constant function. Recall that a function f is a periodic function with period k if $f(x) = f(x + k)$. □

Lemma 2. *When there are two agents, TEF1 allocations always exist only if f is a periodic function with period 2.*

Proof (Sketch). We begin by proving TEF1 allocations may not exist when $\exists T_1, f(1)f(2) \neq f(T_1)f(T_1 + 1)$ through an example. Then we show that a function satisfying $\forall T_1, f(1)f(2) = f(T_1)f(T_1 + 1)$ is a periodic function with period 2, which is a necessary condition to guarantee the existence of TEF1 allocations. The constructed profile is as shown in Table 1, and the detailed proof is in the Appendix. □

Lemma 3. *When there are two agents, TEF1 allocations always exist only if f is a constant function.*

Proof (Sketch). By Lemma 2, we know that TEF1 allocations always exist only if f is a periodic function with period 2. Below, we show through an example that when f is a periodic function with period 2 and $f(1) \neq f(2)$, TEF1 allocations may not exist, meaning that TEF1 allocations always exist only if f is a constant function. The constructed profile is as shown in Table 3, and the detailed proof is in the Appendix. □

Algorithm 1 Returns a TEF2 allocation

Input: $n = 2$ and f with $f(1) = f(2) = f(3) \geq f(4), f(5)$
Output: A TEF2 allocation $\mathbf{A}$

1 Initialize $s \leftarrow 0$, $\mathbf{A}^0 \leftarrow <\emptyset, \emptyset>$, $\mathbf{B} \leftarrow <\emptyset, \emptyset>$
2 **for** $t \leftarrow 1$ ***to*** T ***step*** *2* **do**
3   **if** $u_1(A_1^{t-1}) - u_1(A_1^s) < u_1(A_2^{t-1}) - u_1(A_2^s)$ **then**
4   Let I_i be a_1's favorite item in $\{I_t, I_{t+1}\}$ that maximizes his utility, I_{-i} be the other item.
5         $A_1^{t+1} \leftarrow A_1^{t-1} \cup \{I_i\}$, $A_2^{t+1} \leftarrow A_2^{t-1} \cup \{I_{-i}\}$
6   **else**
7   Let I_i be a_2's favorite item in $\{I_t, I_{t+1}\}$ that maximizes his utility, I_{-i} be the other item.
8         $A_2^{t+1} \leftarrow A_2^{t-1} \cup \{I_i\}$, $A_1^{t+1} \leftarrow A_1^{t-1} \cup \{I_{-i}\}$
9   **end**
10   **if** *a_1's favorite item is different from a_2's favorite item* **then**
11         $B_1 \leftarrow B_1 \cup \{a_1\text{'s favorite item}\}$
12         $B_2 \leftarrow B_2 \cup \{a_2\text{'s favorite item}\}$
13   **end**
14   **if** $u_1(A_1^{t+1}) - u_1(A_1^s) < u_1(A_2^{t+1}) - u_1(A_2^s)$ *and* $u_2(A_2^{t+1}) - u_2(A_2^s) < u_2(A_1^{t+1}) - u_2(A_1^s)$ **then**
15         $A_1^{t+1} \leftarrow A_1^s \cup B_1 \cup (A_2^{t+1} \setminus A_2^s \setminus B_2)$
16         $A_2^{t+1} \leftarrow A_2^s \cup B_2 \cup (A_1^{t+1} \setminus A_1^s \setminus B_1)$
17   **end**
18   **if** $u_1(A_1^{t+1}) - u_1(A_1^s) \geq u_1(A_2^{t+1}) - u_1(A_2^s)$ *and* $u_2(A_2^{t+1}) - u_2(A_2^s) \geq u_2(A_1^{t+1}) - u_2(A_1^s)$ **then**
19         $s \leftarrow t + 1$, $\mathbf{B} \leftarrow <\emptyset, \emptyset>$
20   **end**
21 **end**
22 **if** *T is odd* **then**
23   Arbitrarily assign the last item to an agent.
24 **end**

Next, we try a further relaxed fairness notion, TEF2, and show the relaxed condition on the weight function under which TEF2 allocations always exist. We provide an algorithm returning a TEF2 allocation when $f(1) = f(2) = f(3) \geq f(4), f(5)$ when $n = 2$ (See Algorithm 1). Intuitively, in every two rounds, the algorithm greedily allocates the two items that arrive in the two rounds to the two agents separately, and the one who envies the other agent in the current (partial) allocation is advantaged (chooses an item first) (Line 3 - 9). A counter s is used to record the last round after which $\mathbf{A}^s$ is envy-free (Line 18 - 20), and we swap agents' partial allocation in $\mathbf{A}^t \setminus \mathbf{A}^s$ for some t if both agents envy each other in $\mathbf{A}^t \setminus \mathbf{A}^s$ (Line 14 - 17).

Theorem 4. *When there are two agents and $f(1) = f(2) = f(3) \geq f(4), f(5)$, Algorithm 1 returns a TEF2 allocation in $O(T)$.*

Proof. The running time of the algorithm is easy to verify since there is only one "for" loop in $O(T)$. Thus, we focus on the correctness below.

For any $t \in [T]$, let $r_t < t$ be the last round before t such that $\mathbf{A}^{r_t}$ is EF. This implies that if $\mathbf{A}^{t'} \setminus \mathbf{A}^{r_t}$ is EFd for any $t' \in \{r_t + 1, \cdots, t\}$, $\mathbf{A}^t$ is TEFd. Therefore, it suffices to show that $\mathbf{A}^t \setminus \mathbf{A}^{r_t}$ is EF2 for any $t \in [T]$. We first prove that after each round t with an even difference between t and r_t (Lemma 5), $\mathbf{A}^t \setminus \mathbf{A}^{r_t}$ is EF1 by induction. For odd $t - r_t$, $\mathbf{A}^{t-1} \setminus \mathbf{A}^{r_t}$ is EF1, and we can additionally remove I_t to eliminate the envy, so that $\mathbf{A}^t \setminus \mathbf{A}^{r_t}$ is EF2. Eventually, we obtain a TEF2 allocation $\mathbf{A}$.

Lemma 5. *The allocation $\mathbf{A}^t$ that Algorithm 1 returns is EF1 for any even $t - r_t$.*

Proof. In $\mathbf{A}^{r_t+2} \setminus \mathbf{A}^{r_t}$, each agent receives one item and is trivially EF1. Assume that $\mathbf{A}^{t-2} \setminus \mathbf{A}^{r_t}$ is EF1 for some $t > r_t + 2$ such that $t - r_t$ is even. We show that $\mathbf{A}^t \setminus \mathbf{A}^{r_t}$ is also EF1. Let r'_t be the earliest round after r_t such that $\mathbf{A}^{r'_t} \setminus \mathbf{A}^{r_t}$ is EF. Then we have two cases depending on whether a swap (as Line 15, 16 of the algorithm) happens at round $r_t + 1$.

Case 1: r'_t does not exist or there is no swap at r'_t. Suppose, without loss of generality, that $u_1(A_1^{t-2}) - u_1(A_1^{r_t}) < u_1(A_2^{t-2}) - u_1(A_2^{r_t})$, meaning that a_1 envies a_2. Then a_2 does not envy a_1 since otherwise there is a swap contradicting the definition of r_t. Then a_1 chooses his favorite between I_{t-1} and I_t first. After round t, if a_2 envies a_1, the envy can be eliminated by removing the last item in A_1^t; if a_1 still envies a_2, the envy can be eliminated by removing the same item as for round $t-2$. Notice the weight in the allocation can be $f(1)$, $f(2)$ or $f(3)$. After removing an item, the next item's weight can be $f(1)$, $f(2)$, $f(3)$, $f(4)$ or $f(5)$. Since $f(1) = f(2) = f(3) \geq f(4), f(5)$, the utility after removing an item is even smaller than that after removing the gain of that item, meaning that agents will not benefit from other items by removing an item. Thus, our statement on how agents remove the item is correct, and $\mathbf{A}^t \setminus \mathbf{A}^{r_t}$ is EF1.

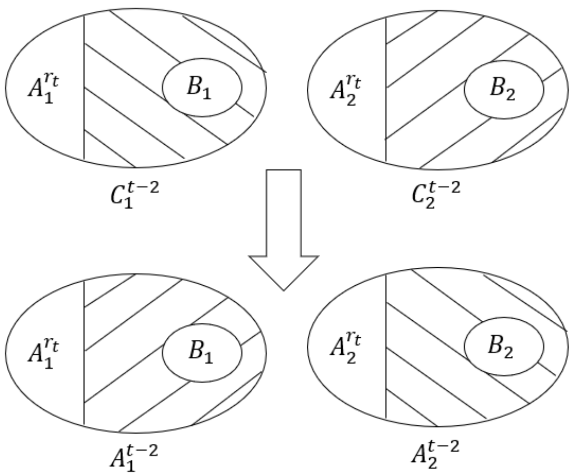

Figure 2: Swapping Operation (Only the shadow parts are swapped)

Case 2: Swap happens at r_t'. Fig. 2 shows how we swap agents' allocation. Note that the swap operation may change some items' weight. However, after swapping, the weight of each item is still $f(1)$, $f(2)$ or $f(3)$. Thus, the utility does not change. Let $f(1) = f(2) = f(3) = c$.

For each $i \in \{t-2, t\}$, let $\mathbf{C}^i \setminus \mathbf{A}^{r_t}$ be the algorithm's partial allocation before the bundle swap. Suppose, without loss of generality, that $u_1(C_1^{t-2}) - u_1(A_1^{r_t}) < u_1(C_2^{t-2}) - u_1(A_2^{r_t})$, meaning that a_1 envies a_2 before swapping, while a_2 does not envy a_1, with $u_2(C_2^{t-2}) - u_2(A_2^{r_t}) \geq u_2(C_1^{t-2}) - u_1(A_1^{r_t})$. Recall that

$$A_1^{t-2} \setminus A_1^{r_t} = C_2^{t-2} \cup B_1 \setminus A_2^{r_t} \setminus B_2$$

$$A_2^{t-2} \setminus A_2^{r_t} = C_1^{t-2} \cup B_2 \setminus A_1^{r_t} \setminus B_1$$

After swapping, the items in $\mathbf{B}$ remain unchanged. Moreover, by $\mathbf{B}$'s definition,

$$\begin{aligned}
&\sum_{I_t \in B_1} v_{1,t} \geq \sum_{I_t \in B_2} v_{1,t} \\
\Rightarrow\ & u_1(C_1^{t-2}) - u_1(A_1^{r_t}) - 2c \times \sum_{I_t \in B_1} v_{1,t} \\
&< u_1(C_2^{t-2}) - u_1(A_2^{r_t}) - 2c \times \sum_{I_t \in B_2} v_{1,t} \\
\Rightarrow\ & u_1(A_1^{t-2}) - u_1(A_1^{r_t}) \\
&= u_1(C_2^{t-2}) - u_1(A_2^{r_t}) - c \times \sum_{I_t \in B_2} v_{1,t} + c \times \sum_{I_t \in B_1} v_{1,t} \\
&> u_1(C_1^{t-2}) - u_1(A_1^{r_t}) - c \times \sum_{I_t \in B_1} v_{1,t} + c \times \sum_{I_t \in B_2} v_{1,t} \\
&= u_1(A_2^{t-2}) - u_1(A_2^{r_t})
\end{aligned}$$

This implies that a_1 does not envy a_2 after swapping at round $t-2$. Then the envy can only occur from a_2 towards a_1. If a_1 and a_2 have different preference on $\{I_{t-1}, I_t\}$, the two items are in $\mathbf{B}$ and remain unchanged. If a_1 and a_2 prefer the same item in $\{I_{t-1}, I_t\}$, the item is assigned to a_1 before swapping and is assigned to a_2 after swapping. Then it is still the agent who envies the other getting his favorite. Thus, it is the same as Case 1, and $\mathbf{A}^t \setminus \mathbf{A}^{r_t}$ is EF1. $\square$

$\square$

Beyond the restriction $f(1) = f(2) = f(3)$, we show the performance on general time-dependent weights through experiments in the Appendix.

5 TEMPORAL PROPORTIONALITY

Regarding proportionality, the sufficient and necessary conditions for the existence of TProp1 allocations are different from those for TEF1 allocations. Since TEF1 implies TProp1, TProp1 allocations always exist if f is a constant function. Based on this observation, we show the sufficient condition that $f(1)$ is the minimum of f.

Theorem 6. *When there are two agents, TProp1 allocations always exist if* $\min_{t \in [T]} f(t) = f(1)$.

Proof. Given an instance where $\min_{t \in [T]} f(t) = f(1)$ holds, we find a TProp1 allocation by constructing another instance with the same agents, items, and valuation, but a different weight function $f'(t) = f(1)$ for all $t \in [T]$. By applying Theorem 1, we find a TEF1 allocation $\mathbf{A}$, which is also TProp1, for the constructed instance. For each agent $i \in \mathbf{N}$ at each time slot t, we have $u_i'(A_i^t \cup \{I\}) \geq \frac{1}{n} u_i'(\mathbf{I}^t)$ for some item $I \notin A_i^t$. Since $\min_{t \in [T]} f(t) = f(1)$, $f(t) \geq f(1)$ and thus for each agent $i \in \mathbf{N}$ at each time slot t, $u_i(A_i^t \cup \{I\}) \geq u_i'(A_i^t)$. Meanwhile, $u_i(\mathbf{I}^t) = \sum_{j=1}^t f(1) v_{i,j} = u_i'(\mathbf{I}^t)$. Therefore, for each agent $i \in \mathbf{N}$ at each time slot t, $u_i(A_i^t \cup \{I\}) \geq \frac{1}{n} u_i(\mathbf{I}^t)$ for some item $I \notin A_i^t$, meaning that the allocation $\mathbf{A}$ is also TProp1 for the given instance. $\square$

To show the necessary condition that $f(1)$ is not the only maximum, we construct an instance showing that no TProp1 allocations exist when $f(1) > \max_{t\in[T],t>1} f(t)$.

Theorem 7. *When there are two agents, TProp1 allocations always exist only if $f(1) \leq \max_{t\in[T],t>1} f(t)$.*

Proof. Assume that T is a large even number, and there are T items, each of which has a value of 1 towards both agents. We show by contradiction that each agent must receive exactly $\frac{t}{2}$ items in any TProp1 allocation at a large even time slot t. Suppose that $\mathbf{A}^t$ is TProp1 such that each agent does not receive exactly $\frac{t}{2}$ items. Then there must be one agent a_i who receives at most $\frac{t}{2} - 1$ items, with utility $u_i(A_i^t \cup \{I_g\}) \leq \frac{1}{2} t f(1)$ even after receiving one more item $I_g \notin A_i^t$, since $f(1) > \max_{t\in[T],t>1} f(t)$. Moreover, the equality does not hold, because after a_i receives one more item, he gets $\frac{1}{2}t$ items in total. If he gains $\frac{1}{2}tf(1)$ utility, he must receive items from I_1 to $I_{\frac{t}{2}}$. At the same time, the other agent a_{3-i} receives at most $2f(1)$ utility even after receiving one more item in the first $\frac{t}{2}$ items. With large t, $2f(1)$ is less than $\frac{1}{4}tf(1)$, the proportional share at the time slot $\frac{t}{2}$, and hence a_{3-i} is not satisfied, breaking TProp1.

Therefore, for a large even t, each agent receives $\frac{t}{2}$ items at time slot t. For $t+2$, each agent receives $\frac{t}{2}+1$ items, meaning that the two new items are assigned to agents separately. The same happens for $t+4, t+6, \cdots$. Therefore, there exists a certain even time slot t^* such that in every two rounds after t^*, the two items are assigned to the two agents separately. Then for each agent, in every four rounds after t^*, the agent receives two items among which at least one item's coefficient is $f(2)$ or $f(3)$. Assume that $\delta = \min(f(1)-f(2), f(1)-f(3))$. After receiving one more item, the utility for each agent is at most

$$
\begin{aligned}
&(\frac{1}{2}t^* + \frac{1}{4}(T-t^*) + 1 + 1)f(1) \\
&+(\frac{1}{4}(T-t^*) - 1)\max(f(2), f(3)) \\
&=(\frac{1}{2}T + 1)f(1) - (\frac{1}{4}(T-t^*) - 1)\delta
\end{aligned}
$$

assuming the items' weight before t^* are all $f(1)$, the extra item's weight is $f(1)$ changing the next item's weight from $f(2)$ to $f(1)$. Since T is large enough, the utility is less than the proportional share $\frac{1}{2}Tf(1)$. Thus, the allocation is not TProp1. $\square$

Similar to envy-freeness, we also study TProp2. We provide an algorithm returning a TProp2 allocation when $f(1) \leq f(2), f(3)$ when $n = 2$. Using a similar greedy approach, in every two rounds, the algorithm allocates the two items to the agents separately, and the one who receives less than the proportional share in the current (partial) allocation is advantaged (chooses an item first) (Line 3 - 11). A counter s is used to record the last round after which $\mathbf{A}^s$ is Prop (Line 20 - 22), and we swap agents' partial allocation in $\mathbf{A}^t \setminus \mathbf{A}^s$ for some t if both agents' utility is less than the proportional share (Line 16 - 19).

Algorithm 2 Returns a TProp2 allocation

Input: $n = 2$ and f with $f(1) \leq f(2), f(3)$
Output: A TEF2 allocation $\mathbf{A}$

1 Initialize $s \leftarrow 0$, $\mathbf{A}^0 \leftarrow <\emptyset, \emptyset>$
2 **for** $t \leftarrow 1$ ***to*** T ***step*** 2 **do**
3   **if** $u_1(A_1^{t-1}) - u_1(A_1^s) < \frac{1}{2}\sum_{j=s+1}^{t-1} f(1)v_{1,j}$ **then**
4   Let I_i be a_1's favorite item in $\{I_t, I_{t+1}\}$ that maximizes his utility, I_{-i} be the other item.
5     $A_1^{t+1} \leftarrow A_1^{t-1} \cup \{I_i\}$
6     $A_2^{t+1} \leftarrow A_2^{t-1} \cup \{I_{-i}\}$
7   **else**
8   Let I_i be a_2's favorite item in $\{I_t, I_{t+1}\}$ that maximizes his utility, I_{-i} be the other item.
9     $A_2^{t+1} \leftarrow A_2^{t-1} \cup \{I_i\}$
10     $A_1^{t+1} \leftarrow A_1^{t-1} \cup \{I_{-i}\}$
11   **end**
12   **if** *a_1's favorite item is different from a_2's favorite item* **then**
13     $B_1 \leftarrow B_1 \cup \{a_1\text{'s favorite item}\}$
14     $B_2 \leftarrow B_2 \cup \{a_2\text{'s favorite item}\}$
15   **end**
16   **if** $u_1(A_1^{t+1}) - u_1(A_1^s) < \frac{1}{2}\sum_{j=s+1}^{t-1} f(1)v_{1,j}$ *and* $u_2(A_2^{t+1}) - u_2(A_2^s) < \frac{1}{2}\sum_{j=s+1}^{t-1} f(1)v_{2,j}$ **then**
17     $A_1^{t+1} \leftarrow A_1^s \cup B_1 \cup (A_2^{t+1} \setminus A_2^s \setminus B_2)$
18     $A_2^{t+1} \leftarrow A_2^s \cup B_2 \cup (A_1^{t+1} \setminus A_1^s \setminus B_1)$
19   **end**
20   **if** $u_1(A_1^{t+1}) - u_1(A_1^s) \geq \frac{1}{2}\sum_{j=s+1}^{t-1} f(1)v_{1,j}$ *and* $u_2(A_2^{t+1}) - u_2(A_2^s) \geq \frac{1}{2}\sum_{j=s+1}^{t-1} f(1)v_{2,j}$ **then**
21     $s \leftarrow t+1$, $\mathbf{B} \leftarrow <\emptyset, \emptyset>$
22   **end**
23 **end**
24 **if** *T is odd* **then**
25   Arbitrarily assign the last item to an agent.
26 **end**

Theorem 8. *When there are two agents and $f(1) \leq f(2), f(3)$, Algorithm 2 returns a TProp2 allocation in $O(T)$.*

Proof. When $f(1) = f(2) = f(3)$, most of the proof is similar to Theorem 5, except that we can ignore the condition for $f(4), f(5)$, since the weight function of each item in the allocation can only be $f(1)$, $f(2)$ or $f(3)$, even after receiving some more items for relaxation. Moreover, Similar to Theorem 6, we can extend the condition $f(1) = f(2) = f(3)$ to $f(1) \leq f(2), f(3)$, as at each time slot agents' utility increase when $f(2), f(3)$ become larger, while the proportional share remains the same. $\square$

Beyond the restriction $f(1) \leq f(2), f(3)$, we show the performance on general time-dependent weights through experiments in the Appendix.

6 MORE THAN TWO AGENTS

In this section, we consider the cases where there are more than two agents. Without the time-dependent weight, while TEF1 allocations always exist when there are two agents, Elkind et al. [2024a] shows that determining the existence of TEF1 allocations is NP-complete. This directly implies the hardness of our problem when f is constant.

Corollary 9. *Given any instance with $n \geq 3$ agents, determining the existence of TEF1 allocations is NP-complete, even if f is constant.*

Beyond hardness, they also provide some polynomial algorithms that return the TEF1 allocation with special valuation profiles. These algorithms are also extendable to our problem. In particular,

Corollary 10. *TEF1 allocations always exist when f is constant and*

- *there are two types of items*
- *agents have generalized binary valuations*
- *agents have single-peaked preferences*

As for the general time-dependent weight f, we find that when the agents are identical, TEF1/TProp1 allocations always exist. First, we introduce the envy graph

Definition 6. *Given an allocation $\mathbf{A}$, the envy graph is a directed graph $G = (\mathbf{N}, E)$, where each vertex represents an agent, and for each pair of agents i, j with $u_i(A_i) < u_i(A_j)$, $(i, j) \in E$.*

Next, we show a property of the envy graph that when agents are identical, any envy graph is acyclic.

Lemma 11. *When agents are identical, any envy graph is acyclic.*

Proof. We prove this property by contradiction. Assume that there is a cycle in the envy graph, referred to as $v_1 \to v_2 \to \cdots \to v_k \to v_1$. Then by definition, we have $u(A_1) < u(A_2) < \cdots < u(A_k) < u(A_1)$, leading to a contradiction. □

Then we show our procedure to find a TEF1 allocation.

Theorem 12. *When agents are identical, TEF1 allocations always exist and can be found in $O(T \log n)$.*

Proof. At time slot t, we build the envy graph for $\mathbf{A}^{t-1}$ and find an agent i who has no incoming edges. Then we allocate I_t to agent i. By Lemma 11, there is no cycle in the envy graph, meaning that such agent i always exists. Finally, the allocation is TEF1.

We show the correctness of this procedure by induction. At time slot 0, each agent receives nothing, and naturally the allocation is TEF1. Assume that $\mathbf{A}^{t-1}$ is EF1. Since agent i has no incoming edges, no one envies agent i in allocation $\mathbf{A}^{t-1}$. After allocating I_t, the envy towards agent i can be removed when i gives up I_t, while i receives an item and will not have additional envy towards others. With other agents unchanged, $\mathbf{A}^t$ is EF1.

As for implementation, we maintain an ordered sequence of agents' utility. The allocation at time slot t is the same as adding a value to the smallest in the sequence. The maintenance can be done in $O(\log n)$ by some data structures such as the balanced tree. Since there are T items, the final complexity is $O(T \log n)$. □

Recall that TEF1 always implies TProp1. Thus, the above procedure also finds a TProp1 allocation.

Corollary 13. *When agents are identical, TProp1 allocations always exist and can be found in $O(T \log n)$.*

7 MULTI-ITEMS PER ROUND

In this section, we focus on cases where multiple items arrive in each round. Let $\mathbf{I}_t = \{I_{t,1}, I_{t,2}, \cdots, I_{t,m_t}\}$ be the set of m_t items arriving in time slot t, $\mathbf{I}^t = \cup_{t'=1}^{t} \mathbf{I}_{t'}$ and $\mathbf{I} = \mathbf{I}^T$. Let $m = |\mathbf{I}| = \sum m_t$ be the total number of items. For allocation, we denote $\mathbf{A}^{p,q}$ as the allocation of $(\cup_{t=1}^{p-1} \mathbf{I_t}) \cup \{I_{p,1}, I_{p,2}, \cdots I_{p,q}\}$.

Algorithm 3 always returns a TEF1 allocation as long as $m_t \geq 2$ for any t, i.e., more than one item arrives in each round. Basically, we allocate one item to each agent at the beginning of each round, guaranteeing that each agent receives something in each round without breaking EF1. This operation leads to the fact that the weight of each item is $f(1)$. Thus, we span the remaining items into a virtual timeline and apply the strategy for two agents without time-dependent weight in He et al. [2019] to obtain the allocation.

Theorem 14. *When there are two agents and $m_t \geq 2$ for any t, Algorithm 3 returns a TEF1 allocation in $O(m)$.*

Proof. Notice that in Algorithm 3, since we allocate at least one item to each agent, the time-dependent weights are always $f(1)$.

To prove the correctness of Algorithm 3. We construct a new instance with constant time-dependent weight $f'(t) = f(1)$ for all $t \in [m]$ on a virtual timeline with $\sum_t m_t$ time slots such that $I_{t,i}$ arrives earlier than $I_{t',j}$ for all $t < t'$ or $t =$

Algorithm 3 Returns a TEF1 allocation

Input: $n = 2$ and $m_t \geq 2$ for any $1 \leq t \leq T$
Output: A TEF1 allocation $\mathbf{A}$

Initialize $s \leftarrow 0$, $s' \leftarrow 0$, $\mathbf{A}^{0,0} \leftarrow <\emptyset, \emptyset>$, $\mathbf{B} \leftarrow <\emptyset, \emptyset>$
for $t \leftarrow 1$ ***to*** T **do**
  if $u_1(A_1^{t-1,m_{t-1}}) - u_1(A_1^{s,s'}) < u_1(A_2^{t-1,m_{t-1}}) - u_1(A_2^{s,s'})$ **then**
    Let I_i be a_1's favorite item in $\{I_{t,1}, I_{t,2}\}$ that maximizes his utility, I_{-i} be the other item.
    $A_1^{t,2} \leftarrow A_1^{t-1,m_{t-1}} \cup \{I_i\}$
    $A_2^{t,2} \leftarrow A_2^{t-1,m_{t-1}} \cup \{I_{-i}\}$
  else
    Let I_i be a_2's favorite item in $\{I, I'\}$ that maximizes his utility, I_{-i} be the other item.
    $A_2^{t,2} \leftarrow A_2^{t-1,m_{t-1}} \cup \{I_i\}$
    $A_1^{t,2} \leftarrow A_1^{t-1,m_{t-1}} \cup \{I_{-i}\}$
  end
  if *a_1's favorite item in $\{I_{t,1}, I_{t,2}\}$ is different from a_2's favorite item* **then**
    $B_1 \leftarrow B_1 \cup \{a_1\text{'s favorite item}\}$
    $B_2 \leftarrow B_2 \cup \{a_2\text{'s favorite item}\}$
  end
  if $u_1(A_1^{t,2}) - u_1(A_1^{s,s'}) < u_1(A_2^{t,2}) - u_1(A_2^{s,s'})$ *and* $u_2(A_2^{t,2}) - u_2(A_2^{s,s'}) < u_2(A_1^{t,2}) - u_2(A_1^{s,s'})$ **then**
    $A_1^{t,2} \leftarrow A_1^{s,s'} \cup B_1 \cup (A_2^{t,2} \setminus A_2^{s,s'} \setminus B_2)$
    $A_2^{t,2} \leftarrow A_2^{s,s'} \cup B_2 \cup (A_1^{t,2} \setminus A_1^{s,s'} \setminus B_1)$
  end
  if $u_1(A_1^{t,2}) - u_1(A_1^{s,s'}) \geq u_1(A_2^{t,2}) - u_1(A_2^{s,s'})$ *and* $u_2(A_2^{t,2}) - u_2(A_2^{s,s'}) \geq u_2(A_1^{t,2}) - u_2(A_1^{s,s'})$ **then**
    $s \leftarrow t,\ s' \leftarrow 2,\ \mathbf{B} \leftarrow <0, 0>$
  end
  for $t' \leftarrow 3$ *to* m_t **do**
    if $u_1(A_1^{t,t'-1}) - u_1(A_1^{s,s'}) < u_1(A_2^{t,t'-1}) - u_1(A_2^{s,s'})$ **then**
      $A_1^{t,t'} \leftarrow A_1^{t,t'-1} \cup \{I_{t,t'}\}$
    else
      $A_2^{t,t'} \leftarrow A_2^{t,t'-1} \cup \{I_{t,t'}\}$
    end
    if $u_1(A_1^{t,t'}) - u_1(A_1^{s,s'}) < u_1(A_2^{t,t'}) - u_1(A_2^{s,s'})$ *and* $u_2(A_2^{t,t'}) - u_2(A_2^{s,s'}) < u_2(A_1^{t,t'}) - u_2(A_1^{s,s'})$ **then**
      $A_1^{t,t'} \leftarrow A_1^{s,s'} \cup B_1 \cup (A_2^{t,t'} \setminus A_2^{s,s'} \setminus B_2)$
      $A_2^{t,t'} \leftarrow A_2^{s,s'} \cup B_2 \cup (A_1^{t,t'} \setminus A_1^{s,s'} \setminus B_1)$
    end
    if $u_1(A_1^{t,t'}) - u_1(A_1^{s,s'}) \geq u_1(A_2^{t,t'}) - u_1(A_2^{s,s'})$ *and* $u_2(A_2^{t,t'}) - u_2(A_2^{s,s'}) \geq u_2(A_1^{t,t'}) - u_2(A_1^{s,s'})$ **then**
      $s \leftarrow t,\ s' \leftarrow t',\ \mathbf{B} \leftarrow <0, 0>$
    end
  end
end

$t', i < j$. The allocation $\mathbf{A}^t$ that Algorithm 3 returns for the original instance equals the allocation $\mathbf{A}^{t,m_t}$ that Algorithm 3 returns for the constructed instance. Thus, showing $\mathbf{A}$ is TEF1 is equivalent to showing $\mathbf{A}^{t,m_t}$ is EF1 for all t, and we apply induction to show that. For convenience, we refer to (t, i) as the time slot that $I_{t,i}$ arrives on the virtual timeline.

$\mathbf{A}^{0,0} = <\emptyset, \emptyset>$ and is trivially EF1. Notice that the strategy to allocate $(I_{t,1}, I_{t,2})$ is the same as in Algorithm 1. With the same argument in Lemma 5, we have $\mathbf{A}^{t,2}$ is EF1 as long as $\mathbf{A}^{t-1,m_{t-1}}$ is EF1. In addition, the strategy to allocate the rest of the items is the same as that in He et al. [2019]. With the same argument as in their proof, we have $\mathbf{A}^{t,i}$ is EF1 as long as $\mathbf{A}^{t,i-1}$ is EF1. Finally, we have $\mathbf{A}^{\mathbf{t},\mathbf{m_t}}$ is EF1. □

Again, since TEF1 implies TProp1, the allocation is also TProp1.

Corollary 15. *When there are two agents and $m_t \geq 2$ for any t, Algorithm 3 returns a TProp1 allocation in $O(m)$.*

8 CONCLUSION

We study the online fair allocation problem with demand-side time-dependent weight, focusing primarily on two fairness concepts: temporal envy-freeness (TEF) and temporal proportionality (TProp). For two agents, we demonstrate that a constant function is both sufficient and necessary to guarantee that TEF1 allocations always exist. Additionally, when $f(1)$ is the minimum among all $f(t)$, TProp1 allocations always exist, while when $f(1)$ is larger than all other $f(t)$, we find a counter-example showing that TProp1 allocations may not exist. Furthermore, we present two polynomial-time algorithms: the first one returns a TEF2 allocation when $f(1) = f(2) = f(3) \geq f(4), f(5)$, and the second one yields a TProp2 allocation when $f(1) \leq f(2), f(3)$. For more than two agents, for a constant f, we show that determining the existence of TEF1 allocations is NP-complete, while TEF1 allocations always exist with specific valuation profiles. For a general f, we provide an $O(T \log n)$ algorithm that always returns a TEF1/TProp1 allocation when agents are identical. For multi-item per round, we also provide a polynomial algorithm that always returns a TEF1/TProp1 allocation.

For future directions, while the hardness of finding a TEF1 allocation has been proven, the complexity of finding a TProp1 allocation remains an open problem. Moreover, it is also interesting to explore other fairness notions such that the allocation is always fair, no matter what the time-dependent weight f is.

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References

Martin Aleksandrov and Toby Walsh. Expected outcomes and manipulations in online fair division. In Gabriele Kern-Isberner, Johannes Fürnkranz, and Matthias Thimm, editors, *KI 2017: Advances in Artificial Intelligence - 40th Annual German Conference on AI, Dortmund, Germany, September 25-29, 2017, Proceedings*, volume 10505 of *Lecture Notes in Computer Science*, pages 29–43. Springer, 2017a. doi: 10.1007/978-3-319-67190-1_3. URL https://doi.org/10.1007/978-3-319-67190-1_3.

Martin Aleksandrov and Toby Walsh. Most competitive mechanisms in online fair division. In Gabriele Kern-Isberner, Johannes Fürnkranz, and Matthias Thimm, editors, *KI 2017: Advances in Artificial Intelligence - 40th Annual German Conference on AI, Dortmund, Germany, September 25-29, 2017, Proceedings*, volume 10505 of *Lecture Notes in Computer Science*, pages 44–57. Springer, 2017b. doi: 10.1007/978-3-319-67190-1_4. URL https://doi.org/10.1007/978-3-319-67190-1_4.

Martin Aleksandrov and Toby Walsh. Pure Nash equilibria in online fair division. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*, pages 42–48, 2017c. doi: 10.24963/ijcai.2017/7. URL https://doi.org/10.24963/ijcai.2017/7.

Martin Aleksandrov and Toby Walsh. Online fair division: A survey. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(09):13557–13562, Apr. 2020. doi: 10.1609/aaai.v34i09.7081. URL https://ojs.aaai.org/index.php/AAAI/article/view/7081.

Martin Aleksandrov, Haris Aziz, Serge Gaspers, and Toby Walsh. Online fair division: Analysing a food bank problem. In Qiang Yang and Michael J. Wooldridge, editors, *Proceedings of the Twenty-Fourth International Joint Conference on Artificial Intelligence, IJCAI 2015, Buenos Aires, Argentina, July 25-31, 2015*, pages 2540–2546. AAAI Press, 2015. URL http://ijcai.org/Abstract/15/360.

Georgios Amanatidis, Haris Aziz, Georgios Birmpas, Aris Filos-Ratsikas, Bo Li, Hervé Moulin, Alexandros A. Voudouris, and Xiaowei Wu. Fair division of indivisible goods: Recent progress and open questions. *Artificial Intelligence*, 322:103965, September 2023. ISSN 0004-3702. doi: 10.1016/j.artint.2023.103965. URL http://dx.doi.org/10.1016/j.artint.2023.103965.

Georgios Amanatidis, Alexandros Lolos, Evangelos Markakis, and Victor Turmel. Online fair division for personalized 2-value instances. In *Algorithmic Game Theory: 18th International Symposium, SAGT 2025, Bath, UK, September 2–5, 2025, Proceedings*, page 209–227, Berlin, Heidelberg, 2025. Springer-Verlag. ISBN 978-3-032-03638-4. doi: 10.1007/978-3-032-03639-1_12. URL https://doi.org/10.1007/978-3-032-03639-1_12.

Siddhartha Banerjee, Chamsi Hssaine, and Sean R. Sinclair. Online fair allocation of perishable resources. *SIGMETRICS Perform. Eval. Rev.*, 51(1):55–56, June 2023. ISSN 0163-5999. doi: 10.1145/3606376.3593558. URL https://doi.org/10.1145/3606376.3593558.

Gerdus Benade, Aleksandr M. Kazachkov, Ariel D. Procaccia, and Christos-Alexandros Psomas. How to make envy vanish over time. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, EC '18, page 593–610, New York, NY, USA, 2018. Association for Computing Machinery. ISBN 9781450358293. doi: 10.1145/3219166.3219179. URL https://doi.org/10.1145/3219166.3219179.

Steven J. Brams and Alan D. Taylor. A procedure for divorce settlements. *Mediation Quarterly*, 13(3):191–205, 2023/01/12 1996.

Nikhil Chandak, Shashwat Goel, and Dominik Peters. Proportional aggregation of preferences for sequential decision making. *Proceedings of the AAAI Conference on Artificial Intelligence*, 38(9):9573–9581, Mar. 2024. doi: 10.1609/aaai.v38i9.28813. URL https://ojs.aaai.org/index.php/AAAI/article/view/28813.

Benjamin Cookson, Soroush Ebadian, and Nisarg Shah. Temporal fair division. *Proceedings of the AAAI Conference on Artificial Intelligence*, 39(13):13727–13734, Apr. 2025. doi: 10.1609/aaai.v39i13.33500. URL https://ojs.aaai.org/index.php/AAAI/article/view/33500.

Edith Elkind, Sonja Kraiczy, and Nicholas Teh. Fairness in temporal slot assignment. In *Algorithmic Game Theory: 15th International Symposium, SAGT 2022, Colchester, UK, September 12–15, 2022, Proceedings*, page 490–507, Berlin, Heidelberg, 2022. Springer-Verlag. ISBN 978-3-031-15713-4. doi: 10.1007/978-3-031-15714-1_28. URL https://doi.org/10.1007/978-3-031-15714-1_28.

Edith Elkind, Alexander Lam, Mohamad Latifian, Tzeh Yuan Neoh, and Nicholas Teh. Temporal fair division of indivisible items, 2024a. URL https://arxiv.org/abs/2410.14593.

Edith Elkind, Tzeh Yuan Neoh, and Nicholas Teh. *Temporal Elections: Welfare, Strategyproofness, and Proportionality*. IOS Press, October 2024b. ISBN 9781643685489.

doi: 10.3233/faia240877. URL http://dx.doi.org/10.3233/FAIA240877.

Edith Elkind, Svetlana Obraztsova, and Nicholas Teh. Temporal fairness in multiwinner voting. *Proceedings of the AAAI Conference on Artificial Intelligence*, 38(20): 22633–22640, Mar. 2024c. doi: 10.1609/aaai.v38i20.30273. URL https://ojs.aaai.org/index.php/AAAI/article/view/30273.

Samiran Ghosh, Malay Banerjee, and Amit K. Chattopadhyay. Effect of vaccine dose intervals: Considering immunity levels, vaccine efficacy, and strain variants for disease control strategy. *PLOS ONE*, 19(9):1–24, 09 2024. doi: 10.1371/journal.pone.0310152. URL https://doi.org/10.1371/journal.pone.0310152.

Jonathan Goldman and Ariel D. Procaccia. Spliddit: Unleashing fair division algorithms. *SIGecom Exch.*, 13(2): 41–46, jan 2015.

Jiafan He, Ariel D. Procaccia, Alexandros Psomas, and David Zeng. Achieving a fairer future by changing the past. In *Proceedings of the Twenty-Eighth International Joint Conference on Artificial Intelligence, IJCAI-19*, pages 343–349. International Joint Conferences on Artificial Intelligence Organization, 7 2019. doi: 10.24963/ijcai.2019/49. URL https://doi.org/10.24963/ijcai.2019/49.

Ian Kash, Ariel D. Procaccia, and Nisarg Shah. No agent left behind: dynamic fair division of multiple resources. In *Proceedings of the 2013 International Conference on Autonomous Agents and Multi-Agent Systems*, AAMAS '13, page 351–358, Richland, SC, 2013. International Foundation for Autonomous Agents and Multiagent Systems. ISBN 9781450319935.

Martin Lackner. Perpetual voting: Fairness in long-term decision making. *Proceedings of the AAAI Conference on Artificial Intelligence*, 34(02):2103–2110, Apr. 2020. doi: 10.1609/aaai.v34i02.5584. URL https://ojs.aaai.org/index.php/AAAI/article/view/5584.

Andrew Mackenzie and Vilmos Komornik. Fairly taking turns. *Games and Economic Behavior*, 142:743–764, 2023. ISSN 0899-8256. doi: https://doi.org/10.1016/j.geb.2023.09.005. URL https://www.sciencedirect.com/science/article/pii/S0899825623001422.

Nicholas Mattei, Abdallah Saffidine, and Toby Walsh. Mechanisms for online organ matching. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*, pages 345–351, 2017. doi: 10.24963/ijcai.2017/49. URL https://doi.org/10.24963/ijcai.2017/49.

Nicholas Mattei, Abdallah Saffidine, and Toby Walsh. Fairness in deceased organ matching. In *Proceedings of the 2018 AAAI/ACM Conference on AI, Ethics, and Society*, AIES '18, page 236–242, New York, NY, USA, 2018. Association for Computing Machinery. ISBN 9781450360128. doi: 10.1145/3278721.3278749. URL https://doi.org/10.1145/3278721.3278749.

Marios Mertzanidis, Alexandros Psomas, and Paritosh Verma. Automating food drop: The power of two choices for dynamic and fair food allocation, 2024. URL https://arxiv.org/abs/2406.06363.

Hugo Steinhaus. The problem of fair division. *Econometrica*, 16:101–104, 1948.

Liansheng Tan and Yongchang Zhang. Optimal resource allocation with principle of equality and diminishing marginal utility in wireless networks. *Wireless Personal Communications*, 84, 09 2015. doi: 10.1007/s11277-015-2655-0.

Shai Vardi, Alexandros Psomas, and Eric Friedman. Dynamic fair resource division. *Math. Oper. Res.*, 47 (2):945–968, May 2022. ISSN 0364-765X. doi: 10.1287/moor.2021.1155. URL https://doi.org/10.1287/moor.2021.1155.

Toby Walsh. Online cake cutting, 2010. URL https://arxiv.org/abs/1007.0614.

Toby Walsh. Allocation in practice. In Carsten Lutz and Michael Thielscher, editors, *KI 2014: Advances in Artificial Intelligence*, pages 13–24, Cham, 2014. Springer International Publishing. ISBN 978-3-319-11206-0.

Toby Walsh. Challenges in resource and cost allocation. In *Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence*, AAAI'15, page 4073–4077. AAAI Press, 2015. ISBN 0262511290.

Yuanyuan Wang and Tianze Wei. Online fair allocations with binary valuations and beyond. *CoRR*, abs/2505.24321, 2025. doi: 10.48550/ARXIV.2505.24321. URL https://doi.org/10.48550/arXiv.2505.24321.

Valentin Zech, Niclas Boehmer, Edith Elkind, and Nicholas Teh. Multiwinner temporal voting with aversion to change. In *Proceedings of the 27th European Conference on Artificial Intelligence (ECAI)*, 2024.

Appendix

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PROOF OF LEMMA 2

Proof. We begin by proving TEF1 allocations may not exist when $\exists T_1, f(1)f(2) \neq f(T_1)f(T_1+1)$ through an example. Then we show that a function satisfying $\forall T_1, f(1)f(2) = f(T_1)f(T_1+1)$ is a periodic function with period 2, which is a necessary condition to guarantee the existence of TEF1 allocations.

Assume that $\exists T_1 \geq 2, f(1)f(2) > f(T_1)f(T_1+1)$. We construct the valuation profile (see Table 1),

Time	*1*	*2*	*3*	$\cdots$	T_1	T_1+1	T_1+2	T_1+3	T_1+4	T_1+5
a_1	$\frac{1}{f(1)}\delta$	$\frac{1}{f(2)}\gamma_1\delta$	$\frac{1}{f(1)}\gamma_2\delta$	$\cdots$	$\frac{1}{f(1)}\gamma_{T_1-1}\delta$	$\frac{1}{f(T_1)}$	$\frac{1}{f(2)}(1+\lambda)$	$\frac{\Delta}{f(1)}$	$\frac{\Delta}{f(3)}(1+\lambda)$	w
a_2	$\frac{\theta}{f(1)}\delta$	$\frac{1}{f(2)}\gamma_1\delta$	$\frac{1}{f(1)}\gamma_2\delta$	$\cdots$	$\frac{1}{f(1)}\gamma_{T_1-1}\delta$	$\frac{1}{f(T_1)}(1+\lambda)$	$\frac{1}{f(2)}$	$\frac{\Delta}{f(1)}(1+\lambda)$	$\frac{\Delta}{f(3)}$	w

Table 1: The constructed valuation profile for Lemma 2

where $\delta \ll 1 \ll \Delta \ll w$ are some positive constants, and $\theta, \gamma_1, \cdots, \gamma_{T_1-1}, \lambda$ satisfy:

$$\sum_{j=1}^{T_1-1} \gamma_j < 1 \tag{1}$$

$$\gamma_j > 0, \qquad \forall 1 \leq j \leq T_1 - 1 \tag{2}$$

$$\frac{f(t)}{f(1)} > \frac{\sum_{i=1}^{t-2} \gamma_i}{\gamma_{t-1}}, \qquad \forall 3 \leq t \leq T_1 \tag{3}$$

$$\sum_{j=1}^{T_1-2} \gamma_j < \theta < \sum_{j=1}^{T_1-1} \gamma_j \tag{4}$$

$$0 < \lambda < \frac{f(1)f(2)}{f(T_1)f(T_1+1)} - 1 \tag{5}$$

We first show that the only TEF1 allocation of the first T_1 items is $(A_i = \{I_1\}, A_{3-i} = \{I_2, I_3, \cdots, I_{T_1}\})$ by induction for some $i \in \{1, 2\}$. Agent a_i receives $A_i^2 = \{I_1\}$ and agent a_{3-i} receives $A_{3-i}^2 = \{I_2\}$ after time slot 2 (It is not hard to find

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that any other allocation of the first two items does not satisfy TEF1). Suppose that agent a_i receives $A_i^t = \{I_1\}$ and agent a_{3-i} receives $A_{3-i}^t = \{I_2, I_3, \cdots, I_t\}$ after time slot t for some $2 \le t \le T_1 - 1$. For time slot $t+1$, if I_{t+1} is allocated to agent a_i,

- suppose agent a_i gives up the item I_1. With constraint (3), the following deduction showing a_{3-i} still envies a_i.

$$\begin{aligned}
&\frac{f(t+1)}{f(1)} > \frac{\sum_{j=1}^{t-1}\gamma_j}{\gamma_t}\\
\Rightarrow\quad & u_{3-i}(A_i^t \cup \{I_{t+1}\} \setminus \{I_1\}) = u_{3-i}(\{I_{t+1}\})\\
&= f((t+1)-0) \times \frac{1}{f(1)}\gamma_t\delta = \frac{f(t+1)}{f(1)}\gamma_t\delta\\
&> (\sum_{j=1}^{t-1}\gamma_j)\delta = u_{3-i}(A_{3-i}^t)
\end{aligned}$$

- suppose agent a_i gives up the item I_{t+1}. With constraint (4), the following deduction showing a_{3-i} still envies a_i.

$$\begin{aligned}
&\theta > \sum_{j=1}^{T_1-2}\gamma_j \ge \sum_{j=1}^{t-1}\gamma_j\\
\Rightarrow\quad & u_{3-i}(A_i^t \cup \{I_{t+1}\} \setminus \{I_{t+1}\}) = u_{3-i}(A_i^t) = u_{3-i}(\{I_1\})\\
&> f(1-0) \times \frac{\theta}{f(1)}\delta = \theta\delta\\
&> (\sum_{j=1}^{t-1}\gamma_j)\delta = u_{3-i}(A_{3-i}^t)
\end{aligned}$$

Therefore, no matter which item a_i gives up, a_{3-i} still envies a_i, and the allocation $(A_i^t \cup \{I_{t+1}\}, A_{3-i}^t)$ does not satisfy TEF1. Hence, I_{t+1} is allocated to agent a_{3-i}.

	$A_1^{T_1+2}$	$A_2^{T_1+2}$
a_1	$f((T_1+1)-1) \times \frac{1}{f(T_1)} = 1$	$f((T_1+2)-T_1) \times \frac{1}{f(2)}(1+\lambda) = 1+\lambda$
a_2	$f((T_1+1)-1) \times \frac{1}{f(T_1)}(1+\lambda) = 1+\lambda$	$f((T_1+2)-T_1) \times \frac{1}{f(2)} = 1$

(a) a_1 receives I_{T_1+1} (We ignore the utility of the first T_1 items, since δ is too small)

	$A_1^{T_1+2}$	$A_2^{T_1+2}$
a_1	$f((T_1+2)-1) \times \frac{1}{f(2)}(1+\lambda) = \frac{f(T_1+1)}{f(2)}(1+\lambda)$	$f((T_1+1)-T_1) \times \frac{1}{f(T_1)} = \frac{f(1)}{f(T_1)}$
a_2	$f((T_1+2)-1) \times \frac{1}{f(2)} = \frac{f(T_1+1)}{f(2)}$	$f((T_1+1)-T_1) \times \frac{1}{f(T_1)}(1+\lambda) = \frac{f(1)}{f(T_1)}(1+\lambda)$

(b) a_2 receives I_{T_1+1} (We ignore the utility of the first T_1 items, since δ is too small)

	$A_1^{T_1+4}$	$A_2^{T_1+4}$
a_1	$f((T_1+3)-(T_1+2)) \times \frac{\Delta}{f(1)} = \Delta$	$f((T_1+4)-(T_1+1)) \times \frac{\Delta}{f(3)}(1+\lambda) = \Delta(1+\lambda)$
a_2	$f((T_1+3)-(T_1+2)) \times \frac{\Delta}{f(1)}(1+\lambda) = \Delta(1+\lambda)$	$f((T_1+4)-(T_1+1)) \times \frac{\Delta}{f(3)} = \Delta$

(c) a_2 receives I_{T_1+2} (We ignore the utility of the first T_1+2 items, since Δ is too large)

Table 2: The agents' utilities in Lemma 2, Case 1

Case 1: $i=1$, i.e., agent a_1 receives the first item. After allocating I_{T_1}, the allocation is envy-free (agent a_1 receives $f(1-0) \times \frac{1}{f(1)}\delta = \delta$ thinking that agent a_2 only receives $f(2-0) \times \frac{1}{f(2)}\gamma_1\delta + \sum_{j=3}^{T_1} f(j-(j-1)) \times \frac{1}{f(1)}\gamma_{j-1}\delta =$

$(\sum_{j=1}^{T_1-1}\gamma_j)\delta < \delta$; agent a_2 receives $(\sum_{j=1}^{T_1-1}\gamma_j)\delta$ thinking that agent a_1 receives $f(1-0)\times\frac{1}{f(1)}\theta\delta = \theta\delta < (\sum_{j=1}^{T_1-1}\gamma_j)\delta)$. Then who receives the item I_{T_1+1} leads to two subcases.

If agent a_1 receives I_{T_1+1}, to fulfill TEF1, agent a_2 will receive I_{T_1+2}. The two agents' utilities are in Table 2a, showing that agent a_1 and agent a_2 envy each other after allocating I_{T_1+2}. Assume that it is agent $a_{i'}$ who receives I_{T_1+3}. Then for agent $a_{3-i'}$, he will envy agent $a_{i'}$,

- even if agent $a_{i'}$ gives up I_{T_1+3}, since they envy each other in allocation $\mathbf{A}^{T_1+2}$;
- even if agent $a_{i'}$ gives up an item other than I_{T_1+3}, since Δ is large enough.

Thus, no matter who receives I_{T_1+3}, it will break TEF1.

If agent a_2 receives I_{T_1+1}, to fulfill TEF1, agent a_1 will receive I_{T_1+2}. The two agents' utilities are in Table 2b. Since $0 < \lambda < \frac{f(1)f(2)}{f(t)f(t+1)} - 1$, agent a_1 will envy agent a_2, while agent a_2 will not envy agent a_1 after allocating I_{T_1+2}. Then I_{T_1+3} will be allocated to agent a_1, since Δ is too large. Again, as Δ is too large, I_{T_1+4} will be allocated to agent a_2. The two agents' utilities are in Table 2c. Similar to allocating I_{T_1+1} to agent a_1, with $w \gg \Delta$, it will break TEF1.

Case 2: $i = 2$**, i.e., agent** a_2 **receives the first item.** After allocating I_{T_1}, the two agents envy each other (agent a_1 receives $(\sum_{j=1}^{T_1-1}\gamma_j)\delta$ thinking that agent a_2 receives $\delta > (\sum_{j=1}^{T_1-1}\gamma_j)\delta$; agent a_2 receives $\theta\delta$ thinking that agent a_1 receives $(\sum_{j=1}^{T_1-1}\gamma_j)\delta > \theta\delta)$. Similar to Case 1, with $\delta \ll 1$, it will break TEF1.

In summary, the constructed instance shows that TEF1 allocations may not exist when $\exists T_1 \geq 2, f(1)f(2) > f(T_1)f(T_1+1)$. On the other hand, we adapt the instance in Table 1 by changing the valuation of I_{T_1+3}, I_{T_1+4} as follows:

Time	$\cdots$	T_1+3	T_1+4	$\cdots$
a_1		$\frac{\Delta}{f(2)}(1+\lambda)$	$\frac{\Delta}{f(2)}$	
a_2		$\frac{\Delta}{f(2)}$	$\frac{\Delta}{f(2)}(1+\lambda)$	

We also update constraint (5) for λ as follows:

$$0 < \lambda < \frac{f(T_1)f(T_1+1)}{f(1)f(2)} - 1,$$

With similar statements, we show that TEF1 allocations may not exist when $\exists T_1 \geq 2, f(1)f(2) < f(T_1)f(T_1+1)$. Thus, TEF1 allocations may not exist when $\exists T_1 \geq 2, f(1)f(2) \neq f(T_1)f(T_1+1)$. In other words, TEF1 allocations always exist only if $\forall T_1 \geq 2, f(1)f(2) = f(T_1)f(T_1+1)$.

Since $f(1)f(2) = f(T_1)f(T_1+1)$ holds for all $T_1 \geq 2$, $f(1)f(2) = f(T_1)f(T_1+1) = f(T_1+1)f(T_1+2)$ implying $f(T_1) = f(T_1+2)$, meaning that f is a periodic function with period 2. Then we can conclude that TEF1 allocations always exist for two agents only if the weight function f is a periodic function with period 2. $\square$

PROOF OF LEMMA 3

Proof. By Lemma 2, we know that TEF1 allocations always exist only if f is a periodic function with period 2. Below, we show through an example that when f is a periodic function with period 2 and $f(1) \neq f(2)$, TEF1 allocations may not exist, meaning that TEF1 allocations always exist only if f is a constant function.

Assume that f is a periodic function with period 2 and $f(1) > f(2)$, implying $\frac{f(2)}{f(1)} < 1$. We construct the valuation profile (See Table 3),

Time	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
a_1	$\frac{1}{f(1)}\delta$	$\frac{\theta}{f(2)}\delta$	$\frac{1}{f(1)}$	$\frac{1}{f(1)}\gamma_1$	$\frac{1}{f(1)}\gamma_2$	$\frac{\Delta}{f(1)}$	$\frac{\Delta}{f(2)}(1+\lambda)$	w
a_2	$\frac{\theta}{f(1)}\delta$	$\frac{1}{f(2)}\delta$	$\frac{\theta}{f(1)}$	$\frac{1}{f(1)}\gamma_1$	$\frac{1}{f(1)}\gamma_2$	$\frac{\Delta}{f(1)}(1+\lambda)$	$\frac{\Delta}{f(2)}$	w

Table 3: The constructed valuation profile for Lemma 3

where $\delta \ll 1 \ll \Delta \ll w$, and $\theta, \gamma_1, \gamma_2, \lambda$ satisfy:

$$\gamma_1 < \theta < \gamma_1 + \gamma_2 < 1 \tag{6}$$

$$\gamma_2 > \gamma_1 \tag{7}$$

$$\frac{\gamma_2}{1-\gamma_1} > \frac{f(2)}{f(1)} \tag{8}$$

$$\lambda > 0 \tag{9}$$

For the first two items, if they are assigned to the same agent, the allocation is not TEF1. If a_1 receives I_2 and a_2 receives I_1, the two agents envy each other so that TEF1 breaks at time slot 3, since $\delta \ll 1$. Thus, a_1 receives I_1 and a_2 receives I_2. Then we have two cases for the rest of the items:

Case 1: a_1 **receives** I_3 Then a_2 receives I_4, and a_2 envies a_1 after allocating the item at time 4. If a_1 receives I_5, the allocation is $\mathbf{A}^5 = (A_1^5 = \{I_1, I_3, I_5\}, A_2^5 = \{I_2, I_4\})$, and a_2 will envy a_1,

- even if a_1 gives up I_1, since δ is too small;
- even if a_1 gives up I_3, since $f(5-1) * \frac{1}{f(1)}\gamma_2 = \frac{f(4)}{f(1)}\gamma_2 = \frac{f(2)}{f(1)}\gamma_2 > f(4-2)\frac{1}{f(1)}\gamma_1 = \frac{f(2)}{f(1)}\gamma_1$;
- even if a_1 gives up I_5, since a_2 already envies a_1 at time 4.

Thus, it must be a_2 who receives I_5.

	A_1^5	A_2^5
a_1	$\frac{f(2)}{f(1)}$	$\frac{f(2)}{f(1)}\gamma_1 + \gamma_2$
a_2	$\frac{f(2)}{f(1)}\theta$	$\frac{f(2)}{f(1)}\gamma_1 + \gamma_2$

Table 4: The agents' utilities in Lemma 3, Case 1 (We ignore the utility of the first two items, since δ is too small)

After time 5, the allocation is $\mathbf{A}^5 = (A_1^5 = \{I_1, I_3\}), A_2^5 = \{I_2, I_4, I_5\})$ and the utilities of the two agents are shown in Table 4. Then, due to constraint (8), we have

$$\frac{\gamma_2}{1-\gamma_1} > \frac{f(2)}{f(1)} \Rightarrow \frac{f(2)}{f(1)} < \frac{f(2)}{f(1)}\gamma_1 + \gamma_2,$$

implying that a_1 envies a_2. Again, due to constraint (6), we have

$$\frac{f(2)}{f(1)}\gamma_1 + \gamma_2 > \frac{f(2)}{f(1)}(\gamma_1 + \gamma_2) > \frac{f(2)}{f(1)}\theta,$$

implying that a_2 does not envy a_1. Since Δ is too large, it must be a_1 who receives I_6. Then a_2 receives I_7, and they envy each other at time 7. Since $\Delta \ll w$, it must break TEF1 at time 8, no matter who receives I_8.

Case 2: a_2 **receives** I_3 Then a_1 receives I_4, and a_1 envies a_2 after allocating the item at time 4. If a_2 receives I_5, the allocation is $\mathbf{A}^5 = (A_1^5 = \{I_1, I_4\}, A_2^5 = \{I_2, I_3, I_5\})$, and a_1 will envy a_2,

- even if a_2 gives up I_2, since δ is too small;
- even if a_2 gives up I_3, since $f(5-2)\frac{1}{f(1)}\gamma_2 = \frac{f(3)}{f(1)}\gamma_2 = \gamma_2 > f(4-1)\frac{1}{f(1)}\gamma_1 = \frac{f(3)}{f(1)}\gamma_1 = \gamma_1$;
- even if a_2 gives up I_5, since a_1 already envies a_2 at time 4.

Thus, it must be a_1 who receives I_5.

After time 5, the allocation is $\mathbf{A}^5 = (A_1^5 = \{I_1, I_4, I_5\}, A_2^5 = \{I_2, I_3\})$ and the utilities of the two agents are shown in Table 5. Since $\theta < \gamma_1 + \gamma_2 < 1$ and δ is too small, they envy each other. Since Δ is too large, it must break TEF1 at time 6, no matter who receives I_6.

In summary, the constructed instance shows that TEF1 allocations may not exist when f is a periodic function with period 2 and $f(1) > f(2)$. On the other hand, we adapt the instance in Table 3 by changing the valuation of I_6, I_7 as follows:

	A_1^5	A_2^5
a_1	$\gamma_1 + \gamma_2$	1
a_2	$\gamma_1 + \gamma_2$	θ

Table 5: The agents' utilities in Case 2 (We ignore the utility of the first two items, since δ is too small)

Time	$\cdots$	*6*	*7*	$\cdots$
a_1		$\frac{\Delta}{f(1)}(1+\lambda)$	$\frac{\Delta}{f(2)}$	
a_2		$\frac{\Delta}{f(1)}$	$\frac{\Delta}{f(2)}(1+\lambda)$	

We also update the constraint (8) for γ_1, γ_2 as follows:

$$\frac{\gamma_2}{\theta - \gamma_1} < \frac{f(2)}{f(1)},$$

With similar statements, we show that TEF1 allocations may not exist when f is a periodic function with period 2 and $f(1) < f(2)$. Thus, TEF1 allocations may not exist when f is a periodic function with period 2 and $f(1) \neq f(2)$. In other words, TEF1 allocations always exist only if f is a periodic function with period 2 and $f(1) = f(2)$, which is a constant function. □

EXPERIMENT

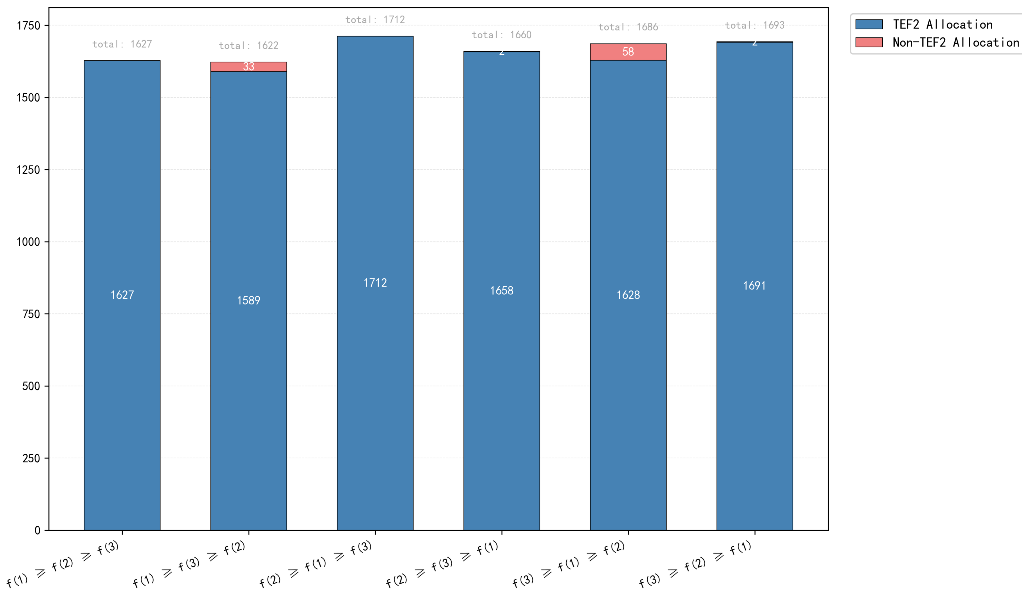
We demonstrate the performance of Algorithm 1 and Algorithm 2 through experiments. The implementations were performed on a computer with an Intel Core i7-8550U processor (4 cores, 8 threads, base frequency 1.80 GHz, turbo boost up to 4.00 GHz), 16.00 GB RAM, and running Windows 10 64-bit operating system.

As for data, we generate 10000 instances for each algorithm. In each instance, we assume 2 agents, 100 items, and the values of items are randomly generated from [0, 100000]. Moreover, the value of the time-dependent weight function is also randomly chosen from [0, 100000].

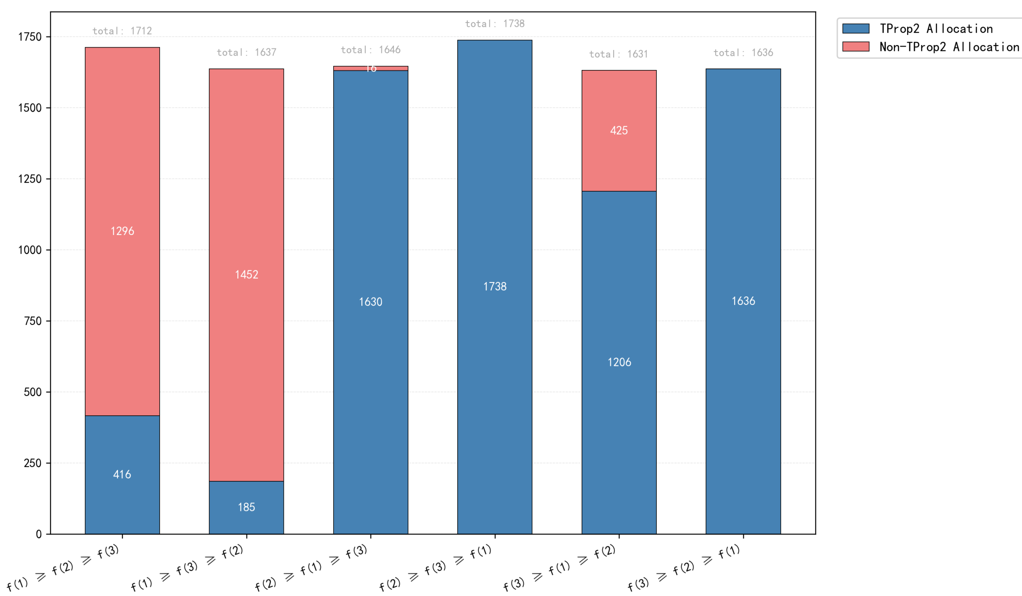
Results For Algorithm 1, Figure 3a shows that we can obtain a TEF2 allocation with high probability, even if $f(1), f(2), f(3)$ are not equal.

For Algorithm 2, Figure 3b shows that we can always obtain a TProp2 allocation when $f(1) \leq f(2), f(3)$, which fits the result of our analysis above. Moreover, the algorithm has a probability of around 87% to return a TProp2 allocation when $f(1)$ is between $f(2), f(3)$. However, When $f(1) \geq f(2), f(3)$, there are only around 18% istances that Algorithm 2 returns TProp2 allocations.

The difference in the two algorithms' performance may be caused by the condition to choose an agent to be advantaged. When the agents' utilities are more than the current proportional share, Algorithm 1 will choose the one who envies the other to be advantaged, while Algorithm 2 will choose agent a_2 to be advantaged. These might be the reasons why Algorithm 1 performs much better than Algorithm 2.



(a) The results of the Algorithm 1



(b) The Results of the Algorithm 2

Figure 3: The Results of Experiments