Tight and near-tight rates of approximation of mixed Nash equilibria by entropy regularization in continuous games

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Abstract

We study how well the quantal response equilibrium (QRE) approximates the mixed Nash equilibrium (MNE) in two-player zero-sum games, as a function of the inverse temperature β. We introduce a reduction framework that decomposes the worst-case Nikaido–Isoda error of the QRE into a sum of two independent single-player Gibbs concentration problems, enabling tight matching bounds across a range of payoff classes. For finite games with $M \times N$ payoff matrices, we establish a tight rate of $\Theta(\beta^{-1}(\log M + \log N))$. For games with α-Hölder-continuous payoffs on the torus, we prove a lower bound of $\Omega((d_x + d_y)/(\alpha\beta))$ and an upper bound of $\mathcal{O}((d_x + d_y)\log(L\beta)/(\alpha\beta))$; whether the logarithmic gap is tight is left as an open problem. For smooth payoffs with sparse non-degenerate MNE, we prove that every individual game achieves a rate of $(d_x+d_y)/(2\beta)+\mathcal{O}(1/\beta^2)$ via Laplace concentration, and that this rate is tight.

1 INTRODUCTION

A fundamental problem in game theory and its applications is the computation of Nash equilibria. In a two-player zero-sum game, one seeks a pair of mixed strategies such that neither player can unilaterally improve their expected payoff. The existence of such *mixed Nash equilibria* (MNE) is guaranteed under broad conditions: for finite games by Nash's theorem [14], and for games with compact strategy spaces and continuous payoffs by Glicksberg's generalization [8]. This framework encompasses a range of formulations arising in machine learning, including generative adversarial networks [9] and multi-agent reinforcement learning [1].

Despite their guaranteed existence, computing MNE is fundamentally hard. Even in two-player finite games, finding an exact Nash equilibrium for nonzero sum games is PPAD-complete [4, 5], and no efficient algorithm is known without additional structural assumptions on the payoff function. A natural and widely studied relaxation is to introduce entropy regularization: rather than solving the original min-max problem, one solves a smoothed version in which each player's objective includes an entropic penalty that encourages exploration over the strategy space. The resulting equilibrium, known as the *quantal response equilibrium* (QRE), was originally introduced in economics to model bounded rationality [11]. From a computational perspective, the strict convexity-concavity induced by the entropy terms makes the QRE uniquely defined and efficiently computable for any fixed regularization strength [3, 6].

It is well known that the QRE converges to an MNE as the regularization vanishes (equivalently, as the inverse temperature $\beta \to \infty$). However, the *rate* of this convergence has received comparatively little attention, particularly from the perspective of lower bounds. Upper bounds on the approximation error - measured by the Nikaido–Isoda duality gap - have been established in several settings: for finite games, the QRE approximates an MNE to within $\mathcal{O}(\beta^{-1}(\log M + \log N))$ where M, N are the action-set sizes [3]; for continuous games with Lipschitz payoffs, analogous bounds involving the Lipschitz constant and the geometry of the strategy space have been obtained [6].

Contributions. In this work, we study tight rates of approximation of MNE by QRE in two-player zero-sum games, focusing on non-asymptotic lower bounds that complement known upper bounds. Our results proceed through a reduction principle (Section 4.2) that decomposes the two-player approximation problem into two independent single-player problems, each involving the approximation of the minimum of a function by its Gibbs measure. This reduction is valid for both upper and lower bounds under a structural condition on the payoff class (Lemmas 2 and 3), and we verify this condition in each setting we consider.

A word on the word "tight." Our bounds match up to constant factors in two of the settings below: the finite case and,

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instance-wise, the smooth case with sparse non-degenerate MNE. In the Hölder-continuous setting our upper and lower bounds match only up to a logarithmic factor in β, and whether that factor is real remains open (Section 6.2); the title reflects this by "near-tight."

We then apply this framework to derive bounds in the following settings:

- **Finite games** (Section 5): For bounded payoff matrices with M and N actions, we show that the worst-case approximation error scales as $\Theta(\beta^{-1}(\log M + \log N))$, matching the known upper bound of [3] with a novel lower bound up to constant factors (Proposition 5).
- **Continuous games with bounded payoffs** (Section 6): We show that for the class of all bounded continuous payoffs, the approximation error does not vanish as $\beta \to \infty$, demonstrating the necessity of quantitative regularity assumptions (Proposition 10).
- **Hölder-continuous payoffs** (Section 6.2): Under an α-Hölder continuity constraint, we establish matching bounds up to a logarithmic factor (Lemmas 12 and 13).
- **Smooth payoffs with sparse non-degenerate MNE** (Section 7): For smooth games whose MNE are finitely supported with nondegenerate curvature, we prove an instance-wise rate of $(d_x + d_y)/(2\beta) + \mathcal{O}(1/\beta^2)$ via Laplace concentration (Proposition 16).

2 RELATED WORKS

Nash equilibria. The existence of mixed Nash equilibria in finite games was established by Nash [14]; Glicksberg [8] extended this to games with compact strategy spaces and continuous payoffs via the Kakutani fixed-point theorem. For continuous games, the structure of MNE is considerably less understood. Karlin [10] showed that certain classes of continuous games - including separable games and bell-shaped games - admit finitely-supported MNE; see also Parrilo [16] and Stein, Ozdaglar, and Parrilo [18] for modern treatments of separable and low-rank games using sum-of-squares and semidefinite programming methods. In particular, [18] showed that separable games always admit finitely-supported Nash equilibria, with each player's support size bounded by the rank of the payoff kernel plus one. Beyond these classes, relatively little is known about the structure of MNE in continuous games. Wang and Chizat [19] also recently introduced sparsity and non-degeneracy assumptions on MNE.

Computational complexity and algorithms. Computing an exact Nash equilibrium is PPAD-complete even for two-player general-sum games [4, 5], motivating the study of tractable relaxations. Entropy regularization produces the quantal response equilibrium (QRE), first introduced by McKelvey and Palfrey [11] for general N-player games. For two-player zero-sum finite games, Cen, Wei, and Chi [3] developed fast policy extragradient methods with linear convergence to the QRE. In the continuous setting, Domingo-Enrich et al. [6] analyzed Langevin descent-ascent dynamics and established convergence to the QRE under a mean-field formulation, and Cai et al. [2] recently proved exponential convergence of min-max Langevin dynamics to QRE under strong convexity-concavity assumptions. Entropy-regularized equilibria also arise naturally as the long-run outcome of no-regret learning dynamics: multiplicative weights update (i.e., entropic mirror descent) applied by both players yields approximate Nash equilibria in zero-sum games [7, 13], and optimistic variants achieve faster $\mathcal{O}(\log T/T)$ convergence [17].

Approximation quality of QRE. While the above works focus on efficiently *computing* QRE, they leave open a more quantitative question: what is the optimal rate at which QRE approximate unregularized MNE as the regularization vanishes? Existing results largely give one-sided guarantees. In finite games, the classical log-sum-exp smoothing inequality [15] yields the upper bound observed by Cen et al. [3], namely that the Nikaido–Isoda error of the QRE is $\mathcal{O}(\beta^{-1}(\log M + \log N))$. In continuous games, Domingo-Enrich et al. [6] derived analogous upper bounds depending on payoff regularity and dimension. These estimates show that QRE converge to MNE, but they do not determine whether the dependence on β, dimension, or action-set size is intrinsic. The purpose of this work is to address this sharpness question by proving lower bounds, and in several settings matching or near-matching rates, for the QRE approximation error.

3 PROBLEM SETTING AND STATEMENT

3.1 PROBLEM SETTING

Two-player zero-sum games. We consider two-player zero-sum games, specified by a triple $(\ell, \mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are the strategy sets of Players 1 and 2 respectively (which may be discrete or continuous), and $\ell : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is the payoff function. When the players adopt mixed strategies $\mu \in \mathcal{P}(\mathcal{X})$ and $\nu \in \mathcal{P}(\mathcal{Y})$ respectively, the expected payoff is $\iint \ell(x, y)\, \mu(\mathrm{d}x)\, \nu(\mathrm{d}y)$.

Mixed Nash equilibria (MNE). A pair $(\mu^*, \nu^*) \in \mathcal{P}(\mathcal{X}) \times \mathcal{P}(\mathcal{Y})$ is called a *mixed Nash equilibrium* (MNE) if neither player can unilaterally improve their objective by deviating. In the zero-sum setting, this is equivalent to the saddle-point condition: $\forall(\mu, \nu)$, $\iint \ell\, \mu^*(\mathrm{d}x)\, \nu(\mathrm{d}y) \le \iint \ell\, \mu(\mathrm{d}x)\, \nu^*(\mathrm{d}y)$. The common value $\rho = \iint \ell\, \mu^*(\mathrm{d}x)\, \nu^*(\mathrm{d}y)$ is the optimal value of the game.

To measure how far a given pair $(\hat{\mu}, \hat{\nu})$ is from being a MNE, the canonical metric is the Nikaido-Isoda (NI) error, a.k.a. duality gap:

$$\mathrm{NI}(\hat{\mu},\hat{\nu}) = \sup_{\mu,\nu} \iint \ell(x,y)\,\hat{\mu}(\mathrm{d}x)\,\nu(\mathrm{d}y) - \int \ell(x,y)\,\mu(\mathrm{d}x)\,\hat{\nu}(\mathrm{d}y). \tag{1}$$

One can verify that $\mathrm{NI}(\hat{\mu},\hat{\nu}) \geq 0$ for all $(\hat{\mu},\hat{\nu})$, with equality if and only if $(\hat{\mu},\hat{\nu})$ is a MNE.

Quantal response equilibria (QRE). For any $\beta > 0$, the QRE of the game $(\ell,\mathcal{X},\mathcal{Y})$ with temperature β^{-1} is defined as the unique saddle point of the strictly convex-concave problem

$$\min_{\mu\in\mathbb{P}(\mathcal{X})} \max_{\nu\in\mathbb{P}(\mathcal{Y})} \iint \ell(x,y)\mu(\mathrm{d}x)\,\nu(\mathrm{d}y) + \frac{1}{\beta}H(\mu) - \frac{1}{\beta}H(\nu) \tag{2}$$

where H denotes the (negative) Shannon entropy. The QRE is equivalently characterized as the unique solution of the fixed-point system

$$\begin{cases} \hat{\mu}(\mathrm{d}x) \propto \exp\big(-\beta \int \ell(x,y)\,\hat{\nu}(\mathrm{d}y)\big)\,\mathrm{d}x \\ \hat{\nu}(\mathrm{d}y) \propto \exp\big(\beta \int \ell(x,y)\,\hat{\mu}(\mathrm{d}x)\big)\,\mathrm{d}y. \end{cases} \tag{3}$$

It is well known that the QRE converges to a MNE as $\beta^{-1} \to 0$ [11, 6]. In this paper we quantify the rate of convergence.

3.2 PROBLEM STATEMENT

For any two-player zero-sum game $(\ell,\mathcal{X},\mathcal{Y})$, denote by NI_ℓ the Nikaido-Isoda error functional and by $(\hat{\mu}_\ell^\beta, \hat{\nu}_\ell^\beta)$ the QRE with temperature β^{-1}. We aim to derive non-asymptotic upper and lower bounds on the minimax rate

$$\varepsilon_{\mathcal{L}}(\overline{\beta}^{-1}) = \inf_{\beta^{-1}\geq\overline{\beta}^{-1}} \sup_{\ell\in\mathcal{L}} \mathrm{NI}_\ell(\hat{\mu}_\ell^\beta, \hat{\nu}_\ell^\beta). \tag{4}$$

The infimum over temperatures $\beta^{-1} \geq \overline{\beta}^{-1}$ is natural, since QREs are efficiently computable for any fixed $\beta^{-1} > 0$ [3, 6].

Upper bounds on $\varepsilon_{\mathcal{L}}(\overline{\beta}^{-1})$ have been obtained in several previous works [3, 6]. The focus of this paper is on deriving matching lower bounds.

4 REDUCTION TO A SINGLE-PLAYER ANALYSIS

A central contribution of this paper is a reduction principle that transforms the two-player QRE approximation problem into two independent single-player Gibbs concentration problems. This reduction is the mechanism that allows us to derive matching upper and lower bounds across the different payoff classes studied later (finite, Hölder, smooth sparse MNE).

At a high level, the Nikaido-Isoda error of a QRE can be written *exactly* as the sum of two excess-risk terms, one for each player. Each term measures how well a Gibbs distribution approximates the minimum of a potential function. Once this decomposition is established, the two-player minimax rate reduces to understanding worst-case Gibbs concentration over appropriate classes of potentials.

We proceed in three steps: (i) formalize the single-player problem, (ii) show how it arises from the NI decomposition at a QRE, and (iii) prove matching upper and lower reductions. Full proofs of all results in this section are given in Appendix A.

4.1 A SINGLE-PLAYER RATE-OF-APPROXIMATION PROBLEM

Fix a set $\mathcal{Z}$ (finite or continuous). Given a potential $V : \mathcal{Z} \to \mathbb{R}$ and inverse temperature $\beta > 0$, define the *Gibbs measure*

$$\mu_V^\beta(\mathrm{d}z) = \frac{e^{-\beta V(z)}\,\mathrm{d}z}{\int e^{-\beta V(z')}\,\mathrm{d}z'} \;\in\; \mathcal{P}(\mathcal{Z}). \tag{5}$$

The Gibbs measure is the unique minimizer of the strictly convex variational problem

$$\min_{\mu\in\mathcal{P}(\mathcal{Z})} \Big\{ \mathbb{E}_\mu[V] + \beta^{-1}H(\mu) \Big\},$$

and therefore represents the entropy-regularized minimizer of V. As $\beta \to \infty$, μ_V^β concentrates on $\arg\min V$.

For a class $\mathcal{C} \subset \mathbb{R}^{\mathcal{Z}}$, define the worst-case single-player approximation error

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) = \sup_{V\in\mathcal{C}} \Big[\mathbb{E}_{z\sim\mu_V^\beta} V(z) - \min V \Big]. \tag{6}$$

Thus $\varepsilon_{\mathcal{C}}(\beta^{-1})$ measures the worst-case excess risk incurred by replacing the exact minimizer of V by its entropy-regularized Gibbs approximation.

Lemma 1 (Monotonicity)**.** *For any $V \in \mathbb{R}^{\mathcal{Z}}$, the function $\beta \mapsto \mathbb{E}_{\mu_V^\beta}[V] - \min V$ is monotonically nonincreasing.*

Proof. Differentiating under the integral gives

$$\frac{d}{d\beta}\mathbb{E}_{\mu_V^\beta}[V] = -\mathrm{Var}_{\mu_V^\beta}(V) \leq 0.$$

□

This simple identity plays an important structural role in the single-player problem. It implies that, for any class $\mathcal{C} \subset \mathbb{R}^{\mathcal{Z}}$, the worst-case Gibbs excess

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) = \sup_{V\in\mathcal{C}} \Big[\mathbb{E}_{\mu_V^\beta} V - \min V \Big]$$

is nondecreasing as a function of the temperature β^{-1}. Consequently, whenever a minimization over admissible temperatures appears in the single-player bounds, the infimum is attained at the smallest allowed temperature.

NI error decomposition at a QRE We now connect the two-player and single-player problems.

Let $(\ell, \mathcal{X}, \mathcal{Y})$ be a zero-sum game and let $(\hat{\mu}^\beta, \hat{\nu}^\beta)$ denote its QRE. By the fixed-point characterization of QRE,

$$\hat{\mu}^\beta = \mu_V^\beta, \qquad \hat{\nu}^\beta = \nu_W^\beta,$$

where the equilibrium potentials are

$$V(x) = \int \ell(x,y)\, \hat{\nu}^\beta(\mathrm{d}y), \quad W(y) = -\int \ell(x,y)\, \hat{\mu}^\beta(\mathrm{d}x).$$

Substituting these expressions into the definition of the Nikaido–Isoda error and splitting the supremum over (μ, ν) yields the exact identity

$$\begin{aligned}\mathrm{NI}(\hat{\mu}^\beta, \hat{\nu}^\beta) = \big[\mathbb{E}_{x\sim\mu_V^\beta} V(x) - \min V\big] \\ + \big[\mathbb{E}_{y\sim\nu_W^\beta} W(y) - \min W\big].\end{aligned} \tag{7}$$

Crucially, the two-player NI error decomposes into a sum of *two independent single-player Gibbs excess risks*. This identity is exact and requires only the zero-sum structure.

Equation (7) is the structural backbone of the paper: all subsequent upper and lower bounds are obtained by bounding the two terms on the right-hand side over appropriate potential classes.

4.2 REDUCTION: UPPER AND LOWER BOUNDS

We now formalize the reduction. The idea is simple: if every QRE potential induced by $\mathcal{L}$ lies in a class $\mathcal{C}_\mathcal{X}$ (resp. $\mathcal{C}_\mathcal{Y}$), then the NI error is bounded by the corresponding single-player worst-case rates. Conversely, if we can embed arbitrary potentials from $\mathcal{C}_\mathcal{X}$ and $\mathcal{C}_\mathcal{Y}$ into a game in $\mathcal{L}$, then the two-player problem is at least as hard as the sum of the two single-player problems.

Lemma 2 (Upper bound reduction)**.** *Let $\mathcal{L} \subset \mathbb{R}^{\mathcal{X}\times\mathcal{Y}}$ and define*

$$\begin{cases}\mathcal{C}_\mathcal{X} = \Big\{\int_y \ell(\cdot, y)\,\nu(\mathrm{d}y) : \ell \in \mathcal{L},\ \nu \in \mathcal{P}(\mathcal{Y})\Big\} \\ \mathcal{C}_\mathcal{Y} = \Big\{-\int_x \ell(x, \cdot)\,\mu(\mathrm{d}x) : \ell \in \mathcal{L},\ \mu \in \mathcal{P}(\mathcal{X})\Big\}.\end{cases} \tag{8}$$

Then

$$\varepsilon_\mathcal{L}(\beta^{-1}) \le \varepsilon_{\mathcal{C}_\mathcal{X}}(\beta^{-1}) + \varepsilon_{\mathcal{C}_\mathcal{Y}}(\beta^{-1}).$$

The intuition is immediate from (7): the QRE potentials V and W necessarily belong to $\mathcal{C}_\mathcal{X}$ and $\mathcal{C}_\mathcal{Y}$, so each term in the NI error is bounded by the corresponding single-player supremum. Thus every upper bound in Sections 5–7 is obtained by bounding $\varepsilon_{\mathcal{C}_\mathcal{X}}$ and $\varepsilon_{\mathcal{C}_\mathcal{Y}}$.

The lower bound direction is more subtle. We show that if $\mathcal{L}$ is rich enough to contain certain product constructions, then any pair of hard single-player instances can be "stitched" into a two-player game whose QRE reproduces them exactly.

Lemma 3 (Lower bound reduction)**.** *Let $\mathcal{C}_\mathcal{X} \subset \mathbb{R}^\mathcal{X}$ and $\mathcal{C}_\mathcal{Y} \subset \mathbb{R}^\mathcal{Y}$ satisfy the inclusion condition: for all $V \in \mathcal{C}_\mathcal{X}$, $W \in \mathcal{C}_\mathcal{Y}$, $\mu \in \mathcal{P}(\mathcal{X})$, $\nu \in \mathcal{P}(\mathcal{Y})$, there exists $c \in \mathbb{R}$ such that*

$$(x,y) \mapsto \frac{1}{c}\big(V(x) - \mathbb{E}_\mu V - c\big)\big(W(y) - \mathbb{E}_\nu W + c\big) \in \mathcal{L}. \tag{9}$$

Then

$$\varepsilon_\mathcal{L}(\beta^{-1}) \ge \varepsilon_{\mathcal{C}_\mathcal{X}}(\beta^{-1}) + \varepsilon_{\mathcal{C}_\mathcal{Y}}(\beta^{-1}).$$

The intuition is as follows: Given near-worst-case potentials V and W, we construct a payoff ℓ whose interaction term is a centered product of V and W. Because of the centering, integrating ℓ against ν_W^β collapses the y-dependence and leaves the x-player facing exactly the potential V (up to constants), and vice versa. Hence the QRE of the constructed game is precisely $(\mu_V^\beta, \nu_W^\beta)$, and the NI error equals the sum of the two single-player excess risks.

The inclusion condition ensures that this product game remains inside $\mathcal{L}$. All lower bounds in Sections 5 and 6 are obtained by verifying that this condition holds for the relevant payoff classes.

Lemma 4. *For any $B > 0$, the inclusion condition* (9) *holds with*

$$\begin{cases}\mathcal{L} = \{\ell : \sup|\ell| \le 4B\}, \\ \mathcal{C}_\mathcal{X} = \{V : \sup_{x,x'} |V(x) - V(x')| \le B\}, \\ \mathcal{C}_\mathcal{Y} = \{W : \sup_{y,y'} |W(y) - W(y')| \le B\}.\end{cases}$$

with $c = B$.

Thus for bounded-payoff games, the two-player minimax rate is exactly the sum of the two single-player oscillation-constrained rates. This observation drives the tight bounds obtained for finite and continuous bounded games.

5 FINITE GAMES

We specialize the reduction framework to finite strategy sets $\mathcal{X} = \{1, \ldots, M\}$ and $\mathcal{Y} = \{1, \ldots, N\}$. In this case the minimax rate can be characterized exactly up to constants.

Proposition 5. *Let $\mathcal{L} = \{\ell \in \mathbb{R}^{M\times N} : \|\ell\|_\infty \le B\}$. For any $\beta > 0$,*

$$\varepsilon_\mathcal{L}(\beta^{-1}) \le \frac{\log M + \log N}{\beta},$$

and for any $\beta \geq \frac{4}{B}\log((M \vee N) - 1)$,

$$\varepsilon_{\mathcal{L}}(\beta^{-1}) \geq \frac{\log(M-1) + \log(N-1)}{2\beta}.$$

The upper bound $\mathcal{O}(\beta^{-1}(\log M + \log N))$ was observed in [3]. The matching lower bound (up to constants) is new. Hence the worst-case NI error scales as

$$\varepsilon_{\mathcal{L}}(\beta^{-1}) = \Theta\left(\frac{\log M + \log N}{\beta}\right).$$

The reduction lemmas convert the two-player problem into oscillation-constrained single-player problems. For

$$\mathcal{C}(N', B') = \{V \in \mathbb{R}^{N'} : \max V - \min V \leq B'\},$$

we obtain the following quantitative equivalence.

Lemma 6 (Finite reduction)**.** *Let* $\mathcal{L} = \{\ell \in \mathbb{R}^{M\times N} : \|\ell\|_\infty \leq B\}$*. Then for every* $\beta > 0$,

$$\begin{aligned}\varepsilon_{\mathcal{C}(M,B/4)}(\beta^{-1}) + \varepsilon_{\mathcal{C}(N,B/4)}(\beta^{-1}) \\ \leq \varepsilon_{\mathcal{L}}(\beta^{-1}) \\ \leq \varepsilon_{\mathcal{C}(M,2B)}(\beta^{-1}) + \varepsilon_{\mathcal{C}(N,2B)}(\beta^{-1}).\end{aligned}$$

Discussion. The upper bound is immediate from Lemma 2. If $\|\ell\|_\infty \leq B$ then for any mixed strategy ν,

$$V(x) = \sum_y \ell(x,y)\nu(y)$$

satisfies $|V(x) - V(x')| \leq 2B$, so $V \in \mathcal{C}(M, 2B)$ (and similarly for W). Thus the NI error is bounded by the sum of two single-player errors with oscillation $2B$.

The lower bound follows from the product construction (Lemmas 3–4). If V and W have oscillation at most $B/4$, then the centered product

$$\ell(x,y) = \frac{1}{B}\big(V(x) - \mathbb{E}_\mu V - B\big)\big(W(y) - \mathbb{E}_\nu W + B\big)$$

has sup-norm at most B, so $\ell \in \mathcal{L}$. This shows that the two-player class is rich enough to embed any pair of oscillation-bounded potentials, yielding the lower inequality. Consequently, the two-player minimax rate is exactly the sum of two single-player oscillation rates up to constant factors.

We now bound the single-player quantity $\varepsilon_{\mathcal{C}}(\beta^{-1})$.

Lemma 7. *For* $\mathcal{C} = \{V \in \mathbb{R}^N : \max V - \min V \leq B\}$,

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) \leq \frac{\log N}{\beta}.$$

Proof. The identity

$$V_i = -\frac{1}{\beta}\log \mu_V^\beta(i) - \frac{1}{\beta}\log\sum_j e^{-\beta V_j}$$

gives

$$\begin{aligned}\mathbb{E}_{\mu_V^\beta}[V] - \min V = -\frac{1}{\beta}H(\mu_V^\beta) \\ - \Big(\frac{1}{\beta}\log\sum_j e^{-\beta V_j} + \min V\Big).\end{aligned} \tag{10}$$

The entropy term is at most $\frac{\log N}{\beta}$, while the second term is nonnegative since $\log\sum_j e^{-\beta V_j} \geq -\beta \min V$. □

This bound is minimax sharp: it is attained when the Gibbs distribution spreads nearly uniformly over many nearly optimal actions.

Remark 8. *If* V *has a fixed gap* $\Delta = V_{(2)} - V_{(1)} > 0$*, then*

$$\mathbb{E}_{\mu_V^\beta}[V] - \min V = \mathcal{O}(\Delta e^{-\beta\Delta}),$$

so the slow $\log N/\beta$ *behavior arises only from near-gapless potentials.*

Lemma 9. *For* $\mathcal{C} = \{V \in \mathbb{R}^N : \max V - \min V \leq B\}$ *and* $\beta \geq \frac{1}{B}\log(N-1)$,

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) \geq \frac{\log(N-1)}{2\beta}.$$

The extremal potential takes three values: 0 at the minimizer, B at some coordinates, and a common interior value x elsewhere. Choosing $x = \beta^{-1}\log(N-1)$ assigns Gibbs mass $(N-1)^{-1}$ to each non-minimal action, balancing their contributions and yielding the stated lower bound. See Appendix B for the exact formula.

Combining Lemmas 6, 7, and 9 establishes Proposition 5.

6 CONTINUOUS GAMES

In this section we consider games whose strategy sets $\mathcal{X}, \mathcal{Y}$ are continuous sets. For simplicity, we take them to be unit high-dimensional toruses: $\mathcal{X} = \mathbb{T}^{d_x}, \mathcal{Y} = \mathbb{T}^{d_y}$ where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. This is a common setting in the literature on MNE computation for continuous games. It is particularly convenient because toruses are compact, without boundaries, and flat (in the sense of Riemannian manifolds). We note that the analysis extends naturally to any compact Riemannian manifold without boundary (e.g. spheres, products of spheres): the key ingredients - compactness of the strategy space, a well-defined volume measure, and the local Euclidean structure used in the Gibbs concentration estimates - are present in that generality.

Before displaying the main results of this section, we start by an example showing that it is necessary to make some quantitative regularity assumptions (boundedness does not suffice), as otherwise the minimax approximation error does not even vanish when $\beta \to \infty$. Full proofs for the Hölder results are given in Appendix C.

6.1 BOUNDED PAYOFFS WITHOUT QUANTITATIVE REGULARITY

Proposition 10. *Let $\mathcal{L}$ be the class of continuous $\ell : \mathbb{T}^{d_x} \times \mathbb{T}^{d_y} \to \mathbb{R}$ with $\|\ell\|_\infty \le B$. For any $\beta > 0$,*

$$\frac{B}{2} \;\le\; \varepsilon_{\mathcal{L}}(\beta^{-1}) \;\le\; 2B.$$

Proof. The upper bound follows from $\mathrm{NI}_\ell(\hat{\mu}, \hat{\nu}) \le 2\|\ell\|_\infty$.

For the lower bound, Lemmas 3 and 4 give

$$\varepsilon_{\mathcal{L}}(\beta^{-1}) \ge \varepsilon_{\mathcal{C}_X}(\beta^{-1}) + \varepsilon_{\mathcal{C}_Y}(\beta^{-1}),$$

where $\mathcal{C}_X$ consists of continuous $V : \mathbb{T}^{d_x} \to \mathbb{R}$ with oscillation at most $B/4$. It suffices to show $\varepsilon_{\mathcal{C}_X}(\beta^{-1}) \ge B/4$.

Fix $x_0 \in \mathbb{T}^{d_x}$ and define

$$V_\delta(x) = \min\{\delta^{-1}\|x - x_0\|,\, B/4\}.$$

As $\delta \to 0$, the Gibbs mass concentrates in an arbitrarily small neighborhood of x_0, while $V_\delta = B/4$ on most of the torus. A direct computation yields $\mathbb{E}_{\mu^\beta_{V_\delta}}[V_\delta] \to B/4$, so $\varepsilon_{\mathcal{C}_X}(\beta^{-1}) \ge B/4$. □

Thus boundedness alone does not guarantee $\varepsilon_{\mathcal{L}}(\beta^{-1}) \to 0$ as $\beta \to \infty$, implying quantitative regularity assumptions are essential.

6.2 HÖLDER-CONTINUOUS PAYOFFS

We now impose α-Hölder regularity for $\alpha \in (0, 1]$, subsuming the Lipschitz case $\alpha = 1$. Let $\mathcal{L}_\alpha$ denote the class of α-Hölder $\ell : \mathbb{T}^{d_x} \times \mathbb{T}^{d_y} \to \mathbb{R}$ with constant L and $\|\ell\|_\infty \le B$. Define

$$\begin{aligned}\mathcal{C}(d, B', L') = \big\{V : \mathbb{T}^d \to \mathbb{R} : \alpha\text{-Hölder}(L'),\\ \sup V - \inf V \le B'\big\}. \quad (11)\end{aligned}$$

Lemma 11 (Hölder reduction)**.** *For any $\alpha \in (0, 1]$, $L > 0$, $B > 0$, and $\beta > 0$,*

$$\begin{aligned}&\varepsilon_{\mathcal{C}(d_x,B/4,L')}(\beta^{-1}) + \varepsilon_{\mathcal{C}(d_y,B/4,L')}(\beta^{-1})\\ &\quad\le \varepsilon_{\mathcal{L}_\alpha}(\beta^{-1})\\ &\quad\le \varepsilon_{\mathcal{C}(d_x,B,L)}(\beta^{-1}) + \varepsilon_{\mathcal{C}(d_y,B,L)}(\beta^{-1}),\end{aligned}$$

where $L' = 2^{\alpha/2-2}L$.

The upper bound follows because averaging an α-Hölder function preserves its Hölder constant. For the lower bound, the product construction (9) must preserve α-Hölder regularity. Writing $a = \|x - x'\|$ and $b = \|y - y'\|$, the inequality

$$a^\alpha + b^\alpha \le 2^{1-\alpha/2}(a^2 + b^2)^{\alpha/2}$$

controls the mixed increments and determines the constant L'.

Lemma 12 (Single-player upper bound)**.** *For $\mathcal{C} = \mathcal{C}(d, B, L)$ and any $\beta > 0$ with $A'\beta \ge 1$, where $A' = \frac{eB(\beta L)^{d/\alpha}}{\omega_d}$,*

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) \le \frac{1 + \log(A'\beta)}{\beta}.$$

The argument lower-bounds the partition function using the thermal radius $r = (\beta L)^{-1/\alpha}$: Hölder continuity gives $V(x) \le 1/\beta$ inside $B(x^*, r)$, so $Z \ge e^{-1}\omega_d r^d$. Splitting $\mathbb{E}[V]$ at a threshold c and optimizing yields the bound.

Lemma 13 (Single-player lower bound)**.** *If $B \ge 2^{2-\alpha}L'\,(d_x \vee d_y)^{\alpha/2}$, then*

$$\varepsilon_{\mathcal{L}_\alpha}(\beta^{-1}) \ge \frac{d_x + d_y}{\alpha\beta}(1 - o(1)) \quad \textit{as } \beta \to \infty.$$

The lower bound is witnessed by the radial cone $V(x) = L'\|x\|^\alpha$. By the substitution $u = \beta L' r^\alpha$, one obtains

$$\mathbb{E}_{\mu^\beta_V}[V] = \frac{d}{\alpha\beta} + o(\beta^{-1}),$$

so the NI error equals $(d_x + d_y)/(\alpha\beta)$ up to lower-order terms.

Remark 14 (Summary and open gap)**.** *Combining Lemmas 12 and 13,*

$$\frac{d_x + d_y}{\alpha\beta}(1 - o(1)) \;\le\; \varepsilon_{\mathcal{L}_\alpha}(\beta^{-1}) \;\lesssim\; \frac{(d_x + d_y)\log(L\beta)}{\alpha\beta}.$$

Whether the true rate is $\Theta((d_x + d_y)/(\alpha\beta))$ or $\Theta((d_x + d_y)\log\beta/(\alpha\beta))$ remains open.

7 SMOOTH PAYOFFS WITH SPARSE NON-DEGENERATE MNE

The minimax results of Sections 5–6 are driven by adversarial, non-smooth constructions in which the Gibbs measure spreads mass across nearly flat regions of the payoff landscape. In such cases, entropy regularization behaves like a global smoothing operator and the worst-case error is governed by large nearly-degenerate sets of near-minimizers.

In contrast, when the payoff is smooth and the MNE are finitely supported and locally well-conditioned, the geometry of the problem changes completely. The entropy-regularized equilibrium no longer spreads mass over extended plateaus, but instead concentrates near isolated quadratic wells. In this regime, classical Laplace asymptotics apply, and the approximation error is governed purely by dimension. In particular, one expects a universal $1/\beta$ rate whose leading constant does not depend on the detailed curvature of the wells.

We formalize this regime below. Throughout this section we assume without loss of generality that $d_x \le d_y$.

Let $\mathcal{L}$ be the set of smooth functions $\ell : \mathbb{T}^{d_x} \times \mathbb{T}^{d_y} \to \mathbb{R}$ satisfying the following structural conditions [20],

- **Sparsity.** Every MNE (μ^*, ν^*) is finitely supported:

$$\mu^* = \sum_i a_i^* \, \delta_{x_i^*}, \qquad \nu^* = \sum_j b_j^* \, \delta_{y_j^*}.$$

  Thus each player randomizes over finitely many isolated actions, and the equilibrium potentials have finitely many minimizers.

- **Non-degeneracy.** For every MNE, define the best-response potentials

$$V^*(x) = \int \ell(x,y)\, \nu^*(\mathrm{d}y),$$
$$W^*(y) = -\int \ell(x,y)\, \mu^*(\mathrm{d}x).$$

  There exist constants $\sigma, \xi, \delta > 0$ such that:

  (i) *Local strong convexity.* Each equilibrium support point is a non-degenerate quadratic minimum:

$$\nabla^2 V^*(x_i^*) \succeq \sigma I_{d_x}, \qquad \nabla^2 W^*(y_j^*) \succeq \sigma I_{d_y}.$$

  (ii) *Spectral gap away from wells.* The potentials are uniformly separated from their minima:

$$V^*(x) - \min V^* \ge \xi \quad \text{outside} \bigcup_i B(x_i^*, \delta),$$

  and similarly for W^*.

Condition (i) guarantees that each equilibrium action is an isolated quadratic well with uniformly positive curvature, ruling out flat directions. Condition (ii) prevents the existence of competing near-minima or extended shallow regions that could attract Gibbs mass. Uniqueness of the MNE is *not* required: multiple isolated equilibria are permitted, provided each satisfies the same curvature and gap conditions.

Remark 15 (On the scope of the assumption)**.** *The assumption has two logically separate parts. The finite-support condition is the substantive restriction: we assume that the MNE under consideration have supports*

$$\mu^* = \sum_i a_i^* \delta_{x_i^*}, \qquad \nu^* = \sum_j b_j^* \delta_{y_j^*}.$$

The non-degeneracy condition is then a local regularity condition at these support points. Namely, the best-response potentials

$$V^*(x) = \int \ell(x,y)\, \nu^*(dy), \qquad W^*(y) = -\int \ell(x,y)\, \mu^*(dx)$$

are required to have positive-definite Hessians at their minimizers, together with a positive gap away from small neighborhoods of those minimizers.

This local condition is stable under small C^2 perturbations of the payoff, as long as the same finitely supported equilibrium structure and the spectral gap persist. In finite-dimensional payoff families, failure of the Hessian condition is expressed by equations such as

$$\det \nabla^2 V^*(x_i^*) = 0 \quad \textit{or} \quad \det \nabla^2 W^*(y_j^*) = 0,$$

and hence corresponds to a lower-dimensional exceptional set. Thus the non-degeneracy requirement should be viewed as ruling out flat directions at the equilibrium support points, rather than as imposing a special algebraic structure on the payoff.

By contrast, finite support is not generic among all smooth continuous games. If an equilibrium measure is supported on a positive-dimensional subset of the torus, the best-response potential may have a manifold of minimizers. In that case Laplace's method no longer yields a dimension-only constant, and the approximation rate can depend on the local geometry of the minimizing set. We leave this regime as a separate open problem in Section 9. The finite-support regime is nevertheless natural and includes several classical families, such as separable and bell-shaped games [10], low-rank games whose support size is controlled by the payoff rank [18], polynomial payoff games with finite separable expansions, and the sin-cos game used in the lower bound below.

Under these assumptions, the QRE potentials V^β and W^β converge in C^k to V^* and W^* as $\beta \to \infty$, and for large β they inherit the same well structure. The NI error then reduces to a sum of Laplace integrals localized around finitely many quadratic minima. The leading term depends only on the dimension and is independent of the precise Hessians.

Proposition 16. *For the class $\mathcal{L}$ defined above, the following hold. Unless otherwise specified, the constants implicit in the $\mathcal{O}$ notation may depend on the parameters defining $\mathcal{L}$.*

1. (Instance-wise tight rate) *For every fixed $\ell \in \mathcal{L}$,*

$$\mathrm{NI}_\ell(\hat\mu_\ell^\beta, \hat\nu_\ell^\beta) \;=\; \frac{d_x + d_y}{2\beta} + \mathcal{O}_\ell\left(\frac{1}{\beta^2}\right), \tag{12}$$

   where the implicit constant in $\mathcal{O}_\ell$ may depend on the instance ℓ (in particular through σ, $\|\ell\|_{C^4}$, d_x, and d_y).

2. (Minimax rate)

$$\frac{d_x + d_y}{2\beta} + \mathcal{O}\left(\frac{1}{\beta^2}\right) \;\le\; \varepsilon_{\mathcal{L}}(\beta^{-1}) \;\le\; \mathcal{O}\left(\frac{(d_x + d_y)\log\beta}{\beta}\right). \tag{13}$$

The expansion (12) follows from Laplace's method; a precise statement tailored to our setting is deferred to Lemma 26 in the appendix. Near each minimizer x_i^*, the potential is approximately quadratic:

$$V^*(x) = \frac{1}{2}(x - x_i^*)^\top H_i(x - x_i^*) + \mathcal{O}(\|x - x_i^*\|^3),$$

where $H_i = \nabla^2 V^*(x_i^*) \succeq \sigma I_{d_x}$. After the rescaling $u = \sqrt{\beta}(x - x_i^*)$, the Gibbs density in each well converges to a Gaussian $\mathcal{N}(0, H_i^{-1})$. The expected value of the quadratic part under this Gaussian is

$$\mathbb{E}_{u\sim\mathcal{N}(0,H_i^{-1})}\big[\frac{1}{2}u^\top H_i\, u\big] = \frac{1}{2}\mathrm{tr}(H_i H_i^{-1}) = \frac{d_x}{2\beta},$$

which, crucially, is independent of the eigenvalues of H_i: the Hessian in the quadratic form cancels against the inverse Hessian in the covariance. When the potential has multiple wells, the Gibbs measure splits its mass among them with weights proportional to $(\det H_i)^{-1/2}$, but since each well contributes the same leading term $d_x/(2\beta)$, the weighted average is unchanged. Higher-order corrections (odd-degree Taylor terms) vanish by symmetry of the Gaussian weight, so the first nonvanishing correction is $\mathcal{O}(\beta^{-2})$. Consequently,

$$\mathbb{E}_{\mu^\beta_{V^*}}[V^*] - \min V^* = \frac{d_x}{2\beta} + \mathcal{O}(\beta^{-2}),$$

and similarly for W^*. The NI decomposition (7) adds the two contributions.

The minimax lower bound in (13) is witnessed by the separable payoff $\ell(x, y) = \sum_{k=1}^{d_x}[\sin(4\pi x_k) + 2\cos(2\pi x_k + 2\pi y_k)] + \sum_{l=1}^{d_y}\sin(4\pi y_l)$, whose QRE potentials are fixed independently of β by a discrete symmetry of the game (see Appendix D for the full verification). The upper bound follows from the Lipschitz result of Section 6, since smooth bounded functions on the torus are automatically Lipschitz.

Remark 17. *The instance-wise rate* (12) *is stronger than a minimax bound: it says that every fixed game in $\mathcal{L}$ individually achieves $(d_x + d_y)/(2\beta)$, not merely that the worst case is controlled.*

Remark 18. *The logarithmic gap in* (13) *reflects that $\mathcal{L}$ allows arbitrarily small curvature $\sigma > 0$; as $\sigma \to 0$, the implicit constant in the $\mathcal{O}_\ell(1/\beta^2)$ remainder blows up. Imposing a uniform lower bound on σ would close the gap and yield a tight $\Theta((d_x + d_y)/\beta)$ minimax rate.*

8 NUMERICAL EXPERIMENTS

We verify the three theoretical regimes numerically. In every case the QRE is computed to high precision and its Nikaido–Isoda error is compared against the predicted rate. Because the plain best-response iteration can cycle at large β, we solve the regularized saddle point with an entropic mirror-prox (extragradient) scheme, which converges linearly for every β. Full code is provided in the supplementary material.

Table 1: Finite games: exact worst-case single-player Gibbs excess (scaled by β; β-independent to the digits shown) versus the upper bound $\log N$ and the lower-bound witness $\log(N-1)/2$.

N	$\beta\,\varepsilon_{\mathcal{C}}$	$\log N$	$\frac{1}{2}\log(N{-}1)$	ratio UB	ratio LB
5	0.718	1.609	0.693	0.446	1.036
10	1.101	2.303	1.099	0.478	1.002
50	2.134	3.912	1.946	0.545	1.097
100	2.629	4.605	2.298	0.571	1.144

Table 2: Two-player games ($M = N = 20$): NI error of the lower-bound witness game matches $\log(N-1)/\beta$ exactly; random games sit far below. Last two columns are the mean and maximum over 80 random draws at $\beta = 50$.

β	witness NI	$\log(N{-}1)/\beta$	random mean	random max
10	0.294444	0.294444	—	—
20	0.147222	0.147222	—	—
50	0.058889	0.058889	0.0283	0.0398
100	0.029444	0.029444	—	—

Finite games. For the single-player class $\mathcal{C}(N, B) = \{V \in \mathbb{R}^N : \max V - \min V \le B\}$ we evaluate the exact worst-case Gibbs excess through the closed-form maximum of Lemma 9 (Eq. (21)), for $N \in \{5, 10, 50, 100\}$ and β from 2 to 500. Table 1 reports the scaled excess $\beta\,\varepsilon_{\mathcal{C}}(\beta^{-1})$ together with its ratio to the upper bound $\log N/\beta$ and to the lower-bound witness value $\log(N-1)/(2\beta)$. Both ratios are essentially constant in β, confirming the $\Theta(\log N/\beta)$ scaling of Proposition 5: the ratio to the upper bound stays in $[0.45, 0.57]$ and the ratio to the lower bound in $[1.00, 1.14]$, so the witness of Lemma 9 is within 14% of the exact worst case (Figure 1, left).

Two-player games. We instantiate the product construction of Lemma 3 from two three-value witness potentials on $N = M = 20$ actions and compute the QRE of the resulting game from scratch. Its Nikaido–Isoda error matches the predicted $\log(N-1)/\beta$ to six digits and the recovered QRE equals the intended pair $(\mu_V^\beta, \nu_W^\beta)$ to within 10^{-10} (Table 2); at $\beta = 50$ we obtain NI $= 0.058889 = \log(19)/50$. By contrast, random 20×20 games with i.i.d. standard-normal entries rescaled to $[-1, 1]$ have much smaller error—mean 0.028 over 80 draws, maximum 0.040, well below the adversarial value 0.058889—confirming that the worst case is genuinely attained by the construction and not by typical games (Figure 1, middle).

Smooth sparse game. For the sin-cos game (29) with $d_x = d_y = 1$ we verify the Laplace rate of Proposition 16. Solving the coupled two-dimensional QRE on a grid confirms the two structural predictions of the proof: the two players' excess terms are equal to machine precision

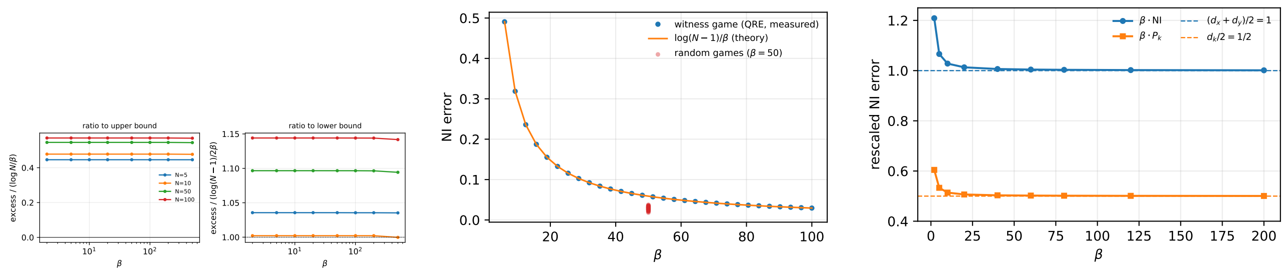

Figure 1: **Left:** finite games—ratio of the exact worst-case single-player excess to the upper bound $\log N/\beta$ and to the lower-bound witness $\log(N-1)/(2\beta)$; both are essentially flat in β. **Middle:** two-player games—the witness game's QRE gap tracks $\log(N-1)/\beta$ exactly, while random 20×20 games (red) sit far below. **Right:** sin-cos game—the rescaled error $\beta \cdot \mathrm{NI}$ and $\beta \cdot P_k$ converge from above to 1 and $\frac{1}{2}$, confirming the Laplace rate.

Table 3: Sin-cos game ($d_x = d_y = 1$): the rescaled NI error and single-player terms converge from above to the Laplace prediction $\beta \cdot \mathrm{NI} \to 1$, $\beta \cdot P_k \to \frac{1}{2}$.

β	5	10	20	40	80	200	limit
$\beta \cdot \mathrm{NI}$	1.066	1.028	1.013	1.006	1.003	1.001	1
$\beta \cdot P_k$	0.533	0.514	0.507	0.503	0.502	0.501	$\frac{1}{2}$

($|P_1 - P_2| \leq 10^{-14}$) and the Fourier coefficient coupling the players vanishes ($|\varphi| \leq 10^{-14}$), the exact mechanism by which the discrete symmetry fixes the QRE potentials independently of β. Evaluating the resulting single-coordinate Gibbs excess by high-accuracy quadrature gives the rescaled error $\beta \cdot \mathrm{NI}$ and $\beta \cdot P_k$ in Table 3: $\beta \cdot \mathrm{NI} \to 1$ and $\beta \cdot P_k \to \frac{1}{2}$ from above, reaching 1.0032 and 0.5016 at $\beta = 80$, in agreement with $(d_x + d_y)/(2\beta) = 1/\beta$ and the $\mathcal{O}(1/\beta^2)$ correction (Figure 1, right).

Together these experiments confirm that the theoretical rates are not only correct but empirically sharp: the finite-game witness is within a small constant of the exact worst case, the two-player product construction attains the worst case to six digits, and the smooth instance-wise constant $(d_x + d_y)/2$ is recovered numerically.

9 DISCUSSION AND CONCLUSION

Our object of study is the QRE as a *solution concept*: the Gibbs measures define the entropy-regularized saddle point, and $\mathrm{NI}_\ell(\hat{\mu}_\ell^\beta, \hat{\nu}_\ell^\beta)$ measures its distance from an exact MNE. The rates nevertheless give practical guidance for choosing the regularization. Since QREs can be computed efficiently in finite games by policy extragradient methods [3] or entropic mirror-prox, and in continuous games by Langevin or MCMC methods [6, 2], a target duality gap ε suggests taking $\beta \asymp \frac{\log M + \log N}{\varepsilon}$ in finite games and $\beta \approx \frac{d_x + d_y}{2\varepsilon}$ in the smooth sparse regime. The matching lower bounds show that these choices are worst-case sharp up to constants.

The same reduction also covers semi-discrete games without new arguments. If $X = \{1, \ldots, M\}$ and $Y = \mathbb{T}^{d_y}$, then the NI decomposition (7) combines the finite and continuous single-player rates. For instance, if $\ell(x, \cdot)$ is α-Hölder, then

$$\varepsilon_{\mathcal{L}}(\beta^{-1}) \asymp \frac{\log M}{\beta} + \frac{d_y}{\alpha\beta},$$

up to the same logarithmic factor appearing in the Hölder upper bound.

Several directions remain open. First, in the Hölder setting, Lemmas 12 and 13 leave the gap

$$\frac{d_x + d_y}{\alpha\beta} \lesssim \varepsilon_{\mathcal{L}_\alpha}(\beta^{-1}) \lesssim \frac{(d_x + d_y)\log(L\beta)}{\alpha\beta}.$$

The cone example $V(x) = L'\|x\|^\alpha$ attains excess $d/(\alpha\beta)$ without the logarithm, suggesting that the upper bound may not be sharp. Second, extending the theory beyond zero-sum games would require new ideas, since the decomposition (7) is what separates the two players into independent Gibbs concentration problems. Third, when the unregularized game has multiple equilibria, the vanishing-temperature QRE path can be viewed as an equilibrium-selection mechanism. This perspective is classical in finite normal-form games [12, 21], and extending it to continuous games with nontrivial equilibrium geometry would complement the rate questions studied here.

Overall, the approximation quality of entropy-regularized equilibria in two-player zero-sum games is governed by single-player Gibbs concentration. Sharper QRE approximation rates therefore reduce to sharper concentration estimates under regularity constraints, most notably in the Hölder setting.

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A PROOFS FOR REDUCTION TO SINGLE-PLAYER (SECTION 4)

Lemma 19 (Upper bound reduction, restated). *Let $\mathcal{L} \subset \mathbb{R}^{\mathcal{X}\times\mathcal{Y}}$ and define $\mathcal{C}_\mathcal{X} = \{\int_y \ell(\cdot, y)\, \nu(\mathrm{d}y);\ \ell \in \mathcal{L},\ \nu \in \mathcal{P}(\mathcal{Y})\}$ and $\mathcal{C}_\mathcal{Y} = \{-\int_x \ell(x, \cdot)\, \mu(\mathrm{d}x);\ \ell \in \mathcal{L},\ \mu \in \mathcal{P}(\mathcal{X})\}$. Then $\varepsilon_\mathcal{L}(\beta^{-1}) \le \varepsilon_{\mathcal{C}_\mathcal{X}}(\beta^{-1}) + \varepsilon_{\mathcal{C}_\mathcal{Y}}(\beta^{-1})$.*

Proof. Fix β and $\ell \in \mathcal{L}$. The QRE potentials V and W defined in (7) belong by construction to $\mathcal{C}_\mathcal{X}$ and $\mathcal{C}_\mathcal{Y}$. The NI decomposition therefore ensures for every ℓ that

$$\mathrm{NI}_\ell(\hat{\mu}^\beta_\ell, \hat{\nu}^\beta_\ell) \le \sup_{V\in\mathcal{C}_\mathcal{X}} \Big[\mathbb{E}_{x\sim\mu^\beta_V} V(x) - \min V\Big] + \sup_{W\in\mathcal{C}_\mathcal{Y}} \Big[\mathbb{E}_{y\sim\nu^\beta_W} W(y) - \min W\Big] \tag{14}$$

$$\le \varepsilon_{\mathcal{C}_X}(\beta^{-1}) + \varepsilon_{\mathcal{C}_Y}(\beta^{-1}). \tag{15}$$

Taking the supremum over $\ell \in \mathcal{L}$ and noting (4) concludes the proof. □

Lemma 20 (Lower bound reduction, restated). *Let $\mathcal{C}_X \subset \mathbb{R}^X$ and $\mathcal{C}_Y \subset \mathbb{R}^Y$ satisfy condition* (9)*. Then $\varepsilon_\mathcal{L}(\beta^{-1}) \ge \varepsilon_{\mathcal{C}_X}(\beta^{-1}) + \varepsilon_{\mathcal{C}_Y}(\beta^{-1})$.*

Proof. We proceed in three stages: first we reduce to a pointwise-in-β inequality, then we construct a hard game instance, then we verify it has the desired properties.

Stage 1. It suffices to show that for every fixed $\beta > 0$,

$$\sup_{\ell\in\mathcal{L}} \mathrm{NI}_\ell(\widehat{\mu}^\beta_\ell, \widehat{\nu}^\beta_\ell) \ \ge\ \varepsilon_{\mathcal{C}_\mathcal{X}}(\beta^{-1}) + \varepsilon_{\mathcal{C}_\mathcal{Y}}(\beta^{-1}). \tag{16}$$

Indeed, assuming (16) holds for every β, taking $\inf_{\bar{\beta}^{-1}\ge\beta^{-1}}$ of the left-hand side yields $\varepsilon_\mathcal{L}(\beta^{-1})$. The right-hand side is unchanged because, by monotonicity of $\beta^{-1} \mapsto \varepsilon_{\mathcal{C}_\mathcal{X}}(\beta^{-1})$ (Lemma 1), the infimum over $\bar{\beta}^{-1} \ge \beta^{-1}$ of the right-hand side is attained at $\bar{\beta}^{-1} = \beta^{-1}$.

Stage 2. Fix $\beta > 0$ and write $\varepsilon_x = \varepsilon_{\mathcal{C}_\mathcal{X}}(\beta^{-1})$, $\varepsilon_y = \varepsilon_{\mathcal{C}_\mathcal{Y}}(\beta^{-1})$. For any $\varepsilon > 0$, by definition of ε_x and ε_y, there exist $V \in \mathcal{C}_\mathcal{X}$ and $W \in \mathcal{C}_\mathcal{Y}$ such that

$$\mathbb{E}_{x\sim\mu^\beta_V} V(x) - \min V \ \ge\ \varepsilon_x - \varepsilon, \qquad \mathbb{E}_{y\sim\nu^\beta_W} W(y) - \min W \ \ge\ \varepsilon_y - \varepsilon.$$

Define the payoff function

$$\ell(x, y) \ =\ \frac{1}{c}\Big[V(x) - \mathbb{E}_{\mu^\beta_V} V - c\Big]\Big[W(y) - \mathbb{E}_{\nu^\beta_W} W + c\Big], \tag{17}$$

where $c \in \mathbb{R}$ is the constant whose existence is guaranteed by assumption (9) (applied with $\mu = \mu^\beta_V$ and $\nu = \nu^\beta_W$). In particular, the equation above implies $\ell \in \mathcal{L}$.

Stage 3. We check that $(\mu^\beta_V, \nu^\beta_W)$ satisfies the fixed-point equation for the game ℓ. For the x-player, compute:

$$-\beta \int_y \ell(x, y)\, \nu^\beta_W(\mathrm{d}y) = -\frac{\beta}{c}\big[V(x) - \mathbb{E}_{\mu^\beta_V} V - c\big] \times \int_y \big[W(y) - \mathbb{E}_{\nu^\beta_W} W + c\big]\, \nu^\beta_W(\mathrm{d}y).$$

The integral on the right evaluates to $\mathbb{E}_{\nu^\beta_W} W - \mathbb{E}_{\nu^\beta_W} W + c = c$, so the expression simplifies to

$$-\beta\big[V(x) - \mathbb{E}_{\mu^\beta_V} V - c\big] = -\beta\, V(x) + \mathrm{const},$$

which confirms that μ^β_V satisfies the x-player fixed-point equation. For the y-player, the argument is symmetric: we compute

$$\beta \int_x \ell(x, y)\, \mu^\beta_V(\mathrm{d}x) = \frac{\beta}{c}\Big[W(y) - \mathbb{E}_{\nu^\beta_W} W + c\Big] \times \int_x \big[V(x) - \mathbb{E}_{\mu^\beta_V} V - c\big]\, \mu^\beta_V(\mathrm{d}x).$$

This integral evaluates to $\mathbb{E}_{\mu^\beta_V} V - \mathbb{E}_{\mu^\beta_V} V - c = -c$, and the expression simplifies to

$$-\frac{\beta}{c}\big[W(y) - \mathbb{E}_{\nu^\beta_W} W + c\big] \times c = -\beta\, W(y) + \mathrm{const},$$

confirming that ν_W^β satisfies the y-player fixed-point equation. Hence $(\mu_V^\beta, \nu_W^\beta)$ is the QRE of the game ℓ.

Hence by the NI decomposition (7),

$$\mathrm{NI}_\ell(\widehat{\mu}_\ell^\beta, \widehat{\nu}_\ell^\beta) = \left[\mathbb{E}_{\mu_V^\beta} V - \min V\right] + \left[\mathbb{E}_{\nu_W^\beta} W - \min W\right] \geq \varepsilon_x + \varepsilon_y - 2\varepsilon.$$

In particular, $\sup_{\ell \in \mathcal{L}} \mathrm{NI}_\ell(\widehat{\mu}_\ell^\beta, \widehat{\nu}_\ell^\beta) \geq \varepsilon_x + \varepsilon_y - 2\varepsilon$. Since $\varepsilon > 0$ was arbitrary, the inequality (16) follows. □

Lemma 21 (Bounded-payoff verification, restated). *For any $B > 0$, let $\mathcal{L} = \{\ell;\ \sup_{x,y} |\ell(x,y)| \leq 4B\}$. Then condition* (9) *holds with $\mathcal{C}_X = \{V;\ \sup_{x,x'} |V(x) - V(x')| \leq B\}$, $\mathcal{C}_Y = \{W;\ \sup_{y,y'} |W(y) - W(y')| \leq B\}$, and $c = B$.*

Proof. For any $V \in \mathcal{C}_X$ and $\mu \in \mathcal{P}(X)$, the oscillation bound implies $|V(x) - \mathbb{E}_\mu V| \leq B$, and similarly for W. Choosing $c = B$ in (9) yields

$$\left|\frac{1}{c}\left(V(x) - \mathbb{E}_\mu V - c\right)\left(W(y) - \mathbb{E}_\nu W + c\right)\right| \leq \frac{1}{|c|}\left|V(x) - \mathbb{E}_\mu V\right|\left|W(y) - \mathbb{E}_\nu W\right| \tag{18}$$

$$+ \left|V(x) - \mathbb{E}_\mu V\right| + \left|W(y) - \mathbb{E}_\nu W\right| + |c| \tag{19}$$

$$\leq \frac{1}{B} \times B \times B + B + B + B = 4B, \tag{20}$$

which implies $\ell \in \mathcal{L}$. □

B PROOFS FOR FINITE GAMES (SECTION 5)

Lemma 22 (Single-player lower bound, restated). *For $\mathcal{C} = \{V \in \mathbb{R}^N;\ \max V - \min V \leq B\}$ and $\beta \geq \frac{1}{B}\log(N-1)$, $\varepsilon_{\mathcal{C}}(\beta^{-1}) \geq \frac{\log(N-1)}{2\beta}$.*

Proof. The proof has two parts: we first derive an exact combinatorial formula for $\varepsilon_{\mathcal{C}}(\beta^{-1})$, then evaluate it at a well-chosen point.

Part 1 (Exact formula). Assume $\beta^{-1} < B$. We claim that

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) \;=\; \max_{0 \leq m \leq N-1}\ \max_{\beta^{-1} \leq x \leq B} \frac{mB\, e^{-\beta B} + (N{-}m{-}1)\, x\, e^{-\beta x}}{1 + m\, e^{-\beta B} + (N{-}m{-}1)\, e^{-\beta x}}. \tag{21}$$

To see this, note that the supremum defining $\varepsilon_{\mathcal{C}}(\beta^{-1})$ can be restricted to $\{V;\ V_i \in [0,B],\ \min V = 0\}$, which is compact, so the supremum is attained at some V. For any index i with $0 < V_i < B$, the first-order stationarity condition yields

$$V_i \;=\; \beta^{-1} + \mathbb{E}_{\mu_V^\beta}[V],$$

so all interior values coincide. Hence V takes at most three values: 0 (at $\arg\min V$), B (at $\arg\max V$), and a single intermediate value $x \in [\beta^{-1}, B)$. Moreover, the objective is decreasing in $|\arg\min V|$, so the optimum is at $|\arg\min V| = 1$. Parametrizing by $m = |\{i : V_i = B\}|$ and the intermediate value x then gives (21).

Part 2 (Evaluation). It remains to produce a good lower bound from (21). Set $m = 0$ and $x = \beta^{-1}\log(N{-}1)$, which is feasible when $\beta \geq \frac{1}{B}\log(N{-}1)$. Substituting:

$$\varepsilon_{\mathcal{C}}(\beta^{-1}) \geq \frac{(N{-}1) \times \beta^{-1}\log(N{-}1) \times (N{-}1)^{-1}}{1 + (N{-}1) \times (N{-}1)^{-1}} = \frac{\log(N-1)}{2\beta}.$$ □

C PROOFS FOR CONTINUOUS GAMES (SECTION 6)

Lemma 23 (Hölder reduction, restated)**.** *For any $\alpha \in (0, 1]$, $L > 0$, $B > 0$, and $\beta > 0$,*

$$\varepsilon_{\mathcal{C}(d_x,B/4,L')}(\beta^{-1}) + \varepsilon_{\mathcal{C}(d_y,B/4,L')}(\beta^{-1}) \;\le\; \varepsilon_{\mathcal{L}_\alpha}(\beta^{-1}) \;\le\; \varepsilon_{\mathcal{C}(d_x,B,L)}(\beta^{-1}) + \varepsilon_{\mathcal{C}(d_y,B,L)}(\beta^{-1}),$$

where $L' = 2^{\alpha/2-2}L$.

Proof. Upper bound. The potentials $V(x) = \int_y \ell(x,y)\,\hat{\nu}^\beta(\mathrm{d}y)$ and $W(y) = -\int_x \ell(x,y)\,\hat{\mu}^\beta(\mathrm{d}x)$ satisfy $|V(x) - V(x')| \le \int |\ell(x,y) - \ell(x',y)|\,\mathrm{d}\hat{\nu}^\beta \le L\|x - x'\|^\alpha$, and similarly for W, so $V \in \mathcal{C}(d_x, B, L)$ and $W \in \mathcal{C}(d_y, B, L)$. Apply Lemma 2.

Lower bound. We verify (9) with $c = B/4$. For $V \in \mathcal{C}(d_x, B/4, L')$, $W \in \mathcal{C}(d_y, B/4, L')$, the function $\ell(x,y) = \frac{1}{c}(V(x) - \mathbb{E}_\mu V - c)(W(y) - \mathbb{E}_\nu W + c)$ satisfies $\|\ell\|_\infty \le B$ by Lemma 4. For α-Hölder regularity, writing $f = V - \mathbb{E}_\mu V - c$ and $g = W - \mathbb{E}_\nu W + c$ with $|f|, |g| \le B/2$:

$$|\ell(x,y) - \ell(x',y')| \le \frac{1}{c}\big(L'\|x - x'\|^\alpha \frac{B}{2} + \frac{B}{2} L'\|y - y'\|^\alpha\big) = 2L'\big(\|x - x'\|^\alpha + \|y - y'\|^\alpha\big).$$

Setting $a = \|x - x'\|$, $b = \|y - y'\|$, $\rho = \sqrt{a^2 + b^2}$, Lagrange multipliers on $\max_{t^2+s^2=1}(t^\alpha + s^\alpha)$ give $t = s = 1/\sqrt{2}$ with maximum $2^{1-\alpha/2}$, so $a^\alpha + b^\alpha \le 2^{1-\alpha/2}\rho^\alpha$. Therefore

$$|\ell(x,y) - \ell(x',y')| \le 2^{2-\alpha/2} L'\,\|(x,y) - (x',y')\|^\alpha = L\,\|(x,y) - (x',y')\|^\alpha,$$

so $\ell \in \mathcal{L}_\alpha$. Apply Lemma 3. □

Lemma 24 (Hölder upper bound, restated)**.** *For $\mathcal{C} = \mathcal{C}(d, B, L)$ and any $\beta > 0$ with $A'\beta \ge 1$ where $A' = eB\,(\beta L)^{d/\alpha}/\omega_d$, $\varepsilon_{\mathcal{C}}(\beta^{-1}) \le \frac{1+\log(A'\beta)}{\beta}$.*

Proof. Set $\min V = 0$ and let x^* be a minimizer. Since V is α-Hölder with $V(x^*) = 0$, we have $V(x) \le L\|x - x^*\|^\alpha$. In particular, on the ball $B(x^*, r)$ with $r = (\beta L)^{-1/\alpha}$, we have $V(x) \le 1/\beta$, giving

$$Z = \int_{\mathbb{T}^d} e^{-\beta V}\,\mathrm{d}x \ge e^{-1}\,\omega_d\,(\beta L)^{-d/\alpha}.$$

For any $c > 0$, splitting the numerator into $\{V \le c\}$ and $\{V > c\}$ yields

$$\mathbb{E}_{\mu_V^\beta}[V] \le c + \frac{Be^{-\beta c}}{Z} \le c + A'\,e^{-\beta c}, \qquad A' = \frac{eB\,(\beta L)^{d/\alpha}}{\omega_d}.$$

The function $f(c) = c + A'e^{-\beta c}$ has $f'(c) = 1 - A'\beta\,e^{-\beta c}$. If $A'\beta \ge 1$ then the minimizer is $c^* = \frac{1}{\beta}\log(A'\beta) > 0$, yielding $f(c^*) = \frac{1+\log(A'\beta)}{\beta}$. □

Lemma 25 (Hölder lower bound, restated)**.** *Provided $B \ge 2^{2-\alpha}\,L'\sqrt{d_x \vee d_y}^{\,\alpha}$, $\varepsilon_{\mathcal{L}_\alpha}(\beta^{-1}) \ge \frac{d_x+d_y}{\alpha\beta}(1 - o(1))$ as $\beta \to \infty$.*

Proof. Take $V(x) = L'\|x\|^\alpha_{\mathbb{T}^{d_x}}$ and $W(y) = L'\|y\|^\alpha_{\mathbb{T}^{d_y}}$. Both are α-Hölder with constant L' (by concavity of $t \mapsto t^\alpha$). The maximum torus distance from the origin is $\sqrt{d}/2$, so $\sup V - \inf V = L'(\sqrt{d_x}/2)^\alpha \le B/4$ by the assumed lower bound on B; thus $V \in \mathcal{C}(d_x, B/4, L')$ and similarly $W \in \mathcal{C}(d_y, B/4, L')$. Via the substitution $u = \beta L' r^\alpha$:

$$\mathbb{E}_{\mu_V^\beta}[V] = \frac{\frac{1}{\beta}\Gamma(d_x/\alpha + 1)}{\Gamma(d_x/\alpha)} + \mathcal{O}(e^{-\beta L'\sqrt{d_x}^{\,\alpha}/2}) = \frac{d_x}{\alpha\beta}(1 - o(1)),$$

and similarly for W. By Lemma (11) and (7), there exists $\ell \in \mathcal{L}_\alpha$ with QRE $(\mu_V^\beta, \nu_W^\beta)$, so $\varepsilon_{\mathcal{L}_\alpha}(\beta^{-1}) \ge \frac{d_x+d_y}{\alpha\beta}(1-o(1))$. □

D PROOFS FOR SMOOTH SPARSE MNE (SECTION 7)

D.1 PROOF OF LEMMA 26 (LAPLACE CONCENTRATION)

Lemma 26 (Laplace concentration, restated)**.** *Let $U : \mathbb{T}^d \to \mathbb{R}$ be smooth with $\min U = 0$. Suppose $\arg\min U = \{z_1, \dots, z_n\}$ is finite with $\nabla^2 U(z_i) \succeq \sigma I_d$ for all i, that $U(x) \geq \xi$ outside $\bigcup_i B(z_i, \delta)$, and that $\|U\|_{C^4} \leq \Lambda$. Then for all $\beta \geq \beta_0(\sigma, \xi, \delta, \Lambda, n, d)$,*

$$\left| \mathbb{E}_{\mu_U^\beta}[U] - \frac{d}{2\beta} \right| \leq \frac{C(\sigma, \Lambda, d)}{\beta^2}. \tag{22}$$

Proof. Write $Z = \int_{\mathbb{T}^d} e^{-\beta U}\, \mathrm{d}x$ and $N = \int_{\mathbb{T}^d} U\, e^{-\beta U}\, \mathrm{d}x$, so that $\mathbb{E}_{\mu_U^\beta}[U] = N/Z$. We decompose both integrals into contributions from the well neighborhoods $\Omega_i = B(z_i, \delta)$ and the exterior $\Omega_{\text{out}} = \mathbb{T}^d \setminus \bigcup_i \Omega_i$:

$$Z = \sum_{i=1}^n Z_i + Z_{\text{out}}, \qquad N = \sum_{i=1}^n N_i + N_{\text{out}},$$

where $Z_i = \int_{\Omega_i} e^{-\beta U}\, \mathrm{d}x$, $N_i = \int_{\Omega_i} U\, e^{-\beta U}\, \mathrm{d}x$, and similarly for the exterior terms.

Exterior estimates. Since $U \geq \xi$ on Ω_{out}, the exterior contributions satisfy

$$0 \leq Z_{\text{out}} \leq e^{-\beta\xi}, \qquad 0 \leq N_{\text{out}} \leq \|U\|_\infty\, e^{-\beta\xi}, \tag{23}$$

so they are exponentially negligible for large β.

Local expansion at well i. Substitute $u = \sqrt{\beta}\,(x - z_i)$, so that $\mathrm{d}x = \beta^{-d/2}\, \mathrm{d}u$. Taylor-expanding the exponent gives

$$\beta\, U\!\left(z_i + \frac{u}{\sqrt{\beta}}\right) = \frac{1}{2}\, u^T H_i\, u + \frac{1}{\sqrt{\beta}}\, P_3^{(i)}(u) + \frac{1}{\beta}\, P_4^{(i)}(u) + R_i(\beta, u), \tag{24}$$

where $H_i = \nabla^2 U(z_i) \succeq \sigma I_d$. The polynomials $P_3^{(i)}$ and $P_4^{(i)}$ are determined by the third and fourth derivatives of U at z_i:

$$P_3^{(i)}(u) = \frac{1}{6} \sum_{a,b,c} \partial_{abc} U(z_i)\, u_a u_b u_c, \qquad P_4^{(i)}(u) = \frac{1}{24} \sum_{a,b,c,e} \partial_{abce} U(z_i)\, u_a u_b u_c u_e,$$

and the remainder satisfies $|R_i(\beta, u)| \leq C_\Lambda\, |u|^5/\beta^{3/2}$ for $|u| \leq \delta\sqrt{\beta}$.

Expansion of Z_i. After the substitution $u = \sqrt{\beta}\,(x - z_i)$, the well contribution to the partition function becomes

$$Z_i = \int_{\Omega_i} e^{-\beta U(x)}\, \mathrm{d}x = \beta^{-d/2} \int_{|u| < \delta\sqrt{\beta}} \exp(-\beta\, U(z_i + u/\sqrt{\beta}))\, \mathrm{d}u.$$

Inserting the Taylor expansion (24) and factoring out the leading Gaussian term:

$$Z_i = \beta^{-d/2} \int_{|u| < \delta\sqrt{\beta}} e^{-\frac{1}{2} u^T H_i u}\, \underbrace{\exp\left(-\frac{1}{\sqrt{\beta}}\, P_3^{(i)}(u) - \frac{1}{\beta}\, P_4^{(i)}(u) - R_i(\beta, u) \right)}_{\text{expand as power series in } \beta^{-1/2}}\, \mathrm{d}u.$$

Expanding the underbraced exponential as $e^{-x} = 1 - x + x^2/2 - \cdots$ and collecting powers of $\beta^{-1/2}$:

$$\exp\left(\frac{-P_3}{\sqrt{\beta}} - \frac{P_4}{\beta} - R_i \right) = 1 - \frac{P_3^{(i)}}{\sqrt{\beta}} + \frac{1}{\beta} \left[\frac{(P_3^{(i)})^2}{2} - P_4^{(i)} \right] + \mathcal{O}\!\left(\frac{1}{\beta^{3/2}} \right),$$

where the $\mathcal{O}(\beta^{-3/2})$ absorbs all terms involving P_3^3, $P_3 P_4$, P_5 (from the remainder R_i), and higher.

We may also extend the domain of integration from $\{|u| < \delta\sqrt{\beta}\}$ to all of $\mathbb{R}^d$: on the complement, $\frac{1}{2} u^T H_i u \geq \frac{\sigma}{2} \delta^2 \beta$, so the extension costs $\mathcal{O}(e^{-\beta\sigma\delta^2/2})$. Combining:

$$Z_i = \beta^{-d/2} \int_{\mathbb{R}^d} e^{-\frac{1}{2} u^T H_i u} \left[1 - \frac{P_3^{(i)}(u)}{\sqrt{\beta}} + \frac{\frac{1}{2} P_3^{(i)}(u)^2 - P_4^{(i)}(u)}{\beta} + \mathcal{O}\!\left(\frac{1}{\beta^{3/2}} \right) \right] \mathrm{d}u.$$

The $\beta^{-1/2}$ term vanishes: $P_3^{(i)}$ is an odd (degree-3) polynomial, and the Gaussian weight $e^{-u^T H_i u/2}$ is even, so

$$\int_{\mathbb{R}^d} P_3^{(i)}(u)\, e^{-\frac{1}{2}u^T H_i u}\, \mathrm{d}u = 0.$$

For the same reason, the $\mathcal{O}(\beta^{-3/2})$ terms also integrate to zero, so the true correction is $\mathcal{O}(\beta^{-2})$. What remains is

$$Z_i = \beta^{-d/2}\, \frac{(2\pi)^{d/2}}{(\det H_i)^{1/2}} \left[1 + \frac{a_i}{\beta} + \mathcal{O}\!\left(\frac{1}{\beta^2}\right)\right], \tag{25}$$

where we used $\int_{\mathbb{R}^d} e^{-\frac{1}{2}u^T H_i u}\, \mathrm{d}u = (2\pi)^{d/2}(\det H_i)^{-1/2}$ for the leading term, and

$$a_i = \frac{(\det H_i)^{1/2}}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \left[\frac{1}{2}\, P_3^{(i)}(u)^2 - P_4^{(i)}(u)\right] e^{-\frac{1}{2}u^T H_i u}\, \mathrm{d}u$$

is bounded by $C(\Lambda, \sigma, d)$.

Expansion of N_i. The same substitution gives

$$N_i = \int_{\Omega_i} U(x)\, e^{-\beta U(x)}\, \mathrm{d}x = \beta^{-d/2} \int_{|u|<\delta\sqrt{\beta}} U\!\left(z_i + \frac{u}{\sqrt{\beta}}\right) e^{-\beta\, U(z_i + u/\sqrt{\beta})}\, \mathrm{d}u.$$

From the Taylor expansion, $U(z_i + u/\sqrt{\beta}) = \frac{1}{2\beta}\, u^T H_i\, u + \mathcal{O}(|u|^3/\beta^{3/2})$ (recall $U(z_i) = 0$ and $\nabla U(z_i) = 0$). Meanwhile, the exponential factor $e^{-\beta U}$ has the same expansion as for Z_i. Extending to $\mathbb{R}^d$ and combining:

$$N_i = \beta^{-d/2} \int_{\mathbb{R}^d} \left[\frac{u^T H_i\, u}{2\beta} + \mathcal{O}\!\left(\frac{|u|^3}{\beta^{3/2}}\right)\right] e^{-\frac{1}{2}u^T H_i u} \left[1 - \frac{P_3^{(i)}(u)}{\sqrt{\beta}} + \mathcal{O}\!\left(\frac{1}{\beta}\right)\right] \mathrm{d}u.$$

Distributing the product, the leading term is

$$\frac{1}{2\beta} \int_{\mathbb{R}^d} (u^T H_i\, u)\, e^{-\frac{1}{2}u^T H_i u}\, \mathrm{d}u.$$

To evaluate this, note that under the Gaussian $\mathcal{N}(0, H_i^{-1})$ with normalizing constant $(2\pi)^{d/2}(\det H_i)^{-1/2}$,

$$\mathbb{E}_{u\sim\mathcal{N}(0,H_i^{-1})}[u^T H_i\, u] = \operatorname{tr}(H_i H_i^{-1}) = d,$$

so the integral equals $d(2\pi)^{d/2}(\det H_i)^{-1/2}$. Similar to above, all correction terms above β^{-2} all vanish due to being odd-degree polynomials integrated against an even weight. The first nonvanishing correction is therefore $\mathcal{O}(1/\beta^2)$, giving

$$N_i = \beta^{-d/2}\, \frac{(2\pi)^{d/2}}{(\det H_i)^{1/2}} \left[\frac{d}{2\beta} + \frac{b_i}{\beta^2} + \mathcal{O}\!\left(\frac{1}{\beta^3}\right)\right], \tag{26}$$

with b_i bounded by $C(\Lambda, \sigma, d)$.

Assembling the ratio. Define $w_i = (\det H_i)^{-1/2}$ and $W = \sum_{i=1}^n w_i$. Combining (25), (26), and the exterior estimates (23):

$$N = \beta^{-d/2}(2\pi)^{d/2} \left[\frac{d}{2\beta}\, W + \frac{1}{\beta^2} \sum_i w_i b_i\right] + \mathcal{O}(e^{-\beta\xi}), \tag{27}$$

$$Z = \beta^{-d/2}(2\pi)^{d/2} \left[W + \frac{1}{\beta} \sum_i w_i a_i\right] + \mathcal{O}(e^{-\beta\xi}). \tag{28}$$

The common prefactor $\beta^{-d/2}(2\pi)^{d/2}$ cancels in the ratio N/Z, and the exponentially small terms are absorbed into $\mathcal{O}(1/\beta^3)$ for β large enough. Using the expansion $(1 + s/\beta)^{-1} = 1 - s/\beta + \mathcal{O}(1/\beta^2)$ with $s = \sum_i w_i a_i / W$:

$$\begin{aligned}
\mathbb{E}_{\mu_U^\beta}[U] = \frac{N}{Z} &= \frac{\frac{d}{2\beta} + \frac{1}{\beta^2} \sum_i w_i b_i / W}{1 + \frac{1}{\beta} \sum_i w_i a_i / W} \\
&= \left[\frac{d}{2\beta} + \frac{1}{\beta^2}\, \frac{\sum_i w_i b_i}{W}\right] \left[1 - \frac{1}{\beta}\, \frac{\sum_i w_i a_i}{W} + \mathcal{O}\!\left(\frac{1}{\beta^2}\right)\right] \\
&= \frac{d}{2\beta} + \frac{1}{\beta^2} \left[\frac{\sum_i w_i b_i}{W} - \frac{d}{2} \cdot \frac{\sum_i w_i a_i}{W}\right] + \mathcal{O}\!\left(\frac{1}{\beta^3}\right).
\end{aligned}$$

The bracketed constant depends on the Hessians and derivatives of U at the minimizers: all controlled by σ, Λ, d but not on β. This establishes (22). $\square$

Remark 27 (Explicit dependence of C)**.** *The constant $C(\sigma, \Lambda, d)$ in Lemma 26 satisfies $C \leq \kappa_d(\Lambda^2/\sigma^3 + \Lambda/\sigma^2)$, where κ_d depends polynomially on d. This follows from bounding $|a_i|$ and $|b_i|$ using Isserlis' theorem for Gaussian moments: the P_4 term contributes $\mathcal{O}(d^4\Lambda/\sigma^2)$ via fourth moments, and the P_3^2 term contributes $\mathcal{O}(d^6\Lambda^2/\sigma^3)$ via sixth moments. The coefficient b_i has the same scaling because H_i in $u^T H_i u$ cancels one covariance factor via $\sum_j (H_i)_{jk}(H_i^{-1})_{ja} = \delta_{ka}$.*

D.2 PROOF OF PROPOSITION 16, LOWER BOUND

Proof of Proposition 16, lower bound in (13). We consider, as an extension of [19] to multivariable toruses, the payoff $\ell : \mathbb{T}^{d_x} \times \mathbb{T}^{d_y} \to \mathbb{R}$ defined by

$$\ell(x, y) = \sum_{k=1}^{d_x} \big[\sin(4\pi x_k) + 2\cos(2\pi x_k + 2\pi y_k)\big] + \sum_{l=1}^{d_y} \sin(4\pi y_l). \tag{29}$$

This couples the first d_x coordinates of each player pairwise through the cosine terms; the remaining $d_y - d_x$ coordinates of Player 2 interact only through their own sine terms. (WLOG $d_x \leq d_y$; as mentioned above we can simply define $\ell \mapsto -\ell$ otherwise.)

Membership in $\mathcal{L}$. We claim that every MNE (μ^*, ν^*) of this game is sparse, with μ^* supported on $\{3/8, 7/8\}^{d_x}$ and ν^* supported on $\{1/8, 5/8\}^{d_y}$, and with all marginals equal to the uniform distribution $\frac{1}{2}\delta_{3/8} + \frac{1}{2}\delta_{7/8}$ (respectively $\frac{1}{2}\delta_{1/8} + \frac{1}{2}\delta_{5/8}$).

To see this, fix any MNE (μ^*, ν^*) and compute the best-response potential for Player 1. The cross-term in coordinate k integrates to

$$\begin{aligned}\int \cos(2\pi x_k + 2\pi y_k)\, \nu_k^*(\mathrm{d}y_k) &= \frac{1}{2}\cos\big(2\pi x_k + \frac{\pi}{4}\big) + \frac{1}{2}\cos\big(2\pi x_k + \frac{5\pi}{4}\big) \\ &= 0,\end{aligned}$$

using $\cos\theta + \cos(\theta + \pi) = 0$, provided the k-th marginal of ν^* is $\frac{1}{2}\delta_{1/8} + \frac{1}{2}\delta_{5/8}$. That this must be the case follows from the indifference condition: for any MNE strategy μ^*, let $p_k = \mu^*(\{x_k = 3/8\})$ be the k-th marginal weight. Then

$$\int \cos(2\pi x_k + 2\pi y_k)\, \mu^*(\mathrm{d}x) = (1 - 2p_k)\, \cos(2\pi y_k - \pi/4).$$

For the opponent to be indifferent between $y_k = 1/8$ and $y_k = 5/8$, this expression must take the same value at both points, which forces $p_k = 1/2$. By symmetry, the ν^*-marginals are also forced to $1/2$–$1/2$.

Hence the best-response potential is

$$V^*(x) = \sum_{k=1}^{d_x} [\sin(4\pi x_k) + 1] + \text{const},$$

with minimizers at $x_k \in \{3/8, 7/8\}$ for each k (where $\sin(4\pi x_k) = -1$). The Hessian at each minimizer is $(4\pi)^2 I_{d_x} \succ 0$, and the gap $\xi > 0$ follows from compactness. An analogous calculation gives $W^*(y) = \sum_l [1 - \sin(4\pi y_l)] + \text{const}$, minimized at $y_l \in \{1/8, 5/8\}$, with the same curvature and gap properties. Every MNE is therefore sparse and non-degenerate, confirming $\ell \in \mathcal{L}$.

Symmetry fixes the QRE potentials. The key observation is that the QRE potentials V^β and W^β are in fact *independent of* β in their spatial dependence, thanks to a discrete symmetry of the game. This eliminates the need for any bootstrap argument relating V^β to V^*.

In particular, the payoff (29) is invariant under $(x_k, y_k) \mapsto (x_k + \frac{1}{2}, y_k + \frac{1}{2})$ for each k:

$$\sin(4\pi(t + \frac{1}{2})) = \sin(4\pi t + 2\pi) = \sin(4\pi t),$$

$$\cos(2\pi(x_k+\frac{1}{2})+2\pi(y_k+\frac{1}{2}))=\cos(2\pi x_k+2\pi y_k+2\pi)=\cos(2\pi x_k+2\pi y_k).$$

Since the QRE is unique (the saddle point of a strictly convex-concave problem), $\hat{\nu}^\beta$ must inherit this symmetry. Consider the first Fourier coefficient of $\hat{\nu}^\beta$ in coordinate k:

$$\varphi_k=\int e^{2\pi i y_k}\,\hat{\nu}^\beta(\mathrm{d}y)=\int e^{2\pi i(y_k+1/2)}\,\hat{\nu}^\beta(\mathrm{d}y)=e^{\pi i}\,\varphi_k=-\varphi_k,$$

so $\varphi_k=0$ for all β. But φ_k is precisely the coefficient that couples x_k to y_k through the cosine term:

$$\begin{aligned}V^\beta(x)&=\int \ell(x,y)\,\hat{\nu}^\beta(\mathrm{d}y)\\&=\sum_{k=1}^{d_x}\sin(4\pi x_k)+2\sum_{k=1}^{d_x}\mathrm{Re}\big[e^{2\pi i x_k}\,\varphi_k\big]+c(\beta)\\&=\sum_{k=1}^{d_x}\sin(4\pi x_k)+c(\beta),\end{aligned}$$

where $c(\beta)=\sum_l\int\sin(4\pi y_l)\,\hat{\nu}^\beta(\mathrm{d}y)+d_x$ is an additive constant independent of x. Since $\mu^\beta_{V^\beta}$ depends on V^β only up to additive constants, the potential governing Player 1's strategy is effectively fixed. By the same argument applied to the $x_k\mapsto x_k+1/2$ symmetry (which kills the first Fourier coefficient of $\hat{\mu}^\beta$ in each coordinate), we also have $W^\beta(y)=-\sum_l\sin(4\pi y_l)+c'(\beta)$.

Laplace asymptotics. With the potentials fixed, Lemma 26 applies directly. Since $V^\beta(x)-\min V^\beta=\sum_k[\sin(4\pi x_k)+1]$ is separable across coordinates, the Gibbs measure $\mu^\beta_{V^\beta}$ factors as a product:

$$\mu^\beta_{V^\beta}(\mathrm{d}x)=\prod_{k=1}^{d_x}\frac{\exp(-\beta[\sin(4\pi x_k)+1])\,\mathrm{d}x_k}{\int_0^1\exp(-\beta[\sin(4\pi t)+1])\,\mathrm{d}t}.$$

Each one-dimensional factor is a Gibbs measure for $\sin(4\pi t)+1$ on $[0,1]$, which has two non-degenerate minimizers at $t=3/8$ and $t=7/8$ with second derivative $(4\pi)^2$. Lemma 26 makes this precise and controls the remainder:

$$\mathbb{E}_{\mu^\beta_{V_k}}[\sin(4\pi x_k)+1]=\frac{1}{2\beta}+\mathcal{O}\Big(\frac{1}{\beta^2}\Big).$$

Summing over $k=1,\dots,d_x$,

$$\mathbb{E}_{\mu^\beta_{V^\beta}}[V^\beta-\min V^\beta]=\frac{d_x}{2\beta}+\mathcal{O}\Big(\frac{1}{\beta^2}\Big),$$

and the identical calculation for W^β yields $d_y/(2\beta)+\mathcal{O}(1/\beta^2)$. By the NI decomposition (Section 4),

$$\mathrm{NI}(\hat{\mu}^\beta,\hat{\nu}^\beta)=\frac{d_x+d_y}{2\beta}+\mathcal{O}\Big(\frac{1}{\beta^2}\Big).$$

Since $\ell\in\mathcal{L}$, we conclude $\varepsilon_{\mathcal{L}}(\beta^{-1})\geq(d_x+d_y)/(2\beta)+\mathcal{O}(1/\beta^2)$. □

D.3 PROOF OF PROPOSITION 16, UPPER BOUND (INSTANCE-WISE)

Proof of Proposition 16, instance-wise rate (12). Fix $\ell\in\mathcal{L}$. Unlike the lower bound, where a discrete symmetry fixed the QRE potentials exactly, we must now handle the fact that $V^\beta(x)=\int\ell(x,y)\,\hat{\nu}^\beta(\mathrm{d}y)$ changes with β for a general game. The argument proceeds by showing that V^β eventually inherits the qualitative structure of V^*, so that Lemma 26 applies with uniform constants.

Convergence of QRE potentials. By compactness of $\mathcal{P}(\mathbb{T}^{d_y})$, the family $(\hat{\nu}^\beta)_{\beta>0}$ has at least one accumulation point ν^* as $\beta\to\infty$, and any such ν^* forms part of an MNE (μ^*,ν^*) (the QRE optimality conditions pass to the limit). Along any subsequence $\beta_n\to\infty$ with $\hat{\nu}^{\beta_n}\to\nu^*$ weakly, the potentials converge in every C^k norm: for any multi-index α,

$$\sup_{x\in\mathbb{T}^{d_x}}\big|\partial^\alpha_x V^{\beta_n}(x)-\partial^\alpha_x V^*(x)\big|=\sup_x\left|\int\partial^\alpha_x\ell(x,y)\,\big[\hat{\nu}^{\beta_n}-\nu^*\big](\mathrm{d}y)\right|\;\longrightarrow\;0,$$

since $\partial_x^\alpha \ell(x, \cdot)$ is continuous on the compact space $\mathbb{T}^{d_y}$ and the convergence is weak against bounded continuous functions; uniformity in x follows from compactness of $\mathbb{T}^{d_x}$.

Applying Lemma 26. By assumption on $\mathcal{L}$, V^* has finitely many minimizers with Hessian eigenvalues $\geq \sigma$ and gap $\geq \xi$. Since $\|V^{\beta_n} - V^*\|_{C^2} \to 0$, for all n large enough V^{β_n} has the same qualitative structure: the same number of minimizers (located near those of V^*), Hessian eigenvalues $\geq \sigma/2$, gap $\geq \xi/2$, and $\|V^{\beta_n}\|_{C^4} \leq \|V^*\|_{C^4} + 1$. These are precisely the hypotheses of Lemma 26, with parameters $(\sigma/2,\ \xi/2,\ \delta/2,\ \|V^*\|_{C^4} + 1,\ n,\ d_x)$ that are all independent of β_n. At each fixed β_n, V^{β_n} is simply a fixed smooth function satisfying these bounds, so Lemma 26 gives

$$\mathbb{E}_{\mu^{\beta_n}_{V^{\beta_n}}}[V^{\beta_n} - \min V^{\beta_n}] = \frac{d_x}{2\beta_n} + \mathcal{O}\left(\frac{1}{\beta_n^2}\right),$$

where the implicit constant is independent of β_n.

From subsequences to the full sequence. The leading term $d_x/(2\beta)$ is universal: it does not depend on which accumulation point ν^* the subsequence converges to, since Lemma 26 yields $d/(2\beta)$ regardless of the specific Hessians, number of wells, or Laplace weights. (This is the key cancellation: the Gaussian moment $\mathbb{E}[u^T H u] = d$ is independent of H as long as $H \succ 0$.)

If there were a subsequence β_n along which the expectation deviated from $d_x/(2\beta_n)$ by more than $\mathcal{O}(1/\beta_n^2)$, we could extract a further convergent subsequence (by compactness of $\mathcal{P}(\mathbb{T}^{d_y})$) and apply the argument above to obtain a contradiction. Hence the bound holds for the full sequence.

The identical analysis applies to W^β, and the NI decomposition gives

$$\mathrm{NI}_\ell(\hat{\mu}^\beta, \hat{\nu}^\beta) = \frac{d_x + d_y}{2\beta} + \mathcal{O}_\ell\left(\frac{1}{\beta^2}\right). \qquad \square$$

Proof of Proposition 16, upper bound in (13). Every smooth bounded function on the compact torus $\mathbb{T}^{d_x} \times \mathbb{T}^{d_y}$ is Lipschitz, with constant $L = \sup \|\nabla \ell\| < \infty$. Therefore $\mathcal{L}$ is contained in the class $\mathcal{L}_1$ of Lipschitz payoffs with $\alpha = 1$ from Section 6. Since $\varepsilon_{\mathcal{L}}(\beta^{-1}) \leq \varepsilon_{\mathcal{L}_1}(\beta^{-1})$ (smaller class, smaller supremum), the bound follows from Lemma 12. $\square$