Solutions to Hartshorne's Algebraic Geometry

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Note: Starred and Formal Schemes questions have been skipped since for the most part we skipped those in class.

Contents

1 I. Varieties

1.1 I.1 x

1.1.1 Ex, I.1.1 g x

1.1. (a) Let Y be the plane curve $y = x^2$ (i.e., Y is the zero set of the polynomial $f = y - x^2$). Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

We have $k[x,y]/(y-x^2) \approx k[x,x^2]$ by plugging in x^2 to y.

1.1.2 b. x g

(b) Let Z be the plane curve $xy = 1$. Show that $A(Z)$ is not isomorphic to a polynomial ring in one variable over k.

$A(z) = k[x,y]/(xy-1) \approx k\left[x, \frac{1}{x}\right]$

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*(c) Let f be any irreducible quadratic polynomial in $k[x,y]$, and let W be the conic defined by f. Show that $A(W)$ is isomorphic to $A(Y)$ or $A(Z)$. Which one is it when?

1.1.3 Ex, I.1.2 x g

1.2. *The Twisted Cubic Curve.* Let $Y \subseteq \mathbf{A}^3$ be the set $Y = \{(t,t^2,t^3) | t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k. We say that Y is given by the *parametric representation* $x = t, y = t^2, z = t^3$.

Since we assume algebraically closed, k is infinite, then to see irreducible, since k is integral domain, if $fg \in I(Y)$, then $f(x,x^2,x^3)\,g(x,x^2,x^3) = 0$.

Thus f or g must be zero, so one of $f, g \in I(Y)$, i.e. it is prime.

Clearly dimension 1 since parametrized by t, the ideal is generated by $y - x^2, z - x^3$ and then $A(Y) = k[x,y,z]/(y-x^2, z-x^3) \approx k[x,x^2,x^3] \approx k[x]$

1.1.4 Ex, I.1.3 x g

1.3. Let Y be the algebraic set in $\mathbf{A}^3$ defined by the two polynomials $x^2 - yz$ and $xz - x$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

If $z = 0$, then $x = 0$ and y is anything. So we have y-axis.

Prime ideal is (z,x)
If $z\neq 0$, then we have $x^2=\alpha y$ for $\alpha\in k$ so we have a parabola.
Prime ideal is $(z-k, y-kx^2)$
If $x=0$, then z is anything, and $y=0$. This is z axis.
Prime ideal is (x,y)

1.1.5 Ex, I.1.4 x g

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1.4. If we identify $\mathbf{A}^2$ with $\mathbf{A}^1 \times \mathbf{A}^1$ in the natural way, show that the Zariski topology on $\mathbf{A}^2$ is not the product topology of the Zariski topologies on the two copies of $\mathbf{A}^1$.

he set $V(y-x)$ is the diagonal it's closed in Zariski topology.

Now a closde base for the product topology on $\mathbb{A}^1\times\mathbb{A}^1$ would be products of sets closed in the Z-topology on both factors. Closed sets in $\mathbb{A}^1$are just points and the whole space. So closed sets in $\mathbb{A}^1\times\mathbb{A}^1$ product topology should be finite unions of horizontal or vertical lines and points. The diagonal is not such.

1.1.6 Ex, I.1.5 x

1.5. Show that a k-algebra B is isomorphic to the affine coordinate ring of some algebraic set in $\mathbf{A}^n$, for some n, if and only if B is a finitely generated k-algebra with no nilpotent elements.

Clearly if B is an affine coordinate ring then it's finitely generated, no nilpotents.
If B is f.g. no nilpotents, let $x_1,...,x_n$ a set of generators.
Then $B=k[x_1,...,x_n]/J$ where J is reduced since no nilpotents.
Thus $I(V(J))=J$ thus B is coordinate ring of $V(I)$.

1.1.7 Ex, I.1.6 x g

1.6. Any nonempty open subset of an irreducible topological space is dense and irreducible. If Y is a subset of a topological space X, which is irreducible in its induced topology, then the closure $\overline{Y}$ is also irreducible.

U dense

Assume to contrary U is not dense.
Let $V=X\backslash\overline{U}$.
Then $X=U^c\cup V^c$, but X was supposed irreducible.
$\Longrightarrow$ U dense.

U irreducible

Assume to contrary that U is not irreducible, $U=Y_1\coprod Y_2$, Y_i closed.
For $X_1,X_2\subset X$ with $Y_i=U\cap X_i$, then $(X_1\cup X_2)\cup U^c=X \implies X$ irreducible.
Contradiction. $\implies U$ irreducible.

closure irreducible

Suppose Y is irreducible, but $\overline{Y}=Y_1\coprod Y_2$.
Then $Y=(Y_1\cap Y)\cup(Y_2\cap Y)\implies Y=(Y_i\cap Y)$ for one of $i=1,2$.
$\overline{Y}$ the smallest closed subset of X containing $Y \implies \overline{Y}=Y_i \implies \overline{Y}$ is irreducible.

1.1.8 Ex, I.1.7 x

1.7. (a) Show that the following conditions are equivalent for a topological space X: (i) X is noetherian; (ii) every nonempty family of closed subsets has a minimal element; (iii) X satisfies the ascending chain condition for open subsets; (iv) every nonempty family of open subsets has a maximal element.

Since it holds from the equivalent conditions for noetherian modules since ideals correspond to submodules.

1.1.9 b. x

(b) A noetherian topological space is *quasi-compact*, i.e., every open cover has a finite subcover.

Apply part (a) to the cover U_1, $U_1 \cup U_2$, $U_1 \cup U_2 \cup U_3, \dots$.

1.1.10 (c) x.

(c) Any subset of a noetherian topological space is noetherian in its induced topology.

follows from part (a).

1.1.11 d x

(d) A noetherian space which is also Hausdorff must be a finite set with the discrete topology.

Argue as in the proof of the baire category theorem.

Ex, I.1.8 x g and below

1.8. Let Y be an affine variety of dimension r in $\mathbf{A}^n$. Let H be a hypersurface in $\mathbf{A}^n$, and assume that $Y \nsubseteq H$. Then every irreducible component of $Y \cap H$ has dimension $r - 1$. (See (7.1) for a generalization.)

Note irreducible components of $Y \cap H$ correspond to minimal prime ideals of height 1.
Now use $dim\ R/\mathfrak{p} + height\ \mathfrak{p} = dim\ R$.

Ex, I.1.9 x g

1.9. Let $\mathfrak{a} \subseteq A = k[x_1, \dots, x_n]$ be an ideal which can be generated by r elements. Then every irreducible component of $Z(\mathfrak{a})$ has dimension $\geqslant n - r$.

Each $f_i \in \mathfrak{a}$ from $i = 1$ to r defines a hypersurface.
Apply the previous excercise r times.

1.1.12 Ex, I.1.10 x

1.10. (a) If Y is any subset of a topological space X, then $\dim Y \leqslant \dim X$.

Any chain of irreducible closeds in Y extends to a chain in X.

1.1.13 (b). x g and I.2.7.a

(b) If X is a topological space which is covered by a family of open subsets $\{U_i\}$, then $\dim X = \sup \dim U_i$.

By (a) $sup\ dim\ U_i \leq dim\ X$.
Let $\{pt\} = X_0 \subset ... \subset X_n$ is a chain of irreducible closed subset of X
Let $U \ni X_0$. By 1.6, $X_i \cap U$ is irreducible and dense in X_i so the strict inclusions are maintained.
Thus $X_0 \cap U \subset ... \subset X_n \cap U$ is a chain and $dim\,(X) \leq dim\ U \leq sup\ dim\ U_i$.

1.1.14 c. x

(c) Give an example of a topological space X and a dense open subset U with $\dim U < \dim X$.

Consider $X = \{0, 1\}$ with open sets $\emptyset, \{1\}, X$.

1.1.15 d. x

(d) If Y is a closed subset of an irreducible finite-dimensional topological space X, and if $\dim Y = \dim X$, then $Y = X$.

If $Y \neq X$, then to any chain in Y we add X to get a longer chain in X so their dimensions are not the same.

1.1.16 e. x

(e) Give an example of a noetherian topological space of infinite dimension.

Let X be the positive integers with closed sets like $\{1, ..., n\}$.

Ex, I.1.11*

*__1.11.__ Let $Y \subseteq \mathbf{A}^3$ be the curve given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x,y,z]$ which cannot be generated by 2 elements. We say Y is *not a local complete intersection*—cf. (Ex. 2.17).

Ex, I.1.12 x.

1.12. Give an example of an irreducible polynomial $f \in \mathbf{R}[x, y]$, whose zero set $Z(f)$ in $\mathbf{A}^2_{\mathbf{R}}$ is not irreducible (cf. 1.4.2).

Consider $y^2 + (x^2 - 1)^2$.
The point is that it only factors over $\mathbb{C}[x, y]$, but it has two real roots.

1.2 I.2 x

1.2.1 I.2.1 x homogenous nullstellensatz

2.1. Prove the "homogeneous Nullstellensatz," which says if $\mathfrak{a} \subseteq S$ is a homogeneous ideal, and if $f \in S$ is a homogeneous polynomial with deg $f > 0$, such that $f(P) = 0$ for all $P \in Z(\mathfrak{a})$ in $\mathbf{P}^n$, then $f^q \in \mathfrak{a}$ for some $q > 0$. [*Hint*: Interpret the problem in terms of the affine $(n+1)$-space whose affine coordinate ring is S, and use the usual Nullstellensatz, (1.3A).]

This follows by looking at the affine cone and using the affine nullstellensatz.

1.2.2 I.2.2 projective containments x

2.2. For a homogeneous ideal $\mathfrak{a} \subseteq S$, show that the following conditions are equivalent:

(i) $Z(\mathfrak{a}) = \varnothing$ (the empty set);
(ii) $\sqrt{\mathfrak{a}}$ = either S or the ideal $S_+ = \bigoplus_{d>0} S_d$;
(iii) $\mathfrak{a} \supseteq S_d$ for some $d > 0$.

Assume (i). Then in $\mathbb{A}^{n+1}$, $Z(\mathfrak{a})$ is either empty or $(0, ..., 0)$ so $\sqrt{\mathfrak{a}}$ is either S or $\oplus_{d>0} S_d$
Assume (ii). If $\sqrt{\mathfrak{a}}$ contains x_i then $x_i \in \mathfrak{a}^m$ for all i.
Since x_i^m divides monomials of degree $m(n+1)$, then $S_{m(n+1)} \supset \mathfrak{a}$.
Assume (iii). If $\mathfrak{a} \supset S_d$ then $x_i^d \in \mathfrak{a}$ have no zeros.

I.2.3

1.2.3 I.2.3.a. containments. x

2.3. (a) If $T_1 \subseteq T_2$ are subsets of S^h, then $Z(T_1) \supseteq Z(T_2)$.

Trivial.

1.2.4 b. x

(b) If $Y_1 \subseteq Y_2$ are subsets of $\mathbf{P}^n$, then $I(Y_1) \supseteq I(Y_2)$.

trivial

1.2.5 c. x

(c) For any two subsets Y_1, Y_2 of $\mathbf{P}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

trivial

1.2.6 d. x

(d) If $\mathfrak{a} \subseteq S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

$I(Z(\mathfrak{a}))$ is the set of f vanishing on the zero set of $\mathfrak{a}$.
By nullstellensatz, such f are in $\sqrt{r}$.

Now let $f \in \sqrt{\mathfrak{a}} = \cap_{P \in Z(\mathfrak{a})} (X_1 - P_1, ..., X_n - P_n)$.
Note that
$Z(\mathfrak{a})$ is the set of P where every $g \in \mathfrak{a}$ vanishes.
So $I(Z(\mathfrak{a}))$ is the set of h which vanish at all P where every $g \in \mathfrak{a}$ vanishes.
So we have the reverse containment.

1.2.7 e. x

(e) For any subset $Y \subseteq \mathbf{P}^n$, $Z(I(Y)) = \bar{Y}$.

part 1 $Z(I(Y)) \subset \overline{Y}$
$Z(I(Y))$ is closed set containing $Y \implies Z(I(Y)) \supset \overline{Y}$, the smallest closed set containing Y.

part 2 $\overline{Y} \subset Z(I(Y))$.
Suppose $p \notin \overline{Y}$ so $I(\overline{Y}) \supset I(\overline{Y} \cup P)$ since some polynomials vanishing on $\overline{Y}$ don't vanish at P.
$\implies P \notin Z(I(Y))$.

I.2.4 a x

2.4. (a) There is a 1-1 inclusion-reversing correspondence between algebraic sets in $\mathbf{P}^n$, and homogeneous radical ideals of S not equal to S_+, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$. *Note*: Since S_+ does not occur in this correspondence, it is sometimes called the *irrelevant* maximal ideal of S.

By 2.3 and 2.2.

1.2.8 b. x g and below

(b) An algebraic set $Y \subseteq \mathbf{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

Suppose that $I(Y)$ is prime.
If $Y = Y_1 \cup Y_2$, then $I(Y) = I(Y_1) \cap I(Y_2) \supset I(Y_1) I(Y_2)$. Thus $I(Y) = I(Y_1)$ or $I(Y_2)$.
If Y is not prime, then there are $ab \in I(Y)$ with both $a, b \notin I(Y)$.
Thus Y is a union of $Y \cap Z(a)$ and $Y \cap Z(b)$ and is not irreducible.

1.2.9 c. x g

(c) Show that $\mathbf{P}^n$ itself is irreducible.

Since $I(\mathbb{P}^n) = 0$ which is prime, so use part (b).

1.2.10 I.2.5 (a) x g

2.5. (a) $\mathbf{P}^n$ is a noetherian topological space.

Irreducible closed chains in $\mathbb{P}^n$ corresponds to ascending chains of primes in $k[x_0, ..., x_n]$ by Ex. 2.3 Note that $k[x_0, ..., x_n]$ is noetherian by hilbert basis theorem.

1.2.11 (b). x

(b) Every algebraic set in $\mathbf{P}^n$ can be written uniquely as a finite union of irreducible algebraic sets, no one containing another. These are called its *irreducible components*.

By proposition 1.5, and part (a).

I.2.6 x

2.6. If Y is a projective variety with homogeneous coordinate ring $S(Y)$, show that $\dim S(Y) = \dim Y + 1$. [*Hint*: Let $\varphi_i : U_i \to \mathbf{A}^n$ be the homeomorphism of (2.2), let Y_i be the affine variety $\varphi_i(Y \cap U_i)$, and let $A(Y_i)$ be its affine coordinate ring. Show that $A(Y_i)$ can be identified with the subring of elements of degree 0 of the localized ring $S(Y)_{x_i}$. Then show that $S(Y)_{x_i} \cong A(Y_i)[x_i, x_i^{-1}]$. Now use (1.7), (1.8A), and (Ex 1.10), and look at transcendence degrees. Conclude also that $\dim Y = \dim Y_i$ whenever Y_i is nonempty.]

$S(Y)_{(x_i)}$ is the coordinate ring of the cone of Y_i if Y_i is nonempty.
So degree 0 part of $S(Y)_{x_i}$ is the coordinate ring of the cone with $x_i = 0$ which is isomorphic to Y_i.
$\implies S(Y)_{x_i} = A(Y_i)\left[x_i, \frac{1}{x_i}\right]$. Comparing transcendence degrees gives the result.

1.2.12 I.2.7 a. x g

2.7. (a) $\dim \mathbf{P}^n = n$.

This follows from I.10.b, using the standard affine cover.

1.2.13 b. x

(b) If $Y \subseteq \mathbf{P}^n$ is a quasi-projective variety, then $\dim Y = \dim \bar{Y}$. [*Hint*: Use (Ex. 2.6) to reduce to (1.10).]

This follows from the proof of 1.10 and using the affine cone.

1.2.14 I.2.8 x g

2.8. A projective variety $Y \subseteq \mathbf{P}^n$ has dimension $n - 1$ if and only if it is the zero set of a single irreducible homogeneous polynomial f of positive degree. Y is called a *hypersurface* in $\mathbf{P}^n$.

Since irreducible homogeneous polynomial correspond to minimal prime ideals of height 1, then using the height / dimension formula gives Y has dimension $n-1$.

Conversely suppose Y has dimension $n-1$.

$\implies dim\ k[Y] = dim\ Y + 1 = n$ (by 1.2.6).

So the ideal of Y can have height at most 1 by the height / dimension formula.

1.2.15 I.2.9 x

2.9. *Projective Closure of an Affine Variety.* If $Y \subseteq \mathbf{A}^n$ is an affine variety, we identify $\mathbf{A}^n$ with an open set $U_0 \subseteq \mathbf{P}^n$ by the homeomorphism φ_0. Then we can speak of $\overline{Y}$, the closure of Y in $\mathbf{P}^n$, which is called the *projective closure* of Y.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

1. $I\left(\overline{Y}\right) \supset \beta\left(I\left(Y\right)\right)$

If $f \in I\left(\overline{Y}\right)$, then $\beta\left(f\right) = f\left(1, x_1, ..., x_n\right)$ vanishes on $Y \subset \mathbb{A}^n \implies f \in I\left(Y\right)$.

2. $\beta\left(I\left(Y\right)\right) \supset I\left(\overline{Y}\right)$.

If homogeneous h vanishes on $\overline{Y}$, and $h\left(1, x_1, ..., x_n\right) = g$, then $h = \beta\left(g\right)$ so $I\left(\overline{Y}\right)$ is generated by $\beta\left(I\left(Y\right)\right)$.

(b) Let $Y \subseteq \mathbf{A}^3$ be the twisted cubic of (Ex. 1.2). Its projective closure $\overline{Y} \subseteq \mathbf{P}^3$ is called the *twisted cubic curve* in $\mathbf{P}^3$. Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, \ldots, f_r$ generate $I(Y)$, then $\beta(f_1), \ldots, \beta(f_r)$ do *not* necessarily generate $I(\overline{Y})$.

1.2.16 I.2.10 x

Sometimes we consider the projective closure $\overline{C(Y)}$ of $C(Y)$ in $\mathbf{P}^{n+1}$. This is called the *projective cone* over Y.

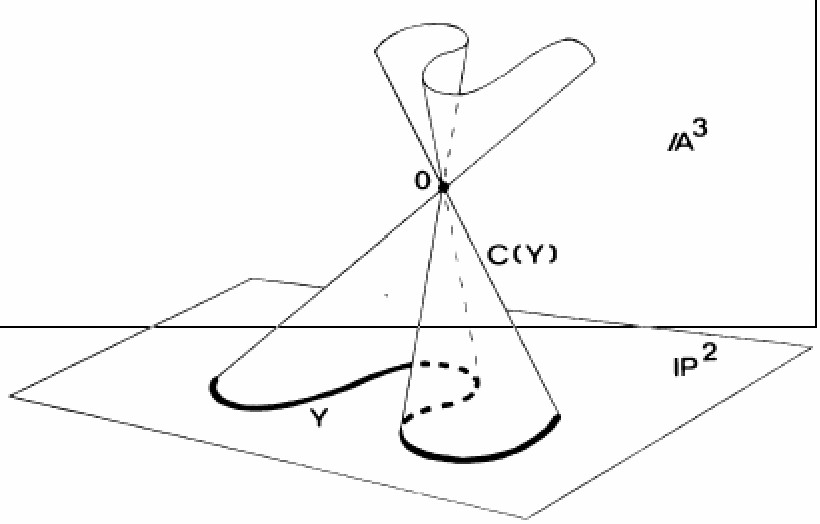

Figure 1. The cone over a curve in $\mathbf{P}^2$.

2.10. *The Cone Over a Projective Variety* (Fig. 1). Let $Y \subseteq \mathbf{P}^n$ be a nonempty algebraic set, and let $\theta: \mathbf{A}^{n+1} - \{(0,\ldots,0)\} \rightarrow \mathbf{P}^n$ be the map which sends the point with affine coordinates $(a_0,\ldots,a_n)$ to the point with homogeneous coordinates $(a_0,\ldots,a_n)$. We define the *affine cone* over Y to be

$$C(Y) = \theta^{-1}(Y) \cup \{(0,\ldots,0)\}.$$

(a) Show that $C(Y)$ is an algebraic set in $\mathbf{A}^{n+1}$, whose ideal is equal to $I(Y)$, considered as an ordinary ideal in $k[x_0,\ldots,x_n]$.

this is clear.

1.2.17 b. x

(b) $C(Y)$ is irreducible if and only if Y is.

Since they have the same ideal. (prime iff the ideal is irreducible was a previous excercise).

1.2.18 c. x g

(c) $\dim C(Y) = \dim Y + 1$.

By 2.6.

1.2.19 I.2.11 x g (use p2)

2.11. *Linear Varieties in* $\mathbf{P}^n$. A hypersurface defined by a linear polynomial is called a *hyperplane*.

(a) Show that the following two conditions are equivalent for a variety Y in $\mathbf{P}^n$:
  (i) $I(Y)$ can be generated by linear polynomials.
  (ii) Y can be written as an intersection of hyperplanes.
  In this case we say that Y is a *linear variety* in $\mathbf{P}^n$.

Trivial.

1.2.20 b. x g

(b) If Y is a linear variety of dimension r in $\mathbf{P}^n$, show that $I(Y)$ is minimally generated by $n - r$ linear polynomials.

Y is intersection of hyperplanes by (a), each corresponding to homogeneous primes of height 1. Each additional hyperplane increases the height of the ideal by 1.

1.2.21 x.

(c) Let Y,Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r + s - n \geqslant 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geqslant r + s - n$. (Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

by projective dimension theorem...

1.2.22 c. x

(c) Let Y, Z be linear varieties in $\mathbf{P}^n$, with $\dim Y = r$, $\dim Z = s$. If $r + s - n \geqslant 0$, then $Y \cap Z \neq \varnothing$. Furthermore, if $Y \cap Z \neq \varnothing$, then $Y \cap Z$ is a linear variety of dimension $\geqslant r + s - n$. (Think of $\mathbf{A}^{n+1}$ as a vector space over k, and work with its subspaces.)

By the projective dimension theorem.

1.2.23 I.2.12 x g and below

2.12. *The d-Uple Embedding.* For given $n, d > 0$, let $M_0, M_1, \ldots, M_N$ be all the monomials of degree d in the $n + 1$ variables $x_0, \ldots, x_n$, where $N = \binom{n+d}{n} - 1$. We define a mapping $\rho_d: \mathbf{P}^n \to \mathbf{P}^N$ by sending the point $P = (a_0, \ldots, a_n)$ to the point $\rho_d(P) = (M_0(a), \ldots, M_N(a))$ obtained by substituting the a_i in the monomials M_j. This is called the d-uple *embedding* of $\mathbf{P}^n$ in $\mathbf{P}^N$. For example, if $n = 1$, $d = 2$, then $N = 2$, and the image Y of the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$ is a conic.

(a) Let $\theta: k[y_0, \ldots, y_N] \to k[x_0, \ldots, x_n]$ be the homomorphism defined by sending y_i to M_i, and let $\mathfrak{a}$ be the kernel of θ. Then $\mathfrak{a}$ is a homogeneous prime ideal, and so $Z(\mathfrak{a})$ is a projective variety in $\mathbf{P}^N$.

θ maps into an integral domain so the kernel is prime.
Any monomial of degree i maps under θ to one of degree $d \cdot i$ which shows that the kernel is homogeneous.

1.2.24 .b. x g and above and below (part d)

(b) Show that the image of ρ_d is exactly $Z(\mathfrak{a})$. (One inclusion is easy. The other will require some calculation.)

If $f \in Ker(\phi)$, then $f(M_0, ..., M_n) = 0$ so f vanishes on $(M_0(a), ..., M_n(a))$ so $Im(v_d) \subset Z(\mathfrak{a})$.
Conversely, if $f \in I(Im(v_d))$, then $f(x) = 0$ for all $x \in Im(v_d)$.
$\implies f(M_0, ..., M_n) = 0$.
Thus $f \in ker\ \phi$. So $ker\ \phi \supset I(Im\ v_d)$.
So $Z(\mathfrak{a}) \subset Im\ v_d$.

1.2.25 c. x

(c) Now show that ρ_d is a homeomorphism of $\mathbf{P}^n$ onto the projective variety $Z(\mathfrak{a})$.

v_d is an injective isomorphism with image equal to $Z(\mathfrak{a})$.

1.2.26 d. x g and above

(d) Show that the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9) is equal to the 3-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

Take the projective closure of the twisted cubic, (x_1, x_1^2, x_1^3)
to get the 3-uple embedding $(x_0^3, x_0^2x_1, x_0x_1^2, x_1^3)$

1.2.27 I.2.13 x g

2.13. Let Y be the image of the 2-uple embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$. This is the *Veronese surface*. If $Z \subseteq Y$ is a closed curve (a *curve* is a variety of dimension 1), show that there exists a hypersurface $V \subseteq \mathbf{P}^5$ such that $V \cap Y = Z$.

A curve in $\mathbb{P}^2$ is defined by $f(x, y, z) = 0$ for homogeneous f.
$f^2 = g(x^2, y^2, z^2, xy, xz, yz)$ for some g.
So $v_2(Z(f)) = V \cap Y = Z$.

1.2.28 I.2.14 x g

2.14. *The Segre Embedding.* Let $\psi: \mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^N$ be the map defined by sending the ordered pair $(a_0, \ldots, a_r) \times (b_0, \ldots, b_s)$ to $(\ldots, a_i b_j, \ldots)$ in lexicographic order, where $N = rs + r + s$. Note that ψ is well-defined and injective. It is called the *Segre embedding*. Show that the image of ψ is a *subvariety* of $\mathbf{P}^N$. [*Hint*: Let the homogeneous coordinates of $\mathbf{P}^N$ be $\{z_{ij} | i = 0, \ldots, r, j = 0, \ldots, s\}$, and let $\mathfrak{a}$ be the kernel of the homomorphism $k[\{z_{ij}\}] \to k[x_0, \ldots, x_r, y_0, \ldots, y_s]$ which sends z_{ij} to $x_i y_j$. Then show that $\operatorname{Im} \psi = Z(\mathfrak{a})$.]

Let $\psi : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^N$ defined by $(a_0, ..., a_r) \times (b_0, ..., b_s) \mapsto (..., a_i b_j, ...)$ in lexicographic order (i.e. $a_i b_j$ is left of $a_k b_l$ iff $i < k$ or $i = k$ and $j < l$, so it's like $(a_0 b_0, a_0 b_1, a_0 b_2, ..., a_1 b_0, a_1 b_1, ...)$) where $N = (r+1)(s+1) - 1 = rs + r + s$. Note that ψ is well-defined and injective. It is called the Segre embedding. Show that the image of ψ is a subvariety of $\mathbb{P}^N$.

Note that any point of $\psi(\mathbb{P}^r \times \mathbb{P}^s)$ satisfies $(..., a_i b_j, ...)(..., a_k b_l, ...) = (..., a_i b_l, ...)(..., a_k b_j, ...)$.

Conversely, if $P \in \mathbb{P}^N$ with coordinates $a_i b_j$ satisfies the above relation, then there is some point Q in $\mathbb{P}^r \times \mathbb{P}^s$ mapping to it. Since we are in projective space, $a_i b_j \neq 0$ for some i, j. In affine space with $a_i b_j = 1$, then we have $a_k b_l = (a_k b_j)(a_i b_l)$. Thus we choose Q to have coordinates $a_k = a_k b_j$ and $b_l = (a_i b_l)$ so that it gets mapped to P under ψ.

In this manner, we know that $\psi(\mathbb{P}^r \times \mathbb{P}^s)$ is described by the vanishing of the equations $(a_i b_j)(a_k b_l) - (a_i b_l)(a_k b_j)$ and is thus a subvariety of $\mathbb{P}^N$.

1.2.29 I.2.15 x g

2.15. *The Quadric Surface in* $\mathbf{P}^3$ (Fig. 2). Consider the surface Q (a *surface* is a variety of dimension 2) in $\mathbf{P}^3$ defined by the equation $xy - zw = 0$.

(a) Show that Q is equal to the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ in $\mathbf{P}^3$, for suitable choice of coordinates.

Recall that $\psi(\mathbb{P}^r \times \mathbb{P}^s)$ is described by the vanishing of the equations $(a_i b_j)(a_k b_l) - (a_i b_l)(a_k b_j)$ and is thus a subvariety of $\mathbb{P}^N$.

1.2.30 b. x

(b) Show that Q contains two families of lines (a *line* is a linear variety of dimension 1) $\{L_t\}, \{M_t\}$, each parametrized by $t \in \mathbf{P}^1$, with the properties that if $L_t \neq L_u$, then $L_t \cap L_u = \varnothing$; if $M_t \neq M_u$, $M_t \cap M_u = \varnothing$, and for all t, u, $L_t \cap M_u =$ one point.

Lines are given by $\psi\left(\mathbb{P}^1 \times \{P\}\right)$ and $\psi\left(\{P\} \times \mathbb{P}^1\right)$.
From the picture of a cone, you can see the required properties.

1.2.31 c. x g

(c) Show that Q contains other curves besides these lines, and deduce that the Zariski topology on Q is not homeomorphic via ψ to the product topology on $\mathbf{P}^1 \times \mathbf{P}^1$ (where each $\mathbf{P}^1$ has its Zariski topology).

Look at the curve $x = y$.

1.2.32 I.2.16 x g

2.16. (a) The intersection of two varieties need not be a variety. For example, let Q_1 and Q_2 be the quadric surfaces in $\mathbf{P}^3$ given by the equations $x^2 - yw = 0$ and $xy - zw = 0$, respectively. Show that $Q_1 \cap Q_2$ is the union of a twisted cubic curve and a line.

If $P = (x, y, z, w) \in Q_1 \cap Q_2$ then $x^2 = yw, xy = zw$
$\implies y^2 w = yx^2 = zwx$
$\implies w = 0 = x$ or $y^2 = xz$.

1.2.33 b. x g

(b) Even if the intersection of two varieties is a variety, the ideal of the intersection may not be the sum of the ideals. For example, let C be the conic in $\mathbf{P}^2$ given by the equation $x^2 - yz = 0$. Let L be the line given by $y = 0$. Show that $C \cap L$ consists of one point P, but that $I(C) + I(L) \neq I(P)$.

Looking at the affine picture, $L \cap C$ is the origin.
But $I(L) + I(C) = (x^2, y) \neq (x, y)$.

1.2.34 I.2.17 x g complete intersections and below

2.17. *Complete intersections.* A variety Y of dimension r in $\mathbf{P}^n$ is a (*strict*) *complete intersection* if $I(Y)$ can be generated by $n - r$ elements. Y is a *set-theoretic complete intersection* if Y can be written as the intersection of $n - r$ hypersurfaces.
(a) Let Y be a variety in $\mathbf{P}^n$, let $Y = Z(\mathfrak{a})$; and suppose that $\mathfrak{a}$ can be generated by q elements. Then show that $\dim Y \geqslant n - q$.

By 1.8, the intersection of q-hypersurfaces has dim at least $n-q$. If it's generated by q elements, then the zero set is the intersection of q hypersurfaces.

1.2.35 b. x g

(b) Show that a strict complete intersection is a set-theoretic complete intersection.

If $I(Y)$ is generated by f_i , then $Y = \cap Z(f_i)$.

1.2.36 starred.

*(c) The converse of (b) is false. For example let Y be the twisted cubic curve in $\mathbf{P}^3$ (Ex. 2.9). Show that $I(Y)$ cannot be generated by two elements. On the other hand, find hypersurfaces H_1, H_2 of degrees 2,3 respectively, such that $Y = H_1 \cap H_2$.

1.3 I.3 x

1.3.1 I.3.1 x g

3.1. (a) Show that any conic in $\mathbf{A}^2$ is isomorphic either to $\mathbf{A}^1$ or $\mathbf{A}^1 - \{0\}$ (cf. Ex. 1.1).

By ex 1.1 since affine varieties are isomorphic iff coordinate rings are.

1.3.2 b. x g

(b) Show that $\mathbf{A}^1$ is *not* isomorphic to any proper open subset of itself. (This result is generalized by (Ex. 6.7) below.)

The coordinate ring of a proper subset has units not in k.

1.3.3 c. x g

(c) Any conic in $\mathbf{P}^2$ is isomorphic to $\mathbf{P}^1$.

A conic has genus 0 by the degree genus, and taking a point gives an embedding to $\mathbb{P}^1$ since the degree is $> 2g = 0$. (You may have to look ahead for this).

1.3.4 d. x g

(d) We will see later (Ex. 4.8) that any two curves are homeomorphic. But show now that $\mathbf{A}^2$ is not even homeomorphic to $\mathbf{P}^2$.

In $\mathbb{P}^2$, and two lines intersect by ex 3.7a, but not in $\mathbb{A}^2$.

1.3.5 e. x g

(e) If an affine variety is isomorphic to a projective variety, then it consists of only one point.

By 3.4, the regular functions on a projective variety are k.
This is only possible for an affine variety if it is a point, by 1.4.4

1.3.6 I.3.2 g bijective, bicontinuous but not isomorphism. x

3.2. A morphism whose underlying map on the topological spaces is a homeomorphism need not be an isomorphism.
(a) For example, let $\varphi : \mathbf{A}^1 \to \mathbf{A}^2$ be defined by $t \mapsto (t^2, t^3)$. Show that φ defines a bijective bicontinuous morphism of $\mathbf{A}^1$ onto the curve $y^2 = x^3$, but that φ is not an isomorphism.

Any inverse is a polynomial in x and y with $(x, y) \to \frac{y}{x}$ since $(t^2, t^3) \to t$.
Note φ is bijective to the cusp and continuous, as it is defined by polynomials.

1.3.7 b. frobenius not isomorphism. x g

(b) For another example, let the characteristic of the base field k be $p > 0$, and define a map $\varphi : \mathbf{A}^1 \to \mathbf{A}^1$ by $t \mapsto t^p$. Show that φ is bijective and bicontinuous but not an isomorphism. This is called the *Frobenius morphism*.

No inverse since it would need $f(t^p) = t$.
Injectivity follows from definition in characteristic p.
Surjectivity is because k is a perfect field, being algebraically closed.

1.3.8 I.3.3a. x

3.3. (a) Let $\varphi : X \to Y$ be a morphism. Then for each $P \in X$, φ induces a homomorphism of local rings $\varphi_P^* : \mathcal{O}_{\varphi(P),Y} \to \mathcal{O}_{P,X}$.

If f is regular, then $f \circ \phi$ is regular on a neighborhood $\phi^{-1}(V)$ of p.
Then we have a map $\mathcal{O}_{\phi(p),Y}$ to $\mathcal{O}_{p,X}$ which is a homomorphism.

1.3.9 b. x

(b) Show that a morphism φ is an isomorphism if and only if φ is a homeomorphism, and the induced map φ_P^* on local rings is an isomorphism, for all $P \in X$.

The fi direction is clear.
Now if φ is an isomorphism, then topologically it is a homeomorphism, and by part (a), the induced map on local rings is an isomorphism.

1.3.10 c. x

(c) Show that if $\varphi(X)$ is dense in Y, then the map φ_P^* is *injective* for all $P \in X$.

If $\phi_p^*(f) = 0$ then f vanishes on the dense set $\phi(X) \cap V$.
Continuity of f implies that it is 0.

1.3.11 I.3.4 x g

3.4. Show that the d-uple embedding of $\mathbf{P}^n$ (Ex. 2.12) is an isomorphism onto its image.

Note that $v_d^{-1} : (x_0 : \cdots : x_n) \mapsto \left(x_0^{d-1}x_0 : x_0^{d-1}x_1 : \cdots : x_0^{d-1}x_n\right)$ which is regular.

1.3.12 I.3.5 x g

3.5. By abuse of language, we will say that a variety "is affine" if it is isomorphic to an affine variety. If $H \subseteq \mathbf{P}^n$ is any hypersurface, show that $\mathbf{P}^n - H$ is affine. [*Hint*: Let H have degree d. Then consider the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$ and use the fact that $\mathbf{P}^N$ minus a hyperplane is affine.

Note that $\mathbb{P}^N - \textit{hyperplane}$ is affine and is the same as $\mathbb{P}^n - H$ under v_d.

1.3.13 I.3.6 x g

3.6. There are quasi-affine varieties which are not affine. For example, show that $X = \mathbf{A}^2 - \{(0,0)\}$ is not affine. [*Hint*: Show that $\mathcal{O}(X) \cong k[x,y]$ and use (3.5). See (III, Ex. 4.3) for another proof.]

Note that an affine variety should have $\mathcal{O}_X$ equal to the coordinate ring.
But regular functions on $\mathbb{A}^2 - (0,0)$ look like $\frac{f}{g}$ with $(f) + (g) = 1$.
g can only vanish along f or at $(0,0)$ and thus has a finite number of zeros.
Thus g is constant so the regular functions are just $k[x,y]$.
Since $k[x,y] \neq \mathbb{A}^2 - \{(0,0)\}$ this is a contradiction.

1.3.14 I.3.7 a x. g

3.7. (a) Show that any two curves in $\mathbf{P}^2$ have a nonempty intersection.

By the projective dimension theorem.

1.3.15 b. x

(b) More generally, show that if $Y \subseteq \mathbf{P}^n$ is a projective variety of dimension $\geqslant 1$, and if H is a hypersurface, then $Y \cap H \neq \varnothing$. [*Hint*: Use (Ex. 3.5) and (Ex. 3.1e). See (7.2) for a generalization.]

Also by the projective dimension theorem

1.3.16 I.3.8 x g

3.8. Let H_i and H_j be the hyperplanes in $\mathbf{P}^n$ defined by $x_i = 0$ and $x_j = 0$, with $i \neq j$. Show that any regular function on $\mathbf{P}^n - (H_i \cap H_j)$ is constant. (This gives an alternate proof of (3.4a) in the case $Y = \mathbf{P}^n$.)

$f \in \mathcal{O}_X$ looks like $f_i/x_i^{deg(f_i)} = f_j/x_j^{deg(f_j)}$.
So $f_i x_j^{deg(f_j)} = f_j x_i^{deg(f_i)}$.
Since $x_i \nmid x_j$ then $x_i^{deg(f_i)} | f_i$ so that $f_i = x_i^{deg(f_i)}$ so the function is constant.

1.3.17 I.3.9 x g

3.9. The homogeneous coordinate ring of a projective variety is not invariant under isomorphism. For example, let $X = \mathbf{P}^1$, and let Y be the 2-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^2$. Then $X \cong Y$ (Ex. 3.4). But show that $S(X) \ncong S(Y)$.

$S(X) = k[x, y]$.
$k[Y] = k[x, y, z] / (xy - z^2)$.
Now look at $\mathfrak{m}/\mathfrak{m}^2$ for $\mathfrak{m} = (x, y, z)$.
It is a 3-dimensional vector space, but there are no such in $S(X)$.

1.3.18 I.3.10 x

3.10. *Subvarieties.* A subset of a topological space is *locally closed* if it is an open subset of its closure, or, equivalently, if it is the intersection of an open set with a closed set.

If X is a quasi-affine or quasi-projective variety and Y is an irreducible locally closed subset, then Y is also a quasi-affine (respectively, quasi-projective) variety, by virtue of being a locally closed subset of the same affine or projective space. We call this the *induced structure* on Y, and we call Y a *subvariety* of X.

Now let $\varphi: X \to Y$ be a morphism, let $X' \subseteq X$ and $Y' \subseteq Y$ be irreducible locally closed subsets such that $\varphi(X') \subseteq Y'$. Show that $\varphi|_{X'}: X' \to Y'$ is a morphism.

Note that locally regular implies regular.

1.3.19 I.3.11 x

3.11. Let X be any variety and let $P \in X$. Show there is a 1-1 correspondence between the prime ideals of the local ring $\mathcal{O}_P$ and the closed subvarieties of X containing P.

This follows from properties of localization of the coordinate ring.

1.3.20 I.3.12 x g

> **3.12.** If P is a point on a variety X, then $\dim \mathcal{O}_P = \dim X$. [*Hint*: Reduce to the affine case and use (3.2c).]

$dim\ X = dim\ A = dim\ A/p + ht\ p = 0 + ht\ p$ (since A/p is a field) $= dim\ \mathcal{O}_p$.

1.3.21 I.3.13 x g

> **3.13.** *The Local Ring of a Subvariety.* Let $Y \subseteq X$ be a subvariety. Let $\mathcal{O}_{Y,X}$ be the set of equivalence classes $\langle U,f \rangle$ where $U \subseteq X$ is open, $U \cap Y \neq \varnothing$, and f is a regular function on U. We say $\langle U,f \rangle$ is equivalent to $\langle V,g \rangle$, if $f = g$ on $U \cap V$. Show that $\mathcal{O}_{Y,X}$ is a local ring, with residue field $K(Y)$ and dimension $= \dim X - \dim Y$. It is the *local ring* of Y on X. Note if $Y = P$ is a point we get $\mathcal{O}_P$, and if $Y = X$ we get $K(X)$. Note also that if Y is not a point, then $K(Y)$ is not algebraically closed, so in this way we get local rings whose residue fields are not algebraically closed.

$\mathcal{O}_{Y,X}$ is clearly a ring.

The set of functions vanishing on Y is the maximal ideal.

Quotienting gives the residue field, which is the invertible functions on Y. Since this is a field, we have confirmed the idea was maximal.

Now $dim\ X = dim\ k\left[X\right]/I\left(Y\right) + ht\ I\left(Y\right) = dim\ Y + ht\ I\left(Y\right) = dim\ Y + dim\ \mathcal{O}_{Y,X}$.

1.3.22 I.3.14 x g (and below) projection from point

> **3.14.** *Projection from a Point.* Let $\mathbf{P}^n$ be a hyperplane in $\mathbf{P}^{n+1}$ and let $P \in \mathbf{P}^{n+1} - \mathbf{P}^n$. Define a mapping $\varphi: \mathbf{P}^{n+1} - \{P\} \to \mathbf{P}^n$ by $\varphi(Q) =$ the intersection of the unique line containing P and Q with $\mathbf{P}^n$
> (a) Show that φ is a morphism.

Let $P = (1, 0, ..., 0)$, $\mathbb{P}^n := (x_0 \neq 0) \subset \mathbb{P}^{n+1}$.

The line through P and $x = (x_0 : \cdots : x_n)$ meets $\mathbb{P}^n$ at $(0 : x_1 : \cdots : x_n)$ which is a morphism in a neighborhood of $\mathbb{P}^n$.

1.3.23 b. x g

> (b) Let $Y \subseteq \mathbf{P}^3$ be the twisted cubic curve which is the image of the 3-uple embedding of $\mathbf{P}^1$ (Ex. 2.12). If t,u are the homogeneous coordinates on $\mathbf{P}^1$, we say that Y is the curve given *parametrically* by $(x,y,z,w) = (t^3,t^2u,tu^2,u^3)$. Let $P = (0,0,1,0)$, and let $\mathbf{P}^2$ be the hyperplane $z = 0$. Show that the projection of Y from P is a cuspidal cubic curve in the plane, and find its equation.

We have $\pi : (t^3, t^2u, tu^2, u^3) \mapsto (t^3, t^2u, u^3)$.

We have $x = t^3$, $y = t^2u$, $z = u^3$.

Recall that the cuspidal cubic is given by $x^2z - y^3$.

Note that plugging in gives: $t^6u^3 - t^6u^3 = 0$ so this is the cuspidal cupic.

1.3.24 I.3.15 x

3.15. *Products of Affine Varieties.* Let $X \subseteq \mathbf{A}^n$ and $Y \subseteq \mathbf{A}^m$ be affine varieties.

(a) Show that $X \times Y \subseteq \mathbf{A}^{n+m}$ with its induced topology is irreducible. [*Hint*: Suppose that $X \times Y$ is a union of two closed subsets $Z_1 \cup Z_2$. Let $X_i = \{x \in X | x \times Y \subseteq Z_i\}$, $i = 1,2$. Show that $X = X_1 \cup X_2$ and X_1, X_2 are closed. Then $X = X_1$ or X_2 so $X \times Y = Z_1$ or Z_2.] The affine variety $X \times Y$ is called the *product* of X and Y. Note that its topology is in general not equal to the product topology (Ex. 1.4).

See Gathman's notes.

1.3.25 b. x

(b) Show that $A(X \times Y) \cong A(X) \otimes_k A(Y)$.

The universal property of fiber product agrees with the universal property of the tensor product for finitely generated algebras.

1.3.26 c. x

(c) Show that $X \times Y$ is a product in the category of varieties, i.e., show (i) the projections $X \times Y \to X$ and $X \times Y \to Y$ are morphisms, and (ii) given a variety Z, and the morphisms $Z \to X$, $Z \to Y$, there is a unique morphism $Z \to X \times Y$ making a commutative diagram

$$\begin{tikzcd} Z \arrow[rr, dashed] \arrow[dr] \arrow[drrr] & & X \times Y \arrow[dl] \arrow[dr] & \\ & X & & Y. \end{tikzcd}$$

Given a a variety Z with morphisms ϕ_x, ϕ_y to X and Y we can take $(\phi_x, \phi_y) : Z \to X \times Y$.

1.3.27 d. x g

(d) Show that $\dim X \times Y = \dim X + \dim Y$.

Suppose that $f(x_1, ..., x_k, y_{k+1}, ..., y_n) = 0$ gives a relation on the combined generators x_i, y_i of X, Y.

On X, $f(-, y_{k+1}, ..., y_n) = 0$ gives a relation on the algebraically independent x generators and a similar thing happens on Y.

Hence f is a trivial relation and the combined generators of X, Y are basis for $X \times Y$.

1.3.28 I.3.16 x g

3.16. *Products of Quasi-Projective Varieties.* Use the Segre embedding (Ex. 2.14) to identify $\mathbf{P}^n \times \mathbf{P}^m$ with its image and hence give it a structure of projective variety. Now for any two quasi-projective varieties $X \subseteq \mathbf{P}^n$ and $Y \subseteq \mathbf{P}^m$, consider $X \times Y \subseteq \mathbf{P}^n \times \mathbf{P}^m$.

(a) Show that $X \times Y$ is a quasi-projective variety.

$X \times Y = (X \times \mathbb{P}^m) \cap (\mathbb{P}^n \times Y) = \pi_n^{-1}(X) \cap \pi_m^{-1}(Y)$ where $\pi_{m,n} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^m, \mathbb{P}^n$ are the projections.

1.3.29 b. x

(b) If X, Y are both projective, show that $X \times Y$ is projective.

similar.

---

*(c) Show that $X \times Y$ is a product in the category of varieties.

1.3.30 I.3.17 x. g

3.17. *Normal Varieties.* A variety Y is *normal at a point* $P \in Y$ if $\mathcal{O}_P$ is an integrally closed ring. Y is *normal* if it is normal at every point.

(a) Show that every conic in $\mathbf{P}^2$ is normal.

By a previous excercise conics in $\mathbb{P}^2$ are isomorphic to $\mathbb{P}^1$ which is smooth.

1.3.31 b. x g

(b) Show that the quadric surfaces Q_1, Q_2 in $\mathbf{P}^3$ given by equations $Q_1 : xy = zw$; $Q_2 : xy = z^2$ are normal (cf. (II. Ex. 6.4) for the latter.)

For Q_1 we can use the jacobian criterion since nonsingular implies normal.

For Q_2 we just need to see that the local ring at the cone point is normal. This can be given by $k[x,y,z]/(z^2 - xy)$. Note that xy is square free, so by II.6.4, we are done.

1.3.32 c. x g

(c) Show that the cuspidal cubic $y^2 = x^3$ in $\mathbf{A}^2$ is not normal.

For curves, normal and nonsingular are equivalent. (Use DIRP)

1.3.33 d. x

(d) If Y is affine, then Y is normal $\Leftrightarrow A(Y)$ is integrally closed.

If A is an integral domain integrally closed then so is each localization. The converse also holds (Atiyah-Macdonald 5.12) Thus X is normal.

Normal is when each local ring is an integrally closed integral domain.

1.3.34 e. x.

(e) Let Y be an affine variety. Show that there is a normal affine variety $\tilde{Y}$, and a morphism $\pi: \tilde{Y} \to Y$, with the property that whenever Z is a normal variety, and $\varphi: Z \to Y$ is a *dominant* morphism (i.e., $\varphi(Z)$ is dense in Y), then there is a unique morphism $\theta: Z \to \tilde{Y}$ such that $\varphi = \pi \circ \theta$. $\tilde{Y}$ is called the *normalization of* Y. You will need (3.9A) above.

Let $\tilde{Y}$ be the affine variety with $k\left[\tilde{Y}\right]$ the normalization of $k[Y]$.

In other words, we take all monic polynomials in $k[Y]$ and all their solutions in $k(Y)$. Corresponding to the inclusion $\theta' : k[Y] \to k\left[\tilde{Y}\right]$ we have $\theta : \tilde{Y} \to Y$.

Note that $k\left[\tilde{Y}\right]$ is finite by 3.9A and unique since the integral closure is unique.

1.3.35 I.3.18.a x

3.18. *Projectively Normal Varieties.* A projective variety $Y \subseteq \mathbf{P}^n$ is *projectively normal* (with respect to the given embedding) if its homogeneous coordinate ring $S(Y)$ is integrally closed.
(a) If Y is projectively normal, then Y is normal.

Assume Y projectively normal.

Then $k[Y]$ is integrally closed.

Since the localization of an integrally closed domain is integrally closed (we used this above), then each local ring is integrally closed.

$\implies Y$ is normal.

1.3.36 b. x g

(b) There are normal varieties in projective space which are not projectively normal. For example, let Y be the twisted quartic curve in $\mathbf{P}^3$ given parametrically by $(x,y,z,w) = (t^4, t^3u, tu^3, u^4)$. Then Y is normal but not projectively normal. See (III, Ex. 5.6) for more examples.

Another criterion for projectively normal is that $H^1(\mathbb{P}^3, I_Y) = 0$.

But

$h^0(\mathbb{P}^3, \mathcal{O}(1)) = \begin{pmatrix} 3+1 \\ 3 \end{pmatrix} = 4 <$

$5 = \begin{pmatrix} 4+1 \\ 1 \end{pmatrix} = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4)) = h^0(Y, \mathcal{O}_Y(1))$.

Also note we have normal since $\mathbb{P}^1$ is nonsingular and twisted quartic is image of $\mathbb{P}^1$.

1.3.37 c. x

(c) Show that the twisted quartic curve Y above is isomorphic to $\mathbf{P}^1$, which is projectively normal. Thus projective normality depends on the embedding.

It is just the image of $v_4 : \mathbb{P}^1 \to \mathbb{P}^3$.

Note that $h^1(\mathbb{P}^1, \mathcal{O}(1)) = 0$.

1.3.38 I.3.19 x

3.19. *Automorphisms of* $\mathbf{A}^n$. Let $\varphi: \mathbf{A}^n \to \mathbf{A}^n$ be a morphism of $\mathbf{A}^n$ to $\mathbf{A}^n$ given by n polynomials $f_1, \ldots, f_n$ of n variables $x_1, \ldots, x_n$. Let $J = \det|\partial f_i / \partial x_j|$ be the *Jacobian* polynomial of φ.
(a) If φ is an isomorphism (in which case we call φ an *automorphism* of $\mathbf{A}^n$) show that J is a nonzero constant polynomial.

If any $f_i \in k$ then φ would not be surjective.
Now computing the derivative of a linear, nonconstant polynomial gives a jacobian in $k^\times$.

**(b) The converse of (a) is an unsolved problem, even for $n = 2$. See, for example, Vitushkin [1].

SKIP

1.3.39 I.3.20 x g and below

3.20. Let Y be a variety of dimension $\geqslant 2$, and let $P \in Y$ be a normal point. Let f be a regular function on $Y - P$.
(a) Show that f extends to a regular function on Y.

Since projective space is proper this works.

1.3.40 b. x g

(b) Show this would be false for $\dim Y = 1$.
See (III, Ex. 3.5) for generalization.

Consider a single variable complex function with a pole.

1.3.41 I.3.21a. x

3.21. *Group Varieties.* A group variety consists of a variety Y together with a morphism $\mu: Y \times Y \to Y$, such that the set of points of Y with the operation given by μ is a group, and such that the inverse map $y \to y^{-1}$ is also a morphism of $Y \to Y$.
(a) The *additive group* $\mathbf{G}_a$ is given by the variety $\mathbf{A}^1$ and the morphism $\mu: \mathbf{A}^2 \to \mathbf{A}^1$ defined by $\mu(a,b) = a + b$. Show it is a group variety.

Note that $\mathbb{A}^1$ is a group under addition, and the inverse map is given by $y \mapsto -y$.

1.3.42 b. x

(b) The *multiplicative group* $\mathbf{G}_m$ is given by the variety $\mathbf{A}^1 - \{0\}$ and the morphism $\mu(a,b) = ab$. Show it is a group variety.

Clear.

1.3.43 c. x

(c) If G is a group variety, and X is any variety, show that the set $\mathrm{Hom}(X, G)$ has a natural group structure.

Since we can add morphisms

1.3.44 d. x

(d) For any variety X, show that $\mathrm{Hom}(X, \mathbf{G}_a)$ is isomorphic to $\mathcal{O}(X)$ as a group under addition.

If $f \in \mathcal{O}(X)$ is regular, then $f : X \to \mathbb{A}^1$ and adding functions is compatible with this.

1.3.45 e. x

(e) For any variety X, show that $\mathrm{Hom}(X, \mathbf{G}_m)$ is isomorphic to the group of units in $\mathcal{O}(X)$, under multiplication.

If $f \in \mathcal{O}(X)^{\times}$, then $f : X \to \mathbb{A}^1 \backslash 0$ and this correspondence preserves multiplication of $f \cdot g$ for $g \in \mathcal{O}(X)^{\times}$.

I.4 x

1.3.46 I.4.1 x g

4.1. If f and g are regular functions on open subsets U and V of a variety X, and if $f = g$ on $U \cap V$, show that the function which is f on U and g on V is a regular function on $U \cup V$. Conclude that if f is a *rational* function on X, then there is a largest open subset U of X on which f is represented by a regular function. We say that f *is defined* at the points of U.

Define a new function which is f on U and g on V.
Continue in this manner until you have defined on all such sets.

1.3.47 I.4.2 x

4.2. Same problem for rational maps. If φ is a rational map of X to Y, show there is a largest open set on which φ is represented by a morphism. We say the rational map *is defined* at the points of that open set.

see 4.1

1.3.48 I.4.3 x g and below

4.3. (a) Let f be the rational function on $\mathbf{P}^2$ given by $f = x_1/x_0$. Find the set of points where f is defined and describe the corresponding regular function.

$f = \frac{x_1}{x_0}$ is defined where $x_0 \neq 0$.
This set isomorphic to $\mathbb{A}^2$, f is projection to first coordinate.

1.3.49 b x. g

(b) Now think of this function as a rational map from $\mathbf{P}^2$ to $\mathbf{A}^1$. Embed $\mathbf{A}^1$ in $\mathbf{P}^1$, and let $\varphi: \mathbf{P}^2 \to \mathbf{P}^1$ be the resulting rational map. Find the set of points where φ is defined, and describe the corresponding morphism.

We can take the projection $(x_0, x_1, x_2) \mapsto (x_0, x_1)$.
Not defined at $(0, 0, 1)$ since $[0, 0] \notin \mathbb{P}^1$.

1.3.50 I.4.4 x g all parts

4.4. A variety Y is *rational* if it is birationally equivalent to $\mathbf{P}^n$ for some n (or, equivalently by (4.5), if $K(Y)$ is a pure transcendental extension of k).
(a) Any conic in $\mathbf{P}^2$ is a rational curve.

By I.3.1.b, conics are $\mathbb{P}^1$.

1.3.51 b. x g

(b) The cuspidal cubic $y^2 = x^3$ is a rational curve.

Define $t \mapsto (t^2, t^3)$ and an inverse $(x, y) \to \frac{x}{y}$ between the cuspidal cupic and $\mathbb{A}^1$
Note $\mathbb{A}^1$ is birational to $\mathbb{P}^1$.

1.3.52 c. x g

(c) Let Y be the nodal cubic curve $y^2z = x^2(x + z)$ in $\mathbf{P}^2$. Show that the projection φ from the point $P = (0,0,1)$ to the line $z = 0$ (Ex. 3.14) induces a birational map from Y to $\mathbf{P}^1$. Thus Y is a rational curve.

The projection gives $(x : y : z) \mapsto (x : y)$, $\mathbb{P}^2 \to Y$.
The inverse is given by $(x : y) \mapsto ((y^2 - x^2)\, x : (y^2 - x^2)\, y : x^3)$, $Y \to \mathbb{P}^2$.
That's $(y^2x - x^3 : y^3 - x^2y : x^3)$
(since $y^2z - zx^2 = z\,(y^2 - x^2) = x^3$ on the curve)

1.3.53 I.4.5 x g

4.5. Show that the quadric surface $Q: xy = zw$ in $\mathbf{P}^3$ is birational to $\mathbf{P}^2$, but not isomorphic to $\mathbf{P}^2$ (cf. Ex. 2.15).

see V.4.1 to see birational, and compute the canonicals to see they are not isomorphic.

1.3.54 I.4.6 x g and below

4.6. *Plane Cremona Transformations.* A birational map of $\mathbf{P}^2$ into itself is called a *plane Cremona transformation.* We give an example, called a *quadratic transformation.* It is the rational map $\varphi:\mathbf{P}^2 \to \mathbf{P}^2$ given by $(a_0,a_1,a_2) \to (a_1a_2,a_0a_2,a_0a_1)$ when no two of a_0,a_1,a_2 are 0.

(a) Show that φ is birational, and is its own inverse.

Let U,V be the sets $\{(x:y:z)\,|xyz\neq 0\}$.
φ maps U to V: $(x,y,z)\mapsto(yz,xz,xy)$
$\varphi^2:(x,y,z)\mapsto(xzxy,yzxy,yzxz)=(x,y,z)$ on U,V.

1.3.55 b. x g

(b) Find open sets $U,V\subseteq\mathbf{P}^2$ such that $\varphi:U\to V$ is an isomorphism.

Let $U=V=\{[x:y:z]\,|xyz\neq 0\}$.
Then $\varphi:U\to V$, and φ^2 is the identity on U.

1.3.56 c. x

(c) Find the open sets where φ and φ^{-1} are defined, and describe the corresponding morphisms. See also (V, 4.2.3).

From V.4.2.3 we see they are defined on the complment of $(1,0,0),(0,1,0),(0,0,1)$.

1.3.57 I.4.7 x

4.7. Let X and Y be two varieties. Suppose there are points $P\in X$ and $Q\in Y$ such that the local rings $\mathcal{O}_{P,X}$ and $\mathcal{O}_{Q,Y}$ are isomorphic as k-algebras. Then show that there are open sets $P\in U\subseteq X$ and $Q\in V\subseteq Y$ and an isomorphism of U to V which sends P to Q.

The isomorphism of local rings induces an isomorphism of $k(X)\approx k(Y)$ via I.3.2 we have a birational map on neighborhoods of P,Q .

1.3.58 I.4.8 x g

4.8. (a) Show that any variety of positive dimension over k has the same cardinality as k. [*Hints*: Do $\mathbf{A}^n$ and $\mathbf{P}^n$ first. Then for any X, use induction on the dimension n. Use (4.9) to make X birational to a hypersurface $H\subseteq\mathbf{P}^{n+1}$. Use (Ex. 3.7) to show that the projection of H to $\mathbf{P}^n$ from a point not on H is finite-to-one and surjective.]

Base Case:
Any curve X is birational to a plane curve so $|X|\leq|\mathbb{P}^2|=|k|$ since $\mathbb{P}^2$ is a 2-manifold.
On the other hand, picking a point not on X and projecting to $\mathbb{P}^1$ gives a surjective morphism from X to $\mathbb{P}^1$ so $|X|\geq|\mathbb{P}^1|=k$.

Thus $|X| = k$.

Inductive Step.
Embed X as a hyperplane in $\mathbb{P}^{n+1}$ by the primitive element theorem.

1.3.59 b. x g

~~(b) Deduce that any two curves over k are ho~~meomorphic (cf. Ex. 3.1).

cf example 3.7, since any two curves have the same cardinality they are homeomorphic in the finite complement topology.

1.3.60 I.4.9 x g

4.9. Let X be a projective variety of dimension r in $\mathbf{P}^n$, with $n \geqslant r + 2$. Show that for suitable choice of $P \notin X$, and a linear $\mathbf{P}^{n-1} \subseteq \mathbf{P}^n$, the projection from P to $\mathbf{P}^{n-1}$ (Ex. 3.14) induces a *birational* morphism of X onto its image $X' \subseteq \mathbf{P}^{n-1}$. You will need to use (4.6A), (4.7A), and (4.8A). This shows in particular that the birational map of (4.9) can be obtained by a finite number of such projections.

We can find a linear space generated by $x_0, ..., x_r$ disjoint from X defines a surjective projection to $\mathbb{P}^r$, and thus an inclusion of function fields $k(X) \hookrightarrow k(\mathbb{P}^r)$.

Then $k(X)$ is a finite algebraic extension of $k(\mathbb{P}^r)$ generated by $x_{r+1}, ..., x_n$.

By theorem of primitive element, $k(X)$ is generated by $\sum a_i x_i$.

We have $X \hookrightarrow \mathbb{P}^n \backslash M \twoheadrightarrow Z\left(\sum a_i x_i\right)$ thus $k(\mathbb{P}^r) \hookrightarrow k(Z(F)) \twoheadrightarrow k(X)$ and there is an open set $U \subset \mathbb{P}^n \backslash M$ where all fibers have cardinality 1 on which the projection is birational.

1.3.61 I.4.10 x g

4.10. Let Y be the cuspidal cubic curve $y^2 = x^3$ in $\mathbf{A}^2$. Blow up the point $O = (0,0)$, let E be the exceptional curve, and let $\tilde{Y}$ be the strict transform of Y. Show that E meets $\tilde{Y}$ in one point, and that $\tilde{Y} \cong \mathbf{A}^1$. In this case the morphism $\varphi: \tilde{Y} \to Y$ is bijective and bicontinuous, but it is not an isomorphism.

Your blown up thing lives on the blow-up surface which is the set of points and lines through origin $(p, \overline{p})$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ (in other words, if we have some lines intersecting, so a singularity, then now we have distinct tangent directions in the blow up).

Using determinant has to be rank one, and the fact that $(x_1, ..., x_n)$ is on the line $(y_1 : ... : y_n)$ only if $(x_1, ..., x_n)$ is a multiple of $(y_1, ..., y_n)$ we see that the blow up surface is described by the conditions $rank\begin{pmatrix} x_1 & \dots & x_n \\ y_1 & \dots & y_n \end{pmatrix} \leq 2$, i.e. all minors vanish, i.e. $x_i y_j - x_j y_i = 0$ for all i, j. Looking at the equations for each patch of $\mathbb{P}^{n-1}$ we can actually see what the blow-up surface looks like. (see figure 3)

It is thus clear the exceptional divisor meets $\tilde{Y}$ at points corresponding to singularities of Y. There is only one such on the cusp.

Now the blow up is the blow up surface and the projection $\pi : \mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$. The projection is a birational map onto $\mathbb{A}^n - 0$. The preimage of 0 is $\pi^{-1}(0) = 0 \times \mathbb{P}^{n-1}$ is the exceptional curve. (To do the same things for your variety, just restrict to your variety).

Since the cusp is not isomorphic to $\mathbb{A}^1$, $\tilde{Y} \ncong \mathbb{A}^1$.

1.4 I.5 x

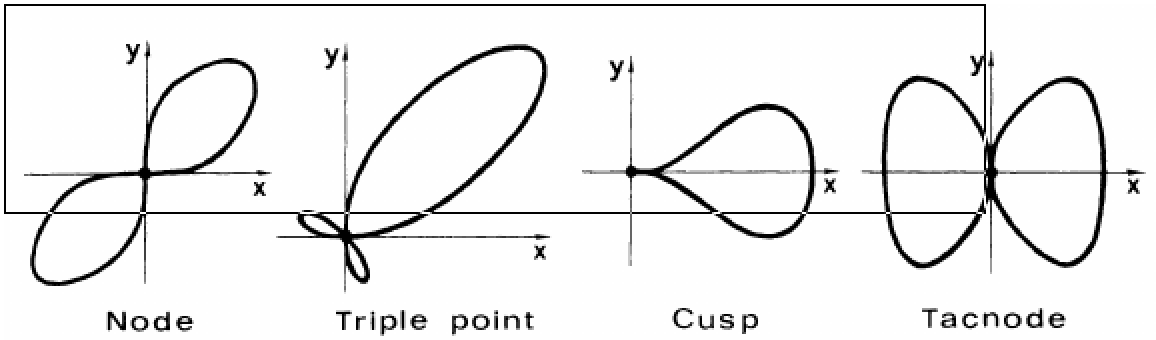

Figure 4. Singularities of plane curves.

1.4.1 I.5.1.a x

5.1. Locate the singular points and sketch the following curves in $\mathbf{A}^2$ (assume char $k \neq 2$). Which is which in Figure 4?
(a) $x^2 = x^4 + y^4$;

The jacobian is 0 at the origin. Graphing gives tacnode.

1.4.2 I.5.1.b. x g what kind of singularities

(b) $xy = x^6 + y^6$;

The jacobian is 0 only at origin.
The lowest multiplicity term is like $xy = 0$ so it's normal crossings and thus the node.

1.4.3 c. x

(c) $x^3 = y^2 + x^4 + y^4$;

Cusp since the lowest terms are $y^2 - x^3$.

1.4.4 d. x g what kind of singularity

(d) $x^2y + xy^2 = x^4 + y^4$.

By plugging in $(t, mt) \mapsto (x, y)$, $t^2m \cdot t + tm^2t^2 = t^4 + m^4t^4$ we can factor out a line with multiplicity three. So it must be a triple point.

1.4.5 I.5.2a. x pinch point x

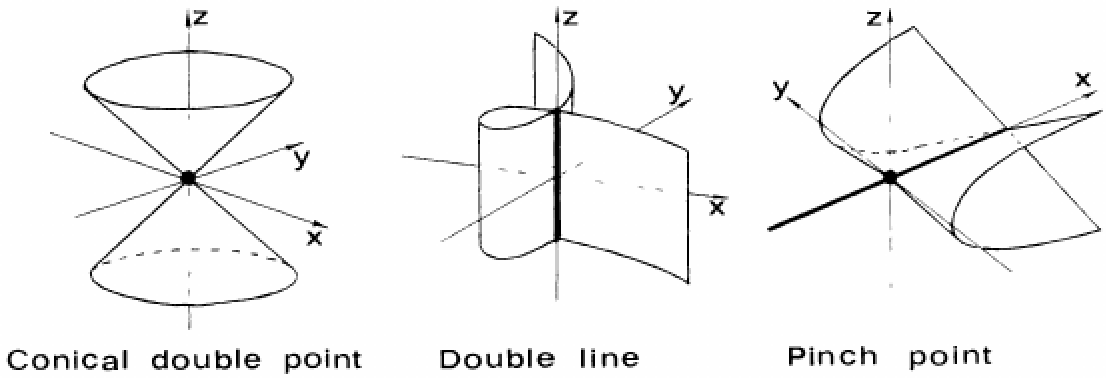

Figure 5. Surface singularities.

5.2. Locate the singular points and describe the singularities of the following surfaces in $\mathbf{A}^3$ (assume char $k \neq 2$). Which is which in Figure 5?
(a) $xy^2 = z^2$;

Checking the jacobian gives singular points along x-axis so we have the pinch point.

1.4.6 b. x conical double point

(b) $x^2 + y^2 = z^2$;

Check jacobian gives singularity at 0.

1.4.7 c. x

(c) $xy + x^3 + y^3 = 0$.

Check jacobian gives singularity along z .
This is double line.

1.4.8 I.5.3 x g

5.3. *Multiplicities.* Let $Y \subseteq \mathbf{A}^2$ be a curve defined by the equation $f(x,y) = 0$. Le $P = (a,b)$ be a point of $\mathbf{A}^2$. Make a linear change of coordinates so that P be comes the point $(0,0)$. Then write f as a sum $f = f_0 + f_1 + \ldots + f_d$, wher f_i is a homogeneous polynomial of degree i in x and y. Then we define the *multi plicity* of P on Y, denoted $\mu_P(Y)$, to be the least r such that $f_r \neq 0$. (Note tha $P \in Y \Leftrightarrow \mu_P(Y) > 0$.) The linear factors of f_r are called the *tangent direction* at P.
(a) Show that $\mu_P(Y) = 1 \Leftrightarrow P$ is a nonsingular point of Y.

A nonsingular point is when at least one of the partials of f is nonzero, so f must have a degree 1 term in x or 1 , so the multilpicity is 1.

1.4.9 b. x g

(b) Find the multiplicity of each of the singular points in (Ex. 5.1) above.

The multiplicity at the origin is the smallest degree term that appears.
To see why this may be so, consider trying to factor out a linear term.

1.4.10 I.5.4 x g

5.4. *Intersection Multiplicity.* If $Y, Z \subseteq \mathbf{A}^2$ are two distinct curves, given by equations $f = 0$, $g = 0$, and if $P \in Y \cap Z$, we define the *intersection multiplicity* $(Y \cdot Z)_P$ of Y and Z at P to be the length of the $\mathcal{O}_P$-module $\mathcal{O}_P/(f,g)$.
(a) Show that $(Y \cdot Z)_P$ is finite, and $(Y \cdot Z)_P \geqslant \mu_P(Y) \cdot \mu_P(Z)$.

Let U an affine neighborhood where P is the only intersection of f, g.
By nullstellensatz, $I_P^r \subset (f,g)$ for some $r > 0$.
As $\mathcal{O}_P = k\left[U\right]_{\mathfrak{a}_P} \implies \mathfrak{m}_P^r \subset (f,g)$.
Note that $\mathcal{O}_P/\mathfrak{m}_P^r$ has finite length since it has a filtration by powers of $\mathfrak{m}$ which are $< r$ and $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is finite dimensional.
Thus $(Y.Z)_P$ is finite.

Note that multiplicity at a point can be described by the lowest term in an equation for the curve.
Comparing with bezout's theorem gives $(Z.Y)_P \geq \mu_P(Y) . \mu_P(Z)$.

1.4.11 b. x g

(b) If $P \in Y$, show that for almost all lines L through P (i.e., all but a finite number), $(L \cdot Y)_P = \mu_P(Y)$.

Bezout since the general element of a linear system of lines through P is going to meet Y transversely.

1.4.12 c. x

(c) If Y is a curve of degree d in $\mathbf{P}^2$, and if L is a line in $\mathbf{P}^2$, $L \neq Y$, show that $(L \cdot Y) = d$. Here we define $(L \cdot Y) = \sum (L \cdot Y)_P$ taken over all points $P \in L \cap Y$, where $(L \cdot Y)_P$ is defined using a suitable affine cover of $\mathbf{P}^2$.

see b.

1.4.13 I.5.5 x

5.5. For every degree $d > 0$, and every $p = 0$ or a prime number, give the equation of a nonsingular curve of degree d in $\mathbf{P}^2$ over a field k of characteristic p.

If the characteristic doesn't divide d , then $x^d + y^d + z^d$.
Else, $xy^{d-1} + yz^{d-1} + zx^{d-1} = 0$.

1.4.14 I.5.6 x g

5.6. *Blowing Up Curve Singularities.*

(a) Let Y be the cusp or node of (Ex. 5.1). Show that the curve $\tilde{Y}$ obtained by blowing up Y at $O = (0,0)$ is nonsingular (cf. (4.9.1) and (Ex. 4.10)).

The cusp was $y^2 - x^3 + x^4 + y^4$.
The blop is defined by $ys - xt$ so on $t = 1$, $ys = x$ and $y^2\left(1 - yt^3 + y^2t^4 + y^2\right) = 0$.
Then $1 - yt^3 + y^3t^4 + y^2$. The jacobian criterion shows it's nonsingular.
On $s = 1$, $y = xt$ and $x^2\left(t^2 - x + x^2 - t^4x^2\right) = 0$. Again using the jacobian criterion gives nonsingularity.

1.4.15 b. x g

(b) We define a *node* (also called *ordinary double point*) to be a double point (i.e., a point of multiplicity 2) of a plane curve with distinct tangent directions (Ex. 5.3). If P is a node on a plane curve Y, show that $\varphi^{-1}(P)$ consists of two distinct nonsingular points on the blown-up curve $\tilde{Y}$. We say that "blowing up P resolves the singularity at P".

Your blown up thing lives on the blow-up surface which is the set of points and lines through origin $(p, \overline{p})$ in $\mathbb{A}^n \times \mathbb{P}^{n-1}$ (in other words, if we have some lines intersecting, so a singularity, then now we have distinct tangent directions in the blow up)

1.4.16 c. x g

(c) Let $P \in Y$ be the tacnode of (Ex. 5.1). If $\varphi: \tilde{Y} \to Y$ is the blowing-up at P, show that $\varphi^{-1}(P)$ is a node. Using (b) we see that the tacnode can be resolved by two successive blowings-up.

Recall the tacnode is defined by $x^2 = x^4 + y^4$.
The blowup surface is defined by $xs - yt$ in $\mathbb{A}^2 \times \mathbb{P}^1$.
On the patch $s = 1$, $x = yt$ and $x^2 = x^4 + y^4$ so $y^2\left(t^2 - y^2t^4 - y^2\right) = 0$.
The lowest degree terms, $t^2 - y^2$ of the strict transform factor linearly, so we get a node.
On $t = 1$, we have $y = xs$ and $x^2 = x^3 + y^4$ so $x^2\left(1 - x - x^2s^4\right)$. The strict is nonsingular.

1.4.17 d. x g

(d) Let Y be the plane curve $y^3 = x^5$, which has a "higher order cusp" at O. Show that O is a triple point; that blowing up O gives rise to a double point (what kind?) and that one further blowing up resolves the singularity.

Note: We will see later (V, 3.8) that any singular point of a plane curve can be resolved by a finite sequence of successive blowings-up.

We can use $xu = yt$ for new coordinates in $\mathbb{A}^2 \times \mathbb{P}^1$.
So $x^3\left(x^2 - u^3\right) = 0$ on $t = 1$, which gives a cusp. One more blowing up gives a smooth one.

1.4.18 I.5.7 x

5.7. Let $Y \subseteq \mathbf{P}^2$ be a nonsingular plane curve of degree >1, defined by the equation $f(x,y,z) = 0$. Let $X \subseteq \mathbf{A}^3$ be the affine variety defined by f (this is the cone over Y; see (Ex. 2.10)). Let P be the point $(0,0,0)$, which is the *vertex* of the cone. Let $\varphi : \tilde{X} \to X$ be the blowing-up of X at P.
(a) Show that X has just one singular point, namely P.

We can just use the jacobian criterion.

1.4.19 b. (important) x g

(b) Show that $\tilde{X}$ is nonsingular (cover it with open affines).

The blow-up hypersurface in $\mathbb{A}^3 \times \mathbb{P}^2$ is defined by $x_1u_2 = x_2u_1$, $x_1u_3 = x_3u_1$, and $x_2u_3 = x_3u_2$.
On $u_2 = 1$, the equations become $x_1 = x_2u_1$, $x_1u_3 = x_3u_1$, and $x_3 = x_2u_3$.
If X has multiplicity d at 0, then $\tilde{X}$ is defined by $f(x_2u_1, x_2, x_2u_3) = 0 = x_2^d f(u_1, 1, u_3)$.
The Jaacobian is just $\left(\frac{\partial f}{\partial u_1} \quad 0 \quad \frac{\partial f}{\partial u_3} \right)$ which has rank 1 as f is nonsingular.
On $u_1 = 1$...
On $u_3 = 0$...

1.4.20 c. x

(c) Show that $\varphi^{-1}(P)$ is isomorphic to Y.

In the above problem we defined $\varphi^{-1}(P)$ by $f(1, u_2, u_3)$, $f(u_1, 1, u_3)$, and $f(u_1, u_2, 1)$.

1.4.21 I.5.8 x

5.8. Let $Y \subseteq \mathbf{P}^n$ be a projective variety of dimension r. Let $f_1, \ldots, f_t \in S = k[x_0, \ldots, x_n]$ be homogeneous polynomials which generate the ideal of Y. Let $P \in Y$ be a point, with homogeneous coordinates $P = (a_0, \ldots, a_n)$. Show that P is nonsingular on Y if and only if the rank of the matrix $\|(\partial f_i/\partial x_j)(a_0, \ldots, a_n)\|$ is $n - r$. [*Hint*: (a) Show that this rank is independent of the homogeneous coordinates chosen for P; (b) pass to an open affine $U_i \subseteq \mathbf{P}^n$ containing P and use the affine Jacobian matrix; (c) you will need Euler's lemma, which says that if f is a homogeneous polynomial of degree d, then $\sum x_i(\partial f/\partial x_i) = d \cdot f$.]

The hint is the answer.

1.4.22 I.5.9 x

5.9. Let $f \in k[x,y,z]$ be a homogeneous polynomial, let $Y = Z(f) \subseteq \mathbf{P}^2$ be the algebraic set defined by f, and suppose that for every $P \in Y$, at least one of $(\partial f/\partial x)(P)$, $(\partial f/\partial y)(P)$, $(\partial f/\partial z)(P)$ is nonzero. Show that f is irreducible (and hence that Y is a nonsingular variety). [*Hint*: Use (Ex. 3.7).]

If $f(P) = g(P)\, h(P) = 0$ then taking $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ and using the product rule shows all partials must vanish. (contradiction)

1.4.23 I.5.10 x g:a,b,c

5.10. For a point P on a variety X, let $\mathfrak{m}$ be the maximal ideal of the local ring $\mathcal{O}_P$. We define the *Zariski tangent space* $T_P(X)$ of X at P to be the dual k-vector space of $\mathfrak{m}/\mathfrak{m}^2$.

(a) For any point $P \in X$, $\dim T_P(X) \geqslant \dim X$, with equality if and only if P is nonsingular.

This is clear geometrically, since the number of independent directions you can go in on the manifold corresponds to the number of tangent directions. A singular point means there are too many tangent directions compared to the one point.

1.4.24 b. x

(b) For any morphism $\varphi: X \to Y$, there is a natural induced k-linear map $T_P(\varphi)$: $T_P(X) \to T_{\varphi(P)}(Y)$.

Since the tangent space at P is just the dual of $\mathfrak{m}_P/\mathfrak{m}_P^2$ we can just take the dual of the natural map $\mathfrak{m}_{\varphi(P)}/\mathfrak{m}_{\varphi(P)}^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2$.

1.4.25 c. x

(c) If φ is the vertical projection of the parabola $x = y^2$ onto the x-axis, show that the induced map $T_0(\varphi)$ of tangent spaces at the origin is the zero map.

Note that on the parabola, $x \in \mathfrak{m}_{(x)}^2$.

1.4.26 I.5.11 x

5.11. *The Elliptic Quartic Curve in* $\mathbf{P}^3$. Let Y be the algebraic set in $\mathbf{P}^3$ defined by the equations $x^2 - xz - yw = 0$ and $yz - xw - zw = 0$. Let P be the point $(x,y,z,w) = (0,0,0,1)$, and let φ denote the projection from P to the plane $w = 0$. Show that φ induces an isomorphism of $Y - P$ with the plane cubic curve $y^2z - x^3 + xz^2 = 0$ minus the point $(1,0,-1)$. Then show that Y is an irreducible nonsingular curve. It is called the *elliptic quartic curve* in $\mathbf{P}^3$. Since it is defined by two equations it is another example of a complete intersection (Ex. 2.17).

Define $\varphi : (x, y, z, w) \mapsto (x, y, z)$.
Let f:y^2*z-x^3+x*z^2;
Let I = I=ideal(x^2 - x*z - y*w)
J=ideal(y*z-x*w-z*w)
f%I=y^2*z-x*y*w-y*z*w;
f%J=-x^3+x*z^2+x*y*w+x*w^2+z*w^2;
Now f%I / J = y
Also f%J / I = -z-x
Hence we should have

$expand\,(y \cdot (y \cdot z - x \cdot w - z \cdot w) + (-z - x) \cdot (x^2 - x \cdot z - y \cdot w))$
$= x\,z^2 + y^2\,z - x^3$
Hence $\varphi\,(Y) \subset Z\,(y^2 z - x^3 + xz^2)$.
Solving for w in Y gives $w = \frac{x^2 - xz}{y}$, $w = \frac{yz}{x+z}$.
We define $\varphi^{-1} : (x, y, z) \mapsto \left(x, y, z, \frac{x^2-xz}{y}\right) = \left(x, y, z. \frac{yz}{x+z}\right)$.
Clearly this is not defined at $(1, 0, -1)$.

1.4.27 I.5.12 x

> **5.12.** *Quadric Hypersurfaces*. Assume char $k \neq 2$, and let f be a homogeneous polynomial of degree 2 in $x_0, \ldots, x_n$.
> (a) Show that after a suitable linear change of variables, f can be brought into the form $f = x_0^2 + \ldots + x_r^2$ for some $0 \leqslant r \leqslant n$.

Any conic is the determinant of a $n \times n$ matrix. Ex.

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} A & B & D \\ B & F & C \\ D & C & G \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Diagonalize this matrix by a lienar transformation.

1.4.28 b. x

> (b) Show that f is irreducible if and only if $r \geqslant 2$.

If $r = 1$ it is clear since sum of squares.

On the other hand, for larger r, any factors must be linear, but then multiplying the two factors together will create terms of higher than degree 2.

1.4.29 c. x

> (c) Assume $r \geqslant 2$, and let Q be the quadric hypersurface in $\mathbf{P}^n$ defined by f. Show that the singular locus $Z = \operatorname{Sing} Q$ of Q is a *linear* variety (Ex. 2.11) of dimension $n - r - 1$. In particular, Q is nonsingular if and only if $r = n$.

Singular locus is where partial derivatives are all 0.
Since it's a conic, partials are linear.
The number of partials taking into account r is $n - r - 1$.
This is because it's the dimension of $dim\ k\,[x_0, ..., x_n] \,/\, (x_0, ..., r_r) - 1$

1.4.30 d. x.

> (d) In case $r < n$, show that Q is a cone with axis Z over a nonsingular quadric hypersurface $Q' \subseteq \mathbf{P}^r$. (This notion of cone generalizes the one defined in (Ex. 2.10). If Y is a closed subset of $\mathbf{P}^r$, and if Z is a linear subspace of dimension $n - r - 1$ in $\mathbf{P}^n$, we embed $\mathbf{P}^r$ in $\mathbf{P}^n$ so that $\mathbf{P}^r \cap Z = \varnothing$, and define the *cone over Y with axis Z* to be the union of all lines joining a point of Y to a point of Z.)

As an example, in $\mathbb{P}^2$ this is obvious since the singular locus of the quadric is a point when taking affine coordinates. So we are looking at the set of lines between something on the plane and a single point. This gives the the cone $yx - z^2 = 0$.

Now let $Z = Sing\ Q$ a linear variety as in C (in $\mathbb{P}^2$ it's just that point).

If Q is a cone, then $Q = x_0^2 + ... + x_r^2$ defines a quadric hypersurface Q' in $\mathbb{P}^r$.

Then $Z = sing\ Q$ is $n - r - 1$ dimensional and linear by (c).

Embed $\mathbb{P}^r$ into $\mathbb{P}^n$ to not intersect $Z \subset \mathbb{P}^n$.

If $b \in Sing\ Q$, then the first r coordinates are 0 by examining the partials.

If $a \in Q$, then a satisfies $x_0^2 + ... + x_r^2 = 0$.

The line between a, b is given by $ta + sb$ with $s, t \in \mathbb{P}^1$.

Note that Q is made up of such lines since points in $\mathbb{P}^n$ lying on Q' must exactly satisfy first r coordinates fit $x_0^2 + ... + x_r^2 = 0$ and last $n - r - 1$ coordinates satisfy whatever.

1.4.31 I.5.13 x

5.13. It is a fact that any regular local ring is an integrally closed domain (Matsumura [2, Th. 36, p. 121]). Thus we see from (5.3) that any variety has a nonempty open subset of normal points (Ex. 3.17). In this exercise, show directly (without using (5.3)) that the set of nonnormal points of a variety is a proper closed subset (you will need the finiteness of integral closure: see (3.9A)).

Assume X affine.

Integral closure is f.g by f_i, i=1,..,n.

$\mathcal{O}_x$ is integrally closed iff image of f_i is in $\mathcal{O}_x$ for all i.

A finite intersection of nonempty opens is nonempty open, and rational functions are defined on such.

The normal locus is therefore nonempty open.

1.4.32 I.5.14 x g

5.14. *Analytically Isomorphic Singularities.*

(a) If $P \in Y$ and $Q \in Z$ are analytically isomorphic plane curve singularities, show that the multiplicities $\mu_P(Y)$ and $\mu_Q(Z)$ are the same (Ex. 5.3).

The isomorphism between local rings must map linear terms to linear terms + higher order terms.

1.4.33 b. x

(b) Generalize the example in the text (5.6.3) to show that if $f = f_r + f_{r+1} + \ldots \in k[[x,y]]$, and if the leading form f_r of f factors as $f_r = g_s h_t$, where g_s, h_t are homogeneous of degrees s and t respectively, and have no common linear factor, then there are formal power series

$$g = g_s + g_{s+1} + \ldots$$
$$h = h_t + h_{t+1} + \ldots$$

in $k[[x,y]]$ such that $f = gh$.

Construct g, h following hartshorne's example.

$f_{r+1} = h_t g_{s+1} + g_s h_{t+1}$ since $s + t = r$ and g_s, h_t generate the maximal ideal of $k[[x, y]]$.

1.4.34 c. x

> (c) Let Y be defined by the equation $f(x, y) = 0$ in $\mathbf{A}^2$, and let $P = (0,0)$ be a point of multiplicity r on Y, so that when f is expanded as a polynomial in x and y, we have $f = f_r +$ higher terms. We say that P is an *ordinary r-fold* point if f_r is a product of r *distinct* linear factors. Show that any two ordinary double points are analytically isomorphic. Ditto for ordinary triple points. But show that there is a one-parameter family of mutually nonisomorphic ordinary 4-fold points.

For double points see part (d).
For triple points write $f = f_3 + h.o.t$, $g = g_3 + h.o.t.$
In $\mathbb{P}^1$ and 3 pairs of lines can be interchanged by a linear transformation, but not for 4 pairs.
Now factoring f_3, g_3 into 3 linear terms we get the result.

1.4.35 d (starred)

> *(d) Assume char $k \neq 2$. Show that any double point of a plane curve is analytically isomorphic to the singularity at $(0,0)$ of the curve $y^2 = x^r$, for a uniquely determined $r \geqslant 2$. If $r = 2$ it is a node (Ex. 5.6). If $r = 3$ we call it a *cusp*; if $r = 4$ a *tacnode*. See (V, 3.9.5) for further discussion.

Show any double point of plane curve is analytically isomorphic to singularity at $(0,0)$ of curve $y^2 = x^r$ for uniquely determined $r \geq 2$.
If $r = 2$, it's a node. If $r = 3$, it's a cusp. If $r = 4$ it's a tacnode. Etc.
(Following Wall - Plane Curve Singularities, Chapter 2)

Change coordinates linearly so that $f(0, y) = y^m A(y)$ where $A(0) \neq 0$.

In this way we may use the Weierstrass preparation theorem to write $f(x, y) = U(x, y)\{y^2 + a(x) + b(x)\}$ where a, b are weierstrass polynomials and U has nonzero constant term. Thus C is given by $y^2 + a(x) + b(x)$.
Via another linear coordinate change we plug in $y = y' - \frac{1}{2}a(x)$, $y^2 = y'^2 - a(x)y' + \frac{1}{4}a(x)^2$
which gives
C is $y'^2 - a(x)y' + \frac{1}{4}a(x)^2 + a(x)y' - \frac{1}{2}a(x)^2 + b(x)$
which is $y'^2 - \frac{1}{4}a(x)^2 + b(x)$ and let $b'(x) = -\frac{1}{2}a(x)^2 + b(x)$ the new constant term in y.
Thus we have reduced to $y'^2 + b'(x) = 0$.
If $b' = 0$ then we have C is $y'^2 = 0$.
Otherwise b' has order k some $k \geq 2$ (so it's a polynomial function of x, and we already have squared terms and we don't have cancellation).
Also the weierstrass preparation gives the constant term of the $a(x)$ are 0.
Now it is a general fact that an order n power series like b' may be written as x'^n for some x' a convergent power series in x of order 1.
Thus changing coordinates to (x', y') we are done.

1.4.36 I.5.15 x g:a,b

5.15. *Families of Plane Curves.* A homogeneous polynomial f of degree d in three variables x, y, z has $\binom{d+2}{2}$ coefficients. Let these coefficients represent a point in $\mathbf{P}^N$, where $N = \binom{d+2}{2} - 1 = \frac{1}{2}d(d+3)$.

(a) Show that this gives a correspondence between points of $\mathbf{P}^N$ and algebraic sets in $\mathbf{P}^2$ which can be defined by an equation of degree d. The correspondence is 1-1 except in some cases where f has a multiple factor.

Note the homogeneous monomials of degree d are like $x^a y^b z^c$ with $a+b+c=d$. There are $\begin{pmatrix} d+2 \\ 2 \end{pmatrix} - 1$ of them so we have a clear correspondence.

1.4.37 b. x

(b) Show under this correspondence that the (irreducible) nonsingular curves of degree d correspond 1-1 to the points of a nonempty Zariski-open subset of $\mathbf{P}^N$. [*Hints:* (1) Use elimination theory (5.7A) applied to the homogeneous polynomials $\partial f/\partial x_0, \ldots, \partial f/\partial x_n$; (2) use the previous (Ex. 5.5, 5.8, 5.9) above.]

If f has no multiple factors, this is clear.

If f is reducible, by elimination theory, points in $\mathbb{P}^N$ with $f \neq 0$, and partials nonzero are in 1-1 correspondence with the nonzero locus of a finite set of polynomials defining an open set in $\mathbb{P}^N$.

1.5 I.6 x

1.5.1 I.6.1 x g:a,b,c

6.1. Recall that a curve is *rational* if it is birationally equivalent to $\mathbf{P}^1$ (Ex. 4.4). Let Y be a nonsingular rational curve which is not isomorphic to $\mathbf{P}^1$.

(a) Show that Y is isomorphic to an open subset of $\mathbf{A}^1$.

As Y is isomorphic to an open subset of projective space, then it is isomorphic to a proper open subset of $\mathbb{P}^1$.

1.5.2 b. x

(b) Show that Y is affine.

Since Y is $\mathbb{A}^1$ minus a finite number of points, $Y = V\left(y\left(x - P_1\right) \cdot \cdots \cdot \left(x - P_n\right)\right)$.

1.5.3 c. x

(c) Show that $A(Y)$ is a unique factorization domain.

By (b), $A(Y) = \mathcal{O}(Y) = k[y]_{(t-P_1, t-P_2, \ldots)}$ which is the localization of a UFD.

1.5.4 I.6.2 x g

6.2. *An Elliptic Curve.* Let Y be the curve $y^2 = x^3 - x$ in $\mathbf{A}^2$, and assume that the characteristic of the base field k is $\neq 2$. In this exercise we will show that Y is not a rational curve, and hence $K(Y)$ is not a pure transcendental extension of k.

(a) Show that Y is nonsingular, and deduce that $A = A(Y) \simeq k[x,y]/(y^2 - x^3 + x)$ is an integrally closed domain.

A nonsingular curve is normal. Thus by the jacobian criterion, A is integrally closed.

1.5.5 b. x

(b) Let $k[x]$ be the subring of $K = K(Y)$ generated by the image of x in A. Show that $k[x]$ is a polynomial ring, and that A is the integral closure of $k[x]$ in K.

As $y^2 \in k\left[x\right]$, thus y satisfies $z^2 - y^2$, thus $y \in \overline{k\left[x\right]}$.
Thus $A \subset \overline{k\left[x\right]}$. On the other hand $k\left[x\right] \subset A$ so $\overline{k\left[x\right]} \subset \overline{A}$.

1.5.6 c. x

(c) Show that there is an automorphism $\sigma : A \to A$ which sends y to $-y$ and leaves x fixed. For any $a \in A$, define the *norm* of a to be $N(a) = a \cdot \sigma(a)$. Show that $N(a) \in k[x]$, $N(1) = 1$, and $N(ab) = N(a) \cdot N(b)$ for any $a,b \in A$.

Writing $f \in A$ as $y \cdot g\left(x\right) + h\left(x\right)$, then $N\left(f\right) = \left(h\left(x\right) + yg\left(x\right)\right)\left(h\left(x\right) - yg\left(x\right)\right) \in k\left[x\right]$.
By easy computation $N\left(1\right) = 1$, $N\left(ab\right) = N\left(a\right) \cdot N\left(b\right)$.

1.5.7 d. x

(d) Using the norm, show that the units in A are precisely the nonzero elements of k. Show that x and y are irreducible elements of A. Show that A is *not* a unique factorization domain.

Units

If a is a unit, then $N\left(a\right)$ has inverse $N\left(a^{-1}\right)$ by (c). so $N\left(a\right) \in k^{\times}$.
If a is nonuit, $a = yf\left(x\right) + g\left(x\right)$ as in (c), then $N\left(a\right) = g\left(x\right)^2 - f\left(x\right)^2\left(x^3 - x\right)$.
If f is nonzero then comparing degrees we see the norm is nonconstant, contradiction.
$\implies g^2$ is constant $\implies$ a is constant.

irreducible

If $x = ab$ is , $N\left(x\right) = N\left(a\right)N\left(b\right) = x^2$ so $N\left(a\right)$ or $N\left(b\right)$ is a linear polynomial if a,b are not in k. Contradiction.

Not UFD

$x|y^2$ and $y \neq ux \implies$ not UFD.

1.5.8 e. x

(e) Prove that Y is not a rational curve (Ex. 6.1). See (II, 8.20.3) and (III, Ex. 5.3) for other proofs of this important result.

Since nontrivial and not UDF then by 6.1, Y is not rational.

1.5.9 I.6.3 x

6.3. Show by example that the result of (6.8) is false if either (a) dim $X \geqslant 2$, or (b) Y is not projective.

For (a) map $\mathbb{A}^2\setminus(0,0)$ to $\mathbb{P}^1$ by $(x,y)\mapsto(x:y)$, for (b) map $\mathbb{P}^1\backslash\infty$ to $\mathbb{A}^1$ by $(x:y)\mapsto x/y$.

1.5.10 I.6.4 x g

6.4. Let Y be a nonsingular projective curve. Show that every nonconstant rational function f on Y defines a surjective morphism $\varphi: Y \to \mathbf{P}^1$, and that for every $P \in \mathbf{P}^1$, $\varphi^{-1}(P)$ is a finite set of points.

f induces a map $Y\to\mathbb{A}^1$ by $x\mapsto f(x)$ and therefore $\varphi:Y\to\mathbb{P}^1$.
As f is nonconstant and Y irreducible, $im(\varphi)=\mathbb{P}^1$ and φ is dominant, so induces $k(Y)\hookrightarrow k(\mathbb{P}^1)$.
If $p\in\mathbb{P}^1$, then $\varphi^{-1}(P)$ is closed by continuity and finite since it's a proper subset of Y with closure not Y as φ is nonconstant.

1.5.11 I.6.5 x

6.5. Let X be a nonsingular projective curve. Suppose that X is a (locally closed) subvariety of a variety Y (Ex. 3.10). Show that X is in fact a closed subset of Y. See (II, Ex. 4.4) for generalization.

Note that the image of a projective variety X under a regular embedding $X\hookrightarrow Y$ is closed.

1.5.12 I.6.6 x g:a,c

6.6. *Automorphisms of* $\mathbf{P}^1$. Think of $\mathbf{P}^1$ as $\mathbf{A}^1 \cup \{\infty\}$. Then we define a *fractional linear transformation* of $\mathbf{P}^1$ by sending $x \mapsto (ax+b)/(cx+d)$, for $a,b,c,d \in k$, $ad-bc \neq 0$.

(a) Show that a fractional linear transformation induces an *automorphism* of $\mathbf{P}^1$ (i.e., an isomorphism of $\mathbf{P}^1$ with itself). We denote the group of all these fractional linear transformations by PGL(1).

Since $ad-bc\neq0$ means the determinant of a 2×2 matrix is nonzero , this matrix therefore has an inverse.
Since FLT is a group action, the inverse matrix gives the inverse of the action.

1.5.13 b. x

> (b) Let Aut $\mathbf{P}^1$ denote the group of all automorphisms of $\mathbf{P}^1$. Show that Aut $\mathbf{P}^1 \simeq$ Aut $k(x)$, the group of k-automorphisms of the field $k(x)$.

For $\varphi \in Aut\left(\mathbb{P}^1\right)$, $(f \mapsto f \circ \varphi) \in Aut\ k\left(x\right)$.
If $\phi \in Aut\ k\left(x\right)$, ϕ induces an auto-birational (is this a word?) map of $\mathbb{P}^1$, a nonsingular curve.

1.5.14 c. x

> (c) Now show that every automorphism of $k(x)$ is a fractional linear transformation, and deduce that $\mathrm{PGL}(1) \to$ Aut $\mathbf{P}^1$ is an isomorphism.

$\varphi \in Aut\ k\left(x\right)$ maps $x \mapsto \frac{f(x)}{g(x)}$ for coprime f, g .
Injectivity of $\varphi \implies f, g$ are linear: $f\left(x\right) = ax + b$, $g\left(x\right) = cx + d$ say.
Since f, g are coprime , then $ad - bc \neq 0$.

1.5.15 I.6.7 x g

> **6.7.** Let $P_1, \ldots, P_r, Q_1, \ldots, Q_s$ be distinct points of $\mathbf{A}^1$. If $\mathbf{A}^1 - \{P_1, \ldots, P_r\}$ is isomorphic to $\mathbf{A}^1 - \{Q_1, \ldots, Q_s\}$, show that $r = s$. Is the converse true? Cf. (Ex. 3.1).

We can extend any map between the two curves to a map $\mathbb{P}^1 \to \mathbb{P}^1$.
Since P_i must map to Q_j thus $r = s$.
The converse is only true for $r \leq 3$ by virtue of the fact that mobius transformations can take at most 3 distinct points in $\mathbb{P}^1$ to any other 3 distinct points.

1.6 I.7 x

1.6.1 I.7.1 x g

> **7.1.** (a) Find the degree of the d-uple embedding of $\mathbf{P}^n$ in $\mathbf{P}^N$ (Ex. 2.12). [Answer: d^n]

Note $\left(\begin{array}{c} dk + n \\ n \end{array}\right) = \frac{(dk)^n}{n!} + \mathcal{O}\left(d^{n-1}k^{n-1}\right)$
This is the dimension of the space of monomials of degree d.

1.6.2 b. x g

> (b) Find the degree of the Segre embedding of $\mathbf{P}^r \times \mathbf{P}^s$ in $\mathbf{P}^N$ (Ex. 2.14). [Answer: $\binom{r+s}{r}$]

The subring of $k\left[x_0, ..., x_r, y_0, ..., y_s\right]$ generated by polynomials of degree $2k$ half x_i's and half y_j's has dim $\left(\begin{array}{c} k + s \\ s \end{array}\right)\left(\begin{array}{c} k + r \\ r \end{array}\right) = \left(\begin{array}{c} r + s \\ r \end{array}\right) \frac{k^{r+s}}{(r+s)!} + \mathcal{O}\left(k^{r+s-1}\right)$.

1.6.3 I.7.2 x g arithmetic genus of projective space.

7.2. Let Y be a variety of dimension r in $\mathbf{P}^n$, with Hilbert polynomial P_Y. We define the *arithmetic genus* of Y to be $p_a(Y) = (-1)^r(P_Y(0) - 1)$. This is an important invariant which (as we will see later in (III, Ex. 5.3)) is independent of the projective embedding of Y.

(a) Show that $p_a(\mathbf{P}^n) = 0$.

Since the hilbert poly of $\mathbb{P}^n$ is $\chi(\mathcal{O}_{\mathbb{P}^n}(k)) = \binom{n+k}{n}$.

Then $p_a(\mathbb{P}^n) = (-1)^r(1-1) = 0$

1.6.4 b. x g

(b) If Y is a plane curve of degree d, show that $p_a(Y) = \frac{1}{2}(d-1)(d-2)$.

By I.7.6.d, $P_H(0) = \binom{0+2}{2} - \binom{0-d+2}{2} = 1 - \binom{2-d}{2} = 1 - (-1)^2\binom{d-1}{2}$.

1.6.5 c. x g important

(c) More generally, if H is a hypersurface of degree d in $\mathbf{P}^n$, then $p_a(H) = \binom{d-1}{n}$.

As in I.7.6.d, we get that $\chi(l) = \binom{n+l}{n} - \binom{n+l-d}{n}$.

Then $p_a(X) = (-1)^{n-1}(1-\chi(0))$ and computing this gives the result.

1.6.6 d. g x

(d) If Y is a complete intersection (Ex. 2.17) of surfaces of degrees a,b in $\mathbf{P}^3$, then $p_a(Y) = \frac{1}{2}ab(a+b-4)+1$.

We have the standard four term exact sequence
$0 \to S(\mathbb{P}^3)_{l-a-b} \to S(\mathbb{P}^3)_{l-a} \otimes S(\mathbb{P}^3)_{l-b} \to S(\mathbb{P}^3)_l \to S(X)_l \to 0$
so that $\chi(l) = \binom{l+3}{l} - \binom{l+3-a}{3} - \binom{l+3-b}{3} + \binom{l+3-a-b}{3}$.
Writing this out and solving gives the solution.

1.6.7 e. x

(e) Let $Y^r \subseteq \mathbf{P}^n$, $Z^s \subseteq \mathbf{P}^m$ be projective varieties, and embed $Y \times Z \subseteq \mathbf{P}^n \times \mathbf{P}^m \to \mathbf{P}^N$ by the Segre embedding. Show that

$$p_a(Y \times Z) = p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z).$$

Note that the hilbert polynomial of $Y \times Z$ is the product of the hilbert polynomials of Y and Z since tensor products multiply dimensions.

1.6.8 I.7.3 x g

> **7.3.** *The Dual Curve.* Let $Y \subseteq \mathbf{P}^2$ be a curve. We regard the set of lines in $\mathbf{P}^2$ as another projective space, $(\mathbf{P}^2)^*$, by taking (a_0,a_1,a_2) as homogeneous coordinates of the line $L: a_0x_0 + a_1x_1 + a_2x_2 = 0$. For each nonsingular point $P \in Y$, show that there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is >1. This is the *tangent line* to Y at P. Show that the mapping $P \mapsto T_P(Y)$ defines a *morphism* of Reg Y (the set of nonsingular points of Y) into $(\mathbf{P}^2)^*$. The closure of the image of this morphism is called the dual curve $Y^* \subseteq (\mathbf{P}^2)^*$ of Y.

The tangent line to a curve Y defined by a polynomial f at $P = (a_0, a_1, a_2)$ is given by $\frac{\partial f}{\partial x}(x - a_0) + \frac{\partial f}{\partial y}(y - a_1) + \frac{\partial f}{\partial z}(z - a_2) = 0$.

If P is nonsingular, then at least one partial is nonzero so it's well-defind hence unique.

If we assume P is at zero and the curve is affine, then the tangent line is given by taking partials and then substituting 0, and so we see the linear term is just the above line.

We further assume $f(x, y) = y + h.o.t$ since and $P = (0, 0,) \in \mathbb{A}^2$ since P is nonsingular.

The only line with higher intersection multiplicity at P is the x-axis, which incidentally is the linear part.

1.6.9 I.7.4 x

> **7.4.** Given a curve Y of degree d in $\mathbf{P}^2$, show that there is a nonempty open subset U of $(\mathbf{P}^2)^*$ in its Zariski topology such that for each $L \in U$, L meets Y in exactly d points. [*Hint*: Show that the set of lines in $(\mathbf{P}^2)^*$ which are either tangent to Y or pass through a singular point of Y is contained in a proper closed subset.] This result shows that we could have defined the degree of Y to be the number d such that almost all lines in $\mathbf{P}^2$ meet Y in d points, where "almost all" refers to a nonempty open set of the set of lines, when this set is identified with the dual projective space $(\mathbf{P}^2)^*$.

Using bezout, lines which meet Y transversely at smooth points meet in degree $Y = d$ points.

By noetherianness, the singular locus is a finite set of points on Y.

The lines meeting one of these are a proper closed subset of $\mathbb{P}^{2*}$.

By 7.3, the tangent lines are contained in a proper closed subset of $\mathbb{P}^{2*}$ ($\subset Y \times \mathbb{P}^1$)

1.6.10 I.7.5 x g:a,b upper bound on multiplicity x

> **7.5.** (a) Show that an irreducible curve Y of degree $d > 1$ in $\mathbf{P}^2$ cannot have a point of multiplicity $\geqslant d$ (Ex. 5.3).

The degree can be read off of the lowest term in the equation if the point is at $(0, 0)$.

If all terms have degree d, then it can't be irreducible of degree > 1.

1.6.11 b. x

> (b) If Y is an irreducible curve of degree $d > 1$ having a point of multiplicity $d - 1$, then Y is a rational curve (Ex. 6.1).

Assume Y is defined by $f(x, y) + g(x, y) = 0$, with $deg\ f = d - 1$, $deg\ g = d$.

If $t = \frac{y}{x}$, $\implies (x,y) \mapsto \left(yt, \frac{-f(t,1)}{g(t,1)}\right)$ which is projection from a point gives an inverse rational map to $\mathbb{A}^1$
.

1.6.12 I.7.6 x Linear Varieties x

7.6. *Linear Varieties.* Show that an algebraic set Y of pure dimension r (i.e., every irreducible component of Y has dimension r) has degree 1 if and only if Y is a linear variety (Ex. 2.11). [*Hint*: First, use (7.7) and treat the case $\dim Y = 1$. Then do the general case by cutting with a hyperplane and using induction.]

If Y has pure dimension r , then by 7.6b, Y is irreducible.
The hilbert polynomial of a linear variety has a leading term which gives linear degree.
If on the other hand Y has degree 1 , then for any hyperplane H not containing Y , $Y \cap H$ has degree 1 and is thus linear.

1.6.13 I.7.7 x

7.7. Let Y be a variety of dimension r and degree $d > 1$ in $\mathbf{P}^n$. Let $P \in Y$ be a nonsingular point. Define X to be the closure of the union of all lines PQ, where $Q \in Y, Q \neq P$.
(a) Show that X is a variety of dimension $r + 1$.

Choose a hyperplane H in $\mathbb{P}^n$ not containing P or Y.
If PQ is a line from with endpoints in Y, then map PQ to the line through Q and the vertex of the cone over Y.
A rational inverse takes the line through the vertex of the cone over Y and Q to PQ.
Note that the cone has dimension $r + 1$.

1.6.14 b. x

(b) Show that $\deg X < d$. [*Hint*: Use induction on $\dim Y$.]

If $dim\ Y = 0$, then Y has d points and X has $d-1$ lines.
If $dim\ Y = r$ and H is a hyperplane containing P but not Y, then by Thm 7.7, 7.6b, $deg\ X \cap H = deg\ X$.
By induction $deg\ X \cap H \leq deg\ Y \cap H \leq deg\ Y = d$.

1.6.15 I.7.8 x contained in linear subspace. x

7.8. Let $Y^r \subseteq \mathbf{P}^n$ be a variety of degree 2. Show that Y is contained in a linear subspace L of dimension $r + 1$ in $\mathbf{P}^n$. Thus Y is isomorphic to a quadric hypersurface in $\mathbf{P}^{r+1}$ (Ex. 5.12).

ex. I.7.7 gives Y is in degee 1 variety of dim $r + 1$ in $\mathbb{P}^n$.
By Ex I.7.6, this is linear, and thus isomorphic to $\mathbb{P}^{r+1}$.

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2 II Schemes

2.1 II.1 x

2.1.1 x II.1.1 Constant presheaf

1.1. Let A be an abelian group, and define the *constant presheaf* associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathscr{A}$ defined in the text is the sheaf associated to this presheaf.

Let $\mathscr{F}$ denote the constant presheaf.
Let $\mathscr{F}^+$ the sheaf associated to this presheaf.
$\mathscr{F}^+(U)$, is then the maps $s : U \to \cup_{P \in U} \mathscr{F}_P$ satisfying certain conditions.
(see
We have a map $A^{pre} \to \mathscr{A}$ by taking $a \in A$ to the constant map $U \to A$.
It is easy to check that this is an isomorphism on the stalks.

2.1.2 II.1.2 x g

1.2. (a) For any morphism of sheaves $\varphi : \mathscr{F} \to \mathscr{G}$, show that for each point P, $(\ker \varphi)_P = \ker(\varphi_P)$ and $(\operatorname{im} \varphi)_P = \operatorname{im}(\varphi_P)$.

Filtered colimits (like the stalk) commute with finite limits (like the kernel).
–see Vakils notes for this terminology
Also since the cokernel is range / image, then image is the kernel of $\mathscr{G} \to coker\ \varphi$.

2.1.3 b. x

(b) Show that φ is injective (respectively, surjective) if and only if the induced map on the stalks φ_P is injective (respectively, surjective) for all P.

x

These follow from part (a).

2.1.4 c. x.

(c) Show that a sequence $\ldots \mathscr{F}^{i-1} \xrightarrow{\varphi^{i-1}} \mathscr{F}^i \xrightarrow{\varphi^i} \mathscr{F}^{i+1} \to \ldots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.

x

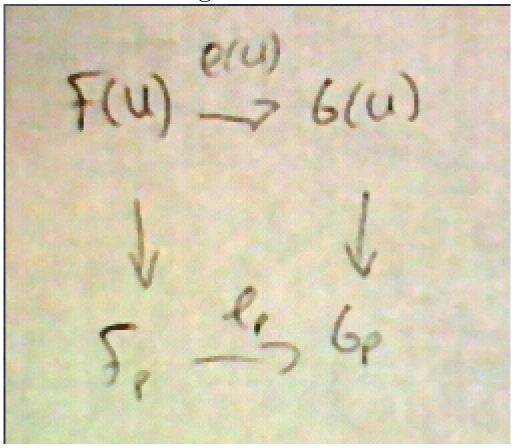
This follows from the definition of exactness and part (a).

2.1.5 II.1.3x surjective condition. x

1.3. (a) Let $\varphi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that φ is surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\varphi(t_i) = s|_{U_i}$ for all i.

To show φ is surjective, we need to show it's surjective on stalks.

We have a diagram:



For some $s_p \in \mathscr{G}_p$, pull back to s.
Now find $t_i \in \mathscr{F}(U_i)$ using the assumptions of the problem.
The converse follows by 1.2.b. Since surjective iff surjective on stalks.

2.1.6 2.1.3.b. x g Surjective not on stalks x

> (b) Give an example of a surjective morphism of sheaves $\varphi: \mathscr{F} \to \mathscr{G}$, and an open set U such that $\varphi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.

Let $\mathscr{F}$ the sheaf of holomorphic functions on $\mathbb{C}\backslash 0$
consider $\varphi : f \to exp(f)$.
Note we can write holomorphic functions locally as a logarithm,
$\implies \varphi$ is surjective on stalks.
Globally this doesn't work

2.1.7 II.1.4 x

> **1.4.** (a) Let $\varphi: \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\varphi(U): \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\varphi^+: \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.

Use 1.2.b, since sheafification preserves stalks.

2.1.8 b. x

> (b) Use part (a) to show that if $\varphi: \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then im φ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

Note that sheafification preserves injective morphisms such as $im^{pre}\,\varphi \hookrightarrow \mathscr{G}$.

2.1.9 x II.1.5

> **1.5.** Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.

2.1.1 is isomorphism via stalks and
Excercise 1.2. b says injective / surjective iff

stalks are injective / surjective.

2.1.10 II.1.6.a x map to quotient is surjective

1.6. (a) Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0.$$

Sheafification is left adjoint (vakil notes), it preserves colimits and thus surjections.
We can also just look at the stalks. $\mathscr{F}_p \to \mathscr{F}/\mathscr{F}'_p$ is surjective then using 1.2, $\mathscr{F} \to \mathscr{F}/\mathscr{F}'$ is surjective.
...

2.1.11 II.1.6.b x

(b) Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

In vakil's notes we see that forgetful functor is right adjoint to sheafification.
Thus sheafification preserves kernels, and so the left most map is injective as presheaf maps.

2.1.12 II.1.7 x

1.7. Let $\varphi: \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.
(a) Show that $\operatorname{im} \varphi \cong \mathscr{F}/\ker \varphi$.

By 1.6 and 1.4.b

2.1.13 II.1.7.b x

(b) Show that $\operatorname{coker} \varphi \cong \mathscr{G}/\operatorname{im} \varphi$.

Again, by 1.6 and 1.4.b.

2.1.14 II.1.8 x g

1.8. For any open subset $U \subseteq X$, show that the functor $\Gamma(U, \cdot)$ from sheaves on X to abelian groups is a left exact functor, i.e., if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is an exact sequence of sheaves, then $0 \to \Gamma(U, \mathscr{F}') \to \Gamma(U, \mathscr{F}) \to \Gamma(U, \mathscr{F}'')$ is an exact sequence of groups. The functor $\Gamma(U, \cdot)$ need not be exact; see (Ex. 1.21) below.

If $0 \to \mathscr{F}' \xrightarrow{\varphi} \mathscr{F} \to \mathscr{F}''$, then $ker\ \varphi = 0$ so $ker\ \varphi(U) = 0$.

2.1.15 II.1.9 x Direct sum of sheaves x g

1.9. *Direct Sum.* Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf. It is called the *direct sum* of $\mathscr{F}$ and $\mathscr{G}$, and is denoted by $\mathscr{F} \oplus \mathscr{G}$. Show that it plays the role of direct sum and of direct product in the category of sheaves of abelian groups on X.

Since the forgetful functor preserves limits and the direct sum is a limit.
(See the first chapter in Vakil's notes)

2.1.16 II.1.10 x Direct Limits x

1.10. *Direct Limit.* Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the *direct limit* of the system $\{\mathscr{F}_i\}$, denoted $\varinjlim \mathscr{F}_i$, to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a direct limit in the category of sheaves on X, i.e., that it has the following universal property: given a sheaf $\mathscr{G}$, and a collection of morphisms $\mathscr{F}_i \to \mathscr{G}$, compatible with the maps of the direct system, then there exists a unique map $\varinjlim \mathscr{F}_i \to \mathscr{G}$ such that for each i, the original map $\mathscr{F}_i \to \mathscr{G}$ is obtained by composing the maps $\mathscr{F}_i \to \varinjlim \mathscr{F}_i \to \mathscr{G}$.

Since sheaficication is left adjoiint and preserves direct limits
(see the first chapter in Vakil's notes)

2.1.17 II.1.11 x

1.11. Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.

Suppose U is open, U_i a finite cover since noetherian, and $U_{ij} = U_i \cap U_j$

Let J be the category whose objects are U_i and U_{ij} , and morphisms are inclusions of $U_i \cap U_j$ in U_i and U_j.

The sheaf axioms is that the limit of the functor $U \mapsto \underset{\rightarrow}{lim}\mathscr{F}_n(U)$ restricted to J must be isomorphic to $\underset{\rightarrow}{lim}\mathscr{F}_n(U)$.

We have $\underset{\leftarrow\ ij}{lim}\left(\underset{\rightarrow\ n}{lim}\ \mathscr{F}_n\right)(U_{ij}) = \underset{\leftarrow\ ij}{lim}\left(\underset{\rightarrow n}{lim}\mathscr{F}_n(U_{ij})\right) =$
$\underset{\rightarrow\ n}{lim}\left(\underset{\leftarrow\ ij}{lim}\ \mathscr{F}_n(U_{ij})\right) = \underset{\rightarrow\ n}{lim}\ \mathscr{F}_n(U) =$
$\underset{\rightarrow\ n}{lim}\ \mathscr{F}_n(U)$.

2.1.18 II.1.12 x

1.12. *Inverse Limit.* Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the presheaf $U \mapsto \varprojlim \mathscr{F}_i(U)$ is a sheaf. It is called the *inverse limit* of the system $\{\mathscr{F}_i\}$, and is denoted by $\varprojlim \mathscr{F}_i$. Show that it has the universal property of an inverse limit in the category of sheaves.

As in the previous.

2.1.19 II.1.13 Espace Etale x

> **1.13.** *Espace Étalé of a Presheaf.* (This exercise is included only to establish the connection between our definition of a sheaf and another definition often found in the literature. See for example Godement [1, Ch. II, §1.2].) Given a presheaf $\mathscr{F}$ on X, we define a topological space $\text{Spé}(\mathscr{F})$, called the *espace étalé* of $\mathscr{F}$, as follows. As a set, $\text{Spé}(\mathscr{F}) = \bigcup_{P \in X} \mathscr{F}_P$. We define a projection map $\pi : \text{Spé}(\mathscr{F}) \to X$ by sending $s \in \mathscr{F}_P$ to P. For each open set $U \subseteq X$ and each section $s \in \mathscr{F}(U)$, we obtain a map $\bar{s} : U \to \text{Spé}(\mathscr{F})$ by sending $P \mapsto s_P$, its germ at P. This map has the property that $\pi \circ \bar{s} = \text{id}_U$, in other words, it is a "section" of π over U. We now make $\text{Spé}(\mathscr{F})$ into a topological space by giving it the strongest topology such that all the maps $\bar{s} : U \to \text{Spé}(\mathscr{F})$ for all U, and all $s \in \mathscr{F}(U)$, are continuous. Now show that the sheaf $\mathscr{F}^+$ associated to $\mathscr{F}$ can be described as follows: for any open set $U \subseteq X$, $\mathscr{F}^+(U)$ is the set of *continuous* sections of $\text{Spé}(\mathscr{F})$ over U. In particular, the original presheaf $\mathscr{F}$ was a sheaf if and only if for each U, $\mathscr{F}(U)$ is equal to the set of all continuous sections of $\text{Spé}(\mathscr{F})$ over U.

Suppose $s \in \mathscr{F}^+(U)$,

we have $\bar{s} : U \to Spec(\mathscr{F})$ sending $P \mapsto s_P$ which is continuous (by strongest topology ...) and we need s to be continuous.

If $V \subset Spe(\mathscr{F})$ is open and $P \in s^{-1}(V)$, $\implies s(P) \in \mathscr{F}_P \implies P \in U$.

Let U' an open neighborhood of P, $t \in \mathscr{F}(U')$ such that $s|_{U'} = t$ is continuous.

Then $s|_{U'}^{-1}(V) = t^{-1}(V)$ is an open neighobrohood of P contained in $s^{-1}(V)$ (by srongest topology ..)

Hence each point in $s^{-1}V$ has an open neighborhood contained in the preimage so s is continuous.

Conversely if $s : U \to Spec(\mathscr{F})$ is continuous , V is open, and $t \in \mathscr{F}(V)$ then for $x \in t^{-1}(s(U))$ we must have $s(x) = t(x)$ so there is an open W with $s|_W = t|_W$.

Thus $W \subset t^{-1}(s(U))$.

Since we are using the strongest topology such that t is continuous, then $t^{-1}s(U)$ is open in U for $t \in \mathscr{F}(U)$ $\implies s(U)$ is open in $Spe(\mathscr{F})$.

Now if $x \in U$, then $s(x)$ is equal to a germ (t, W) in $\mathscr{F}_x$.

Continuity of s gives $s^{-1}(t(W))$ is open (by the same reasoning as above $t(W)$ is open)

so on an open $W' \subset W$ we have $t|_{W'} = s|_{W'}$, and thus s locally gives a section of $\mathscr{F}$.

Thus s gives a section of $\mathscr{F}^+$.

2.1.20 II.1.14 x g

> **1.14.** *Support.* Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The *support* of s, denoted $\text{Supp}\, s$, is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that $\text{Supp}\, s$ is a closed subset of U. We define the *support* of $\mathscr{F}$, $\text{Supp}\, \mathscr{F}$, to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.

If P is not in the support, then the germ of s in the stalk at p is zero.

Thus there is a neighborhood where s vanishes. Hence the complement of the support is open.

Note that 19.b. gives an an example of non-closed support.

2.1.21 II.1.15 Sheaf Hom x: g

1.15. *Sheaf $\mathscr{H}om$.* Let $\mathscr{F}, \mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set $\operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ of morphisms of the restricted sheaves has a natural structure of abelian group. Show that the presheaf $U \mapsto \operatorname{Hom}(\mathscr{F}|_U, \mathscr{G}|_U)$ is a sheaf. It is called the *sheaf of local morphisms* of $\mathscr{F}$ into $\mathscr{G}$, "sheaf hom" for short, and is denoted $\mathscr{H}om(\mathscr{F}, \mathscr{G})$.

presheaf

Let U open.

Clearly $Hom\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$ is an abelian group and the obvious restriction maps give a presheaf $F : U \mapsto Hom\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$.

identity

Let $\{U_i\}$ an open cover of U.

If $s \in F(U) = Hom\left(\mathscr{F}|_U, \mathscr{G}|_U\right)$ satisfies $s|_{U_i} = 0$ for all i , then since $\mathscr{F}$ is a sheaf, there is $f \in \mathscr{F}(U)$ with $s(f) = 0$ on U.

Hence $s|_U = 0$.

glueing

similar.

2.1.22 II.1.16 g Flasque Sheaves x

1.16. *Flasque Sheaves.* A sheaf $\mathscr{F}$ on a topological space X is *flasque* if for every inclusion $V \subseteq U$ of open sets, the restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$ is surjective.

(a) Show that a constant sheaf on an irreducible topological space is flasque. See (I, §1) for irreducible topological spaces.

All the restriction morphisms are identity.

2.1.23 b. x

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ is flasque, then for any open set U, the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ of abelian groups is also exact.

Exc. 1.8 gives left exactness, so we just need to show $\mathscr{F}(U) \twoheadrightarrow \mathscr{F}''(U)$.

If $t \in \mathscr{F}''(U)$, then by surjectivity of $\mathscr{F} \to \mathscr{F}''$, there is an open cover U_i of U on which t lifts to elements $t_i \in \mathscr{F}(U_i)$.

On $U_i \cap U_j$, $t_i - t_j = r_{ij} \in \mathscr{F}'(U_i \cap U_j)$.

Since $\mathscr{F}'$ has surjective restrictions, we can extend r_{ij} to $r'_{ij} \in \mathscr{F}'(U_i)$.

$\left(t_i - r'_{ij}\right)|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$ so if we replace t_i by $t_i - r'_{ij}$ then we have defined a lifting of t on $U_i \cap U_j$.

2.1.24 c. x

(c) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then $\mathscr{F}''$ is flasque.

5 lemma

2.1.25 d. x g

> (d) If $f: X \to Y$ is a continuous map, and if $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is a flasque sheaf on Y.

Since continuous maps preserve inclusions.

2.1.26 e. x sheaf of discontinuous sections

> (e) **Let $\mathscr{F}$ be any sheaf on X. We define a new sheaf $\mathscr{G}$, called the sheaf of *discontinuous sections* of $\mathscr{F}$ as follows. For each open set $U \subseteq X$, $\mathscr{G}(U)$ is the set of**
>
> **maps $s: U \to \bigcup_{P \in U} \mathscr{F}_P$ such that for each $P \in U$, $s(P) \in \mathscr{F}_P$. Show that $\mathscr{G}$ is a flasque sheaf, and that there is a natural injective morphism of $\mathscr{F}$ to $\mathscr{G}$.**

Let $I \subset J$ be sets of points in open sets $U \subset X$.
This inclusion gives a natural categorical surjection $\prod_{P \in J} \mathscr{F}_P \twoheadrightarrow \prod_{Q \in I} \mathscr{F}_Q$.
For $U \subset X$ open, define $\mathscr{F}(U) \to \mathscr{G}(U)$ by $x \mapsto (P \mapsto x_P)$.
We show the kernel is trivial.
If $P \mapsto x_p$ is zero for $x \in \mathscr{F}(U)$, then there is a neighborhood U_P of P such that $x|_{U_P} = 0$.
The U_P cover U so by the identity sheaf axiom, $x = 0$.

2.1.27 II.1.17 x g skyscraper sheaves (important)

> **1.17.** *Skyscraper Sheaves.* Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \{P\}^-$, and 0 elsewhere, where $\{P\}^-$ denotes the closure of the set consisting of the point P. Hence the name "skyscraper sheaf." Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\{P\}^-$, and $i: \{P\}^- \to X$ is the inclusion.

Taking limits of open sets containing P , we see the stalk at $Q \in \{P\}^-$ is A and taking limits of $Q \notin \{P\}^-$ we see the stalk is 0 elsewhere.
Now if $P \notin U$ then $i^{-1}(U) = \emptyset$ so $i_*(A)(U) = 0$.
If $P \in U$, then $i_*(A)(U) = A(i^{-1}(U)) = A\left(\{P\}^-\right) = \Gamma\left(A, \{P\}^-\right) = A$.
Hence $i_*(A)$ is the skyscraper sheaf.

2.1.28 II.1.18 x g Adjoint property of f^(-1)

> **1.18.** *Adjoint Property of f^{-1}.* Let $f: X \to Y$ be a continuous map of topological spaces. Show that for any sheaf $\mathscr{F}$ on X there is a natural map $f^{-1}f_*\mathscr{F} \to \mathscr{F}$, and for any sheaf $\mathscr{G}$ on Y there is a natural map $\mathscr{G} \to f_*f^{-1}\mathscr{G}$. Use these maps to show that there is a natural bijection of sets, for any sheaves $\mathscr{F}$ on X and $\mathscr{G}$ on Y,
>
> $$\mathrm{Hom}_X(f^{-1}\mathscr{G}, \mathscr{F}) = \mathrm{Hom}_Y(\mathscr{G}, f_*\mathscr{F}).$$
>
> Hence we say that f^{-1} is a *left adjoint* of f_*, and that f_* is a *right adjoint* of f^{-1}.

Suppose that $\varphi \in Hom_X(f^{-1}\mathscr{G}, \mathscr{F})$.
If $U \subset X$ is open, then we have an induced map $\varphi_U : lim_{V \supset f(U)}\mathscr{G}(V) \to \mathscr{F}(U)$.

If $W \subset Y$ is open, then we define $\sigma(\varphi) : \mathscr{G}(W) \overset{\approx}{\mapsto} \underset{\rightarrow}{lim}\mathscr{G}(V) \mapsto \mathscr{F}(f^{-1}(W)) = f_*(\mathscr{F})(W)$, $\sigma : Hom_X(f^{-1}\mathscr{G}, \mathscr{F}) \rightarrow Hom_Y(\mathscr{G}, f_*\mathscr{F})$.

conversely

Let $\psi : \mathscr{G} \rightarrow f_*\mathscr{F}$ a sheaf morphism.

For $V \subset Y$ open, with $f(U) \subset V$, then $U \subset f^{-1}(V)$ so we have maps $\mathscr{G}(V) \rightarrow \mathscr{F}(f^{-1}(V)) \rightarrow \mathscr{F}(U)$.

The restriction maps and taking the limit over $f(U) \subset V$ give $\underset{\rightarrow}{lim}\mathscr{G}(V) \rightarrow \mathscr{F}(U)$ for each U .

Using the sheaf axioms gives a sheaf map $\tau(\psi) : f^{-1}(\mathscr{G}) \mapsto \mathscr{F}$, $\tau : Hom_Y(\mathscr{G}, f_*(\mathscr{F})) \rightarrow Hom_X(f^{-1}(\mathscr{G}), \mathscr{F})$. Note that $\sigma \circ \tau = id$ and $\tau \circ \sigma = id$.
[1]

2.1.29 II.1.19 x g Extending a Sheaf by Zero (important?)

1.19. *Extending a Sheaf by Zero.* Let X be a topological space, let Z be a closed subset, let $i: Z \rightarrow X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j: U \rightarrow X$ be its inclusion.

(a) Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$. Hence we call $i_*\mathscr{F}$ the sheaf obtained by extending $\mathscr{F}$ by zero outside Z. By abuse of notation we will sometimes write $\mathscr{F}$ instead of $i_*\mathscr{F}$, and say "consider $\mathscr{F}$ as a sheaf on X," when we mean "consider $i_*\mathscr{F}$."

If $P \notin Z$, then we can find an open neighborhood V containing P, not intersecting Z.
Thus $(i_*\mathscr{F})(V) = 0$.
If $j : P \hookrightarrow Z$, then $(i_*\mathscr{F})_P = \Gamma(i^*j^*(i_*\mathscr{F})) = \Gamma(j^*\mathscr{F}) = \mathscr{F}_P$.

2.1.30 b. g x extending by zero

(b) Now let $\mathscr{F}$ be a sheaf on U. Let $j_!(\mathscr{F})$ be the sheaf on X associated to the presheaf $V \mapsto \mathscr{F}(V)$ if $V \subseteq U$, $V \mapsto 0$ otherwise. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$. We call $j_!\mathscr{F}$ the sheaf obtained by *extending $\mathscr{F}$ by zero* outside U.

If $P \notin U$, then since the stalk $(j_!(\mathscr{F}))_P$ is index by opens containing P , we see that it is zero.
If $P \in U$, and $(s, V) \in \mathscr{F}_P$, then $P \in V' \subset U$ and on $\mathscr{F}_P$, $(s, V) = (s|_{V'}, V')$.

2.1.31 c. x

(c) Now let $\mathscr{F}$ be a sheaf on X. Show that there is an exact sequence of sheaves on X,

$$0 \rightarrow j_!(\mathscr{F}|_U) \rightarrow \mathscr{F} \rightarrow i_*(\mathscr{F}|_Z) \rightarrow 0.$$

This follows from a,b

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[1] http://sierra.nmsu.edu/morandi/notes/sheafcohomology.pdf

2.1.32 II.1.20 x Subsheaf with Supports

1.20. *Subsheaf with Supports.* Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support (Ex. 1.14) is contained in Z.
(a) Show that the presheaf $V \mapsto \Gamma_{Z \cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is called the subsheaf of $\mathscr{F}$ with supports in Z, and is denoted by $\mathscr{H}^0_Z(\mathscr{F})$.

identity.
Follows since $\mathscr{F}$ is a sheaf.
gluing.
If U is open, let U_i be an open cover.
Suppose that $s_i \in \Gamma_{Z \cap U_i}(U_i, \mathscr{F}|_{U_i})$ satisfies $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$.
As $\mathscr{F}$ is a sheaf there is $s \in \mathscr{F}(U)$ with $s|_{U_i} = s_i$.
Suppose $P \in U \backslash Z$.
Pick i with $P \in U_i$ so that $s|_{U_i} = s_i$ and thus $s_P = (s_i)_P = 0$.
Thus we know s has support inside Z so $s \in \Gamma_{Z \cap U}(U, \mathscr{F}|_U)$.

2.1.33 b. x

(b) Let $U = X - Z$, and let $j: U \to X$ be the inclusion. Show there is an exact sequence of sheaves on X

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U).$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

By (a) and definition of $\mathscr{H}^0_Z(\mathscr{F})$, injectivity on the left is clear.
For open V, $\mathscr{F}(U \cap V) = \mathscr{F}|_U(U \cap V) = \mathscr{F}|_U(j^{-1}(V)) = j_*\mathscr{F}|_U(V)$ so we obtain the second map on open V,
hence we have $\mathscr{F} \to j_*(\mathscr{F}|_U)$.
Note that if $\mathscr{F}$ is flasque, then the restrictions are surjective so we will have surjectivity on the second term.

2.1.34 II.1.21 g sheaf of ideals x

1.21. *Some Examples of Sheaves on Varieties.* Let X be a variety over an algebraically closed field k, as in Ch. I. Let $\mathcal{O}_X$ be the sheaf of regular functions on X (1.0.1).
(a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf. It is called the *sheaf of ideals* $\mathscr{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

glueing
If U_i is an open cover of U and $f_i \in \mathscr{I}_Y(U_i)$ satisfy $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then we can find $f \in \mathcal{O}_X(U)$ such that $f|_{U_i} = f_i$.
For $P \in Y \cap U$ choose i with $P \in U_i$.
$f|_{U_i} = f_i$ so that $f(P) = f_i(P) = 0$ since $f_i \in \mathscr{I}_Y(U_i)$.
So f vanishes at P, and since P was arbitrary, $f \in \mathscr{I}_Y(U)$.
identity

Follows since its a subpressheaf of a sheaf (see Vakil's notes)

2.1.35 b. x g

(b) If Y is a subvariety, then the quotient sheaf $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i: Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

If U is open, $f \in \mathcal{O}_X(U)$, then $f|_{Y\cap U}$ gives a section of $\mathcal{O}_Y(U \cap Y) = i_*(U \cap Y)$.
Define $\varphi : \mathcal{O}_X \to i_*\mathcal{O}_Y$ to be this restriction.
For $P \notin Y$, $(i_*\mathcal{O}_Y)_P$ is zero.
If $P \in Y$, then $g \in (i_*\mathcal{O}_Y)_P$ has no pole at P and thus there is $h \in \mathcal{O}_{X,P}$ with $\varphi_P(h) = g$, φ_P being the induced map on stalks.
Thus φ_P is surjective.
We therefore have an exact sequence $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$.

2.1.36 c. x g

(c) Now let $X = \mathbf{P}^1$, and let Y be the union of two distinct points $P,Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$,

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathscr{F} \to 0.$$

Show however that the induced map on global sections $\Gamma(X,\mathcal{O}_X) \to \Gamma(X,\mathscr{F})$ is not surjective. This shows that the global section functor $\Gamma(X,\cdot)$ is not exact (cf. (Ex. 1.8) which shows that it is left exact).

WLOG assume $P = (0, 1)$ and $Q = (1, 1)$.
On the open set $x_1 \neq 0$, then $P = 0$, $Q = 1$.
By definition of the skyscraper sheaves, $\mathscr{F}(U) = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q(U) = 0$ for $U \not\ni P, Q$, and $\mathscr{I}_Y(U) = \mathcal{O}_X(U)$.
This gives exactness away from P, Q .
We also have $\mathcal{O}_{X,Q} = \left\{\frac{f}{g}|g(1) \neq 0\right\}$, $\mathscr{I}_{Y,Q} = \left\{\frac{f}{g}|g(q) \neq 0, f(1) = 0\right\}$, and the map given by evaluation at 1 gives an exact sequence
$0 \to \mathscr{I}_{Y,Q} \to \mathcal{O}_{X,Q} \to \mathscr{F}_Q \to 0$.
The same thing happens at P.
On global sections however, we have $0 \to 0 \to k \to k \oplus k \to 0$.

2.1.37 d. x g

(d) Again let $X = \mathbf{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P\in X} i_P(I_P)$, where I_P is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf (Ex. 1.17) given by I_P at the point P.

If$f \in \mathcal{O}_X(U)$, then f is given by a system of regular functions f_i on U_i such that $f_i|_{U_i\cap U_j} = f_j|_{U_i\cap U_j}$ and globally f is a rational function such that $f|_{U_i} = f_i$.
The f_i define a section of $\mathscr{K}(U)$.
Given $f \in \mathscr{K}(U)$, then quotient defines $\overline{f} \in \sum i_P(\mathscr{K}/\mathcal{O}_P)$.
On stalks, we have a sequence $0 \to \mathcal{O}_P \to K \to K/\mathcal{O}_P \to 0$ which is clearly exact.

2.1.38 e. x

(e) Finally show that in the case of (d) the sequence

$$0 \to \Gamma(X, \mathcal{O}) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K} \ \mathcal{O}) \to 0$$

is exact. (This is an analogue of what is called the "first Cousin problem" in several complex variables. See Gunning and Rossi [1, p. 248].)

In the case of (d) we have $X = \mathbb{P}^1$.
note that $H^1 = 0$ by cech cohomology.

2.1.39 II.1.22 x g Glueing sheaves (important)

1.22. *Glueing Sheaves.* Let X be a topological space, let $\mathfrak{U} = \{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism $\varphi_{ij}: \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$ such that (1) for each i, $\varphi_{ii} = \mathrm{id}$, and (2) for each i, j, k, $\varphi_{ik} = \varphi_{jk} \ \varphi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i: \mathcal{F}|_{U_i} \xrightarrow{\sim} \mathcal{F}_i$ such that for each i, j, $\psi_j = \varphi_{ij} \ \psi_i$ on $U_i \cap U_j$. We say loosely that $\mathcal{F}$ is obtained by *glueing* the sheaves $\mathcal{F}_i$ via the isomorphisms φ_{ij}.

For $i_j : U_j \hookrightarrow X$, $i_{jk} : U_j \cap U_k \hookrightarrow X$ define we have morphisms induced by restriction:
$i_{j*}(\mathscr{F}_j) \to i_{jk*}(\mathscr{F}_j|U_{jk})$.
The inverse limit of the sheaves in the above morphism has morphisms $\mathscr{F} \to i_{j*}(\mathscr{F}_j)$.
On stalks we have $\mathscr{F}|_{U_j} \to \mathscr{G}_j|_{U_j} = \mathscr{F}_j$
So the idea is to define $\mathscr{F}$ to be the above inverse limit.

2.2 II.2

2.2.1 II.2.1 x

2.1. Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.

As topological spaces, they are clearly the same, as the primes of A_f are the primes of A not containing f. The isomorphism of sheaves is from 2.2. b

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2.2.2 II.2.2 x g induced scheme structure.

2.2. Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme. We call this the *induced scheme structure* on the open set U, and we refer to $(U, \mathcal{O}_X|_U)$ as an *open subscheme* of X.

For a cover $U_i = Spec\ A_i$ of X,
we can intersect U_i's and cover the intersections with affine opens.
By II.2.1, then the affine opens are $Spec\ A_f$.
So it's a scheme.

2.2.3 II.2.3 a. x reduced is stalk local x

2.3. *Reduced Schemes.* A scheme $(X, \mathcal{O}_X)$ is *reduced* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.

$(X, \mathcal{O}_X)$ is reduced $\implies$ $\mathcal{O}_X(U)$ no nilpotents.
For $P \in X$, let (U, s) an element of the stalk.
If (U, s) is nilpotent, then find a neighborhood V of P and n such that $s^n = 0$ on V.
But $\mathcal{O}_X(V)$ no nilpotents, so $(V, s|_V) = (U, s) = 0$.

Suppose each stalk no nilpotents.
If $s \in \mathcal{O}_X(U)$ has $s^n = 0$, $n > 0$, then the germ of s^n is zero at each point in U.
Then the stalk of s must vanish at each such point (since no nilpotents)
But then $s = 0$ by separatedness.

2.2.4 b. reduced scheme x

(b) Let $(X, \mathcal{O}_X)$ be a scheme. Let $(\mathcal{O}_X)_{red}$ be the sheaf associated to the presheaf $U \mapsto \mathcal{O}_X(U)_{red}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathcal{O}_X)_{red})$ is a scheme. We call it the reduced scheme associated to X, and denote it by X_{red}. Show that there is a morphism of schemes $X_{red} \to X$, which is a homeomorphism on the underlying topological spaces.

First suppose $X = Spec\ A$ is affine, and let $A_{red} = A/nil(A)$.
Since $nil(A)$ is the intersection of primes of A, then $sp\ Spec(A) = sp\ Spec\ A_{red}$.
We have a cover of open affines given by $\mathcal{O}_{Spec(A_{red})}(D(f)) \approx (A/nil(A))_f$.
Thus on each basic open affine U, $\mathcal{O}_{Spec(A_{red})}|_U \approx \mathcal{O}_{(Spec\ A)_{red}}|_U$ since localization commutes with quotient.
Thus $Spec(A_{red}) \approx (X, (\mathcal{O}_X)_{red})$ since we have a cover of basic open affines.
If X is a scheme, then cover X with open affine schemes $Spec\ A_i$.
For the morphism $X_{red} \to X$, we can glue the morphisms induced by $A_i \to A_i/nil(A_i)$ from quotienting.

2.2.5 c. x

(c) Let $f: X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g: X \to Y_{red}$ such that f is obtained by composing g with the natural map $Y_{red} \to Y$.

Define $g: X \to Y_{red}$ by taking the continuous map $g = f$ and the sheaf map $\mathcal{O}_{Y_{red}}(U) \to g_*\mathcal{O}_X$ from (b), i.e. induced by the affine ring homomorphisms on global sections to the reduction. These are unique since they factor uniquely through the reduction. Patching together gives $X \to Y_{red} \to Y$.

2.2.6 II.2.4 x

2.4. Let A be a ring and let $(X, \mathcal{O}_X)$ be a scheme. Given a morphism $f: X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^{\#}: \mathcal{O}_{\operatorname{Spec} A} \to f_* \mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map

$$\alpha: \operatorname{Hom}_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\mathfrak{Rng}}(A, \Gamma(X, \mathcal{O}_X)).$$

Show that α is bijective (cf. (I, 3.5) for an analogous statement about varieties).

Let U_i an affine cover of X.
We have by hypothesis $A \to \Gamma(X, \mathcal{O}_X)$, and by restriction $\Gamma(X, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_{U_i})$.
Thus we obtain a map $U_i \approx Spec\ (\Gamma(U_i, \mathcal{O}_{U_i})) \to Spec\ A$.
By glueing we obtain an inverse map $X \to Spec\ A$ to α.

2.2.7 II.2.5 x g

2.5. Describe Spec **Z**, and show that it is a final object for the category of schemes, i.e., each scheme X admits a unique morphism to Spec **Z**.

$Spec\ \mathbb{Z}$ is (0) (open) and (p) closed. (since prime is maximal in commutative pid)
Since rings have unique morphism to $\mathbb{Z}$,
and morphisms are in 1-1 correspondence with ring homomorphisms
(for affine scheme) we're done.

2.2.8 II.2.6 x g

2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

$Spec\ 0$ has no points since there are no prime ideals.
Since there are no points, there is a unique morphism of topological spaces to any X.
Since there is a unique trivial map from 0 to any X, then it's initial.

2.2.9 II.2.7 x g

2.7. Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the *residue field* of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of Spec K to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

given a point + inclusion gives morphism
$i : Spec\ K \to X$, given by identifying the point of $Spec\ K$ with one $x \in X$ gives a continuous morphism.
$i_* \mathcal{O}_{Spec\ K}$ is skyscraper sheaf with ring of sections K.
If U is open, $U \ni x$, define $i^\sharp : \mathcal{O}_X \to i_* \mathcal{O}_{Spec\ K}$ by $\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x) \to K = i_* \mathcal{O}_{Spec\ K}(U)$.
given morphism gives point + inclusion

Clearly $\mathfrak{p} = i((0))$ is the point, and if $\mathfrak{p} \in U = Spec\ A$, then if $\phi : A \to K$ is the corresponding ring homomorphism, then $\mathfrak{p}$ is the kernel of ϕ , so $k(x) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to K$ gives the inclusion.

2.2.10 II.2.8 x g Dual numbers + Zariski Tangent Space

2.8. Let X be a scheme. For any point $x \in X$, we define the *Zariski tangent space* T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the *ring of dual numbers* over k. Show that to give a k-morphism of Spec $k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, *rational over* k (i.e., such that $k(x) = k$), and an element of T_x.

Since the assertion is local, we assume X is affine.
Zariski Tangent Space corresponds to derivations
First we show that $T_x \approx Der(\mathcal{O}_P, k)$, the vector space of derivations.
We have $k \overset{c\mapsto c}{\to} \mathcal{O}_P \overset{f\mapsto f(P)}{\to} k$ is the identity map $\implies \mathcal{O}_P \approx k \oplus \mathfrak{n}_P$, $f \leftrightarrow (f(P), f - f(P))$.
If $D : \mathcal{O}_P \to k$ is a derivation, then D is zero on k and $\mathfrak{n}_P^2$ by the product rule.
Therefore D defines a k-linear map $\mathfrak{n}_P/\mathfrak{n}_P^2 \to k$.
On the other hand if $f : \mathfrak{n}_P/\mathfrak{n}_P^2 \to \mathcal{O}_P$, then $\mathcal{O}_P \overset{f\mapsto(df)_P}{\to} \mathfrak{n}_P/\mathfrak{n}_P^2 \to k$ defines a derivation.
Derivations Correspond to Dual Numbers
If $\alpha : \mathcal{O}_P \to k[X]/(X)^2$ is a local homomorphism of k-algebras, then $\alpha(a) = a_0 + D_\alpha(a)\epsilon$, $\epsilon = X + (X^2)$.
As α is a homomorphism of k-algebras, $a \mapsto a_0$ is the quotient map $\mathcal{O}_P \to \mathcal{O}_P/\mathfrak{m} = k$.
Also $\alpha(ab) = (ab)_0 + D_\alpha(ab)\epsilon$ and $\alpha(a)\alpha(b) = (a_0 + D_\alpha(a)\epsilon)(b_0 + D_\alpha(b)\epsilon) =$
$a_0b_0 + \epsilon(a_0D_\alpha(b) + b_0D_\alpha(a))$ so that D_α satisfies the product rule, and thus gives a derivation $\mathcal{O}_P \to k$.
On the other hand all derivations arise in this manner.
[2]

2.2.11 II.2.9 x g Unique Generic Point (Important)

2.9. If X is a topological space, and Z an irreducible closed subset of X, a *generic point* for Z is a point ζ such that $Z = \{\zeta\}^-$. If X is a scheme, show that every (nonempty) irreducible closed subset has a unique generic point.

First we reduce to the affine case, $Z = Spec\ A$, $A = k[x_1, ..., x_n]/\mathfrak{a}$ a f.g. algebra .
If $U \subset Z$ is open and $\zeta \in U$ with $\{\zeta\}^- = U$ then $\{\zeta\}^- = Z$ since Z is irreducible.
As Z is irreducible, $\implies nil(\mathfrak{a})$ has a unique minimal prime whose closure is X.

2.2.12 II.2.10 x g

2.10. Describe Spec $\mathbf{R}[x]$. How does its topological space compare to the set $\mathbf{R}$? To $\mathbf{C}$?

Irreducibles give points corresponding to prime ideals.
Such are either (0), $(x - a)$ for $a \in \mathbb{R}$, or $(x^2 + ax + b)$ for an irreducible quadratic.
(0) is the generic point and the other types of points are maximal ideals.
The closed sets are finite collections of points.

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[2] Milne, AG

2.2.13 II.2.11 x g (Spec Fp important)

2.11. Let $k = \mathbf{F}_p$ be the finite field with p elements. Describe Spec $k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Spec $k[x] = \{0\} \cup \{(f)\}$, f an irreducible monic polynomial.
The residue field of a point corresponding to one of the f's of degree d is the finite field with p^d elements.
If f is one such, then the isomorphism $\mathbb{F}_p[x]/(f(x)) \approx \mathbb{F}_{p^d}$ gives $\alpha \in \mathbb{F}_{p^n}$.
Conversely, given $\alpha \in \mathbb{F}_{p^n}$ not contained in a subfield, gives a minimal polynomial of degree d, $\prod_{i=0}^{d-1}\left(x - \alpha^{p^i}\right)$
.

Thus we count elements of $\mathbb{F}^{p^d}$ not contained in any subfield, and this number is given by the mobius inversion formula. (see Apostol intro analytic number theory).

2.2.14 II.2.12 x g Glueing Lemma

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2.12. *Glueing Lemma.* Generalize the glueing procedure described in the text (2.3.5) as follows. Let $\{X_i\}$ be a family of schemes (possible infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure (Ex. 2.2). Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij}: U_{ij} \to U_{ji}$ such that (1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and (2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$. Then show that there is a scheme X, together with morphisms $\psi_i: X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}. We say that X is obtained by *glueing* the schemes X_i along the isomorphisms φ_{ij}. An interesting special case is when the family X_i is arbitrary, but the U_{ij} and φ_{ij} are all empty. Then the scheme X is called the *disjoint union* of the X_i, and is denoted $\coprod X_i$.

Define an equivalence relation by $x \sim y$ if $x \in U_{ij} \subset X_i$, $y \in U_{ji} \subset X_j$ and $\phi_{ij}x = y$. Let $X = \coprod X_i/\sim$ with the quotient topology.
Now glue the sheaves $\psi_* \mathcal{O}_{X_i}$ using I.1.22 to get $\mathcal{O}_X$.
Then $(X, \mathcal{O}_X)$ clearly satisfies (1) and (2).
For (3), note that $\psi(U_{ij}) \subset \psi_i(X_i) \cap \psi_j(X_j)$ and conversely that if $x \in \psi_i(X_i) \cap \psi_j(X_j)$, then there are $x_i \in X_i$, $x_j \in X_j$ with $x_i \sim x_j$. Thus $x \in \psi_i U_{ij} \implies \psi(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$.
(4) is similar.

2.2.15 II.2.13 x quasicompact vs noetherian. a

2.13. A topological space is *quasi-compact* if every open cover has a finite subcover.
(a) Show that a topological space is noetherian (I, §1) if and only if every open subset is quasi-compact.

If X is noetherian, then any open $U \subset X$ is noetherian (I.1.7c) and I.7.b gives that U is quasi-compact.
If every open U is quasi-compact, and we have an ascending chain of opens $U_1 \subset U_2 \subset ...$ (descending chain of closeds), then $U = \bigcup U_i$ is covered by a finite subset of U_i.
Then the chain must stabilize.

2.2.16 b. x

(b) If X is an affine scheme, show that sp(X) is quasi-compact, but not in general noetherian. We say a scheme X is *quasi-compact* if sp(X) is.

Let $\{U\}_i$ an open cover. $V_i = X \backslash U_i$ define ideals $I_i \subset \Gamma(\mathcal{O}_X, X)$.
$\bigcup U_i = X \implies 1 = \sum a_i f_i$ with $f_i \in I_i$ and the sum is finite.
non-noetherian affine scheme is $Spec\ k[x_1, x_2, ...]$.

2.2.17 c. x

(c) If A is a noetherian ring, show that sp(Spec A) is a noetherian topological space.

A decreasing sequence of closed subsets corresponds to an increasing sequence of ideals.

2.2.18 d. x

(d) Give an example to show that sp(Spec A) can be noetherian even when A is not.

Find a space with one prime ideal but which has an increasing chain of ideals. (e.g. p-adic integers or $k[x_1, x_2, ...] / (x_1^2, x_2^2, ...)$.

2.2.19 II.2.14 x

2.14. (a) Let S be a graded ring. Show that Proj $S = \varnothing$ if and only if every element of S_+ is nilpotent.

If every element of S_+ is nilpotent then $\mathfrak{p} \supset S_+$ since every prime ideal contains every nilpotent. Thus $Proj\ S = \emptyset$.

If $Proj\ S = \emptyset$, and $s \in S_+$, for any prime ideal $\mathfrak{p} \subset S$, then $\sum_{d \geq 0} \mathfrak{p} \cap S_d \subset \mathfrak{p}$ is prime. Now $D_+(s) = \emptyset$ so all homogeneous prime ideals contain $s \implies s$ nilpotent.

2.2.20 b. x

(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \text{Proj } T | \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of Proj T, and show that φ determines a natural morphism $f : U \to$ Proj S.

Suppose $S_+ \neq 0$. If $\mathfrak{p} \subset U$ is prime, there is $f \in S_+$ with $\varphi(f) \notin \mathfrak{p}$.
Thus there is a homogeneous component f_i with $\varphi(f_i) \notin \mathfrak{p}$.
Thus there is a principal open $\mathfrak{p} \in D_+(\varphi(f_i)) \subset U$.
These principal opens cover U, and since U is a union of opens, U is open in $Proj\ T$.
For the morphism, define $f : p \mapsto \varphi^{-1}(\mathfrak{p})$, $f : U \to Proj\ S$.
This takes closed sets to closed sets, and is thus continuous.
A sheaf morphism is given by $S_{\varphi^{-1}\mathfrak{p}} \to T_{\mathfrak{p}}$.

2.2.21 c. x

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

Let $\mathfrak{p} \in Proj\ T$, $\mathfrak{p} \supset \varphi(S_+)$. If $t \in T_e$, since φ_d is an iso for $d \geq d_0$, we can find $s \in S_{ed_0}$ with $\varphi_{ed_0 s = t^{d_0}}$. Thus $\varphi_{ed_0} s = t^{d_0} \in \mathfrak{p}$ which is prime, thus must contain t. As $\mathfrak{p} \subset T_+$, then $\mathfrak{p}$ can't be contained in $Proj\ T$ by definition of $Proj\ T$ as the set of homogeneoeus prime ideals not containing $T_+ = \oplus_{d>0} T_d$. so it's not in $\varphi(S_+)$.

Next we show f is surjective. If $\mathfrak{p} \in Proj\ S$, let $\mathfrak{q} = \sqrt{\langle \varphi \mathfrak{p} \rangle}$ the radical of the homogeneous ideal generated by $\varphi\mathfrak{p}$ the image of φ. Note that $\varphi^{-1}\mathfrak{q} \supset \mathfrak{p}$. On the other hand if $a \in \varphi^{-1}\mathfrak{q}$, then $\varphi a^n \in \langle \varphi \mathfrak{p} \rangle$ so $\varphi a^n = \sum b_i \varphi s_i$ for $b_i \in T$ and $s_i \in \mathfrak{p}$. For large m, every monomial in b_i is in $T_{\geq d_0}$ so we get an isomorphism $T_d \approx S_d$ for large d. Thus $\left(\sum b_i s_i\right)^m$ is a polynomial in φs_i with coefficients in $c_j \in S$ where $\mathfrak{p}$ lives. Thus $\varphi a^{nm} \in \varphi \mathfrak{p}$ so $a^{nm} \in \mathfrak{p}$ so $a \in \mathfrak{p}$. Thus in total $\varphi^{-1}\mathfrak{q} = \mathfrak{p}$. Using similar reasoning, we can see that $\mathfrak{q}$ is prime so in fact we get f is surjective.

Next we need injective. If $f(\mathfrak{p}) = f(\mathfrak{q})$, then $\varphi^{-1}\mathfrak{p} = \varphi^{-1}\mathfrak{q}$. For $t \in \mathfrak{p}$, for large enough d, then there is $s \in S$ with $\varphi s = t^{d_0}$ by assumption. Then $\varphi s = t^{d_0} \in \mathfrak{q}$ As $\mathfrak{q}$ is prime it contains t thus $\mathfrak{p} \subset \mathfrak{q}$ and by symmetry $\mathfrak{p} = \mathfrak{q}$.

Now consider the induced map on structure sheaves. As $D_+(s) = D_+(s^i)$ cover $Proj\ S$ for $i \geq d_0$, then $f^{-1} D_+(s^i) = D_+(t) \subset Proj\ T$ for some t. We want that $S_{(s^i)} \to T_{(t)}$ is an isomorphism. Let $s = s^i$. If $\frac{f}{s^n} \mapsto 0$, then $0 = t^m \varphi f = \varphi(s^m) \varphi f$ for large m, so $s^m f \in ker\ \varphi$. For large enough powers of $s^m f$, $S_d \to T_d$ is an isomorphism so $s^m f = 0$ so $\frac{f}{s^n} = 0$ so we know $S_{(s)} \to T_{(t)}$ is injective. By taking large degrees we can show easily surjective.

2.2.22 d. x

(d) Let V be a projective variety with homogeneous coordinate ring S (I, §2). Show that $t(V) \cong \operatorname{Proj} S$.

By II.4.10

2.2.23 II.2.15x g (important)

2.15. (a) Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.

Let $P \in t(V)$ closed. Residue field has transcendence degree 0 in an algebraically closed so it's k.

If $P \in t(V)$ has residue field k, but is not closed, then irreducible closed subset Z corresponding to P has dimension $\geq 1 \implies$ tr.deg ≥ 1 contradiction.

2.2.24 b. x

(b) If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

$f^\sharp : \mathcal{O}_Y \to f_* \mathcal{O}_X$ gives a morphism of residue fields $k(f(P)) \to k(P)$.
$k \hookrightarrow k(f(P)) \hookrightarrow k(P) = k$.

2.2.25 c. x

(c) Now show that if V, W are any two varieties over k, then the natural map

$$\mathrm{Hom}_{\mathfrak{Var}}(V,W) \to \mathrm{Hom}_{\mathfrak{Sch}/k}(t(V),t(W))$$

is bijective. (Injectivity is easy. The hard part is to show it is surjective.)

The natural map is given by $\varphi \mapsto \varphi^*$.

By (b) closed points map to closed points, so $\varphi^*(\mathfrak{p}) = \varphi(\mathfrak{p})$.

If Y is an irreducible subvariety, $\varphi^*(Y) = \varphi(Y)$.

The maps on schemes over k are extensions of $\varphi : V \to W$ and so we get injectivity.

For surjectivity, if $\varphi^* : t(v) \to t(W)$, then φ^* takes closed points to closed points, and thus define $\varphi := \varphi^*|_V$.

For regularity of φ , let $\varphi(P) = Q$ and choose $U = \mathit{Spec}\ A \ni \mathfrak{p}$.

Then $\mathfrak{p} \in U' = \mathit{spec}\ A' \subset f^{-1}(U)$.

Thus $f|_{U'} : \mathit{Spec}\ A' \to \mathit{Spec}\ A$ is induced by the map on rings, and this gives regularity.

2.2.26 II.2.16 x

2.16. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

(a) If $U = \mathrm{Spec}\, B$ is an open *affine* subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

We trivially have $D(\overline{f}) = U \cap X_f$ so $U \cap X_j$ is open in U .

($D(\overline{f}) = \{x \in U : \overline{f} \notin x\} = \{x \in U : \overline{f}_x \notin \mathfrak{m}_x\}$)

Furthermore, an affine open cover $\{U_i\}$ of X , we have $X_f = \bigcup_i (U_i \cap X_f)$ is the union of opens.

2.2.27 b. x

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$. [*Hint*: Use an open affine cover of X.]

By quasi-compactness, find an finite open cover of $U_i = \mathit{Spec}\ A_i$.

Then $a|_{U_i \cap X_f} = a|_{\mathit{Spec}(A_i)_f}$ is zero for ecah i.

Thus $f^{n_i} a = 0$ in A_i for some n_i (by theorem).

For a large enough n, then $f^n a = 0$ in each $\mathit{Spec}\ A_i$.

By the sheaf axioms, then $f^n a = 0$.

2.2.28 c. x

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. (This hypothesis is satisfied, for example, if $\mathrm{sp}(X)$ is noetherian.) Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

If $U_i = Spec\ A_i$ $b|_{X_f \cap U_i} = \frac{b_i}{f^N}$ for each i.
On the overlaps, $b_i - b_j|_{U_i \cap U_j}$ vanishes $\implies f^M (b_i - b_j) = 0$.
Using the sheaf axiom, lift $f^M b_i$ on U_i to a global section s on X.
$s - f^{N+M} b$ restricts to $f^M b_i - f^M b_i = 0$ on $U_i \cap X_f$
$\implies f^{n+M} b$ is the restriction of a global section.

2.2.29 d. x

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Let $\varphi : A_j \to \Gamma\left(X_f, \mathcal{O}_{X_f}\right)$ be $\varphi : a/f^n \mapsto \frac{a|_{X_f}}{f^n|_n}$.
The kernel is trivial since $a/f^n = 0$ implies it's zero on A_f .
For a section s on X_f , $f^m s$ is the restriction of a global section.
This gives surjectivity.

2.2.30 II.2.17 Criterion for affineness x

2.17. *A Criterion for Affineness.*
(a) Let $f: X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

The condition $f^{-1}(U_i) \approx U_i$ implies for open $V \subset X$, $f(V) = \bigcup f\left(V \cap f^{-1}(U_i)\right)$ is open.
$\implies f$ is homeo.
Since any $p \in U_i$ some i , the map on stalks is an iso.
Together we have a homeomorphism and an isomorphism and gluing along double intersections of the U_i will give an isomorphism of schemes.

2.2.31 b. x

(b) A scheme X is affine if and only if there is a finite set of elements $f_1, \ldots, f_r \in A = \Gamma(X, \mathcal{O}_X)$, such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A. [*Hint:* Use (Ex. 2.4) and (Ex. 2.16d) above.]

setup
If A is affine, we let $f_1 = 1$.
Conversely, suppose $f_1, ..., f_r$ generate the unit ideal.
Since f_i generate A, $D(f_i)$ cover $Spec\ A$.
We need to show that $D(f_i) \approx Spec\ A_f$ is isomorphic to $X_f \approx Spec\ A_i$ so that by (a), $X \to Spec\ A$ is an isomorphism.
Consider $\varphi_i : \Gamma(X, \mathcal{O}_X)_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_X)$.
injective
If $\frac{a}{f_i^n} \in ker\varphi_i$, then $\frac{a}{f_i^n} = 0$ on $Spec\ (A_j)_{f_i} = X_{f_i} \cap X_{f_j}$ in the domain.
Hence $f^{n_j} a = 0$ in A_j for some n_j , and for a large N, $f_i^N a = 0 = \frac{a}{f^n}$.
surjective
If $a \in im(\varphi_i)$, then as $\mathcal{O}_X\left(X_{f_i f_j}\right) = (A_j)_{f_i} \implies a|_{X_{f_j f_i}} = \frac{b_j}{f_i^{n_j}}$ for $b_j \in A_j$.
Choose $N \gg n_j$ for all n_j so on $X_{f_i f_j f_k}$ we have $b_j - b_k = f_i^N a - f_i^N a = 0$.
Thus $f_i^{m_{jk}} (b_j - b_k) = 0$ on $X_{f_j f_k}$. For $M \gg m_{jk}$ all j, k , there is thus $f_i^M b_j$ in X_{f_j} agreeing with $f_i^{N+M} a$ on X_{f_i} .

Using the global generation lemma, gives a global section d restricting to $f_i^{N+M}a$ on X_{f_i} . Then $\frac{d}{f^{N+M}}$ maps to a by φ_i .

2.2.32 II.2.18 g x

2.18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

Since the nilradical is the intersection of the prime ideals.
If f is nilpotent, then it's in all the prime ideals so it vanishes at every $\mathfrak{p}$.

2.2.33 b. g x

(b) Let $\varphi: A \to B$ be a homomorphism of rings, and let $f: Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e., $f(Y)$ is dense in X.

Injective
If the sheaf map is injective then $A \approx \Gamma(X, \mathcal{O}_X) \to \Gamma(X, f_*\mathcal{O}_Y) \approx B$ is injective.
If $A \hookrightarrow B$ is injective, then for $\mathfrak{p} \in Spec\ A$, $(f_*\mathcal{O}_{Spec\ B})_{\mathfrak{p}} \approx B \otimes_A A_{\mathfrak{p}}$ as it is the colimit of $\mathcal{O}_{Spec\ B}$ evaluated on $D(a)$, $a \notin \mathfrak{p}$.
Thus we have an injective morphism $A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}$.
Dominant
The largest open set not intersecting the image is covered by $D(f)$ with $f \in \phi^{-1}\mathfrak{p}$, $\mathfrak{p} \in Spec\ B$.
For each f , $\phi f \in \mathfrak{p}$ for all $\mathfrak{p} \in Spec\ B$ so ϕf is in their intersection and is thus nilpotent.
Injectivity of ϕ implies f is nilpotent and thus $D(f)$ is empty.

2.2.34 c. x g (important)

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

By ring theory, primes of A containing $I = ker\ \varphi$ correspond to primes of A/I.
Now $D(f)$ pulls back to $D(f+I)$ in $Spec(A/I)$.
This shows that the map is open, since we have a distinguished base.
The map is clearly continuous, and checking the stalk gives surjectivity.

2.2.35 d. g x

(d) Prove the converse to (c), namely, if $f: Y \to X$ is a homeomorphism onto a closed subset, and $f^{\#}: \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then φ is surjective. [*Hint:* Consider $X' = \operatorname{Spec}(A/\ker \varphi)$ and use (b) and (c).]

We have $f^{\#}$ is surjective on each stalk so if $b \in B$, there is $\frac{a_i}{f_i^{n_i}} \in A_{f_i}$ on a principle open set $D(f_i) \ni \mathfrak{p}_i$ mapping to the germ of b at each $\mathfrak{p}_i \in Spec\ A$.

By quasi-compactness, $Spec\ A = \bigcup_{i=1}^{n} D(f_i)$ so $1 = \sum g_i f_i^N$, $g_i \in A$ and $b = \sum g_i f_i^N b = \sum g_i f_i^{N'} a_i \in im\ \varphi$.

2.2.36 II.2.19 x g

2.19. Let A be a ring. Show that the following conditions are equivalent:

(i) Spec A is disconnected;
(ii) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$ (these elements are called *orthogonal idempotents*);
(iii) A is isomorphic to a direct product $A_1 \times A_2$ of two nonzero rings.

(i) $\implies$ (iii) Suppose $Spec\ A = U \coprod V$, for two closed sets $U = Spec\ A/I$, $V = spec\ A/J$.
Then $Spec\ A + Spec\ A/I \times A/J$ so $A = A/I \times A/J$.
(iii) $\implies$ (ii) for $e_1 = (1,0)$, $e_2 = (0,1)$.
(ii) $\implies$ (i) For any $\mathfrak{p}$, $e_1 e_2 = 0$ so either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$.
Then $V((e_i)), V((e_2))$ covers $Spec\ A$.
If $\mathfrak{p} \in V((e_1)) \cap V((e_2))$ then $1 = e_1 + e_2 \in \mathfrak{p} \implies \mathfrak{p} = A \implies Spec\ A = V((e_1)) \coprod V((e_2))$.

2.3 II.3 x

2.3.1 II.3.1 x

3.1. Show that a morphism $f: X \to Y$ is locally of finite type if and only if for *every* open affine subset $V = \text{Spec } B$ of Y, $f^{-1}(V)$ can be covered by open affine subsets $U_j = \text{Spec } A_j$, where each A_j is a finitely generated B-algebra.

Suppose $f: X \to Y$ is locally of finite type.

Then there is $V_i = Spec\ B_i$ a covering of Y by open affine subschemes such that $f^{-1}V_i$ is covered by open affines $Spec\ A_{ij}$, each A_{ij} an f.g. B_i-algebra.

The intersections $V_i \cap B$ are open in B_i, as well as the union of the open sets $Spec\ (B_i)_{f_{ij}}$.

If f_{ij} is considered as an element of A_{ij} under $B_i \to A_{ij}$, then $\varphi^{-1} Spec\ (B_i)_{f_{ij}} = Spec\ (A_{ij})_{f_{ik}}$, and thus each $(A_{ij})_{f_{ik}}$ is an f.g. $(B_i)_{f_{ik}}$-algebra.

Now cover $Spec\ B$ with open affines $Spec\ C_i$ whose preimages are covered with open affines D_{ij}, each D_{ij} an f.g. C_i-algebra.

For $\mathfrak{p} \in Spec\ B$, $\mathfrak{p} \in Spec\ C_i$ for some i. So $\mathfrak{p} \in B_{g_\mathfrak{p}} \subset Spec\ C_i$.

If $g_\mathfrak{p}$is associated with its image under $B \to C_i \to D_{ij}$, then $Spec\,(C_i)_{g_\mathfrak{p}} \approx Spec\ B_{g_\mathfrak{p}}$, and taking the preimage gives $Spec\ (D_{ij})_{g_\mathfrak{p}}$.

$(D_{ij})_{g_\mathfrak{p}}$ is an f.g. $B_{g_\mathfrak{p}}$-algebra, hence an $f.g.$ B-algebra, and the $Spec\ (D_{ij})_{g_\mathfrak{p}}$ cover $Spec\ B$.

2.3.2 II.3.2 x

3.2. A morphism $f: X \to Y$ of schemes is *quasi-compact* if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for *every* open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

First note that if a topological space has a finite cover made up of q.c. open sets, then it is q.c. For if U_i is an open cover with each U_i q.c. and V_j is a cover for X, then $V_j \cap U_i$ is an open cover of U_i which has a finite subcover. ...

Now suppose f is quasi-compact, $Spec\ B_i$ is an open affine cover of Y and $f^{-1}Spec\ B_i$ is q.c.

If $Spec\ V \subset Y$ is an arbitrary open affine, then each intersection $Spec\ B_i \cap Spec\ C$ is covered by basic open affines for $Spec\ B_i$.

Then $Spec\ B_i$ cover X so also $Spec\ C$.

By II.2.13.b, there exists a finite subcover of $Spec\ C$, of the for $D(b_k)$, $b_k \in B_{i_k}$.

By quasicompactness of f, cover each $f^{-1}\ Spec\ B_i$ with a finite number of $Spec\ A_{ij}$. The preimage of $D(b_k)$ in $Spec\ A_{i_k j}$ is $Spec\ (A_{i_k j})_{b_k}$ and the union of these is a cover of $f^{-1} Spec\ C$ by open affines. By II.2.13b, the open affines are quasi-compact, so by the first paragraph, we are done.

2.3.3 II.3.3 x

> **3.3.** (a) Show that a morphism $f: X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

If f is ft and qc, then by definition we can cover Y with $Spec\ B_i$ and $f^{-1}(Spec\ B_i)$ is covered by a finite number of $Spec\ A_{ij}$. By a previous excercise, the $Spec\ A_{ij}$ are qc. Combining quasicompact covers, gives qc on the whole space.

2.3.4 b. x

> (b) Conclude from this that f is of finite type if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

By the previous exercises in this section.

2.3.5 c. x

> (c) Show also if f is of finite type, then for *every* open affine subset $V = \operatorname{Spec} B \subseteq Y$, and for *every* open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Suppose finite type, cover $f^{-1}(V)$ with $U_i = Spec\ A_i$, A_i an f.g. B -algebra.

Using quasicompactness, let $Spec\ (A_i)_{g_i}$ a finite cover of U by principal opens basic in U_i. Each A_i is an $f-g$ B algebra, $\implies$ $(A_i)_{g_i} = A_{f_i}$ some f_i is an f.g. B-algebra $\implies$ A is a finitely generated B-algebra.

2.3.6 II.3.4 x

> **3.4.** Show that a morphism $f: X \to Y$ is finite if and only if for *every* open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

Suppose f is finite. Let $U = f^{-1}V = f^{-1}Spec\ B$

There is at least one affine cover by $V_i = Spec\ B_i$ of Y such that each preimage $f^{-1}V_i = U_i = Spec\ A_i$ is affine with each A_i an f.g. B_i-module.

We can cover the intersections $U \cap U_i$ with distinguished opens $D(f_{ij}) = (B_i)_{f_{ij}}$, and the preimage of $D(f_{ij}) = Spec\,(A_i)_{f_{ij}}$, f_{ij} is associated with its image in A_i .

Since A_i is an f.g. B_i-module, then $(A_i)_{f_{ij}}$ is an f.g. $(B_i)_{f_{ij}}$ -module by basic localization properties.

We have a cover of V by principal open affines $Spec\ B_{g_i}$ with preimages $Spec\ C_i$, C_i an f.g. B_i-module.

By exc. II.2.17, since $Spec\ B$ is affine, by exc 2.2.13b, it is q.c.

Thus there are a finite number of $Spec\ B_{g_i}$ covering V.
Thus $\sum g_i r_i = 1$ so the image in $\Gamma(U, \mathcal{O}_U)$ generates the unit ideal.
$f^{-1} Spec\ B_{g_i} = U_{f(g_i)}$, so by affine criterion, $U = Spec\ A$ some f.g. B-algebra.
Note that if $f_1, ..., f_n \in B$ generate the unit ideal and A_{f_i} is an f.g. B_{f_i}-module for each i , then A is actually finitely generated as a B-module.

2.3.7 II.3.5 x g

3.5. A morphism $f : X \to Y$ is *quasi-finite* if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.
(a) Show that a finite morphism is quasi-finite.

Let $\mathfrak{p}$ a point in Y , we want to show the preimage is a finite number of prime ideals.
Since the assertion is local, by finiteness we assume $X = Spec\ A$, $Y = Spec\ B$, A is an f.g. B-module.
$A \otimes_B k(\mathfrak{p})$ is an f.g. $k(p)$-module, and a field-module is a vector space.
A vector space is artinian, so there are a finite number of prime ideals in $A \otimes_B k(\mathfrak{p})$.

2.3.8 b. x g

(b) Show that a finite morphism is *closed*, i.e., the image of any closed subset is closed.

A subset of a topological space is closed iff it is closed in every element of an open cover.
Thus we assume $X = spec\ A$, $Y = spec\ B$, with A an f.g. B-module.
$f(X)$ is closed if the complement is open.
Thus we want to show if $y \in f(X)^c$, then there is $g \in k[Y]$ with $g(y) = 1$ and $f(X) \subset Z(g)$.
Let $A = k[Y]$,$B = k[X]$, $\mathfrak{m}$ the maximal ideal of A correspnoding to y.
The nullstellensatz gives since $y \notin f(X)$, then $f^*(\mathfrak{m}) B = B$ so by Nakayama, $k[X]$ annihilates $f^*(g)$.

2.3.9 c. x

(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

You can check that $Spec\ k[t, t^{-1}] \oplus k\left[t, (t-1)^{-1}\right] \to Spec\ k[t]$ is finite-type, finite fibers, surjective but not module-finite.

2.3.10 II.3.6 x g Function Field

3.6. Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. It is called the *function field* of X, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

If $U = Spec\ A$ is an open affine subset of X, then by definition, A is an integral domain so (0) is a prime ideal.
$V(I)$ contains (0) iff (0) contains I so the closure of (0) is $V((0)) = Spec\ A$.
By uniqueness, (0) is the generic point η of X .
$\mathcal{O}_X(U)_{(0)} = \mathcal{O}_\eta$ is the fraction field of $\mathcal{O}_X(U)$.

2.3.11 II.3.7 x

3.7. A morphism $f: X \to Y$, with Y irreducible, is *generically finite* if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f: X \to Y$ is *dominant* if $f(X)$ is dense in Y. Now let $f: X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite. [*Hint*: First show that the function field of X is a finite field extension of the function field of Y.]

First we show that $k(X)$ is a finite field extension of $k(Y)$ as in the hint.

Choose an open affine $Spec\ B = V \subset Y$, and an open affine $Spec\ A = U \subset f^{-1}V$ such that A is an f.g. B-algebra.

Since X is irreducible, so is U, so A is integral.

Since A is finitely generated over B, so is $k(B) \otimes_B A \approx B^{-1}A$.

Noether normalization, says there is an integer n such that $B^{-1}A$ is finite over $k(B)[t_1, ..., t_n]$.

Since $B^{-1}A$ is integral over $k(B)[t_1, ..., t_n]$, the induced morphism of affine schemes is surjective.

By going up, $Spec\ B^{-1}A \to Spec\ k(B)[t_1, ..., t_n]$ is surjective so that $n = 0$ and $B^{-1}A$ is integral over $k(B)$.

Finite-type gives that $B^{-1}A$ is finite over $k(B)$. Clearing denominators, $k(B^{-1}A) = k(A)$ is finite over $k(B)$.

Now we show for the affine case and leave the patching.

Let $X = Spec\ A$, $Y = Spec\ B$ assume $\{a_i\}$ generate A over B.

As elements of $k(A)$, the a_i satisfy $f_i(a_i) = 0$ for $f_i \in k(B)$, since $k(A)$ is finite over $k(B)$.

Clearing denominators of f_i gives g_i with coefficients in B.

If b is the product of the leading coefficients of these polynomials, then image of g_i in B_b, A_b are monic.

So A_b is f.g. over B_b Hence A_b is integral over B_b , and is thus an f.g. B_b-module. Then $U = D(b)$

2.3.12 II.3.8 x Normalization

3.8. *Normalization.* A scheme is *normal* if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \text{Spec } A$ of X, let $\tilde{A}$ be the integral closure of A in its quotient field, and let $\tilde{U} = \text{Spec } \tilde{A}$. Show that one can glue the schemes $\tilde{U}$ to obtain a normal integral scheme $\tilde{X}$, called the *normalization* of X. Show also that there is a morphism $\tilde{X} \to X$ having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f: Z \to X$, f factors uniquely through $\tilde{X}$. If X is of finite type over a field k, then the morphism $\tilde{X} \to X$ is a finite morphism. This generalizes (I, Ex. 3.17).

Normalization

Let U and V two open affine subschemes of X.

Let $\tilde{U} = Spec\ \tilde{A}$, $\tilde{V} = Spec\ \tilde{B}$.

In order to glue, we must find an isomorphism $\varphi : U' \to V'$ where U' is the inverse image of $U \cap V$ in $\tilde{U}$ and V' is the inverse image of $U \cap V$ in $\tilde{V}$.

Assume WLOG that U, V are open affines on some common affine scheme $W = Spec\ C$, $A = C_f$, $B = C_g$, with $f, g \in C$.

By localizing minimal polynomials we find, $\tilde{A}_f$ is integral over A_f .

If u belongs to integral closure of A_f , then u is root of monic polynomial h with coefficients in A_f . Clearing denominators, $f^l u \in \tilde{A}$ for some l.

Thus if $\tilde{A}$ is the integral closure of A , then $\tilde{A}_f$ is the integral closure of A_f .

Thus we can glue $\tilde{U}$ to get a scheme $\tilde{X}$, and the inclusions $A \hookrightarrow \tilde{A}$ induce $\tilde{U} \to U \subset X$ so there is an induced morphism $\tilde{X} \to X$.

Dominant Morphism / factors uniquely

If there is a dominant morphism from a normal scheme$Z \to X$, then for open affine U, and preimage Z_U we have a dominant morphism $Z_U \to U$.

Assume WLOG X, Z are affine.

We want to show that if $f : A \hookrightarrow \tilde{A}$ and and $g : A \to \tilde{B}$ is any ring homomorphism, then we have a morphism $\tilde{A} \to \tilde{B}$.

We have a morphism $\tilde{A} \to frac\left(\tilde{B}\right)$, and as an element of the image is integral over $im\,(A)$, then it is integral over $\tilde{B}$.

Thus $im\left(\tilde{A}\right)$ lies in $\tilde{B}$ as $\tilde{B}$ is integrally closed.

X of finite type

If X is finite type, then we want to show the morphism is finite. But the integral closure of a finitely generated k-algebra A is a finitely generated A-module.

2.3.13 II.3.9 x g Topological Space of a Product

3.9. *The Topological Space of a Product.* Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology (I, Ex. 1.4). Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set.

(a) Let k be a field, and let $\mathbf{A}_k^1 = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbf{A}_k^1 \times_{\operatorname{Spec} k} \mathbf{A}_k^1 \cong \mathbf{A}_k^2$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

If $A^1 \times A^1 \approx Spec\ k\,[x] \otimes k\,[x] \approx Spec\ k\,[x, y]$.

Note that $\mathfrak{p} = (x - y) \in sp\ Spec\ k\,[x, y]$ is sent to (0) by the projections, but $(0) \neq (x, y)$.

On the other hand, $((f)\,, (g)) \in sp\ Spec\ k\,[x] \times sp\ Spec\ k\,[y]$ maps to (f) and (g) by the projections.

2.3.14 b. g x

(b) Let k be a field, let s and t be indeterminates over k. Then $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$.

Note that $k\,(s) \times k\,(t)$ is $S^{-1}k\,[s, t]$ where S is generated by products of irreducible polynomials in s and irreducible polynomials in t.

Thus, elements of $k\,(s) \otimes k\,(t)$ are written as
$\frac{1}{c(s) \otimes d(t)} \left(\sum a_i\,(s) \otimes b_i\,(t)\right)$ for $a_i, c \in k\,[x]$ and $b_i, d \in k\,[t]$.
In other words, the holomorphic functions with poles along horizontal and vertical lines.

Points of $Spec\ k\,(s) \otimes_k k\,(t)$ are points of $Spec\ k\,[s, t]$ that aren't in the preimage of the projections (i.e. poles along horizontal and vertical lines).

Now take the induced structure sheaf.

2.3.15 II.3.10 x g Fibres of a morphism

3.10. *Fibres of a Morphism.*
(a) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\mathrm{sp}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

Note that $X_y \approx X \times_Y Spec\ k(y) \approx f^{-1}(V) \times_{Spec\ A} Spec\ k(y)$ where $y \in V = Spec\ A \subset Y$.
If $f^{-1}(V) = \bigcup Spec\ B_i$, then $f^{-1}(V) \times_{Spec\ A} Spec\ k(y) \approx \bigcup Spec\,(B_i \otimes_A k(y))$. $\star$
Now if $y = \mathfrak{p} \in Spec\ A \implies$
$Spec\left(B \otimes_A (A/\mathfrak{p})_{\mathfrak{p}}\right) \approx Spec\ \ (B_{\mathfrak{p}} \otimes_A A/\mathfrak{p}) \approx Spec\ \ (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}})$.
$B_{\mathfrak{p}} \approx \left\{\frac{b}{d} | d \notin f(p), d \in f(A)\right\} \implies Spec\ B_p = \{\mathfrak{q} \in Spec\ B | f^{-1}(\mathfrak{q}) \subset \mathfrak{p}\}$.
Hence $Spec\ \ (B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}) \approx \{\mathfrak{q} \in Spec\ B | f^{-1}(y) \subset \mathfrak{p}, \mathfrak{q} \supset f(\mathfrak{p})\} = f^{-1}(\mathfrak{p})$.
By $\star$ we have $sp\ X_y \approx f^{-1}(\mathfrak{p})$.

2.3.16 b. x

(b) Let $X = \operatorname{Spec} k[s,t]/(s - t^2)$, let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed.)

a is nonzero
$X_y = X_a \approx Spec\ k[s,t]/(s - t^2) \times_{Spec\ k[x]} Spec\ k(a) \approx$
$\approx Spec\ \ (k[s,t]/(s - t^2) \otimes k[s]/(s - a)) \approx Spec\ k[s,t]/(s - t^2, s - a)$.
$s = t^2 = a$ so that the elements are $a_0 + a_1 t$.
As $t^2 = a$, if $a \neq 0$, then we have $k[s,t]/(s - t^2, s - a) \approx k \oplus k$ by
$(1,0) \leftrightarrow \frac{1}{2\sqrt{a}}t + \frac{1}{2}$ and $(0,1) \leftrightarrow -\frac{1}{2\sqrt{2}}t + \frac{1}{2}$.
Note that $k \oplus k$ has two points. Each point has residue field k.
a is zero
If $a = 0$, then $k[s,t]/(s - t^2, s - a) \approx k[t]/(t^2)$ which is the dual numbers.
generic point.
$X_\eta \approx Spec\ k[s,t]/(s - t^2) \otimes_k k(s) \approx Spec\ k(s)[t]/(s - t^2)$.
$k(s)[t]/(s - t^2)$ is a field and $s - t^2$ has degree 2 in t.

2.3.17 II.3.11 x g Closed subschemes

3.11. *Closed Subschemes.*
(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

Denote $Y' = Y \times_X X'$, and let $g : X' \to X$ be any morphism.
First replace X' with an affine open neighborhood U' of $f'(Y')$ by basic closed immersion conditions. Similarly, assume $U' \subset g^{-1}(U)$, where $U \subset Y$ is an affine open.
If $U' = Spec\ A'$ and $U = Spec\ A$, then since f is a closed immersion, $f^{-1}(U) \approx Spec\ A/I$.
Thus $f'^{-1}(U') \approx Spec\ \ (A' \otimes_A A/I) \approx Spec\ \ (A'/IA')$ so that $f' : Y' \to X'$ is a closed immersion.

2.3.18 b. (Starred)

*(b) If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$. [*Hints*: First show that Y can be covered by a finite number of open affine subsets of the form $D(f_i) \cap Y$, with $f_i \in A$. By adding some more f_i with $D(f_i) \cap Y = \varnothing$, if necessary, show that we may assume that the $D(f_i)$ cover X. Next show that $f_1, \ldots, f_r$ generate the unit ideal of A. Then use (Ex. 2.17b) to show that Y is affine, and (Ex. 2.18d) to show that Y comes from an ideal $\mathfrak{a} \subseteq A$.] *Note*: We will give another proof of this result using sheaves of ideals later (5.10).

2.3.19 c. x

(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

Suppose firstX, Y are affine. Let $f : Y' \to X$ a closed immersion.

As a map on topological spaces, $f : Y' \to Y \to X$ gives $sp\,(Y)' \approx sp\,(Y) \approx sp\,(V\,(\mathfrak{a})) \subset sp\,(X)$.

For an open $U \subset V\,(\mathfrak{a}) \subset X$, since $Y = V\,(\mathfrak{a})$, U open in Y, then $\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_{Y'}$ extends to $\mathcal{O}_X \twoheadrightarrow f_*\mathcal{O}_{Y'} \twoheadrightarrow f_*\mathcal{O}_Y$.

If X, Y are not affine, then glue together open affines to achieve the result.

2.3.20 d. x g scheme-theoretic image

(d) Let $f: Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the *scheme-theoretic image* of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Suppose Z reduced. By (c), f factors through the reduced induced structure on $\overline{f\,(Z)}$. If Z is non-reduced, we have a factorization $Z \to Z_{red} \to X$, and the scheme-theoretic image is given by the the closure of image of Z_{red}.

2.3.21 II.3.12 x Closed subschemes of Proj S

3.12. *Closed Subschemes of* Proj S.

(a) Let $\varphi: S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Ex. 2.14) is equal to Proj T, and the morphism $f: \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.

Clearly $\varphi\,(S_+) = T_+$ so that $\{\mathfrak{p} \in Proj\ Y | \mathfrak{p} \not\supset \varphi\,(S_+)\} = U = Proj\ T$.

We have $T \approx S/ker\ \varphi$ and homogeneoeus prime ideals of $S/ker\ \varphi$ correspond to homogeneoeus ideals of S which contain $ker\ \varphi$ hence $f\,(Proj\ T) = f\,(Proj\ S/ker\ \varphi) \approx V\,(ker\ \varphi)$.

On stalks, we have $S_{(\varphi^{-1}(x))} \to T_{(x)}$ induced by φ which is surjective since φ is . Thus the sheaf homomorphism is surjective.

By closedness and surjectivity, we have a closed immersion.

2.3.22 b. x

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as image of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

We will see later (5.16) that every closed subscheme of X comes from a homogeneous ideal I of S (at least in the case where S is a polynomial ring over S_0).

Let $\varphi : S/I' \to S/I = (S/I') / \oplus_{i=1}^{d_0} I_i$ the natural projection homomorphism.

φ is a graded homomorphism of graded rings, with φ_d the identity for $d \geq d_0$.

By exc. II.2.14c, φ induces an isomorphism $f : Proj\ S/I \to Proj\ S/I'$.

2.3.23 II.3.13 x g Properties of Morphisms of Finite Type

3.13. *Properties of Morphisms of Finite Type.*

(a) A closed immersion is a morphism of finite type.

Suppose $f : Y \to X$ is a closed immersion.

Let $U_i = Spec\ A_i$ be an open affine cover of X.

Then $f^{-1}U_i \to U_i$ is a closed immersion, and by exc. II.3.11.b., $f^{-1}U_i = Spec\ B_i$ for some finitely generated algebra B_i.

We have a surjection $(A_i)_{\mathfrak{p}} \to (B_i)_{f^{-1}(\mathfrak{p})}$ on all the localizatons at prime ideals, thus $A_i \to B_i$ is surjective (See Liu chapter 1). Hence each B_i is an f.g. A_i-module.

2.3.24 b. x g

(b) A quasi-compact open immersion (Ex. 3.2) is of finite type.

Let $i : U \to X$ a q.c. open immersion.

Let $Spec\ A_i$ an open affine cover of X.

i restricts to open immersions $U_i \to Spec\ A$.

Each U_i is covered by basic open affines $D(f_{ij}) \approx Spec\ (A_i)_{f_{ij}}$.

Each $(A_i)_{f_{ij}}$ is a finitely generated A_i algebra.

Thus i is locally of finite type

By exc II.3.3.a, i is finite type.

2.3.25 c. x

(c) A composition of two morphisms of finite type is of finite type.

This is follows from the definitions.

2.3.26 d. x

(d) Morphisms of finite type are stable under base extension.

Suppose that $f : Y \to X$ is finite type we want to show that the projection $X' \times_X Y \to X'$ is finite type.

- If X', X, Y are affine, then $A \otimes_C B$ is an f.g. B-algebra if A is an $f.g.$ C-algebra.
- If X', X are both affine, then any finite open affine cover $U_i \subset Y$ gives a finite open affine cover $U_i \times_X X'$ of $Y \times_X X'$, so by the first case we finish.
- If X is affine, then let V_i an open affine cover of Y' . Each $V_i \times_X Y$ is finite type over V_i . Since they cover X' , and $V_i \times_X Y$ is the preimage of V_i then $X' \times_X Y$ is finite type over X' .
- If X is covered by open affines $U_i = Spec\ A_i$ and $g : X' \to X$, then $g^{-1}U_i \times_{U_i} f^{-1}U_i$ is finite type over $g^{-1}U_i$ by previous case. Thus $g^{-1}U_i \times_X Y \to g^{-1}U_i$ is finite type. So f' is finite type on an open cover of X' and thus is finite type.

2.3.27 e. x

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.

$X \times_S Y \to S$ can be factored $X \times_S Y \to Y \to S$.
This is finite type since $X \to S$ is finite type, and thus by base extension (d), $X \times_S Y \to Y$ is finite type.
The second map is finite type by assmption.
By (c) the composition is finite type.

2.3.28 f. x

(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and if f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.

Pick $C \subset Z$, $Spec\ B \subset g^{-1}(Spec\ C)$, $X \supset Spec\ A \subset f^{-1}(Spec\ B)$ nonempty open sets.
By exc 3.3c, as $Spec\ A \subset h^{-1}(Spec\ C)$, then A is an f.g. C-algebra, and there is a morphism $C \to B \to A$

.

If $\{a_i\}_{i=1}^n$ generate A as a $C-$-algebra, then $C[x_1, ..., x_n] \twoheadrightarrow A$.
Since this map factors through $B[x_1, ..., x_n]$, then $B[x_1, ..., x_n] \twoheadrightarrow A$ is surjective.
Thus A is an f.g. B -algebra.

Now use quasicompactness on a cover $Spec\ C_i$ of Z.

2.3.29 g. x

(g) If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.

For $V_i = Spec\ B_i$ a finite affine cover of Y, by finite type hypothesis, there are $U_{ij} = Spec\ A_{ij}$ a finite cover of $f^{-1}V_i$ each A_{ij} a finitely gnerated B_i-algebra.
Y noetherian $\implies$ Each B_i is noetherian $\implies$ by Hilbert basis, A_{ij} are noetherian $\implies$ X is locally noetherian.

Consider a finite open affine cover $\{U_i\}$ of X.
f finite type $\implies$ q.c. by exc I.3.3.a.
Thus $f^{-1}U_i$ is q.c. exc I.3.2.
If $\{V_j\}$ is an open cover of Y , taking preimages gives an open cover of $f^{-1}U_i$ for each i.
Each such $f^{-1}U_i$ is q.c., so there is a finite cover.
The union of the subcovers is finite and is still a cover.
Thus X is q.c., hence noetherian.

2.3.30 II.3.14 x g

3.14. If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.

We must show every open set in a basis contains a closed point.
Every affine open set contains a closed point, since it's an f.g. algebra.
A closed point is closed in the whole subscheme since closed points correspond to points where $k(x)/k$ is finite.

example

Spec $k[X]_{()} = \{0, (x)\}$ Since (x) is a closed point and 0 is not.

2.3.31 II.3.15 x

3.15. Let X be a scheme of finite type over a field k (not necessarily algebraically closed).

(a) Show that the following three conditions are equivalent (in which case we say that X is *geometrically irreducible*).

(i) $X \times_k \bar{k}$ is irreducible, where $\bar{k}$ denotes the algebraic closure of k. (By abuse of notation, we write $X \times_k \bar{k}$ to denote $X \times_{\mathrm{Spec}\, k} \mathrm{Spec}\, \bar{k}$.)
(ii) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.
(iii) $X \times_k K$ is irreducible for every extension field K of k.

See Liu, section 3.2.2.

2.3.32 b. x

(b) Show that the following three conditions are equivalent (in which case we say X is *geometrically reduced*).

(i) $X \times_k \bar{k}$ is reduced.
(ii) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.
(iii) $X \times_k K$ is reduced for all extension fields K of k.

2.3.33 c. x

(c) We say that X is *geometrically integral* if $X \times_k \bar{k}$ is integral. Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.

2.3.34 II.3.16 x g Noetherian Induction

3.16. *Noetherian Induction.* Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y. (In particular, $\mathscr{P}$ must hold for the empty set.) Then $\mathscr{P}$ holds for X.

Suppose there are closed subsets where $\mathscr{P}$ doesn't hold.
Since X is noetherian, there is a smallest on Z.
Since Z is minimal, there can be no proper closed subsets of Z not satisfying $\mathscr{P}$.
But then we have a contradiction as Z it self must satisfy Z.

2.3.35 II.3.17 x Zariski Spaces

3.17. *Zariski Spaces.* A topological space X is a *Zariski space* if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point (Ex. 2.9).

For example, let R be a discrete valuation ring, and let $T = \operatorname{sp}(\operatorname{Spec} R)$. Then T consists of two points $t_0 =$ the maximal ideal, $t_1 =$ the zero ideal. The open subsets are $\varnothing$, $\{t_1\}$, and T. This is an irreducible Zariski space with generic point t_1.

(a) Show that if X is a noetherian scheme, then $\operatorname{sp}(X)$ is a Zariski space.

Note by 3.1.1 $sp(X)$ is noetherian, so we need to show each closed irreducible subset has a unique generic point.
For a closed irreducible Z , and open U either U contains the generic point or does not intersect Z .
Thus the result holds iff it holds for an open affine U.

Suppoes then that X is affine.
Then irreducible closed subsets correspond to, by the nullstellensatz to prime radical ideals.
Let $\mathfrak{p}$ the generic point for $V(\mathfrak{p})$.
If $\mathfrak{p}, \mathfrak{q}$ are two generic points for a closed determined by an ideal I , then $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{I} = \sqrt{\mathfrak{q}} = \mathfrak{q}$.

2.3.36 b. x g

(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. We call these *closed points.*

Minimal closed subsets are irreducible, and thus has a unique generic point by definition of a Zariski space. For another point in the minimal closed set, by minimality, the closure is the whole thing. By uniqueness of generic point, we are done.

2.3.37 c. x g

(c) Show that a Zariski space X satisfies the axiom T_0: given any two distinct points of X, there is an open set containing one but not the other.

Let $x \neq y \in X$, $U = X \setminus \{x\}^-$.
If $y \in U$ we are done, else $y \in \{x\}^-$, but then y is the generic point of $\{x\}^-$ so $x = y$.

2.3.38 d. x g

points of X, there is an open set containing one but not the other.

(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.

If $\eta \notin U$, then $\eta \in U^c$, closed. By irreducibleness, $X = \overline{\{\eta\}}$.
So U is empty.

2.3.39 e. x g specialization

nonempty open subset of X.

(e) If x_0, x_1 are points of a topological space X, and if $x_0 \in \{x_1\}^-$, then we say that x_1 *specializes* to x_0, written $x_1 \rightsquigarrow x_0$. We also say x_0 is a *specialization* of x_1, or that x_1 is a *generization* of x_0. Now let X be a Zariski space. Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points. (We say closed subsets are *stable under specialization.*) Similarly, open subsets are *stable under generization.*

Let $X = \cup Z_i$ the expression of X as union of maximal irreducible closed subsets.

Let η_i the generic point of Z_i and $\eta_i \in \{x_i\}^-$.
Then $Z_i \subset \{x_i\}^-$ and since the Z_i are maximal, $Z_i = \{x_i\}^-$.
Generic points of irreducible closed subsets are unique by previous $\implies \eta_i = x_i$.
$\implies \eta_i$ is maximal.

Conversely, suppose η maximal.
Then $\eta \in \{\eta_i\}^-$ and $\eta = \eta_i$.

The "also" part
If Z is a closed subset and $z \in Z$ is a point, then since $\{z\}^-$ is the smallest closed subset containing z , and Z contains z , $\implies \{z\}^- \subset Z$.

2.3.40 f. x

(f) Let t be the functor on topological spaces introduced in the proof of (2.6). If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha: X \to t(X)$ is a homeomorphism.

noetherian
The lattice of closed subsets of $t(X)$ is the same as the lattice of closed subsets of $X \implies t(x)$ is noetherian.

unique generic points
If η a generic point in a closed irreducible subset Z of X.
The closure $\{\eta\}^-$ in $t(X)$ is the smallest closed subset of X containing η .
η is a closed subset of $X \implies$ that $\{\eta\}^- = \eta$.
If η' is a generic point for $\{\eta\}^-$ then $\{\eta\}^- = \{\eta'\}^- \implies \eta = \eta'$.

"furthermore" part
If X is a zariski space, then there is 1-1 correspondence between points and irreducible closed subsets.
Hence $\alpha : X \to t(X)$ is a bijection on udnerlying sets.

The inverse is clearly continuous .

2.3.41 II.3.18x Constructible Sets

3.18. *Constructible Sets.* Let X be a Zariski topological space A *constructible subset* of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every open subset is in $\mathfrak{F}$, (2) a finite intersection of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the complement of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.

(a) A subset of X is *locally closed* if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.

Consider $\coprod Z_i \cap U_i \subset X$ finite disjoint union locally closed.
Suppose that this coproduct satisfies 1,2,3.
Conditions 1 and 3 imply closed subsets of X are in $\mathfrak{F}$.
Conditions 2 and 3 imply finite unions of elements of $\mathfrak{F}$ are in $\mathfrak{F}$.
Thus if $Z_i \cap U_i$ are djsoint, then $\coprod Z_i \cap U_i = \cup Z_i \cap U_i \in \mathfrak{F}$.

Consider a collection $\mathfrak{F}'$ of such coproducts. We want to show they satisfy 1,2,3.
1 is since $U \cap X = U$ and X is closed.
2 just take intersections of them and get another similar coproduct
3 is by induction.
Thus any such $\mathfrak{F}' = \mathfrak{F}$.

On the other hand, let $\mathfrak{F}_n \subset \mathfrak{F}$ the collection of subsets of X which can be written as finite disjoint union of n locally closed, thus $\cup_n \mathfrak{F}_n = \mathfrak{F}$.
The itnersection of elements of $\mathfrak{F}_n$ and $\mathfrak{F}_m$ is in $\mathfrak{F}$ as in 2 above.
If $S \in \mathfrak{F}_1$, $S = U \cap Z$ and $S^c = (U \cap Z)^c = U^c \cup Z^c = U^c \coprod (Z^c \cap U)$ in $\mathfrak{F}$.
We proceed by induction showing that $S \in \mathfrak{F}_n$ satisfies complements.
$S \in \mathfrak{F}_n$, $S = S_{n-1} \coprod S_1$, and $S^c = S_{n-1}^c \cap S_1^c$.
But S_{n-1}^c and S_1^c are in $\mathfrak{F}$ by induction, and their intersection is in their by 2.
Thus we have shown that $\mathfrak{F}$ is the locally closed coproducts as aabove.

2.3.42 b. x

(b) Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.

Suppose $S \in \mathfrak{F}$ is constructible. Let η the generic point and suppose it's in S
$\overline{S} \supset \{\eta\}^- = X$. $\implies$ S is dense.

Suppose that $S \in \mathfrak{F}$ is dense. $S = \coprod_{i=1}^n Z_i \cap U_i$ by part (a).
Note that $\cup Z_i \supset S$ so $\cup Z_i \supset \overline{S}$ since the Z_i are closed, and the union is finite.
Thus $\cup Z_i \supset \overline{S} = X$.
X irreducible $\implies$ $Z_i = X$ some i.
Thus $S = U_n \coprod \left(\coprod_{i=1}^{n-1} Z_i \cap U_i\right)$.
As X is a zariski space, by exc II.3.17.d the generic point is contained in every nonempty open subset of X.
So U_n contains generic point so S contains generic point.

2.3.43 c. x

(c) A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.

closed $\implies$ constructible and stable under specialization is clear.

Suppose S is constructible, thus by (a) $S = \coprod_{i=1}^{n} Z_i \cap U_i$ and is stable under specialization.
if η_i is generic point of irreducible comonent of Z_i intersecting U_i nontrivially.
S stable under specialization implies S contains every point in $\{x\}^-$.
$\implies$ S contains every point of every irreducible component of each Z_i .
$\implies$ $S \supset \bigcup Z_i$.
Consider $x \in S$. Suppose $x \in Z_i$ (it is for some i).
$\implies$ $S \subset \bigcup Z_i$.
$\implies$ $S = \bigcup Z_i$ since we've shown both containments.
$\implies$ S is closed as the Z_i are .

2.3.44 d. x

(d) If $f: X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.

A constructive set looks like $\coprod_{i=1}^{n} Z_i \cap U_i$
Note $f^{-1}\left(\coprod Z_i \cap U_i\right) = \coprod f^{-1} Z_i \cap f^{-1} U_i$.
By continuity, the preimages $f^{-1} Z_i$ and $f^{-1} U_i$ are closed and open respectively.

2.3.45 II.3.19 x

3.19. The real importance of the notion of constructible subsets derives from the following theorem of Chevalley—see Cartan and Chevalley [1, exposé 7] and see also Matsumura [2, Ch. 2, §6]: let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particular, $f(X)$, which need not be either open or closed, is a constructible subset of Y. Prove this theorem in the following steps.

(a) Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral noetherian schemes, and f is a dominant morphism.

reduce to affine

We want to show that it is possible to reduce (we don't actually have to show the result in this case).
If $\{V_i\}$ is an affine cover of Y, and $\{U_{ij}\}$ is an affine cover for each $f^{-1}(V_i)$, then if $f(U_{ij})$ is constructible each i, j , then $f(X) = \cup f(U_{ij})$ is constructible, so we assume X, Y are affine.

reduce to irreducible

If $\{V_i\}$ are irreducible components of Y, and $\{U_{ij}\}$ irreducible components of $f^{-1}(V_i)$, then if $f(U_{ij})$ is constructible for each i, j , then $f(X) = \cup f(U_{ij})$ is constructible, so we can assume X, Y are irreducible.

reduce to integral

WLOG we can assume reduced topologically, so irreducible + reduced gives integral.

reduce to dominant

Suppose $f(X)$ is constructible for each dominant morphism.
We have induced morphism $f' : X \to \overline{f(X)}$ which is also dominant so $f'(X)$ is constructible.

Thus $f'(X) \subset \overline{f(X)}$ is $\coprod U_i \cap Z_i$ by (a) of previous problem.
$\overline{f(X)}$ closed $\implies Z_i$closed in Y.
$U_i = V_i \cap \overline{f(X)}$ for $V_i \subset Y$ open under the induced topology.
Then $f(X) = \coprod U_i \cap Z_i = \coprod V_i \cap \overline{f(X)} \cap Z_i$ is constructible.

2.3.46 b (starred)

*(b) In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\varphi: A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\varphi(a) \neq 0$, then φ extends to a homomorphism φ' of B into K, such that $\varphi'(b) \neq 0$. [*Hint*: Prove this algebraic result by induction on the number of generators of B over A. For the case of one generator, prove the result directly. In the application, take $b = 1$.]

2.3.47 c. x

(c) Now use noetherian induction on Y to complete the proof.

By (b), $\exists a \in A$ with $D(a) \subset f(X)$.
We show $f(X) \cap V(a)$ is constructible in Y, assuming it's nonempty
Assume $V(a) = Spec\ (A/(a))$ so we have an induced map $f' : Spec\ B/aB \to Spec\ A/(a)$ with image $f(X) \cap V(a)$.
$A \to B$ injective $\implies A/(a) \to B/aB$ injective $\implies f'$ dominant.
As both rings are noetherian, $(a) = \cap \mathfrak{p}_i$, $\mathfrak{p}_i$ primary ideals by primary decomposition.
$\sqrt{\mathfrak{p}_i}$ are prime and $\sqrt{(a)} = \cap \sqrt{\mathfrak{p}_i} \implies V(a) = \cup V(\mathfrak{p}_i)$.
For each $\sqrt{\mathfrak{p}_i} B$, we have $B/\mathfrak{q}_j \to Spec\ A/\sqrt{\mathfrak{p}_i}$ for $\mathfrak{q}_j \in Spec\ B$.
Each image contains a nonemprt subset by (b), and hence is constructible in $V(\mathfrak{p}_i)$ by noetherian induction.
A locally closed subset of $V(\mathfrak{p}_i)$ is also a locally closed subset of $Spec\ B$,
$\implies$ images of $Spec\ B/\mathfrak{q}_i \to Spec$ are constructible in $Spec\ B$.

2.3.48 d. x g

(d) Give some examples of morphisms $f: X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be either open or closed.

We can map $\mathbb{A}^1 \to \mathbb{P}^2$ by $x \mapsto (x, 1, 0)$.
$f(\mathbb{A}^1_k)$ is neither open nor closed since $(x, 1, 0)$ nor its complement are varieties.

2.3.49 II.3.20 x g Dimension

3.20. *Dimension.* Let X be an integral scheme of finite type over a field k (not necessarily algebraically closed). Use appropriate results from (I, §1) to prove the following.

(a) For any closed point $P \in X$, $\dim X = \dim \mathcal{O}_P$, where for rings, we always mean the Krull dimension.

$$dim\ X = dim\ A = ht\ \mathfrak{m} + dim\ A/\mathfrak{m} = ht\ \mathfrak{m}A_{\mathfrak{m}} = dim\mathcal{O}_P\ .$$

2.3.50 b. x g

(b) Let $K(X)$ be the function field of X (Ex. 3.6). Then $\dim X = \text{tr.d. } K(X)/k$.

This follows from Thm 1.8A .

2.3.51 c. x

(c) If Y is a closed subset of X, then $\text{codim}(Y,X) = \inf\{\dim \mathcal{O}_{P,X} | P \in Y\}$.

$codim\ (Y,X) = codim\ (spec\ A/I, Spec\ A)$
$= inf_{\mathfrak{p}\supset I}\, (Spec\ A/I, Spec\ A) = inf_{\mathfrak{p}\supset I} ht\, (\mathfrak{p})$
$= inf_{\mathfrak{p}\in Y} dim\ \mathcal{O}_{\mathfrak{p},X}$.

2.3.52 d. x g

(d) If Y is a closed subset of X, then $\dim Y + \text{codim}(Y,X) = \dim X$.

If Y is irreducible, this is 1.8.A.

If Y is reducible, and $Z \subset Y$ is an irreducible closed subset of largest dimension, then $dim\ Y + codim\ (Y,X) = dim\ Z + dim\,(Z,X) = dim\ X$.

2.3.53 e. x g

(e) If U is a nonempty open subset of X, then $\dim U = \dim X$.

Since they have the same function fields, we can use (d).

2.3.54 f. x

(f) If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.

$dim\ X' = Dim\,(X \times_k x') = dim\ X + dim\ k = dim\ X$ since a field has dimension 0

2.3.55 II.3.21 x

3.21. Let R be a discrete valuation ring containing its residue field k. Let $X = \text{Spec } R[t]$ be the affine line over Spec R. Show that statements (a), (d), (e) of (Ex. 3.20) are false for X.

(e), the nonempty open set $Spec\ R\,[t]_u \subset Spec\ R\,[t]$ with $\mathfrak{m}_R = (u)$.
dimensions are 1 and 2 respectively.

(a)
Now consider the maximal ideal $(ut-1)$.

Then $R[t]/(ut-1) \approx Q(R)$ as $t = u^{-1}$ modulo $(ut-1)$.
$R[t]$ is factorial domain, so principal prime ideals have height 1.
Hence $P = (ut-1)$ is a closed point where $dim\ \mathcal{O}_P < dim\ X$.

(d). If $Y = V(P)$, then $0+1 \neq 2$. so $dim\ Y + codim\ Y, X \neq dim\ X$.

2.3.56 II.3.22* (Starred)

***3.22.** *Dimension of the Fibres of a Morphism.* Let $f: X \to Y$ be a dominant morphism of integral schemes of finite type over a field k.
(a) Let Y' be a closed irreducible subset of Y, whose generic point η' is contained in $f(X)$. Let Z be any irreducible component of $f^{-1}(Y')$, such that $\eta' \in f(Z)$, and show that $\operatorname{codim}(Z,X) \leqslant \operatorname{codim}(Y',Y)$.

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(b) Let $e = \dim X - \dim Y$ be the *relative dimension* of X over Y. For any point $y \in f(X)$, show that every irreducible component of the fibre X_y has dimension $\geqslant e$. [*Hint*: Let $Y' = \{y\}^-$, and use (a) and (Ex. 3.20b).]

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(c) Show that there is a dense open subset $U \subseteq X$, such that for any $y \in f(U)$, $\dim U_y = e$. [*Hint*: First reduce to the case where X and Y are affine, say $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Then A is a finitely generated B-algebra. Take $t_1, \ldots, t_e \in A$ which form a transcendence base of $K(X)$ over $K(Y)$, and let $X_1 = \operatorname{Spec} B[t_1, \ldots, t_e]$. Then X_1 is isomorphic to affine e-space over Y, and the morphism $X \to X_1$ is generically finite. Now use (Ex. 3.7) above.]

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(d) Going back to our original morphism $f: X \to Y$, for any integer h, let E_h be the set of points $x \in X$ such that, letting $y = f(x)$, there is an irreducible component Z of the fibre X_y, containing x, and having $\dim Z \geqslant h$. Show that (1) $E_e = X$ (use (b) above); (2) if $h > e$, then E_h is not dense in X (use (c) above); and (3) E_h is closed, for all h (use induction on $\dim X$).

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(e) Prove the following theorem of Chevalley—see Cartan and Chevalley [1, exposé 8]. For each integer h, let C_h be the set of points $y \in Y$ such that $\dim X_y = h$. Then the subsets C_h are constructible, and C_e contains an open dense subset of Y.

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2.3.57 II.3.23 x

3.23. If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in (I, Ex. 3.15, 3.16), and if t is the functor of (2.6), then $t(V \times W) = t(V) \times_{\operatorname{Spec} k} t(W)$.

By II.4.6.d, $t(V) \times_k t(W)$ is separated.

k is algebraically closed so by 4.10, $t(V) \times_k t(W)$ is integral and finite type.
So this is an integral sepated scheme of finite type over an algebraically closed field k.
Thus a variety.
Thus $t(V) \times_k t(W) = t(Y)$ for a variety Y.
Then Y must be $V \times W$ by the universal property of t.

2.4 II.4 x stopped g'ing here

2.4.1 II.4.1 x g Nice example valuative crit

4.1. Show that a finite morphism is proper.

Let $f : X \to Y$ finite.
Properness is local on the base and f is finite so take X, Y affine.
$f : spec\ B \to Spec\ A$.
If R is an arbitrary valuation ring with quotient field K , consider

$$\begin{tikzcd} Spec\ K \arrow[r, "v"] \arrow[d, "i"] & Spec\ B \arrow[d, "f"] \\ Spec\ R \arrow[r, "u"] \arrow[ur, "?"] & Spec\ A \end{tikzcd}$$

This corresponds to

$$\begin{tikzcd} K & B \arrow[l, "v"'] \arrow[dl, "?"] \\ R \arrow[u, "i"] & A \arrow[l, "u"] \arrow[u, "f"'] \end{tikzcd}$$

in terms of rings.
$A \to B$ finite and B integral over $A \implies u(A) \hookrightarrow v(V)$ is integral (Atiyah Mac p 60)
R a valuation ring $\implies R$ integrally closed.
$u(A) \subset R$ and R integrally closed $\implies v(B) \subset R$.
Now the result follows by valuative crit of properness.

2.4.2 II.4.2 x

4.2. Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated. [*Hint*: Consider the map $h: X \to Y \times_S Y$ obtained from f and g.]

Let U dense open on X where f, g agree. We have

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U \arrow[d] \\ Z \arrow[r] \arrow[d] & X \arrow[d, "f{,}g"] \\ Y \arrow[r, "\Delta"] & Y \times_S Y \end{tikzcd}$$

where the middle is pullback of δ .
Y separated $\implies \Delta$ is closed immersion.
II.3.11 $\implies$ closed immersions stable under base extension $\implies Z \to X$ is closed immersion.
f, g agree on $U \implies im(U)$ is contained in diagonal, and so topolgocaily the pullback is U.

$\implies U \to X$ factors through Z .

Image is closed subset of X. U dense, $\implies sp\ Z = sp\ X$.

$Z \to X$ is a closed immersion $\implies \mathcal{O}_X \to \mathcal{O}_Z$ is surjective.

For $X \supset V = Spec\ A$ open affine, $Z|_V \to X|_V = V$ is a closed immersion.

$\implies Z|_V$ is affine, homeomorphic to V , and therefore $\approx Spec\ A/I$.

As $Spec\ A/I \to Spec\ A$ is a homeomorphism, then I is contained in the nilradical, which is 0 as A is X is reduced by assumption.

Hence $Z|_V = Z$ as schemes, $\implies Z \approx X$.

hence f, g agree on all of X so they are in fact the same.

(a)X nonreduced counterexample

Let $X = Y = Spec\ k[x,y] / (x^2, xy)$

If f is identity, and g maps x to 0, then f, g agree on the complement of the origin. However, the map on global sections disagree.

(b) Y is non-separated counterexample.

Let X, Y be the affine line with two origins.

Let f, g be the distinct open inclusions of the affine line which agree outside of the double origin, but send the origin to different places.

2.4.3 II.4.3 x g

> **4.3.** Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.

Consider

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r, "\delta"] & X \times_S X \end{tikzcd} .$$

X separated over $S \implies \delta$ is a closed immersion by definition.

By exc II.3.11, closed immersions are stable under base extension.

Hence $U \cap V \to U \times_S V$ is a closed immersion.

As $U \times_S V$ is affine (defined by a tensor of f.g. algebrae), then $U \cap V \to U \times_S V$ is a closed immersion into an affine scheme, hence $U \cap V$ is affine by exc II.3.11.b.

Nonseparated

Consider affine plane with two origins, and let U, V be the two distinct affine planes . $U \cap V$ is $\mathbb{A}^2 - \{0\}$ which is not affine.

2.4.4 II.4.4 x

> **4.4.** Let $f: X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure (Ex. 3.11d) is proper over S. We refer to this result by saying that "the image of a proper scheme is proper." [*Hint*: Factor f into the graph morphism $\Gamma_f: X \to X \times_S Y$ followed by the second projection p_2, and show that Γ_f is a closed immersion.]

image is closed

Note that as $Z \to S$ is proper and $Y \to S$ is separated, by II.4.8.e, $Z \to Y$ is proper so $f(Z)$ is closed.

In order to show $f(Z)$ is proper, we must, by definition, show separated, finite-type, and universally closed.

separated

Note the diagonal $Y \to Y \times_S Y$ is a closed immerson by separatedness of Y. By base extension $f(Z) \to f(Z) \times_S f(Z)$ is a closed immersion.

finite type

Note that closed subschemes of finite type schemes are finite type.

universally closed.

Let $T \to S$ another morphism. Base extension of $Z \to f(Z)$ gives $f' : T \times_S Z \to T \times_S f(Z)$. If $x \in T \times_S f(Z)$, then under the base extension this corresponds to $x' \in f(Z)$ with $k(x') \subset k(x)$. Surjectivity of $Z \to f(Z)$ implies there is $x'' \in Z$ with $k(x'') \supset k(x')$. If $k(x), k(x'') \subset k$ then we have morphisms $Spec\ k \to T \times_S f(z)$ and $spec\ k \to Z$ which agree on $f(Z)$ and thus lift to $spec\ k \to T \times_S Z$, so there is a point in $T \times_S Z$ mapping to x. Hence $T \times_S Z \to T \times_S f(Z)$ is surjective.

If $W \subset T \times_S f(Z)$ is closed, then $(f')^{-1} W$ is closed and $s' \circ f'\left((f')^{-1} W\right)$ is closed. f' surjective implies $f'\left((f')^{-1}(W)\right) = W$ so that $s' \circ f'\left((f')^{-1} W\right) = s'(W)$. Thus $T \times_S f(Z)$ is closed in T.

2.4.5 II.4.5 x g center is unique by valuative criterion

4.5. Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k (see I, §6) has *center* x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.

(a) If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.

Let R a valuation ring on K with center at x.

Then $\mathcal{O}_{x,X} \subset R \subset K$ and $\mathfrak{m}_R$ lies over $\mathfrak{m}_x$ in $\mathcal{O}_{x,X}$.

Thus we have a diagram:

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r] \arrow[ur] & Spec\ k \end{tikzcd}$$

where $U = Spec\ K$, $T = Spec\ R$.

Comparing with the valuative criterion of separatedness shows that the diagonal morphism is unique, i.e. the inclusion $\mathcal{O}_{x,X} \subset R \subset K$ is unique, i.e. the center at x is unique.

2.4.6 b. x

(b) If X is proper over k, then every valuation of K/k has a unique center on X.

see part a.

2.4.7 (starred)

*(c) Prove the converses of (a) and (b). [*Hint*: While parts (a) and (b) follow quite easily from (4.3) and (4.7), their converses will require some comparison of valuations in different fields.]

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2.4.8 d. x

(d) If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$. This result generalizes (I, 3.4a). [*Hint*: Let $a \in \Gamma(X, \mathcal{O}_X)$, with $a \notin k$. Show that there is a valuation ring R of K/k with $a^{-1} \in \mathfrak{m}_R$. Then use (b) to get a contradiction.]

Note. If X is a variety over k, the criterion of (b) is sometimes taken as the definition of a complete variety.

As in the hint, let $a \in \Gamma(X, \mathcal{O}_x)$, $a \notin k$.
Let b be the image of $a \in k$.
k is algebraically closed $\implies$ b is transcendental over k $\implies$ $k[b^{-1}]$ is a polynomial ring.
Consider $k[b^{-1}]_{(b^{-1})}$, a local ring contained in K.
Let $R \subset K$ be a valuation ring dominating $k[b^{-1}]_{(b^{-1})}$.
$\mathfrak{m}_R \cap k[b^{-1}]_{(b^{-1})} = (b^{-1}) \implies b^{-1} \in \mathfrak{m}_R$.
By the valuative criterion of properness, we have a unique map $\Gamma(X, \mathcal{O}_X) \to R$:

$$\begin{tikzcd} K & \Gamma(X, \mathcal{O}_X) \arrow[l] \arrow[dl] \\ R \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

Then $b \in R$ so $v_R(b) > 0$.
Then $b^{-1} \in \mathfrak{m}_R$ so $v_R(b^{-1}) > 0$.
But the valuation of a unit should be 0 so this is a contradiction which resulted from assuming $a \notin k$.

2.4.9 II.4.6 x g

4.6. Let $f: X \to Y$ be a proper morphism of affine varieties over k. Then f is a finite morphism. [*Hint*: Use (4.11A).]

Let $f : Spec\ A \to Spec\ B$ finite.
Let $\varphi : B \to A$ be the map on global sections, a ring morphism.
Let $K = k(A)$.
By the valuative criterion of properness, we have, for $K \supset R \supset \varphi(B)$, R valuation ring as in 4.5,

$$\begin{tikzcd} Spec\ K \arrow[r] \arrow[d] & Spec\ A \arrow[d] \\ Spec\ R \arrow[r] \arrow[ur] & Spec\ B \end{tikzcd}$$

By II.4.11A, the integral closure of $\varphi(B)$ in K is the intersection of all valuation rings of K containing $\varphi(B)$.

The map $Spec\ R \to Spec\ A$ gives an inclusion of A in every such valuation ring, and thus A is integral over B.

As f is finite type, f is therefore finite.

2.4.10 II.4.7 x R-scheme

4.7. *Schemes Over* **R**. For any scheme X_0 over **R**, let $X = X_0 \times_{\mathbf{R}} \mathbf{C}$. Let $\alpha : \mathbf{C} \to \mathbf{C}$ be complex conjugation, and let $\sigma : X \to X$ be the automorphism obtained by keeping X_0 fixed and applying α to **C**. Then X is a scheme over **C**, and σ is a *semi-linear* automorphism, in the sense that we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\sigma"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} \mathbf{C} \arrow[r, "\alpha"] & \operatorname{Spec} \mathbf{C}. \end{tikzcd}$$

Since $\sigma^2 = \mathrm{id}$, we call σ an *involution.*

(a) Now let X be a separated scheme of finite type over **C**, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$, there is an open affine subset containing both of them. (This last condition is satisfied for example if X is quasi-projective.) Show that there is a unique separated scheme X_0 of finite type over **R**, such that $X_0 \times_{\mathbf{R}} \mathbf{C} \cong X$, and such that this isomorphism identifies the given involution of X with the one on $X_0 \times_{\mathbf{R}} \mathbf{C}$ described above.

This follows from Milne AG, theorem 16.35.

2.4.11 b. x

For the following statements, X_0 will denote a separated scheme of finite type over **R**, and X, σ will denote the corresponding scheme with involution over **C**.

(b) Show that X_0 is affine if and only if X is.

X_0 affine implies $X_0 \times_{\mathbb{R}} \mathbb{C} \approx X$ is affine.
If $X = Spec\ A$ is affine, then $X_0 = Spec\ A^{\sigma}$, A^{σ} being fixed by the involution.

2.4.12 c. x

(c) If X_0, Y_0 are two such schemes over **R**, then to give a morphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions, i.e., $f \circ \sigma_X = \sigma_Y \circ f$.

Suppose we have f that commutes with σ .
If $X = Spec\ A$, $Y = Spec\ B$, then we have an induced morphism $A^{\sigma} \to B^{\sigma}$.
This gives $X_0 \to Y_0$.
X, Y not affine, then take cover of X by open affines U_i preserved by σ .
For each i tlet V_{ij} an open affine cover of $f^{-1}U_i$ and preserved by σ .
If $\pi : Y \to Y_0$ is the projection , this is affine by (b).
Then we can glue $\pi (V_{ij}) \to \pi (U_i)$ to get a morphism $Y_0 \to X_0$.

2.4.13 d. x

volutions, i.e., $f \quad \sigma_X = \sigma_Y$.

(d) If $X \cong \mathbf{A}^1_{\mathbf{C}}$, then $X_0 \cong \mathbf{A}^1_{\mathbf{R}}$.

By case II of (e)

2.4.14 e. x

(e) If $X \cong \mathbf{P}^1_{\mathbf{C}}$, then either $X_0 \cong \mathbf{P}^1_{\mathbf{R}}$, or X_0 is isomorphic to the conic in $\mathbf{P}^2_{\mathbf{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^2 = 0$.

We proceed by cases.

If σ has no closed fixed points, then for $x \in X \approx \mathbb{P}^1$ a closed point, let $U = X \setminus \{x, \sigma x\}$.

Let f send $(x, \sigma x)$ to $(0, \infty)$. Assume x, and σx are 0 and ∞ so that $U \approx Spec\ \mathbb{C}[t, t^{-1}]$.

The lift of σ is $\mathbb{C}$ -semilinear and σ induces an invertible semilinaer $\mathbb{C}$-algebra homomorphism on $\mathbb{C}[t, t^{-1}]$.

The element t is sent under σ to at^k for $k \in \mathbb{Z}$. As $\sigma^2 = 1$, then $k \approx \pm 1$.

If $k = 1$, then σ would fix $\mathbb{C}[t]_{(t)}$ so k must be -1.

$t\sigma t = a$ is fixed by σ. Since σ acts by conjugation, then $a \in \mathbb{R}$.

If a is positive, then the ideal $(t - \sqrt{a})$ is fixed. Thus $a \in \mathbb{R}_{\leq 0}$.

Now change coordinates from t to $\frac{1}{\sqrt{-a}}$, so the involution becomes $t \mapsto -t^{-1}$.

Writing $t = \frac{Z}{X}$, $-t^{-1} = \frac{Z}{Y}$, we have isomorphisms $\frac{\mathbb{C}\left[\frac{Y}{X}, \frac{Z}{X}\right]}{\left(\frac{Y}{X} + \left(\frac{Z}{X}\right)^2\right)} \approx \mathbb{C}[-t]$, and $\frac{\mathbb{C}\left[\frac{X}{Y}, \frac{Z}{Y}\right]}{\left(\frac{X}{Y} + \left(\frac{Z}{Y}\right)^2\right)} \approx \mathbb{C}[t^{-1}]$, $-t^{-1} = \frac{Z}{Y}$, σ switches $\frac{X}{Z}$ and $\frac{Y}{Z}$ and conjugates scalars.

Writing $U = \frac{1}{2}(X + Y)$, $V = \frac{i}{2}(Y - X)$, we get

$\mathscr{Q} \approx Proj \frac{\mathbb{C}[X,Y,Z]}{(U^2+V^2+Z^2)} \approx Proj \frac{\mathbb{C}[X,Y,Z]}{(XY+Z^2)} \approx \mathbb{P}^1_{\mathbb{C}} \approx X$ where σ acts by conjugating scalars.

Thus $X_0 \approx \mathscr{Q}_0 \approx Proj \frac{\mathbb{R}[X,Y,Z]}{(U^2+V^2+Z^2)}$.

If on the other hand σ has at least one fixed point, then σ restricts to a semilinear automorphism of the complement of the fixed point, which is an open set $Spec\ \mathbb{C}[t] \subset \mathbb{P}^1_{\mathbb{C}}$. Since σ is invertible, t gets sent to something of the form $at + b$. B changing coordinates to $s = ct + d$ with $\sigma s = s$, we have a σ -invariant isomorphism $X \approx \mathbb{P}^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

2.4.15 II.4.8 x

4.8. Let $\mathscr{P}$ be a property of morphisms of schemes such that

(a) a closed immersion has $\mathscr{P}$;

(b) a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;

(c) $\mathscr{P}$ is stable under base extension.

Then show that:

(d) a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;

Let $f : X \to Y$ and $g : A \to B$ two morphisms having $\mathscr{P}$.

We want to show that $f \times g$ has $\mathscr{P}$.

By base change, $X \times A \to Y \times A$ has $\mathscr{P}$.

Also by base change, $Y \times A \to Y \times B$ has $\mathscr{P}$.

By composition, $X \times A \to Y \times A \to Y \times B$ has $\mathscr{P}$.

But this is $f \times g$.

2.4.16 e. x

(e) if $f: X \to Y$ and $g: Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;

Note this morphism first in the vertical left hand side of the following diagram:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \\
X \times_Z Y \arrow[r] \arrow[d] \arrow[dr] & Y \times_Z Y \\
Y \arrow[dr] & X \arrow[d] \\
& Z
\end{tikzcd}$$

Note top right has it by separated since closed immersion

Note the top left is the base change since $X \times_X X \approx X$ and $X \times_X (Y \times_Z Y) \approx X \times_Z Y$.

LHS bottom is also base change.

2.4.17 f. x

(f) If $f: X \to Y$ has $\mathscr{P}$, then $f_{red}: X_{red} \to Y_{red}$ has $\mathscr{P}$.

[*Hint*: For (e), consider the graph morphism $\Gamma_f: X \to X \times_Z Y$ and note that it is obtained by base extension from the diagonal morphism $\Delta: Y \to Y \times_Z Y$.]

Consider the fiber product:

$$\begin{tikzcd}
X_{red} \arrow[dr, "\Gamma_{f_{red}}"] \arrow[drr, "id"] \arrow[ddr, "f_{red}"'] & & \\
& Y_{red} \times_Y X_{red} \arrow[r] \arrow[d] & X_{red} \arrow[d] \\
& Y_{red} \arrow[r] & Y
\end{tikzcd}$$

$X_{red} \to X \to Y$(top and righ) is a composition of a closed immersion and morphism with $\mathscr{P}$, so it has $\mathscr{P}$.

Thus $Y_{red} \times_Y X_{red}$ is a base change of morphism with $\mathscr{P}$ so has it by assumption.

Note if $\Gamma_{f_{red}}$, the graph, has $\mathscr{P}$, then f_{red} is a composition of morphisms with property $\mathscr{P}$.

But the graph is the following base change

$$\begin{tikzcd}
X_{red} \arrow[r] \arrow[d, "\Gamma"] & Y_{red} \arrow[d] \\
X_{red} \times_Y Y_{red} \arrow[r] & Y_{red} \times_Y Y_{red}
\end{tikzcd}$$

As $Y_{red} \times_Y Y_{red} = Y_{red}$ and $\Delta = id$, then Δ is closed immersion and Γ thus has property $\mathscr{P}$.

2.4.18 II.4.9 x g important - used stein factorization

4.9. Show that a composition of projective morphisms is projective. [*Hint*: Use the Segre embedding defined in (I, Ex. 2.14) and show that it gives a closed immersion $\mathbf{P}^r \times \mathbf{P}^s \to \mathbf{P}^{rs+r+s}$.] Conclude that projective morphisms have properties (a)–(f) of (Ex. 4.8) above.

Let $X \to Y \to Z$ projective. We have

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[dr] & \mathbb{P}^r \times Y \arrow[r, "id\times g'"] \arrow[d] & \mathbb{P}^r \times \mathbb{P}^s \times Z \arrow[d] \\
 & Y \arrow[r, "g'"] \arrow[dr, "g"] & \mathbb{P}^s \times Z \arrow[d] \\
 & & Z
\end{tikzcd}$$

f' , g' and $id \times g'$ are closed immersions.

Now using segre embedding, $\mathbb{P}^r \times \mathbb{P}^s \times Z \to Z$ factors like

$\mathbb{P}^r \times \mathbb{P}^s \times Z \to \mathbb{P}^{rs+r+s} \times Z \to Z$

Since segre embedding is closed immersion, then have closed immersion $X \to \mathbb{P}^{rs+r+s}$ which factors as $g \circ f$.

2.4.19 II.4.10 Chow's Lemma (starred)

***4.10.** *Chow's Lemma.* This result says that proper morphisms are fairly close to projective morphisms. Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g: X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.

(a) Reduce to the case X irreducible.

2.4.20 b part of starred

(b) Show that X can be covered by a finite number of open subsets $U_i, i = 1, \ldots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.

2.4.21 c. part of starred

(c) Let $U = \bigcap U_i$, and consider the map

$$f: U \to X \times_S P_1 \times_S \cdots \times_S P_n$$

deduced from the given maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure (Ex. 3.11d) $\overline{f(U)}$. Let $g: X' \to X$ be the projection onto the first factor, and let $h: X' \to P = P_1 \times_S \ldots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

2.4.22 d. part of starred

(d) Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.

2.4.23 II.4.11 x

4.11. If you are willing to do some harder commutative algebra, and stick to noetherian schemes, then we can express the valuative criteria of separatedness and properness using only *discrete* valuation rings.

(a) If $\mathcal{O},\mathfrak{m}$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$. Prove this in the following steps. By taking a polynomial ring over $\mathcal{O}$, reduce to the case where L is a *finite* extension field of K. Then show that for a suitable choice of generators $x_1,\ldots,x_n$ of $\mathfrak{m}$, the ideal $\mathfrak{a} = (x_1)$ in $\mathcal{O}' = \mathcal{O}[x_2/x_1,\ldots,x_n/x_1]$ is not equal to the unit ideal. Then let $\mathfrak{p}$ be a minimal prime ideal of $\mathfrak{a}$, and let $\mathcal{O}'_{\mathfrak{p}}$ be the localization of $\mathcal{O}'$ at $\mathfrak{p}$. This is a noetherian local domain of dimension 1 dominating $\mathcal{O}$. Let $\tilde{\mathcal{O}}'_{\mathfrak{p}}$ be the integral closure of $\mathcal{O}'_{\mathfrak{p}}$ in L. Use the theorem of Krull–Akizuki (see Nagata [7, p. 115]) to show that $\tilde{\mathcal{O}}'_{\mathfrak{p}}$ is noetherian of dimension 1. Finally, take R to be a localization of $\tilde{\mathcal{O}}'_{\mathfrak{p}}$ at one of its maximal ideals.

Let $y_1,...,y_n$ transcendental elements of L over K such that L is finite over $K(y_1,...,y_n)$.

Thus extension of $\mathfrak{m}$ in $\mathcal{O}[y_1,...,y_n]$ is not the whole ring so we localize at a prime ideal lying over $\mathfrak{m}$.

Assume WLOG that L is finite field extension of K.

Let $x_1,...,x_n$ a system of parameters for $\mathfrak{m}$.

As $x_1,...,x_n$ are algebraically independent over K, the extension of $\mathfrak{m}$ in $\mathcal{O}[x_2,...,x_n]_{(x_1)}$ is (x_1), which is not the whole ring.

Let $\mathfrak{p}$ a minimal prime lying over (x_1).

By KPIT, $\mathfrak{p}$ has height 1.

If B is the localization of $\mathcal{O}'$ at $\mathfrak{p}$, then B is a noetherian local domain of dimension 1.

By DIRP, the integral closure of B in L is noetherian, dimension 1.

Localizing the integral closure at a maximal ideal gives a DVR in L dominating $\mathcal{O}$.

2.4.24 b. x

(b) Let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Show that f is separated (respectively, proper) if and only if the criterion of (4.3) (respectively, (4.7)) holds for all *discrete* valuation rings.

By part (a), we only have to consider discrete valuation rings. Thus this follows from Thm I.6.1A

2.4.25 II.4.12 x Examples of Valuation Rings

4.12. *Examples of Valuation Rings.* Let k be an algebraically closed field.

(a) If K is a function field of dimension 1 over k (I, §6), then every valuation ring of K/k (except for K itself) is discrete. Thus the set of all of them is just the abstract nonsingular curve C_K of (I, §6).

Let $R \subset K$ be a valuation ring. If $\mathfrak{m}_R$ is principal, then by thm I.6.2A the valuation ring is discrete.

For $t \in \mathfrak{m}_R$, and $(t) \neq \mathfrak{m}_R$, let $s \in \mathfrak{m}_R \backslash (t)$.

If t is not transcendental then $\sum_{i=0}^{n} a_i t^i = 0$ with a constant term, then $a_0 = t \sum a_i t^{i-1}$ and so $a_0 \in (t)$ which must be K. Contradiction.

Thus, since K has dimension 1, and t is transcendental, K is a finite algebraic extension of $k(t)$.

As $s \notin (t)$, then s is algebraic over k. Thus $\sum a_i s^i = 0$.

Thus $a_0 = s\sum a_i s^{i-1}$, and $a_0 = \frac{f(t)}{g(t)}$ so $\frac{f(t)}{g(t)} = s\sum a_i s^{i-1}$ so $f(t) = g(t)\, s\sum a_i s^{i-1}$ and thus $f(t) \in (s) \subset \mathfrak{m}_R \setminus (t)$.

As $t \in \mathfrak{m}_R$, then $a_0 = 0$ or else, $a_0 \in \mathfrak{m}_R$.

Thus $t \in (s)$.

If $(s) = \mathfrak{m}_R$ the we are done, other wise take an ascending chain and use noetherianness.

2.4.26 b. (1) x

(b) If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.

(1) If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1 on X.

Let $U = Spec\ A$ be an open affine. Then x_1 corresponds to a prime ideal $\mathfrak{p} \subset A$ of height 1 and $\mathcal{O}_{X,x_1} \approx A_{\mathfrak{p}}$, a noetherian local ring of dimension 1. As X is nonsingular, and a curve so normal then so is A, $A_{\mathfrak{p}}$. By DIRP, $A_{\mathfrak{p}}$ is a DVR, which must have center x_1.

2.4.27 (2) x

(2) If $f: X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation ring of K/k with center at the closed point x_0 on X.

If X' is smooth, then by (1), R is DVR.

f induces an inclusion $\mathcal{O}_{X,x} \hookrightarrow R$, so R dominates $\mathcal{O}_{X,x_0}$.

Recall this means R has center x_0.

2.4.28 (3) x

(3) Let $x_0 \in X$ be a closed point. Let $f: X_1 \to X$ be the blowing-up of x_0 (I, §4) and let $E_1 = f^{-1}(x_0)$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2: X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}(x_1)$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{x_{i+1},X_{i+1}}$ dominates $\mathcal{O}_{x_i,X_i}$. Let $R_0 = \bigcup_{i=0}^{\infty} \mathcal{O}_{x_i,X_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k by (I, 6.1A). Show that R is a valuation ring of K/k, and that it has center x_0 on X. When is R a discrete valuation ring?

Note. We will see later (V, Ex. 5.6) that in fact the R_0 of (3) is already a valuation ring itself, so $R_0 = R$. Furthermore, every valuation ring of K/k (except for K itself) is one of the three kinds just described.

This is clear.

2.5 II.5 x

2.5.1 II.5.1 g x

5.1. Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank We define the *dual* of $\mathscr{E}$, denoted $\check{\mathscr{E}}$, to be the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$.
(a) Show that $(\check{\mathscr{E}})^\vee \cong \mathscr{E}$.

Let $\varphi : \mathscr{E} \to \mathscr{H}_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$ be defined by evaluation.

If U is open and $V \subset U$, then for $s \in \mathscr{E}(U)$ define $t \in hom_{\mathcal{O}_X}(\mathscr{E}(V), \mathcal{O}_X(V)) \to \mathcal{O}_X(V)$ by evaluation at $s|_V$.

If $\mathscr{E}$ is locally free, then this is an isomorphism on the stalks.

2.5.2 (b) g x

(b) For any $\mathcal{O}_X$-module $\mathscr{F}$, $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes_{\mathcal{O}_X} \mathscr{F}$.

Let U be an open set where $\mathscr{E}$ is gree.

Define $\varphi : \mathscr{H}om_{\mathcal{O}_X}\left(\mathscr{E}|_U, \mathcal{O}_{X|U}\right) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U) \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}|_U, \mathscr{F}|_U)$ by mapping
$f \otimes a \mapsto (x \mapsto f(x)a)$.

Define $\psi : \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F})|_U \to [\mathscr{H}om(\mathscr{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathscr{F}]|_U$ by mapping
$f \mapsto \sum_{i=1}^n e_i^* \otimes f(e_i)$.

Now check that these are inverse bijective homomorphisms.

2.5.3 (c) x g

(c) For any $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \mathrm{Hom}_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$.

We take the sheafification of $Hom(M \otimes N, P) \approx Hom(M, \mathscr{H}om(N, P))$, AM p 28

2.5.4 (d) x g Projection Formula

(d) (*Projection Formula*). If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) \cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.

We have$f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathcal{O}_Y^n) \approx f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^n) \approx f_*(\mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X)^n$, since right adjoints commute with limits, lefts with colimits (finite direct sums).

$\approx f_*(\mathscr{F})^n \approx f_*(\mathscr{F}) \otimes \mathcal{O}_Y^n \approx f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.

Now if $\mathscr{E}$ is locally free, procedd in the same manner on an open cover where $\mathscr{E}|_{U_i}$ is free, and then glue the results.

2.5.5 II.5.2 (a) x

5.2. Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.
(a) To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R K \to L$.

Spec R has two nontrivial open subsets, the total space and the generic point.

By definition, $\mathscr{F}$ is an $\mathcal{O}_X$ module, if $\mathscr{F}$ is an $\mathcal{O}_X(X) = R$-module M and there is a $\mathcal{O}_X(U) = K$ -module L and a restriction morphism $M \to L_R$, L_R being L considered as an R-module.

Restriction and extension of scalars are adjoint, so the R-module homomorphism represented by restriction gives the K-module homomorphism of extension $M \otimes_R K \to L$ by adjunction.

2.5.6 (b) x

(b) That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.

If $\mathscr{F}$ is q.c. , then locally $\mathscr{F} \approx \tilde{M}$.

R has a unique closed point and the neighborhood is the whole space.

Thus $\mathscr{F} \approx \tilde{M}$ globally.

Thus $L \approx \mathscr{F}(U) \approx M_{(0)} \approx M \otimes_R K$.

On the other hand, if $\rho : M \otimes_R K \to L$ is an isomorphism,

By (a), we know $\mathscr{F} \approx \tilde{M}$ iff $L \approx M_0$. But $L \approx M \otimes_R K \approx M_{(0)}$.

2.5.7 II.5.3 x g

5.3. Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $\tilde{}$ and Γ are adjoint, in the following sense: for any A-module M, and for any sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$, there is a natural isomorphism

$$\operatorname{Hom}_A(M, \Gamma(X, \mathscr{F})) \cong \operatorname{Hom}_{\mathcal{O}_X}(\tilde{M}, \mathscr{F}).$$

Clearly $f : \tilde{M} \to \mathscr{F}$ gives a map on global sections $\tilde{M}(X) \to \mathscr{F}(X)$, which is $M \to \Gamma(X, \mathscr{F})$.

On the other hand, given $f : M \to \Gamma(X, \mathscr{F})$, define $f^\sharp$ on the distingushed base, $D(f)$ by $f^\sharp|_{D(f)}\left(\frac{m}{g}\right) \mapsto \frac{f(m)}{g}$. Glueing gives an $f^\sharp = f$ on X. Thus $f \mapsto f^\sharp \in Hom\left(\tilde{M}, \mathscr{F}\right)$ is injective. If $f^\sharp$ induces f , then clearly f induces $f^\sharp$, so that $f \mapsto f^\sharp$ is surjective.

2.5.8 II.5.4 x

5.4. Show that a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ on a scheme X is quasi-coherent if and only if every point of X has a neighborhood U, such that $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathscr{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank. (These properties were originally the definition of quasi-coherent and coherent sheaves.)

First note that locally free implies q.c.

Basically if we have locally free, then we know $\mathscr{F}|_{U_i} \approx (\mathcal{O}_X)^K$ some index set K.

By II.5.2, $\tilde{\mathcal{O}_X^K} \approx \left(\tilde{\mathcal{O}_X}\right)^K$, and since $\mathcal{O}_X$ is itself a sheaf, and the associated sheaf is uniquely isomorphic as a sheaf, $\mathcal{O}_X \approx \tilde{\mathcal{O}}_X$.

Now suppose $\mathscr{F}$ is q.c. Let U a neighborhood of a point, $U = Spec\ A$, such that $\mathscr{F}|_U \approx \tilde{M}$. (This is thm II.5.4). Folliwing Eisenbud, page 17, let $m_{\alpha_{\alpha \in A}}$ a generating set for the $\mathcal{O}_{X|U}$-module M (we can at least take the m_α to be the elements of M). Let B index the kernel so that $\mathcal{O}_{X|U}^B \xrightarrow{\psi} \mathcal{O}_{X|U}^A \xrightarrow{\gamma} M \to 0$ is exact.

As $M \approx coker(\psi) \approx \mathcal{O}^A_{X|U}/im(\psi) \approx \mathcal{O}^A_{X|U}/ker(\gamma) \approx \mathcal{O}^A_{X|U}/\mathcal{O}^B_{X|U}$, the same sequence as above, sheafified, is exact by II.5.2.a. Note that $\mathcal{O}^{\tilde{A}}_{X|U}$ and $\mathcal{O}^{\tilde{B}}_{X|U}$ are free by above.

As exact functors preserve cokernels, we have the result in one direction.

Conversely, if $\mathscr{F}|_U$, $U = Spec\ A$ is the cokernel of a morphism of free sheaves on U, then it is a cokernel of locally free sheaves on U. We know that locally free implies q.c., thus it is a cokernel of q.c. sheaves on U. Now using II.5.7, the cokernel of q.c. is q.c. so $\mathscr{F}|_U$ is q.c.

2.5.9 II.5.5 (a) x g

5.5. Let $f: X \to Y$ be a morphism of schemes.
(a) Show by example that if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.

Let $f : X = Spec\ k(t) \to Y = Spec\ k$.

Note that $\mathcal{O}_{Spec\ k(t)}$ is coherent on $k(t)$, but $k(t)$ is not finitely generated as a k-module, since it contains $1, t, t^2, ...$

Thus the pushforward is not coherent.

2.5.10 b. x g closed immersion is finite.

(b) Show that a closed immersion is a finite morphism (§3).

Let $f : Y \to X$ closed immersion.

For an open affine cover U_i of X, the restrictions are closed immersions $f : U_i \to U_i$. (by definition of closed immersion).

By Ex, II.3.11.b, these are $Spec(A_i/I_i) \to Spec(A_i)$ and each A_i/I_i is f.g.

2.5.11 (c) x g

(c) If f is a finite morphism of noetherian schemes, and if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ is coherent on Y.

Let $\{Spec\ B_i\}$ an open affine cover of Y.

f finite implies $f^{-1}Spec\ B_i \approx Spec\ A_i$, A_i an f.g. B_i-module.

$\mathscr{F}$ coherent, and X noetherian $\implies$ $\mathscr{F} \approx \tilde{M}_i$ an f.g. A_i-module.

By II.5.2.d, $f_*\mathscr{F}|_{Spec\ B_i} \approx ({}_{B_i}M_i)^\sim$, and ${}_{B_i}M_i$ is an f.g B_i-module.

2.5.12 II.5.6 (a) x g

5.6. *Support.* Recall the notions of support of a section of a sheaf, support of a sheaf, and subsheaf with supports from (Ex. 1.14) and (Ex. 1.20).
(a) Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathscr{F} = \tilde{M}$. For any $m \in M = \Gamma(X, \mathscr{F})$, show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$, where $\operatorname{Ann} m$ is the *annihilator* of $m = \{a \in A | am = 0\}$.

$\mathfrak{p} \in V(Ann\ m) \implies \mathfrak{p} \supset Ann\ m \implies sm \neq 0$ for $s \notin \mathfrak{p} \implies m_\mathfrak{p} \neq 0 \implies \mathfrak{p} \in Supp\ m$.

$\mathfrak{p} \in Supp\ m \implies sm = 0$ some $s \notin \mathfrak{p} \implies \mathfrak{p} \not\supset Ann\ m \implies p \notin V(Ann\ m)$.

2.5.13 (b) x g

(b) Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$.

Let m_i a set of generators for M.
$Ann\ M = \cap Ann\ m_i$.
$Supp\ \mathscr{F} \approx Supp\ \tilde{M} \approx \{\mathfrak{p} \in Spec\ A | M_{\mathfrak{p}} \neq 0\}$.
Then $M_{\mathfrak{p}} \neq 0$ iff $m_i \neq 0$ in $M_{\mathfrak{p}}$ for some i iff $Ann\,(m_i) \subset \mathfrak{p}$ iff $V\,(Ann\ M) \ni \mathfrak{p}$

2.5.14 (c) x g

(c) The support of a coherent sheaf on a noetherian scheme is closed.

The support is the union of supports of sheaf of each element.
On an an open affine cover U_i with $\mathscr{F}|_{U_i} \approx \tilde{M}_i$, then by (b), the support is closed on U_i , and thus on X.

2.5.15 (d) x

(d) For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by $\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^{\sim} \cong \mathscr{H}^0_Z(\mathscr{F})$, where $Z = V(\mathfrak{a})$ and $\mathscr{F} = \tilde{M}$. [*Hint*: Use (Ex. 1.20) and (5.8) to show a priori that $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. Then show that $\Gamma_{\mathfrak{a}}(M) \cong \Gamma_Z(\mathscr{F})$.]

Let $U = X - Z$, $j : U \hookrightarrow X$ the inclusion.
Let $U = V\,(\mathfrak{a})^c$.
exc II.1.20b gives $0 \to \mathscr{H}^0_Z\,(\mathscr{F}) \to \mathscr{F} \to j_*\mathscr{F}$.
Thm I.5.8.c gives $j_*\mathscr{F}$ is q.c.,
As $\mathscr{H}^0_Z\,(\mathscr{F})$ is kernel of q.c. sheaves, $\mathscr{H}^0_Z\,(\mathscr{F})$ is q.c.
$\Gamma_{\mathfrak{a}}\,(M)^{\sim} \approx \mathscr{H}^0_Z\,(\mathscr{F})$ iff $\Gamma_{\mathfrak{a}}\,(M) \approx \Gamma_Z\,(\mathscr{F})$.
Note $m \in \Gamma_Z\,(\mathscr{F})$ iff $Supp\ m \subset V\,(\mathfrak{a})$.
From a previous excercise, this is equivalent to $V\,(Ann\ m) \subset V\,(\mathfrak{a})$.
By nullstellants equivaltn to $\sqrt{\mathfrak{a}} \subset \sqrt{Ann\ m}$.
By noetherian equivalent to $\mathfrak{a}^n \subset Ann\ m$.
By definition equivalent to $m \in \Gamma_{\mathfrak{a}}\,(m)$.

2.5.16 (e) x

(e) Let X be a noetherian scheme, and let Z be a closed subset. If $\mathscr{F}$ is a quasi-coherent (respectively, coherent) $\mathscr{O}_X$-module, then $\mathscr{H}^0_Z(\mathscr{F})$ is also quasi-coherent (respectively, coherent).

Since X is noetherian, if $\mathscr{F}|_{U_i} \approx \tilde{M}_i$ for $U_i = spec\ A$, $Z = Spec\ A/\mathfrak{a}_i$ an open affine cover, then by (d), $\mathscr{H}^0_Z\,(\mathscr{F})\,|_{U_i} \approx \Gamma_{\mathfrak{a}_i}\,(M_i)^{\sim}$.

2.5.17 II.5.7 x

5.7. Let X be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.
(a) If the stalk $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighborhood U of x such that $\mathscr{F}|_U$ is free.

Let $X = Spec\ A$, $\mathscr{F} = \tilde{M}$, M is generated by $m_1, ..., m_n$.
We have $\mathscr{F}_x \approx M_{\mathfrak{p}} \approx A_{\mathfrak{p}} x_1 + ... + A_{\mathfrak{p}} x_n$, $\mathfrak{p} \in Spec\ A$, and x_i sections on a principal open set $D(f)$.
In $M_{\mathfrak{p}}$, write the image of m_i as $\frac{a_{i,1}}{g_{i,1}} x_1 + ... + \frac{a_{i,n}}{g_{i,n}} x_n$.
Writing $g = \prod_{i,j} g_{i,j}$, we see that m_i are spanned by x_i on $D(fg)$.
If $h = fg$, then $M_h = A_h x_1 + ... + A_h x_n$.
x_i linearly independent in $M_{\mathfrak{p}} \implies x_i$ linearly independent in $M_h \implies \mathscr{F}|_{D(h)} \approx \tilde{M}_h$ is a finite direct sum.

2.5.18 b. x

(b) $\mathscr{F}$ is locally free if and only if its stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.

If $\mathscr{F}$ is locally free, then by definition stalks are free.
If the stalks are all free, then each point has a neigborhood .. use part (a)

2.5.19 (c) x

(c) $\mathscr{F}$ is invertible (i.e., locally free of rank 1) if and only if there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$. (This justifies the terminology invertible: it means that $\mathscr{F}$ is an invertible element of the monoid of coherent sheaves under the operation $\otimes$.)

If $\mathscr{F}$ is invertible, then $\mathscr{H}om(\mathscr{F}, \mathcal{O}_X) \otimes \mathscr{F} \approx \mathcal{O}_X$ via the evaluation morphism is surjective, since $\mathscr{F}$ is locally free rank 1, it is an isomorphism.

Conversely, suppose $\mathscr{F} \otimes \mathscr{G} \approx \mathcal{O}_X$.
For $x \in X$, then $\left(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x\right) \otimes_{\mathcal{O}_{X,x}} k(x) \approx \left(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)\right) \otimes_{k(x)} \left(\mathscr{G}_x \otimes_{\mathcal{O}_{X,x}} k(x)\right) \approx k(x)$.
Thus $\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x), \mathscr{G}_x \otimes k(x)$ are dimension 1 .
Let U be a set where $\mathscr{F}|_U \approx \tilde{M}$, and $\mathscr{G} \approx \tilde{N}$, $\mathfrak{p} \approx \mathfrak{m}_x$.
$\mathscr{F}, \mathscr{G}$ coherent $\implies M, N$ are f.g, so from a set of generators of $\mathscr{F}_x \otimes k(x) \approx M_{\mathfrak{p}} \otimes k(x)$ we obtain a set of generators for $M_{\mathfrak{p}}$ via nakayama.
As $\mathscr{F}_x \otimes k(x)$ is a one-dimensional vector space, $M_{\mathfrak{p}}$ is generated by $m \in M$ as an $A_{\mathfrak{p}}$-module, and $N_{\mathfrak{p}}$ is generated by n.
As $M_{\mathfrak{p}} \otimes N_{\mathfrak{p}}$ is therefore generated by $m \otimes n$.
Define $f : A_{\mathfrak{p}} \to M_{\mathfrak{p}}$ by $\frac{a}{s} \mapsto \frac{a}{s} m$, and an inverse by $\frac{m}{s} \mapsto \frac{m}{s} \otimes n \mapsto \frac{a}{s}$.
This morphism gives $\mathscr{F}_x \approx \mathcal{O}_{X,x}$.

2.5.20 II.5.8 x

5.8. Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function

$$\varphi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x),$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following results.

(a) The function φ is *upper semi-continuous*, i.e., for any $n \in \mathbf{Z}$, the set $\{x \in X | \varphi(x) \geqslant n\}$ is closed.

Using properties of the induced topology, we only need to check that $\Phi(n) = \{x \in X : \varphi(x) \geq n\}$ is closed on open affines of a cover.

Let $x \in \Phi(n)^c$ so that $m = dim_{k(p)} M_p/\mathfrak{p}M_\mathfrak{p} < n$, $\mathfrak{p} \in Spec\ A = X$, $\mathfrak{p}$ correpsonds to x, and $M = \Gamma(X, \mathscr{F})$. By Nakayama's lemma, $M_\mathfrak{p}$ is generated by less than n elements $m_i \in M$ as well.

Let n_i a generating set for M.

In $M_\mathfrak{p}$, $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$, and if $s = \prod s_{ij}$, then $sn_i = \sum a'_{ij} m_j$ for some a'_{ij} .

By definition of localization $s \notin \mathfrak{p}$, so that $\mathfrak{p} \in D(s)$.

If $\mathfrak{q} \in D(s)$ then $s \notin \mathfrak{q} \implies s$ invertible in $A_\mathfrak{q} \implies n_i = \sum \frac{a'_{ij}}{s} m_j$ there.

n_i generate $M_\mathfrak{q}$, and since m_j generate n_i , then m_j generate M_q .

Thus M_q is generated by $< n$ elements, and thus so is $M_\mathfrak{q}/\mathfrak{q}M_\mathfrak{q}$.

Thus $\mathfrak{q} \in \Phi(n)^c$, so that $D(s) \subset \Phi(n)^c$ as $\mathfrak{q}$ was arbitrary, is an open neighborhood of $\mathfrak{p}$. Thus $\Phi(n)^c$ is open.

2.5.21 (b) x

(b) If $\mathscr{F}$ is locally free, and X is connected, then φ is a constant function.

If $\mathscr{F}$ is locally free then every point has a neighborhood where φ is constant.

Then $\phi(n) = \{x : \varphi(x) \geq n\}$ is a union of open sets and therefore open, but also closed by (a), thus the whole space or empty since X is connected.

Pick some n such that $\phi(n)$ is empty. Now for one larger n...

2.5.22 (c) x

(c) Conversely, if X is reduced, and φ is constant, then $\mathscr{F}$ is locally free.

Let $x \in X$, $x \in U = Spec\ A$, and $\mathfrak{p}$ correspond to x.

Let M correspond to $\mathscr{F}|_{Spec\ A}$, a finitely generated A module, as $\mathscr{F}$ is assumed coherent.

Let $n = dim\ M_\mathfrak{p} \otimes k(\mathfrak{p})$, and choose, by Nakayama's lemma, a set of n generators, m_i for $M_\mathfrak{p}$.

Then we can write a finite set of generators n_i for M in $M_\mathfrak{p}$ as $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$. For $s = \prod s_{ij}$, we have a short exact sequence

$0 \to ker\ \varphi \to A_s^{\oplus n} \xrightarrow{\varphi} M_s \to 0$

which also holds on $A_\mathfrak{q}$, $\mathfrak{q} \in D(s)$.

φ constant by assumption, implies each $M_\mathfrak{q} \otimes k(\mathfrak{q})$ has dimension n , so $k(\mathfrak{q}) \otimes ker\ \varphi = 0$ for all such $\mathfrak{q} \in D(s)$,

Thus for any $y \in ker\ \varphi$, y is the sum of elements of $\mathfrak{q}A_s$ for all $\mathfrak{q} \in D(s)$, which are therefore in the nilradical. But A_s is reduced as X is reduced so $ker\ \varphi = 0$, and thus M_s is free.

2.5.23 II.5.9 x

5.9. Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \operatorname{Proj} S$.
(a) Show that there is a natural homomorphism $\alpha: M \to \Gamma_*(\tilde{M})$.

$m \in M_d$ has degree zero in $M(d)_{(f)} = \Gamma(D_+(f), M(n)^\sim)$, and thus defines a section on each $D_+(f)$.

These sections agree on intersections and give a global section, so we obtain $\alpha : M \to \Gamma_*\left(\tilde{M}\right)$, which is a homomorphism of groups.

If $s \in S_e$, $m \in M_d$, then $s\alpha(m) \in \Gamma_*\left(\tilde{M}\right)$ is defined as the image of $m \otimes s$ in $\Gamma(X, M(d)^\sim \otimes \mathcal{O}_X(e)) \approx \Gamma(X, M(d+e)^\sim)$.

Thus α gives a morphism of graded modules.

2.5.24 (b) x

(b) Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees, i.e., there is a $d_0 \in \mathbf{Z}$ such that for all $d \geqslant d_0$, $\alpha_d: M_d \to \Gamma(X, \tilde{M}(d))$ is an isomorphism. [*Hint*: Use the methods of the proof of (5.19).]

(EGA) Note that $\Gamma\left(X, \tilde{M}(d)\right)$ is a quasi-finitely generated graded S-module. Define $\alpha_n : M_n \to \Gamma(X, M(n)^\sim)$ by $m \mapsto m/1$. Note that $M \to \Gamma_*\left(\tilde{M}\right)$ and $\Gamma_*(\mathscr{F})^\sim \to \mathscr{F}$ are adjoint functors with counit $\epsilon : \Gamma_*(\mathscr{F}) \to \mathscr{F}$ given by $\epsilon_{D_+(s)}\left(\frac{m}{s}\right) \mapsto \nu\left(\frac{1}{s} \cdot m|_{D_+(s)}\right)$, where $\nu : \mathscr{F}(0) \to \mathscr{F}$ is the canonical isomorphism. As the composite of $\tilde{\alpha}$ with ϵ is the identity, then $\tilde{\alpha}$ is the unit. Note that $\mathscr{F}$ quasi-coherent implies the counit is an isomorphism by thm II.5.15. Note further that as the twisting functor is exact, a morphism of quasi-finitely generated modules is a quasi-isomorphism iff the morphism on associated modules is an isomorphism.

2.5.25 (c) x

(c) With the same hypotheses, we define an equivalence relation $\approx$ on graded S-modules by saying $M \approx M'$ if there is an integer d such that $M_{\geqslant d} \cong M'_{\geqslant d}$. Here $M_{\geqslant d} = \bigoplus_{n \geqslant d} M_n$. We will say that a graded S-module M is *quasi-finitely generated* if it is equivalent to a finitely generated module. Now show that the functors $\tilde{}$ and Γ_* induce an equivalence of categories between the category of quasi-finitely generated graded S-modules modulo the equivalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.

By (b), M is equivalent to $\Gamma_*\left(\tilde{M}\right)$ if M is finitely generated. By II.5.15, $\Gamma_*(\mathscr{F})^\sim$ is isomorphic to $\mathscr{F}$ for $\mathscr{F}$ quasicoherent.

Thus we want to show that for q quasi-finitely generated graded S-module M, then $\tilde{M}$ is coherent, and for coherent sheaf $\mathscr{F}$, $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

suppose q-f- generated

Suppose first that M is quasi-finitely generated. Let M' an f.g. S-module with $M_{\geq d} \approx M'_{>d}$ for some d. Then for any $f \in S_1$, $M_{(f)} \approx M'_{(f)}$, since $\frac{m}{f^n} = \frac{mf^d}{f^{n+d}}$. M' finitely generated implies $M'_{(f)}$ is finitely generated.

As S is generated by S_1 as an S_0-algebra, then $M_{(f)}$ cover $X = Proj\ S$ for various f. On such a cover $\tilde{M}$ is locally equivalent to a coherent sheaf.

suppose coherent

If $\mathscr{F}$ is coherent, by thm II.5.17, $\mathscr{F}(n)$ is generated by a finite number of global sections for large enough n . If M' is the submodule of $\Gamma_*(\mathscr{F})$ generated by these sections, then $\tilde{M}' \hookrightarrow \Gamma_*\tilde{(\mathscr{F})} \approx \mathscr{F}$ via th II.5.15.

By exactness of twist we get $\tilde{M'(n)} \hookrightarrow \mathscr{F}(n)$ which is an isomorphism since $\mathscr{F}(n)$ is gbgs in M' . Tensoring with $\mathcal{O}(-n)$ gives $\tilde{M}' \approx \mathscr{F}$.

As M' is f.g, then by (b) for large enough d, $M_d \approx \Gamma(X, \mathscr{F}(d))$ which shows quasi-finite generation.

2.5.26 II.5.10 x

5.10. Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \operatorname{Proj} S$. We have seen that a homogeneous ideal I in S defines a closed subscheme of X (Ex. 3.12), and that conversely every closed subscheme of X arises in this way (5.16).

(a) For any homogeneous ideal $I \subseteq S$, we define the *saturation* $\bar{I}$ of I to be $\{s \in S \mid \text{for each } i = 0, \ldots, r, \text{ there is an } n \text{ such that } x_i^n s \in I\}$. We say that I is *saturated* if $I = \bar{I}$. Show that $\bar{I}$ is a homogeneous ideal of S.

I is clearly an ideal.

Write $\overline{I} \ni s = s_0 + ... + s_k$ with each s_i homogeneous of degree i .

x_i homogeneous of degree 1 $\implies$ $x_i s_k$ is homogeneous of degree $n+k$.

I homogeneous ideal and $x_i^n s \in I$ $\implies x_i^n s_k \in I$.

$\implies s_k \in \overline{I}$.

$\implies \overline{I}$ homogeneous.

2.5.27 (b) x

(b) Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.

Suppose that I_1 and I_2 define the same closed subscheme of X.

By thm II.5.9, they define the same q.c. sheaf of ideals $\mathscr{I}$ on X.

If $s \in I_1$ is homogeneous of degree d , then $\frac{s}{x_i^d}$ is a section of $\mathscr{I}(D_+(x_i))$.

As I_1 and I_2 define the same ideal sheaf, then for each i , there is $t_i \in I_2$, homogeneous of degree d with $\frac{s}{x_i^d} = \frac{t_i}{x_i^d}$, which implies $x_i^{n_i}(s - t_i) = 0$ for some n_i. Since $t_i \in I_2$, so is $x_i^{n_i} t_i = x_i^{n_i} s$, thus s is in the saturation of I_2, hence $I_2 \subset \overline{I}_1$. Since the operation of saturation is idempotent, $\overline{I_2} = \overline{I}_1$.

2.5.28 (c) x

(c) $\boxed{\text{If } Y \text{ is any closed subscheme of } X, \text{ then the ideal } \Gamma_*(\mathscr{I}_Y) \text{ is saturated}}$ Hence it is the largest homogeneous ideal defining the subscheme Y.

Let $s \in \overline{\Gamma_*(\mathscr{I}_Y)}$, bar stands for saturation.

By definition, for each i there is n with $x_i^n s \in \Gamma_*(\mathscr{I}_Y)$

Choose N larger than all such n.

We claim that $s|_{U_i}$ is in $\Gamma(U_i, \mathscr{I}_Y(d))$.

As $x_i^N s \in \Gamma(X, \mathscr{I}_Y(d+N))$, $x_i^{-n} \otimes x_i^n s \in \Gamma(U_i, \mathscr{I}_Y(d+N) \otimes \mathcal{O}(-N)) \approx \Gamma(U_i, \mathscr{I}_Y(d))$ with image s.

2.5.29 (d) x

(d) There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.

Homogeneous ideals of S correspond to q.c. sheaves of ideals via $\Gamma_*(-)$ and sheafification.

Quasi-coherent sheaves of ideals correspond, 1-1, via taking the ideal and Prop II.5.9 to closed subschemes of X.

As there is a unique saturated homogeneous ideal in the preimage of each sheafification by (b), then by (c) we get a bijection as Γ_* is the inverse.

2.5.30 II.5.11 x g

5.11. Let S and T be two graded rings with $S_0 = T_0 = A$. We define the *Cartesian product* $S \times_A T$ to be the graded ring $\bigoplus_{d \geq 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) \cong X \times_A Y$, and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.

The Cartesian product of rings is related to the *Segre embedding* of projective spaces (I, Ex. 2.14) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1 over A, corresponding to a projective embedding $X \hookrightarrow \mathbf{P}_A^r$, and if $y_0, \ldots, y_s$ is a set of generators for T_1, corresponding to a projective embedding $Y \hookrightarrow \mathbf{P}_A^s$, then $\{x_i \otimes y_j\}$ is a set of generators for $(S \times_A T)_1$, and hence defines a projective embedding $\operatorname{Proj}(S \times_A T) \hookrightarrow \mathbf{P}_A^N$, with $N = rs + r + s$. This is just the image of $X \times Y \subseteq \mathbf{P}^r \times \mathbf{P}^s$ in its Segre embedding.

Let $\alpha_0, ..., \alpha_r$ and $\beta_0, ..., \beta_s$ be the generators of the A-modules S and T, respectively. Then $\alpha_i \otimes \beta_j$ become generators of $S_1 \otimes_A T_1$ and $S \times_A T = A[\alpha_i \otimes \beta_j]$. As $S \times_A T_{(\alpha_i \otimes \beta_j)} \approx S_{(\alpha_i)} \otimes_A T_{(\beta_j)}$ for all $0 \leq i \leq r$, $0 \leq j \leq s$, then $D_+(\alpha_i \otimes \beta_j) \approx Spec\ S_{(\alpha_i)} \times_A Spec\ T_{(\beta_j)} \approx D_+(\alpha_i) \times D_+(\beta_j)$. Thus $Proj\ S \times_A T \approx X \times_A Y$.

The second property follows from the fact that II.5.12.c, and the universal property of the cartesian product.

2.5.31 II.5.12 x g

5.12. (a) Let X be a scheme over a scheme Y, and let $\mathscr{L}, \mathscr{M}$ be two very ample invertible sheaves on X. Show that $\mathscr{L} \otimes \mathscr{M}$ is also very ample. [*Hint*: Use a Segre embedding.]

Nakai moisheazon. Or take the segre product of the two closed embeddings.

2.5.32 (b) x g

(b) Let $f: X \to Y$ and $g: Y \to Z$ be two morphisms of schemes. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathscr{L} \otimes f^*\mathscr{M}$ is a very ample invertible sheaf on X relative to Z.

By assumption, there is a closed immersion $i : X \to \mathbb{P}_Y^{n_1}$ with $i^*\left(\mathcal{O}_{\mathbb{P}_Y^{n_1}}(1)\right) \approx \mathscr{L}$, and a closed immersion $j : Y \to \mathbb{P}_Z^{n_2}$ with $j^*\mathcal{O}_{\mathbb{P}_Z^{n_2}}(1) \approx \mathscr{M}$. Consider the composition $X \xrightarrow{i} \mathbb{P}^{n_1} \times Y \xrightarrow{id \times j} \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times Z \xrightarrow{\psi \times id} \mathbb{P}^N \times Z$

with $N = n_1n_2 + n_1 + n_2$ and ψ the segre embedding. We have $\phi^* \left(\mathcal{O}_{\mathbb{P}^N_Z}(1)\right) \approx \mathscr{L} \otimes f^*\mathscr{M}$ which is what we wanted to show.

2.5.33 II.5.13 x

5.13. Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d > 0$, let $S^{(d)}$ be the graded ring $\bigoplus_{n \geq 0} S_n^{(d)}$ where $S_n^{(d)} = S_{nd}$. Let $X = \operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)} \cong X$, and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.

This construction is related to the d-uple *embedding* (I, Ex. 2.12) in the following way. If $x_0, \ldots, x_r$ is a set of generators for S_1, corresponding to an embedding $X \hookrightarrow \mathbf{P}_A^r$, then the set of monomials of degree d in the x_i is a set of generators for $S_1^{(d)} = S_d$. These define a projective embedding of $\operatorname{Proj} S^{(d)}$ which is none other than the image of X under the d-uple embedding of $\mathbf{P}_A^r$.

As S is generated by S_1 over S_0 , then $S^{(d)}$ is generated by $S_1^{(d)} = S_d$ over S_0 . Thus the open sets $D_+(f)$, $f \in S_d$ cover both $Proj\ S$ and $Proj\ S^{(d)}$. The identity map $\frac{s}{f^n} \mapsto \frac{s}{f^n}$ identifies $S_{(f)}$ and $S_{(f)}^{(d)}$ so that $Spec\ S_{(f)} \approx Spec\ S_{(f)}^{(d)}$, and glueing gives $Proj\ S \approx Proj\ S^{(d)}$. The same maps give $S(d)_{(f)} \approx S^{(d)}(1)_{(f)}$ so that $\mathcal{O}(1)$ and $\mathcal{O}_X(d)$ correspond.

2.5.34 II.5.14 x

5.14. Let A be a ring, and let X be a closed subscheme of $\mathbf{P}_A^r$. We define the *homogeneous coordinate ring* $S(X)$ of X for the given embedding to be $A[x_0, \ldots, x_r]/I$, where I is the ideal $\Gamma_*(\mathscr{I}_X)$ constructed in the proof of (5.16). (Of course if A is a field and X a variety, this coincides with the definition given in (I, §2)!) Recall that a scheme X is *normal* if its local rings are integrally closed domains. A closed subscheme $X \subseteq \mathbf{P}_A^r$ is *projectively normal* for the given embedding, if its homogeneous coordinate ring $S(X)$ is an integrally closed domain (cf. (I, Ex. 3.18)). Now assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbf{P}_k^r$. Show that for some $d > 0$, the d-uple embedding of X is projectively normal, as follows.

(a) Let S be the homogeneous coordinate ring of X, and let $S' = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure. [*Hint*: First show that X is integral. Then regard S' as the global sections of the sheaf of rings $\mathscr{S} = \bigoplus_{n \geq 0} \mathcal{O}_X(n)$ on X, and show that $\mathscr{S}$ is a sheaf of integrally closed domains.]

As $\mathcal{O}_{x,X}$ are integral domains, then X is reduced.

Since X is by assumption normal, X is irreducible, and S is a domain.

If $\mathscr{L}$ is the sheaf $\oplus_{n\geq 0}\mathcal{O}_X(n)$, then $\mathscr{L}_\mathfrak{p} = \oplus_{n\geq 0} S(n)_{(\mathfrak{p})} = \left\{\frac{s}{f} \in S | deg\ s \geq deg\ f\right\}$. Then $\mathscr{L}_\mathfrak{p}$ is integrally closed since an elemeent integral over $\mathscr{L}_\mathfrak{p}$ is integral over $S_\mathfrak{p}$ which is integrally closed since X is normal. On the other hand only elements which have positive degree can be integral over $\mathscr{L}_\mathfrak{p}$.

Taking global sections is left exact so $\Gamma(X, \mathscr{L}) \approx \oplus_{n\geq 0}\Gamma(X, \mathcal{O}_X(n)) = S'$, which is integrally closed. As in thm II.5.19, S' is contained in the integral closure of S, so $S' = \overline{S}$.

2.5.35 (b) x

> (b) Use (Ex. 5.9) to show that $S_d = S'_d$ for all sufficiently large d.

By 5.9b, since $\tilde{S} \approx \mathcal{O}_X$.

2.5.36 (c) x

> (c) Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.

For $d \gg 0$, $S_{nd} = S'_{nd}$ by (b).
If $s \in K\left(S^{(d)}\right)$ is integral over $S^{(d)}$, it lies in $S'^{(d)}$, the integral closure of $S^{(d)}$.
Hence the homogeneous coordinate ring is integrally closed.

2.5.37 (d) x

> (d) As a corollary of (a), show that a closed subscheme $X \subseteq \mathbf{P}^r_A$ is projectively normal if and only if it is normal, and for every $n \geqslant 0$ the natural map $\Gamma(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$ is surjective.

X projectively normal $\implies$ S integrally closed by definition.
$\implies$ $S \approx S'$, the integral closure.
$\implies$ $S_n = \Gamma\left(X, \mathcal{O}_X(n)\right)$ for all n by (a).
If $T = A\left[x_0, ..., x_r\right]$ then $T_n = \Gamma\left(\mathbb{P}^r_A, \mathcal{O}_{\mathbb{P}^r_A}(n)\right)$ by (a) and this surjectcs onto $\Gamma\left(X, \mathcal{O}_X(n)\right)$.
If $\Gamma\left(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)\right) \twoheadrightarrow \Gamma\left(X, \mathcal{O}_X(n)\right)$, then $S = S'$ when S is normal by (a).

2.5.38 II.5.15 x Extension of Coherent Sheaves

> **5.15.** *Extension of Coherent Sheaves.* We will prove the following theorem in several steps: Let X be a noetherian scheme, let U be an open subset, and let $\mathscr{F}$ be a coherent sheaf on U. Then there is a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_U \cong \mathscr{F}$.
> (a) On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves. We say a sheaf $\mathscr{F}$ is the *union* of its subsheaves $\mathscr{F}_\alpha$ if for every open set U, the group $\mathscr{F}(U)$ is the union of the subgroups $\mathscr{F}_\alpha(U)$.

If X is affine, then a q.c. sheaf is a module, and a coherent sheaf is an f.g. module.
Note that an A-module is a union of its finitely generated submodules.

2.5.39 (b) x

> (b) Let X be an affine noetherian scheme, U an open subset, and $\mathscr{F}$ coherent on U. Then there exists a coherent sheaf $\mathscr{F}'$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$. [*Hint*: Let $i: U \to X$ be the inclusion map. Show that $i_*\mathscr{F}$ is quasi-coherent, then use (a).]

Using II.5.8.c, $i_*\mathscr{F}$ is q.c. By (a) $i_*\mathscr{F} = \bigcup \mathscr{G}_\alpha$, $\mathscr{G}_\alpha \approx N_\alpha^\sim$ is a coherent subsheaf of $i_*\mathscr{F}$. Since X is noetherian, this union has a maximal element, $\bigcup N_\alpha^\sim \approx i^*\mathscr{F}'$. But then $\mathscr{F}'$ is a coherent subsheaf of $i_*\mathscr{F}$ and $\mathscr{F}'|_U \approx \mathscr{F}$.

2.5.40 (c) x

(c) With $X, U, \mathscr{F}$ as in (b), suppose furthermore we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that we can find $\mathscr{F}'$ a coherent subsheaf of $\mathscr{G}$, with $\mathscr{F}'|_U \cong \mathscr{F}$. [*Hint*: Use the same method, but replace $i_*\mathscr{F}$ by $\rho^{-1}(i_*\mathscr{F})$, where ρ is the natural map $\mathscr{G} \to i_*(\mathscr{G}|_U)$.]

Consider $\rho^{-1}(i_*\mathscr{F}) \subset \mathscr{G}$ which is the pullback of q.c., thus is q.c. As $\rho^{-1}(i_*\mathscr{F})|_U \approx \mathscr{F}$, then as in (b), we find $\mathscr{F}'|_U \approx \mathscr{F}$.

2.5.41 (d) x

(d) Now let X be any noetherian scheme, U an open subset, $\mathscr{F}$ a coherent sheaf on U, and $\mathscr{G}$ a quasi-coherent sheaf on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that there is a coherent subsheaf $\mathscr{F}' \subseteq \mathscr{G}$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$. Taking $\mathscr{G} = i_*\mathscr{F}$ proves the result announced at the beginning. [*Hint*: Cover X with open affines, and extend over one of them at a time.]

Let $U_1, ..., U_n$ be an open affine cover of X. Using (b), (c), extend $\mathscr{F}|_{U_1 \cap U}$ to a coherent sheaf $\mathscr{F}' \subset \mathscr{G}|_{U_1}$, and glue $\mathscr{F}$ and $\mathscr{F}'$ on the open set $U_1 \cap U$ to get a coherent sheaf $\mathscr{F}$ on $U \cap U_1$. Now do the same thing but with $X' = U_i$ and $U' = U_i \cap (U \cup U_1)$. And repeat.

2.5.42 e. x

(e) As an extra corollary, show that on a noetherian scheme, any quasi-coherent sheaf $\mathscr{F}$ is the union of its coherent subsheaves. [*Hint*: If s is a section of $\mathscr{F}$ over an open set U, apply (d) to the subsheaf of $\mathscr{F}|_U$ generated by s.]

If s is a section of $\mathscr{F}$, we apply d to $\mathscr{F}|_U$ generated by s.
This gives for each open U and $s \in \mathscr{F}(U)$ a coherent subsheaf $\mathscr{G}$ of $\mathscr{F}$ where $s \in \mathscr{G}(U)$.
Now take the union of the $\mathscr{G}$.

2.5.43 II.5.16 xc a. g Tensor Operations on Sheaves

5.16. *Tensor Operations on Sheaves.* First we recall the definitions of various tensor operations on a module. Let A be a ring, and let M be an A-module. Let $T^n(M)$ be the tensor product $M \otimes \ldots \otimes M$ of M with itself n times, for $n \geqslant 1$. For $n = 0$ we put $T^0(M) = A$. Then $T(M) = \bigoplus_{n \geqslant 0} T^n(M)$ is a (noncommutative) A-algebra, which we call the *tensor algebra* of M. We define the *symmetric algebra* $S(M) = \bigoplus_{n \geqslant 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $S(M)$ is a commutative A-algebra. Its component $S^n(M)$ in degree n is called the nth *symmetric product* of M. We denote the image of $x \otimes y$ in $S(M)$ by xy, for any $x, y \in M$. As an example, note that if M is a free A-module of rank r, then $S(M) \cong A[x_1, \ldots, x_r]$.

We define the *exterior algebra* $\bigwedge(M) = \bigoplus_{n \geqslant 0} \bigwedge^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge(M)$ is a *skew commutative* graded A-algebra. This means that if $u \in \bigwedge^r(M)$ and $v \in \bigwedge^s(M)$, then $u \wedge v = (-1)^{rs} v \wedge u$ (here we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2(M)$ is denoted by $x \wedge y$). The nth component $\bigwedge^n(M)$ is called the nth *exterior power* of M.

Now let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-modules. We define the *tensor algebra*, *symmetric algebra*, and *exterior algebra* of $\mathscr{F}$ by taking the sheaves associated to the presheaf, which to each open set U assigns the corresponding tensor operation applied to $\mathscr{F}(U)$ as an $\mathcal{O}_X(U)$-module. The results are $\mathcal{O}_X$-algebras, and their components in each degree are $\mathcal{O}_X$-modules.

(a) Suppose that $\mathscr{F}$ is locally free of rank n. Then $T^r(\mathscr{F})$, $S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.

$\mathscr{F}$ locally free of rank n implies there are $e_1, ..., e_n$ such that on some open cover $\{U\}$, $\mathscr{F}(U) \approx \mathcal{O}_X(U)_{e_1}|_U \oplus \cdots \oplus \mathcal{O}_X(U)\, e_n|_U$, i.e. we can the e_i are basis global sections.

Then the presheaves $U \mapsto T^r\mathscr{F}(U)$, $U \mapsto S^r\mathscr{F}(U)$, $U \mapsto \Lambda^r\mathscr{F}(U)$ are free with basis global sections $\{e_{i_1} \otimes \cdots \otimes e_{i_r} | 1 \leq i_1, ..., i_r \leq n\}$, $\{e_{i_1} \cdots e_{i_r} | 1 \leq i_1 \leq ... \leq i_r \leq n\}$, and $\{e_{i_1} \wedge ... \wedge e_{i_r} | 0 < i_1 < ... < i_r < n\}$ respectively. Note that free presheaves are sheaves, and on the cover U , each of the above presheaves is free with the rank of the basis. Each basis can be calculated to have the required dimension.

2.5.44 (b) xc g

(b) Again let $\mathscr{F}$ be locally free of rank n. Then the multiplication map $\bigwedge^r\mathscr{F} \otimes \bigwedge^{n-r}\mathscr{F} \to \bigwedge^n\mathscr{F}$ is a perfect pairing for any r, i.e., it induces an isomorphism of $\bigwedge^r\mathscr{F}$ with $(\bigwedge^{n-r}\mathscr{F})^\vee \otimes \bigwedge^n\mathscr{F}$. As a special case, note if $\mathscr{F}$ has rank 2, then $\mathscr{F} \cong \mathscr{F}^\vee \otimes \bigwedge^2\mathscr{F}$.

Let $e_1, ..., e_n$ be basis elements.

The pairing defined by $\omega \otimes \lambda \mapsto \omega \wedge \lambda$ gives an isomorphism $\mathcal{O}_X \to \Lambda^n\mathscr{F}$, $f \mapsto f(e_1 \wedge ... \wedge e_n)$.

If λ is a global section of $\Lambda^{n-r}\mathscr{F}$, then λ defines a morphism $\Lambda^r\mathscr{F} \to \Lambda^n\mathscr{F} \approx \mathcal{O}_X$ by $\omega \mapsto \omega \wedge \lambda$.

On the other hand given a morphism of $\mathcal{O}_X$-modules $\Lambda^r\mathscr{F} \to \Lambda^n\mathscr{F} \approx \mathcal{O}_X$, we have a morphism on global sections $\varphi : \Gamma(X, \lambda^r\mathscr{F}) \to \Gamma(X, \Lambda^n\mathscr{F}) \approx \Gamma(X, \mathcal{O}_X)$. Then a global section of $\Lambda^{n-r}\mathscr{F}$ is defined by $\sum (-1)^{ki} \varphi(e_{i_1} \wedge ... \wedge e_{i_r})\, e_{j_1} \wedge ... \wedge e_{j_{n-r}}$, where j_k are elements of $\{1, ..., n\} \setminus \{i_l\}$.

The operations defined in the two preceeding paragraphs are inverses so $\Lambda^r\mathscr{F} \approx (\Lambda^{n-r}\mathscr{F})^* \otimes \Lambda^n\mathscr{F}$.

2.5.45 (c) cx

(c) Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathscr{F})$,

$$S^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \ldots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

for each p.

Let U be an open set where the sheaves $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ are free.
Note that $\mathscr{F}|_U \approx \mathscr{F}'|_U \oplus \mathscr{F}''|_U$ implies that
$S^r\mathscr{F}|_U \approx \oplus_{i=0}^r (S^i\mathscr{F}'|_U \otimes S^{r-i}\mathscr{F}''|_U)$.
Set $F^{r+1} = 0$ as in the hypothesis, and assume that we have chosen, F^i such that $F^i/F^{i+1} \approx S^i\mathscr{F}'|_U \otimes S^{r-i}\mathscr{F}''|_U$, $i \geq j$.
Suppose that x_i are a basis of $\mathscr{F}'|_U$ and y_i are a basis for $\mathscr{F}''|_U$.
If $y_i + c_i$, $c_i \in \mathscr{F}'|_U$ are another basis for $\mathscr{F}''|_U$, then $x_i \otimes (y_j + c_j) \mapsto x_i y_j + x_i c_j \equiv x_i y_j mod\ F^j = S^r\mathscr{F}'|_U$.
Thus the lift of a basis for $S^r\mathscr{F}|_U/F$ is independent of splitting, and so we define F^{j-1} to be such a lift.

2.5.46 (d) x

(d) Same statement as (c), with exterior powers instead of symmetric powers. In particular, if $\mathscr{F}', \mathscr{F}, \mathscr{F}''$ have ranks n', n, n'' respectively, there is an isomorphism $\bigwedge^n \mathscr{F} \cong \bigwedge^{n'} \mathscr{F}' \otimes \bigwedge^{n''} \mathscr{F}''$.

As in (c).

2.5.47 (e) xc

(e) Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathcal{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathscr{F}$, i.e., $f^*(S^n(\mathscr{F})) = S^n(f^*\mathscr{F})$ etc.

For $n = 0$ this is clear.
Note that $T^n f^*\mathscr{F} \approx f^*\mathscr{F} \otimes_{\mathcal{O}_X} T^{n-1} f^*\mathscr{F}$ by definition.
That is $(f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X) \otimes_{\mathcal{O}_X} f^* T^{n-1}\mathscr{F}$ by definition.
That is $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} f^* T^{n-1}\mathscr{F}$ by rules of tensor.
That is $f^{-1}\mathscr{F} \otimes_{f^{-1}\mathcal{O}_Y} (f^{-1}\mathscr{F}^{\otimes n-1}) \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$ by induction.
Which is $f^* T^n \mathscr{F}$ as colimits (f^{-1})commute with left adjoints $(\otimes\ )$.
Thus tensor algebra commutes with pullback.

Suppose $\mathscr{I}$ is the subsheaf defined by
$0 \to \mathscr{I} \to T^n\mathscr{F} \to S^n\mathscr{F} \to 0$.
Pullbacks are left adjoint thus right exact, and tensor commutes with pullback so we get
$0 \to f^*\mathscr{I} \to f^*T^n\mathscr{F} \to f^*S^n\mathscr{F} \to 0$.
By above, this is $0 \to f^*\mathscr{I} \to T^n f^*\mathscr{F} \to f^*S^n\mathscr{F} \to 0$.
But then $S^n f^*\mathscr{F} \approx f^* S^n \mathscr{F}$.
Dido Λ.

2.5.48 II.5.17 x Affine Morphisms

5.17. *Affine Morphisms.* A morphism $f:X \to Y$ of schemes is *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

(a) Show that $f:X \to Y$ is an affine morphism if and only if for *every* open affine $V \subseteq Y, f^{-1}(V)$ is affine. [*Hint*: Reduce to the case Y affine, and use (Ex. 2.17).]

Suppose that $f : X \to Y$ is affine. Let $\{V\}_i$ an open affine cover of Y such that $f^{-1}V_i$ is affine for all i . Given another open affine subset $V \subset Y$, then $V \cap V_i$ is covered by $D(d_{ij})$ which are distinguished on V_i.

Let $A_i = \Gamma(V_i, \mathcal{O}_Y)$, $B_i = \Gamma(f^{-1}V_i, \mathcal{O}_X)$ denote the rings of sections on these affine sets.

Then $f|_{f^{-1}V_i} : f^{-1}V_i \to V_i$ is induced by $\phi : A_i \to B_i$ and $f^{-1}|_{f^{-1}V_i} D(f_{ij}) = D(\phi f_{ij})$ is also affine.

The open sets $D(f_{ij})$ for a cover of V with affine preimages.

Thus $f|_{f^{-1}V_i}$ is affine. Thus we have reduced to the case where $Y = Spec\ B$ is affine and f is affine.

By definition, there is an open cover $Spec\ B_i$ of Y where the preimages $f^{-1}Spec\ B_i$ are affine subschemes of X. Let $\{D(f_i)\}$ be a refinement to distinguished open affines with affine preimages.

As Y is affine, it is quasi-compact, so there is a finite subcover of $\{D(f_i)\}$. Thus $\sum_{i=1}^{n} a_i f_i = 1$ and $f^{-1}D(f_i)$ are open affines. so $\sum a_i f^\sharp(f_i) = 1 \in \Gamma(X, \mathcal{O}_X)$, where $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$. Furthermore, restricting to an open affine cover of X gives that $X_{g_i} = f^{-1}D(f_i)$.

Thus we have shown that X is affine. As open immersions are perserved by base change, then morphisms between affine schemes are preserved under base change, thus the preimage $f^{-1}U = U \times_X Y$ with U affine is affine.

2.5.49 (b) x g

. *Affine Morphisms.* A morphism $f:X \to Y$ of schemes is *affine* if there is an open affine cover $\{V_i\}$ of Y such that $f^{-1}(V_i)$ is affine for each i.

(b) An affine morphism is quasi-compact and separated. Any finite morphism is affine.

Suppose $f : X \to Y$ is affine. Let V_i an open affine cover of Y.

By (a), $f^{-1}V_i$ is affine, and thus quasi-compact.

Thus f is quasi-compact.

By 4.1, each restriction $f^{-1}(V_i) \to V_i$ is affine, and so by thm II.4.1, these restrictions are separated. The diagonal $X \to X \times_Y X$ factors through the restrictions, and is thus separated.

Note by definition that a finite morphism is proper and affine.

2.5.50 (c) x g (used for stein factorization)

(c) Let Y be a scheme, and let $\mathscr{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras (i.e., a sheaf of rings which is at the same time a quasi-coherent sheaf of $\mathcal{O}_Y$-modules). Show that there is a unique scheme X, and a morphism $f:X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathscr{A}(V)$, and for every inclusion $U \hookrightarrow V$ of open affines of Y, the morphism $f^{-1}(U) \hookrightarrow f^{-1}(V)$ corresponds to the restriction homomorphism $\mathscr{A}(V) \to \mathscr{A}(U)$. The scheme X is called **Spec** $\mathscr{A}$. [*Hint*: Construct X by glueing together the schemes $\operatorname{Spec} \mathscr{A}(V)$, for V open affine in Y.]

We will define X as the $Spec\ \mathscr{A}(U)$ glued together.

If $U = Spec\ A$, $V = Spec\ B$ have nonempty intersection, then we can cover $U \cap V$ by open sets $W_i = Spec\ C_i$ distinguished in both U and V.

Since $\mathscr{A}$ is an $\mathcal{O}_Y$-module, then there is a restriction $\rho_{UW} : \mathscr{A}(U) \to \mathscr{A}(W)$.

W_i being distinguished in U, V , $\implies$ C is a localization of A and B, and by quasi-coherence, $\mathscr{A}(W)$ is a localization of $\mathscr{A}(U)$ and $\mathscr{A}(V)$. Thus we identify $\mathscr{A}(U), \mathscr{A}(V)$ along $\mathscr{A}(W)$. Let $g : Spec\ \mathscr{A}(U) \to U$, and $h : Spec\ \mathscr{A}(V) \to V$ be the induced morphisms.

The isomorphisms given by the distinguished covering of $U \cap V$ glue together to give an isomorphism $g^{-1}(U \cap V) \approx h^{-1}(U \cap V)$ and agree on triple overlaps since the restriction maps on a sheaf do. Glueing together gives X.

Now we wish to define $f : X \to Y$. We glue the maps $\mathscr{A}(U) \to U$ on all open affines since they are compatible on the overlaps. If $U \subset V \subset Y$, then $f^{-1}(U) \to f^{-1}(V)$ comes from the restriction morphism ρ_{VU} as above.

For uniqueness, suppose there is another such scheme X', with a morphism $f' : X' \to Y$. But then glueing morphisms on $Spec\ \mathscr{A}(U)$ gives an isomorphism $X' \to X$.

2.5.51 d. x

(d) If $\mathscr{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : X = \mathbf{Spec}\ \mathscr{A} \to Y$ is an affine morphism, and $\mathscr{A} \cong f_* \mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism, then $\mathscr{A} = f_* \mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \mathbf{Spec}\ \mathscr{A}$.

f is affine since for each open affine $U \subset Y$, $f^{-1}(U) \approx Spec\ \mathscr{A}(U)$.

If $U \subset Y$ is open, then $f_* \mathcal{O}_X(U) = \mathcal{O}_X(f^{-1}(U))$.

If U is affine or contained in an open affine, then this is clearly $\mathscr{A}(U)$.

Otherwise, cover Y with open affines U_i .

Then $\mathcal{O}_X(f^{-1}(U \cap U_i)) \approx \mathscr{A}(U \cap U_i)$, and patching gives an isomorphism on U.

Thus $\mathscr{A} \approx f_* \mathcal{O}_X$ since it is true on any open set.

On the other hand, if $f : X \to Y$ is affine, then $\mathscr{A} = f_* \mathcal{O}_X$ satisfies for any open set $U \subset Y$, $\mathscr{A}(U) = \mathcal{O}_X(f^{-1}(U))$. Thus there is a morphism $\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}(U))$ so $\mathscr{A}(U)$ has the structure of an $\mathcal{O}_Y(U)$-module.

If $V \subset U$, then $\mathcal{O}_X(f^{-1}(U)) \to \mathcal{O}_X(f^{-1}(V))$ is an $\mathcal{O}_X(U)$-module homomorphism, so $\mathscr{A}$ is an $\mathcal{O}_Y$ -module which we claim is quasi-coherent as an $\mathcal{O}_Y$-algebra.

If $U = Spec\ A \subset Y$ is affine, then by (a) $f^{-1}(U) = Spec\ B$ is an affine, where B is an A-module.

As $\mathscr{A}|_U \approx \tilde{B}$, then $\mathscr{A}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$ -algebras.

We next claim that $X \approx Spec\ \mathscr{A}$. If $V \subset U$ is open and affine, then $f^{-1}(V) \to f^{-1}(U)$ is induced from the ring map $\mathscr{A}(U) \to \mathscr{A}(V)$. Then $X \approx Spec\ \mathscr{A}$ by (c).

2.5.52 e. x

(e) Let $f : X \to Y$ be an affine morphism, and let $\mathscr{A} = f_* \mathcal{O}_X$. Show that f_* induces an equivalence of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathscr{A}$-modules (i.e., quasi-coherent $\mathcal{O}_Y$-modules having a structure of $\mathscr{A}$-module). [*Hint*: For any quasi-coherent $\mathscr{A}$-module $\mathscr{M}$, construct a quasi-coherent $\mathcal{O}_X$-module $\tilde{\mathscr{M}}$, and show that the functors f_* and $\tilde{}$ are inverse to each other.

Let $\mathscr{M}$ q.c. as in the hint.

If $U, V \subset Y$ are open and affine , then $U \cap V$ is covered by open sets that are distinguished in both U and V.

The correspondence of sections between elements of localized modules on the intersections $U \cap V$ gives an isomorphism between $\mathscr{M}(U)$ and $\mathscr{M}(V)$ on $U \cap V$.

By exc II.1.22, we can glue the $\mathcal{O}_X(f^{-1}(U))$-modules $\mathscr{M}(U)^{\sim}$ as U ranges over all open affines of Y to get an $\mathcal{O}_X$-module $\mathscr{M}^{\sim}$.

Following the hint, we claim that $\sim$ and f_* are inverse to each other hand thus give an equivalence of categories.

Let $\mathscr{F}$ a q.c. $\mathcal{O}_X$ -module. Then $(f_*\mathscr{F})^{\sim}$ is isomorphic to $\mathscr{F}$ on an open affine cover, and so by thm II.5.5, $(f_*\mathscr{F})^{\sim} \approx \mathscr{F}$.

On the other hand, $f_*\tilde{\mathscr{M}} \approx \mathscr{M}$.

2.5.53 II.5.18 x Vector Bundles

5.18. *Vector Bundles.* Let Y be a scheme. *A (geometric) vector bundle* of rank n over Y is a scheme X and a morphism $f: X \to Y$, together with additional data consisting of an open covering $\{U_i\}$ of Y, and isomorphisms $\psi_i: f^{-1}(U_i) \to \mathbf{A}^n_{U_i}$, such that for any i, j, and for any open affine subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbf{A}^n_V = \operatorname{Spec} A[x_1, \ldots, x_n]$ is given by a *linear* automorphism θ of $A[x_1, \ldots, x_n]$, i.e., $\theta(a) = a$ for any $a \in A$, and $\theta(x_i) = \sum a_{ij} x_j$ for suitable $a_{ij} \in A$.

An *isomorphism* $g: (X, f, \{U_i\}, \{\psi_i\}) \to (X', f', \{U'_i\}, \{\psi'_i\})$ of one vector bundle of rank n to another one is an isomorphism $g: X \to X'$ of the underlying schemes, such that $f = f' \circ g$, and such that X, f, together with the covering of Y consisting of all the U_i and U'_i, and the isomorphisms ψ_i and $\psi'_i \circ g$, is also a vector bundle structure on X.

(a) Let $\mathscr{E}$ be a locally free sheaf of rank n on a scheme Y. Let $S(\mathscr{E})$ be the symmetric algebra on $\mathscr{E}$, and let $X = \mathbf{Spec}\, S(\mathscr{E})$, with projection morphism $f: X \to Y$. For each open affine subset $U \subseteq Y$ for which $\mathscr{E}|_U$ is free, choose a basis of $\mathscr{E}$, and let $\psi: f^{-1}(U) \to \mathbf{A}^n_U$ be the isomorphism resulting from the identification of $S(\mathscr{E}(U))$ with $\mathcal{O}(U)[x_1, \ldots, x_n]$. Then $(X, f, \{U\}, \{\psi\})$ is a vector bundle of rank n over Y, which (up to isomorphism) does not depend on the bases of $\mathscr{E}_U$ chosen. We call it the *geometric vector bundle associated* to $\mathscr{E}$, and denote it by $\mathbf{V}(\mathscr{E})$.

(b) For any morphism $f: X \to Y$, a *section* of f over an open set $U \subseteq Y$ is a morphism $s: U \to X$ such that $f \circ s = \mathrm{id}_U$. It is clear how to restrict sections to smaller open sets, or how to glue them together, so we see that the presheaf $U \mapsto \{$set of sections of f over $U\}$ is a sheaf of sets on Y, which we denote by $\mathscr{S}(X/Y)$. Show that if $f: X \to Y$ is a vector bundle of rank n, then the sheaf of sections $\mathscr{S}(X/Y)$ has a natural structure of $\mathcal{O}_Y$-module, which makes it a locally free $\mathcal{O}_Y$-module of rank n. [*Hint*: It is enough to define the module structure locally, so we can assume $Y = \operatorname{Spec} A$ is affine, and $X = \mathbf{A}^n_Y$. Then a section $s: Y \to X$ comes from an A-algebra homomorphism $\theta: A[x_1, \ldots, x_n] \to A$, which in turn determines an ordered n-tuple $\langle \theta(x_1), \ldots, \theta(x_n) \rangle$ of elements of A. Use this correspondence between sections s and ordered n-tuples of elements of A to define the module structure.]

(c) Again let $\mathscr{E}$ be a locally free sheaf of rank n on Y, let $X = \mathbf{V}(\mathscr{E})$, and let $\mathscr{S} = \mathscr{S}(X/Y)$ be the sheaf of sections of X over Y. Show that $\mathscr{S} \cong \mathscr{E}^{\check{}}$, as follows. Given a section $s \in \Gamma(V, \mathscr{E}^{\check{}})$ over any open set V, we think of s as an element of $\mathrm{Hom}(\mathscr{E}|_V, \mathcal{O}_V)$. So s determines an $\mathcal{O}_V$-algebra homomorphism $S(\mathscr{E}|_V) \to \mathcal{O}_V$. This determines a morphism of spectra $V = \mathbf{Spec}\, \mathcal{O}_V \to \mathbf{Spec}\, S(\mathscr{E}|_V) = f^{-1}(V)$, which is a section of X/Y. Show that this construction gives an isomorphism of $\mathscr{E}^{\check{}}$ to $\mathscr{S}$.

(d) Summing up, show that we have established a one-to-one correspondence between isomorphism classes of locally free sheaves of rank n on Y, and isomorphism classes of vector bundles of rank n over Y. Because of this, we sometimes use the words "locally free sheaf" and "vector bundle" interchangeably, if no confusion seems likely to result.

For this excercise I will give the natural 1-1 correspondence between vector bundles and locally-free sheaves over an algebraically closed field.

Assume E is a vector bundle. Define a sheaf $\mathscr{E}$ by letting $\mathscr{E}(U)$ be the set of sections of $\mathscr{E}$ over U. This gives a module structure on the fiber by adding together sections or multiplying them with functions. We also have a restriction map since if $U \subset V$ and $s \in \mathscr{E}(U)$, then $s|_V$ is a section on V so there is a map $\mathscr{E}(U) \to \mathscr{E}(V)$ satisfying the requirements for the restriction morphism of a sheaf.

Now let U_i be a covering on which E satisfies $\pi^{-1}(U_i) \approx U_i \times \mathbb{A}^r$. Write a section as $s = \sum f_i x_i$ where x_i are the coordinate sections $x_i : U_i \to U_i \times \mathbb{A}^r$ definde by $x_i : p \mapsto (p, (0, ..., 1, ..., 0))$ with 1 in the i^{th} place. Then $\mathscr{E}(U_i) \to (\mathcal{O}(U_i))^r$ defined by $s \to (f_1, ..., f_r)$ gives an isomorphism so that $\mathscr{E}$ is locally free.

Conversely, suppose that $\mathscr{E}$ is a locally free sheaf. Define E to be $\{(P,t) \,|\, P \in Y, t \in \mathscr{E}_P/\mathfrak{m}_P\mathscr{E}_P\}$, where $\mathfrak{m}_P$ is the maximal ideal of the local ring $\mathcal{O}_P$. Define a projection $\pi : E \to Y$ by projecting to the first coordinate.

On an open covering U_i where $\mathscr{E}(U_i) \approx (\mathcal{O}_Y(U_i))^r$, then $\mathscr{E}_P/\mathfrak{m}_P\mathscr{E}_P \approx \mathbb{A}^r$ since k is algebraically closed. Thus we have trivializations φ_i

On $U_i \cap U_j$ the transition functions of $(\mathcal{O}_Y(U_i \cap U_j))^r \approx \mathscr{E}(U_i \cap U_j) \approx \mathscr{E}(U_i \cap U_j) \approx (\mathcal{O}_Y(U_i \cap U_j))^r$ are given by matrices and reducing modulo the maximal ideal gives an element of the general linear group so that we have transition functions $\varphi_j \varphi_i^{-1} = (id, \phi_{ij})$ where ϕ_{ij} are invertible matrices.

If f is a mpa of locally free sheaves then reduce modulo P on the fiber to get a map of bundles.

If g is a map of vector bundles, then composing with a sections gives a morphism of locally free sheaf which is compatible with restrictions.

2.5.54 b. x

see part a

2.5.55 c. x

see part a

2.5.56 d. x

see part a

2.6 II.6 x Divisors

2.6.1 II.6.1 x g

6.1. Let X be a scheme satisfying (*). Then $X \times \mathbf{P}^n$ also satisfies (*), and $\mathrm{Cl}(X \times \mathbf{P}^n) \cong (\mathrm{Cl}\, X) \times \mathbf{Z}$.

By thm II.6.6, $X \times \mathbb{P}^1$ is noetherian integral and separated and further $X \times \mathbb{A}^1$ is regular in codimension 1. As two $X \times \mathbb{A}^1$ cover $X \times \mathbb{P}^1$, then $X \times \mathbb{P}^n$ satisfies (*).

Now consider the exact sequence $\mathbb{Z} \xrightarrow{i} Cl\left(X \times \mathbb{P}^1\right) \xrightarrow{j} Cl\left(X\right) \to 0$ of thm II.6.5. where $i : n \mapsto nZ$, and Z corresponds to $\pi_2^{-1}\infty \subset X \times \mathbb{P}^1$ and $j : Cl\left(X \times \mathbb{P}^1\right) \to Cl\left(X \times \mathbb{A}^1\right) \approx cl\left(X\right)$. Note that j is split by the map $k : cl\left(X\right) \to cl\left(X \times \mathbb{P}^1\right)$ sending $\sum n_i Z_i$ to $\sum n_i \pi_1^{-1} Z_i$.

We claim that i is also split. The map k sends $\xi \mapsto \xi - kj\xi$ in the kernel of j so in the image of i by exactness. We claim that i is injective. If $nZ \sim 0$ for $n \in \mathbb{Z}$, then Z is $\pi_2^{-1}0$, $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$.

On $X \times \mathbb{A}^1$, then Z is embedded at the origin as X. Thus the local ring of Z in $K\left(t\right)$ is $K\left[t\right]_{(t)}$. $nZ \sim 0 \implies \exists f \in K\left(t\right)$ with $v_Z\left(f\right) = n$ and $v_Y\left(f\right) = 0$ for every other prime divisor Y. Thus $f = \frac{t^n g(t)}{h(t)}$, $g, h \in K\left[t\right]$ and $t \nmid g\left(t\right), h\left(t\right)$.

If the degrees of g, h are 0, then changing coordinates $t \mapsto t^{-1}$, we get $v_Y\left(f\right) = -n$, Y a copy of X embedded at the origin or infinity which is opposite to Z. If g or h has positive degree, then it has an irreducible factor in $K\left[t\right]$ corresponding to a prime divisor $p_2^{-2}x$ for $x \in \mathbb{P}^1$ with f $v_{p_2^{-1}x}\left(f\right) \neq 0$. Thus there is no rational function $\left(f\right) = nZ$ so i is injective.

As both i and j are split, $Cl\left(X \times \mathbb{P}^1\right) \approx Cl\left(X\right) \times \mathbb{Z}$.

2.6.2 II.6.2 (starred) Varieties in Projective Space

***6.2.** *Varieties in Projective Space.* Let k be an algebraically closed field, and let X be a closed subvariety of $\mathbf{P}^n_k$ which is nonsingular in codimension one (hence satisfies (*)). For any divisor $D = \sum n_i Y_i$ on X, we define the *degree* of D to be $\sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of Y_i, considered as a projective variety itself (I, §7).

(a) Let V be an irreducible hypersurface in $\mathbf{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$. They all have codimension 1 by (I, Ex. 1.8). For each i, let f_i be a local equation for V on some open set U_i of $\mathbf{P}^n$ for which $Y_i \cap U_i \neq \varnothing$, and let $n_i = v_{Y_i}(\bar{f}_i)$, where $\bar{f}_i$ is the restriction of f_i to $U_i \cap X$. Then we define the *divisor* $V.X$ to be $\sum n_i Y_i$. Extend by linearity, and show that this gives a well-defined homomorphism from the subgroup of $\mathrm{Div}\, \mathbf{P}^n$ consisting of divisors, none of whose components contain X, to $\mathrm{Div}\, X$.

MISS

(b) If D is a principal divisor on $\mathbf{P}^n$, for which $D.X$ is defined as in (a), show that $D.X$ is principal on X. Thus we get a homomorphism $\mathrm{Cl}\, \mathbf{P}^n \to \mathrm{Cl}\, X$.

MISS

(c) Show that the integer n_i defined in (a) is the same as the intersection multiplicity $i(X, V; Y_i)$ defined in (I, §7). Then use the generalized Bézout theorem (I, 7.7) to show that for any divisor D on $\mathbf{P}^n$, none of whose components contain X,

$$\deg(D.X) = (\deg D) \cdot (\deg X).$$

MISS

(d) If D is a principal divisor on X, show that there is a rational function f on $\mathbf{P}^n$ such that $D = (f).X$. Conclude that $\deg D = 0$. Thus the degree function defines a homomorphism $\deg: \mathrm{Cl}\, X \to \mathbf{Z}$. (This gives another proof of (6.10), since any complete nonsingular curve is projective.) Finally, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Cl}\, \mathbf{P}^n & \longrightarrow & \mathrm{Cl}\, X \\ \cong \downarrow \deg & & \downarrow \deg \\ \mathbf{Z} & \xrightarrow{\cdot(\deg X)} & \mathbf{Z} \end{array},$$

and in particular, we see that the map $\mathrm{Cl}\, \mathbf{P}^n \to \mathrm{Cl}\, X$ is injective.

MISS

II.6.3 Cones (starred)

***6.3.** *Cones.* In this exercise we compare the class group of a projective variety V to the class group of its cone (I, Ex. 2.10). So let V be a projective variety in $\mathbf{P}^n$, which is of dimension $\geqslant 1$ and nonsingular in codimension 1. Let $X = C(V)$ be the affine cone over V in $\mathbf{A}^{n+1}$, and let $\bar{X}$ be its projective closure in $\mathbf{P}^{n+1}$. Let $P \in X$ be the vertex of the cone.

(a) Let $\pi: \bar{X} - P \to V$ be the projection map. Show that V can be covered by open subsets U_i such that $\pi^{-1}(U_i) \cong U_i \times \mathbf{A}^1$ for each i, and then show as in (6.6) that $\pi^*: \mathrm{Cl}\, V \to \mathrm{Cl}(\bar{X} - P)$ is an isomorphism. Since $\mathrm{Cl}\, \bar{X} \cong \mathrm{Cl}(\bar{X} - P)$, we have also $\mathrm{Cl}\, V \cong \mathrm{Cl}\, \bar{X}$.

MISS

(b) We have $V \subseteq \bar{X}$ as the hyperplane section at infinity. Show that the class of the divisor V in $\mathrm{Cl}\, \bar{X}$ is equal to π^* (class of $V.H$) where H is any hyperplane of $\mathbf{P}^n$ not containing V. Thus conclude using (6.5) that there is an exact sequence

$$0 \to \mathbf{Z} \to \mathrm{Cl}\, V \to \mathrm{Cl}\, X \to 0,$$

where the first arrow sends $1 \mapsto V.H$, and the second is π^* followed by the restriction to $X - P$ and inclusion in X. (The injectivity of the first arrow follows from the previous exercise.)

MISS

(c) Let $S(V)$ be the homogeneous coordinate ring of V (which is also the affine coordinate ring of X). Show that $S(V)$ is a unique factorization domain if and only if (1) V is projectively normal (Ex. 5.14), and (2) $\mathrm{Cl}\, V \cong \mathbf{Z}$ and is generated by the class of $V.H$.

MISS

(d) Let $\mathcal{O}_P$ be the local ring of P on X. Show that the natural restriction map induces an isomorphism $\mathrm{Cl}\, X \to \mathrm{Cl}(\mathrm{Spec}\, \mathcal{O}_P)$.

MISS

2.6.3 II.6.4 x

6.4. Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a *square-free* nonconstant polynomial, i.e., in the unique factorization of f into irreducible polynomials, there are no repeated factors. Let $A = k[x_1, \ldots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring. [*Hint*: The quotient field K of A is just $k(x_1, \ldots, x_n)[z]/(z^2 - f)$. It is a Galois extension of $k(x_1, \ldots, x_n)$ with Galois group $\mathbf{Z}/2\mathbf{Z}$ generated by $z \mapsto -z$. If $\alpha = g + hz \in K$, where $g,h \in k(x_1, \ldots, x_n)$, then the minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. Now show that α is integral over $k[x_1, \ldots, x_n]$ if and only if $g,h \in k[x_1, \ldots, x_n]$. Conclude that A is the integral closure of $k[x_1, \ldots, x_n]$ in K.]

In $K = frac\ A$ we have $\frac{1}{g+zh}\frac{g-zh}{g-zh} = \frac{g-zh}{g^2-fh^2}$ since $z^2 = f$ in A.

Let $B = k[x_1, ..., x_n]$ and $L = frac(B)$

Thus every element of K can be wrriten as $g' + zh'$ with g', h' quotients of polynomials. Thus $K = L[z]/(z^2 - f)$ a degree 2 extension of L which is galois. If $\alpha = g + hz \in K$, then α has minimal polynomial $X^2 - 2gX + (g^2 - h^2 f)$ so that α is integral over $k[x_1, ..., x_n]$ iff the coefficients $2g, (g^2 - h^2 f)$ are in B. Which happens iff $2g, h^2 f \in B$.

If α is integral over B, then $g \in B$ and thus $h^2 f \in B$. If h has a nontrivial denominator, then $h^2 f \notin B$ since f is square-free so that $h \in B$ and thus $\alpha \in A$.

If, on the other hand, $\alpha \in A$ then $2g, h^2 f \in B$ so α is integral over B thus A is the integral closure of B so is integrally closed.

II.6.5 (Starred) Quadric Hypersurfaces (starred)

***6.5.** *Quadric Hypersurfaces.* Let char $k \neq 2$, and let X be the affine quadric hypersurface Spec $k[x_0, \ldots, x_n]/(x_0^2 + x_1^2 + \ldots + x_r^2)$—cf. (I, Ex. 5.12).

(a) Show that X is normal if $r \geqslant 2$ (use (Ex. 6.4)).

(b) Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \ldots + x_r^2$. Now imitate the method of (6.5.2) to show that:

(1) If $r = 2$, then Cl $X \cong \mathbf{Z}/2\mathbf{Z}$;
(2) If $r = 3$, then Cl $X \cong \mathbf{Z}$ (use (6.6.1) and (Ex. 6.3) above);
(3) If $r \geqslant 4$ then Cl $X = 0$.

part of starred

(c) Now let Q be the projective quadric hypersurface in $\mathbf{P}^n$ defined by the same equation. Show that:

(1) If $r = 2$, Cl $Q \cong \mathbf{Z}$, and the class of a hyperplane section $Q.H$ is twice the generator;
(2) If $r = 3$, Cl $Q \cong \mathbf{Z} \oplus \mathbf{Z}$;
(3) If $r \geqslant 4$, Cl $Q \cong \mathbf{Z}$, generated by $Q.H$.

part of starred

(d) Prove Klein's theorem, which says that if $r \geqslant 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbf{P}^n$ such that $V \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection. (First show that for $r \geqslant 4$, the homogeneous coordinate ring $S(Q) = k[x_0, \ldots, x_n]/(x_0^2 + \ldots + x_r^2)$ is a UFD.)

part of starred

- so wts irreducible hypersurface on a quadric is complete intersection with $r \geq 4$.
  - $k[x_0, ..., x_n] / (x_0^2 + ... + x_r^2)$ is a UFD why??
    -
  - UFD implies?
    - divisors are principle in UFD...
    - so Y would be a divisor on the UFD Q, and thus principal ideal.. generated by a single element.
  - a complete intersection...
  - so codim of Y is going to be... well Q has codim 1, and Y codim 1 on Q, so codim 2 in total? so the homogenous ideal would be generated by two elements: $(x_0^2 + ... + x_r^2)$ and the thing from the principal ideal....

2.6.4 II.6.6 x g

6.6. Let X be the nonsingular plane cubic curve $y^2z = x^3 - xz^2$ of (6.10.2).
(a) Show that three points P,Q,R of X are collinear if and only if $P + Q + R = 0$ in the group law on X. (Note that the point $P_0 = (0,1,0)$ is the zero element in the group structure on X.)

Assume P, Q, R are on the line L.
By bezout, $L \cap X = \{P, Q, R\}$.
Now arguing as in example II.6.10 as $L \sim (z = 0) \implies P + Q + R \sim 3P_0$.
$\implies (P - P_0) + (Q - P_0) + (R - P_0) \sim (P_0 - P_0) = 0$.
Conversely, if $P + Q + R = 0$ in the group law, and assume WLOG P, Q, R are distinct.
Consider L through P and Q.
Let T be the (from bezout) third intersection point with X.
Then $P + Q + T \sim 3P_0$ geometrically.
Thus $P + Q + T = 0$ in the group law on X so $R = -P - Q = T$.

2.6.5 (b) x g

(b) A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.

Assume P has order 2 in group law.
By exc I.7.3, $T_P(X)$ intersects X at P with multiplicity > 1.
If $P = P_0$, then tangent line passes through P_0 clearly.
Otherwise, $T_P(X)$ intersects X in three points by bezout, so in one additional point besides P, named Q.
Thus $2P + Q = 0$ since they are on a line, and by previous.
As P has order 2, then $Q = 0$, so the tangent line passes through P_0.

On the other hand, suppose $T_P(X)$ passes through P_0 .
P_0 is identity on group law so P has order 2 if equal to P_0.
Otherwise, $T_P(X)$ intersects X in P, P, P_0 .
Thus $2P + P_0 = 0$ so P has order 2.

2.6.6 (c) x g

(c) A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point. (An *inflection point* of a plane curve is a nonsingular point P of the curve, whose tangent line (I, Ex. 7.3) has intersection multiplicity $\geqslant 3$ with the curve at P.)

P an inflection point $\implies T_P(X).X = 3$ by bezout and definition of inflection point.
Thus $3P = 0$ in the group law.

Conversely, if $3P = 0$, by part (a) there is an L such that $L \cap X = P$, $L.X = 3$.

2.6.7 (d). x

(d) Let $k = \mathbf{C}$. Show that the points of X with coordinates in $\mathbf{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?

By mordell-weil theorem.
Generators are $(0,1,0)$, $(1,0,1)$, $(-1,0,1)$, $(0,0,1)$.

2.6.8 II.6.7 (starred)

***6.7.** Let X be the nodal cubic curve $y^2z = x^3 + x^2z$ in $\mathbf{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0, $\mathrm{CaCl}^\circ X$, is naturally isomorphic to the multiplicative group $\mathbf{G}_m$.

2.6.9 II.6.8 (a) x g (use easier method)

6.8. (a) Let $f: X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \mapsto f^*\mathscr{L}$ induces a homomorphism of Picard groups, $f^*: \operatorname{Pic} Y \to \operatorname{Pic} X$.

Method 1. (easier)
By II.5.2.e, f^* takes locally free sheaves to locally free sheaves of the same rank.
Locally we have $f^*(\mathscr{L} \otimes M) \approx f^*\left(\tilde{M} \otimes \tilde{N}\right) \approx f^*\left(M \tilde{\otimes}_B N\right) \approx$
$(M \otimes B \otimes A)^\sim \approx ((M \otimes A) \otimes (N \otimes A))^\sim$ using II.5.2.
This is $f^*\left(\tilde{M}\right) \otimes f^*\left(\tilde{N}\right) \approx f^*(\mathscr{L}) \otimes f^*(\mathscr{M})$.
So f^* preserves the group structure.

Method 2 (which I will follow for subsequent parts)
Notation:
For a morphism $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces and $V \subset Y$, let f_V map $\left(f^{-1}(V), \mathcal{O}_X|_{f^{-1}(V)}\right) \to \left(V, \mathcal{O}_{Y|V}\right)$.
Let ϵ the counit adjunction natural transformation.

Let η the unit adjunction natural transformation.
Let $h^A = Hom\left(A, -\right)$.

We have $Hom_{\mathcal{O}_X}\left(f^*F, G\right) \approx Hom_{\mathcal{O}_Y}\left(G, f_*F\right)$ by adjunction.
Then (using the module version of Hartshorne II.5.1.c),
$Hom_{\mathcal{O}_X}\left(f^*M \otimes_{\mathcal{O}_X} f^*N, P\right) \approx Hom_{\mathcal{O}_X}\left(f^*N, Hom_{\mathcal{O}_X}\left(f^*M, P\right)\right)$.
By adjunction, this is $\approx Hom_{\mathcal{O}_Y}\left(N, f_*Hom_{\mathcal{O}_X}\left(f^*M, P\right)\right)$.

For any $\psi \in Hom_{\mathcal{O}_X}\left(f^*M, P\right)|_{f^{-1}(V)}$ where MV is open in Y, we have that $\left(f_V\right)_* \psi \in Hom_{\mathcal{O}_Y}\left(f_*f^*M, f_*P\right)|_{f^{-1}(V}$
We thus have a canonical morphism $\kappa : f_*Hom_{\mathcal{O}_X}\left(f^*M, P\right) \to Hom_{\mathcal{O}_Y}\left(f_*f^*M, f_*P\right)$. Then $\epsilon \circ \kappa : f_*Hom_{\mathcal{O}_X}\left(f^*M, F\right.$
$Hom_{\mathcal{O}_Y}\left(M, f^*P\right)$ gives a natural transformation. This commutes with the usual adjunction and hence gives a natural isomorphism. (a natty ice for short)
Now we have
$Hom_{\mathcal{O}_Y}\left(N, f_*Hom_{\mathcal{O}_X}\left(f^*M, P\right)\right) \approx Hom_{\mathcal{O}_\mathcal{Y}}\left(N, Hom_{\mathcal{O}_Y}\left(M, f_*P\right)\right) \approx$
$Hom_{\mathcal{O}_Y}\left(M \otimes N, f_*P\right) \approx Hom_{\mathcal{O}_X}\left(f^*\left(M \otimes N\right), P\right)$.
Yoneda's lemma states that $Nat\left(h^A, F\right) \approx F\left(A\right)$, in other words, the natural transformations from h^A to $F : C \to set$ (a functor) are in 1-1 correspondence with the elements of $F\left(A\right)$. Hence $Nat\left(Hom_{\mathcal{O}_X}\left(f^*\left(M \otimes N\right), -\right)\right.$
$Hom_{\mathcal{O}_X}\left(f^*\left(M \otimes N\right), f^*M \otimes f^*N\right) \approx Hom_{\mathcal{O}_X}\left(f^*\left(M \otimes N\right), f^*\left(M \otimes N\right)\right)$.

2.6.10 (b) x

(b) If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^*\colon \mathrm{Cl}\ Y \to \mathrm{Cl}\ X$ defined in the text via the isomorphisms of (6.16).

(continuing the notation from part a).

Theorem II.6.16 states: If X is a noetherian, integral, separated, locally factorial scheme then there is a natural isomorphism $Cl\left(X\right) \approx Pic\left(X\right)$. We are given X, Y nonsingular curves. By definition, X, Y are separated and integral and finite over an algebraically closed field. As they are finite over a field $k = spec\left(k\right)$, then by II.3.13.g, we have that X, Y are noetherian. Also by definition they are regular of dimension 1. Using II.3.20, and I.6.2.A, all the local rings are DVR's, which are then UFD's, which by Lang(algebra) XII.6 are factorial. Thus we can use II.6.16 on X, Y.

Notation: $Cl\left(X\right) := Div\left(X\right)/prin\left(X\right)$ the group of divisors mod principal divisors. This is isomorphic to $Pic\left(X\right)$ by above. $CaCl\left(X\right) :=$ the group of cartier divisor classes modulo principal diviors (divisors of functions) by 6.11 in the case of this problem, $Div\left(X\right) \approx CaCl\left(X\right)$ and principals correspond to principals under this isomorphism.

Machinery:

- Using the proof of 6.11, we can convert a Weil divisor in $Cl\left(X\right)$ to a cartier divisor in $CaCl\left(X\right)$. Let $D \in Cl\left(X\right)$. For any point in X, D_x must be principal by II.6.2. Thus $D_x = \left(f_x\right)$, $f_x \in K$. We can find an open neighborhood U_x so that D and $\left(f_x\right)$ has the same restriction to U_x .

- In order to convert a Weil divisor to a Cartier divisor, let $\left\{\left(U_i, f_i\right)\right\}$ denote the cartier divisor, $\left\{U_i\right\}$ is an open cover of X. Basically, we just let $Cl\left(X\right) \ni D = \sum v_Y\left(f_i\right)Y$ on X taken over prime divisors Y on X. By 6.11, the sum is finite. Note that on the intersections, it is well-defined by linear equivalence since $v_Y\left(\frac{f_i}{f_j}\right) = 0$.

- (6.13) To convert from a Cartied divisor D to an invertible sheaf, $\mathcal{O}_{U_i} \to \mathscr{L}\left(D\right)|_{U_i}$ defined by $1 \mapsto \frac{1}{f_i}$ is an isomorphism.

- (6.13) To convert from the invertible sheaf $\mathscr{L}\left(D\right)$ to D , let f_i on U_i be the inverse of a local generator.

- (Ha page 137) $f^*: Cl(Y) \to Cl(X)$ is defined by $f^*Q = \sum_{f(P)=Q} v_P(t) \cdot P$ where $t$is a local parameter (element of $K(Y)$ with $v_Q(t) = 1$). f is a finite morphism, so this is a closed sum. Compute first for points Q and then extend by linearity to $f^*: Cl(Y) \to Cl(X)$.

Let $Q \in Y$ and then Q considered as a subscheme (using II.3.2.6) is closed, integral and codimension 1 (every $\mathcal{O}_Q(U)$ is integrally closed and the curves are assumed to have codimension 1 local rings). Let $t \in \mathcal{O}_Q$ be a local parameter at Q, i.e. t is an element of $K(Y)$ with $v_Q(t) = 1$.

Now let U_Q be a neighborhood such that t is only 0 at Q. We can do this since there are only finitely many points where t has a pole or zero. (See Ha I.6.5 or II.6.1 to deal with the poles and to deal with the zeros look at $\frac{1}{f}$). Then $D = \{(U_Q, t), (Y - U_Q, 1)\} \in CaCl$ is the associated Cartier divisor, and by definition,

$f^*D = \sum_{f(P)=Q, P\in f^{-1}(U_Q)} v_P(t \circ f) \cdot P + \sum_{f(P)=Q, P\in f^{-1}(Y-U_Q)} v_P(t\circ f) \cdot P$ is the pullback to X. Going back to Cartier divisor and then picard group we get: $f^*D \approx \{(f^{-1}(U_Q, t\circ f), (f^{-1}(Y - U_Q), 1))\} \approx \mathscr{L}(f^*D)|_{f^{-1}(U_Q)}$ which is $\frac{1}{t\circ f}\mathcal{O}_Y|_{f^{-1}(U_Q)}$ and $\mathscr{L}(f^*D)|_{f^{-1}(Y-U_Q)} \approx \mathcal{O}_Y|_{f^{-1}(Y-U_Q)}$.

example from silverman.

Let $f: \mathbb{P}^1 \to \mathbb{P}^1$ be defined by $\phi([X, Y]) = \left[X^2(X-Y)^2, Y^5\right]$. Let $Q = [0, 1]$ which ramifies into $\phi^{-1}Q = \{[0, 1], [1, 1]\}$. Then we can let $t([x, y]) = [x, x]$ so that $t(P) = t([0, 1]) = [0, 0]$ which has $v_P(t) = 1$. Then $f^*Q = \sum_{P\in\phi^{-1}(Q)} v_P(t) P = v_{[0,1]}([x, x])[0, 1] + v_{[1,1]}([x, x])[1, 1] = [0, 1] + 0[1, 1]$.

$D = \left\{\left(B\left([0, 1], \frac{1}{4}\right), t\right), \left(1, Y - B\left([0, 1], \frac{1}{4}\right)\right)\right\}$.

What is $D^{-1} \in CaCl(Y)$? We must have it be linearly equivalent D to a principal divisor. So we could write it as $D^{-1} = \left\{\left(U_Q, \frac{1}{t}\right), (Y - U_Q, 1)\right\}$. We can then calculate the associated invertible sheaf $\mathscr{L}(D^{-1}) \in Pic(Y)$. This sheaf is given by $\mathscr{L}(D^{-1})|_{U_Q} = t\mathcal{O}_Y|_{U_Q}$ and $\mathscr{L}(D^{-1})|_{X-U_Q} = \mathcal{O}_Y|_{X-U_Q}$. Now we must pull back the invertible sheaf $\mathscr{L}(D^{-1})$. I want to show that $f^*\mathscr{L}(D^{-1}) \otimes_{\mathcal{O}_X} \mathscr{L}(f^*D) \approx \mathcal{O}_X$. Note that restricted to $f^{-1}(Y - U_Q)$this is clearly the case since we have $f^*\mathscr{L}(D^{-1}) \otimes_{\mathcal{O}_X} \mathscr{L}(f^*D) \approx$

$f^{-1}\mathcal{O}_Y|_{f^{-1}(Y-U_Q)} \otimes_{f^{-1}\mathcal{O}_Y|_{f^{-1}(Y-U_Q)}} \mathcal{O}_X \otimes \mathcal{O}_X$ and using a tensor simplification this is just $\mathcal{O}_X$. Restricted to $f^{-1}(U_Q)$ (use Grillet Abstract Algebra XI.4.6) we also have

$f^{-1}t\mathcal{O}_Y|_{U_Q} \otimes_{f^{-1}\mathcal{O}_Y|_{U_Q}} \mathcal{O}_X \otimes \frac{1}{t}\mathcal{O}_X \approx \mathcal{O}_X$. Since $f^*\mathscr{L}(D^{-1})$ is inverse to $\mathscr{L}(f^*D)$, the pullback homomorphisms correspond.

2.6.11 (c) x

> (c) If X is a locally factorial integral closed subscheme of $\mathbf{P}^n_k$, and if $f: X \to \mathbf{P}^n$ is the inclusion map, then f^* on Pic agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).

The homomorphisms on divisor class grups in ex 6.2 is defined as follows: Let V be an irreducible hypersurface in $\mathbb{P}^n$ which does not contain X, and let Y_i be the irreducible components of $V \cap X$ (X is a closed subvariety nonsingular in codimension one). Let f_i be a local equation for V on some open set U_i of $\mathbb{P}^n$ for which $Y_i \cap U_i \neq \emptyset$. Let $n_i = v_{Y_i}\left(\overline{f_i}\right)$ where $\overline{f_i}$ is the restriction of f_i to $U_i \cap X$. Then the divisor $V.X$ is $\sum n_i Y_i = \sum v_{Y_i}\left(\overline{f_i}\right) = \sum v_{Y_i}(f_i|_{U_i\cap X})$.

Assume X is not contained in the hyperplane $x_0 = 0$ (this is V) whose cartier divisor is $H = \left\{\left(D_+\left(x_i, \frac{x_i}{x_i}\right)\right)\right\}$. The associated sheaf $\mathscr{L}(H)$ satisfies $\mathscr{L}(H)|_{D_+(x_0)} = \frac{x_i}{x_i}\mathcal{O}_{\mathbb{P}^n}|_{D_+(x_i)}$ by definition. The pullback sheaf $f^*\mathscr{L}(H) \approx f^{-1}\mathscr{L}(H) \otimes_{f^{-1}\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_X$ satisfies $f^*\mathscr{L}(H)|_{f^{-1}D_+(x_i)} = \frac{x_i}{x_0}\mathcal{O}_X$ (same logic as in (b)) associated cartier is $\left\{\left(f^{-1}(D_+(x_0)), f^*\frac{x_0}{x_1}\right)\right\} = \left\{\left(D_+(x_i) \cap X, \frac{x_0}{x_1}\right)\right\}$ (f is just inclusion in this case) . The Weil divisor associated here is $\sum_{D_+(x_i)} \sum_{P\in D_+(x_i)} v_P\left(\frac{x_0}{x_i}\right) P$ which is the same as taking f^* in the other way.

2.6.12 II.6.9 (starred) Singular Curves (starred)

***6.9.** *Singular Curves.* Here we give another method of calculating the Picard group of a singular curve. Let X be a projective curve over k, let $\tilde{X}$ be its normalization, and let $\pi: \tilde{X} \to X$ be the projection map (Ex. 3.8). For each point $P \in X$, let $\mathcal{O}_P$ be its local ring, and let $\tilde{\mathcal{O}}_P$ be the integral closure of $\mathcal{O}_P$. We use a * to denote the group of units in a ring.

(a) Show there is an exact sequence

$$0 \to \bigoplus_{P \in X} \tilde{\mathcal{O}}_P^*/\mathcal{O}_P^* \to \operatorname{Pic} X \xrightarrow{\pi^*} \operatorname{Pic} \tilde{X} \to 0.$$

[*Hint*: Represent Pic X and Pic $\tilde{X}$ as the groups of Cartier divisors modulo principal divisors, and use the exact sequence of sheaves on X

$$0 \to \pi_*\mathcal{O}_{\tilde{X}}^*/\mathcal{O}_X^* \to \mathcal{K}^*/\mathcal{O}_X^* \to \mathcal{K}^*/\pi_*\mathcal{O}_{\tilde{X}}^* \to 0.]$$

MISS

(b) Use (a) to give another proof of the fact that if X is a plane cuspidal cubic curve, then there is an exact sequence

$$0 \to \mathbf{G}_a \to \operatorname{Pic} X \to \mathbf{Z} \to 0,$$

and if X is a plane nodal cubic curve, there is an exact sequence

$$0 \to \mathbf{G}_m \to \operatorname{Pic} X \to \mathbf{Z} \to 0.$$

MISS

2.6.13 II.6.10 x g The Grothendieck Group K(X)

6.10. *The Grothendieck Group $K(X)$.* Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group generated by all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.

(a) If $X = \mathbf{A}_k^1$, then $K(X) \cong \mathbf{Z}$.

If $\mathscr{F}$ is a coherent sheaf, then $\mathscr{F}$ corresponds to an f.g. $k[t]$ -module M.

Let $k[t]^{\oplus n} \to k[t]^{\oplus m} \to M \to 0$ be a presentation of M, where the first morphism is injective, as $k[t]$ is a PID. (commutative algebra fact)

This gives an exact sequence $0 \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m} \to \mathscr{F} \to 0$.

Thus in $K(X)$ we have $\gamma(\mathscr{F}) \approx (m-n)\gamma(\mathcal{O}_X)$.

Thus the map $\mathbb{Z} \to K(X)$, $n \mapsto n\gamma(\mathcal{O}_X)$ is surjective.

This morphism splits via the rank homomorphism from (b).

2.6.14 (b) x

(b) If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the *rank* of $\mathscr{F}$ to be $\dim_K \mathscr{F}_\xi$, where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism rank : $K(X) \to \mathbf{Z}$.

Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an s.e.s. of coherent sheaves.

Localizing at the generic point, ξ, gives an s.e.s. as well, and so $dim_K \mathscr{F}_\xi = dim_K \mathscr{F}''_\xi + dim_K \mathscr{F}'_\xi$ and thus the rank homorphism is well-defined.

Note further that $\gamma(\mathcal{O}_X) \mapsto 1$ so that rank is surjective.

2.6.15 (c) x

(c) If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0,$$

where the first map is extension by zero, and the second map is restriction. [*Hint*: For exactness in the middle, show that if $\mathscr{F}$ is a coherent sheaf on X, whose support is contained in Y, then there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \ldots \supseteq \mathscr{F}_n = 0$, such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is an $\mathcal{O}_Y$-module. To show surjectivity on the right, use (Ex. 5.15).]

For further information about $K(X)$, and its applications to the generalized Riemann–Roch theorem, see Borel–Serre [1], Manin [1], and Appendix A.

Surjectivity on the right is easy since if $\mathscr{F}$ is coherent on $X - Y$, then by exc II.5.15, $\mathscr{F}$ is extendable to a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_{X-Y} \approx \mathscr{F}$.

Now suppose $\mathscr{F}$ is in the kernel of the second map. **Claim**: there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supset \mathscr{F}_i \supset \ldots \supset \mathscr{F}_n = 0$ such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y.

Let $i : Y \hookrightarrow X$. The functors i^* and i_* are adjoint so there is a natural morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F} : Coh(X) \to Coh(X)$. Let $U = Spec\ (A)$ an open set where $\mathscr{F}|_U = M^\sim$. Then $Y \cap Spec\ A \approx Spec\ A/I$ so that η is induced by $M \to M/IM$ on U. Thus η is surjective. Let $\mathscr{F}_0 = \mathscr{F}$ and define $\mathscr{F}_j$ inductively as $\mathscr{F}_j = ker(\mathscr{F}_{j-1} \to i_* i^* \mathscr{F}_j)$. By definition, each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y so we have a filtration satisfying the requried conditions.

We must now show this filtration is finite. We have $\mathscr{F}_j|_U = I^j M$. The support of $M^\sim$ is contained in $Spec\ A/I = V(I)$ so by exc II.5.6.b, $\sqrt{Ann\ M} \supset \sqrt{I} \supset I$. A noetherian implies I is finitely generated. Thus there is N with $Ann\ M \supset I^N$ (as in exc II.5.6.d). Thus $0 = I^N M$. As X is noetherian, we may cover X with finitely map U and choose a maximum such N to see the filtration is finite. Thus we have proven the claim

Now we have $\gamma(\mathscr{F}_i) = \gamma(\mathscr{F}_{i+1}) + \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$ in $K(X)$ so $\gamma(\mathscr{F}) = \sum_{i=0}^{n-1} \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$. Thus $\gamma(\mathscr{F})$ is in the image of $K(Y) \to K(X)$. The other half is obvious.

II.6.11 (Starred) The Grothendieck Group of Nonsingular Curve (starred)

*6.11. *The Grothendieck Group of a Nonsingular Curve.* Let X be a nonsingular curve over an algebraically closed field k. We will show that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$, in several steps.

(a) For any divisor $D = \sum n_i P_i$ on X, let $\psi(D) = \sum n_i \gamma(k(P_i)) \in K(X)$, where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the structure sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = \gamma(\mathcal{O}_D)$. Then use (6.18) to show that for any D, $\psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.

(b) For any coherent sheaf $\mathscr{F}$ on X, show that there exist locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$. Let $r_0 = \operatorname{rank} \mathscr{E}_0$, $r_1 = \operatorname{rank} \mathscr{E}_1$, and define $\det \mathscr{F} = (\bigwedge^{r_0}\mathscr{E}_0) \otimes (\bigwedge^{r_1}\mathscr{E}_1)^{-1} \in \operatorname{Pic} X$. Here $\bigwedge$ denotes the exterior power (Ex. 5.16). Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it gives a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally show that if D is a divisor, then $\det(\psi(D)) = \mathscr{L}(D)$.

starred

(c) If $\mathscr{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 \to \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$, where $\mathscr{T}$ is a torsion sheaf. Conclude that if $\mathscr{F}$ is a sheaf of rank r, then $\gamma(\mathscr{F}) - r\gamma(\mathscr{O}_X) \in \operatorname{Im} \psi$.

starred

(d) Using the maps ψ, det, rank, and $1 \mapsto \gamma(\mathscr{O}_X)$ from $\mathbf{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbf{Z}$.

starred

2.6.16 II.6.12 x g:1st paragraph

6.12. Let X be a complete nonsingular curve. Show that there is a unique way to define the *degree* of any coherent sheaf on X, $\deg \mathscr{F} \in \mathbf{Z}$, such that:

(1) If D is a divisor, $\deg \mathscr{L}(D) = \deg D$;
(2) If $\mathscr{F}$ is a *torsion sheaf* (meaning a sheaf whose stalk at the generic point is zero), then $\deg \mathscr{F} = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$; and
(3) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\deg \mathscr{F} = \deg \mathscr{F}' + \deg \mathscr{F}''$.

Let D a divisor. As $K(X) \approx Pic\ X \oplus \mathbb{Z}$ then and an invertible sheaf corresponds to a weil divisor written as $\sum n_i P_i$ gives an integer $\sum n_i$ we have a map $K(X) \to \mathbb{Z}$ which satisfies (1) for the degree. (3) is clear by definition of grothendieck group.

Now suppose $\mathscr{F}$ is a torsion sheaf, $\gamma(\mathscr{F}) \approx \gamma(\mathcal{O}_D)$, $D = \sum n_i P_i$. The stalk of $\mathcal{O}_D$ at P_i is k^{n_i} which has length n_i as a k-module.

k -algebraically closed implies $k \hookrightarrow \mathcal{O}_{P_i}$ and $\mathcal{O}_{P_i}/\mathfrak{m}_P \approx k$. Thus a filtration of k^{n_i} as an $\mathcal{O}_{P_i}$-module can be extended to a k-filtration. Thus the afore-mentioned stalk has length as an $\mathcal{O}_{P_i}$ module at most equal to it's length as a k-module.

Now suppose that there is a maximal k-filtration of k^{n_i} . Such a filtration has simple quotients.

Let $M \approx \langle a \rangle$ a simple nonzero module, $M \approx Ann\ A \approx \mathcal{O}_{P_i}$, and $Ann\ a \subset \mathfrak{m}_P$ so that $\mathfrak{m}_P/Ann\ a = 0$ as this is a submodule of M. Thus $M \approx \mathfrak{m}_P/Ann\ a \approx k$ so that the filtration has simple quotients as an $\mathcal{O}_{P_i}$-module. This is the opposite inequality.

To see uniqueness, if a sheaf has rank 0, then it is torsion, so (2) gives uniqueness, for rank 1, (1) gives uniqueness, for rank $n \geq 2$, then use an exact sequence and induction.

2.7 II.7 x Projective Morphisms

2.7.1 II.7.1 x g

7.1. Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f: \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism. [*Hint*: Reduce to a question of modules over a local ring by looking at the stalks.]

By another excercise, we can check that f is an isomorphism on the stalks, which since the sheaves are invertible, are local rings.

Suppose $f : A \to A$ is a surjective morphism of A-modules, A being a local ring.

Let $f(c) = 1$. Then $cf(1) = 1$ so $f(1) = c^{-1}$.

Then an inverse to f is given by $a \mapsto a \cdot c$.

2.7.2 II.7.2 x

7.2. Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leqslant m$. Show that the corresponding morphisms $\varphi: X \to \mathbf{P}_k^n$ and $\psi: X \to \mathbf{P}_k^m$ differ by a suitable linear projection $\mathbf{P}^m - L \to \mathbf{P}^n$ and an automorphism of $\mathbf{P}^n$, where L is a linear subspace of $\mathbf{P}^m$ of dimension $m - n - 1$.

Write $s_i = \sum a_{ij} t_j$.

Choose a_{ij} to fill out an invertible matrix.

Then $u_i = \sum a_{ij} x_i$ are global sections of $\mathcal{O}(1)$ on $\mathbb{P}^m$ and $\phi^* u_i = \phi^* \sum a_{ij} x_j = \sum a_{ij} \phi^* x_j = \sum a_{ij} t_j = s_i$.

If $L = Z(u_1, ..., u_n)$, then $\rho : \mathbb{P}^m \backslash L \to \mathbb{P}^n$ satisfies $\rho \circ \phi = \psi$ by II.7.1.

Note that L is a linear subspace, and ρ is the linear projection, and the a_{ij} define an automorphism.

2.7.3 II.7.3 x

7.3. Let $\varphi: \mathbf{P}_k^n \to \mathbf{P}_k^m$ be a morphism. Then:

(a) either $\varphi(\mathbf{P}^n) = pt$ or $m \geqslant n$ and $\dim \varphi(\mathbf{P}^n) = n$;

Following Hartshorne II.7, note that $\varphi : \mathbb{P}^n \to \mathbb{P}^m$ is determined by an invertible sheaf $\mathscr{L}$ and $m+1$ global sections on $\mathbb{P}^n$ which are $s_i = \varphi^*(x_i)$ for x_i homogeneous coordinates of $\mathbb{P}^m$. We have further that $\mathscr{L} = \varphi^* \mathcal{O}_{\mathbb{P}^m}(1)$.

We know all the invertible sheaves on $\mathbb{P}^n$ are $\mathcal{O}_{\mathbb{P}^n}(d)$, $d \in \mathbb{Z}$.

If $d \leq 0$, then there are at most only trivial sections so that φ must be constant.

On the other hand, if $d \geq 1$, then using example 7.8.3, the sections s_i are homogeneous polynomials of degree d. Id est, $s_i = x_0^{i_0} \cdots x_m^{i_m}$, where $\sum i_j = d$. Thus suggests the d-uple embedding.

The d-uple embedding v sends the degree d homogeneous polynomial $u_0^{i_0} \cdots u_m^{i_m}$ to the linear form $u_{i_0 \cdots i_m} \in \mathbb{P}^N$, where $N = \begin{pmatrix} n+d \\ d \end{pmatrix} - 1$. Note dimension here comes from fact that there are $\begin{pmatrix} d+n \\ n \end{pmatrix}$ different homogeneous polynomials of degree d in $\mathbb{P}^n$. Thus we have embedded $\mathcal{O}_{\mathbb{P}^n}(d)$ in $\mathcal{O}_{\mathbb{P}^N}(1)$. As the d-uple embedding is a closed immersion, dimension is preserved from $\mathbb{P}^n$ to $\mathbb{P}^N$. $v(\mathbb{P}^n)$ is thus generated by n linearly independent linear forms L_j corresponding to the s_i .

If E is defined by $L_j = 0$, $\forall j$, then the projection $\mathbb{P}^N - E \to \mathbb{P}^n$ is finite. After an isomorphism, we can ensure that $x_i \in \mathbb{P}^m$ pulls back to s_i thus regaining φ via a decomposition ϕ .

Every φ can be obtained in this manner. If $n < m$, then as the maps are finite, $dim(\mathbb{P}^n) = dim(\varphi(\mathbb{P}^n))$. As the composition of finite morphisms is finite and finite maps have finite fibers, the fibers must be finite.

On the other hand, if $m < n$, we know that L is generated by the global sections $s_i = \varphi^*(x_i)$ yet now there are $< n$. As we know all invertible sheaves are $\mathcal{O}(d)$, and we know the number of generating sections for $\mathcal{O}(d)$ (not $(n+1)!$), so φ must in this case be constant.

2.7.4 b. x

(b) in the second case, φ can be obtained as the composition of (1) a d-uple embedding $\mathbf{P}^n \to \mathbf{P}^N$ for a uniquely determined $d \geqslant 1$, (2) a linear projection $\mathbf{P}^N - L \to \mathbf{P}^m$, and (3) an automorphism of $\mathbf{P}^m$. Also, φ has finite fibres.

See above

2.7.5 II.7.4 a. x g

7.4. (a) Use (7.6) to show that if X is a scheme of finite type over a noetherian ring A, and if X admits an ample invertible sheaf, then X is separated.

If X admits ample invertible sheaf $\mathscr{L}$, then Xadmits closed immersion to some $\mathbb{P}^n$.
Projective space is separated over spec A , and thus $X \to Spec\ A$ is separated.

2.7.6 b. x

(b) Let X be the affine line over a field k with the origin doubled (4.0.1). Calculate Pic X, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on X.

Let X the affine line over k with origin doubled.
Let U_0, U_1 be the two open copies of affine line.
$Pic\ X$ is then the set of pairs $(\mathscr{L}, \mathscr{L}')$ of invertible sheaves on $\mathbb{A}^1$ which have equal restriction to $\mathbb{A}^1 \setminus \{0\}$.
By thm II.6.2, II.6.16, $Pic\ \mathbb{A}^1 = 0$ so any $\mathscr{N} \approx (\mathscr{L}, \mathscr{M}) \approx (\mathcal{O}_{\mathbb{A}^1}, \mathcal{O}_{\mathbb{A}^1})$.
Claim $Pic\ X = \mathbb{Z}$ with elements $(\mathcal{O}_{\mathbb{A}^1}, \mathscr{L}(n \cdot 0))$.
For $\mathscr{N} \in Pic\ X$, $\mathscr{N}$ as $\mathscr{N}$ is isomorphic to the structure sheaf, $\mathscr{N}$ is determined by $\mathcal{O}_{U_0}|_{U_{10}} \approx \mathscr{N}|_{U_{10}} \approx \mathcal{O}_{U_1}|_{U_{10}}$.
By II.6.16, $Pic\ U_{01} = 0$ and so $\mathscr{L}_{U_{01}} \approx \mathcal{O}_{U_{01}}$ and therefore the isomorphism is an automorphism of $k[x, x^{-1}]$ as a module over itself. Such automorphisms correspond to the polynomials ax^n for a unique $n \in \mathbb{Z}$.
The $\mathscr{N}$ is determined by this isomorphism so that $Pic\ X \approx \mathbb{Z}$.
Write $\mathscr{L}_n$ for $\mathscr{N}$. Then to give a global sections of $(\mathcal{O}_{\mathbb{A}^1}, \mathscr{L}_n)$ it is equivalent to give $f \in \Gamma(\mathbb{A}^1, \mathcal{O}_{\mathbb{A}^1})$ and $g \in \Gamma(\mathbb{A}^1, \mathscr{L}_n)$ such that $f|_{\mathbb{A}^1 \setminus \{0\}} = g|_{\mathbb{A}^1 \setminus \{0\}}$. Thus $f = g$ and therefore the global sections lie in the intersection which is $k[t]$ for $n \geq 0$ or $(t^{-1}) \subset k[t]$ for $n < 0$. Thus an element of $k[t]$ and of $t^{-1}k[t]$ which agree on $U_0 \cap U_1$. An element of $t^{-n}k[t]$ must have a homogeneous component of nonnegative degree so if $n > 0$, then the local ring at the origin of U_1 is not generated by a global esction. Thus $\mathscr{L}_n$, $n > 0$ are not gbgs. If $n < 0$ clearly $\mathscr{L}_n$ are not gbgs. Finally $\mathscr{L}_0$ is gbgs. As $\mathscr{L}_n \otimes \mathscr{L}_m \approx \mathscr{L}_{m+n}$ we see that there are no ample $\mathscr{L}_n$ since large powers will not be gbgs.

2.7.7 II.7.5 x g

7.5. Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.
(a) If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is ample.

Note the tensor of sheaves which are gbgs is gbgs.
Thus $\mathscr{M}^n$ is gbgs.
For arbitrary $\mathscr{F} \in coh(X)$, $\mathscr{F} \otimes \mathscr{L}^m$ is gbgs $n \gg 0$.
Then $\mathscr{F} \otimes (\mathscr{L} \otimes \mathscr{M})^m \approx (\mathscr{L} \otimes \mathscr{L}^m) \otimes \mathscr{M}^m$ is gbgs.

2.7.8 b. x g

(b) If $\mathscr{L}$ is ample and $\mathscr{M}$ is arbitrary then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.

As $\mathscr{M}$ is at least coherent, $\mathscr{L}^m \otimes \mathscr{M}$ is gbgs some m.
For arbitrary $\mathscr{F} \in Coh(X)$, $\mathscr{F} \otimes \mathscr{L}^n$ is gbgs.
Thus $\mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L}^p) \approx (\mathscr{F} \otimes \mathscr{L}^m) \otimes (\mathscr{M} \otimes \mathscr{L}^n) \otimes \mathscr{L}^{p-m-n}$ for large enough p.

2.7.9 c. x g

(c) If $\mathscr{L}, \mathscr{M}$ are both ample, so is $\mathscr{L} \otimes \mathscr{M}$.

The geometric definition of ample is that it intersects every curve positively.

2.7.10 d. x g

(d) If $\mathscr{L}$ is very ample and $\mathscr{M}$ is generated by global sections, then $\mathscr{L} \otimes \mathscr{M}$ is very ample.

Let $\varphi_{\mathscr{L}}$, $\varphi_{\mathscr{M}}$ the corresponding morphisms to $\mathbb{P}^n$, $\mathbb{P}^m$ respectively, where $\varphi_{\mathscr{L}}^*(\mathcal{O}(1)) = \mathscr{L}$ and $\varphi_{\mathscr{M}}^*(\mathcal{O}(1)) = \mathscr{M}$.

The product $\varphi_{\mathscr{L}} \times \varphi_{\mathscr{M}}$ in the segre embedding satisfies $\varphi^*(\mathcal{O}(1)) = \mathscr{L} \otimes \mathscr{M}$ which is an immersion since so is $\varphi_{\mathscr{L}}$.

2.7.11 x g ample large multiple is very ample. x

(e) If $\mathscr{L}$ is ample, then there is an $n_0 > 0$ such that $\mathscr{L}^n$ is very ample for all $n \geqslant n_0$.

If $\mathscr{L}^m$ is very ample and $\mathscr{L}^d$ is gbgs for all $d > d_0$, then take $m + d_0$ and use part (d) .

2.7.12 II.7.6 x g The Riemann Roch Problem

7.6. *The Riemann–Roch Problem.* Let X be a nonsingular projective variety over an algebraically closed field, and let D be a divisor on X. For any $n > 0$ we consider the complete linear system $|nD|$. Then the Riemann–Roch problem is to determine $\dim|nD|$ as a function of n, and, in particular, its behavior for large n. If $\mathscr{L}$ is the corresponding invertible sheaf, then $\dim|nD| = \dim \Gamma(X, \mathscr{L}^n) - 1$, so an equivalent problem is to determine $\dim \Gamma(X, \mathscr{L}^n)$ as a function of n.

(a) Show that if D is very ample, and if $X \hookrightarrow \mathbf{P}_k^n$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim|nD| = P_X(n) - 1$, where P_X is the *Hilbert polynomial* of X (I, §7). Thus in this case $\dim|nD|$ is a polynomial function of n, for n large.

Using the given embedding, we associate $\mathscr{L}$ with $S(1)^{\sim}$, where S is the homogeneous coordinate ring of the variety.

Usingexc II.5.9.b, $S_n \to \Gamma(X, S(n)^{\sim}) \approx \Gamma(X, \mathscr{L}^n)$ is an isomorphism for large enough n.

By definition, we have $dim\ |nD| = dim\ \Gamma(X, \mathscr{L}^n) - 1 = dim\ S_n - 1 = \phi(n) - 1$, where ϕ is the hilbert function of X.

Now recall that the hilbert function equals the hilbert polynomial on integers for large enough n.

2.7.13 b. x

(b) If D corresponds to a torsion element of Pic X, of order r, then $\dim|nD| = 0$ if $r|n$, -1 otherwise. In this case the function is periodic of period r.

It follows from the general Riemann- Roch theorem that $\dim|nD|$ is a polynomial function for n large, whenever D is an *ample* divisor. See (IV, 1.3.2), (V, 1.6), and Appendix A. In the case of algebraic surfaces, Zariski [7] has shown for any effective divisor D, that there is a finite set of polynomials $P_1, \ldots, P_r$, such that for all n sufficiently large, $\dim|nD| = P_{i(n)}(n)$, where $i(n) \in \{1, 2, \ldots, r\}$ is a function of n.

Suppose D is torsion of order r. If $r|n$, $nD = 0$ so nD is trivial. Thus $dim\,|nD| = 0$. If $r \nmid n$, and there is an effective divisor E lienarly equivalent to nD , then $0 \sim rnD \sim rE > 0$ contradiction. Thus $dim\,|nD| = -1$.

2.7.14 II.7.7 x g Some Rational Surfaces

7.7. *Some Rational Surfaces.* Let $X = \mathbf{P}_k^2$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.

(a) The complete linear system $|D|$ gives an embedding of $\mathbf{P}^2$ in $\mathbf{P}^5$, whose image is the Veronese surface (I, Ex. 2.13).

Let $\phi : \mathbb{P}^2 \to \mathbb{P}^5$ correspond to the linear system $|D|$ since there are 6 global sections. As in thm II.7.1, define $\phi : \mathbb{P}^2_{s_i} \to D_+(y_i)$, s_i the $(i+1)^{th}$ basis vector of $|D|$. If $s_0 = x_0^2$ the morphism from $\mathbb{P}^2_{s_0}$ is given by $Spec\ \left[\frac{x_1}{x_0}, \frac{x_2}{x_0}\right] \to Spec\ \left[\frac{y_1}{y_0}, ..., \frac{y_5}{y_0}\right]$ which on global sections is $\left(\frac{y_1}{y_0}, ..., \frac{y_5}{y_0}\right) \mapsto \left(\frac{x_2^2}{x_1^2}, \frac{x_0 x_1}{x_1^2}, \frac{x_0 x_2}{x_1^2}, \frac{x_1 x_2}{x_1^2}\right)$. As this agrees with v_2, and on each other open set we have similar agreement, then by exc II.4.2, the morphisms agree.

2.7.15 b. x

(b) Show that the subsystem defined by $x^2, y^2, z^2, y(x-z), (x-y)z$ gives a closed immersion of X into $\mathbf{P}^4$. The image is called the *Veronese surface* in $\mathbf{P}^4$. Cf. (IV, Ex. 3.11).

Recall closed immersion iff separates points and tangent lines.

claim: points are separated by the linear system.

Let $P_0 = (a_0 : b_0 : c_0)$, and $P_1 = (a_1 : b_1 : c_1)$ two points.

If $a_0 = 0$ and $a_1 \neq 0$, then x^2 separates points.

If $a_0 = a_1 = 0$, then the sections are y^2, z^2, yz which generate $\mathcal{O}_{\mathbb{P}^1}(2)$ which is very ample. Similarly for the other coordinate hyperplanes. Thus we assume that the distinct points are not contained in any coordinate hyperplanes.

Thus consider two points $(a, b), (c, d)$ on $D(x)$.

Then $y^2 - a^2(1)$ and $z^2 - b^2(1)$ separate all points except at the roots $(c, d) = (\pm a, \pm b)$. For $(-a, -b)$ we can separate points with $y - yz - (a - ab)(1)$ and similarly for other cases.

Claim: tangent lines are separated.

On $z = 1$ we have $1, x^2, y^2, xy - y, x - y$. Assume $P = (a, b)$. If $a \neq 0$, then $x - y - (a - b)(1)$ and $x^2 - a^2(1)$ have no tangent lines in common. If $b \neq 0$, then $x - y - (a - b)(1)$ and $y^2 - b^2(1)$ have no tangent lines in common. In the last case, if $a = b = 0$, then $xy - y$ and $x - y$ have different tangent lines at the origin. Thus tangent lines are separated. On $y = 1$ the situation is similar. On $x = 1$ we have $1, y^2, z^2, y - yz, z - yz$ and $y - yz$ and $z - yz$ have different tangent lines at the origin.

2.7.16 c. x

(c) Let $\mathfrak{d} \subseteq |D|$ be the linear system of all conics passing through a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbf{P}^4$. Furthermore, if we blow up P, to get a surface $\tilde{X}$, then this map extends to give a closed immersion of $\tilde{X}$ in $\mathbf{P}^4$. Show that $\tilde{X}$ is a surface of degree 3 in $\mathbf{P}^4$, and that the lines in X through P are transformed into straight lines in $\tilde{X}$ which do not meet. $\tilde{X}$ is the union of all these lines, so we say $\tilde{X}$ is a *ruled surface* (V, 2.19.1).

Let $P \in \mathbb{P}^2$ be given by $\langle x_0, x_1 \rangle$ so on $D_+(x_0), D_+(x_1)$, and $D_+(x_2) - P \subset \mathbb{P}^2$, the linear system $\mathfrak{d}$ with basis vectors $x_0^2, x_1^2, x_0 x_z, x_1 x_2, x_0 x_2$ maps U homeomorphically onto an open subset of closed subvariety $V = V(y_2 y_3 - y_0 y_4, y_1 y_3 - y_2 y_4)$.

The image of $\tilde{X}$ is closed and $U = \pi^{-1}$ is dense in $\tilde{X}$ so the closure $\overline{U}$ is the image of $\tilde{X}$.

A global section y_0 of $\mathcal{O}(1)$ corresponds to the divisor $V(y_0, y_1, y_2) + V(y_0, y_2, y_3) + V(y_0, y_3, y_4)$ which has degree 3.

Any line of X through P can be written as $ax_0 + bx_1 = 0$.

It's image in $U \subset \mathbb{P}^2$ has as closure the line $V(ay_0 + by_2, ay_1 + by_2, ay_4 + by_3)$. Two distinct such lines do not meet since the ratio of their coefficients are different.

2.7.17 II.7.8 x sections vs quotient invertible sheaves

7.8. Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbf{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 correspondence between *sections* of π (i.e., morphisms $\sigma : X \to \mathbf{P}(\mathscr{E})$ such that $\pi \circ \sigma = \mathrm{id}_X$) and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.

By 7.12, we have a 1-1 correspondence between sections of π and surjections $\mathscr{E} \to \mathscr{L} \to 0$.

2.7.18 II.7.9 x g

7.9. Let X be a regular noetherian scheme, and $\mathscr{E}$ a locally free coherent sheaf of rank $\geqslant 2$ on X.
(a) Show that Pic $\mathbf{P}(\mathscr{E}) \cong$ Pic $X \times \mathbf{Z}$.

Define $\alpha : Pic\ X \times \mathbb{Z} \to Pic\ \mathbb{P}(\mathscr{E})$ by $(\mathscr{L}, n) \mapsto (\pi^*\mathscr{L}) \otimes \mathcal{O}(n)$.

Let r be the rank of $\mathscr{E}$. If $x \in X$, $x \in U = Spec\ A$, where $\mathscr{E}|_U \approx \mathcal{O}_X|_U^r$ then we have $\pi^{-1}U = \mathbb{P}_U^{r-1}$ so there is an embedding $\mathbb{Z} \approx \mathbb{P}_{k(x)}^{r-1} \to \mathbb{P}_U^{r-1} \to \mathbb{P}(\mathscr{E})$. As $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(n)|_U \approx \mathcal{O}_U(n)$, then this gives a left inverse to $\mathbb{Z} \to Pic\ \mathbb{P}(\mathscr{E})$.

Claim: α is injective.

Suppose $\pi^*\mathscr{L} \otimes \mathcal{O}(n) \approx \mathcal{O}_{\mathbb{P}(\mathscr{E})}$. By II.7.11, $\pi_*(\pi^*\mathscr{L} \otimes \mathcal{O}_n) \approx \mathcal{O}_X$, so by the projection formula, $\mathscr{L} \otimes \pi_*\mathcal{O}(n) \approx \mathcal{O}_X$. As $\pi_*\mathcal{O}(n)$ is the degree n part of the symmetric algebra, and rank $\mathscr{E} \geq 2$, then $n = 0$ so $\mathscr{L} \approx \mathcal{O}_X$.

Claim α is surjective.

If U_i gives an open cover of X on which $\mathscr{E}$ is trivial, and each U_i is integral and separated, (as X is regular, and affine schemes are separated) , then $V_i := \mathbb{P}(\mathscr{E}|_{U_i}) \approx U_i \times \mathbb{P}^{r-1}$ cover $\mathbb{P}(\mathscr{E})$.

As X is regular so are U_i thus satisfy (*) so by exc II.6.1, $Pic\ V_i \approx Pic\ U_i \times \mathbb{Z}$.

For $\mathscr{L} \in Pic\ \mathbb{P}(\mathscr{E})$, we obtain an element $\mathcal{O}_i(n_i) \otimes \pi_i^*\mathscr{L}_i \in Pic\ V_i \approx Pic\ U_i \times \mathbb{Z}$ together with transition isomorphisms $\alpha_{ij} : (\mathcal{O}_i(n_i) \otimes \pi_i^*\mathscr{L}_i)|_{V_{ij}} \to \left(\mathcal{O}_j(n_j) \otimes \pi_j^*\mathscr{L}_j\right)|_{V_{ji}}$ satisfying the cocycle condition. Using the projection formula gives $\pi_*\alpha_{ij} : \pi_*\mathcal{O}_i(n_i)|_{V_{ij}} \otimes \mathscr{L}_i \to \pi_*\mathcal{O}_j(n_j)|_{V_{ij}} \otimes \mathscr{L}_j$. As in II.7.11, $n_i = n_j$ and by definition of $\mathbb{P}(\mathscr{E})$, $\mathcal{O}_j(n)|_{V_{ij}} = \mathcal{O}_{ij}(n)$ so we have $\mathcal{O}_{ij}(n) \otimes \pi_i^*\mathscr{L}_i|_{V_{ij}} \approx \mathcal{O}_{ij}(n) \otimes \pi_j^*\mathscr{L}_j|_{V_{ij}}$. Tensoring with $\mathcal{O}_{ij}(-n)$ and using the projection formula with II.7.11 gives isomorphisms $\mathscr{L}_i|_{U_{ij}} \approx \mathscr{L}_j|_{U_{ij}}$ satisfying the cocycle condition. Glueing gives a sheaf $\mathscr{M}$ such that $\pi^*\mathscr{M} \otimes \mathcal{O}(n) \approx \mathscr{L}$ on any component of X.

2.7.19 b. x

(b) If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbf{P}(\mathscr{E}) \cong \mathbf{P}(\mathscr{E}')$ (over X) if and only if there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.

Suppose that $f : \mathbb{P}(\mathscr{E}) \approx \mathbb{P}(\mathscr{E}')$. By (a), $f^*\mathcal{O}'(1) \approx \mathcal{O}(1) \otimes \pi^*\mathscr{L}$ for $\mathscr{L} \in Pic\ X$. Using exc projection and thm II.7.11, $\mathscr{E}' = \pi'_*(\mathcal{O}'(1)) = \pi_*(\mathcal{O}(1) \otimes \pi^*\mathscr{L}) = \pi_*\mathcal{O}(1) \otimes \mathscr{L} = \mathscr{E} \otimes \mathscr{L}$.

On the other hand if $\mathscr{E}' \approx \mathscr{E} \otimes \mathscr{L}$, then by thm II.7.11.b, there is a surjection $\pi^*\mathscr{E}' \approx \pi^*\mathscr{E} \otimes \pi^*\mathscr{L} \twoheadrightarrow \mathcal{O}(1) \otimes \pi^*\mathscr{L}$, and thus a map $\mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ by thm II.7.12. An inverse is given by considering $\mathscr{E} \approx \mathscr{E}' \otimes \mathscr{L}^{-1}$.

2.7.20 II.7.10 x P^n Bundles over a Scheme

7.10. $\mathbf{P}^n$-*Bundles Over a Scheme.* Let X be a noetherian scheme.
(a) By analogy with the definition of a vector bundle (Ex. 5.18), define the notion of a *projective n-space bundle* over X, as a scheme P with a morphism $\pi : P \to X$ such that P is locally isomorphic to $U \times \mathbf{P}^n$, $U \subseteq X$ open, and the transition automorphisms on Spec $A \times \mathbf{P}^n$ are given by A-linear automorphisms of the homogeneous coordinate ring $A[x_0, \ldots, x_n]$ (e.g., $x_i' = \sum a_{ij}x_j$, $a_{ij} \in A$).

See Gathmann's notes.

2.7.21 b. x g

(b) If $\mathscr{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbf{P}(\mathscr{E})$ is a $\mathbf{P}^n$-bundle over X.

X is covered by open affines $U_i = Spec\ A_i$ where $\mathscr{E} \approx \mathcal{O}|_{U_i}^{\oplus(n+1)}$.

Let $\pi : \mathbb{P}(\mathscr{E}) \to X$ the projection.

Then $\pi^{-1}U_i \approx Proj\ \mathscr{I}(\mathscr{E})(U_i) \approx Proj\ \mathscr{I}\left(\mathcal{O}_{U_i}^{\oplus(n+1)}\right)(U_i)$

$\approx \mathscr{I}\left(\mathcal{O}|_U^{\oplus(n+1)}\right)(U) \approx Proj\ A[x_0, ..., x_n] \approx \mathbb{P}_U^n$.

Thus we have the trivializations needed for a $\mathbb{P}^n$ -bundle.

On $V = U_i \cap U_j$, by definition of $\mathbb{P}(\mathscr{E})$ we have an automorphisms $\psi = \psi_j \circ \psi_i^{-1}$ defined via $\mathcal{O}_{U_i}^{n+1}|_V \approx \mathcal{O}_{U_j}^{n+1}|_V$ coming from the restriction morphisms $\mathscr{E}(U_i) \to \mathscr{E}(V) \leftarrow \mathscr{E}(U_j)$.

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*(c) Assume that X is regular, and show that every $\mathbf{P}^n$-bundle P over X is isomorphic to $\mathbf{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X. [*Hint*: Let $U \subseteq X$ be an open set such that $\pi^{-1}(U) \cong U \times \mathbf{P}^n$, and let $\mathscr{L}_0$ be the invertible sheaf $\mathcal{O}(1)$ on $U \times \mathbf{P}^n$. Show that $\mathscr{L}_0$ extends to an invertible sheaf $\mathscr{L}$ on P. Then show that $\pi_*\mathscr{L} = \mathscr{E}$ is a locally free sheaf on X and that $P \cong \mathbf{P}(\mathscr{E})$.] Can you weaken the hypothesis "X regular"?

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2.7.22 d. x

(d) Conclude (in the case X regular) that we have a 1-1 correspondence between $\mathbf{P}^n$-bundles over X, and equivalence classes of locally free sheaves $\mathscr{E}$ of rank $n+1$ under the equivalence relation $\mathscr{E}' \sim \mathscr{E}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on X.

This follows from (b), (c), and exc II.7.9.

2.7.23 II.7.11 x

7.11. On a noetherian scheme X, different sheaves of ideals can give rise to isomorphic blown up schemes.

(a) If $\mathscr{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathscr{I}^d$ for any $d \geqslant 1$ gives a scheme isomorphic to the blowing up of $\mathscr{I}$ (cf. Ex. 5.13).

By exc II.5.13, we have $Proj\ \oplus_{n\geq0} \mathscr{I}(U)^{nd} \approx Proj\ \oplus_{n\geq0} \mathscr{I}(U)^n$.

Now if $\varphi : T \to S$ is a morphism of graded rings, then as in II.5.13, we have a commutative diagram:

$$\begin{array}{ccc} Proj\ S & \longrightarrow & Proj\ T \\ \downarrow & & \downarrow \\ Proj\ S^{(d)} & \longrightarrow & Proj\ T^{(d)} \end{array}$$

so the given isomorphism is natural and thus on intersections $U_i \cap U_j$ we have isomorphisms.

Now glueing gives the result.

2.7.24 b. x

(b) If $\mathscr{I}$ is any coherent sheaf of ideals, and if $\mathscr{J}$ is an invertible sheaf of ideals, then $\mathscr{I}$ and $\mathscr{I} \cdot \mathscr{J}$ give isomorphic blowings-up.

By lemma II.7.9

2.7.25 c. x

(c) If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f: f^{-1}U \to U$ is an isomorphism. Then $\mathscr{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.

If $f: Z \to X$ is birational, then $E = Exc(f) = Z - f^{-1}(U)$ is a divisor (see Debarre Higher Dimensional Algebraic Geometry 1.40). Pushing forward to X the corresponding ideal sheaf gives the required $\mathscr{I}$.

2.7.26 II.7.12 x g

7.12. Let X be a noetherian scheme, and let Y, Z be two closed subschemes, neither one containing the other. Let $\tilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathscr{I}_Y + \mathscr{I}_Z$). Show that the strict transforms $\tilde{Y}$ and $\tilde{Z}$ of Y and Z in $\tilde{X}$ do not meet.

If $P \in \tilde{Y} \cap \tilde{Z}$, then $\pi(P)$ is contained in an open affine $U = Spec\ A \subset X$.
Then $\pi^{-1}U = Proj \bigoplus_{d \geq 0} (I_Y + I_Z)^d$ with $I_Y = \mathscr{I}_Y(U), I_Z = \mathscr{I}_Z(U)$.
Then $Y \cap U = Spec\ A/I_Y$, $Z \cap U = Spec\ A/I_Z$.
Then $\pi^{-1}(U \cap Y) = Proj \bigoplus_{d \geq 0} ((I_Y + I_Z)(A/I_Y))^d \subset \tilde{Y}$, and $\pi^{-1}(U \cap Z) = ...$.
The map $\pi^{-1}(U \cap Y) \to \pi^{-1}(U)$ is given by $\oplus_{d \geq 0} (I_Y + I_Z)^d \to \oplus_{d \geq 0} ((I_Y + I_Z)(A/I_Y))^d$ and $\pi^{-1}(U \cap Z) \to \pi^{-1}(U)$ is given by ....
Then the kernel of each of these ring homomorphisms is $\oplus_{d \geq 0} I_Y^d$, $\oplus_{d \geq 0} I_Z^d$.
A nonempty intersection gives a homogeneous prime ideal of $\oplus_{d \geq 0} (I_Y + I_Z)^d$ containing $\oplus_{d \geq 0} I_Y^d$, $\oplus_{d \geq 0} I_Z^d$. This is a contradiction.

2.7.27 II.7.13 A Complete Nonprojective Variety *

***7.13.** *A Complete Nonprojective Variety.* Let k be an algebraically closed field of char $\neq 2$. Let $C \subseteq \mathbf{P}_k^2$ be the nodal cubic curve $y^2 z = x^3 + x^2 z$. If $P_0 = (0,0,1)$ is the singular point, then $C - P_0$ is isomorphic to the multiplicative group $\mathbf{G}_m = \operatorname{Spec} k[t,t^{-1}]$ (Ex. 6.7). For each $a \in k$, $a \neq 0$, consider the translation of $\mathbf{G}_m$ given by $t \mapsto at$. This induces an automorphism of C which we denote by φ_a.

Now consider $C \times (\mathbf{P}^1 - \{0\})$ and $C \times (\mathbf{P}^1 - \{\infty\})$. We glue their open subsets $C \times (\mathbf{P}^1 - \{0, \infty\})$ by the isomorphism $\varphi: \langle P,u \rangle \mapsto \langle \varphi_u(P),u \rangle$ for $P \in C, u \in \mathbf{G}_m = \mathbf{P}^1 - \{0,\infty\}$. Thus we obtain a scheme X, which is our example. The projections to the second factor are compatible with φ, so there is a natural morphism $\pi: X \to \mathbf{P}^1$.

(a) Show that π is a proper morphism, and hence that X is a complete variety over k.

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(b) Use the method of (Ex. 6.9) to show that $\operatorname{Pic}(C \times \mathbf{A}^1) \cong \mathbf{G}_m \times \mathbf{Z}$ and $\operatorname{Pic}(C \times (\mathbf{A}^1 - \{0\})) \cong \mathbf{G}_m \times \mathbf{Z} \times \mathbf{Z}$. [*Hint:* If A is a domain and if $*$ denotes the group of units, then $(A[u])^* \cong A^*$ and $(A[u,u^{-1}])^* \cong A^* \times \mathbf{Z}$.]

MISS

(c) Now show that the restriction map $\mathrm{Pic}(C \times \mathbf{A}^1) \to \mathrm{Pic}(C \times (\mathbf{A}^1 - \{0\}))$ is of the form $\langle t,n \rangle \mapsto \langle t,0,n \rangle$, and that the automorphism φ of $C \times (\mathbf{A}^1 - \{0\})$ induces a map of the form $\langle t,d,n \rangle \mapsto \langle t,d+n,n \rangle$ on its Picard group.

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(d) Conclude that the image of the restriction map $\mathrm{Pic}\, X \to \mathrm{Pic}(C \times \{0\})$ consists entirely of divisors of degree 0 on C. Hence X is not projective over k and π is not a projective morphism.

MISS

2.7.28 II.7.14 x g

7.14. (a) Give an example of a noetherian scheme X and a locally free coherent sheaf $\mathscr{E}$, such that the invertible sheaf $\mathscr{O}(1)$ on $\mathbf{P}(\mathscr{E})$ is *not* very ample relative to X.

Consider $X = \mathbb{P}(\mathcal{O}(-1)) \approx \mathbb{P}^1$.

If $\mathcal{O}(1)$ on X was very ample, then it would give a closed immersion to $\mathbb{P}^n$, where the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ would be $\mathcal{O}_{\mathbb{P}^1}(-1)$, which is not effective, contradicting negativity.

2.7.29 b. x

(b) Let $f: X \to Y$ be a morphism of finite type, let $\mathscr{L}$ be an ample invertible sheaf on X, and let $\mathscr{S}$ be a sheaf of graded $\mathscr{O}_X$-algebras satisfying (†). Let $P = \mathbf{Proj}\, \mathscr{S}$, let $\pi: P \to X$ be the projection, and let $\mathscr{O}_P(1)$ be the associated invertible sheaf. Show that for all $n \gg 0$, the sheaf $\mathscr{O}_P(1) \otimes \pi^* \mathscr{L}^n$ is very ample on P relative to Y. [*Hint*: Use (7.10) and (Ex. 5.12).]

$\mathscr{L}$ is ample relative to U , and for $n > 0$, $\mathscr{L}^n$ is v.a. on X relative to Y.

If $\pi : P \to X$ is projection, then by thm II.7.10, for large enough m, $\mathcal{O}_P(1) \otimes \pi^* \mathscr{L}^m$ is very ample on P relative to X. By exc II.5.12, for n fixed, and $m \gg 0$, $\mathcal{O}_P(1) \otimes \mathscr{L}^{m+n}$ is very ample on P relative to Y.

2.8 II.8 x Differentials

2.8.1 II.8.1 x

8.1 Here we will strengthen the results of the text to include information about the sheaf of differentials at a not necessarily closed point of a scheme X.

(a) Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = B/\mathfrak{m}$ of B is a separably generated extension of k. Then the exact sequence of (8.4A),

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the left also. [*Hint*: In copying the proof of (8.7), first pass to $B/\mathfrak{m}^2$, which is a complete local ring, and then use (8.25A) to choose a field of representatives for $B/\mathfrak{m}^2$.]

Injectivity of the first map is equivalent to surjectivity of $hom_{k(B)}\left(\Omega_{B/k} \otimes k(B), k(B)\right) \approx Der_k(B, k(B)) \to hom_k(\mathfrak{m}/\mathfrak{m}^2, k(B))$.

Note that if $d : B \to k(B)$ is a derivation, then $\delta^*(d)$ is the restriction to $\mathfrak{m}$ with $d(\mathfrak{m}^2) = 0$ by the product rule.

If $h \in Hom(\mathfrak{m}/\mathfrak{m}^2, k(B))$ and $b = c + \lambda \in B$, $\lambda \in k(B), c \in \mathfrak{m}$ (using thm II.8.25A), then define $db = h(c \bmod \mathfrak{m}^2)$.

Then d is a $k(B)$ derivation and $\delta^*(d) = h$.

2.8.2 b. x

(b) Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B +$ tr.d. $k(B)/k$.

Suppose $\Omega_{B/k}$ is free of rank$= dim\ B + tr.d.\ k(B)/k$.

By thm II.8.6.a, $dim\ \Omega_{k(B)/k} = tr.d.\ k(B)/k$, so using

$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$ from (a), $dim\ \mathfrak{m}/\mathfrak{m}^2 = dim\ B$ so B is regular.

Conversely, suppose B is regular.

Then similarly, $dim_{k(B)}\Omega_{B/k} \otimes k(B) = dim\ B + tr.d.\ k(B)/k$.

Recalling the proof of thm II.8.8, we just need to show that $dim_K\Omega_{B/k}$ is $dim\ B + tr.d.k(B)/k$.

We have $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$ using thm II.8.2A.

As k is perfect, by thm I.4.8A, K is separably generated so $dim_K\Omega_{K/k} = tr.d.K/k$ by thm II.8.6.A.

Thus $dim_K\Omega_{B/k} \otimes K \approx tr.d.\ K/k$.

As $B = A_{\mathfrak{p}}$ for $\mathfrak{p} \in Spec\ A$, then $Frac\ A = Frac\ B$ and $ht\ \mathfrak{p} = dim\ B$.

By thm I.1.8A, $tr.d.\ tr.d.K/k = dim\ A = ht\ \mathfrak{p} + dim\ A/\mathfrak{p} =$
$dim\ B + dim\ A/\mathfrak{p} = dim\ B + tr.d.Frac(A/\mathfrak{p})/k =$
$dim\ B + tr.d.\ k(B)$.

2.8.3 c. x

(c) Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necessarily closed, show that the local ring $\mathcal{O}_{x,X}$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

Let $x \in U = Spec\ A$. By (b), $\mathcal{O}_{X,x}$ is a regular local ring iff $\Omega_{A_{\mathfrak{p}}/k} \approx \Omega_{A/k} \approx \left(\Omega_{X/k}\right)_x$ is free of rank equal to $dim\ A_{\mathfrak{p}} + tr.d.\ k(A_{\mathfrak{p}})/k = dim\ A = dim\ X$.

2.8.4 d. x g

(d) Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X | \mathcal{O}_x \text{ is a regular local ring}\}$ is an open dense subset of X.

By II.8.16, U is dense since it contains any open dense subset $V \subset X$ which is smooth.

If $x \in U$, $\Omega_{X/k}$ is locally free by (c), so b exc II.5.7.a, there is an open neighborhood $W \ni x$ where $\Omega_{X/k}|_W$ is free of rank n.

At each $w \in W$ the stalks are free of rank n , so by part (c) $w \in U$.

2.8.5 II.8.2 x

8.2. Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathscr{O}_X \to \mathscr{E}$ giving rise to an exact sequence

$$0 \to \mathscr{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is also locally free. [*Hint*: Use a method similar to the proof of Bertini's theorem (8.18).]

Define $Z \subset X \times V$ by $\{(x,s) \,|s_x \in \mathfrak{m}_x \mathscr{E}_x\}$ and define $p_1, p_2 : X \times V|_Z$ as the projections.

A fiber of p_1 over x_0 consists of all sections vanishing at x_0 which is the kernel of $V \otimes_k k(x_0) \to \mathscr{E}_{x_0} \otimes_{\mathcal{O}_{x_0}} k(x_0) \approx \mathscr{E}_{x_0} \otimes_{\mathcal{O}_{x_0}} \mathcal{O}_{x_0}/\mathfrak{m}_{x_0} \approx \mathscr{E}_{x_0}/\mathfrak{m}_{x_0}\mathscr{E}_{x_0}$. As $\mathscr{E}$ is gbgs, then this map is surjective.

Since $\mathscr{E}$ is rank r, $dim\ V - dim\ ker = rk\mathscr{E}_{x_0} = r$ by rank nullity.

Thus $dim\ ker = dim\ V - r$. Thus $dim\ Z = dim\ X + dim\ V - r$.

But then as $r > n$ we must have $dim\ Z < dim\ V$ so p_2 is not surjective.

For the second part if s therefore satisfies $s_x \notin \mathfrak{m}_x\mathscr{E}_x$ then $\mathcal{O}_x \to \mathscr{E}$ is given by $\times s$. By exc II.5.7.b, the cokernel of this map is locally free of rank $rk\ \mathscr{E} - 1$.

2.8.6 II.8.3 x g Product Schemes

8.3. *Product Schemes.*

(a) Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X \times_S Y/S} \cong p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

II.8.10 says (writing S for Y and Y for Y') that if $f : X \to S$ is a morphism, and $g : Y \to S$ is another morphism, and $f' : X \times_S Y \to Y$ is obtained by base extension, then $\Omega_{X \times_S Y/Y} \approx p_1^*\left(\Omega_{X/S}\right)$ and also $\Omega_{X \times_S Y/X} \approx p_2^*\left(\Omega_{Y/S}\right)$.

II.8.11 gives that (writing S for Z, X for Y, and $X \times_S Y$ for S)

$\Omega_{X \times_S Y/Y} \approx p_1^*\left(\Omega_{X/S}\right) \to \Omega_{X \times_S Y/S} \to \Omega_{X \times_S Y/X} \approx p_2^*\left(\Omega_{Y/S}\right) \to 0$

and similarly,

$p_2^*\left(\Omega_{Y/S}\right) \to \Omega_{X \times Y/S} \to p_1^*\Omega_{X/S} \to 0$

are exact sequences.

From here, you just need to show, using both sequences, that the first sequence splits, so that $\Omega_{X \times Y/S} \approx p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$.

2.8.7 b. x g

(b) If X and Y are nonsingular varieties over a field k, show that $\omega_{X \times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.

Let $dim\ (X) = m$, $dim\ (Y) = n$.

Using definition of the canonical sheaf, $\omega_{X \times Y} = \Lambda^{m \times n}\Omega_{X \times Y}$ which is $\Lambda^{mn}\left(p_1^*\left(\Omega_X\right) \oplus p_2^*\left(\Omega_Y\right)\right)$.

Now using exc II.5.16.d, e, and thm II.8.15, we see that

$\Lambda^{mn}\left(p_1^*\left(\Omega_X\right) \oplus p_2^*\left(\Omega_Y\right)\right) \approx \Lambda^m p_1^*\left(\Omega\right) \otimes \Lambda^n p_2^*\left(\Omega\right) \approx$

$p_1^*\left(\Lambda^m\Omega_X\right) \otimes p_2^*\left(\Lambda^n\Omega_Y\right)$.

2.8.8 c. x g

(c) Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I, Ex. 7.2). This shows that the arithmetic genus and the geometric genus of a nonsingular projective variety may be different.

From exc I.7.2.b, $p_a(Y) = 1$. From (e), with Y^3 we get $p_a(Y \times Y) = -1$.
Now using ex. II.8.20.3 gives $\mathcal{O}_Y \approx \omega_Y$. Using (b) gives $\omega_{Y\times Y} \approx p_1^*\mathcal{O}_Y \otimes p_2^*\mathcal{O}_Y$.
This is $\mathcal{O}_{Y\times Y}$ by definition.
As Y is projective, thus complete, thus proper, we can use exc II.4.5.d to see that $\Gamma(Y \times Y, \mathcal{O}_{Y\times Y}) = k$ which has dimension over k equal to 1. By definition geometric genus is $p_g = dim_k\, \Gamma(X, \omega_X)$ which is 1.

2.8.9 II.8.4 x Complete Intersections in Pn

8.4. *Complete Intersections in* $\mathbf{P}^n$. A closed subscheme Y of $\mathbf{P}^n_k$ is called a (*strict, global*) *complete intersection* if the homogeneous ideal I of Y in $S = k[x_0, \ldots, x_n]$ can be generated by $r = \operatorname{codim}(Y, \mathbf{P}^n)$ elements (I, Ex. 2.17).

(a) Let Y be a closed subscheme of codimension r in $\mathbf{P}^n$. Then Y is a complete intersection if and only if there are hypersurfaces (i.e., locally principal subschemes of codimension 1) $H_1, \ldots, H_r$, such that $Y = H_1 \cap \ldots \cap H_r$ *as schemes*, i.e., $\mathscr{I}_Y = \mathscr{I}_{H_1} + \ldots + \mathscr{I}_{H_r}$. [*Hint*: Use the fact that the unmixedness theorem holds in S (Matsumura [2, p. 107]).]

If $I_Y = (f_1, ..., f_r)$ then $Y = \cap_{i=1}^r H_i$, $H_i = Z(f_i)$ as in chapter 1.
On the other hand, if $Y = \cap H_i$, is an intersection of integral hypersurfaces, then as H_i are irreducible (I_{H_i}) is a prime ideal in the coordinate ring $S = k[x_0, ..., x_n]$ of $\mathbb{P}^n$.
Thus $I_{H_{i+1}}$ is not a zero divisor mod I_H so that $(I_{H_1}, ..., I_{H_r})$ forms a regular sequence and the ideal is contained in I_Y.
As $S/(I_{H_1}, ..., I_{H_r})$ has degree $\sum deg\ H_i$ by bezout, then $(I_{H_1}, ..., I_{H_r}) = I \cap J$ where $codim\ J > 2$.
Using the unmixedness theorem, primary components of $(I_{H_1}, ..., I_{H_r})$ are $codim \le 1$ thus $J = \emptyset$ so $I_Y = (I_{H_1}, ..., I_{H_r})$.

2.8.10 b. x g

(b) If Y is a complete intersection of dimension $\geqslant 1$ in $\mathbf{P}^n$, and if Y is normal, then Y is projectively normal (Ex. 5.14). [*Hint*: Apply (8.23) to the affine cone over Y.]

As Y is normal, then $Sing\ Y$, $Sing\ C(Y)$ have codimension ≥ 2. Thus the homogeneous coordinate ring of $S(C(Y))$ is integrally closed by thm II.8.23.b. And thus so is $S(Y)$. Thus Y is projectively normal.

2.8.11 c. x g

If Y is a complete intersection of dimension $\geqslant 1$ in $\mathbf{P}^n$, and if Y is normal,

(c) With the same hypotheses as (b), conclude that for all $l \geqslant 0$, the natural map $\Gamma(\mathbf{P}^n, \mathcal{O}_{\mathbf{P}^n}(l)) \to \Gamma(Y, \mathcal{O}_Y(l))$ is surjective. In particular, taking $l = 0$, show that Y is connected.

Since Y is projectively normal, the natural map is surjective by definition.
When $l = 0$, then $k \twoheadrightarrow \Gamma(Y, \mathcal{O}_Y)$ gives $dim\ \Gamma(Y, \mathcal{O}_Y) \leq 1$, so the number of components is ≤ 1 .

2.8.12 d. x g

(d) Now suppose given integers $d_1, \ldots, d_r \geqslant 1$, with $r < n$. Use Bertini's theorem (8.18) to show that there exist nonsingular hypersurfaces $H_1, \ldots, H_r$ in $\mathbf{P}^n$, with deg $H_i = d_i$, such that the scheme $Y = H_1 \cap \ldots \cap H_r$ is irreducible and nonsingular of codimension r in $\mathbf{P}^n$.

Assume that k is algebraically closed so we can use Bertini.
By bertini, the general element of a free linear series is smooth.
Thus, if H is a hyperplane, $|dH|$ has a smooth element H' of degree d.
Now $\mathbb{P}^n|_{H'} \approx \mathbb{P}^{n-1}$. If we repeat this r times, we get a copy of $\mathbb{P}^{n-r}$ which is nonsingular and irreducible and codimension r.

2.8.13 e. x g

(e) If Y is a nonsingular complete intersection as in (d), show that $\omega_Y \cong \mathcal{O}_Y(\sum d_i - n - 1)$.

Suppose $Y = H_1 \cap ... \cap H_n$.
Adjunction formula gives $\omega_{H_1} \approx \mathcal{O}_{\mathbb{P}^n}(-n-1) \otimes \mathcal{O}_{H_1}(d_1) \approx \mathcal{O}_{H_1}(d_1 - n - 1)$ where $d_i = deg\ H_i$, and $\mathcal{O}_{\mathbb{P}^n}(-n-1)$ represents the canonical of $\mathbb{P}^n$.
By thm II.8.20, we $\omega_{H_1 \cap H_2} \approx \omega_{H_1} \otimes \mathcal{O}_{H_1 \cap H_2}(H_1.H_2)$.
Note that $H_1.H_2$ corresponds to $\mathcal{O}_{H_1}(d_2)$ and thus in total
$\omega_{H_1 \cap H_2} \approx \mathcal{O}_{H_1 \cap H_2}(d_1 + d_2 - n - 1)$.
Now repeat this step by step.

2.8.14 f. x g

(f) If Y is a nonsingular hypersurface of degree d in $\mathbf{P}^n$, use (c) and (e) above to show that $p_g(Y) = \binom{d-1}{n}$. Thus $p_g(Y) = p_a(Y)$ (I, Ex. 7.2). In particular, if Y is a nonsingular plane curve of degree d, then $p_g(Y) = \frac{1}{2}(d-1)(d-2)$.

By adjunction $K_Y \approx (-n-1+d)\, H$.
Taking global sections of $0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Y \to 0$ gives, by part (c) a s.e.s. on global sections, thus $h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d-n-1)) = h^0(Y, \mathcal{I}_Y(d-n-1)) + h^0(Y, \mathcal{O}_Y(d-n-1))$.
Note that the LHS is $\binom{n+d-n-1}{n}$.
Recall $p_g = h^0(Y, \omega_Y) \approx h^0(Y, \mathcal{O}_Y(d-n-1))$.
Now $deg\ Y = d > d-n-1$ so that there are no sections vanishing on Y of degree $d-n-1$ so $h^0(Y, \mathcal{I}_Y(d-n-1)) = 0$.

2.8.15 g. x g

(g) If Y is a nonsingular curve in $\mathbf{P}^3$, which is a complete intersection of nonsingular surfaces of degrees d,e, then $p_g(Y) = \frac{1}{2}de(d + e - 4) + 1$. Again the geometric genus is the same as the arithmetic genus (I, Ex. 7.2).

Note that for a curve we have $p_a(Y) = p_g(Y)$.

We have an exact sequence

$0 \to S(\mathbb{P}^3)^{(l-d-e)} \to S(\mathbb{P}^3)^{(l-d)} \oplus S(\mathbb{P}^3)^{(l-e)} \to S(\mathbb{P}^3)^{(l)} \to S(Y)^{(l)} \to 0$

where $P \mapsto (gP, fP)$ and $(P,Q) \mapsto fP - gQ$ and the third map is obvious.

Now $\chi(l) = \binom{l+3}{3} - \binom{l+3-d}{3} - \binom{l+3-e}{3} + \binom{l+3-d-e}{3}$ and thus $\chi(0) = 1 - \binom{3-d}{3} = \binom{3-e}{3} + \binom{3-d-e}{3}$.

2.8.16 II.8.5 x g Relative Canonicals Important!

8.5. *Blowing up a Nonsingular Subvariety.* As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geqslant 2$, let $\pi : \tilde{X} \to X$ be the blowing-up of X along Y, and let $Y' = \pi^{-1}(Y)$.

(a) Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \tilde{X}$, and $\mathbf{Z} \to \operatorname{Pic} X$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \tilde{X} \cong \operatorname{Pic} X \oplus \mathbf{Z}$.

By thm II.8.24, $\tilde{X}$ is nonsingular so that $Pic\left(\tilde{X}\right) \approx cl\left(\tilde{X}\right)$.

Thm II.6.5 gives $\mathbb{Z} \to Pic\left(\tilde{X}\right) \to Pic(U) \approx Pic(X) \to 0$, $U = \tilde{X} \backslash Y'$.

(The second equality is since exceptional divisors are contracted).

By thm II.8.24, $\mathcal{O}_{\tilde{X}}(-nY) \approx \mathcal{O}_{Y'}(-n)$ so if $\mathcal{O}_{\tilde{X}}(nY')$ is trivial, then $n = 0$ so that we actually have an s.e.s. $0 \to \mathbb{Z} \to Pic\left(\tilde{X}\right) \to Pic(X) \to 0$.

This is split by π^* .

2.8.17 b. x g

(b) Show that $\omega_{\tilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$. [*Hint*: By (a) we can write in any case $\omega_{\tilde{X}} \cong f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M}$ on X, and some integer q. By restricting to $\tilde{X} - Y' \cong X - Y$, show that $\mathscr{M} \cong \omega_X$. To determine q, proceed as follows. First show that $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Then take a closed point $y \in Y$ and let Z be the fibre of Y' over y. Then show that $\omega_Z \cong \mathcal{O}_Z(-q-1)$. But since $Z \cong \mathbf{P}^{r-1}$, we have $\omega_Z \cong \mathcal{O}_Z(-r)$, so $q = r - 1$.]

By (a) we write $\omega_{\tilde{X}} \approx f^*(\mathscr{L}) \oplus \mathcal{O}_{\tilde{X}}(nY')$, $\mathscr{L}$ invertible.

If $j : U \hookrightarrow \tilde{X}$ then j^*f^* is identity on $Pic(X) \approx Pic(U)$, $U = \tilde{X} \backslash Y'$.

Then $j^*(\omega_{\tilde{X}}) \approx \omega_U \implies \mathscr{L} \approx \omega_X$.

Using adjunction $\omega_{Y'} \approx \omega_{\tilde{X}} \otimes \mathcal{O}_{Y'}(Y') \approx f^*\omega_X \otimes \mathcal{O}_{Y'}((n+1)Y')$

At a closed $y \in Y'$, let Y'_y a fiber. By exc II.8.3.b,

$\omega_{Y'_y} \approx \pi_1^*\omega_y \otimes \pi_2^*\omega_{Y'} \approx \pi^*\mathcal{O}_y \otimes \pi_2^*(f^*\omega_X \otimes \mathcal{O}_{Y'}(-n-1))$

$\approx \mathcal{O}_y \otimes \pi_2^*\mathcal{O}_{Y'}(-n-1) \approx \mathcal{O}_{Y_y}(-n-1)$.

As $Y'_y \approx \mathbb{P}^{r-1}$ then $\omega_{Y'_y} \approx \mathcal{O}_{Y'_y}(-n)$ so $n = r - 1$.

2.8.18 II.8.6 x Infinitesimal Lifting Property

8.6. *The Infinitesimal Lifting Property.* The following result is very important in studying deformations of nonsingular varieties. Let k be an algebraically closed field, let A be a finitely generated k-algebra such that Spec A is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact sequence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f: A \to B$. Then there exists a k-algebra homomorphism $g: A \to B'$ making a commutative diagram

$$\begin{tikzcd} & 0 \arrow[d] \\ & I \arrow[d] \\ & B' \arrow[d] \\ A \arrow[ur, dashed, "g"] \arrow[r, "f"] & B \arrow[d] \\ & 0 \end{tikzcd}$$

We call this result the *infinitesimal lifting property* for A. We prove this result in several steps.

(a) First suppose that $g: A \to B'$ is a given homomorphism lifting f. If $g': A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of $\operatorname{Hom}_A(\Omega_{A/k}, I)$. Note that since $I^2 = 0$, I has a natural structure of B-module and hence also of A-module. Conversely, for any $\theta \in \operatorname{Hom}_A(\Omega_{A/k}, I)$, $g' = g + \theta$ is another homomorphism lifting f. (For this step, you do not need the hypothesis about Spec A being nonsingular.)

Suppose first $g : A \to B'$ lifts f ...
As $g - g'$ lifts $f - f = 0$, then θ lies in $I \subset B'$
Further $\theta : 1 \mapsto 0$. Thus θ is 0 on k.
Then $\theta(ab) = g(ab) - g'(ab) =$
$g(a)g(b) - g'(a)g'(b)$.
Now add $(g'(a)g(b) - g'(a)g(b))$ and factor the $\theta(ab)$ into
$g(b)\theta(a) + g'(a)\theta(b)$.
Thus θ satisfies the liebniz rule.

Conversely, if $\theta \in Hom_A(\Omega_{A/k,I}, I)$ then $\theta \circ d : A \to I \hookrightarrow B'$ gives a k-linear morphism.
Note that $0 \to I \to B' \to B \to 0$ exact implies this is in the kernel of $B' \to B$.
Thus $g + \theta$ is a k-linear homomorphism lifting f.
A simple computation shows that $g(ab) + \theta(ab) = (g(a) + \theta(b))(g(b) + \theta(b))$ so that $g + \theta$ is actually a k-algebra homomorphism.

2.8.19 b. x

(b) Now let $P = k[x_1, \ldots, x_n]$ be a polynomial ring over k of which A is a quotient, and let J be the kernel. Show that there does exist a homomorphism $h: P \to B'$ making a commutative diagram,

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

and show that h induces an A-linear map $\bar{h}: J/J^2 \to I$.

A k-homomorphisms h is determined by the images of the x_i.
For each i , let b_i be a lift of $f(x_i)$ in B'.
Thus define $h : x_i \to b_i$ as a k-algebra homomorphism.
For $a \in P$, if $a \in J$, then $h(a)$ is 0 by commutativity, so that $h(a) \in I$.
Also if $a \in J^2$ then $h(a) \in I^2 = 0$ so our map descends to $\overline{h} : J/J^2 \to I$.
As h preserves multiplication $\overline{h}$ is A-linear.

2.8.20 c. x

(c) Now use the hypothesis Spec A nonsingular and (8.17) to obtain an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0.$$

Show furthermore that applying the functor $\operatorname{Hom}_A(\cdot, I)$ gives an exact sequence

$$0 \to \operatorname{Hom}_A(\Omega_{A/k}, I) \to \operatorname{Hom}_P(\Omega_{P/k}, I) \to \operatorname{Hom}_A(J/J^2, I) \to 0.$$

Let $\theta \in \operatorname{Hom}_P(\Omega_{P/k}, I)$ be an element whose image gives $\bar{h} \in \operatorname{Hom}_A(J/J^2, I)$. Consider θ as a derivation of P to B'. Then let $h' = h - \theta$, and show that h' is a homomorphism of $P \to B'$ such that $h'(J) = 0$. Thus h' induces the desired homomorphism $g: A \to B'$.

By II.8.17, II.8.3A, we have an exact sequence on global sections $0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$.
A nonsingular implies $Ext^i\left(\Omega_{A/k}, I\right) = 0$ for $i > 0$.
Taking the LES associated to the dual gives therefore a surjection $hom\left(\Omega_{P/k} \otimes A, I\right) \twoheadrightarrow hom\left(J/J^2, I\right)$.
There is therefore $\theta : \omega_{P/k} \to I$ with image $\overline{h}$ by (b).
If we define θ' as $P \to \Omega_{P/k} \to I \to B'$ this gives a k-derivation.
If $h' = h - \theta$, and $b \in J$, then $h'(b) = h(b) - \theta(b) = \overline{h(b)} - \overline{h}(b) = 0$.
Thus h' gives a morphism $g : A \to B'$ lifting f by (a).

---

also HS def theory

2.8.21 II.8.7 x

8.7. As an application of the infinitesimal lifting property, we consider the following general problem. Let X be a scheme of finite type over k, and let $\mathscr{F}$ be a coherent sheaf on X. We seek to classify schemes X' over k, which have a sheaf of ideals $\mathscr{I}$ such that $\mathscr{I}^2 = 0$ and $(X', \mathcal{O}_{X'}/\mathscr{I}) \cong (X, \mathcal{O}_X)$, and such that $\mathscr{I}$ with its resulting structure of $\mathcal{O}_X$-module is isomorphic to the given sheaf $\mathscr{F}$. Such a pair $X', \mathscr{I}$ we call an *infinitesimal extension* of the scheme X by the sheaf $\mathscr{F}$. One such extension, the *trivial* one, is obtained as follows. Take $\mathcal{O}_{X'} = \mathcal{O}_X \oplus \mathscr{F}$ as sheaves of abelian groups, and define multiplication by $(a \oplus f) \cdot (a' \oplus f') = aa' \oplus (af' + a'f)$. Then the topological space X with the sheaf of rings $\mathcal{O}_{X'}$ is an infinitesimal extension of X by $\mathscr{F}$.

The general problem of classifying extensions of X by $\mathscr{F}$ can be quite complicated. So for now, just prove the following special case: if X is affine and nonsingular, then any extension of X by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one. See (III, Ex. 4.10) for another case.

Let $A'/I \approx A$, $I \approx M$, $I^2 = 0$.

We must show that $A' \approx (A \oplus M, +)$ where $+$ is defined by $(a, m)\,(a', m') \mapsto (aa', am' + a'm)$.

The infinitesimal lifting property gives a morphism $A \to A'$ and thus we have $A \oplus M \approx A'$ by associating M with I as an A-module.

If $a \in A$ then $(a, 0)\,(a', m') = (aa', am')$ by the A-module structure on A , and since $M \approx I$.

If $m \in M \approx I$, then $(0, m)\,(a', m') = (0, a'm)$ since $mm' \in I^2$.

2.8.22 II.8.8 x

8.8. Let X be a projective nonsingular variety over k. For any $n > 0$ we define the nth *plurigenus* of X to be $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$. Thus in particular $P_1 = p_g$. Also, for any q, $0 \leqslant q \leqslant \dim X$ we define an integer $h^{q,0} = \dim_k \Gamma(X, \Omega^q_{X/k})$ where $\Omega^q_{X/k} = \bigwedge^q \Omega_{X/k}$ is the sheaf of regular q-forms on X. In particular, for $q = \dim X$, we recover the geometric genus again. The integers $h^{q,0}$ are called *Hodge numbers*.

Using the method of (8.19), show that P_n and $h^{q,0}$ are *birational* invariants of X, i.e., if X and X' are birationally equivalent nonsingular projective varieties, then $P_n(X) = P_n(X')$ and $h^{q,0}(X) = h^{q,0}(X')$.

As in II.8.19.

2.9 II.9 Formal Schemes - skip

II.9.1

9.1. Let X be a noetherian scheme, Y a closed subscheme, and $\hat{X}$ the completion of X along Y. We call the ring $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$ the ring of *formal-regular functions* on X along Y. In this exercise we show that if Y is a connected, nonsingular, positive-dimensional subvariety of $X = \mathbf{P}^n_k$ over an algebraically closed field k, then $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$.

(a) Let $\mathscr{I}$ be the ideal sheaf of Y. Use (8.13) and (8.17) to show that there is an inclusion of sheaves on Y, $\mathscr{I}/\mathscr{I}^2 \hookrightarrow \mathcal{O}_Y(-1)^{n+1}$.

MISS

(b) Show that for any $r \geqslant 1$, $\Gamma(Y, \mathscr{I}^r/\mathscr{I}^{r+1}) = 0$.

MISS

(c) Use the exact sequences

$$0 \to \mathscr{I}^r/\mathscr{I}^{r+1} \to \mathcal{O}_X/\mathscr{I}^{r+1} \to \mathcal{O}_X/\mathscr{I}^r \to 0$$

and induction on r to show that $\Gamma(Y, \mathcal{O}_X/\mathscr{I}^r) = k$ for all $r \geqslant 1$. (Use (8.21Ae).)

MISS

(d) Conclude that $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}}) = k$. (Actually, the same result holds without the hypothesis Y nonsingular, but the proof is more difficult—see Hartshorne [3, (7.3)].)

MISS

II.9.2

9.2. Use the result of (Ex. 9.1) to prove the following geometric result. Let $Y \subseteq X = \mathbf{P}_k^n$ be as above, and let $f: X \to Z$ be a morphism of k-varieties. Suppose that $f(Y)$ is a single closed point $P \in Z$. Then $f(X) = P$ also.

MISS

II.9.3

9.3. Prove the analogue of (5.6) for formal schemes, which says, if $\mathfrak{X}$ is an affine formal scheme, and if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules, and if $\mathfrak{F}'$ is coherent, then the sequence of global sections

$$0 \to \Gamma(\mathfrak{X}, \mathfrak{F}') \to \Gamma(\mathfrak{X}, \mathfrak{F}) \to \Gamma(\mathfrak{X}, \mathfrak{F}'') \to 0$$

is exact. For the proof, proceed in the following steps.

(a) Let $\mathfrak{I}$ be an ideal of definition for $\mathfrak{X}$, and for each $n > 0$ consider the exact sequence

$$0 \to \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}' \to \mathfrak{F}'' \to 0.$$

Use (5.6), slightly modified, to show that for every open affine subset $\mathfrak{U} \subseteq \mathfrak{X}$, the sequence

$$0 \to \Gamma(\mathfrak{U}, \mathfrak{F}'/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}') \to \Gamma(\mathfrak{U}, \mathfrak{F}'') \to 0$$

is exact.

MISS

(b) Now pass to the limit, using (9.1), (9.2), and (9.6). Conclude that $\mathfrak{F} \cong \varprojlim \mathfrak{F}/\mathfrak{I}^n\mathfrak{F}'$ and that the sequence of global sections above is exact.

MISS

II.9.4

9.4. Use (Ex. 9.3) to prove that if

$$0 \to \mathfrak{F}' \to \mathfrak{F} \to \mathfrak{F}'' \to 0$$

is an exact sequence of $\mathcal{O}_{\mathfrak{X}}$-modules on a noetherian formal scheme $\mathfrak{X}$, and if $\mathfrak{F}', \mathfrak{F}''$ are coherent, then $\mathfrak{F}$ is coherent also.

MISS

II.9.5

9.5. If $\mathfrak{F}$ is a coherent sheaf on a noetherian formal scheme $\mathfrak{X}$, which can be generated by global sections, show in fact that it can be generated by a finite number of its global sections.

MISS

II.9.6

9.6. Let $\mathfrak{X}$ be a noetherian formal scheme, let $\mathfrak{I}$ be an ideal of definition, and for each n, let Y_n be the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{I}^n)$. Assume that the inverse system of groups $(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies the Mittag–Leffler condition. Then prove that $\operatorname{Pic} \mathfrak{X} = \varprojlim \operatorname{Pic} Y_n$. As in the case of a scheme, we define $\operatorname{Pic} \mathfrak{X}$ to be the group of locally free $\mathcal{O}_{\mathfrak{X}}$-modules of rank 1 under the operation $\otimes$. Proceed in the following steps.

(a) Use the fact that $\ker(\Gamma(Y_{n+1}, \mathcal{O}_{Y_{n+1}}) \to \Gamma(Y_n, \mathcal{O}_{Y_n}))$ is a nilpotent ideal to show that the inverse system $(\Gamma(Y_n, \mathcal{O}^*_{Y_n}))$ of units in the respective rings also satisfies (ML).

MISS

(b) Let $\mathfrak{F}$ be a coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-modules, and assume that for each n, there is some isomorphism $\varphi_n \colon \mathfrak{F}/\mathfrak{I}^n\mathfrak{F} \cong \mathcal{O}_{Y_n}$. Then show that there is an isomorphism $\mathfrak{F} \cong \mathcal{O}_{\mathfrak{X}}$. Be careful, because the φ_n may not be compatible with the maps in the two inverse systems $(\mathfrak{F}/\mathfrak{I}^n\mathfrak{F})$ and $(\mathcal{O}_{Y_n})$! Conclude that the natural map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is injective.

MISS

(c) Given an invertible sheaf $\mathscr{L}_n$ on Y_n for each n, and given isomorphisms $\mathscr{L}_{n+1} \otimes \mathcal{O}_{Y_n} \cong \mathscr{L}_n$, construct maps $\mathscr{L}_{n'} \to \mathscr{L}_n$ for each $n' \geqslant n$ so as to make an inverse system, and show that $\mathfrak{L} = \varprojlim \mathscr{L}_n$ is a coherent sheaf on $\mathfrak{X}$. Then show that $\mathfrak{L}$ is locally free of rank 1, and thus conclude that the map $\operatorname{Pic} \mathfrak{X} \to \varprojlim \operatorname{Pic} Y_n$ is surjective. Again be careful, because even though each $\mathscr{L}_n$ is locally free of rank 1, the open sets needed to make them free might get smaller and smaller with n.

MISS

(d) Show that the hypothesis "$(\Gamma(Y_n, \mathcal{O}_{Y_n}))$ satisfies (ML)" is satisfied if either $\mathfrak{X}$ is affine, or each Y_n is projective over a field k.

Note: See (III, Ex. 11.5–11.7) for further examples and applications.

MISS

3 III Cohomology

3.1 III.1

no questions.

3.2 III.2 x

3.2.1 III.2.1.a x g

2.1. (a) ~~Let $X = \mathbf{A}_k^1$ be the affine line over an infinite field k. Let P,Q~~ be distinct closed points of X, and let $U = X - \{P,Q\}$. Show that $H^1(X, \mathbf{Z}_U) \neq 0$.

We have an s.e.s. $0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_P\mathbb{Z} \oplus i_Q\mathbb{Z} \to 0$.
(see Ex.2.1.17)
LES is $0 \to \Gamma(X, \mathbb{Z}) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \cdots$.
Since $\mathbb{Z}$ cannot surject to $\mathbb{Z}^2$, $H^1 \neq 0$.

*(b) More generally, let $Y \subseteq X = \mathbf{A}_k^n$ be the union of $n+1$ hyperplanes in suitably general position, and let $U = X - Y$. Show that $H^n(X, \mathbf{Z}_U) \neq 0$. Thus the result of (2.7) is the best possible.

MISS

3.2.2 III.2.2 x flasque resolution g

2.2. Let $X = \mathbf{P}_k^1$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathcal{K} \to \mathcal{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

By II.21.d, $\mathscr{K}/\mathscr{O}$ is $\sum i_P(I_P)$, the sum of skyscraper sheaves.
The constant sheaf is flasque as $\mathbb{P}^1$ is connected. (use II.1.16)
Applying the LES in cohomology gives the desired vanishing in cohomology.

3.2.3 III.2.3 x Cohomology with Supports

2.3. *Cohomology with Supports* (Grothendieck [7]). Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support in Y (II, Ex. 1.20).

(a) Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathfrak{Ab}(X)$ to $\mathfrak{Ab}$.

We denote the right derived functors of $\Gamma_Y(X, \cdot)$ by $H_Y^i(X, \cdot)$. They are the *cohomology groups of X with supports* in Y, and coefficients in a given sheaf.

Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ an s.e.s.

We have $\Gamma_Y(X, \mathscr{F}') \subset \Gamma_Y(X, \mathscr{F})$ and we need to show exactness on the right. Let s be in the kernel of the second map. This s gives rise to an element in the kernel of $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$. By left exactness of Γ , there exists s' mapping to s. We want to show $s'_x = 0$ for $x \in X \backslash Y$. This follows by checking the stalks.

3.2.4 b. x Flasque Global sections are exact

(b) If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, with $\mathscr{F}'$ flasque, show that

$$0 \to \Gamma_Y(X,\mathscr{F}') \to \Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(X,\mathscr{F}'') \to 0$$

is exact.

We know from (a) it's left exact.
So we need to show the RHS is surjective.
For $s \in \Gamma_Y(X, \mathscr{F}'')$, $s \in \Gamma(X, \mathscr{F}'')$.
We have exactness on open sets by II.1.16.b.
Thus choose $t \in \Gamma(X, \mathscr{F})$ in preimage of s.
At $x \in U = X \backslash Y$, then $t_x \mapsto s_x = 0$, so by exactness of stalks, we have $a_x \in \mathscr{F}'_x$ mapping to t_x.
Now find a small neighborhood U_a and a section a mapping to $t|_{U_a}$.
Find an open cover of such U_a.
note that the u, t restricted to such an open cover agree
Using sheaf axioms and flasqueness we find a global a mapping to t on U.
$t - a \mapsto s$, and on U, $t - a$ is zero so $t - a$ is supported on Y.

3.2.5 III.2.2.c x

(c) Show that if $\mathscr{F}$ is flasque, then $H^i_Y(X,\mathscr{F}) = 0$ for all $i > 0$.

See the proof of Prop. III.2.5, Flasque Vanishing Theorem

3.2.6 x

(d) If $\mathscr{F}$ is flasque, show that the sequence

$$0 \to \Gamma_Y(X,\mathscr{F}) \to \Gamma(X,\mathscr{F}) \to \Gamma(X - Y,\mathscr{F}) \to 0$$

is exact.

easy
Apply the global sections functor to the sequence of II.1.20.b.

3.2.7 x

(e) Let $U = X - Y$. Show that for any $\mathscr{F}$, there is a long exact sequence of cohomology groups

$$\begin{aligned} 0 &\to H^0_Y(X,\mathscr{F}) \to H^0(X,\mathscr{F}) \to H^0(U,\mathscr{F}|_U) \to \\ &\to H^1_Y(X,\mathscr{F}) \to H^1(X,\mathscr{F}) \to H^1(U,\mathscr{F}|_U) \to \\ &\to H^2_Y(X,\mathscr{F}) \to \dots . \end{aligned}$$

Let $\mathscr{F}^\bullet$ be an injective resolution of $\mathscr{F}$.
Now apply the sequence of part (d) to $\mathscr{F}$ to get the long exact sequence.

3.2.8 excision x

(f) *Excision.* Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathscr{F}$,

$$H^i_Y(X,\mathscr{F}) \cong H^i_Y(V,\mathscr{F}|_V).$$

There is an isomorphism $\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(V,\mathscr{F}|_V)$ (to see this it may be helpful to consider the espace etale of $\mathscr{F}$) where V is an open subset containing Y.

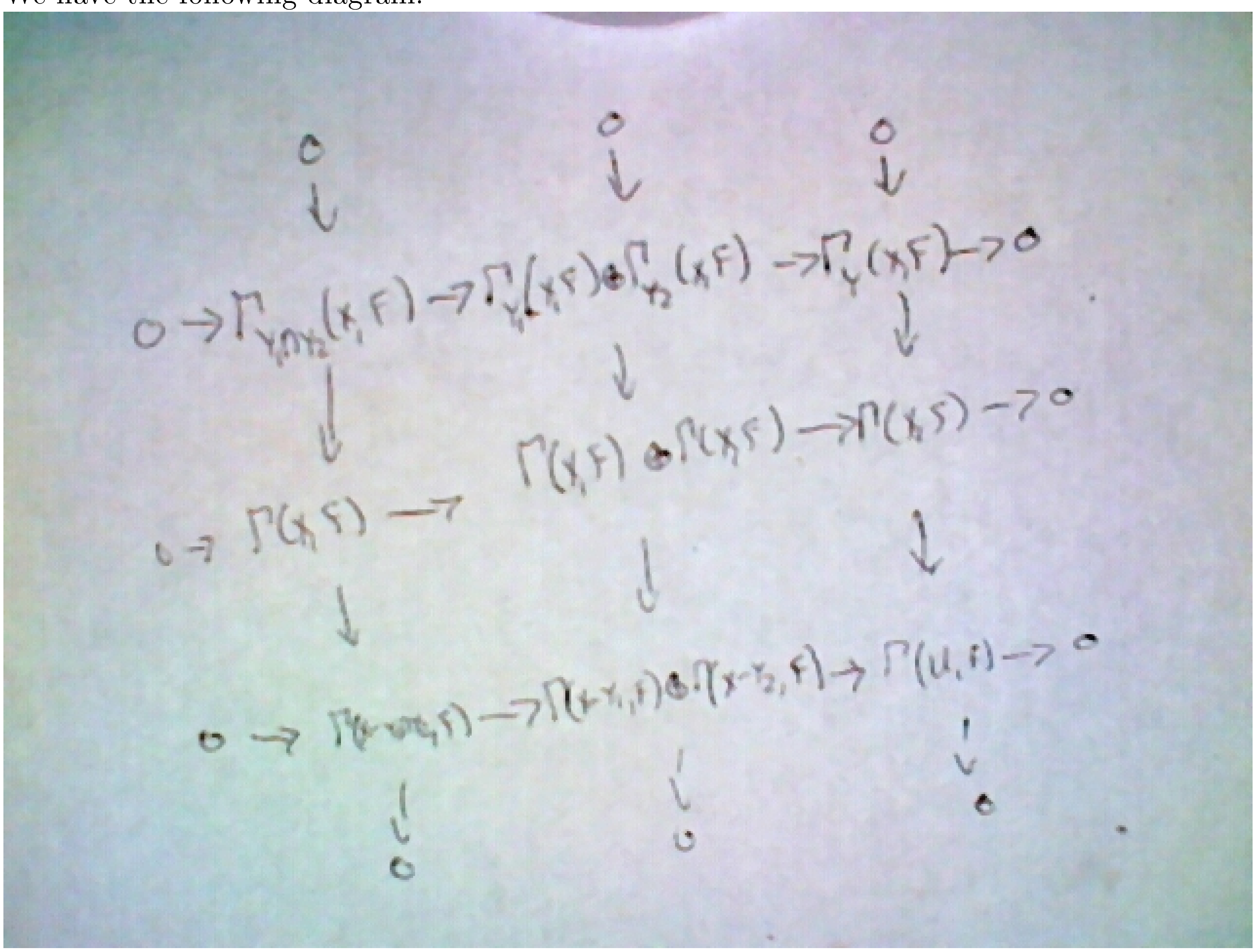
Now if I^i is an injective resolution for $\mathscr{F}$, then $I^i|_V$ is an injective resolution for $\mathscr{F}|_V$, so the stated isomorphism gives an isomorphism in cohomology.

3.2.9 II.2.4 x Mayer-Vietoris

2.4. *Mayer Vietoris Sequence.* Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports

$$\ldots \to H^i_{Y_1\cap Y_2}(X,\mathscr{F}) \to H^i_{Y_1}(X,\mathscr{F}) \oplus H^i_{Y_2}(X,\mathscr{F}) \to H^i_{Y_1\cup Y_2}(X,\mathscr{F}) \to$$
$$\to H^{i+1}_{Y_1\cap Y_2}(X,\mathscr{F}) \to \ldots.$$

We have the following diagram:



The columns are exact by flasqueness and III.2.7.d. The middle row is obviously exact, and the bottom row is exact by sheaf axioms. Using the 9 lemma or a spectral sequence gives the top row exact. Now take the LES of the top row.

3.2.10 III.2.5 x

2.5. Let X be a Zariski space (II, Ex. 3.17). Let $P \in X$ be a closed point, and let X_P be the subset of X consisting of all points $Q \in X$ such that $P \in \{Q\}^-$. We call X_P the *local space* of X at P, and give it the induced topology. Let $j: X_P \to X$ be the inclusion, and for any sheaf $\mathscr{F}$ on X, let $\mathscr{F}_P = j^*\mathscr{F}$. Show that for all i, $\mathscr{F}$, we have

$$H^i_P(X,\mathscr{F}) = H^i_P(X_P,\mathscr{F}_P).$$

Since H^i results from taking cohomology of the global sections functor, we claim that $\Gamma_P(X,\mathscr{F}) \approx \Gamma_P(X_P,\mathscr{P}_P)$.

The morphism $\Gamma(X,\mathscr{F}) \to \Gamma(X_P,\mathscr{F}_P) = \underset{\leftarrow\, p\in U}{lim} \mathscr{F}(U) = \mathscr{F}_P$ induces a morphism on the localization $f : \Gamma_P(X,\mathscr{F}) \to \Gamma_P(X_P,\mathscr{F}_P)$.

We just need this to be a bijection.

Suppose that $f(t) = f(s)$ in $\Gamma_P(X_P,\mathscr{F}_P)$. Thus $s_P = t_P$ since f sends elements to their germs. Since they agree on every stalk, then $s = t$. Thus f is injective.

Now suppose $s \in \Gamma_P(X_P,\mathscr{F}_P) = \mathscr{F}_P$.

Thus we can find a neighborhood $U \ni P$ and $s_U \in \mathscr{F}(U)$, by definition of compatible germs, such that s_U represents s.

If necessary shrink U so that $(s_U)_Q = 0$ for $Q \neq P$.

Then if $V = X \backslash P$,$s_U|_{U\cap V} = 0$ so that s_U and 0 glue to give a global section with support in P. Thus f is surjective.

3.2.11 III.2.6 x

2.6. Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha\in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective. [*Hints*: First show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and for every subsheaf $\mathscr{R} \subseteq \mathbf{Z}_U$, and for every map $f: \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map of $\mathbf{Z}_U \to \mathscr{I}$. Secondly, show that any such sheaf $\mathscr{R}$ is finitely generated, so any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ factors through one of the $\mathscr{I}_\alpha$.]

Note that a sheaf $\mathscr{I}$ is injective iff for each open set $U \subset X$ and for every subsheaf $\mathscr{R} \subset \mathbb{Z}_U$ and for every map $f : \mathscr{R} \to \mathscr{I}$, there exists an extension of f to a map $\mathbb{Z}_U \to \mathscr{I}$. This is essentially the definition of injective in any algebra book.

Now suppose $\mathscr{R} \subset \mathbb{Z}_U$. If $U = \coprod U_i$ is a decomposition of U into connected components, then by noetherianity of X, $\bigcup U_i = U_n$.

For each i, $\mathscr{R}(U_i) \subset \mathbb{Z}_U(U_i) = \mathbb{Z}$ are subgroups generated by s_i so that finitely many s_i generate $\mathscr{R}$.

For a map $f : \mathscr{R} \to \underset{\rightarrow}{lim}\mathscr{I}_a$, then $f(s_i) = t_i \in \mathscr{I}_{a_i}(U_i)$.

This is a direct system, so the morphism factors as $\mathscr{R} \to \mathscr{I}_b \to \underset{\rightarrow}{lim}\mathscr{I}_a$ for some b.

If $U \subset X$ and $\mathscr{R} \subset \mathbb{Z}_U$, then f factors through $f_b : \mathscr{R} \to \mathscr{I}_b$.

By the hint, $\mathscr{I}_b$ is injective gives f_b extends to $\mathbb{Z}_U \to \mathscr{I}_b$ and thus we give an extension $\mathbb{Z}_U \to \mathscr{I}_b \to \underset{\rightarrow}{lim}\mathscr{I}_a$ of f. So $\underset{\rightarrow}{lim}\ \mathscr{I}_a$ is injective.

3.2.12 x III.2.7a g Cohomology of circle

2.7. Let S^1 be the circle (with its usual topology), and let $\mathbf{Z}$ be the constant sheaf $\mathbf{Z}$.
(a) Show that $H^1(S^1,\mathbf{Z}) \simeq \mathbf{Z}$, using our definition of cohomology.

It's a lot easier to just use hurewicz theorem since $\pi_1(S^1)$ is $\mathbb{Z}$.

3.2.13 b. x

(b) Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathscr{R}) = 0$.

If $\mathscr{D}$ is the sheaf of all real-valued functions, then we get a LES.

$0 \to H^0(S^1, \mathscr{R}) \to H^0(S^1, \mathscr{D}) \xrightarrow{a} H^0(S^1, \mathscr{D}/\mathscr{R}) \to H^1(S^1, \mathscr{R}) \to 0$ as $H^1(S^1, \mathscr{D}) = 0$ since $\mathscr{D}$ is flasque.

Let $s = \{(U_i, s_i)\}_{i=1}^n \in H^0(S^1, \mathscr{D}/\mathscr{R})$ since S^1 is compact, and write $r_i = s_{i+1} - s_i$ extend by zero so r_i is defined on U_i. Wlog by shrinking assume $(U_i \cap U_{i+1}) \cap (U_{i+1} \cap U_{i+2}) = \emptyset$.

Thus if $r = \{(U_i, r_i)\}$ then on $U_i \cap U_j$, $r_i - r_{i+1} = r_i$.

Thus $r \in H^0(S^1, \mathscr{D}/\mathscr{R})$.

Now let $t_i = s_i + r_i$, $t = \{(U_i, t_i)\}$ so $t : S^1 \to \mathbb{R}$, and $t \in H^0(S^1, \mathscr{D})$ since $t_i|_{U_i \cap U_{i+1}} = s_i + s_{i+1} - s_i = s_{i+1} = t_{i+1}|_{U_i \cap U_{i+1}}$. Also t is mapped to itself in $H^0(S^1, \mathscr{D}/\mathscr{R})$ so that t is in the image of α .

If $r' \in H^0(S^1, \mathscr{D})$ satisfies $r'|_{U_i \cap U_{i+1}} = r_i$ on $U_i \cap U_{i+1}$ and 0 elsewhere, then $r' \overset{a}{\mapsto} r$ so r is in the image of a. Thus $s = t - r$ is in the image of a. So a is surjective.

3.3 III.3 x Cohomology of a Noetherian Affine Scheme

3.3.1 III.3.1 x

3.1. Let X be a noetherian scheme. Show that X is affine if and only if X_{red} (II. Ex. 2.3) is affine. [*Hint*: Use (3.7), and for any coherent sheaf $\mathscr{F}$ on X, consider the filtration $\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \ldots$, where $\mathscr{N}$ is the sheaf of nilpotent elements on X.]

Suppose X is affine. Then $X_{red} = Spec(A/N)$, N is the nilradical of $A = \Gamma(X, \mathcal{O}_X)$. So $X_{red} = Spec(A/N)$.

Now suppose X_{red} is affine. If X is dimension 0, then each point is in an affine neighborhood so X is affine.

Now let $\mathscr{N}$ be the sheaf of nilpotents on X, by noetherianity, $\mathscr{N}^d = 0$ for $d \geq m$.

Consider the $\mathcal{O}_{X_{red}} - module$ $\mathscr{G}_d = \mathscr{N}^d \cdot \mathscr{F} / \mathscr{N}^{d+1} \cdot \mathscr{F}$. By theorem III.3.7, $H^1(X, \mathscr{G}_d) = H^1(X_{red}, \mathscr{G}_d) = 0$ so we have a surjection

$0 = H^1\left(X, \mathscr{N}^{d+1}\mathscr{F}\right) \twoheadrightarrow H^1\left(X, \mathscr{N}^d \mathscr{F}\right)$.

By induction, $H^1\left(X, \mathscr{N}^k \mathscr{F}\right) = 0$ for $k < d$.

Thus $H^1(X, \mathscr{G}_d) = 0$ so $\mathscr{F}$ is affine.

By thm 3.7, this is equivalent to X affine.

3.3.2 III.3.2 x

3.2. Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.

If X is affine, then by exc II.3.11.b, every irreducible component is affine.

If every irreducible component is affine, and Y_1, Y_2 is an arbitrary closed subscheme, Y_2 an irreducible component of X, then consider $0 \to \mathscr{I}_{Y_1 \cup Y_2} \to \mathscr{I}_{Y_1} \to i_* \mathscr{I}_{Y_1 \cap Y_2} \to 0$, $i : Y_2 \hookrightarrow X$. We have by thm III.3.7, $H^1(X, i_* \mathscr{I}_{Y_1 \cap Y_2}) = H^1(Y_2, \mathscr{I}_{Y_1 \cap Y_2}) = 0$ so there is a surjection $H^1(X, \mathscr{I}_{Y_1 \cup Y_2}) \twoheadrightarrow H^1(X, \mathscr{I}_{Y_1})$. Continuing in this manner step-by-step, we achieve a surjection $H^1\left(X, \mathscr{I}_{Y_1 \cup Y_2 \cup \ldots \cup Y_n}\right) \twoheadrightarrow H^1\left(X, \mathscr{I}_{Y_1}\right)$. Eventually, $Y_1 \cup \ldots \cup Y_n = X$ so that $0 = H^1\left(X, \mathscr{I}_X\right) \twoheadrightarrow H^1\left(X, \mathscr{I}_{Y_1}\right)$ and thus by thm III.3.7, X is affine.

3.3.3 III.3.3 $\Gamma_{\mathfrak{a}}$ is left exact. x

3.3. Let A be a noetherian ring, and let $\mathfrak{a}$ be an ideal of A.
(a) Show that $\Gamma_{\mathfrak{a}}(\cdot)$ (II, Ex. 5.6) is a left-exact functor from the category of A-modules to itself. We denote its right derived functors, calculated in $\mathfrak{Mod}(A)$, by $H^i_{\mathfrak{a}}(\cdot)$.

Consider $0 \to N \to E \to M \to 0$.
We have $0 \to \Gamma_{\mathfrak{a}}(N) \to \Gamma_{\mathfrak{a}}(E)$ as $\Gamma_{\mathfrak{a}}(P) \subset P$ for A-mdoules P.
If $e \in ker\left(\Gamma_{\mathfrak{a}}(E) \to \Gamma_{\mathfrak{a}}(M)\right)$, then $\mathfrak{a}^t e = 0$ some t. (defintion of $\Gamma_{\mathfrak{a}}$).
Since global sections are left exact, then there is $n \mapsto e$
Then $\mathfrak{a}^t n \mapsto 0$.So $n \in \Gamma_{\mathfrak{a}}(N)$.
Thus we have shown that the kernel of the second map is contained in the image of the first map.
The opposite inclusion is clear.

3.3.4 b. x

(b) Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \tilde{M}),$$

where $H^i_Y(X, \cdot)$ denotes cohomology with supports in Y (Ex. 2.3).

We show that $\Gamma_a(\cdot) = \Gamma_Y(X, \tilde{\cdot})$.
Let M arbitrary. If $m \in \Gamma_a(M)$ then $a^n m = 0$ for some n.
If $\mathfrak{p} \in X$ is not in Y, then $\mathfrak{p} \not\supset \mathfrak{a}$ so there is $a \in \mathfrak{a}$ not in $\mathfrak{p}$.
Then $a^n m \notin \mathfrak{p}$ so $a^n m = 0$ and thus $m = 0$ in $M_{\mathfrak{p}}$.
Thus $m \in \Gamma_Y(X, M^{\sim})$.
Next suppose $m \in \Gamma_Y\left(X, \tilde{M}\right)$.
Thus $Supp\ m = V(Ann\ m) \subset V(\mathfrak{a})$ and thus $\sqrt{Ann\ m} \supset \mathfrak{a}$ by thm II.2.1 .
A is noetherian, so $\mathfrak{a} = (f_1, ..., f_n)$ and $f_i^{n_i} \in \sqrt{Ann\ m}$ and thus $f_i^{n_i j_i} \in Ann\ m$.
If $N = \prod n_i j_i$ then $f_i^N \in Ann\ m$ for all i.
Choosing a large enough N' , then $\mathfrak{a}^{N'} \subset Ann\ m$ and thus $m \in \Gamma_{\mathfrak{a}}(M)$.

3.3.5 x

(c) For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.

$\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) \subset H^i_{\mathfrak{a}}(M)$ by definition.
Let $x \in H^i_{\mathfrak{a}}(M)$.
Take Γ_a of an injective resolution I_i for M to see that H^i_a is a quotient of $\Gamma_a(I_i)$.
Thus $H^i_a(M) \subset \Gamma_a(H^i_a(M))$.

3.3.6 III.3.4 x Cohomological Interpretation of Depth

3.4. *Cohomological Interpretation of Depth.* If A is a ring, $\mathfrak{a}$ an ideal, and M an A-module, then $depth_{\mathfrak{a}} M$ is the maximum length of an M-regular sequence $x_1, \ldots, x_r$, with all $x_i \in \mathfrak{a}$. This generalizes the notion of depth introduced in (II, §8).

(a) Assume that A is noetherian. Show that if $\operatorname{depth}_{\mathfrak{a}} M \geqslant 1$, then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated. [*Hint*: When M is finitely generated, both conditions are equivalent to saying that $\mathfrak{a}$ is not contained in any associated prime of M.]

Suppose $depth_{\mathfrak{a}} M \geq 1$ then there is $x \in a$ such that x is not a zero-divisor for M. Then neither is x^n for any n. Thus $\mathfrak{a}^n$ can not annihilate any element so $\Gamma_{\mathfrak{a}}(M) = 0$.

Now suppose $\Gamma_{\mathfrak{a}}(M) = 0$ for M finitely generated. For $m \in M$ and $n \geq 0$ then there is an $x \in \mathfrak{a}^n$ with $xm \neq 0$.

Then $\mathfrak{a}$ is not contained in any associated prime so by prime avoidance, $\mathfrak{a}$ is not in the union of associated primes of $\mathfrak{p}$ which is the set of zero divisors of M. Thus $depth_{\mathfrak{a}} M \geq 1$.

3.3.7 b. x

(b) Show inductively, for M finitely generated, that for any $n \geqslant 0$, the following conditions are equivalent:

(i) $\operatorname{depth}_{\mathfrak{a}} M \geqslant n$;
(ii) $H^i_{\mathfrak{a}}(M) = 0$ for all $i < n$.

For more details, and related results, see Grothendieck [7].

Suppose the statement is true for n.
Choose M with $depth_{\mathfrak{a}} M \geq n+1$.
If $x_1, ..., x_{n+1} \in \mathfrak{a}$ is an M-regular sequence then we have
$\cdots \to H^{n-1}_{\mathfrak{a}}(M/x_1M) \to H^n_{\mathfrak{a}}(M) \overset{\times x_1}{\to} H^n_{\mathfrak{a}}(M) \to \cdots$.
By induction, $H^{n-1}_{\mathfrak{a}}(M/x_1M) = 0$.
Thus the induced map $\times x_1$ should be injective, thus by exc III.3.3.c, $H^n_{\mathfrak{a}}(M) = 0$.

On the other hand if $H^i_{\mathfrak{a}}(M) = 0$ for $i < n+1$, then the LES we have $H^i_{\mathfrak{a}}(M/x_1M) = 0$ for $i < n$.
Now by induction, $depth_{\mathfrak{a}} M/x_1M \geq n-1 \implies depth_{\mathfrak{a}} M \geq n$.

3.3.8 III.3.5 x

3.5. Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:

(i) $\operatorname{depth} \mathcal{O}_P \geqslant 2$;
(ii) if U is any open neighborhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.

This generalizes (I, Ex. 3.20), in view of (II, 8.22A).

For $U \ni P$, every section of $\mathcal{O}_X$ over $U-P$ extends to a section of $\mathcal{O}_X$ over U iff $\Gamma(U, \mathcal{O}_X) \approx \Gamma(U-P, \mathcal{O}_X)$ which, computing cohomology, by exc III.2.3(e) is the same as $H^0_P(U, \mathcal{O}_X|_U) \approx H^1_P(U, \mathcal{O}_X|_U) = 0$. By exc

III.2.5 this is equivalent to $H^0_P(Spec\ \mathcal{O}_P, \mathcal{O}_{Spec\ \mathcal{O}_P}) \approx H^1_P(Spec\ \mathcal{O}_P, \mathcal{O}_{Spec\ \mathcal{O}_P}) = 0$. Now use exc III.3.3.b, and exc III.3.4 to see this is equivalent to $depth_{\mathfrak{m}}\mathcal{O}_P \geq 2$.

3.3.9 III.3.6 x

3.6. Let X be a noetherian scheme.
(a) Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.

$\mathscr{G}$ is constructed by covering X with open affines $U_i = Spec\ A_i$ for $i = 1, n$ and let $\mathscr{F}|_{U_i} \approx \tilde{M}_i$. If M_i is embedded in an injective A_i-module I_i , then for each i, let $f_i : U_i \hookrightarrow X$ and define $\mathscr{G} = \oplus f_{i*}\left(\tilde{I}_i\right)$.

To prove injective, given $0 \to \mathscr{F}' \to \mathscr{F}$ and $\mathscr{F}' \to \mathscr{G}$ we want to lift this to a morphism $\mathscr{F} \to \mathscr{G}$.

Note that by injectivity of I_i, any $f : \mathscr{F}' \to f_{i*}\tilde{I}_i$ gives $\mathscr{F}'|_{U_i} \to \tilde{I}_i$ which lifts to $\overline{f} : \mathscr{F}|_{U_i} \to \tilde{I}_i$. Thus $f_{i*}\tilde{I}_i$ is injective now the lifts commute with direct sums.

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*(b) Show that any injective object of $\mathfrak{Qco}(X)$ is flasque. [*Hints*: The method of proof of (2.4) will *not* work, because $\mathscr{C}_U$ is not quasi-coherent on X in general. Instead, use (II, Ex. 5.15) to show that if $\mathscr{I} \in \mathfrak{Qco}(X)$ is injective, and if $U \subseteq X$ is an open subset, then $\mathscr{I}|_U$ is an injective object of $\mathfrak{Qco}(U)$. Then cover X with open affines . . .]

starred

3.3.10 part c. x

(c) Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.

Using part (b), we see injective resolutions are flasque.

3.3.11 III.3.7 x

3.7. Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.
(a) For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \tilde{M}) \cong \varinjlim_n \operatorname{Hom}_A(\mathfrak{a}^n, M).$$

Note that A is noetherian, so $\mathfrak{a} = (f_1, ..., f_n)$ is f.g.

Also U is covered by $D(f_i)$ so $s \in \Gamma\left(U, \tilde{M}\right)$ is $\sum \frac{m_i}{f_i} a_i \in \oplus M_{f_i}$.

On the other hand if $\sum \frac{s_i}{f_i} a_i \in \oplus M_{f_i}$ and is sent to 0 by localization at f_j , then it's actually in $\Gamma\left(U, \tilde{M}\right)$.

If $\phi : \mathfrak{a}^r \to M$ define $f(\phi)$ by $\left(\frac{\phi(f_1^r)}{f_1^r}, ..., \frac{\phi(f_n^r)}{f_n^r}\right)$. This defines a section, and it's well-defined by calculation.

Thus we have an induced $g : \underset{\to\ n}{lim}\ hom_A(\mathfrak{a}^r, M)$

Note that if $g(\phi = 0)$,then $\frac{\phi(f_i^r)}{f_i^r} = 0 \in M_{f_i}$ so $f_i^{s_i}\phi(f_i^r) = 0 \in M$ some s_i . Choose one s to work for all of them. Thus if $\mathfrak{a}^{n(s+r)+1}$ is generated by f_i^{s+r} so that $\phi\left(f_i^{s+r}\right) = 0$. So g is injective.

Now suppose $f \in \Gamma\left(U, \tilde{M}\right)$, f defines $\left(\frac{m_1}{f_1^{r_1}}, ..., \frac{m_n}{f^{r_n}}\right)$.
Since $f \in \Gamma\left(U, \tilde{M}\right)$, then $(f_i f_j)^{s_{ij}}(f_i m_j - f_j m_i) = 0 \in M$.
If $s > s_{ij}$, then define $m_i' = f_i^s m_i$ and if $r > r_i$ then $\left(f_i^{r+s} f - m_i'\right)|_{D(f_j)} = 0$ so $f_i^{r+s} f = m_i'$ on U.
For $R > n(r+s)$, $\mathfrak{a}^R$ is generated by f_i^{r+s}.
Define $\phi : \mathfrak{a}^R \to M$ sending $\sum a_i f_i^{r+s}$ to $(\sum a_i m_i')|_U$.
Check that this is well-defined and the image of ϕ is $\left(\frac{m_1'}{f_1^{r+s}}, ..., \frac{m_n'}{f_n^{r+s}}\right) = f$. This gives surjectivity.

3.3.12 b. x

(b) Apply this in the case of an injective A-module I, to give another proof of (3.4).

Let $U \supset V$ with $U = X - V(\mathfrak{a})$ and $V = X - V(\mathfrak{b})$.
As in (a), assume $\mathfrak{a}, \mathfrak{b}$ are radical.
Thus $V(\mathfrak{a}) \subset V(\mathfrak{b}) \implies \mathfrak{b} \subset \mathfrak{a}$.
Thus $\mathfrak{b}^n \subset \mathfrak{a}^n$ so that $Hom_A(\mathfrak{a}^n, I) \to Hom_A(\mathfrak{b}^n, I)$ is surjective.
Thus $\underset{\rightarrow}{lim} Hom_A(\mathfrak{a}^n, I) \twoheadrightarrow \underset{\rightarrow}{lim}\ Hom_A(\mathfrak{b}^n, I)$ so by (a), $I^\sim$ is flasque.

3.3.13 III.3.8 x Localization not injective non noetherian.

3.8. Without the noetherian hypothesis, (3.3) and (3.4) are false. Let $A = k[x_0, x_1, x_2, \ldots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \ldots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.

If $I \to I_{x_0}$ is surjective, find $m \in I$ with $x_0^n(x_0 m - 1) = 0$.
Thus $x_0^{n+1} m = x_0^n$.
Multiplying both sides by x_{n+1} shows that $x_0^n x_{n+1} = 0$.
(use the given relation).
However this ring doesn't have this relation.

3.4 III.4 x Cech Cohomology

3.4.1 III.4.1 x g pushforward cohomology affine morphism

4.1. Let $f: X \to Y$ be an affine morphism of noetherian separated schemes (II, Ex. 5.17). Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geqslant 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_* \mathscr{F}).$$

[*Hint*: Use (II, 5.8).]

Consider $\{V_i\}$ an affine cover of Y.
Since f is affine, $f^{-1}(V_i)$ are affine cover of X.
By separatedness and an excercise in (II.3?) the intersections of the preimages are affine.
Since $\mathscr{F}$ is q.c., on the intersections of the preimages, $\mathscr{F}|_{f^{-1}(V_i)\cap ...} \approx \tilde{M}$.
Since f is affine, the pushforwards of such are also of $\tilde{M}'$'s (see ex II.5.17?)
Now taking cech complexes, and applying III.4.5, the result follows.

3.4.2 III.4.2 x

4.2. Prove Chevalley's theorem: Let $f:X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.

(a) Let $f:X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha:\mathcal{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomorphism at the generic point of Y.

Apply $\mathscr{H}om\,(;,\mathscr{F})$ to α gives a morphism $\mathscr{H}om\,(f_*\mathscr{M},\mathscr{F}) \to \mathscr{H}om\,(\mathcal{O}_Y^r,\mathscr{F})$ which is an isomorphism at the generic point.

We have an isomorphism $\mathscr{H}om\,(\mathcal{O}_Y^r,\mathscr{F}) \approx \mathscr{F}$.

By exc II.5.17, since $\mathscr{H}om\,(f_*\mathscr{M},\mathscr{F})$ is q.c., there is a q.c. $\mathscr{G}$ with $\mathscr{H}om\,(f_*\mathscr{M},\mathscr{F}) \approx f_*\mathscr{G}$.

3.4.3 b. x

(b) For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta: f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y. [*Hint*: Apply $\mathscr{H}om(\cdot,\mathscr{F})$ to α and use (II, Ex. 5.17e).]

(following http://mathramble.wordpress.com/2013/03/14/chevalleys-theorem/)

Let L the function field of X and K the function field of Y. The morphism f gives an inclusion $K \hookrightarrow L$. f finite implies there is a basis $\{e_1,...,e_r\}$ for L over K, where e_j is represented by a $s_j \in \Gamma(U,\mathcal{O}_X)$. If $\mathscr{E}_j$ is the coherent sheaf $s_j \cdot \mathcal{O}_{U_j}$, and $\tau_j : U_j \hookrightarrow X$, then $(\tau_j)_*(\mathscr{E}_j)$ is quasi-coherent since U_j are noetherian, and since f is finite this sheaf is coherent.

For $\mathscr{M} = \oplus(\tau_j)_*(\mathscr{E}_j)$, the generators e_j of $f_*\mathscr{M}$ give a morphism $\alpha : \mathcal{O}_Y^r \to f_*\mathscr{M}$ which is an isomorphism $K^r \approx L$ at the generic point of Y.

Now take $\beta = \mathscr{H}om\,(\alpha,\mathscr{F})$ so that $\beta : \mathscr{H}om\,(f_*\mathscr{M},\mathscr{F}) \to \mathscr{H}om\,(\mathcal{O}_Y^r,\mathscr{F})$. Note that $\mathscr{H}om\,(\mathcal{O}_Y^r,\mathscr{F}) \approx \mathscr{F}^r$ and $\mathscr{H}om\,(f_*\mathscr{M},\mathscr{F})$ is an $f_*\mathcal{O}_X$-module. By exc II.5.17, for f affine, $f_* : \mathfrak{Coh}\,(\mathcal{O}_Y) \approx \mathfrak{Qcoh}\,(f_*\mathcal{O}_X)$ so that $\mathscr{H}om\,(f_*\mathscr{M},\mathscr{F}) \approx f_*\mathscr{G}$, $\mathscr{G}$ a coherent $\mathcal{O}_X$ -module. Thus $\beta : f_*\mathscr{G} \to \mathscr{F}^r$. As taking $\mathscr{H}om$ commutes with taking stalks, then β is an isomorphism at the generic point.

3.4.4 c. x

(c) Now prove Chevalley's theorem. First use (Ex. 3.1) and (Ex. 3.2) to reduce to the case X and Y integral. Then use (3.7), (Ex. 4.1), consider ker β and coker β, and use noetherian induction on Y.

By a previous excercise, we may assume X, Y are irreducible. Suppose that Y is not affine, and let Σ (by the previous excercise) are not affine. Let $Z \hookrightarrow X$ a minimal element of Σ, Z is reduced. f is finite, so WLOG let $f := f|_{f^{-1}(Z)}$ since finite morphisms are stable under base-change. Thus as Z is minimal, and we are base changing from Y to Z, we can assume that proper closed subschemes of Y are affine.

If $\mathscr{F} \in \mathfrak{Coh}\,(X)$, then by (b) we can find $\mathscr{G} \in \mathfrak{Coh}\,(X)$ and $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point. If $\mathscr{D} = ker\ \beta$ then $\mathscr{D} \in \mathfrak{Qcoh}\,(Supp\ \mathscr{D})$. Since Z is minimal $Supp\ \mathscr{D}$ is affine, so by thm III.3.7, $\mathscr{D}$ is acyclic. As a finite morphism is affine, then $f_*\mathscr{G}$ is acyclic. Taking LES in cohomology of $0 \to \mathscr{D} \to f_*\mathscr{G} \to \mathscr{F}^r \to 0$ and using induction gives $\mathscr{F}^r$, hence $\mathscr{F}$ is acyclic so that Y is affine.

3.4.5 III.4.3 g nice x

4.3. Let $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$, and let $U = X - \{(0,0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j | i,j < 0\}$. In particular, it is infinite-dimensional. (Using (3.5), this provides another proof that U is not affine—cf. (I, Ex. 3.6).)

Let $U_x = Spec\ k\left[x, y, x^{-1}\right]$, $U_y = Spec\ k\left[x, y, y^{-1}\right]$. Then $U_{xy} = U_x \cap U_y = Spec\ k\left[x, y, x^{-1}, y^{-1}\right]$. The cech complex is therefore

$0 \to k\left[x, y, x^{-1}\right] \oplus k\left[x, y, y^{-1}\right] \to k\left[x, y, x^{-1}, y^{-1}\right] \to 0$.

The differential is given by $d\left(f_1, f_2\right) = f_1 - f_2$.

Then $H^0\left(U, \mathcal{O}_U\right) \approx ker\ d \approx k\left[x, y\right]$.

H^1 is $k\left[x, y, x^{-1}, y^{-1}\right] / \left\{\sum a x^i y^j \mid i \geq 0\ or\ j \geq 0\right\}$.

Thus H^1 is generated by monomials with negative degree in both x and y.

3.4.6 III.4.4 x

4.4. On an arbitrary topological space X with an arbitrary abelian sheaf $\mathscr{F}$, Čech cohomology may not give the same result as the derived functor cohomology. But here we show that for H^1, there is an isomorphism if one takes the limit over all coverings.

(a) Let $\mathfrak{U} = (U_i)_{i \in I}$ be an open covering of the topological space X. A *refinement* of $\mathfrak{U}$ is a covering $\mathfrak{V} = (V_j)_{j \in J}$, together with a map $\lambda: J \to I$ of the index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$, show that there is a natural induced map on Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F}).$$

The coverings of X form a partially ordered set under refinement, so we can consider the Čech cohomology in the limit

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}).$$

If $p \geq 0$, then for each $(p+1)$-tuple $j \in Js$, there is an induced morphism from restriction $\mathscr{F}\left(U_{\lambda(j)}\right) \to \mathscr{F}\left(V_j\right)$ and hence an induced morphism $C^p\left(\mathfrak{U}, \mathscr{F}\right) \to C^p\left(\mathfrak{V}, \mathscr{F}\right)$. After some manipulation of indices, we find that for any i, , $\alpha \in C^p$ $\left(\lambda^{i+1} d\alpha\right)_j = \left(d\alpha\right)_{\lambda(j_0)\ldots\lambda(j_{p+1})} |_{V_{j_0 \cdots j_{p+1}}} = \ldots = \left(d\lambda^i \alpha\right)_{j_0 \ldots j_{p+1}}$. This gives a commutative square

$$\begin{array}{ccc} C^p\left(\mathfrak{U}, \mathscr{F}\right) & \xrightarrow{d} & C^{p+1}\left(\mathfrak{U}, \mathscr{F}\right) \\ \downarrow{\scriptstyle \lambda^i} & & \downarrow{\scriptstyle \lambda^{i+1}} \\ C^p\left(\mathfrak{V}, \mathscr{F}\right) & \xrightarrow{d} & C^{p+1}\left(\mathfrak{V}, \mathscr{F}\right) \end{array}$$

and thus a morphism $C^\bullet\left(\mathfrak{U}, \mathscr{F}\right) \to C^\bullet\left(\mathfrak{V}, \mathscr{F}\right)$.

3.4.7 b. x

(b) For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering

$$\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$$

are compatible with the refinement maps above.

Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$ which gives a unique up to homotopy map $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$. The λ^i from (a) are induced by maps of chain complex, and by uniqueness, $C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{V}, \mathscr{F}) \to \mathscr{I}^\bullet$ is homotopic to $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ and since homotopic maps give the same thing on cohomology, we have

$$\begin{tikzcd} \check{H}(\mathfrak{U}, \mathscr{F}) \arrow[r] \arrow[dr] & \check{H}^i(\mathfrak{V}, \mathscr{F}) \arrow[d] \\ & H^i(X, \mathscr{F}) \end{tikzcd}$$

3.4.8 c. x

(c) Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map

$$\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism. [*Hint*: Embed $\mathscr{F}$ in a flasque sheaf $\mathscr{G}$, and let $\mathscr{R} = \mathscr{G}/\mathscr{F}$, so that we have an exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{R} \to 0.$$

Define a complex $D^\cdot(\mathfrak{U})$ by

$$0 \to C^\cdot(\mathfrak{U}, \mathscr{F}) \to C^\cdot(\mathfrak{U}, \mathscr{G}) \to D^\cdot(\mathfrak{U}) \to 0.$$

Then use the exact cohomology sequence of this sequence of complexes, and the natural map of complexes

$$D^\cdot(\mathfrak{U}) \to C^\cdot(\mathfrak{U}, \mathscr{R}),$$

and see what happens under refinement.]

Weibel 5.8.3 gives a spectral sequence $\check{H}^r(X, H^s(\mathscr{F})) \to H^{r+s}(X, \mathscr{F})$. Now $H^0(X, H^1(\mathscr{F})) = 0$ by Milne, Etale 10.5.

Hence $\check{H}^0(X, H^s(\mathscr{F})) = 0$ so we have the required equality.

3.4.9 III.4.5 x

4.5. For any ringed space $(X, \mathcal{O}_X)$, let Pic X be the group of isomorphism classes of invertible sheaves (II, §6). Show that Pic $X \cong H^1(X, \mathcal{O}_X^*)$, where $\mathcal{O}_X^*$ denotes the sheaf whose sections over an open set U are the units in the ring $\Gamma(U, \mathcal{O}_X)$, with multiplication as the group operation. [*Hint*: For any invertible sheaf $\mathscr{L}$ on X, cover X by open sets U_i on which $\mathscr{L}$ is free, and fix isomorphisms $\varphi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$. Then on $U_i \cap U_j$, we get an isomorphism $\varphi_i^{-1} \circ \varphi_j$ of $\mathcal{O}_{U_i \cap U_j}$ with itself. These isomorphisms give an element of $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$. Now use (Ex. 4.4).]

Since this is the case that's important to me, I will assume $\mathscr{L}$ is a holomorphic line bundle. (You can recall from an excercise in chapter 2 that vector bundles correspond to locally free sheaves so line bundles to invertible sheaves).

Let U_i an open covering of X on which $\mathscr{L}$ is trivial.

Then $g_{ab} : (U_a \cap U_b) \times \mathbb{C} \to (U_a \cap U_b) \times \mathbb{C}$ gives a section of $\mathcal{O}^*$ such that $g_{ab}g_{ba} = 1$ and $g_{ab}g_{bc}g_{ca} = 1$. In the language of cech cohomology, a cocycle in $\tau \in C^1(U, \mathcal{O}^*)$ must be a collection of sections in $\tau_{ij} \in \mathcal{O}^*(U_i \cap U_j)$ such that $\tau_{23} - \tau_{13} + \tau_{12}|_{U_1 \cap U_2 \cap U_3} = 0$, or in multiplicative notation (since we are dealing with $\mathcal{O}^*$, $\tau_{23}\tau_{31}\tau_{12} = 1$. Thus $\{g_{ab}\}$ is a cocycle in $C^1(U, \mathcal{O}^*)$ and thus gives a cohomology class in $H^1(X, \mathcal{O}^*)$.

Now we want to show that if $\mathscr{L} \approx \mathscr{M}$, then $\mathscr{L} \otimes \mathscr{M}^*$ gives a cech coboundary. Recall that to have a cech coboundary τ , we need $\sigma \in C^0(U, \mathcal{O}^*)$ represented by a collection $\sigma_i \in \mathcal{O}^*(U_i)$ such that $\delta\sigma_{ij} = (\sigma_j - \sigma_i)|_{U_i \cap U_j} \in \mathcal{O}^*(U_i \cap U_j)$ and such that $\delta\sigma = \tau$ or in multiplicative notation we need $\frac{\sigma_j}{\sigma_i} = \tau_{ij}$. In other words, τ is in the image of the boundary map.

So if $\mathscr{L} \approx \mathscr{M}$, then $\mathscr{L} \otimes \mathscr{M}^*$ is trivial, so there are transition functions $\{\tau_{ab} := g_{ab}/h_{ab}\}$ for $\mathscr{L} \otimes \mathscr{M}^*$ which give a nowhere vanishing section σ. Let $\sigma_a : U_a \to \mathbb{C}^*$ the restriction of σ. On $U_a \cap U_b$ we therefore have $\frac{g_{ab}}{h_{ab}} \cdot \sigma_a = \sigma_b$ or $\frac{g_{ab}}{h_{ab}} =: \tau_{ab} = \frac{\sigma_b}{\sigma_a}$ so τ_{ab} defines a cech coboundary. Thus two invertible sheaves are the same iff their difference is a cech coboundary.

3.4.10 III.4.6 x

4.6. Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathscr{I}$ be a sheaf of ideals with $\mathscr{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathscr{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,

$$0 \to \mathscr{I} \to \mathcal{O}_X^* \to \mathcal{O}_{X_0}^* \to 0,$$

where $\mathcal{O}_X^*$ (respectively, $\mathcal{O}_{X_0}^*$) denotes the sheaf of (multiplicative) groups of units in the sheaf of rings $\mathcal{O}_X$ (respectively, $\mathcal{O}_{X_0}$); the map $\mathscr{I} \to \mathcal{O}_X^*$ is defined by $a \mapsto 1 + a$, and $\mathscr{I}$ has its usual (additive) group structure. Conclude there is an exact sequence of abelian groups

$$\ldots \to H^1(X, \mathscr{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X, \mathscr{I}) \to \ldots .$$

Use the stalks to check exactness.

Take the long exact sequence, and then use III.4.5

3.4.11 III.4.7 x

4.7. Let X be a subscheme of $\mathbf{P}_k^2$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d. (Do not assume f is irreducible.) Assume that $(1,0,0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitly, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1,$$

$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

The standard cover of $\mathbb{P}^2$ consisting of $U_i = D(x_i)$ gives an open cover of X as well. Closed subschemes of affine schemes are affine, so this is an affine cover. Removing $U_0 \cap X$ from the cover we still have an affine cover.

Writing $u = \frac{x_0}{x_1}$, $v = \frac{x_2}{x_1}$, $x = \frac{x_0}{x_2}$, and $y = \frac{x_1}{x_2}$ We have a cech complex:
$\frac{k[u,v]}{f(u,1,v)} \oplus \frac{k[x,y]}{f(x,y,1)} \to \frac{k[x,y,y^{-1}]}{f(x,y,1)}$ where $(g(u,v), h(x,y)) \mapsto (gxy^{-1}, y^{-1}) - h(x,y)$.

For (g,h) in the kernel, $g - h \in (f(x,y,1))$ so $g - h = f'f$ for $f' \in k[x,y,y^{-1}]$. Since $(1,0,0) \notin X$ then $f(x,y,1) = \sum_{0\le i\le d, 0\le j\le d} a_{ij}x^iy^j$ with $a_{0d} = 1$ (so when we plug in $(1,0,0)$ to f it doesn't give 0). Write $f' = f_0 + f_1 + f_2$ where the monomial terms x^iy^j of f_0 have $i \le -d - j$, of f_1 have $j \ge 0$, and of f_2 have $i > -d - j$ so that the monomial spanning the image in $\frac{k[u,v]}{f(u,1,v)}$ of $f_0 f$ and the image of $f_1 f$ in $\frac{k[x,y]}{f(x,y,1)}$ overlap at the constant term.

Note that $f_2 f$ is in the image of the boundary of the cech complex so either $i+d \leq -j$ or $j \geq 0$. But then $g = f_0 f + g_0$ and $h = -f_1 f + h_0$ where g_0, h_0 are constants. Thus (g,h) gives the same element in the kernel as if g,h are constant and thus as if $g = h$. Hence the kernel is (a,a) for $a \in k$ so that $dim\ H^0(X, \mathcal{O}_X) = 1$.

Now f in the cokernel is a polynomial in $k[x,y,y^{-1}]$. Any monomial $x^i y^j$ with $j \geq 0$ is zero in the cokernel since there is $(0, x^i, y^j)$ mapping to it. Similarly, if $j \geq i$, then $(u^i v^{j-i}, 0)$ maps to it and it is likewise 0. Thus write $f = \sum_{j<0,,I} a_{ij} x^i y^j$. $(1,0,0)$ is not a point so $f(x_0, x_1, x_2) = \tilde{f} + a_0 x_0^d$ for a_0 nonzero. (Thus when we plug in $(1,0,0)$ it won't satisfy $f(..) = 0$). Assume a_0 is 1 since scaling by units doesn't change anything. Thus we have $x^d = -\tilde{f}(x,y,1)$ where $\tilde{f}(x,y,1)$ satisfies $0 \leq i < d$, $0 \leq j$. Thus we can rephrase the cokernel as the sums of monomials $a_{ij} x^i y^j$ with $1 \leq i < d$ and $-i < j < 0$. Thus $dim\ H^1(X, \mathcal{O}_X) \leq \frac{1}{2}(d-1)(d-2)$ since that is how many there are. Note that polynomials with $1 \leq i < d$ and $-i < j < 0$ are not in the image of the boundary map by what we have just said. On the other hand, if the are in $(f(x,y,1)) = \left(x^d + \tilde{f}(x,y,1)\right)$, then they should have some $i \geq d$. Thus $dim\ H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2)$.

3.4.12 III.4.8 x cohomological dimension

4.8. *Cohomological Dimension* (Hartshorne [3]). Let X be a noetherian separated scheme. We define the *cohomological dimension* of X, denoted $\mathrm{cd}(X)$, to be the least integer n such that $H^i(X, \mathscr{F}) = 0$ for all quasi-coherent sheaves $\mathscr{F}$ and all $i > n$. Thus for example, Serre's theorem (3.7) says that $\mathrm{cd}(X) = 0$ if and only if X is affine. Grothendieck's theorem (2.7) implies that $\mathrm{cd}(X) \leqslant \dim X$.

(a) In the definition of $\mathrm{cd}(X)$, show that it is sufficient to consider only coherent sheaves on X. Use (II, Ex. 5.15) and (2.9).

Suppose that $cd(X) = n$ and assume that $\mathscr{F}$ is q.c.

By exc II.5.15.a, $\mathscr{F} = \underset{\rightarrow}{lim} \mathscr{F}_a$, $\mathscr{F}_a$ are coherent subsheaves of $\mathscr{F}$.

By thm III.2.9 we get $H^i(X, \mathscr{F}) \approx H^i\left(X, \underset{\rightarrow}{lim} \mathscr{F}_a\right) \approx \underset{\rightarrow}{lim} H^i(X, \mathscr{F}_a) \approx \underset{\rightarrow}{lim}\ 0$ for $i > n$.

3.4.13 b. x

(b) If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X. Use (II, 5.18).

Thm II.5.18, gives that $\mathscr{F} \in \mathfrak{Coh}(\mathscr{X})$ is a quotient of a finite rank locally free sheaf, $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F}$, $\mathscr{E}$ is fiite rank locally free. This gives a LES in cohomology $\cdots \to H^i(X, \mathscr{E}) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{G}) \to H^{i+1}(X, \mathscr{E}) \to \cdots$.

For $i > n$ if the theorem holds for locally free, then the outer terms of above are 0 so the inner terms are equal. By Grothendieck's theorem, $H^i(X, \mathscr{G}) = 0$ for $i > dim\ X$ so $H^i(X, \mathscr{F}) = 0$ for $i > n$.

3.4.14 c x.

(c) Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that $\mathrm{cd}(X) \leqslant r$.

Using 4.5, since X is separated, we can compute using cech complex of $r+1$ affines.

Then $C^{r+1} = 0$ and so $H^i(X, \mathscr{F}) = 0$ for $i > r$.

*(d) If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude (independently of (2.7)) that $\mathrm{cd}(X) \leqslant \dim X$.

starred

3.4.15 e. x

(e) Let Y be a set-theoretic complete intersection (I, Ex. 2.17) of codimension r in $X = \mathbf{P}_k^n$. Show that $\operatorname{cd}(X - Y) \leqslant r - 1$.

By exc I.2.17, $Y = H_1 \cap ... \cap H_r$. By exc I.3.5, $X - H_i$ is affine, for each i, and thus $X - Y = (X - H_1) \cup ... \cup (X - H_r)$ which is covered by r open affines. Using (c), then $cd\,(X - Y) \leq r - 1$.

3.4.16 III.4.9 x

4.9. Let $X = \operatorname{Spec} k[x_1,x_2,x_3,x_4]$ be affine four-space over a field k. Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\bar{Y}$ in $\mathbf{P}_k^4$ is also not a set-theoretic complete intersection. [*Hints*: Use an affine analogue of (Ex. 4.8e). Then show that $H^2(X - Y, \mathcal{O}_X) \neq 0$, by using (Ex. 2.3) and (Ex. 2.4). If $P = Y_1 \cap Y_2$, imitate (Ex. 4.3) to show $H^3(X - P, \mathcal{O}_X) \neq 0$.]

Via exc III.4.8.e, we can show $H^2\left(X - Y, \mathcal{O}_{X-Y}\right) \neq 0$. If $Z \subset X$ is closed, then by exc III.2.3.d, we have an exact sequence $H^i\left(X, \mathcal{O}_X\right) \to H^i\left(X - Z, \mathcal{O}_{X-Z}\right) \to H_Z^{i+1}\left(X, \mathcal{O}_X\right) \to H^{i+1}\left(X, \mathcal{O}_X\right)$. Using 3.8, $H^i\left(X, \mathcal{O}_X\right) = H^{i+1}\left(X, \mathcal{O}_X\right)$ so the middle terms are equal. In this manner, $H^2\left(X - Y, \mathcal{O}_{X-Y}\right) \approx H_Y^3\left(X, \mathcal{O}_X\right)$. and same for Y_1 .

Mayer-vietoris gives $H_{Y_i}^3\left(X, \mathcal{O}_X\right) \oplus H_{Y_2}^3\left(X, \mathcal{O}_X\right) \to H_Y^3\left(X, \mathcal{O}_X\right) \to H_{Y_1 \cap Y_2}^4\left(X, \mathcal{O}_X\right) \to H_{Y_1}^4\left(X, \mathcal{O}_X\right) \oplus H_{Y_2}^4\left(X, \mathcal{O}_X\right)$. Now $X - Y_1$ is a complete intersection of codimension 2 and thus $cd\left(X - Y_1\right) \leq 1$ by (e) of the last exercise. Thus $H_{Y_1}^3\left(X, \mathcal{O}_X\right) = H^2\left(X - Y_2, \mathcal{O}_{X-Y_1}\right) = 0$. Similarly, $H^2\left(X - Y_2, \mathcal{O}_{X-Y_2}\right) = H^3\left(X - Y_2, \mathcal{O}_{X-Y_2}\right) = 0$. The mayer-vietoris sequence gives us that $H_Y^3\left(X, \mathcal{O}_X\right) \approx H_{Y_1 \cap Y_2}^4\left(X, \mathcal{O}_X\right)$.

If $P = Y_1 \cap Y_2 = \{(0,0,0,0,0)\}$ then by mayer vietoris, we must show that $H_P^4\left(X, \mathcal{O}_X\right) = H^3\left(X - P, \mathcal{O}_{X-P}\right) \neq 0$. By cech cohomology on the cover $U_i = D\left(x_i\right)$ using the complex $k\left[x_1, x_2, x_3, x_4, x_1^{-1}\right] \oplus \cdots \oplus k\left[x_1, x_2, x_3, x_4, x_4^{-1}\right] \xrightarrow{d_0}$ $k\left[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}\right] \oplus \cdots \oplus k\left[x_1, x_2, x_3, x_4, x_3^{-1}, x_4^{-1}\right] \xrightarrow{d_1} \cdots$ $k\left[x_1, x_2, x_3, x_4, x_1^{-1}, x_2^{-1}, x_3^{-1}, x_4^{-1}\right]$ we have $H^3\left(X - P, \mathcal{O}_{X-P}\right) = \left\{x_1^i x_2^j x_3^k x_4^l : i, j, k, l < 0\right\} \neq 0$.

3.4.17 III.4.10 (starred)

***4.10.** Let X be a nonsingular variety over an algebraically closed field k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathscr{F}$ (II, Ex. 8.7) up to isomorphism, and the group $H^1(X, \mathscr{F} \otimes \mathscr{T})$, where $\mathscr{T}$ is the tangent sheaf of X (II,§8). [*Hint*: Use (II, Ex. 8.6) and (4.5).]

MISS

3.4.18 III.4.11 x

4.11. This exercise shows that Čech cohomology will agree with the usual cohomology whenever the sheaf has no cohomology on any of the open sets. More precisely, let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\mathfrak{U} = (U_i)$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \ldots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$, that $H^k(V, \mathscr{F}|_V) = 0$. Then prove that for all $p \geqslant 0$, the natural maps

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more general result.

Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ an injective resolution of $\mathscr{F}$. We will compute the spectral sequence associated to $E_0^{p,q} = \prod_{i_0<\ldots<i_p} \mathscr{I}^p \left(U_{i_0\cdots i_q}\right)$. For any open U and i, the sheaf $\mathscr{I}^i|_U$ is injective and the restriction $0 \to \mathscr{F}|_U \to \mathscr{I}^\bullet|_U$ is an injective resolution of $\mathscr{F}|_U$. Thus $E_1^{p,q} := H^p\left(E_0^{\bullet,q}\right)$ are $C^q\left(\mathscr{F}, \mathfrak{U}\right)$ for $p = 0$ and 0 otherwise by assumption, i.e. the cohomology of $\mathscr{F}|_U$. $E_0^{0,q} \to E_0^{q+1}$ give are the usual differentials on $C^q\left(\mathscr{F}, \mathfrak{U}\right)$ and thus $E_2^{p,q} := H^q\left(E_1^{p,\bullet}\right)$ is $\check{H}^q\left(\mathscr{F}, \mathfrak{U}\right)$ for $p = 0$ and 0 otherwise.

Next we will compute the spectral sequence going in the counterclockwise direction. This direction will give the cech cohomology of $\mathscr{I}^p$. $\mathscr{I}$ are flasque by thm III.2.4, so $H^q\left(E_0^{p,\bullet}\right) = \Gamma\left(X, \mathscr{J}^p\right)$ for $q = 0$ and 0 else. On the next page we get , $H^p\left(E_1^\bullet, q\right) = H^q\left(X, \mathscr{F}\right)$ for $q = 0$ and 0 otherwise.

Thus the cohomology of the total complex is isomorphic to both $H^\bullet\left(X, \mathscr{F}\right)$ and $\check{H}^\bullet\left(\mathfrak{U}, \mathscr{F}\right)$ by computing first clockwise and then counterclockwise.

3.5 III.5 x Cohomology_Of_Projective_Space

3.5.1 III.5.1 x g

5.1. Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. We define the *Euler characteristic* of $\mathscr{F}$ by

$$\chi(\mathscr{F}) = \sum(-1)^i \dim_k H^i(X, \mathscr{F}).$$

If

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Let ϕ^i be the map induced on cohomology between $\mathscr{F}'$ and $\mathscr{F}$, and ψ^i the map for $\mathscr{F} \to \mathscr{F}''$ and δ^i the coboundary.

$\chi\left(\mathscr{F}\right) = \sum_{i=0}^n \left(-1\right)^i dim\ H^i\left(X, \mathscr{F}\right) =$
$\sum_{i=0}^n \left(-1\right)^i \left(dim\ ker\delta^i + dim\ ker\psi^i\right) =$
$\sum \left(-1\right)^i \left(dim\ ker\delta^i + dim\ ker\psi^i + dim\ ker\phi^i - dim\ ker\phi^i\right) =$
$\sum \left(-1\right)^i \left(dim\ ker\phi^i + dim\ ker\psi^i\right) +$
$\sum \left(-1\right)^i \left(dim\ ker\delta^i - dim\ ker\phi^i\right) =$
$\chi\left(\mathscr{F}'\right) + \chi\left(\mathscr{F}''\right)$.

3.5.2 III.5.2 x

5.2. (a) Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbf{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbf{Z}$. We call P the *Hilbert polynomial* of $\mathscr{F}$ with respect to the sheaf $\mathcal{O}_X(1)$. [*Hints*: Use induction on dim Supp $\mathscr{F}$, general properties of numerical polynomials (I, 7.3), and suitable exact sequences

$$0 \to \mathscr{R} \to \mathscr{F}(-1) \to \mathscr{F} \to \mathscr{Q} \to 0.]$$

First replace $\mathscr{F}$ by $j_*\mathscr{F}$ so that we may assume $X = \mathbb{P}_k^r$. Dimensions of cohmology are preserved under change of base by the flat base change theorem, so we assume k is algebraically closed, and thus infinite. We induct on the dimension of the support of $\mathscr{F}$. If this dimension is 0, then $P(z) = 0$ satisfies the requirements.

Now we want to find $x \in \Gamma(X, \mathcal{O}_X(1))$ such that the map $\mathscr{F}(-1) \xrightarrow{x} \mathscr{F}$ is injective (or just injective on an affine base). For such injectivity, we can consider a finitely generated module over a noetherian ring, and by primary decomposition theorems, we must avoid finitely many asociated primes to get injectivity (zero divisors are set of union of associated primes). Thus we want to find a hyperplane missing all of the finitely many associated primes of the finitely many elements of the open cover, since our field is infinite.

Note for example if all hyperplanes pass through finitely many points, and we have an open affine such as *Spec Sym* V^* with basis $x_0, ..., x_n \in V^*$, $x_n = \xi$ gives the spectrum of $k[x_0, ..., x_{n-1}]$. Here hyperplanes are given by elements of V^* which are k-linear combinations of $x_0, ..., x_{n-1}$ and $x_n = 1$. For ax_n, $a \in k$ if all hyperplanes pass through finitely many points, then one of these points contains a nonzero multiple of $x_n = 1$ which contradicts the fact that this point should be a prime ideal of the ring.

In this manner we achieve an $x \in \Gamma(X, \mathcal{O}_X(1))$ such that $\mathscr{F}(-1) \xrightarrow{x} \mathscr{F}$ is injective. Therefore we achieve an exact sequence $0 \to \mathscr{F}(-1) \xrightarrow{x} \mathscr{F} \to \mathscr{G} \to 0$. This gives (since by a previous problem Euler characteristic is additive on S.E.S. that $\chi(X, \mathscr{F}(m)) = \chi(X, \mathscr{G}(m)) + \chi(X, \mathscr{F}(m-1))$. Thus we are done if the support of $\mathscr{G}$is smaller than the support of $\mathscr{F}$. But since $\mathscr{G}_p \approx \mathscr{F}_p / x\mathscr{F}_p$ is trivial if x is a unit in $x \notin \mathfrak{p}$, we have $supp\ \mathscr{G} = supp \mathscr{F} \cap V(x)$ so by Hauptidealsatz, $dim\ Supp\mathscr{G} = dim\ Supp\mathscr{F} - 1$.

3.5.3 b. x g

(b) Now let $X = \mathbf{P}_k^r$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \ldots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (I, §7).

By III.5.2, $H^i(X, \mathscr{F}(n)) = 0$ for $i > 0$ and $n \gg 0$.
Thus $\chi(\mathscr{F}(n)) = dim(H^0(X, \mathscr{F}(n))) = dim\ M_n$.

3.5.4 III.5.3a. x Arithmetic genus

5.3. *Arithmetic Genus.* Let X be a projective scheme of dimension r over a field k. We define the *arithmetic genus* p_a of X by

$$p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1).$$

Note that it depends only on X, not on any projective embedding.

(a) If X is integral, and k algebraically closed, show that $H^0(X,\mathcal{O}_X) \cong k$, so that

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X,\mathcal{O}_X).$$

In particular, if X is a curve, we have

$$p_a(X) = \dim_k H^1(X,\mathcal{O}_X).$$

[*Hint*: Use (I, 3.4).]

Projective $\implies$ integral $\implies$ variety (II.4.10) $\implies$ $H^0(X,\mathcal{O}_X) = k$ (I.3.4.a)

3.5.5 III.5.3.b. x

(b) If X is a closed subvariety of $\mathbf{P}_k^r$, show that this $p_a(X)$ coincides with the one defined in (I, Ex. 7.2), which apparently depended on the projective embedding.

So $(-1)^r(\chi(\mathcal{O}_X) - 1)$ is the new one , cf $(-1)^r(P_Y(0) - 1)$ for the old one. Now just use III.5.2.a and note that $\chi(\mathcal{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$.

3.5.6 c. x g Important genus is birational invariant for curves!!

(c) If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a *birational* invariant. Conclude that a nonsingular plane curve of degree $d \geqslant 3$ is not rational. (This gives another proof of (II, 8.20.3) where we used the geometric genus.)

Using, for instance, the valuative criterion, a birational map between projective curves gives an isomorphism, since whereever the map is not defined, we can extend the map. Thus birational $\implies$ isomorphic $\implies$ arithmetic genus is the same. Now using genus degree, $g = \frac{1}{2}(d-1)(d-2) \geq \frac{1}{2}(3-1)(3-2) = \frac{1}{2}(2\cdot)1 = 1 > 0$ for a plane curve of degree ≥ 3. But a conic, which is rational by chapter 1 is degree 2 and by $g = \frac{1}{2}(d-1)(d-2)$ has genus 0.

III.5.4 x g

5.4. Recall from (II, Ex. 6.10) the definition of the Grothendieck group $K(X)$ of a noetherian scheme X.

(a) Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X. Show that there is a (unique) additive homomorphism

$$P: K(X) \to \mathbf{Q}[z]$$

such that for each coherent sheaf $\mathcal{F}$ on X, $P(\gamma(\mathcal{F}))$ is the Hilbert polynomial of $\mathcal{F}$ (Ex. 5.2).

Note that the hilbert polynomial is additivity on short exact sequences. Thus the map taking a coherent sheaf to its hilbert polynomial is compatible with the structure of the grothendieck group.

3.5.7 b. x

(b) Now let $X = \mathbf{P}_k^r$. For each $i = 0, 1, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that

(1) $K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0, \ldots, r\}$, and
(2) the map $P: K(X) \to \mathbf{Q}[z]$ is injective.

[*Hint*: Show that (1) $\Rightarrow$ (2). Then prove (1) and (2) simultaneously, by induction on r, using (II, Ex. 6.10c).]

First we show (1) $\implies$ (2).
For a linear embedding $i : \mathbb{P}^i \hookrightarrow \mathbb{P}^r$, then $\mathcal{O}_{L_i} = i_* \mathcal{O}_{\mathbb{P}^i}$.
Now $\chi(\mathcal{O}_{\mathbb{P}^i}(m)) = \begin{pmatrix} i+m \\ i \end{pmatrix}$ so $\sum a_i \gamma(\mathcal{O}_{L_i}) \mapsto \sum a_i \begin{pmatrix} i+m \\ i \end{pmatrix}$ under P.
Anything in the kernel of this map must have all $a_i = 0$ by induction on the coefficients.
So P is injective.

Now we do (1) by induction, using that (1) $\implies$ (2) for smaller r. Note $r = 0$ is the trivial base case.
Then exc II.6.10 gives us a right exact sequence $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to K(\mathbb{P}^r - \mathbb{P}^{r-1}) \to 0$ from extension by zero.
Suppose L_i satisfies $L_i \subset L_{r-1}$ for $i < r$.
P factors as $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to \mathbb{Q}[z]$ which is injective for $i < r$ by induction, so that $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r)$ is injective.
Thus by assumption of (1) and induction, $K(\mathbb{P}^r)$ has a subgroup $\mathbb{Z}^r$ with basis $\mathcal{O}_{L_i}$, $i = 0, ..., r-1$ which is the kernel of the second map $K(\mathbb{P}^r) \to K(\mathbb{P}^r - \mathbb{P}^{r-1}) \approx K(\mathbb{A}^r) \approx \gamma(\mathcal{O}_{\mathbb{A}^r}) \cdot \mathbb{Z}$ via the method of exc II.6.10.
Thus $K(\mathbb{P}^r)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^r$.
By basic properties of projective modules, $Ext^1(\mathbb{Z}, \mathbb{Z}) = 0$ which means only extensions are trivial so that $K(\mathbb{P}^r) \approx \mathbb{Z}^{r+1}$ which is generated by $\gamma(\mathcal{O}_{L_i})$ for $i = 0, ..., r$.

3.5.8 III.5.5 x g

5.5. Let k be a field, let $X = \mathbf{P}_k^r$, and let Y be a closed subscheme of dimension $q \geqslant 1$, which is a complete intersection (II, Ex. 8.4). Then:
(a) for all $n \in \mathbf{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective. (This gives a generalization and another proof of (II, Ex. 8.4c), where we assumed Y was normal.)

We induct on the codimension $r = n - q$ of Y. If the codimension of Y is 0, then $H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is clearly surjective.
Now suppose that for all complete intersections Y' of codimension $0 \le i \le r-1$ we have shown that $H^0(X, \mathcal{O}_X(n)) \to H^0(Y', \mathcal{O}_{Y'}(n))$ is surjective. Assume that Y has codimension r. Since Y is a complete intersection, then using II.8.4, $Y = H_1 \cap ... \cap H_s$ where H_i are hypersurfaces. Then $Z = H_1 \cap ... \cap H_{s-1}$ is also a complete intersection. Z has codimension less than r, so by the induction hypothesis, $H^0(X, \mathcal{O}_X(n)) \to H^0(Z, \mathcal{O}_Z(n))$ is surjective. Thus we only have to show that $H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective.
If $H_s$has degree d then there is an exact sequence of sheaves:

$0 \to \mathcal{O}_Z(n-d) \to \mathcal{O}_Z(n) \to \mathcal{O}_Y(n) \to 0$ and taking cohomology gives

$0 \to H^0(Z, \mathcal{O}_Z(n-d)) \to H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n)) \to ...$

By (c) all higher cohomology is 0 so the H^0 form a short exact sequence.

Hence $H^0(Z, \mathcal{O}_Z(n)) \to H^0(Y, \mathcal{O}_Y(n))$ is surjective so that $H^0(X, \mathcal{O}_X(n)) \to H^0(X, \mathcal{O}_X(n))$ is surjective.

3.5.9 b. x g

(b) Y is connected;

By III.5.3.a, $H^0(X, \mathcal{O}_X) \approx k$.

By part a. of this problem, then $H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is surjective.

By III.2.6.1, the cohomology groups are k-modules.

Thus if $H^0(Y, \mathcal{O}_Y) \neq 0$, then $H^0(Y, \mathcal{O}_Y) = k$.

The rank of $H^0(Y, \mathcal{O}_Y)$ counts the number of connected components so there is only one component.

3.5.10 c. x g complete intersection cohomology.

(c) $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbf{Z}$;

If Y has codim 0, then $Y = \mathbb{P}^n_k$. The result is then III.5.1.b.

Suppose we have proven $H^i(Y, \mathcal{O}_Y(n)) = 0$ up to codimension $r-1$.

By II.8.4, Y is a complete intersection of codimension r in $\mathbb{P}^n_k$ iff there are hypersurfaces $H_1, ..., H_r$ such that $Y = H_1 \cap ... \cap H_r$. Assume $Y = H_1 \cap ... \cap H_r$ each H_i has degree d_i .

Then $H_1 \cap ... \cap H_i$ is a c.i. for $1 \le i \le r$, so setting $Z = H_1 \cap ... \cap H_{r-1}$, then Z is a c.i.

We have the short exact sequence of (a):

$0 \to \mathcal{O}_Z(n-d) \to \mathcal{O}_Z(n) \to \mathcal{O}_Y(n) \to 0$ and associated long exact sequence:

$\cdots \to H^i(Z, \mathcal{O}_Z(n-d)) \to H^i(Z, \mathcal{O}_Z(n)) \to H^i(Y, \mathcal{O}_Y(n)) \to ...$

Using induction, since the left hand terms are zero, the right hand terms are zero for relevant i , and the result follows.

3.5.11 d. x g

(d) $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.
[*Hint*: Use exact sequences and induction on the codimension, starting from the case $Y = X$ which is (5.1).]

Recall $p_a(X) = (-1)^r(\chi(\mathcal{O}_X) - 1)$, $\chi(\mathscr{F}) = \sum (-1)^i \dim_k H^i(X, \mathscr{F})$.

Then $p_a(Y) = (-1)^r(\chi(\mathcal{O}_Y) - 1) =$

$(-1)^q \left[\left(\sum (-1)^i \dim_k H^i(Y, \mathcal{O}_Y)\right) - 1\right] =$

$(-1)^q \left[\left((-1)^0 \dim_k H^0(Y, \mathcal{O}_Y) - 1\right) + (-1)^q \dim_k H^q(Y, \mathcal{O}_Y)\right] =$

$\left[(-1)^q(\dim_k k - 1) + (-1)^{2q} \dim_k H^q(Y, \mathcal{O}_Y)\right] =$

$[0 + \dim_k H^q(Y, \mathcal{O}_Y)] =$

$\dim_k H^q(Y, \mathcal{O}_Y)$.

3.5.12 III.5.6 x curves on a nonsingular quadric

5.6. *Curves on a Nonsingular Quadric Surface.* Let Q be the nonsingular quadric surface $xy = zw$ in $X = \mathbf{P}_k^3$ over a field k. We will consider locally principal closed subschemes Y of Q. These correspond to Cartier divisors on Q by (II, 6.17.1). On the other hand, we know that Pic $Q \cong \mathbf{Z} \oplus \mathbf{Z}$, so we can talk about the *type* (a,b) of Y (II, 6.16) and (II, 6.6.1). Let us denote the invertible sheaf $\mathscr{L}(Y)$ by $\mathcal{O}_Q(a,b)$. Thus for any $n \in \mathbf{Z}$, $\mathcal{O}_Q(n) = \mathcal{O}_Q(n,n)$.

(a) Use the special cases $(q,0)$ and $(0,q)$, with $q > 0$, when Y is a disjoint union of q lines $\mathbf{P}^1$ in Q, to show:

(1) if $|a - b| \leqslant 1$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
(2) if $a,b < 0$, then $H^1(Q,\mathcal{O}_Q(a,b)) = 0$;
(3) If $a \leqslant -2$, then $H^1(Q,\mathcal{O}_Q(a,0)) \neq 0$.

For (3) we will consider the LES associated to $0 \to \mathcal{O}_Q(-q,0) \to \mathcal{O}_Q \to \mathcal{O}_Y \to 0$.

H^0

By chapter 1, $H^0(Q,\mathcal{O}_Q) = k$ and thus $H^0(Q,\mathcal{O}_Q(-q,0)) = 0$ since the only constants in k vanishing on Y (note $\mathcal{O}_Q(-q,0)$ is the ideal sheaf of Y) are constants. As Y is a disjoint union of $\mathbb{P}^1$'s, $H^0(Q,\mathcal{O}_Y) = k^{\oplus q}$.

H^1

Ex III.5.5.b, gives $H^1(Q,\mathcal{O}_Q) = 0$, since it's a disjoint union, and from known cohomology of $\mathbb{P}^n$ (see the version in stacks), $H^1(Q,\mathcal{O}_Y) = 0$ since it is generated by monomials with negative degree.

H^2

By serre duality, and using the that $p_q(Q) = 0$ from exc I.7.2, then $H^2(Q,\mathcal{O}_Q) = 0$. Thus using exactness we can finish the sequence:

H^0: $0 \to 0 \to k \to k^{\oplus q} \xrightarrow{\delta}$

H^1: $k^{\oplus(q-1)} \to 0 \to 0 \xrightarrow{\delta}$

H^2: $H^2(Q,\mathcal{O}_Q(-q,0)) \to 0 \to H^2(Q,\mathcal{O}_Y) \xrightarrow{\delta} 0$

Clearly all H^2 terms are 0.

Thus $H^1(Q,\mathcal{O}_Q(a,0)) = k^{\oplus(-a-1)} \neq 0$, $a \leq -2$

For (1)

Let $a \in \mathbb{Z}$ anc consider the LES associated to $0 \to \mathcal{O}_{\mathbb{P}^3}(-2+a) \to \mathcal{O}_{\mathbb{P}^3}(a) \to \mathcal{O}_Q(a) \to 0$.

From the known cohomology of $\mathbb{P}^3$, and exactness we find $H^1(\mathcal{O}_Q(a)) = 0$

Now consider the LES associated to $0 \to \mathcal{O}_Q(a-1,a) \to \mathcal{O}_Q(a) \to \mathcal{O}_Y(a) \to 0$.

Since $H^1(\mathcal{O}_Q(a)) = 0$ and the surjection $H^0(Q,\mathcal{O}_Q(a)) \to H^0(Y,\mathcal{O}_Y(a))$ follows from considering the degree a part of the coordinate rings for the quadric, and Y which is writen as a quotient of Q, $Y = Proj(k[x,y,z,w]/(xy - zw, x, z))$

Exactness gives $H^1(\mathcal{O}_Q(a-1,a)) = 0$ and similarly $H^1(\mathcal{O}_Q(a,a-1)) = 0$

For (2) consider the LES associated to $0 \to \mathcal{O}_Q(-a,-a-n) \to \mathcal{O}_Q(-a) \to \mathcal{O}_Y(-a) \to 0$. Now use exactness and the fact that Y is a disjoint union of copies of $\mathbb{P}^1$.

3.5.13 b. x

(b) Now use these results to show:

(1) if Y is a locally principal closed subscheme of type (a,b), with $a,b > 0$, then Y is connected;

(2) now assume k is algebraically closed. Then for any $a,b > 0$, there exists an irreducible nonsingular curve Y of type (a,b). Use (II, 7.6.2) and (II, 8.18).

(3) an irreducible nonsingular curve Y of type (a,b), $a,b > 0$ on Q is projectively normal (II, Ex. 5.14) if and only if $|a-b| \leqslant 1$. In particular, this gives lots of examples of nonsingular, but not projectively normal curves in $\mathbf{P}^3$. The simplest is the one of type (1,3), which is just the rational quartic curve (I, Ex. 3.18).

(1). The LES associated to I_Y has $H^0(I_Y) = 0$, $H^0(Q, \mathcal{O}_Q) = k$, and by (a), $H^1(Q, \mathcal{I}_Y) = H^1(Q, \mathcal{O}_Q(-a,-b)) = 0$. Now use exactness of $0 \to 0 \to k \to H^0(\mathcal{O}_Y) \to 0$.

(2) Thm II.7.6.2 says that $\mathcal{O}_Q(-a,-b)$ is associated to a closed immersion $\mathbb{P}^1 \times \mathbb{P}^1 = Q \to \mathbb{P}^a \times \mathbb{P}^b$. By Bertini, we can find H in $\mathbb{P}^a \times \mathbb{P}^b$ which is a nonsingular hyperplane section of the embedding. Pulling back gives a smooth curve Y of type (a,b) in $Q \subset \mathbb{P}^3$. Y is ample hence connected by lefschetz hyperplane theorem and thus irreducible by Bertini.

(3) Note that $Q \approx \mathbb{P}^1 \times \mathbb{P}^1$ is locally isomorphic to $\mathbb{A}^1 \times \mathbb{A}^1 \approx \mathbb{A}^2$ which is normal. Thus by II.8.4, since Q is a complete intersection, Q is projectively normal, i.e. $H^1(Q, \mathcal{I}_Y(n)) = 0,\ n \geq 0$. Consider the LES associated to $0 \to \mathcal{I}_Y(n) \to \mathcal{O}_Q(n) \to \mathcal{O}_Y(n)$. Note that $\mathcal{I}_Y(n) \approx \mathcal{O}_Q(-a,-b)(n) \approx \mathcal{O}_Q(n-a, n-b)$. For $|a-b| = |n-a-(n-b)| \leq 1$, we have $H^1(Q, \mathcal{O}_Q(-a,-b)(n)) = 0$ by (a) so Y is projectively normal. For $|a-b| > 1$, then $\mathcal{O}_Q(-a,-b)(a) = \mathcal{O}_Q(0, a-b)$ and now use (a).

3.5.14 c. x

(c) If Y is a locally principal subscheme of type (a,b) in Q, show that $p_a(Y) = ab - a - b + 1$. [*Hint*: Calculate Hilbert polynomials of suitable sheaves, and again use the special case $(q,0)$ which is a disjoint union of q copies of $\mathbf{P}^1$. See (V, 1.5.2) for another method.]

WLOG Y looks like $Y = Y_1 \coprod Y_2$ which is a copies of one $\mathbb{P}^1$ and b copies of the other $\mathbb{P}^1$ making up Q. Note I_{Y_2} is flat so we have an SES $0 \to \mathcal{I}_{Y_1} \otimes \mathcal{I}_{Y_2} \to \mathcal{I}_{Y_2} \to \mathcal{O}_{Y_1} \otimes \mathcal{I}_{Y_2} \to 0$ which is $0 \to \mathcal{O}_Q(-a,-b) \to \mathcal{O}_Q(0,-b) \to \mathcal{O}_Y \otimes \mathcal{O}_Q(0,-b) \to 0$. There are no global sections in cohomology by (a) as $a,b > 0$ and similarly no H^1 of the first term by (a).

As in previous part, $H^1(Q, \mathcal{O}_Q(0,-b)) = k^{\oplus(b-1)}$ and $H^2(Q, \mathcal{O}_Q(0,-b)) = 0$. By the method of (a), $H^1(Q, \mathcal{O}_Y \otimes \mathcal{O}_Q(0,-b)) \approx k^{\oplus a(b-1)}$. Thus we have an LES in cohomology

H^1 :$0 \to h^{\oplus(b-1)} \to k^{\oplus a(b-1)} \to$

H^2: $H^2(Q, \mathcal{O}_Q(-a,-b)) \to 0 \to 0$.

Now $\chi(\mathcal{O}_Q(-a-b)) = h^2(Q, \mathcal{O}_Q(-a,-b)) = a(b-1) - (b-1) = ab - a - b + 1$.

3.5.15 III.5.7 x g

5.7. Let X (respectively, Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.

(a) If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i: Y \to X$ is the inclusion.

For an arbitrary coherent sheaf $\mathscr{F}$, by III.5.3, $H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n)$ is 0 for n sufficiently large.
Thus $0 = H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n) \approx H^i(X, i_*(\mathscr{F} \otimes i^*\mathscr{L}^n)) \approx H^i(Y, \mathscr{F} \otimes i^*\mathscr{L}^n)$ via the projection formula.
By AMVCCQ it follows that $i^*\mathscr{L}$ is ample.

3.5.16 x g ample iff red is ample

(b) $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{red} = \mathscr{L} \otimes \mathcal{O}_{X_{red}}$ is ample on X_{red}.

$\mathscr{L}$ ample $\implies \mathscr{L}_{red}$ is ample follows from part (d).
Assume $\mathscr{L}_{red}$ is ample on X_{red}. For $\mathscr{F} \in \mathfrak{Coh}(X)$, let $\mathscr{N}$ the nilradical of $\mathcal{O}_X$.
There is a filtration $\mathscr{F} \supset \mathscr{N} \supset \mathscr{N}^2\mathscr{F} \supset \cdots \supset \mathscr{N}^r \cdot \mathscr{F} = 0$.
The quotients $\mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}$ are coherent $\mathcal{O}_{X_{red}}$-modules so that since $\mathscr{L}_{red}$ is ample, $H^j(X, [\mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F}] \otimes$
0 for $j > 0$ and m sufficiently large.
By decreasing induction using the LES associated to $0 \to \mathscr{N}^{i+1}\mathscr{F} \to \mathscr{N}^i\mathscr{F} \to \mathscr{N}^i\mathscr{F}/\mathscr{N}^{i+1}\mathscr{F} \to 0$ we find that $H^j(X, \mathscr{N}^i\mathscr{F} \otimes \mathscr{L}^m) = 0$ for $j > 0$ and m sufficiently large.

3.5.17 c. x g

(c) Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

Let $X = X_1 \cup ... \cup X_r$ be all the irreducible components. Assume $\mathscr{L}|_{X_i}$ is ample for each i. If $\mathscr{F} \in \mathfrak{Coh}(X)$ and $\mathcal{I}$ is the ideal sheaf of X_1 in X, then we have an exact sequence
$0 \to \mathcal{I}\mathscr{F} \to \mathscr{F} \to \mathscr{F}/\mathcal{I}\mathscr{F} \to 0$.
The first term is supported on $X_2 \cup ... \cup X_r$ and the last term on X_1.
By induction on the number of irreducible components, then $H^j(X, \mathcal{I}\mathscr{F} \otimes \mathscr{L}^m) = H^j(X, (\mathscr{F}/\mathcal{I}\mathscr{F}) \otimes \mathscr{L}^m) = 0$ for $j > 0$ and $m \gg 0$.
By the exact sequence, $H^j(X, \mathscr{F} \otimes \mathscr{L}^{\otimes m}) = 0$ for $j > 0$ and $m \gg 0$.
The reverse direction is a consequence of (d).

3.5.18 d. x g finite pullbacks ampleness

(d) Let $f: X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X. [*Hints*: Use (5.3) and compare (Ex. 3.1, Ex. 3.2, Ex. 4.1, Ex. 4.2). See also Hartshorne [5, Ch. I §4] for more details.]

Suppose that $\mathscr{L}$ is ample on X. For $\mathscr{F} \in \mathfrak{Coh}(X)$, $R^jf_*(\mathscr{F} \otimes f^*\mathscr{L}^m) = 0$ for $j > 0$ by finiteness of f. Thus by exc III.8.1, $H^i(X, \mathscr{F} \otimes f^*\mathscr{L}^m) \approx H^i(Y, f_*\mathscr{F} \otimes \mathscr{L}^m) = 0$ by ampleness of $\mathscr{L}$.
On the other hand, suppose $f^*\mathscr{L}$ is ample on Y.
Then we can write $\int_W c_1(f^*\mathscr{L})^{dim\ W} = deg(W \to V) \cdot \int_V c_1(\mathscr{L})^{dim\ V}$ by the projection formula, so by Nakai-Moishezon we are done.

3.5.19 III.5.8.a x g

5.8. Prove that every one-dimensional proper scheme X over an algebraically closed field k is projective.
(a) If X is irreducible and nonsingular, then X is projective by (II, 6.7).

As X is proper, it's separated, thus complete. By nonsingularity, and thm II.6.7, then X is projective.

3.5.20 b. x

(b) If X is integral, let $\tilde{X}$ be its normalization (II, Ex. 3.8). Show that $\tilde{X}$ is complete and nonsingular, hence projective by (a). Let $f: \tilde{X} \to X$ be the projection. Let $\mathscr{L}$ be a very ample invertible sheaf on $\tilde{X}$. Show there is an effective divisor $D = \sum P_i$ on $\tilde{X}$ with $\mathscr{L}(D) \cong \mathscr{L}$, and such that $f(P_i)$ is a nonsingular point of X, for each i. Conclude that there is an invertible sheaf $\mathscr{L}_0$ on X with $f^*\mathscr{L}_0 \cong \mathscr{L}$. Then use (Ex. 5.7d), (II, 7.6) and (II, 5.16.1) to show that X is projective.

To see nonsingular, note that normal $\implies$ regular in codimension 1.

Completeness follows as in (a) Thus we have projective.

Assume $f : \tilde{X} \to X$ is the projection.

For the next part, suppose that $\mathscr{L}$ is very ample on $\tilde{X}$, then $\mathscr{L}$ gives an embedding $\mathscr{L} \hookrightarrow \mathbb{P}^n$ and so $\mathscr{L}$ looks like $\mathscr{L}(D)$ for an intersection of $\tilde{X}$ with a hyperplane in $\mathbb{P}^n$. By Bertini or some such we can choose this hyperplane missing a finite set of points, and the singular locus is a finite set of points. The existence of such a very ample $\mathscr{L}$ implies that X is projective since it gives an embedding into $\mathbb{P}^m$.

3.5.21 c. x g

(c) If X is reduced, but not necessarily irreducible, let $X_1, \ldots, X_r$ be the irreducible components of X. Use (Ex. 4.5) to show Pic $X \to \bigoplus$ Pic X_i is surjective. Then use (Ex. 5.7c) to show X is projective.

Consider two components. From a previous example $Pic(X) \approx H^1(X, \mathcal{O}_X^*)$. Taking cohomology og the s.e.s $1 \to \mathcal{O}_X^* \to \mathcal{O}_{X_1}^* \times \mathcal{O}_{X_2}^* \to \mathcal{O}_{X_1 \cap X_2}^* \to 1$ gives

$1 \to k^* \to \mathcal{O}^*(X_1) \times \mathcal{O}^*(X_2) \to^* \mathcal{O}(X_1 \cap X_2) \to Pic(X) \to Pic(X_1) \times Pic(X_2) \to 1$ since the skyscraper is supported at points so has $cd = 0$.

Surjectivity of the map $\mathcal{O}^*(X_1) \times \mathcal{O}^*(X_2) \to^* \mathcal{O}(X_1 \cap X_2)$ implies our desired surjectiity.

Now an induction argument will give for $r > 2$.

Now by (b) the X_i are projective so have very ample sheaves, and pulling back these very ample sheaves gives very ample by 5.7.c which gives a projective embedding.

3.5.22 d. x

(d) Finally, if X is any one-dimensional proper scheme over k, use (2.7) and (Ex. 4.6) to show that Pic $X \to$ Pic X_{red} is surjective. Then use (Ex. 5.7b) to show X is projective.

The second part is clear. For the first part, see Bosch, Neron Models, 9.2.

3.5.23 III.5.9 x g Nonprojective Scheme

5.9. *A Nonprojective Scheme.* We show the result of (Ex. 5.8) is false in dimension 2. Let k be an algebraically closed field of characteristic 0, and let $X = \mathbf{P}_k^2$. Let ω be the sheaf of differential 2-forms (II, §8). Define an infinitesimal extension X' of X by ω by giving the element $\xi \in H^1(X, \omega \otimes \mathscr{T})$ defined as follows (Ex. 4.10). Let x_0, x_1, x_2 be the homogeneous coordinates of X, let U_0, U_1, U_2 be the standard open covering, and let $\xi_{ij} = (x_j/x_i)d(x_i/x_j)$. This gives a Čech 1-cocycle with values in Ω_X^1, and since dim $X = 2$, we have $\omega \otimes \mathscr{T} \cong \Omega^1$ (II, Ex. 5.16b). Now use the exact sequence

$$\ldots \to H^1(X, \omega) \to \text{Pic } X' \to \text{Pic } X \xrightarrow{\delta} H^2(X, \omega) \to \ldots$$

of (Ex. 4.6) and show δ is injective. We have $\omega \cong \mathcal{O}_X(-3)$ by (II, 8.20.1), so $H^2(X, \omega) \cong k$. Since char $k = 0$, you need only show that $\delta(\mathcal{O}(1)) \neq 0$, which can be done by calculating in Čech cohomology. Since $H^1(X, \omega) = 0$, we see that Pic $X' = 0$. In particular, X' has no ample invertible sheaves, so it is not projective.

The only parts of the hint which is not clear is that $\delta(\mathcal{O}(1)) \neq 0$ which we need in order to show that δ is injective. The point is that cohomology of $X = \mathbb{P}^2$ is known (use duality since it's $\omega = \mathcal{O}_X(-3)$ to get $H^2(X, \mathcal{O}_X(-3)) \approx H^0(X, \mathcal{O}_X) \approx k$. Again by duality, $H^1(X, \omega) = 0$.

Thus we have $0 \to Pic\ X' \to Pic\ X \to \delta\ k \to ???$.

So if we show that δ is injective, then $Pic\ X' = 0$.

Since $\mathcal{O}(1)$ is a generator, then if it's nonzero, $\langle \mathcal{O}(1) \rangle$ is nonzero so we get injectivity.

Essentially we should compute cech cohomology of X w'r't' ω .

Our cech complex looks like $0 \to k[t_0, t_1, t_2]_{(t_0)}\, d\,()$

For a possibly easier example of a nonprojective surface see http://math.stanford.edu/~vakil/0506-216/216class4 for a much easier example of showing (ex 5.8) is false in dimension 2.

3.5.24 III.5.10 x g

5.10. Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \ldots \to \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0, such that for all $n \geqslant n_0$, the sequence of global sections

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \ldots \to \Gamma(X, \mathscr{F}^r(n))$$

is exact.

If $r = 3$, then we have a short eact sequence. Now the result follows from Serre vanishing applied to the LES.

By induction suppose the result holds for $r - 1$.

We can break $0 \to \mathscr{F}^1 \to \cdots \to \mathscr{F}^{n-2} \xrightarrow{f} \mathscr{F}^{n-1} \to \mathscr{F}^n \to 0$.

Thus we can split into two sequences:

$0 \to \mathscr{F}^1 \to \cdots \to \mathscr{F}^{n-2} \xrightarrow{f} coker\ f \to 0$ and $0 \to coker\ f \to \mathscr{F}^{n-1} \to \mathscr{F}^n \to 0$.

Now use induction hypothesis on both of these sequences. ..

3.6 III.6 x Ext Groups and Sheaves

3.6.1 III.6.1x

6.1. Let $(X, \mathscr{O}_X)$ be a ringed space, and let $\mathscr{F}', \mathscr{F}'' \in \mathfrak{Mod}(X)$. An *extension* of $\mathscr{F}''$ by $\mathscr{F}'$ is a short exact sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

in $\mathfrak{Mod}(X)$. Two extensions are *isomorphic* if there is an isomorphism of the short exact sequences, inducing the identity maps on $\mathscr{F}'$ and $\mathscr{F}''$. Given an extension as above consider the long exact sequence arising from $\operatorname{Hom}(\mathscr{F}'', \cdot)$, in particular the map

$$\delta : \operatorname{Hom}(\mathscr{F}'', \mathscr{F}'') \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}'),$$

and let $\xi \in \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ be $\delta(1_{\mathscr{F}''})$. Show that this process gives a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$, and elements of the group $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$. For more details, see, e.g., Hilton and Stammbach [1, Ch. III].

I solve this in SummerStudyChallenge2.pdf under the section "Ext" (it was a problem given in Dan's class).

3.6.2 III.6.2.a. x

6.2. Let $X = \mathbf{P}^1_k$, with k an infinite field.

(a) Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathscr{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathscr{O}_V \to k(x) \to 0$, where $x \in X$ is a closed point, V is an open neighborhood of x, and $\mathscr{O}_V = j_!(\mathscr{O}_X|_V)$, where $j: V \to X$ is the inclusion.]

Solution: Suppose to the contrary that $P \in \mathfrak{Mod}(X)$ is projective and there is a surjective map $P \to \mathcal{O}_X \to 0$.

First I recall some definitions. For any $x \in X$, $k(x)$ denotes the residue field $\mathcal{O}_{X,x}/m_x$. We can make a sheaf out of $k(x)$ using the skyscraper sheaf so

$$i_P(k(x))(U) = \begin{cases} k(x) & P \in U \\ 0 & P \notin U \end{cases}.$$

For $P = x$, I will abuse the notation and write $k(x)$ for the sheaf $i_x(k(x))$. Recall that $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, for $j : V \hookrightarrow X$ the inclusion of an open subset, is the sheaf defined by

$$\mathcal{O}_V(U) = \begin{cases} \mathcal{O}_X|_V(U) & U \subset V \\ 0 & U \not\subset V \end{cases}$$

where $O_X|_V$ is the sheaf $j^{-1}O_X = \underset{V \supset j(U)}{lim} O_X(V)$.

Now that we have made some definitions, the proof will proceed as follows: (1) use the assumed projectivity of P to create a diagram of sheaves involving an arbitrary open set U, (2) show that this diagram cannot commute at the level of the abelian group $P(U)$ whenever $P(U) \to k(x)(U)$ is nonzero. As $P \to O_X \to 0$ is a surjection, surely there must be some U with $P(U) \to k(x)(U)$ not the zero map. Thus we will get a contradiction to the assumption that $P \to \mathcal{O}_X \to 0$ is a surjection.

If $U \subset X$ is arbitrary and $x \in U$, then we can choose a distinguished open neighborhood $V \ni x$ properly contained in U. Clearly $k(x) \to 0$ makes a surjective sheaf morphism. Since $\mathcal{O}_{X|V}$ is affine, then $\mathcal{O}_{X|V}(U)$ on the induced topology for V is the set of functions $s : U \to \coprod_{p \in U} A_p$. Also, if nonzero, $k(x)(U)$ will be

$O_{X,x}/m_x \cong A_x/m_x$. So take the map defined by projection of s to A_p composed with the quotient map (or perhaps in the other order). This defines a morphism of sheafs since we have commutativity of

$$\begin{tikzcd} O_V(U) \arrow[r] \arrow[d] & k(x)(U) \arrow[d, "restriction"] \\ O_V(V') \arrow[r] & k(x)(V') \end{tikzcd} .$$

So now we have

$$\mathcal{O}_V \to k(x) \to 0$$

and

$$P \to \mathcal{O}_X \to 0$$

surjective (check the stalks). Also,

$$\mathcal{O}_X \to k(x) \to 0$$

and therefore

$$P \to k(x) \to 0$$

are surjective. We therefore have

$$\begin{tikzcd} & \mathcal{O}_V \arrow[d] & \\ P \arrow[ur] \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

commutative since P is projective. If we evaluate this diagram of sheaves at U, this is

$$\begin{tikzcd} & \mathcal{O}_V(U) = 0 \arrow[d] & \\ P(U) \arrow[ur] \arrow[r] & k(x)(U) = k(x) \arrow[r] & 0 \end{tikzcd}$$

so that whenever $P(U) \to k(x)(U)$ is nonzero, the diagram will not commute. Since U has been chosen arbitrarily, this contradicts the definition of $P \to k(x)$ being surjective since $k(x) \neq 0$.

3.6.3 b. x

(b) Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$. [*Hint*: Consider surjections of the form $\mathscr{L} \to \mathscr{L} \otimes k(x) \to 0$, where $x \in X$ is a closed point, and $\mathscr{L}$ is an invertible sheaf on X.]

Solution: Assume $P \xrightarrow{\varphi} \mathcal{O}_X \to 0$. By definition, the $ker(\varphi)$ is a subsheaf of P. Also, the kernel of a morphism of $\mathcal{O}_X$ modules is an $\mathcal{O}_X$-module by definition. Using 1.6, 1.7, and the definitions of a quotient of $\mathcal{O}_X$-modules, we see that

$$0 \to ker(\varphi) \to P \xrightarrow{\varphi} \mathcal{O}_X \to 0$$

is exact. In the case of $P \in Coh(X)$, we note that by II.5.7, $ker(\varphi)$ will be coherent. Also, in the case that $P \in Qcoh(X)$, then we can use II.5.15 to write P as an ascending union of coherent sheaves, $P = \bigcup_i P_i$ and get an exact sequence

$$0 \to ker(\varphi) \to P_i \to \mathcal{O}_X \to 0$$

for some i since $\mathcal{O}_X = \bigcup_i \varphi(P_i)$, which is an ascending union.

In any case, now we take the associated long exact sequence and get:

$$0 \to H^0(ker(\varphi)) \to H^0(P) \to H^0(\mathcal{O}_X) \to H^1(ker(\varphi)) \to \cdots .$$

Using III.5.2, since $ker(\varphi)$ will be coherent, we can find n such that $H^1(ker(\varphi)(n)) = H^1(P(n)) = 0$. Thus $H^0(P(n)) \to H^0(\mathcal{O}_X(n))$ is a surjection. Using III.2.2 and III.1.4, this implies that $\Gamma(P(n)) \to \Gamma(\mathcal{O}_X(n))$ is a surjection.

Recall that twisting a skyscraper sheaf gives an isomorphic skyscraper sheaf. Since $\mathcal{O}_X \to k(x) \to 0$ is exact (look at the stalks), we get a commutative diagram:

$$\begin{tikzcd} & \mathcal{O}_X(-n-1) \arrow[d] & \\ P \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

via the composition $P \to \mathcal{O}_X \to k(x)$. Using projectivity and twisting by n, since twisting is an exact function, we get the diagram:

$$\begin{tikzcd} & \mathcal{O}_X(-1) \arrow[d] & \\ P(n) \arrow[ur] \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

The global section x_0^n in $\Gamma[\mathcal{O}_X(n)]$ is nonzero at x, so since $\Gamma[P(n)] \to \Gamma[\mathcal{O}_X(n)]$ is surjective, we can find $s \in \Gamma[P(n)]$ which maps to $P(n)$. However there are no global sections in $\mathcal{O}_X(-1)$ so the diagram will not commute.

3.6.4 III.6.3.a. x

6.3. Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.
(a) If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geqslant 0$.

Since $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent iff for every open affine subset $U = Spec\ A$ $\mathscr{E}xt^i_X(\mathscr{F}, \mathscr{G})|_U = \mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U)$ by thm III.6.2, we assume $X = Spec\ A$ is affine. Then $\mathscr{F}$, $\mathscr{G}$ will correspond to finitely generate A-modules M, N. $\mathscr{E}xt^i_X\left(\tilde{M}, \tilde{N}\right) = Ext^i_A(M,N)\tilde{}$ by exc III.6.7. Thus $\mathscr{E}xt^i_X\left(\tilde{M}, \tilde{N}\right)$ is q.c. Since M is f.g. and A is noetherian, there is a resolution of M by finite rank free A-modules A^{n_i}. Then $Ext^i_A(M,N) = h^i(hom_A(A^{n_\bullet}, N))$ by definition of ext, which is $h^i(N^{n_\bullet})$. The N^{n_i} are f.g. since N is so that $h^i(N^{n_\bullet}) = Ext^i_A(M,N)$. Thus $\mathscr{E}xt^i_X\left(\tilde{M}, \tilde{N}\right) = Ext^i_A(M,N)^\sim$ is q.c.

3.6.5 b. x

(b) If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geqslant 0$.

by (a) proof.

3.6.6 III.6.4 x

6.4. Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. In this case we say $\mathfrak{Coh}(X)$ has *enough locally frees*. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$, show that the δ-functor $(\mathscr{E}xt^i(\cdot, \mathscr{G}))$, from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$, is a contravariant universal δ-functor. [*Hint*: Show $\mathscr{E}xt^i(\cdot, \mathscr{G})$ is coefface-able (§1) for $i > 0$.]

By thm III.1.3.A, we need to show $\mathscr{E}xt^i(\cdot\mathscr{G})$ is coeffaceable. Assume to make things easier that all $\mathscr{F} \in \mathfrak{Coh}(X)$ is the quotient of a locally free sheaf of finite rank, $\mathscr{L}$. The we can reduce to showing that $\mathscr{E}xt^i(\mathscr{L},\mathscr{G}) = 0$ for $\mathscr{L}$ locally free, finite rank. By thm III.6.2, we need only show $\mathscr{E}xt^i(\mathscr{L}|_{U_i},\mathscr{G}|_{U_i}) = 0$ for a collecton $U_i = Spec\ A_i$ covering with $\mathscr{L}|_{U_i} = \oplus_{j=1}^n \mathcal{O}_{U_i}$. For an injective resolution $0 \to \mathscr{G}|_{U_i} \to \mathscr{I}^\bullet$, we have $\mathscr{E}xt^i\left(\oplus_{j=1}^n \mathcal{O}_{U_i},\mathscr{G}|_{U_i}\right) = h^i\left(\oplus_{j=1}^n \mathscr{H}om(\mathcal{O}_{U_i},\mathscr{J}^\bullet)\right) =$
$\oplus_{j=1}^n h^i(\mathscr{H}om(\mathcal{O}_{U_i},\mathscr{I}^\bullet)) = \oplus_{j=1}^n h^i(\mathscr{I}^\bullet) = 0$, $i > 0$.

3.6.7 III.6.5 x

6.5. Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees (Ex. 6.4). Then for any coherent sheaf $\mathscr{F}$ we define the *homological dimension* of $\mathscr{F}$, denoted $\mathrm{hd}(\mathscr{F})$, to be the least length of a locally free resolution of $\mathscr{F}$ (or $+\infty$ if there is no finite one). Show:
(a) $\mathscr{F}$ is locally free $\Leftrightarrow \mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;

Suppose $\mathscr{F}$ is locally free finite rank. By thm III.6.5, $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for $i > 0$ and $\mathscr{G} \in \mathfrak{Mod}(X)$.

On the other hand, if $\mathscr{E}xt^1(\mathscr{F},\mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$, Thm III.6.8 gives $0 = \mathscr{E}xt^1(\mathscr{F},\mathscr{G})_x \approx Ext^1_{\mathcal{O}_x}(\mathscr{F}_x,\mathscr{G}_x)$ on the stalks. Thus $\mathscr{F}_x$ is projective and f.g. + projective gives free, so $\mathscr{F}_x$ is free $\implies \mathscr{F}$ is locally free by exc II.5.7.b.

3.6.8 b. x

(b) $\mathrm{hd}(\mathscr{F}) \leqslant n \Leftrightarrow \mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;

If $hd(\mathscr{F}) \leq n$ then there is a locally free resolution of $\mathscr{F}$ of length n. Using thm III.6.5 we can find $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$, $i > n$.

Suppose $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for $i > n$ and $\mathscr{G} \in \mathfrak{Mod}(X)$.

For $n = 0$, argue as in (a).

$\mathscr{F}$ is a quotient of a locally free sheaf: $0 \to \mathscr{H} \to \mathscr{E} \to \mathscr{F} \to 0$, $\mathscr{H}$ the kernel.

The LES $\cdots \to \mathscr{E}xt^n(\mathscr{E},\mathscr{G}) \to \mathscr{E}xt^n(\mathscr{H},\mathscr{G}) \to \mathscr{E}xt^{n+1}(\mathscr{F},\mathscr{G}) \to \cdots$ gives, using (a), that $\mathscr{E}xt^n(\mathscr{H},\mathscr{G}) \approx \mathscr{E}xt^{n+1}(\mathscr{F},\mathscr{G})$. Thus for $i > n-1$, $\mathscr{E}xt^i(\mathscr{H},\mathscr{G}) = 0$, $i > n-1$. The inductive hypothesis gives $hd\ \mathscr{H} \leq n-1$. Thus $\mathscr{H}$ has a locally free resolution of length $n-1$. Composing this with $\mathscr{H} \to \mathscr{E}$ gives the desired resolution of length n.

3.6.9 c. x

(c) $\mathrm{hd}(\mathscr{F}) = \sup_x \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$.

Let $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ be a locally free resolution of length n.

This gives a projective resolution on the stalks of length $\leq n$.

Thus $hd(\mathscr{F}) \geq sup_x pd_{\mathcal{O}_x}\mathscr{F}_x$.

If $hd(\mathscr{F}) > sup_x pd\mathcal{O}_x\mathscr{F}_x$, then by thm III.6.10A, $Ext^i(\mathscr{F}_x, N) = 0$ for all x, all $i \geq hd\mathscr{F}$, and all $\mathcal{O}_x$-modules $\mathscr{G}$.

Thus $\mathscr{E}xt^i(\mathscr{F},\mathscr{G}) = 0$ for $i \geq hd\mathscr{F}$ and $\mathcal{O}_X$-modules $\mathscr{G}$ which is a contradiction by (b).

3.6.10 III.6.6.a x

6.6. Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.

(a) M is projective if and only if $\operatorname{Ext}^i(M,A)=0$ for all $i>0$. [*Hint*: Use (6.11A) and descending induction on i to show that $\operatorname{Ext}^i(M,N)=0$ for all $i>0$ and all finitely generated A-modules N. Then show M is a direct summand of a free A-module (Matsumura [2, p. 129]).]

M projective implies $Ext^i(M,A)=0, i>0$ by properties of projective and $hom(-,A)$.

On the other hand, if $Ext^i(M,A)=0$ for $i>0$, and N is an f.g. A-module, then we have an exact sequence $0\to K\to A^n\to N\to 0$, and we get a LES in ext. Note that $Ext^i(M,A^n)=0$ by computing via induction on $0\to A^{n-1}\to A^n\to A\to 0$.

Note that $Ext^i(M,N)=0, i>dim\ A$ for all f.g. A-modules N by thm III.6.11.A. Using the isomorphisms $Ext^{i-1}(M,N)\to Ext^i(M,K)$ we see the same is true for $i-1$. By induction the same holds for $i\geq 1$.

Note that $Ext^1(M,K)=0$ so $Ext^0(M,A^n)\to Ext^0(M,M)$ is surjective. Thus $M\to M$ factors as $M\to A^n\to M$ so M is a direct summand of A^n so M is projective.

3.6.11 b. x

(b) Use (a) to show that for any n, pd $M\leqslant n$ if and only if $\operatorname{Ext}^i(M,A)=0$ for all $i>n$.

For $pd\ M\leq N$, and projective resolution of length $\leq n$ computes $Ext^i(M,A)$ which must therefore be zero for $i>n$.

Conversely, suppose $Ext^i(M,A)=0$, $i>n$. We will proceed by induction on n. If $n=0$, then $pd\ M\leq 0$ by (a).

Otherwise,assume $Ext^i(M,A)=0$ for $i>n-1$ implies $pd\ M\leq n-1$.

M is finitely generated so there is an exact sequence $0\to N\to A^k\to M\to 0$ some k which gives $\to Ext^{i-1}\left(A^k,A\right)\to Ext^{i-1}(N,A)\to Ext^i(M,A)\to\cdots$ in LES of ext.

A^k free implies $Ext^i\left(A^k,A\right)=0$, $i>0$ so $Ext^{i-1}(N,A)\approx Ext^i(M,A)$ for $i>1$.

Thus $Ext^i(N,A)=0$, $i>n-1$ therefore gives $pd\ N\leq n-1$ by induction.

Thus N has a projective resolution of length $n-1$ which gives by the S.e.S. above, a projective resolution of M of length n.

3.6.12 III.6.7 x

6.7. Let $X=\operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then

$$\operatorname{Ext}^i_X(\tilde{M},\tilde{N})\cong\operatorname{Ext}^i_A(M,N)$$

and

$$\mathscr{E}xt^i_X(\tilde{M},\tilde{N})\cong\operatorname{Ext}^i_A(M,N)^{\sim}.$$

Let A^{n_i} a finite free resolution of M. Then $Ext^i_A(M,N)$ are the left derived functors of $h^i\left(hom_A\left(A^{n\bullet},N\right)\right)$. $\sim$ is an exact equivalence so by III.1.4 is a universal δ-functor. Exactness of $hom_A(A^n,\cdot)$ gives this is also a universal δ-functor since A-mod has enough injectives and $hom_A(\cdot,I)$ is exact and thus $h^i\left(hom_A\left(A^{n\bullet},\cdot\right)\right)$ are effaceable. On 0-degree terms we have $Ext^0_X\left(\tilde{M},\tilde{N}\right)\approx hom_A(M,N)$ and $h^0\left(hom_A\left(A^{n\bullet},N\right)\right)\approx hom_A(M,N)$.

Using A^{n_i} gives a finite free resolution of $\tilde{M}$ so by thm III.6.5 we compute $\mathscr{E}xt$. M noetherian and $f.g$ gives $hom_A(M,N)^{\sim}\approx\mathscr{H}om\left(\tilde{M},\tilde{N}\right)$. Thus $\mathscr{E}xt^i\left(\tilde{M},\tilde{N}\right)\approx h^i\left(\mathscr{H}om\left(\tilde{A}^{n\bullet}\right)\right)\approx h^i\left(hom_A\left(A^{n\bullet},N\right)^{\sim}\right)\approx$

$h^i\left(hom_A\left(A^{n\bullet}, N\right)\right)^\sim \approx Ext^i_A\left(M, N\right)^\sim.$

3.6.13 III.6.8 x

6.8. Prove the following theorem of Kleiman (see Borelli [1]): if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).

(a) First show that open sets of the form X_s, for various $s \in \Gamma(X, \mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X. [*Hint*: Given a closed point $x \in X$ and an open neighborhood U of x, to show there is an $\mathscr{L}, s$ such that $x \in X_s \subseteq U$, first reduce to the case that $Z = X - U$ is irreducible. Then let ζ be the generic point of Z. Let $f \in K(X)$ be a rational function with $f \in \mathcal{O}_x$, $f \notin \mathcal{O}_\zeta$. Let $D = (f)_\infty$, and let $\mathscr{L} = \mathscr{L}(D)$, $s \in \Gamma(X, \mathscr{L}(D))$ correspond to D (II, §6).]

Let $x \in X$, and U an open neighborhood. If $Z = X - U$, $Z = Z_1 \cup \cdots \cup Z_n$ the irreducible components. We reduce to the case Z is irreducible since we can take the product of sections in each component . Thus we can assume Z corresponds to a prime weil divisor. By thm II.6.11 this gives a cartier divisor D given by $\{(U_i, f_i)\}$, $\frac{f_i}{f_j} \in \mathcal{O}^*_X(U)$, and $f_i \in \mathfrak{m}_{Z'}\mathcal{O}_{X,Z'}$ iff $Z' = Z$. By thm II.6.13, the $\frac{1}{f_i}$ generate an invertible sheaf $\mathscr{L}(D)$ and $f_i f_i^{-1} \in \Gamma(U_i, \mathscr{L}(D))$ glue to give $s \in \Gamma(X, \mathscr{L}(D))$ under the $f f_i^{-1} \leftrightarrow f$. As $s|_{U_i} \leftrightarrow f_i$ then $X_s = U$ so $x \in X_s \subset U$.

3.6.14 b. x

(b) Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.

If $\mathscr{F} \in \mathfrak{Coh}(X)$, then $U_i = Spec\ A_i$ covers X with $\mathscr{F}|_{U_i} \approx \tilde{M}_i$ for f.g. A_i-module M_i.

Thus $\mathscr{F}_{U_i}$ is generated by a finite number of $m_{ij} \in M_i = \Gamma(U_i, \mathscr{F}|_{U_i})$.

Now take a refinement of the cover U_i given by $X_{s_{ik}} \subset U_i$, $s_{ik} \in \Gamma(X, \mathscr{L}_{ik})$ for some $\mathscr{L}_{ij}$.

By thm II.5.14, $s_{ik}^{n_{ij}} m_{ij}$ extends to a global section of $\mathscr{L}^{n_{ik}} \otimes \mathscr{F}$.

The global section gives a morphism $\mathcal{O}_X \to \mathscr{L}^{n_{ik}} \otimes \mathscr{F}$, twisting gives a morphism to $\mathscr{F}$ and taking a direct sum of the morphisms gives a morphism to $\mathscr{F}$.

On $X_{s_{ij}}$, m_{ij} is in the image of $\mathscr{L}^{-n_{ij}} \to \mathscr{F}$ which gives surjectivity.

3.6.15 III.6.9 x

6.9. Let X be a noetherian, integral, separated, regular scheme. (We say a scheme is *regular* if all of its local rings are regular local rings.) Recall the definition of the *Grothendieck group* $K(X)$ from (II, Ex. 6.10). We define similarly another group $K_1(X)$ using locally free sheaves: it is the quotient of the free abelian group generated by all locally free (coherent) sheaves, by the subgroup generated by all expressions of the form $\mathscr{E} - \mathscr{E}' - \mathscr{E}''$, whenever $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ is a short exact sequence of locally free sheaves. Clearly there is a natural group homomorphism $\varepsilon: K_1(X) \to K(X)$. Show that ε is an isomorphism (Borel and Serre [1, §4]) as follows.

(a) Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}_. \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution

$$0 \to \mathscr{E}_n \to \ldots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0.$$

Exc III.6.8 gives coherent sheafs are quotients of locally free sheaves of finite rank. Thus $\mathfrak{Coh}(X)$ has enough locally free sheaves so by exc III.6.5.c, we have $hd\mathscr{F} = sup_x pd_{\mathcal{O}_x}\mathscr{F}_x$.

Regularity of X gives that $pd\ \mathscr{F}_x \leq dim\ \mathcal{O}_{X,x} \leq dim\ X$ via exc III.6.11.A .

Thus $hd\ \mathscr{F} = sup_x pd_{\mathcal{O}_x}\mathscr{F}_x \leq dim\ X$.

Thus there is a finite locally free resolution of $\mathscr{F}$.

3.6.16 b. x

(b) For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}_. \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum(-1)^i\gamma(\mathscr{E}_i)$ in $K_1(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K(X)$ to $K_1(X)$, and finally, that it is an inverse to ε.

This is given in Borel, Serre - Theoreme de Riemann-Roch.

3.6.17 III.6.10 x Duality for Finite Flat Morphism

6.10. *Duality for a Finite Flat Morphism.*

(a) Let $f: X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathscr{G}$ (II, Ex. 5.17e).

Let $\mathscr{F}$ represent $f_*\mathcal{O}_X \approx N^{\sim} \in \mathfrak{Coh}(Y)$ and $\mathscr{G} \approx M^{\sim} \in \mathfrak{Qco}(Y)$.

Write a presentation $\mathcal{O}_Y^m \to \mathcal{O}_Y^n \to \mathscr{F} \to 0$.

Applying $\mathscr{H}om_Y(-, \mathscr{G})$ gives $\mathscr{H}om_X(\mathscr{F}, \mathscr{G})$ as the kernel of a map $\mathscr{G}^n \to \mathscr{G}^m$ of quasi-coherent sheaves. (This functor is left exact)

$Hom_X(\mathscr{F}, \mathscr{G}) \approx \mathscr{H}om(N^{\sim}, M^{\sim}) \in \mathfrak{Qco}(Y)$.

Thus by thm II.5.5, $Hom_X(N^{\sim}, M^{\sim}) \approx Hom(N, M)^{\sim}$.

-

3.6.18 b. x

(b) Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G}).$$

We have a map $\alpha : f_*\mathscr{H}om_X\left(\mathscr{F}, f^!G\right) \to \mathscr{H}om_Y\left(f_*\mathscr{F}, f_*f^!\mathscr{G}\right)$ which is defined by $\phi \in Hom_{\mathcal{O}_X|f^{-1}U}\left(\mathscr{F}|_{f^{-1}(U)}, \ldots\right.$ maps to $\psi \in Hom_{\mathcal{O}_Y|_U}\left(f_*\mathscr{F}, f_*f^!\mathscr{G}\right)$, where $\psi_W : \mathscr{F}\left(f^{-1}W\right) \to f^!\mathscr{G}\left(f^{-1}W\right)$ is defined by $\phi_{f^{-1}W}$ for open $W \subset U$.

$\sim$ and f_* give an equivalence of categories, and thus $f_*f^!\mathscr{G} \approx \mathscr{H}om_Y\left(f_*\mathcal{O}_X, \mathscr{G}\right)$ so $\beta : \mathscr{H}om_Y\left(f_*\mathscr{F}, f_*f^!\mathscr{G}\right) \overset{\approx}{\to} \mathscr{H}om_Y\left(f_*\mathscr{F}, \mathscr{H}om_Y\left(f_*\mathcal{O}_X, \mathscr{G}\right)\right)$.

As $f_*\mathcal{O}_X$ is an $\mathcal{O}_Y$ -algebra, and we have an evaluation map $\mathscr{H}om_Y\left(f_*\mathcal{O}_X, \mathscr{G}\right) \to \mathscr{H}om_Y\left(\mathcal{O}_Y, \mathscr{G}\right) \approx \mathscr{G}$, and thus $\gamma : \mathscr{H}om\left(f_*\mathscr{F}, \mathscr{H}om_Y\left(f_*\mathcal{O}_X, \mathscr{G}\right)\right) \to \mathscr{H}om_Y\left(f_*\mathscr{F}, \mathscr{G}\right)$.

Now $\delta := \alpha \to \beta \to \gamma$ gives a natural morphism $f_*\mathscr{H}om_X\left(\mathscr{F}, f^!\mathscr{G}\right) \to \mathscr{H}om_Y\left(f_*\mathscr{F}, \mathscr{G}\right)$. Suppose we are mapping between open affines $Y = Spec\ A, X = Spec\ B$ so that $\mathscr{F} = \tilde{M}$ and $G = \tilde{N}$, where M, N are f.g. B, A modules respectively. Then δ maps $\phi \in Hom_B\left(M, Hom_A\left(B, N\right)\right) \to Hom_A\left(M \otimes_B B, N\right)$ by $\phi \mapsto \left(m \otimes 1 \mapsto \phi\left(m\right)\left(1\right)\right)$. By previous problem, this is an isomorphism. Thus δ is locally an isomorphism.

3.6.19 c. x

(c) For each $i \geqslant 0$, there is a natural map

$$\varphi_i : \mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G}).$$

[*Hint*: First construct a map

$$\mathrm{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \mathrm{Ext}^i_Y(f_*\mathscr{F}, f_*f^!\mathscr{G}).$$

Then compose with a suitable map from $f_*f^!\mathscr{G}$ to $\mathscr{G}$.]

By assumption $\mathscr{F} \in \mathfrak{Coh}\left(X\right)$, $\mathscr{G} \in \mathfrak{Qcoh}\left(Y\right)$.

The map from (b) given by $f_*\mathscr{H}om_X\left(\mathscr{F}, \mathscr{H}\right) \to \mathscr{H}om_Y\left(f_*\mathscr{F}, f_*\mathscr{H}\right)$ is natural in $\mathscr{H}$.

Therefore we have a natural transformation. Composing it with Γ gives

$Hom_X\left(\mathscr{F}, -\right) \to Hom_Y\left(f_*\mathscr{F}, f_*-\right) \qquad \star.$

Note that $Hom_Y\left(f_*\mathscr{F}, f_*-\right)$ is the pushforward f_*, composed with the 0-part of $Ext^i_Y\left(f_*\mathscr{F}, -\right)$ which is a universal δ -functor.

By properties / definition of universal δ fucntor, $Hom_Y\left(f_*\mathscr{F}, f_*-\right)$ is that same 0-part.

On the other hand, $Hom_X\left(\mathscr{F}, -\right)$ is the 0part of the derived functor which is the universal delta functor $Ext^i_X\left(\mathscr{F}, -\right)$.

Thus $\star$ gives a map of δ functors $Ext^i_X\left(\mathscr{F}, -\right) \to Ext^i_Y\left(f_*\mathscr{F}, f_*-\right)$ which is the one desired by the hint after plugging in $f^!\mathscr{G}$.

By the technique of (b), we have a natural map $f_*f^!\mathscr{G} \approx \mathscr{H}om\left(f_*\mathcal{O}_X, \mathscr{G}\right) \to \mathscr{G}$. By functoriality, this gives a natural map $Ext^i_Y\left(f_*\mathscr{F}, f_*f^!\mathscr{G}\right) \to Ext^i_Y\left(f_*\mathscr{F}, \mathscr{G}\right)$ and composing this with $\star$ gives the map we need.

3.6.20 d. x

(d) Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y (this is equivalent to saying f flat—see §9). Show that φ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y. [*Hints*: First do $i = 0$. Then do $\mathscr{F} = \mathcal{O}_X$, using (Ex. 4.1). Then do $\mathscr{F}$ locally free. Do the general case by induction on i, writing $\mathscr{F}$ as a quotient of a locally free sheaf.]

First assume $\mathscr{F} = \mathcal{O}_X$. Then by thm III.6.3.c, $Ext_X^i\left(\mathcal{O}_X, f^!\mathscr{G}\right) \approx H^i\left(X, f^!\mathscr{G}\right)$.

As f is affine, exc III.4.1 implies $H^i\left(X, f^!\mathscr{G}\right) \approx H^i\left(Y, f_*f^!\mathscr{G}\right) \approx H^i\left(Y, \mathscr{H}om\left(f_*\mathcal{O}_X, \mathscr{G}\right)\right)$. As $f_*\mathcal{O}_X$ is locally free, then by exc II.5.1.b, $H^i\left(Y, \mathscr{H}om_Y\left(f_*\mathcal{O}_X, \mathscr{G}\right)\right) \approx H^i\left(Y, \left(f_*\mathcal{O}_X\right)^\vee \otimes \mathscr{G}\right)$.

By thm III.6.3, thus $H^i\left(Y, \left(f_*\mathcal{O}_X\right)^\vee \otimes \mathscr{G}\right) \approx Ext_Y^i\left(f_*\mathcal{O}_X, \mathscr{G}\right)$. Composing these isomorphisms gives φ_i. WLOG we have shown for $\mathscr{F}$ locally free finite rank.

Note $\mathfrak{Coh}(X)$ has enough locally frees and thus there is a locally free sheaf $\mathscr{E}$ and an s.e.s. $0 \to \mathscr{R} \to \mathscr{E} \to \mathscr{F} \to 0$ for some kernel $\mathscr{R}$. The right adjoint of f_* is $f^!$ and so $0 \to f_*\mathscr{R} \to f_*\mathscr{E} \to f_*\mathscr{F} \to 0$ is exact by flatness of f_*. The maps $Hom_X\left(\mathscr{F}, f^!\mathscr{G}\right) \to Hom_Y\left(f_*\mathscr{F}, \mathscr{G}\right)$ give a morphism of the two LES in ext. By previous, the degree 0 maps are isomorphisms. Also since $\mathscr{E}$ is locally free, the map $Ext_X^1\left(\mathscr{E}, f^!\mathscr{G}\right) \to Ext_Y^1\left(f_*\mathscr{E}, f^!\mathscr{G}\right)$ is an isomorphism. By the 5-lemma, therefore, map $Ext_X^1\left(\mathscr{F}, f^!\mathscr{G}\right) \to Ext_Y^1\left(f_*\mathscr{F}, f^!\mathscr{G}\right)$ is an isomorphism, and by similar logic, $Ext_X^1\left(\mathscr{R}, f^!\mathscr{G}\right) \approx Ext_Y^1\left(f_*\mathscr{R}, f^!\mathscr{G}\right)$. We can repeat this argument in higher degrees.

3.7 III.7 x Serre Duality Theorem

3.7.1 III.7.1 x g Special Case Kodaira Vanishing

7.1. Let X be an integral projective scheme of dimension $\geqslant 1$ over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$. (This is an easy special case of Kodaira's vanishing theorem.)

Suppose to the contrary that $\mathscr{L}^\vee$ has a global section s.

Let Z be the vanishing set of s.

let C some curve intersecting Z.

Then by ampleness of $\mathscr{L}$, C intersects both $\mathscr{L}$ and $\mathscr{L}^\vee$ positively.

3.7.2 III.7.2 x

7.2. Let $f: X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.

(a) Show that $f^!\omega_Y^\circ$ is a dualizing sheaf for X, where $f^!$ is defined as in (Ex. 6.10).

By thm II.8.11, we have $f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$.

Ω_X is locally free rank n and Ω_Y is locally free rank n so $f^*\Omega_Y$ is locally free rank n since $f^*\mathcal{O}_Y \approx \mathcal{O}_X$. Since f is proper, $K(X)/K(Y)$ is a finite separable extension, which gives $\Omega_{K(X)/K(Y)} = 0$ (basically since a separable minimal polynomial has every element β a minimal poly P with$P(\beta) = 0$ but $dP(\beta) \neq 0$, but $dP(\beta) = (d\beta) \cdot P(\beta)$ by the product rule, which is 0 so $d\beta$ must be 0.). Thus $\Omega_{X/Y}$ is a torsion sheaf.

This gives (for some kernel) an exact sequence at any $P \in X$:

$0 \to \mathscr{K}_P \to \mathcal{O}_P^n \to \mathcal{O}_P^n \to \left(\Omega_{X/Y}\right)_P \to 0$.

Note $\mathscr{K}_P$ is torsion, since after tensoring with $K(X)$, we get

$dim_{K(X)}\mathscr{K}_P \otimes K(X) = -n + m + dim_{K(x)}\left(\Omega_{X/Y}\right)_P \otimes K(X) = 0.$

Since $\mathscr{K}_P \subset \mathcal{O}_P^n$, where $\mathcal{O}_P$ is a domain, then $\mathscr{K} = 0$. Thus we have $0 \to f^*\Omega_Y \to \Omega_X \to \Omega_{X/Y} \to 0$. Hence $\Omega_X^n \approx f^*\Omega_Y^n \otimes \mathscr{L}(R)$ $\quad\star$ where R is the ramification divisor $\Omega_{X/Y}$.

On the other hand, the trace map $t : f_*\mathcal{O}_X \to \mathcal{O}_Y$ gives $t' : \mathcal{O}_X \to f^!\mathcal{O}_{\mathcal{Y}}$ whose cokernel $\mathscr{F}$ fits in an exact sequence $0 \to \mathcal{O}_X \to f^!\mathcal{O}_Y \to \mathscr{F} \to 0$. Taking highest exterior powers and using that the sequence splits since $\mathscr{F}$ is invertible gives $f^!\mathcal{O}_Y \approx \mathscr{L}(R)$. Thus $f^!\mathcal{O}_Y \otimes f^*\Omega_Y^n \approx \mathscr{L}(R) \otimes f^*\Omega_Y^n$. This gives $f^!\Omega_Y^n \approx f^*\Omega_Y^n \otimes \mathscr{L}(R)$. Combining with $\star$ gives that $\omega_X \approx f^!\omega_Y$.

3.7.3 III.7.3 x Cohomology of differentials on Pn

7.3. Let $X = \mathbf{P}_k^n$. Show that $H^q(X, \Omega_X^p) = 0$ for $p \neq q$, k for $p = q$, $0 \leqslant p, q \leqslant n$.

Consider the filtration $\mathscr{F}^p$ for $\wedge^r\mathcal{O}(-1)^{n+1}$ from exc II.5.16(d) with $\mathscr{F}^p/\mathscr{F}^{p+1} \approx \Omega^p \otimes \wedge^{r-p}\mathcal{O}$. Note $\wedge^{r-p}\mathcal{O} \approx 0$, $r - p \neq 0, 1$ and $\wedge^{r-p}\mathcal{O} \approx \mathcal{O}$ for $r - p = 0, 1$ then $\mathscr{F}^p \approx \mathscr{F}^{p+1}$, $p \neq r, r-1$. Thus we have $\wedge^r\mathcal{O}(-1)^{n+1} \supset \mathscr{F}^r \supset \mathscr{F}^{r+1} = 0$. Then $\mathscr{F}^r/\mathscr{F}^{r+1} = \mathscr{F}^r$ is isomorphic to $\Omega^r \otimes \wedge^{r-r}\mathcal{O} \approx \Omega^r$. The quotient $\mathscr{F}^{r-1}/\mathscr{F}^r \approx \wedge^r\left(\mathcal{O}(-1)^{n+1}\right)/\Omega^r$ is $\Omega^{r-1} \otimes \wedge^{r-(r-1)}\mathcal{O} \approx \Omega^{r-1}$. Thus we have an exact sequence $0 \to \Omega^r \to \wedge\mathcal{O}(-1)^{n+1} \to \Omega^{r-1}$. Now $\wedge\mathscr{L}^{\oplus m} \approx (\mathscr{L}^{\otimes r})^{\oplus\binom{r}{m}}$ (one way of showing this is to take a trivializing cover, choose a local basis, and then look at the transition morphisms) and so our exact sequence is

$0 \to \Omega^r \to \mathcal{O}(-r)^{\oplus N} \to \Omega^{r-1} \to 0$ for suitable N that we don't care about. This gives rise to a long exact on cohomology. Since $H^i(X, \mathcal{O}(-r)) = 0$ for $i < n$ or $r < n+1$ (by thm III.5.1), we therefore have isomorphisms $H^i(X, \Omega^r) \approx H^{i-1}(X, \Omega^{r-1})$ for $1 \leq i$ if $r < n+1$. If $r \geq n+1$ then we still have isomorphisms but only for $1 \leq i < n$.

Now we know that $H^0(X, \Omega^0) \approx H^0(X, \mathcal{O}_X) \approx k$ (thm III.5.1) and so using these isomorphisms we see that $H^i(X, \Omega^i) \approx k$ for $0 \leq i \leq n$. Again, using thm III.5.1, we know the cohomology of $\Omega^n \approx \mathcal{O}(-n-1)$, and in particular, that $H^i(X, \Omega^n) \approx 0$ for $i < n$. Using our isomorphisms above, this tells us that $H^i(X, \Omega^r)$ in the region $i \leq r, 0 \leq r \leq n$. All that remains to show is the region $i > r$, $0 \leq i \leq n$ and this follows from Corollary III.7.13.

3.7.4 III.7.4 (starred)

***7.4.** *The Cohomology Class of a Subvariety.* Let X be a nonsingular projective variety of dimension n over an algebraically closed field k. Let Y be a nonsingular subvariety of codimension p (hence dimension $n - p$). From the natural map $\Omega_X \otimes \mathcal{O}_Y \to \Omega_Y$ of (II, 8.12) we deduce a map $\Omega_X^{n-p} \to \Omega_Y^{n-p}$. This induces a map on cohomology $H^{n-p}(X, \Omega_X^{n-p}) \to H^{n-p}(Y, \Omega_Y^{n-p})$. Now $\Omega_Y^{n-p} = \omega_Y$ is a dualizing sheaf for Y, so we have the trace map $t_Y : H^{n-p}(Y, \Omega_Y^{n-p}) \to k$. Composing, we obtain a linear map $H^{n-p}(X, \Omega_X^{n-p}) \to k$. By (7.13) this corresponds to an element $\eta(Y) \in H^p(X, \Omega_X^p)$, which we call the *cohomology class* of Y.

(a) If $P \in X$ is a closed point, show that $t_X(\eta(P)) = 1$, where $\eta(P) \in H^n(X, \Omega^n)$ and t_X is the trace map.

MISS

(b) If $X = \mathbf{P}^n$, identify $H^p(X, \Omega^p)$ with k by (Ex. 7.3), and show that $\eta(Y) = (\deg Y) \cdot 1$, where $\deg Y$ is its *degree* as a projective variety (I, §7). [*Hint*: Cut with a hyperplane $H \subseteq X$, and use Bertini's theorem (II, 8.18) to reduce to the case Y is a finite set of points.]

MISS

(c) For any scheme X of finite type over k, we define a homomorphism of sheaves of abelian groups $d\log: \mathcal{O}_X^* \to \Omega_X$ by $d\log(f) = f^{-1}df$. Here $\mathcal{O}^*$ is a group under multiplication, and Ω_X is a group under addition. This induces a map on cohomology $\operatorname{Pic} X = H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X)$ which we denote by c—see (Ex. 4.5).

MISS

(d) Returning to the hypotheses above, suppose $p = 1$. Show that $\eta(Y) = c(\mathscr{L}(Y))$, where $\mathscr{L}(Y)$ is the invertible sheaf corresponding to the divisor Y.

See Matsumura [1] for further discussion.

MISS

3.8 III.8 x Higher Direct Images of Sheaves

3.8.1 III.8.1 x g Leray Degenerate Case

8.1. Let $f: X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geqslant 0$,

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F}).$$

(This is a degenerate case of the Leray spectral sequence—see Godement [1, II, 4.17.1].)

Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of sheaves on X.
Then $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$ is an injective resolution on Y.
By hypothesis, $R^i f_*(\mathscr{F}) = 0$ so this second resolution is exact.
The cohomology of $\mathscr{F}$ is the cohomology of the complex $\Gamma(X, \mathscr{I}^\bullet)$ which is isomorphic to the complex $\Gamma(Y, f_*\mathscr{F}^\bullet)$, and thus the required isomorphism.

3.8.2 III.8.2 x g

8.2. Let $f: X \to Y$ be an affine morphism of schemes (II, Ex. 5.17) with X noetherian, and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypotheses of (Ex. 8.1) are satisfied, and hence that $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for each $i \geqslant 0$. (This gives another proof of (Ex. 4.1).)

Since $f^{-1}(affine)$ is affine, then, using III.8.1, $H^i\left(f^{-1}(U), \mathscr{F}|_{f^{-1}(U)}\right) = 0 \implies R^i f_*\mathscr{F} = 0$ for $i > 0$.

3.8.3 III.8.3 x g Projection Formula derived

8.3. Let $f : X \to Y$ be a morphism of ringed spaces, let $\mathscr{F}$ be an $\mathcal{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the *projection formula* (cf. (II, Ex. 5.1))

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}.$$

Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ an injective resolution of $\mathscr{F}$.
Exc II.5.1.d gives an isomorphism of chain complexes
$f_* (\mathscr{I}^\bullet \otimes f^*\mathscr{E}) \approx f_* (\mathscr{I}^\bullet) \otimes \mathscr{E} \qquad \star$.
As $f^*\mathscr{E}$ is locally free, $0 \to \mathscr{F} \otimes f^*\mathscr{E} \to \mathscr{I}^\bullet \otimes f^*\mathscr{E}$ gives an injective resolution by exc III.6.7.
Taking cohomology of LHS of $\star$ therefore gives $R^i f_* (\mathscr{F} \otimes f^*\mathscr{E})$.
Note that $R^i f_* (\mathscr{F})$ is the cokernel of $f_*\mathscr{I}^{i-1} \to ker\left(f_*\mathscr{I}^i \to f_*\mathscr{I}^{i+1}\right)$.
Tensoring by locally free $\mathscr{E}$ is exact so $R^i f_* (\mathscr{F}) \otimes \mathscr{E}$ are cohomology of RHS of $\star$.

3.8.4 III.8.4 x

8.4. Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geqslant 1$. Let $X = \mathbf{P}(\mathscr{E})$ (II, §7), with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.

(a) Then $\pi_*(\mathcal{O}(l)) \cong S^l(\mathscr{E})$ for $l \geqslant 0$, $\pi_*(\mathcal{O}(l)) = 0$ for $l < 0$ (II, 7.11); $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and all $l \in \mathbf{Z}$; and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.

For $U_i = Spec\ A_i$ a cover of X on which $\mathscr{E}$ is free, we have $\pi^{-1}(U_i) \approx \mathbb{P}^n_{A_i}$.
Hence $H^j\left(\pi^{-1}(U_i), \mathcal{O}(l)|_{\pi^{-1}(U_i)}\right) \approx H^j\left(\mathbb{P}^n_{A_i}, \mathcal{O}(l)|_{\pi^{-1}(U_i)}\right)$ which is 0 by the known cohomology of $\mathbb{P}^n$, in degrees between 1 and $n-1$.
Then $R^j\pi_*\mathcal{O}(l) = 0$, $0 < j < n$ by thm III.8.1.
Similarl reasoning gives, $R^n\pi_*\mathcal{O}(l) = 0$, $l > -n-1$.

3.8.5 b. x

(b) Show there is a natural exact sequence

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0,$$

cf. (II, 8.13), and conclude that the *relative canonical sheaf* $\omega_{X/Y} = \wedge^n\Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1}\mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$ (cf. (7.1.1)).

By thm II.7.11.b, we have a natural surjection $\pi^*\mathscr{E} \to \mathcal{O}_X(1)$. This gives an s.e.s. $0 \to \mathscr{F} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O}_X \to 0$ after twisting. If $U = spec\ A$ is an open affine subscheme of Y where $\mathscr{E}$ is isomorphic to $\mathcal{O}_Y^{n+1}$, then $\pi^{-1}U \approx \mathbb{P}^n_A$ and the restriction of this exact sequence is
$0 \to \mathscr{F}|_{\mathbb{P}^n_A} \to \mathcal{O}(-1)_X|_{\mathbb{P}^n_A} \to \mathcal{O}_X|_{\mathbb{P}^n_A} \to 0$ which is the exact sequence from thm II.8.13. Thus we have isomorphisms $\mathscr{F}|_{\mathbb{P}^n_A} \approx \Omega_{\mathbb{P}^n_A/U}$. These isomorphisms are compatible with restrictions to smaller affine subsets and so we obtain a global isomorphisms $\mathscr{F} \approx \Omega_{X/Y}$.

The isomorphism $\wedge^n\Omega_{X/Y} \approx (\pi^* \wedge^{n+1}\mathscr{E})(-n-1)$ results from exc II.5.16. If we then cover X with open subsets of the form $U_i = \mathbb{P}^n_{A_i}$, $Spec\ A_i$ are opens of Y on which $\mathscr{E} \approx \mathcal{O}_Y^{n+1}$ (and so $\pi^{-1}U \approx \mathbb{P}^n_A$), then restricting to these we get isomorphisms $\omega_{X/Y}|_{\pi^{-1}(U)} \approx \mathcal{O}_{\pi^{-1}(U)}(-n-1)$ via the isomorphisms just mentioned. Thus we have $R^n\pi_*\left(\omega_{X/Y}\right)|_{Spec\ A} \approx R^n\pi_*\left(\Omega_{X/Y}|_{\mathbb{P}^n_A}\right) \approx H^n\left(\mathbb{P}^n_A, \omega_{\mathbb{P}^n_A/A}\right)^\sim \approx A^\sim \approx \mathcal{O}_{Spec\ A}$ (by thm's III.8.2, III.8.5, and III.5.1.) Since these isomorphisms are all natural, we obtain the desired isomorphism $R^n\pi_*\left(\omega_{X/Y}\right) \approx \mathcal{O}_Y$.

3.8.6 c. x

(c) Now show, for any $l \in \mathbf{Z}$, that

$$R^n\pi_*(\mathcal{O}(l)) \cong \pi_*(\mathcal{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee .$$

The map $\pi^*\mathscr{E} \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}(1)$ gives $\pi^*\mathscr{E}(-1) \to \mathcal{O}_{\mathbb{P}(\mathscr{E})}$ which is surjective from a locally free rank $n+1$ sheaf. Thus we can extend to an exact koszul complex:

$0 \to \pi^*(\wedge^{n+1}\mathscr{E})(-n-1) \to \cdots \pi^*(\wedge^2\mathscr{E})(-2) \to \pi^*\mathscr{E}(-1) \to \mathcal{O}_{\mathbb{P}(\mathscr{E})} \to 0.$

There is a spectral sequence with E_2 page, $E_2^{p,q} = H^p(L^{\bullet,q})$ and $L^{-i-q} = R^q\pi_*(\pi^* \wedge^i \mathscr{E}(-i)) \approx \wedge^i\mathscr{E} \otimes_{\mathcal{O}_S} R^q\pi_*\left(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-i)\right)$ which converges to 0. As $\mathscr{E}$ is locally free, then locally $X \to Y$ is identified with $\mathbb{P}^n_Y \to Y$. By previous parts, and the known cohomology of $\mathbb{P}^n$ we find $L^{-i,q}$ is 0 except when $i = q = 0$ or $i = n+1$ and $q = n$. Thus $E_2^{p,q} = 0$ except when $(p,q) = (0,0)$ and $(p,q) = (-n-1,n)$. Thus $E_2^{0,0} = \pi_*\mathcal{O}_{\mathbb{P}(\mathscr{E})} \approx \mathcal{O}_Y$ by definition. and $E_2^{-n-1,n} \approx \wedge^{n+1}\mathscr{E} \otimes_{\mathcal{O}_Y} R^n\pi_*\left(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-n-1)\right)$. The lone nontrivial differential d_{n+1} gives an isomorphism $d_{n+1}^{0,0} : \mathcal{O}_Y \to \wedge^{n+1}\mathscr{E} \otimes R^n\pi_*\left(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-n-1)\right)$ which is an isomorphism by convergence of the sequence. Hence $\wedge^{n+1}\mathscr{E} = \left[R^n\pi_*\left(\mathcal{O}_{\mathbb{P}(\mathscr{E})}(-n-1)\right)\right]^{-1}$. In conjunction with (a), (b), , this gives $l \geq -n-1$.

For $l < -n-1$, consider the map $\pi^*\left(S^{-l+n+1}\mathscr{E}\right) \otimes_{\mathcal{O}_X} \mathcal{O}_X(l) \to \mathcal{O}_X(-n-1)$. The projection formula gives a map $S^{-l+n+1}(\mathscr{E}) \otimes_Y R^n\pi_*(\mathcal{O}_X(l)) \to R^n\pi_*\mathcal{O}_X(-n-1) = (\wedge^{n+1}\mathscr{E})^{-1}$. Combined with known cohomology and derived cohomology of $\mathbb{P}^n$, this gives a perfect pairing between $R^n\pi_*(\mathcal{O}_X(l))$ and $S^{-l+n+1}(\mathscr{E}) \otimes \wedge^{n+1}(\mathscr{E})$.

3.8.7 d. x

(d) Show that $p_a(X) = (-1)^n p_a(Y)$ (use (Ex. 8.1)) and $p_g(X) = 0$ (use (II, 8.11)).

First I attempt to show that $p_a(X) = (-1)^n p_a(Y)$.

So $p_a(Y) = (-1)^n(\chi\mathcal{O}_Y - 1) = (-1)^n(h^0(\mathcal{O}_Y) - h^1(\mathcal{O}_Y) + ... - h^n(\mathcal{O}_Y) - 1)$

$p_a(X) = (-1)^{2n}(h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + ... - h^{2n-1}(\mathcal{O}_X) - 1)$.

By exc 8.1 $p_a(X) = (-1)^{2n}(h^0(\mathcal{O}_Y) - h^1(\mathcal{O}_Y) + ... - h^{2n-1}(\mathcal{O}_Y) - 1)$

For dimension reasons

$p_a(X) = (-1)^{2n}(h^0(\mathcal{O}_Y) - h^1(\mathcal{O}_Y) + ... - h^n(\mathcal{O}_Y) - 1)$

Note that $(-1)^{2n}/(-1)^n = (-1)^n$

Thus $p_a(X) = (-1)^n p_a(Y)$.

Next I want to show that $p_g(X) = 0$. Recall that geometric genus of projective space is 0 (example III.8.20.1). Recall that geometric genus is defined as dimension of the global sections of ω_X . There is a canonical isomorphism $\omega_{X/K}|_U \approx \omega_{U/K}$ and since X is locally projective space, we can cover X by sets U_i such that $\omega_X|_U$ has no sections, thus we see that $dim\Gamma(X, \omega_X) = 0$.

3.8.8 e. x

(e) In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g$, $p_g = 0$, and irregularity g (7.12.3). This kind of surface is called a *geometrically ruled surface* (V, §2).

clear from part (c).

3.9 III.9 x Flat Morphisms

3.9.1 III.9.1 x

9.1. A flat morphism $f:X \to Y$ of finite type of noetherian schemes is open, i.e, for every open subset $U \subseteq X$, $f(U)$ is open in Y. [*Hint*: Show that $f(U)$ is constructible and stable under generization (II, Ex. 3.18) and (II, Ex. 3.19).]

The induced morphism $U \to Y$ is also finite type, so assume $U = X$.

Exc II.3.18 gives $f(X)$ is constructible, so closed under generization would imply open.

WTS if $y' \in Y$ is a generization of $y \in f(X)$ then we can find x' mapping to y' .

If $V = Spec\ B$ is an open neighborhood of y, then $V \ni y'$ and $f^{-1}SPec\ B \to Spec\ B$ is finite-type and flat.

If $x \mapsto y$, and $U = Spec\ A \ni y$, then A is a flat B-module by thm II.9.1.A(d).

Let $g : B \to A$ the induced ring homomorphism and using going-up theorem for g.

3.9.2 III.9.2 x twisted cubic

9.2. Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.

We can write the twisted cubic as $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$ which is projection from $(0, 0, 1, 0)$. This is the projection of the family (t^3, t^2u, atu^2, u^3) onto the $z \neq 0$ plane. The cusp at $(0, 0, 0, 1)$ is on $w \neq 0$ where X_a is given by $(x, y, z) = (t^3, t^2, at)$.

Eliminating t gives $k[a, x, y, z]/I$, $I = (y^3 - x^2, z^2 - a^2y, z^3 - a^3x, zy - ax, zx - ay^2)$.

At $a = 0$ we get $I_0 = (y^3 - x^2, z^2, zx, zy)$.

Thus the 0 fiber has suppose $y^3 - x^2$ in $Spec\ k[x, y]$.

For $\mathfrak{p}$ with $x \notin \mathfrak{p}$, then $z \in \mathfrak{p}$ since $xz = 0 \in \mathfrak{p}$.

Thus the local rings are reduced.

At $\mathfrak{p} = (x, y)$, $z \neq 0$ gives a nilpotent.

3.9.3 III.9.3 x g

9.3. Some examples of flatness and nonflatness.

(a) If $f:X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.

I want to show finite + surjective + nonsingular gives flat. By exc III.10.9 (we don't need this the current excercise to prove III.10.9) we get that a surjective morphism is flat iff the fibers have the same dimension. So I need to show the fibers have the same dimension. Note that a finite morphism is quasi-finite by exc II.3.5 so all the fibers are 0-dimensional. Thus f is flat.

3.9.4 b. x

(b) Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x,y]$ and $X = \operatorname{Spec} k[x,y,z,w]/(z,w) \cap (x + z, y + w)$.

If x is in the intersection of the two planes, the assumption is that f is flat, which gives $\mathcal{O}_{x,X}$ is a finite rank free $\mathcal{O}_{f(x),Y} - module$ by thm III.9.1.A(f).

Then $\mathcal{O}_{x,X}/\mathfrak{m}_{f(x),Y}\mathcal{O}_{x,X} \approx k$.

If $\mathcal{O}_{x,X}$ is finite rank free $\mathcal{O}_{f(x),Y}$-module, then by the isomorhpsim, it has rank one.

Thus $g : \mathcal{O}_{f(x),Y} \approx \mathcal{O}_{x,X}$ as $\mathcal{O}_{f(x),Y}$-modules.

Let $f = g(1)$ so that $z = hf$, $h \in \mathcal{O}_{f(x),Y}$.

But this contradicts hf .

3.9.5 c. x

(c) Again let $Y = \operatorname{Spec} k[x,y]$, but take $X = \operatorname{Spec} k[x,y,z,w]/(z^2,zw,w^2,xz-yw)$. Show that $X_{red} \cong Y$, X has no embedded points, but that f is not flat.

Note for affine scheme $X = Spec\ A$, $X_{red} = Spec\,(A_{red})$.

To see $X_{red} = Y$,

```
sage: P.<x,y,z,w> = QQ[]
sage: I = Ideal(z^2,z*w,w^2,x*z-y*w)
sage: I.radical()
Ideal (w, z) of Multivariate Polynomial Ring in x, y, z, w over Rational Field
```

So an embedded point is a nilpotent element at a singular point.

To find singular points

```
i1 : R = QQ[x,y,z,w];

i2 : I=ideal(z^2,z*w,w^2,x*z-y*w)

                2           2
o2 = ideal (z , z*w, w , x*z - y*w)

o2 : Ideal of R

i3 : jacobian I

o3 = {1} | 0  0 0  z  |
     {1} | 0  0 0  -w |
     {1} | 2z w 0  x  |
     {1} | 0  z 2w -y |

              4         4
o3 : Matrix R  <--- R
```

So clearly at (x, y, z, w) there's a singular point. There are no nilpotents in that local ring however. Another singular point is at (x, y, z)since there, the rank is less than 2, which is $n - dim\,(X)$. ($dim\,(X) = 2$ since nilpotents don't affect dimension). However, at this point, there are no nilpotents, since just y is there. All other points the jacobian has rank at least 2. Thus no embedded points. So we need to quotient $k\,[x, y, z, w]\,/\,(z^2, zw, w^2, xz - yw)$ by the nilradical.

So let f be the reduction map, $f : X \to X_{red}$ so it's not quite specified, but probably it's the reduction map $f : X \to X_{red}$. So it's clear that the dimension of the fibers changes. Note the fibers usually have dimension 2 when x, y not zero, when one of them is zero, dimension 1, (since reduction doesn't change dimension), and when both are 0 has dimension 0 . But this contradicts thm III.9.10.

3.9.6 III.9.4 x open nature of flatness

9.4. *Open Nature of Flatness.* Let $f: X \to Y$ be a morphism of finite type of noetherian schemes. Then $\{x \in X | f \text{ is flat at } x\}$ is an open subset of X (possibly empty)—see Grothendieck [EGA IV$_3$, 11.1.1].

This follows immediately from Matusumura thm 24.3 which states:

let A a noetherian ring, B an f.g. A-algebra, and M a finite B-module. Set $U = \{P \in Spec\ B | M_P \text{ is flat over } A\}$; then U is open in $Spec\ B$.

3.9.7 III.9.5 x Very Flat Families

9.5. *Very Flat Families.* For any closed subscheme $X \subseteq \mathbf{P}^n$, we denote by $C(X) \subseteq \mathbf{P}^{n+1}$ the projective cone over X (I, Ex. 2.10). If $I \subseteq k[x_0, \ldots, x_n]$ is the (largest) homogeneous ideal of X, then $C(X)$ is defined by the ideal generated by I in $k[x_0, \ldots, x_{n+1}]$.

(a) Give an example to show that if $\{X_t\}$ is a flat family of closed subschemes of $\mathbf{P}^n$, then $\{C(X_t)\}$ need not be a flat family in $\mathbf{P}^{n+1}$.

Consider the flat family X_t defined by $(1:0:0)$, $(0:1:0)$, and $(1:1:t)$ in $\mathbb{P}^2$.

These points are only on a line together at $t = 0$.

For $t \neq 0$, then $I_{X_t} = \langle xz - txy, yz - txy, z^2 - t^2xy \rangle$.

If Y is the closure of I_{X_t} in $\mathbb{A}^3 \times \mathbb{A}^1$, then $I_Y = \langle xz - txy, yz - txy, z^2 - t^2xy, x^2y - xy^2 \rangle$.

Y is the closure of a flat family over a smooth by one dimensional base so it is flat.

Note that $\{C(X_t)\}$ is the fiber $\{Y_t\}$.

On the other hand, for $t = 0$, X_0 lies on $z = 0$ so $I_{C(X_0)}$ has $z = 0$.

However, Y_0 has no such linear terms.

3.9.8 b. x

(b) To remedy this situation, we make the following definition. Let $X \subseteq \mathbf{P}^n_T$ be a closed subscheme, where T is a noetherian integral scheme. For each $t \in T$ let $I_t \subseteq S_t = k(t)[x_0, \ldots, x_n]$ be the homogeneous ideal of X_t in $\mathbf{P}^n_{k(t)}$. We say that the family $\{X_t\}$ is *very flat* if for all $d \geqslant 0$,

$$\dim_{k(t)}(S_t/I_t)_d$$

is independent of t. Here $(\ \ )_d$ means the homogeneous part of degree d.

So basically all we need to do is compute relate the hilbert polynomial of $X_{(t)}$ with $dim_k(S_l/I_l)$. so hilbert polynomial gives $dim_k(S/I)_l$. Since grading commutes with the quotient in this case, we are done.

3.9.9 c. x

(c) If $\{X_t\}$ is a very flat family in $\mathbf{P}^n$, show that it is flat. Show also that $\{C(X_t)\}$ is a very flat family in $\mathbf{P}^{n+1}$, and hence flat.

This should be clear from (a), and from (b) since the problems cooked up in (b) cannot occur in the case that the dimensions of the graded parts are always constant over T. Also just recall the definition of the hilbert polynomial, and that the hilbert polynomial constant gives flatness.

3.9.10 d. x

(d) If $\{X_{(t)}\}$ is an algebraic family of projectively normal varieties in $\mathbf{P}^n_k$, parametrized by a nonsingular curve T over an algebraically closed field k, then $\{X_{(t)}\}$ is a very flat family of schemes.

By thm 9.11, we already get a flat family. The only difference here is we are assuming projectively normal varieties instead of just normal. We know the hilbert polynomials are the same and we know by projectively normal that the higher parts of the hilbert polynomial are all 0 or equivalent to the one from projective space.

3.9.11 III.9.6 x

9.6. Let $Y \subseteq \mathbf{P}^n$ be a nonsingular variety of dimension $\geqslant 2$ over an algebraically closed field k. Suppose $\mathbf{P}^{n-1}$ is a hyperplane in $\mathbf{P}^n$ which does not contain Y, and such that the scheme $Y' = Y \cap \mathbf{P}^{n-1}$ is also nonsingular. Prove that Y is a complete intersection in $\mathbf{P}^n$ if and only if Y' is a complete intersection in $\mathbf{P}^{n-1}$. [*Hint*: See (II, Ex. 8.4) and use (9.12) applied to the affine cones over Y and Y'.]

This is Proposition 5.2.2.5 in Migliore, pp 129 Intro to Liason theory and deficiency modules.

3.9.12 III.9.7 x

9.7. Let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/t^2$ be the ring of dual numbers, and define an *infinitesimal deformation* of Y *as a closed subscheme* of X, to be a closed subscheme $Y' \subseteq X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathcal{N}_{Y/X})$, where

$$\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

First we do the affine case. Consider $I \subset A$, $I' \subset A[t]$. Suppose $Spec\ A[t]/I'$ is an infinitseimal deformation of $Spec\ A/I$ in $Spec\ A$. Then $t^2 \in I'$ since $A[t]/I'$ is a D-algebra. Furthermore, the image of I' is I since the composition $(Spec\ A[t]/I') \otimes_D k \to A/I$ is an isomorphism. Finally, the kernel of the composite morphism $A \to A[t]/I' \xrightarrow{t} A[t]/I'$ is contained in I', by the criterion of thm III.9.1.a - note that every element of $A[t]/I' \otimes_D (t)$ is $a \otimes t$. The converse of each of these facts also clearly holds. Thus $Spec\ A[t]/I'$ is an infinitesimal deformation of $Spec\ A/I$ in $Spec\ A$ iff (a) $t^2 \in I'$ (b) under the map, $A[t] \to A$ sending t to 0 , the image of I' is I, and (c) the kernel of the composite morphism $A \to A[t]/I' \xrightarrow{t} A[t]/I'$ is contained in I' .

Let A be a ring , $I \subset A$ an ideal, and $\phi \in hom_A(I/I^2, A/I)$. Define $I' \subset A[t]$ by the set of polynomials $a_0 + a_1 t + \cdots + a_n t^n \in A[t]$ such that $a_0 \in I$ and $\phi(a_0) = a_1$ or 0 in A/I. Then (a), (b), (c) give an infinitesimal deformation of $Spec\ A/I$ in $Spec\ A$.

On the other hand, if we have an infinitesimal deformation of $Spec\ A/I$ in $Spec\ A$, then we can define a morphism $\phi \in hom_{A/I}(I/I^2, A/I)$. Given $a \in I$, by condition (b), the set of elements $a + bt \in I'$ is nonempty. Define $\phi : I/I^2 \to A/I$ by $\phi(a) = b$. This choice of ϕ is well defined by (c) and (b). Note that ϕ is A/I linear since for $(ax + by) + zt$, $x + x't$ and $y + y't \in I'$, then $ax + ax't$ and $by + by't$ are in I' and thus $(ax + by) + zt - (ax + ax't) - (by + by't) = (z - ax' + by')t \in I'$ so $z - ax' - by' \in I$ by (b), (c).

Thus we have an isomorphism between $hom_{A/I}(I/I^2, A/I)$ and the set of infinitesimal deformations of $Spec(A/I)/Spec\ A$. Note that for ideals $I \subset A$ and $J \subset B$, and $\psi : A \to B$ with $\psi^{-1}J \subset I$, we have a commutative square

$$\begin{array}{ccc} hom_{A/I}\left(I/I^2, A/I\right) & \xrightarrow{\sim} & [Spec\,(A/I)\,/Spec\ A]^\star \\ \downarrow & & \downarrow \\ hom_{B/J}\left(J/J^2, B/J\right) & \xrightarrow{\sim} & [Spec\,(B/J)\,/Spec\ B]^\star \end{array}$$, $\star$ meaning the infinitesimal deformations of that space.

In the general space we therefore have nice restrictions and glueing works.

III.9.8 (starred)

***9.8.** Let A be a finitely generated k-algebra. Write A as a quotient of a polynomial ring P over k, and let J be the kernel:

$$0 \to J \to P \to A \to 0.$$

Consider the exact sequence of (II, 8.4A)

$$J/J^2 \to \Omega_{P/k} \otimes_P A \to \Omega_{A/k} \to 0.$$

Apply the functor $\mathrm{Hom}_A(\cdot, A)$, and let $T^1(A)$ be the cokernel:

$$\mathrm{Hom}_A(\Omega_{P/k} \otimes A, A) \to \mathrm{Hom}_A(J/J^2, A) \to T^1(A) \to 0.$$

Now use the construction of (II, Ex. 8.6) to show that $T^1(A)$ classifies infinitesimal deformations of A, i.e., algebras A' flat over $D = k[t]/t^2$, with $A' \otimes_D k \cong A$. It follows that $T^1(A)$ is independent of the given representation of A as a quotient of a polynomial ring P. (For more details, see Lichtenbaum and Schlessinger [1].)

MISS

3.9.13 III.9.9 x rigid example

9.9. A k-algebra A is said to be *rigid* if it has no infinitesimal deformations, or equivalently, by (Ex. 9.8) if $T^1(A) = 0$. Let $A = k[x,y,z,w]/(x,y) \cap (z,w)$, and show that A is rigid. This corresponds to two planes in $\mathbf{A}^4$ which meet at a point.

By previous excercise we want a surjective morphism $hom_A\left(\Omega_{P/k} \otimes A, A\right) \to hom_A\left(J/J^2, A\right)$, $P = k\left[x, y, z, w\right]$, and $A = P/J$. Now $\Omega_{P/k}$ and $\Omega_{P/k} \otimes A$ are generated by dx, dy, dz, dw and their images respectively, and $hom_A\left(\Omega_{P/k} \otimes A, A\right)$ is generated by the duals dx^*, dy^*, dz^*, dw^* . J is generated by xz, xw, yz, yw as a P-module, and these elements give generators of J/J^2 as an A-module. Thus $\phi \in hom_A\left(J/J^2, A\right)$ is determined by the value on on xz, xw, yz, yw so we can define $\psi \in hom_A\left(J/J^2, A\right)$ by giving the value on xz, xw, yz, yw .

Now $J/J^2 \to \Omega_{P/k} \otimes A$ sends f to $df \otimes 1$ and any morphism $hom_A\left(\Omega_{P/k} \otimes A, A\right) \to hom_A\left(J/J^2, A\right)$ is a linear transformation defined by where it sends generators. Note that $d\left(dx\right) = zdx + xdz$, and we can find all the other images similar using the liebniz rule. If $\gamma \in hom_A\left(\Omega_{P/k} \otimes A, A\right)$ sends dx to 1 and other generators to zero, then γ is mapped to $(z, w, 0, 0)$ by the linear transformation. If γ' sends dy to 1 and all other generators to 0, then γ' is mapped to $(0, 0, z, w)$. In a similar manner, we can determine the image of other generating morphisms, and so the linear transformation is given by

$$\begin{pmatrix} z & w & 0 & 0 \\ 0 & 0 & z & w \\ x & 0 & y & 0 \\ 0 & x & 0 & y \end{pmatrix}.$$

Consider $(b_1, b_2, b_3, b_4) \in hom_A\left(J/J^2\right) \subset A^4$ where b_1 is the image of xz , b_2 image of xw , b_3 image of yz, and b_4 image of yw . Note that xz, xw, yz, yw are zero in A so multiplying by x or y kills terms with z

or w but sends x^iy^j to $x^{i+1}y^j$ or x^iy^{j+1} respectively. Thus $b_1 = \frac{x}{y}b_3 + b_1'$ where $b_1' \in (z,w)\,k[z,w]$. In total, $b_1 = \frac{z}{w}b_2 + b_1''$ with $b_1'' \in (x,y)\,k[x,y]$. Thus $b_1 = \frac{z}{w}b_2 + \frac{x}{y}b_3$. Similarly, $b_2 = \frac{x}{y}b_4 + \frac{w}{z}b_1$, $b_3 = \frac{y}{x}b_1 + \frac{z}{w}b_4$, $b_4 = \frac{y}{x}b_2 + \frac{w}{z}b_3$ so that (b_1, b_2, b_3, b_4) is in the image of $hom_A\left(\Omega_{P/k} \otimes A, A\right) \to hom_A\left(J/J^2, A\right)$.

3.9.14 III.9.10 x g

9.10. A scheme X_0 over a field k is *rigid* if it has no infinitesimal deformations.
(a) Show that $\mathbf{P}_k^1$ is rigid, using (9.13.2).

Infinitesimal deformations correspond to $H^1(X, \mathscr{T}_X)$.

Now using the fact that $K_{\mathbb{P}^1} = -2H \implies \mathscr{T}_X = \mathcal{O}(2)$ and computing cohomology using the euler exact sequence gives the result.

3.9.15 b. x

(b) One might think that if X_0 is rigid over k, then every global deformation of X_0 is locally trivial. Show that this is not so, by constructing a proper, flat morphism $f: X \to \mathbf{A}^2$ over k algebraically closed, such that $X_0 \cong \mathbf{P}_k^1$, but there is no open neighborhood U of 0 in $\mathbf{A}^2$ for which $f^{-1}(U) \cong U \times \mathbf{P}^1$.

Use III.9.9 and just write our family with a characteristic function. $\delta_{0,0}(a,b)\cdot[x^2+yz]+(1-\delta_{0,0}(a,b))\cdot[z^2]$ where $\delta_{0,0}(a,b)$ is the characteristic function of $(0,0)$. So in particular, we get a nonsingular conic at $0,0$ and everywhere else a singular conic. But does that even give a morphism? so suppose we take the closed set $x=0$ in $\mathbb{A}^2$. Then pulling back gives $V((1-\delta_{0,0}(a,b))\cdot[z^2]) \cup V(\delta_{0,0}(a,b)\cdot[x^2+yz])$ union of closeds... hmm it seems to make sense. Ok I'm going with it.

---

*(c) Show, however, that one can trivialize a global deformation of $\mathbf{P}^1$ after a flat base extension, in the following sense: let $f: X \to T$ be a flat projective morphism, where T is a nonsingular curve over k algebraically closed. Assume there is a closed point $t \in T$ such that $X_t \cong \mathbf{P}_k^1$. Then there exists a nonsingular curve T', and a flat morphism $g: T' \to T$, whose image contains t, such that if $X' = X \times_T T'$ is the base extension, then the new family $f': X' \to T'$ is isomorphic to $\mathbf{P}_{T'}^1 \to T'$.

MISS

3.9.16 III.9.11 x interesting g

9.11. Let Y be a nonsingular curve of degree d in $\mathbf{P}_k^n$, over an algebraically closed field k. Show that

$$0 \leqslant p_a(Y) \leqslant \tfrac{1}{2}(d-1)(d-2).$$

[*Hint*: Compare Y to a suitable projection of Y into $\mathbf{P}^2$, as in (9.8.3) and (9.8.4).]

For $\mathbb{P}^2$ it's clear. else embed into $\mathbb{P}^3$, then put into $\mathbb{P}^2$ using IV.3.10. Now $p_a(Y) = \frac{1}{2}(d-1)(d-2) - nodes$ and use the fact that genus is birational invariant.

3.10 III.10 x Smooth Morphisms

3.10.1 III.10.1 x regular != smooth always

10.1. Over a nonperfect field, smooth and regular are not equivalent. For example, let k_0 be a field of characteristic $p > 0$, let $k = k_0(t)$, and let $X \subseteq \mathbf{A}_k^2$ be the curve defined by $y^2 = x^p - t$. Show that every local ring of X is a regular local ring, but X is not smooth over k.

Let $X = Spec\ R$, $R = k[x,y]/(f)$, $f = y^2 - x^p + t$. f defines an irreducible polynomial so X is irreducible and dimension 1 over k. Clearly $E = K[x]/(x^p - t)$ gives a field extension of degree p which is inseparable since any k, x, y is a $p - th$ root of some element of E. Note that for τ a p^{th} root of t, then $\mathfrak{p} = (x - \tau, y)$ is not regular over $\overline{k}$ as $dim\ \mathfrak{p}/\mathfrak{p}^2 = 2 > dim\ R \otimes_K \overline{K} = 1$. However for $p > 0$ away from $\mathfrak{p}$ computing the jacobian shows X/K is smooth. Now use thm 10.2.

3.10.2 III.10.2 x g

10.2. Let $f: X \to Y$ be a proper, flat morphism of varieties over k. Suppose for some point $y \in Y$ that the fibre X_y is smooth over $k(y)$. Then show that there is an open neighborhood U of y in Y such that $f: f^{-1}(U) \to U$ is smooth.

This is local so we can assume X, Y is affine. Note the map on tangent spaces. Suppose f is smooth at x. In this case, the sequence given by II.8.12, is exact on the left at x. I.e. we have

$0 \to \mathscr{I}_x/\mathscr{I}_x^2 \to j^*\Omega_{Y/k,x} \to \Omega^1_{X/k,x} \to 0.$

There are functions $g_1, ..., g_n$ around x with dg_i forming a basis around $\Omega^1_{X/k,x}$. The g_i define $g : U \to \mathbb{A}^n_k$ for an open set containing x which is etale since the dg_i are linearly independent. Thus we f factors as an etale map together with projection to Y.

3.10.3 III.10.3 x

10.3. A morphism $f: X \to Y$ of schemes of finite type over k is *étale* if it is smooth of relative dimension 0. It is *unramified* if for every $x \in X$, letting $y = f(x)$, we have $\mathfrak{m}_y \cdot \mathcal{O}_x = \mathfrak{m}_x$, and $k(x)$ is a separable algebraic extension of $k(y)$. Show that the following conditions are equivalent:

(i) f is étale;
(ii) f is flat, and $\Omega_{X/Y} = 0$;
(iii) f is flat and unramified.

Clearly flat and unramified is the same as smooth of relative dimension 0.

Suppose f is flat and unramified. Then $\Omega_{X/Y} \otimes k(x) = 0$ so by Nakayama, $\Omega_{X/Y}$ is 0 at any stalk. Thus (iii) implies (ii)

Now suppose (ii). Consider $f^*\Omega_{Y/k} \to \Omega_{X/k} \to \Omega_{X/Y} \to 0$. At x this is

$\left(\mathfrak{m}_{f(x)}/\mathfrak{m}^2_{f(x)}\right) \otimes_{k(y)} k(x) \to \mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega_{X/Y,x} \otimes k(x) \to 0$ where the last term is 0. Thus the middle is surjective and nakayama gives $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$. Since we are looking locally, consider a homomorphism $A \to B$ of f.g. k-algebrais with $\Omega_{B/A} = 0$. If $\mathfrak{p} \subset A$ is prime, this gives a point in Y. For $S = B \otimes_A k(\mathfrak{p})$, then $\Omega_{S/k(\mathfrak{p})} = 0$ by base extension. The primes of S are the preimages of $\mathfrak{p}$ by $Spec(B) \to Spec(A)$. If $\mathfrak{q}$ is in the preimage then $\Omega_{(S/\mathfrak{q})/k(\mathfrak{p})} = 0$. By the logic used in advanced conditions for a closed immersion, $k(\mathfrak{p}) \subset (S/\mathfrak{q})$ is of transcendence degree 0 and separably generated, and because it is finitely generated it must be finite separable. Thus it is unramified, hence etale. Thus (ii) implies (i).

Now suppose (i). By the previous paragraph, unramified $\implies \Omega_{X/Y}=0$. Thus we consider preimages. By base change we assume Y is integral. Let $X_1\cup\cdots\cup X_n=X$ the irreducible decomposition of X. If $U_i=X\backslash\left(X_1\cup\cdots\cup\hat{X}_i\cup...\cup X_n\right)$, then $dim\left(U_i\right)=dim\left(X_i\right)$ and wlog we assume $U_i\approx Spec\ (B)$, and $f\left(U_i\right)\subset V=Spec\left(A\right)$. This gives a flat extension $i:A\to B$ of f.g. k-algebras with $\Omega_{B/A}=0$. Then i is injective since for s mapping to 0, $A\overset{\times s}{\to}A$ is 0 after tensoring by $-\otimes B$ contradicting flatness. If $\mathfrak{n}$ is the nilradical of B,then $A\to B/\mathfrak{n}$ is injective, $B/\mathfrak{n}$ is a domain and thus as in the previous paragraph, with $\mathfrak{p}=(0)\subset A$ and $\mathfrak{q}=(0)\subset B/\mathfrak{n}$, we find $K\left(S\right)/K\left(A\right)$ is a finite extension which gives $dim\left(B/\mathfrak{n}\right)=dim\left(A\right)$. Thus (i) $\implies$ (iii)

3.10.4 III.10.4 x

10.4. Show that a morphism $f:X\to Y$ of schemes of finite type over k is étale if and only if the following condition is satisfied: for each $x\in X$, let $y=f(x)$. Let $\hat{\mathcal{O}}_x$ and $\hat{\mathcal{O}}_y$ be the completions of the local rings at x and y. Choose fields of representatives (II, 8.25A) $k(x)\subseteq\hat{\mathcal{O}}_x$ and $k(y)\subseteq\hat{\mathcal{O}}_y$ so that $k(y)\subseteq k(x)$ via the natural map $\hat{\mathcal{O}}_y\to\hat{\mathcal{O}}_x$. Then our condition is that for every $x\in X$, $k(x)$ is a separable algebraic extension of $k(y)$, and the natural map

$$\hat{\mathcal{O}}_y\otimes_{k(y)}k(x)\to\hat{\mathcal{O}}_x$$

is an isomorphism.

Assume it's etale. We are looking locally so we can assume $X=Spec\ A$, $Y=Spec\ B$. From exc III.10.3 get that it's flat and unramified. Thus using Matsumura, pp 74, the induced maps $\mathfrak{m}_A^n/\mathfrak{m}_A^{n+1}\to\mathfrak{m}_B^n/\mathfrak{m}_B^{n+1}$ are isomorphisms. The reverse is in atiyah macdonals. Thus at the limit, the maps on completions are isomorpisms. The reverse is in Atiyah Macdonald. Alternatively, use Liu prop 3.26.

3.10.5 III.10.5 x etale neighborhood x

10.5. If x is a point of a scheme X, we define an *étale neighborhood* of x to be an étale morphism $f:U\to X$, together with a point $x'\in U$ such that $f(x')=x$. As an example of the use of étale neighborhoods, prove the following: if $\mathscr{F}$ is a coherent sheaf on X, and if every point of X has an étale neighborhood $f:U\to X$ for which $f^*\mathscr{F}$ is a free $\mathcal{O}_U$-module, then $\mathscr{F}$ is locally free on X.

For $x\in X$, $f\left(x'\right)=x$, let $r=dim_{k(x)}\mathscr{F}_x\otimes k\left(x\right)$ which is $rk\ f_*\mathscr{F}$ at $x'\in U$.
After possibly shrinking U, we get a an exact sequenc $0\to kernel\to\mathcal{O}_X^r\to\mathscr{F}\to 0$.
f flat implies $0\to f^*kernel\to\mathcal{O}_U^r\to f^*\mathscr{F}\to 0$ is exact, and similarly if we localize at x'.
By assumption $\left(f^*\mathscr{F}\right)_{x'}$ is free, thus flat so the sequence
$0\to f^*kernel_x\otimes k\left(x\right)\to\mathcal{O}_{U,x}^r\otimes k\left(x\right)\to f^*\mathscr{F}_x\otimes k\left(x\right)\to 0$ is exact.
The dimensions on the right are the same so the kernel must be 0 by nakayama.
Thus $\mathcal{O}_U^r\to\mathscr{F}$ is an isomorphism.

3.10.6 III.10.6 x g Etale Cover of degree 2. x

10.6. Let Y be the plane nodal cubic curve $y^2 = x^2(x+1)$. Show that Y has a finite étale covering X of degree 2, where X is a union of two irreducible components, each one isomorphic to the normalization of Y (Fig. 12).

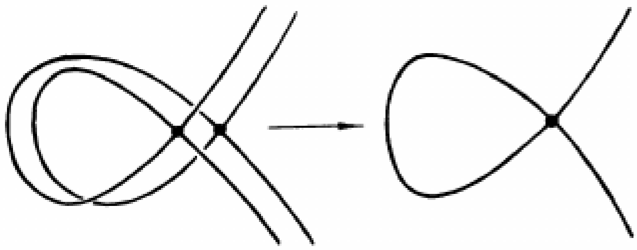

Figure 12. A finite étale covering.

So recall the normalization I think is just like a parabola, (it separates the two branches passing through the node) so I'm pretty sure its $R[t]$ with the map $t \to (t^2-1, t(t^2-1))$. In particular, I compute the normalization in Summer Problems 2012. Now we use this form the cover: $Spec\ k[s,t]\left(t^2-\left(s^2-1\right)^2\right) \to Spec\ k[x,y]/\left(y^2-x^2(x+1)\right)$, $x \mapsto (s^2-1)$ and $y \mapsto st$. (This is two parabolas joined together).

The two components are the two parabolas $k[s,t]/(t-(s^2-1))$, and $k[s,t]/(t+(s^2-1))$. To see that the cover is etale, check that it gives an isomorphism on the tangent cones. The tangent cone is geometrically the union of (tangent lines) two branches of C at each node. But then geometrically, it is clear this gives an isomorphism. Thus we have an etale cover.

3.10.7 III.10.7 x Serre's linear system with moving singularities

10.7. (*Serre*). *A linear system with moving singularities.* Let k be an algebraically closed field of characteristic 2. Let $P_1, \ldots, P_7 \in \mathbf{P}_k^2$ be the seven points of the projective plane over the prime field $\mathbf{F}_2 \subseteq k$. Let $\mathfrak{d}$ be the linear system of all cubic curves in X passing through $P_1, \ldots, P_7$.

(a) $\mathfrak{d}$ is a linear system of dimension 2 with base points $P_1, \ldots, P_7$, which determines an inseparable morphism of degree 2 from $X - \{P_i\}$ to $\mathbf{P}^2$.

Let T denote the base locus consisting of $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$, $P_4 = (1,1,0)$, $P_5 = (1,0,1)$, $P_6 = (1,1,1)$, and $P_7 = (1,1,1)$. A generic cubic in $\mathbb{P}^2$ is the zero set of $V(f)$ of $f(x,y,z) = a_1x^3 + a_2x^2y + a_3x^2z + a_4y^3 + a_5xyz + a_6xz^2 + a_7xy^2 + a_8z^3 + a_9y^2z + a_{10}yz^2$.

For $P_1, P_2, P_3 \in V(f)$, we must have a_1, a_4, a_8 are zero. $P_4 \in V(f)$ mean $a_2 = a_7$, $P_5 \in V(f)$ gives $a_3 = a_6$, $P_6 \in V(f)$ means $a_9 = a_{10}$, and $P_7 \in V(f)$ means $a_5 = 0$. Thus a cubic in $\mathfrak{D}$ looks like $a(x^2y+xy^2) + b(x^2z+xz^2) + c(y^2z+yz^2)$. Thus $\mathfrak{d}$ is generated by 3 cubics C_i with intersection T so there is a morphism $g : \mathbb{P}^2 \backslash T \to \mathbb{P}^2$ given by $(x,y,z) \mapsto (x^2y+xy^2, x^2z+xz^2, y^2z+yz^2)$.

We can show inseparability locally on one of the affines $D_+(x), D_+(y), D_+(z)$. For example on $D_+(z)$ we have coordinates $s = x/z$, $t = y/z$ and g is given by $(x,y,1) \mapsto \left(\frac{x^2y+xy^2}{y^2+y}, \frac{x^2+x}{y^2+y}\right)$. On function fields, we have $h : k(s,t) \hookrightarrow k(x,y)$, $s \mapsto \frac{x+y}{y+1}x$, $t \mapsto \frac{x+1}{y+1}\frac{x}{y}$. Note that $y \cdot h(t) + +h(s) = \frac{x+1+x+y}{y+1}x = x$. In $h(s)$ we get $0 = h(s) + h(s) = h(s) + \frac{x+y}{y+1}x =$
$\frac{h(s)(y+1)+y^2h(t)^2+yj(ts)+y^2h(t)+yh(ts)+h(s)^2+yh(s)}{y+1}$.

Thus$y^2h(t)(h(t)+1) + h(s)(h(s)+1) = 0$ and we can find a minimal polynomial $y^2+c = 0$, c a function of $h(s), h(t)$. Then u^2+c is a minimal polynomial that is inseparable of degree 2.

3.10.8 b. x

(b) Every curve $C \in \mathfrak{d}$ is singular. More precisely, either C consists of 3 lines all passing through one of the P_i, or C is an irreducible cuspidal cubic with cusp $P \neq$ any P_i. Furthermore, the correspondence $C \mapsto$ the singular point of C is a 1-1 correspondence between $\mathfrak{d}$ and $\mathbf{P}^2$. Thus the singular points of elements of $\mathfrak{d}$ move all over.

The singular points are given by the partial derivatives $0 = \frac{\partial f}{\partial x}$, $0 = \frac{\partial f}{\partial y}$, and $0 = \frac{\partial f}{\partial z}$, which are $ay^2+bz^2 = 0$, $ax^2 + cz^2 = 0$, and $bx^2 + cy^2 = 0$. Thus $\sqrt{a}y = \sqrt{b}z$, $\sqrt{a}x = \sqrt{c}z$, and $\sqrt{b}x = \sqrt{c}y$. There is a singular point $S = \left(\sqrt{c}, \sqrt{b}, \sqrt{a}\right)$. Each P_i is singular for only one of the cubics in $\mathfrak{d}$. P_1 lies on $yz\,(y+z)$, P_2 is singular on $xz\,(x+z)$, P_3 is singular on $xy\,(x+z)$, ... Each of these equations results from choosing a, b, c in $\mathbb{F}^2$ which is the same as choosing a union of three lines. Thus these relations give all the cubics with a singular point in the base locus. Note that two different sets of a, b, c give different singular points so $\mathfrak{d} \to \mathbb{P}^2_k$ is a bijection so the singularities of $\mathfrak{d}$ are moving.

III.10.8 x

10.8. *A linear system with moving singularities contained in the base locus (any characteristic).* In affine 3-space with coordinates x, y, z, let C be the conic $(x-1)^2 + y^2 = 1$ in the xy-plane, and let P be the point $(0,0,t)$ on the z-axis. Let Y_t be the closure in $\mathbf{P}^3$ of the cone over C with vertex P. Show that as t varies, the surfaces $\{Y_t\}$ form a linear system of dimension 1, with a moving singularity at P. The base locus of this linear system is the conic C plus the z-axis.

geometrically this is fairly obvious.

III.10.9 x

10.9. Let $f: X \to Y$ be a morphism of varieties over k. Assume that Y is regular, X is Cohen–Macaulay, and that every fibre of f has dimension equal to $\dim X - \dim Y$. Then f is flat. [*Hint*: Imitate the proof of (10.4), using (II, 8.21A).]

So recall a variety: integral separated scheme finite type over algebraically closed field k. Also flatness is local so we just need to show that $X \supset Spec\ A \to Spec\ R \subset Y$ is flat. Or that $R \to A$ is flat. So A is a local noetherian R-algebra, R is regular by assumption We have $dim\ X = dim\ Y + dim\ fiber$. So really I just need to show that $dim\ fiber$ is $dim\ A/PA$. But $X_y = Spec\,(A \otimes k\,(P)) = Spec\,(A_p/PA_P)$ and dimension works with localization. So we're good. Now this follows from Eisenbud 18.16.b which says if A is CM then A is flat over R iff $dim\ A = dim\ R + dim\ A/PA$.

3.11 III.11 x Theorem On Formal Functions

3.11.1 III.11.1 x g higher derived cohomology of plane minus origin.

11.1. Show that the result of (11.2) is false without the projective hypothesis. For example, let $X = \mathbf{A}^n_k$, let $P = (0, \ldots, 0)$, let $U = X - P$, and let $f: U \to X$ be the inclusion. Then the fibres of f all have dimension 0, but $R^{n-1} f_* \mathcal{O}_U \neq 0$.

Since $R^{n-1} f_* \mathcal{O}_U$ is the sheaf associated to $V \mapsto H^i\left(f^{-1}(V), \mathcal{O}_U|_{f^{-1}(V)}\right)$, we can just compute the cohomology of $\mathbb{A}^n_k - \{0\}$. WLOG let $n = 2$, then take the open cover $U_x = Spec\, k\,[x, y, x^{-1}]$, and $U_y = Spec\, k\,[x, y, y^{-1}]$. The cech complex is

$0 \to k[x, y, x^{-1}] \oplus k[x, y, y^{-1}] \to k[x, y, x^{-1}, y^{-1}] \to 0 \to \ldots$.
So the first cohomology group is linear combinations of monomials of negative degree.
Clearly the fibers have dim 0 on the inclusion.

3.11.2 III.11.2 x g

11.2. Show that a projective morphism with finite fibres (= quasi-finite (II, Ex. 3.5)) is a finite morphism.

Let $f : X \to Y$. This result is local on the base, thus we assume Y is affine. Since X is projective, $X \subset Y \times \mathbb{P}^n$, and we are projecting to Y. By induction, and blowing up a point of $\mathbb{P}^n$ we can assume $n = 1$. If $y \in Y$, then since f has finite fibers, we can find $z \in y \times \mathbb{P}^1$, $z \in Y \times \mathbb{P}^1 \backslash X$. This shows that we can take a smaller open affine and call that X.

Thus assume X is affine, defined by a polynomial in x with coefficients in $A(Y)$, $f(x) = a_n x^n + \cdots + a_0$. Localizing at a_n gives a quotient of $A[x]/(f)$ which is finitely generated by $1, x, ..., x^{n-1}$.

3.11.3 III.11.3 x

11.3. Let X be a normal, projective variety over an algebraically closed field k. Let $\mathfrak{d}$ be a linear system (of effective Cartier divisors) without base points, and assume that $\mathfrak{d}$ is *not composite with a pencil*, which means that if $f : X \to \mathbf{P}_k^n$ is the morphism determined by $\mathfrak{d}$, then $\dim f(X) \geqslant 2$. Then show that every divisor in $\mathfrak{d}$ is connected. This improves Bertini's theorem (10.9.1). [*Hints*: Use (11.5), (Ex. 5.7) and (7.9).]

(Kleiman)

Suppose that $\mathfrak{d}$ is reducible. I claim that $\mathfrak{d}$ is composite with a pencil.

By Bertini, $\mathfrak{d}$ has no variable singular points outside the base locus so a general member of $\mathfrak{d}$ has distinct components. Fix a general member U of $\mathfrak{d}$ with minimal number of components and assume the number of components is $d \geq 2$ by contrapositive, so that $\mathfrak{d}$ is disconnected / reducible. Let P be a subsystem which contains U, with no fixed components, and which is parametrized by a line. A general member of P has d components and the i^{th} component is a 1-parameter family P_i .

We can factor P so that each factor has coefficients which are coordinates of a generic point of a curve in projective space parametrizing P_i. Then P_i is a linear pencil since the degree of the curve gives the number of hypersurfaces in P_i which pass through a general point of projective space. But since only one member of P intersects x, this degree is 1.

Note that each of the P_i must be equal to each other. If U_1 is a general member of P_1 , and a general $x \in U_1$, then for each i, x must lie in some $U_i \in P_i$. Thus U_i must be a component of U since P is a pencil. Thus U_i must equal U_1 and thus $U_1 \in P_i$. Thus P_i are all the same. Thus all components of U belong to P_1 so $\mathfrak{d}$ system is composite with a pencil.

3.11.4 III.11.4 x Principle of Connectedness

11.4. *Principle of Connectedness.* Let $\{X_t\}$ be a flat family of closed subschemes of $\mathbf{P}_k^n$ parametrized by an irreducible curve T of finite type over k. Suppose there is a nonempty open set $U \subseteq T$, such that for all closed points $t \in U$, X_t is connected. Then prove that X_t is connected for all $t \in T$.

Since this question is stable under base change, wlog assume T is normalized. f flat and projective gives $f_*\mathcal{O}_X$ torsion free. T smooth gives $f_*\mathcal{O}_X$ locally free. f has connected fibers over U gives $f_*\mathcal{O}_X$ has rank one on U and thus everywhere. Thus $f_*\mathcal{O}_X$ gives an invertible sheaf. Note that the global sections of $f_*\mathcal{O}_X$ are the same as the global sections of $\mathcal{O}_X$. Thus $f_*\mathcal{O}_X \approx \mathcal{O}_T$. By a theorem in this section, the fibers are connected.

III.11.5*

***11.5.** Let Y be a hypersurface in $X = \mathbf{P}_k^N$ with $N \geqslant 4$. Let $\hat{X}$ be the formal completion of X along Y (II, §9). Prove that the natural map $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is an isomorphism. [*Hint*: Use (II, Ex. 9.6), and then study the maps $\operatorname{Pic} X_{n+1} \to \operatorname{Pic} X_n$ for each n using (Ex. 4.6) and (Ex. 5.5).]

Skip

III.11.6 - Skip (formal schemes)

-

11.6. Again let Y be a hypersurface in $X = \mathbf{P}_k^N$, this time with $N \geqslant 2$.

(a) If $\mathscr{F}$ is a locally free sheaf on X, show that the natural map

$$H^0(X,\mathscr{F}) \to H^0(\hat{X},\hat{\mathscr{F}})$$

is an isomorphism.

MISS

(b) Show that the following conditions are equivalent:

(i) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there exists a coherent sheaf $\mathscr{F}$ on X such that $\mathfrak{F} \cong \hat{\mathscr{F}}$ (i.e., $\mathfrak{F}$ is *algebraizable*);

(ii) for each locally free sheaf $\mathfrak{F}$ on $\hat{X}$, there is an integer n_0 such that $\mathfrak{F}(n)$ is generated by global sections for all $n \geqslant n_0$.

[*Hint*: For (ii) $\Rightarrow$ (i), show that one can find sheaves $\mathscr{E}_0, \mathscr{E}_1$ on X, which are direct sums of sheaves of the form $\mathcal{O}(-q_i)$, and an exact sequence $\hat{\mathscr{E}}_1 \to \hat{\mathscr{E}}_0 \to \mathfrak{F} \to 0$ on $\hat{X}$. Then apply (a) to the sheaf $\mathscr{H}om(\mathscr{E}_1,\mathscr{E}_0)$.]

MISS

(c) Show that the conditions (i) and (ii) of (b) imply that the natural map $\operatorname{Pic} X \to \operatorname{Pic} \hat{X}$ is an isomorphism.

Note. In fact, (i) and (ii) always hold if $N \geqslant 3$. This fact, coupled with (Ex. 11.5) leads to Grothendieck's proof [SGA 2] of the Lefschetz theorem which says that if Y is a hypersurface in $\mathbf{P}_k^N$ with $N \geqslant 4$, then $\operatorname{Pic} Y \cong \mathbf{Z}$, and it is generated by $\mathcal{O}_Y(1)$. See Hartshorne [5, Ch. IV] for more details.

MISS

III.11.7 -Skip (formal schemes)

11.7. Now let Y be a curve in $X = \mathbf{P}_k^2$.

(a) Use the method of (Ex. 11.5) to show that $\operatorname{Pic} \hat{X} \to \operatorname{Pic} Y$ is surjective, and its kernel is an infinite-dimensional vector space over k.

MISS

(b) Conclude that there is an invertible sheaf $\mathfrak{L}$ on $\hat{X}$ which is not algebraizable.

MISS

(c) Conclude also that there is a locally free sheaf $\mathfrak{F}$ on $\hat{X}$ so that no twist $\mathfrak{F}(n)$ is generated by global sections. Cf. (II, 9.9.1)

MISS

III.11.8 x higher derived is 0 in a neighborhood

11.8. Let $f: X \to Y$ be a projective morphism, let $\mathscr{F}$ be a coherent sheaf on X which is flat over Y, and assume that $H^i(X_y, \mathscr{F}_y) = 0$ for some i and some $y \in Y$. Then show that $R^i f_*(\mathscr{F})$ is 0 in a neighborhood of y.

We will show that $\left(R^i f_* \left(\mathscr{F}\right)\right)_y^\wedge$ ($\wedge$ for completion) is 0.
By the formal function theorem, this is equivalent to $H^i\left(X_y, \mathscr{F} \otimes \mathcal{O}_y/\mathfrak{m}_y^k\right) = 0$ for all k.
Note that $H^i\left(X_y, \mathscr{F} \otimes \mathcal{O}_y/\mathfrak{m}_y\right) = 0$ by assumption.
We also have
$0 \to \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1} \to \mathcal{O}_y/\mathfrak{m}_y^{k+1} \to \mathcal{O}_y/\mathfrak{m}_y^k \to 0$ and since $\mathscr{F}$ is flat, then by long exact sequence and induction, we just have to show that $H^i\left(X_y, \mathscr{F} \otimes \mathfrak{m}_y^k/\mathfrak{m}_y^{k+1}\right) = 0$.
Since $\mathfrak{m}_y^k/\mathfrak{m}_y^{k+1}$ is a direct sum of copies of $\mathcal{O}_y/\mathfrak{m}_y$, since the cohomology commutes with the direct product.

3.12 III.12 x Semicontinuity

3.12.1 III.12.1 x g upper semi-continuous tangent dimension

12.1. Let Y be a scheme of finite type over an algebraically closed field k. Show that the function

$$\varphi(y) = \dim_k(\mathfrak{m}_y/\mathfrak{m}_y^2)$$

is upper semicontinuous on the set of closed points of Y.

This is intuitively clear as $\mathfrak{m}/\mathfrak{m}^2$ is the number of tangent directions.
proof:
Since this is a local result, assume Y is some affine variety.
The tangent space in this case is the kernel of the linear transformation given by the jacobian matrix of the polynomials in the ideal of Y.
Since the rank function on matrices is upper-semicontinuous, the result follows.

3.12.2 III.12.2 x

12.2. Let $\{X_t\}$ be a family of hypersurfaces of the same degree in $\mathbf{P}^n_k$. Show that for each i, the function $h^i(X_t, \mathcal{O}_{X_t})$ is a constant function of t.

Let $f \in R[x_1, ..., x_n]$ a polynomial of degree d, $\mathfrak{p} \in Spec\ R$, and $k = quot\,(R/\mathfrak{p})$. Let $\overline{f}$ the reduction of f mod $\mathfrak{p}$. By a change of coordinates, $\overline{f}$ can be written in weierstrass form with respect to x_n: $\overline{f} = \overline{a}x_n^d + p_{d-1}x_n^{d-1} + \cdots + p_0$, where p_j are polynomials in the other variables, and $0 \neq \overline{a} \in k$. If $\frac{r_i}{s_i} \in k$ are coefficients giving the change of coordinates are in R, then we can accomplish the change of coordinates on the open set $D\left(a \prod s_i\right) \subset Spec\ R$ containing $\mathfrak{p}$ where a is invertible. Over the ring R' attained by adjoining $\frac{1}{a}\prod s_i$ to R, we have $R'[x_1, ..., x_n] / (f)$ is a free $R'[x_1, ..., x_{n-1}]$ module with basis $1, x_n, ..., x_n^{d-1}$ which is therefore a free R'-module. This gives the affine case, as free gives us flatness of the morphism to T. The affine case covers $\mathbb{P}^n$.

3.12.3 III.12.3 x Rational Normal Quartic

12.3. Let $X_1 \subseteq \mathbf{P}^4_k$ be the *rational normal quartic curve* (which is the 4-uple embedding of $\mathbf{P}^1$ in $\mathbf{P}^4$). Let $X_0 \subseteq \mathbf{P}^3_k$ be a nonsingular rational quartic curve, such as the one in (I, Ex. 3.18b). Use (9.8.3) to construct a flat family $\{X_t\}$ of curves in $\mathbf{P}^4$, parametrized by $T = \mathbf{A}^1$, with the given fibres X_1 and X_0 for $t = 1$ and $t = 0$.

Let $\mathscr{I} \subseteq \mathcal{O}_{\mathbf{P}^4 \times T}$ be the ideal sheaf of the total family $X \subseteq \mathbf{P}^4 \times T$. Show that $\mathscr{I}$ is flat over T. Then show that

$$h^0(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0 \end{cases}$$

and also

$$h^1(t, \mathscr{I}) = \begin{cases} 0 & \text{for } t \neq 0 \\ 1 & \text{for } t = 0. \end{cases}$$

This gives another example of cohomology groups jumping at a special point.

This question has a typo which makes it not quite work. See
http://mathoverflow.net/questions/90260/trouble-with-semicontinuity
for details.

3.12.4 III.12.4x

12.4. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $f: X \to Y$ be a flat projective morphism whose fibres are all integral schemes. Let $\mathscr{L}, \mathscr{M}$ be invertible sheaves on X, and assume for each $y \in Y$ that $\mathscr{L}_y \cong \mathscr{M}_y$ on the fibre X_y. Then show that there is an invertible sheaf $\mathscr{N}$ on Y such that $\mathscr{L} \cong \mathscr{M} \otimes f^*\mathscr{N}$. [*Hint*: Use the results of this section to show that $f_*(\mathscr{L} \otimes \mathscr{M}^{-1})$ is locally free of rank 1 on Y.]

Suppose that $\mathscr{F}$ is an invertible sheaf on X which is trivial on the fibers X_y. I claim that $f_*\mathscr{F} = \mathscr{G}$ is invertible on Y with $f^*\mathscr{G} = \mathscr{F}$.

By Grauert, $\pi_*\mathscr{F}$ is locally free rank 1 (call this $\mathscr{G}$) and $\mathscr{G} \otimes k(y) \to H^0(X_y, \mathscr{F}_y)$ is an isomorphism. The natural map $f^*\mathscr{G} = f^*f_*\mathscr{G} \to \mathscr{G}$ is an isomorphism since it is surjective on the fibers.

Now if $\mathscr{L}_y \approx \mathscr{M}_y$ then $\mathscr{L}_y \otimes \mathscr{M}_y^{-1}$ is trivial on the fibers so we are done.

3.12.5 III.12.5 x Picard Group of projective bundle

12.5. Let Y be an integral scheme of finite type over an algebraically closed field k. Let $\mathscr{E}$ be a locally free sheaf on Y, and let $X = \mathbf{P}(\mathscr{E})$—see (II, §7). Then show that $\operatorname{Pic} X \cong (\operatorname{Pic} Y) \times \mathbf{Z}$. This strengthens (II, Ex. 7.9).

We will map $\mathscr{F} \times m \in Pic\ Y \times \mathbb{Z}$ to $p^*\mathscr{F} \otimes \mathcal{O}_X(m)$.

Injective:
Suppose that $p^*\mathscr{F} \otimes \mathcal{O}_X(m) \approx \mathcal{O}_X$.
Then $p_*\mathcal{O}_X \approx \mathcal{O}_Y$ by thm II.7.11
Using projection, $p_*(\mathcal{O}_X(m) \otimes p^*\mathscr{F}) \approx \mathcal{O}_Y$
So $(p_*\mathcal{O}_X(m)) \otimes \mathscr{F} \approx \mathcal{O}_Y$.
$\mathscr{F}$ invertible so $(p_*\mathcal{O}_X(m)) \approx \mathscr{F}^{-1}$.
By thm II.7.11, for $m \leq 0$ we're done.
If $m > 0$, then $p_*\mathcal{O}_X(m) \approx S^m(\mathscr{E})$ which has rank $\begin{pmatrix} n+m+1 \\ n-1 \end{pmatrix}$.
So if rank is > 1, we are done for $m > 0$ since the rank will be too big to have $S^m(\mathscr{E}) \otimes \mathscr{F} \approx \mathcal{O}_Y$
If $\mathscr{E}$ is invertible, then we have $p_*\mathcal{O}_X(m) = \mathcal{O}_Y$ and so $\mathscr{F}$ must still be $\mathcal{O}_Y$ (see 11.12 in 3284 book).

Surjective.
Consider $\mathscr{M} \in Pic(X)$.
The restriction to the X_y , $\mathscr{M}_y$ is an invertible sheaf on $\mathbb{P}^n$ and thus is $\mathcal{O}_{\mathbb{P}^n}(m)$ for some m.
Thus for a preimage we consider $\mathscr{M} \otimes \mathcal{O}_X(-m)$.
Note this is effective and the restriction is trivial for every y.
This follows by semicontinuity and since the euler characteristic is locally constant. By (hirzebruch) riemann-roch we can find that degree 0 is locally constant so degree 0 + effective gives us a sheaf which is the same on each fiber. Now use the previous excercise.

III.12.6*

***12.6.** Let X be an integral projective scheme over an algebraically closed field k, and assume that $H^1(X, \mathcal{O}_X) = 0$. Let T be a connected scheme of finite type over k.
(a) If $\mathscr{L}$ is an invertible sheaf on $X \times T$, show that the invertible sheaves $\mathscr{L}_t$ on $X = X \times \{t\}$ are isomorphic, for all closed points $t \in T$.

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(b) Show that $\operatorname{Pic}(X \times T) = \operatorname{Pic} X \times \operatorname{Pic} T$. (Do *not* assume that T is reduced!) Cf. (IV, Ex. 4.10) and (V, Ex. 1.6) for examples where $\operatorname{Pic}(X \times T) \neq \operatorname{Pic} X \times \operatorname{Pic} T$. [*Hint*: Apply (12.11) with $i = 0,1$ for suitable invertible sheaves on $X \times T$.]

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4 IV Curves

4.1 IV.1 x Riemann_Roch_Theorem

alternative

4.1.1 x IV.1.1 g Regular except at a point

1.1. Let X be a curve, and let $P \in X$ be a point. Then there exists a nonconstant rational function $f \in K(X)$, which is regular everywhere except at P.

Let $Q \neq P$. and let $D = 2P - Q$.
Choose n such that $deg\,(nD) > max\,\{2g-2, g, 1\}$.
By R.R. and speciality of nD, $h^0\,(nD) = n + 1 - g$.
So $nD - (f) \sim D'$ for some effective D'.
Then $D' - 2nP + nQ \sim (f)$
and so since D' is effective, f has a pole at P.

4.1.2 x IV.1.2 g Regular Except pole at Points

1.2. Again let X be a curve, and let $P_1, \ldots, P_r \in X$ be points. Then there is a rational function $f \in K(X)$ having poles (of some order) at each of the P_i, and regular elsewhere.

Let $Q \in X - \{P_1, ..., P_r\}$.
Let $D = (P_1 + ... + P_r - (r-1)\,Q)$, $n > max\,\{2g-2, g, 1\}$.
r.r gives $h^0\,(nD) = n + 1 - g \geq 1$.
By definition of linear series, $\exists f \in K\,(X)$, $nD - (f) = D' \geq 0$.
i.e. $D' + n\,(-P_1 - ... - P_r + (r-1)\,Q) = (f) \star$
Now if D' cancels no poles, then (f) is our function by $\star$
Else D' cancels a pole,
Note since $deg\,(nD) = n$ thus $deg\,(D') = deg\,(nD)$.
So D' only cancels one pole (since effective). Thus $D' = nP_i$.
Using Excercise IV.1.1, find g which is regular except for a pole at P_i.
To avoid cancelling, find N greater than the order of g at each P_i.
Then $f^N g$ will be regular everywhere except at each P_i.

4.1.3 x IV.1.3 g Nonproper Curve is affine

1.3. Let X be an integral, separated, regular, one-dimensional scheme of finite type over k, which is *not* proper over k. Then X is affine. [*Hint*: Embed X in a (proper) curve $\bar{X}$ over k, and use (Ex. 1.2) to construct a morphism $f: \bar{X} \to \mathbf{P}^1$ such that $f^{-1}(\mathbf{A}^1) = X$.]

Remark II.4.10.2 (e), says that every variety can be embedded as an open dense subset of a complete variety.
So embed X in such a complete variety.
By 1.6.10 embed X as an open subset of a complete curve $\overline{X}$.
Then $\overline{X} \backslash X = \{P_1, ..., P_r\}$ since it's closed.
By Excercise IV.1.2, "regular except at P_i", there is a section f with no poles except at P_i.
f gives a finite morphism to $\mathbb{P}^1$ from $\overline{X}$.
By finiteness of the morphism, $f^{-1}\,(\mathbb{A}^1) = X$ is affine.
Embed X in a proper variety over k.

4.1.4 IV.1.4 x

1.4. Show that a separated, one-dimensional scheme of finite type over k, none of whose irreducible components is proper over k, is affine. [*Hint*: Combine (Ex. 1.3) with (III, Ex. 3.1, Ex. 3.2, Ex. 4.2).]

By III.3.1, we only need to show for a reduced scheme.
By III.3.2, we only need to show for an irreducible component of a reduced scheme.
Thus we assume X is integral.
By II.2.4, since the image of a proper scheme is proper, the normalization $\tilde{X}$ is not proper.
By IV.1.3, $\tilde{X}$ is affine. (not proper then affine)
By III.4.2, Chevalley's theorem, since we have a finite surjective with $\tilde{X}$ affine, then X is affine.

4.1.5 IV.1.5 x g Dimension less than degree

1.5. For an effective divisor D on a curve X of genus g, show that $\dim|D| \leqslant \deg D$. Furthermore, equality holds if and only if $D = 0$ or $g = 0$.

Using riemann roch,
$dim\,|D| = h^0\,(D) - 1 = deg\ D - g + h^0\,(K - D)$
Since D is effective, it's
$\leq deg\ D - g + h^0\,(K)$
Note the canonical has $h^0\,(K) - h^0\,(K - K) = 2g - 2 + 1 - g = g$.
$= deg\ D - g + g$ since the canonical has $h^0\,(K) = g$.
Note that equality holds when $h^0\,(K - D) = h^0\,(K)$ so when $D = 0$ or $g = 0$.

4.1.6 IV.1.6 x g finite morphism to $\mathbb{P}^1$

1.6. Let X be a curve of genus g. Show that there is a finite morphism $f: X \to \mathbf{P}^1$ of degree $\leqslant g + 1$. (Recall that the *degree* of a finite morphism of curves $f: X \to Y$ is defined as the degree of the field extension $[K(X):K(Y)]$ (II, §6).)

Choose $g + 1$ points P_i on X.
By IV.1.2, there is a nonconstant function f in $k\,(X)$ with poles at P_i and regular elsewhere.
Since f is nonconstant, By Theorem II.6.8 f is finite.
Since $f^{-1}\,(\infty)$ consists of the poles P_i, then $deg\ f \leq g + 1$.

4.1.7 IV.1.7 x g

1.7. A curve X is called *hyperelliptic* if $g \geqslant 2$ and there exists a finite morphism $f: X \to \mathbf{P}^1$ of degree 2.

(a) If X is a curve of genus $g = 2$, show that the canonical divisor defines a complete linear system $|K|$ of degree 2 and dimension 1, without base points. Use (II, 7.8.1) to conclude that X is hyperelliptic.

If X has genus 2, then $deg\ K_X = 2g - 2 = 2$ and $dim\,|K_X| = g - 1 = 1$.

To show base point free, we need $dim\,|K_X - P| = dim\,|K_X| - 1$ for any point.
Note that $|P| - |K_X - P| = 1 + 1 - g = 0$ and $|P| = 0$ since if it was 1, there would be a degree 1 morphism to $\mathbb{P}^1$.

By remark II.7.8.1, a linear system without basepoints gives a morphism to $\mathbb{P}^n$. (in this case to $\mathbb{P}^1$ since $dim\,|K_X| = 1$). Since $deg\ K_X = 2$, we have a degree 2 morphism to $\mathbb{P}^1$ so X is hyperelliptic.

4.1.8 b. x g

(b) Show that the curves constructed in (1.1.1) all admit a morphism of degree 2 to $\mathbf{P}^1$. Thus there exist hyperelliptic curves of any genus $g \geqslant 2$.
Note. We will see later (Ex. 3.2) that there exist nonhyperelliptic curves. See also (V, Ex. 2.10).

Let X a curve on the quadric corresponding to divisor of degree $(g+1, 2)$. Denote $p_2 : X \to \mathbb{P}^1$ the second projection from $Q = \mathbb{P}^1 \times \mathbb{P}^1$. This projection is non-constant and thus finite by thm II.6.8. By thm II.6.9, for a point, $deg\ p_2^*(P) = deg p_2 \cdot deg\ P$ so $2 = deg\ p_2$.

4.1.9 IV.1.8 x g arithmetic genus of a singular curve

1.8. *p_a of a Singular Curve.* Let X be an integral projective scheme of dimension 1 over k, and let $\tilde{X}$ be its normalization (II, Ex. 3.8). Then there is an exact sequence of sheaves on X,

$$0 \to \mathcal{O}_X \to f_*\mathcal{O}_{\tilde{X}} \to \sum_{P \in X} \tilde{\mathcal{O}}_P/\mathcal{O}_P \to 0,$$

where $\tilde{\mathcal{O}}_P$ is the integral closure of $\mathcal{O}_P$. For each $P \in X$, let $\delta_P = \operatorname{length}(\tilde{\mathcal{O}}_P/\mathcal{O}_P)$.

(a) Show that $p_a(X) = p_a(\tilde{X}) + \sum_{P \in X} \delta_P$. [*Hint*: Use (III, Ex. 4.1) and (III, Ex. 5.3).]

$\tilde{X}$ nonsingular projective by Leray spectral sequence that $H^0(f_*\mathcal{O}_{\tilde{X}}) = k$. As $\sum_{p \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P$ is a direct sum of flasque, then exc III.4.1, $H^1(X, f_*\mathcal{O}_{\tilde{X}}) \approx H^1\left(\tilde{X}, \mathcal{O}_{\tilde{X}}\right)$. Thus we have an s.e.s.

$0 \to H^0\left(X, \sum_{P \in X} \tilde{\mathcal{O}_P}/\mathcal{O}_P\right) \to H^1(X, \mathcal{O}_X) \to H^1\left(\tilde{X}, \mathcal{O}_{\tilde{X}}\right) \to 0$.

Now by exc III.5.3, $p_a(X) = p_a\left(\tilde{X}\right) + \sum_{P \in X} dim_k \tilde{\mathcal{O}_P}/\mathcal{O}_P = p_a\left(\tilde{X}\right) + \sum \delta_P$.

4.1.10 (b) x g Genus 0 is nonsingular.

(b) If $p_a(X) = 0$, show that X is already nonsingular and in fact isomorphic to $\mathbf{P}^1$. This strengthens (1.3.5).

$\delta_P = 0$ by the formula from the question.
Thus local rings are normal.
By DIRP, its nonsingular.
By 3.1.5, it's $\mathbb{P}^1$

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*(c) If P is a node or an ordinary cusp (I, Ex. 5.6, Ex. 5.14), show that $\delta_P = 1$. [*Hint*: Show first that δ_P depends only on the analytic isomorphism class of the singularity at P. Then compute δ_P for the node and cusp of suitable plane cubic curves. See (V, 3.9.3) for another method.]

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IV.1.9 (star)

*1.9. *Riemann–Roch for Singular Curves.* Let X be an integral projective scheme of dimension 1 over k. Let X_{reg} be the set of regular points of X.

(a) Let $D = \sum n_i P_i$ be a divisor with support in X_{reg}, i.e., all $P_i \in X_{reg}$. Then define $\deg D = \sum n_i$. Let $\mathscr{L}(D)$ be the associated invertible sheaf on X, and show that

$$\chi(\mathscr{L}(D)) = \deg D + 1 - p_a.$$

4.1.11 x g difference of very amples.

(b) Show that any Cartier divisor on X is the difference of two very ample Cartier divisors. (Use (II, Ex. 7.5).)

Choose $n > 0$ such that $D + mL$ is globally generated where L is some very ample divisor.
By II.7.5, $(D + mL) + L = D + (m+1) L = B$ is very ample.
Then $D = B - (m+1) L$ is the difference of very amples.

4.1.12 x g Invertible sheaves are $\mathscr{L}(D)$

(c) Conclude that every invertible sheaf $\mathscr{L}$ on X is isomorphic to $\mathscr{L}(D)$ for some divisor D with support in X_{reg}.

By (b), assume very ample cartier.
Choose a hyperplane which doesn't not intersect the singular locus of X.

4.1.13 x. Alternative riemann-roch

(d) Assume furthermore that X is a locally complete intersection in some projective space. Then by (III, 7.11) the dualizing sheaf ω_X is an invertible sheaf on X, so we can define the *canonical divisor* K to be a divisor with support in X_{reg} corresponding to ω_X. Then the formula of (a) becomes

$$l(D) - l(K - D) = \deg D + 1 - p_a.$$

note that LCI by II.8.32
Thus by III.7.6
, $H^1(X, \mathscr{L}(D)) \approx Ext^0(\mathscr{L}(D), \omega_X^0) \approx$
$Ext^1(\mathcal{O}_X, \omega_X \otimes \mathscr{L}(-D)) \approx H^0(X, \omega_X^0 \otimes \mathscr{L}(-D))$.
Now use the Riemann Roch formula from part (a).

4.1.14 IV.1.10 g x

1.10. Let X be an integral projective scheme of dimension 1 over k, which is locally complete intersection, and has $p_a = 1$. Fix a point $P_0 \in X_{reg}$. Imitate (1.3.7) to show that the map $P \to \mathscr{L}(P - P_0)$ gives a one-to-one correspondence between the points of X_{reg} and the elements of the group Pic X. This generalizes (II, 6.11.4) and (II, Ex. 6.7).

By exc IV.1.9.d, $deg\ K_X = p_a - 1 = 0$.
Now let D be a divisor of degree 0.
By exc IV.1.9.c, applied to $D + P_0$ gives an invertible sheaf $\mathscr{L}(D')$ where D' has support in X_{reg} .
Then $deg(K_X - D - P_0) = 0 - 1$ so it has no sections so $dim\,|D + P_0| = D'$.

4.2 IV.2 x Hurwitz Theorem

4.2.1 IV.2.1x g projective space simply connected

~~**2.1.** Use (2.5.3) to show that $\mathbf{P}^n$ is simply connecte~~d.

First I will retype the case for $\mathbb{P}^1$ since I forget how it goes.

Let $f : X \to \mathbb{P}^1$ an etale cover. Assume X is connected. Then X is smooth over k since f is etale (obvious), and X is proper over k since f is finite (exc II.4.1 or something). So X is a curve (note connected and regular imply irreducible - i'll take your word for it). Since f is etale, f is separable, so we apply hurwitz theorem. f unramified, then the ramification divisor $R = 0$ so $2g(X) - 2 = n(-2)$.

$g(X) \geq 0$ so this only happens for $g(X) = 0$, and $n = 1 \implies X = \mathbb{P}^1$.

Now assume by induction that we know $\mathbb{P}^i$ are all simply connected, $i < n$. Let $H = \mathbb{P}^{n-1}$ a simply connected hyperplane in $\mathbb{P}^n$. Suppose $f : X \to \mathbb{P}^n$ an etale cover. Pulling back H gives f^*H ample, so it's connected by lefschetz hyperplane or thm III.7.9.

So we want to show no nontrivial etale coverings.
We proceed by induction.
For a base case we have 2.5.3.
Now let $H \approx \mathbb{P}^{n-1}$ a hyperplane in $\mathbb{P}^n$.
Suppose there is a nontrivial etale covering $f : X \to \mathbb{P}^n$.
Pulling back H gives f^*H ample since f is finite, by III.7.9, it's connected.
Thus $f|_H$ is an isomorphism. So f is degree 1. So f is an isomorphism.

4.2.2 IV.2.2 x g classification of genus 2 curves

2.2. *Classification of Curves of Genus* 2. Fix an algebraically closed field k of characteristic $\neq 2$.

(a) If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbf{P}^1$ of degree 2 (Ex. 1.7). Show that it is ramified at exactly 6 points, with ramification index 2 at each one. Note that f is uniquely determined, up to an automorphism of $\mathbf{P}^1$, so X determines an (unordered) set of 6 points of $\mathbf{P}^1$, up to an automorphism of $\mathbf{P}^1$.

$|K_X|$ gives a finite morphism to $\mathbb{P}^1$ of degree $2g - 2 = 2$.
Then using Hurwitz,
$2g - 2 = 2(-2) + deg\ R$ so $deg\ R = 6$.

Thus for any branch point, we have have 6 ramification points with ramification index 2 (2 is the degree of the map f).

4.2.3 b. x g

(b) Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x - \alpha_1) \cdots (x - \alpha_6)$. Let $f: X \to \mathbf{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbf{P}^1$, and nowhere else. (Cf. (II, Ex. 6.4).)

Projection from f onto the x coordinate is ramified (branched) at values of x which have one value of z. Thus there are 6 ramification points given by the α_i . By hurwitz, $deg\ R = 6$ and the genus of $X = 2$.

Now suppose that there is a divisor giving a degree 2 map to $\mathbb{P}^1$. So Consider $\mathcal{O}_X\left(D^\vee \otimes K_X\right)$. Note that $|D| - |K_X - D| = 2 + 1 - 2 = 1$ so that $|K_X - D|$ has dimension 0.

Note that $|K_X - D| - |D| = deg\left(K_X - D\right) - 1$ so that $K_X - D$ has degree 0. Thus $K_X - D$ is trivial. so $K_X = D$.

4.2.4 c. x

(c) Using (I, Ex. 6.6), show that if P_1, P_2, P_3 are three distinct points of $\mathbf{P}^1$, then there exists a unique $\varphi \in \operatorname{Aut} \mathbf{P}^1$ such that $\varphi(P_1) = 0$, $\varphi(P_2) = 1$, $\varphi(P_3) = \infty$. Thus in (a), if we order the six points of $\mathbf{P}^1$, and then normalize by sending the first three to $0, 1, \infty$, respectively, we may assume that X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$, where $\beta_1, \beta_2, \beta_3$ are three distinct elements of k, $\neq 0, 1$.

We can just find linear fractional transformations sending $P_1 \mapsto 0$, $P_2 \mapsto 1$, $P_3 \mapsto \infty$. Each case is given in Rudin 14.3.

4.2.5 .x

(d) Let Σ_6 be the symmetric group on 6 letters. Define an action of Σ_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3$ of k, $\neq 0, 1$, as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given element $\sigma \in \Sigma_6$, then renormalize as in (c) so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta'_1, \beta'_2, \beta'_3$.

nothing to do here.

Are you trying to make me do group theory?

4.2.6 conclusion. x

(e) Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3$ of k, $\neq 0, 1$, modulo the action of Σ_6 described in (d). In particular, there are many non-isomorphic curves of genus 2. We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbf{A}_k^3$ modulo a finite group.

Clear from parts a-d.

4.2.7 IV.2.3 x inflection points gauss map

2.3. *Plane Curves.* Let X be a curve of degree d in $\mathbf{P}^2$. For each point $P \in X$, let $T_P(X)$ be the tangent line to X at P (I, Ex. 7.3). Considering $T_P(X)$ as a point of the dual projective plane $(\mathbf{P}^2)^*$, the map $P \to T_P(X)$ gives a morphism of X to its *dual curve* X^* in $(\mathbf{P}^2)^*$ (I, Ex. 7.3). Note that even though X is nonsingular, X^* in general will have singularities. We assume char $k = 0$ below.

(a) Fix a line $L \subseteq \mathbf{P}^2$ which is not tangent to X. Define a morphism $\varphi : X \to L$ by $\varphi(P) = T_P(X) \cap L$, for each point $P \in X$. Show that φ is ramified at P if and only if either (1) $P \in L$, or (2) P is an *inflection point* of X, which means that the intersection multiplicity (I, Ex. 5.4) of $T_P(X)$ with X at P is $\geqslant 3$. Conclude that X has only finitely many inflection points.

Suppose first that $P \in L$. WLOG assume P is the origin in $\mathbb{A}^2$ and L is the line $y = 0$, and T_p is $x = 0$. For $Q = (a, b) \in X$, then T_Q is $\left\{\frac{\partial f}{\partial x}|_Q (x-a) + \frac{\partial f}{\partial y}|_Q (y-b) = 0\right\}$. $\varphi(Q)$ can be found by setting $y = 0$ and solving for x. This gives $\frac{\partial f}{\partial x}|_Q x = \frac{\partial f}{\partial y}|_Q b + \frac{\partial f}{\partial x}|_Q a$ and dividing by $\frac{\partial f}{\partial x}|_Q$ gives $x = \frac{\frac{\partial f}{\partial y}|_Q b}{\frac{\partial f}{\partial x}|_Q} + a$.

If t is a local parameter at $0 \in \mathbb{A}^1$, then $\varphi^*(t) = \frac{\frac{\partial f}{\partial y} \cdot y}{\frac{\partial f}{\partial x}} + x$. Then on the y-axis, which is T_P, $\frac{\partial f}{\partial y}(0) = 0$ and $\varphi(0) = 0$ so x vanishes at 0 to order ≥ 2 . Since $\frac{\partial f}{\partial y} \cdot y \in \mathfrak{m}_0^2$, and $\frac{\partial f}{\partial x} \neq 0$, then $\varphi^*(t) \in \mathfrak{m}_0^2$ which gives φ ramified at 0.

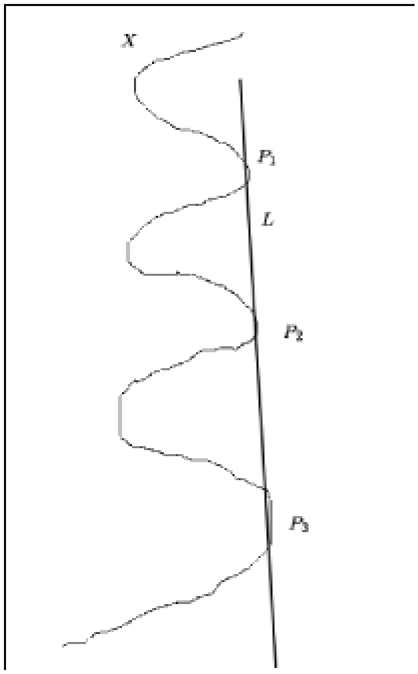
On the other hand if $P \notin L$, again let $P = (0,0)$ in $\mathbb{A}^2$, T_p be the line $x = 0$, but this time set L to be the line at infinity. For $Q \in X$, then the tangent at $Q = (a, b)$ is $\frac{\partial f}{\partial x}|_Q (x - az) + \frac{\partial f}{\partial y}(y - bz) = 0$ since we take the projective tangent line this time. Q is mapped to the slope of its tangent line, which is the intersect of the tangent line and L which is the line at infinity, given by $z = 1$. Thus $\varphi : X \to \mathbb{P}^1$ maps $Q \mapsto \left(-\frac{\partial f}{\partial y}|_Q : \frac{\partial f}{\partial x}|_X\right)$. Note that $\frac{\partial f}{\partial x}|_{\{(0,0)\}} \neq 0$ near P so we have $\varphi : X \to \mathbb{A}^1$, $Q \mapsto -\frac{\partial f}{\partial y}|_Q / \frac{\partial f}{\partial x}|_Q$. $\varphi(0) = 0$ so X has no constant term. We write $f(x, y) = ax + by + cx^2 + dxy + ey^2 + \ldots$. If t is the local coordinate at 0 (for the y), then $\varphi^*(t) \in \mathfrak{m}_0^2 \iff \frac{\partial f}{\partial y} \in \mathfrak{m}_0^2 \iff b + dx + 2ey \in \mathfrak{m}_0^2|_{x=0} \iff b + 2ey \in \mathfrak{m}_0^2$ which means f restricted to $x = 0$ has degree ≥ 3 in y (so it's only the higher order terms). Which is the same as intersection mult of f with $x = 0$ is ≥ 3 or 0 an inflection point.

Note that Hurwitz shows the degree of the ramification divisor is finite, so X has a finite number of inflection points.

4.2.8 b. x multiple tangents.

(b) A line of $\mathbf{P}^2$ is a *multiple tangent* of X if it is tangent to X at more than one point. It is a *bitangent* if it is tangent to X at exactly two points. If L is a multiple tangent of X, tangent to X at the points $P_1, \ldots, P_r$, and if none of the P_i is an inflection point, show that the corresponding point of the dual curve X^* is an *ordinary r-fold point*, which means a point of multiplicity r with distinct tangent directions (I, Ex. 5.3). Conclude that X has only finitely many multiple tangents.

So recall the gauss map which takes a point $x \in C$ to the coefficients of the definition equation of the tangent line to x in $\mathbb{P}^{2\vee}$.

Now consider the picture 

So if P_1 and P_2 and P_3 all map to the same point on C^*, then we have a multiple point on the dual curve. i.e., in small neighborhoods on C^* corresponding to neighborhoods around each P_i there are distinct branches, but then at the P_i there is an intersection. Now if there was an inflection point, note that an inflection point on C corresponds to where two tangents come together. On the dual curve this would correspond to where two transverse branches become less and less transverse, and finally they are the same, i.e. this is a cusp. Since we don't have any such inflection points, all of the tangents at the point of singularity must be distinct.

In local coordinates, we have an inflection point when the hessian curve intersects the curve. We write this as $\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = 0$ in local coordinates. On the other hand we have a cusp when $f_x(0,0) = 0$, $f_y(0,0) = 0$ and $f_{xx}(0,0)\,p^2 + 2pqf_{xy}(0,0) + f_{yy}(0,0)\,q^2$ is the square of a linear factor and a node otherwise. This condition can be rephrased as the discriminant of the polynomial $\left[2^2 f_{xy}(0,0)^2 - 4f_{xx}(0,0)\,f_{yy}(0,0)\right] = 0$ for a cusp, and non-zero otherwise. which is the same as the condition that the above determinant be 0. Thus inflection points correspond to cusps, and since we don't have any we have ordinary multiple points. Now since there are only finitely many singular points there must be only finitely many multiple tangents.

4.2.9 c. x g

(c) Let $O \in \mathbf{P}^2$ be a point which is not on X, nor on any inflectional or multiple tangent of X. Let L be a line not containing O. Let $\psi: X \to L$ be the morphism defined by projection from O. Show that ψ is ramified at a point $P \in X$ if and only if the line OP is tangent to X at P, and in that case the ramification index is 2. Use Hurwitz's theorem and (I, Ex. 7.2) to conclude that there are exactly $d(d-1)$ tangents of X passing through O. Hence the degree of the dual curve (sometimes called the *class* of X) is $d(d-1)$.

WLOG assume $O = (0,0) \in \mathbb{A}^2$, $P = (0,1) \in \mathbb{A}^2$, L the line at infinity which doesn't contain O. Let ψ the projection from O, $\psi : (x,y) \mapsto (x:y)$. We can define $\psi : U \to D(y)$ by $(x,y) \mapsto x/y$ where U is a neighborhood of P. Thus $\psi(P) = 0$. Note that ψ is ramified at P when $\psi^*(t) = \frac{x}{y} \in \mathfrak{m}_P^2$, t a local parameter of 0. If $y \neq 0$, ramification is therefore equivalent to $x \in \mathfrak{m}_P^2$, or the line $x = 0$ being tangent to X at P.

Hurwitz gives $(d-1)(d-2) - 2 = -2d + deg\ R$. Thus $R = d(d-1)$, and R is reduced since 0 is not on an inflection point or tangent line. Thus the number of tangent lines is $d(d-1) = deg\ R$.

4.2.10 d. x g

(d) Show that for all but a finite number of points of X, a point O of X lies on exactly $(d+1)(d-2)$ tangents of X, not counting the tangent at O.

Let O be a point not at any of the finite number of inflections or multiple tangents. If $\psi : X \to \mathbb{P}^1$ is projection from O , $deg\ \psi = d{-}1$ (recall X is a curve of degree d). Thus by hurwitz, $2g{-}2 = -2d{+}2{+}deg\,(R)$, so by genus degree in $\mathbb{P}^2$, gives $(d-1)\,(d-2){+}2d{-}4 = deg\ R$ so by $allroots\,((x-1)\cdot(x-2)+2\cdot x-4, x) = [x=-1.0, x=2.0]$ we have $deg\ R = (d+1)\,(d-2)$.

4.2.11 e. x g

(e) Show that the degree of the morphism φ of (a) is $d(d-1)$. Conclude that if $d \geqslant 2$, then X has $3d(d-2)$ inflection points, properly counted. (If $T_P(X)$ has intersection multiplicity r with X at P, then P should be counted $r-2$ times as an inflection point. If $r=3$ we call it an *ordinary inflection point*.) Show that an ordinary inflection point of X corresponds to an ordinary cusp of the dual curve X^*.

Note that $\varphi^{-1}(P) = \{Q \in X | P \in T_Q(X)\}$. For P not an inflection or on a multiple tangent line, then (c) gives $\sharp\varphi^{-1}(P) = d\,(d-1)$ so $deg\varphi = d\,(d-1)$. Hurwitz gives that if $deg\ R = 3d^2 - 5d$. Since $P \notin L$, we can add an extra d ramification points by (1) of a.

4.2.12 f. x g

(f) Now let X be a plane curve of degree $d \geqslant 2$, and assume that the dual curve X^* has only nodes and ordinary cusps as singularities (which should be true for sufficiently general X). Then show that X has exactly $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents. [*Hint*: Show that X is the normalization of X^*. Then calculate $p_a(X^*)$ two ways: once as a plane curve of degree $d(d-1)$, and once using (Ex. 1.8).]

The map $\varphi : X \to X^*$ is finite and birational. X is normal so by the universal property of normalization, X is the normalization of X^*. Then

$p_a\,(X^*) = \frac{1}{2}\,(d\,(d-1)-1)\,(d\,(d-1)-2)$,

$p_a\,(X^*) = p_a\,(X) + \sharp$ inflections $+\ \sharp$ bitangents

Plugging in $p_a\,(X) = \frac{1}{2}\,(d-1)\,(d-2)$, and inflections is $3d\,(d-2)$, and solving for the number of bitangents gives it.

4.2.13 g. x g

(g) For example, a plane cubic curve has exactly 9 inflection points, all ordinary. The line joining any two of them intersects the curve in a third one.

Since a plane cubic has degree 3, then by (e), there are $3\cdot 3\,(3-2) = 9$ inflection points. Since $r=3$, these are ordinary. An inflection point is where there is multiplicity 3 or greater. Since all are ordinary, the multiplicity is exactly 3. Now choose coordinates x, y, z such that $y=0, z=0$ are the tangents through the inflection points at $(0,0,1)\,,(0,1,0)$. Thus the cubic is $yz\,(ax+by+cz)+dx^3 = 0$ by computing the intersection with the hessian. The third flex is therefore $x=0$. Note that $x=0$ is the line joining the two points.

4.2.14 h. x

(h) A plane quartic curve has exactly 28 bitangents. (This holds even if the curve has a tangent with four-fold contact, in which case the dual curve X^* has a tacnode.)

A plane quartic has degree 4 (we assume nonsingular in this chapter). Now plug in to the formula from (f).

4.2.15 IV.2.4 x g Funny curve in characteristic p

2.4. *A Funny Curve in Characteristic p.* Let X be the plane quartic curve $x^3y + y^3z + z^3x = 0$ over a field of characteristic 3. Show that X is nonsingular, every point of X is an inflection point, the dual curve X^* is isomorphic to X, but the natural map $X \to X^*$ is purely inseparable.

To check singularities, we use jacobian criterion.

Partials are $f_z = y^3$, $f_x = z^3$, $f_y = x^3$. Since $(0,0,0)$ is not in projective space, then there is no point where all partials are zero, so it's nonsingular.

To check inflection points, we compute the hessian:
$$\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{pmatrix} = \begin{pmatrix} 6xy & 3x^2 & 0 \\ 0 & 6yz & 3y^2 \\ 3z^2 & 0 & 6zx \end{pmatrix}.$$
This is 0 in characteristic 3, so every point is an inflection point.

Tangent line at $P = (x_0, y_0, z_0)$ is $f_x \cdot (x - x_0) + f_y \cdot (y - y_0) + f_z \cdot (z - z_0) = 0$. This is $z_0^3 (x - x_0) + x_0^3 (y - y_0) + y_0^3 (z - z_0) = 0$. This is $z_0^3 x + x_0^3 y + y_0^3 z = 0$ since $z_0^3 x_0 + x_0^3 y_0 + y_0^3 z_0$ lies on X. Thus the gauss map is the frobenius. The function field morphism is thus purely inseparable and finite. Thus by thm IV.2.5? $X \approx X^*$.

4.2.16 IV.2.5 x Automorphisms f a curve in genus >= 2

2.5. *Automorphisms of a Curve of Genus $\geqslant 2$.* Prove the theorem of Hurwitz [1] that a curve X of genus $g \geqslant 2$ over a field of characteristic 0 has at most $84(g-1)$ automorphisms. We will see later (Ex. 5.2) or (V, Ex. 1.11) that the group $G = \operatorname{Aut} X$ is finite. So let G have order n. Then G acts on the function field $K(X)$. Let L be the fixed field. Then the field extension $L \subseteq K(X)$ corresponds to a finite morphism of curves $f: X \to Y$ of degree n.

(a) If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}f(P)$ consists of exactly n/r points, each having ramification index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's theorem implies that
$$(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^{s} (1 - 1/r_i).$$

Let $P \in X$ a ramification point, $e_p = r$. If $y \in Y$ is a branch point, and x_i, $i = 1, ..., s$ are the points of X lying over y, then these form an orbit of GX. Thus the x_i 's have conjugate stabilizers. Thus the number of points in this orbit is the index of the stabilizer which has order $|G|/r$. Thus at x, f has multiplicity r.

By Hurwitz, $2p_a(X) - 2 = |G|(2p_a(Y) - 2) + \sum_{i=1}^{s} \frac{|G|}{r_i}(r_i - 1)$ and rearranging this gives the desired equation.

4.2.17 b. x

(b) Since $g \geqslant 2$, the left hand side of the equation is >0. Show that if $g(Y) \geqslant 0$, $s \geqslant 0$, $r_i \geqslant 2$, $i = 1, \ldots, s$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s} (1 - 1/r_i) > 0,$$

then the minimum value of this expression is $1/42$. Conclude that $n \leqslant 84(g-1)$. See (Ex. 5.7) for an example where this maximum is achieved.

If $p_a(Y) \geq 1$ and $R = \sum\left(1 - \frac{1}{r_i}\right) = 0$, then $p_a(Y) \geq 2$, by (a), so $|G| \leq p_a(X) - 1$.

If $p_a(Y) \geq 1$ and $R = \sum\left(1 - \frac{1}{r_i}\right) > 0$, then $R \geq \frac{1}{2}$, so $2p_a(Y) - 2 + r \geq \frac{1}{2}$ and thus $|G| \leq 4(p_a(X) - 1)$.

If $p_a(Y) = 0$, then the equation from (a) is $2p_a(X) - 2 = |G|(-2 + R)$ so $R > 2$. Using some arithmetic, if $R = \sum_{i=1}^{s}\left(1 - \frac{1}{r_i}\right) > 2$, then $R \geq 2\frac{1}{42}$ so $R - 2 \geq \frac{1}{42}$ so $|G| \leq 84(g-1)$.

4.2.18 IV.2.6 x g pushforward of divisors

2.6. *f_* for Divisors.* Let $f: X \to Y$ be a finite morphism of curves of degree n. We define a homomorphism $f_*: \operatorname{Div} X \to \operatorname{Div} Y$ by $f_*(\sum n_i P_i) = \sum n_i f(P_i)$ for any divisor $D = \sum n_i P_i$ on X.

(a) For any locally free sheaf $\mathscr{E}$ on Y, of rank r, we define $\det \mathscr{E} = \wedge^r \mathscr{E} \in \operatorname{Pic} Y$ (II, Ex. 6.11). In particular, for any invertible sheaf $\mathscr{M}$ on X, $f_*\mathscr{M}$ is locally free of rank n on Y, so we can consider $\det f_*\mathscr{M} \in \operatorname{Pic} Y$. Show that for any divisor D on X,

$$\det(f_*\mathscr{L}(D)) \cong (\det f_*\mathscr{O}_X) \otimes \mathscr{L}(f_*D).$$

Note in particular that $\det(f_*\mathscr{L}(D)) \neq \mathscr{L}(f_*D)$ in general! [*Hint*: First consider an effective divisor D, apply f_* to the exact sequence $0 \to \mathscr{L}(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0$, and use (II, Ex. 6.11).]

Let $\mathscr{E}$ a locally free sheaf on Y of rank r. Define $\det \mathscr{E} = \wedge^r \mathscr{E} \in \operatorname{Pic} Y$. For an invertible sheaf $\mathscr{M}$ on X, $f_*\mathscr{M}$ is locally free rank n on Y. Thus we have $\det f_*\mathscr{M} \in \operatorname{Pic} Y$. If D is a divisor on X, then since $f: X \to Y$ finite, we assume X and Y are affine. Then $\mathscr{L}(-D)$ is q.c, so by thm III.8.1 (degenerate leray), $R^1 f_*\mathscr{L}(-D) = 0$. Then from the s.e.s. of $\mathscr{L}(-D)$ we get $0 \to f_*\mathscr{L}(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$.

If $D \geq 0$, then thm II.6.11.b gives $\det f_*\mathscr{L}(-D) \approx \det f_*\mathcal{O}_X \otimes (\det f_*\mathcal{O}_D)^{-1}$. Then $f_*\mathcal{O}_D \approx \oplus_{i=1}^{n}\mathcal{O}_{f_*D}$, so $\det f_*\mathcal{O}_D = \det \mathcal{O}_{f_*D} = \mathscr{L}(f_*D)$. Thus $\det f_*\mathcal{O}_D^{-1} = \mathscr{L}(-f_*D)$. If D is arbitrary, write $D = D_1 - D_2$ the difference of two effective divisors. Now look at the s.e.s. $0 \to \mathscr{L}(D) \to \mathscr{L}(-D_2) \to \mathcal{O}_{D_1} \to 0$. If we apply f_* and take determinants we get $f_*\mathscr{L}(D)$.

4.2.19 b. x g

(b) Conclude that f_*D depends only on the linear equivalence class of D, so there is an induced homomorphism $f_*: \operatorname{Pic} X \to \operatorname{Pic} Y$. Show that $f_* \circ f^*: \operatorname{Pic} Y \to \operatorname{Pic} Y$ is just multiplication by n.

Note that $\mathscr{L}(D)$ only depends on linear equvialence class.
Furthermore, $det\ f = n$ so pullback of a point gives a degree n divisor. Thus $f_* f^*$ multiplies by n.

4.2.20 c. x

(c) Use duality for a finite flat morphism (III, Ex. 6.10) and (III, Ex. 7.2) to show that
$$\det f_* \Omega_X \cong (\det f_* \mathcal{O}_X)^{-1} \otimes \Omega_Y^{\otimes n}.$$

Exc III.7.2.a gives $f^! \Omega_Y = \Omega_X$.
Exc III.6.10.a. gives $f_* \Omega_X = Hom_Y (f_* \mathcal{O}_X, \Omega_Y) = (f_* \mathcal{O}_X)^* \otimes \Omega_Y$.
Now take the determinants of both sides and note that $(f_* \mathcal{O}_X)^*$ is locally free rank n.

4.2.21 d. x branch divisor

(d) Now assume that f is separable, so we have the ramification divisor R. We define the *branch divisor* B to be the divisor $f_* R$ on Y. Show that
$$(\det f_* \mathcal{O}_X)^2 \cong \mathscr{L}(-B).$$

Note $K_X \sim f^* K_Y + R$.
Thus $f_* K_X \sim n K_Y + B$.
Thus $\mathscr{L}(-B) \approx \Omega_Y^{\otimes n} \otimes \mathscr{L}(f_* K_X)^{-1}$.
By (a) and (b), $\mathscr{L}(f_* K_X)^{-1} \approx det\ f_* \mathcal{O}_X \otimes det\,(f_* \Omega_X)^{-1}$, and $\mathscr{L}(-B) \approx (det\ f_* \mathcal{O}_X)^2$.

4.2.22 IV.2.7 x Etale Covers degree 2

2.7. *Étale Covers of Degree* 2. Let Y be a curve over a field k of characteristic $\neq 2$. We show there is a one-to-one correspondence between finite étale morphisms $f : X \to Y$ of degree 2, and 2-torsion elements of Pic Y, i.e., invertible sheaves $\mathscr{L}$ on Y with $\mathscr{L}^2 \cong \mathcal{O}_Y$.

(a) Given an étale morphism $f : X \to Y$ of degree 2, there is a natural map $\mathcal{O}_Y \to f_* \mathcal{O}_X$. Let $\mathscr{L}$ be the cokernel. Then $\mathscr{L}$ is an invertible sheaf on Y, $\mathscr{L} \cong \det f_* \mathcal{O}_X$, and so $\mathscr{L}^2 \cong \mathcal{O}_Y$ by (Ex. 2.6). Thus an étale cover of degree 2 determines a 2-torsion element in Pic Y.

A stalk of $f_* \mathcal{O}_X$ is a rank 2 free module over the stalk of $\mathcal{O}_Y$.
Thus a stalk of $f_* \mathcal{O}_X$ is isomorphic to the stalk of $\mathcal{O}_Y$.
Thus $\mathscr{L}$ is invertible.
Now $\mathscr{L} \approx det\ \mathscr{L} \approx det\ f_* \mathcal{O}_X \otimes (det\ \mathcal{O}_Y)^{-1} \approx det\ f_* \mathcal{O}_X$ via the sequence
$0 \to \mathcal{O}_Y \to f_* \mathcal{O}_X \to \mathscr{L} \to 0$.
Thus $\mathscr{L}^2 = \mathscr{L}(-B) = \mathcal{O}_Y$.

4.2.23 b. x

(b) Conversely, given a 2-torsion element $\mathscr{L}$ in Pic Y, define an $\mathcal{O}_Y$-algebra structure on $\mathcal{O}_Y \oplus \mathscr{L}$ by $\langle a,b \rangle \cdot \langle a',b' \rangle = \langle aa' + \varphi(b \otimes b'), ab' + a'b \rangle$, where φ is an isomorphism of $\mathscr{L} \otimes \mathscr{L} \to \mathcal{O}_Y$. Then take $X = \mathbf{Spec}(\mathcal{O}_Y \oplus \mathscr{L})$ (II, Ex. 5.17). Show that X is an étale cover of Y.

Let $X \to Y$ be the morphism f given by exc II.5.17.d. Then f is affine and finite and thus X is integral, separated, finite type over k, $dim\ X = 1$. X is a curve. X is smooth since normal. The function field is a degree 2 extension so by exc III.10.3, f is etale.

4.2.24 c. x

(c) Show that these two processes are inverse to each other. [*Hint*: Let $\tau: X \to X$ be the involution which interchanges the points of each fibre of f. Use the *trace map* $a \mapsto a + \tau(a)$ from $f_*\mathcal{O}_X \to \mathcal{O}_Y$ to show that the sequence of $\mathcal{O}_Y$-modules in (a)

$$0 \to \mathcal{O}_Y \to f_*\mathcal{O}_X \to \mathscr{L} \to 0$$

is split exact.

The give sequence from (a) has a section $\sigma \mapsto (\sigma + \tau\sigma)/2$, $f_*\mathcal{O}_X \to \mathcal{O}_Y$.
Thus $f_*\mathcal{O}_X \approx \mathcal{O}_Y \oplus \mathscr{L}$ so $X \approx Spec(\mathcal{O}_Y \oplus \mathscr{L})$ by exc III.5.17.

4.3 IV.3 Embeddings In Projective Space

alt:

4.3.1 IV.3.1 x g

3.1. If X is a curve of genus 2, show that a divisor D is very ample $\Leftrightarrow$ deg $D \geqslant 5$. This strengthens (3.3.4).

If $D \geq 5$, then thm 3.2 gives D is very ample.
Now suppose D is very ample.
Thus $l(D) = l(D - P - Q) \geq 2$.
Since $g = 2$, $dim\,|D| \neq 1$ so $l(D) \neq 2$.
Thus $l(D) > 2$.
Now by cases:
If $deg(D) \leq 1$, then by exc IV.1.5, $l(D) \leq deg(D) + 1 \leq 2$
If $deg(D) = 2$ then $l(D) = l(K - D) + 1 < l(K) + 1 = 2$.
If $deg(D) = 3$, then $l(K - D) = 0$ so $l(D) = 2$.
If $deg(D) = 4$ then $l(D) = 3$, then thm 3.2 gives D is generated.
Thus $|D|$ gives a morphism to $\mathbb{P}^2$ so it's a plane curve.
Then $g = \frac{1}{2}(4-1)(d-2) = 3 \neq 2$.
Since we have eliminated the impossible, whatever remains, however improbable, must be the truth.

4.3.2 IV.3.2 x g :a,b,c

3.2. Let X be a plane curve of degree 4.
(a) Show that the effective canonical divisors on X are exactly the divisors $X.L$, where L is a line in $\mathbf{P}^2$.

Let $D = X.L$.
We have $p_a(X) = 3$, so $l(K) = 3$, $deg(K) = 4$.
By bezout, $deg(D) = 4$.
$dim\,|L| = 2$ since L is determined by 2 points.

Thus $l(D) = 3$.
Then $l(K - D) = l(D) + g - deg(D) - 1 = 1$ by riemann roch and above.
Now $deg(K - D) = 4 - 4 = 0$.
Thus by exc IV.1.5 we are done.

4.3.3 b. x g

(b) If D is any effective divisor of degree 2 on X, show that $\dim|D| = 0$.

Let D effective divisor of degree 2 on X, $D = P + Q$.
K is very ample so gives an embedding to $\mathbb{P}^2$.
Note thus $dim\,|K| = 2$.
If l is the line through P, Q, then by bezout it hits X , degree 4, at 2 other points, say R, S.
By (a) we assume $K = P + Q + R + S$. Then $dim\,|D| = dim\,|K| - 2 = 2 - 2 = 0$ since K is very ample.

4.3.4 c. x g

(c) Conclude that X is not hyperelliptic (Ex. 1.7).

Part (b) shows we can't have $dim\ \ |D| = 1$ and $deg\ D = 2$.

4.3.5 IV.3.3 x g

3.3. If X is a curve of genus $\geqslant 2$ which is a complete intersection (II, Ex. 8.4) in some $\mathbf{P}^n$, show that the canonical divisor K is very ample. Conclude that a curve of genus 2 can never be a complete intersection in any $\mathbf{P}^n$. Cf. (Ex. 5.1).

Suppose X is $\cap H_i$ of hypersurfi.
By exc II.8.4.d, K is a multiple of hyperplane divisor, so $\mathscr{L}(K) \approx \mathcal{O}_X(nH) \approx \mathcal{O}_X(n)$ for some $n > 0$ since $2g - 2 > 0$.
Then $|K|$ induces d-uple embedding so K is very ample.
If $g = 2$, then degree $K = 2g - 2 = 2$ and so K is not very ample by exc IV.3.1.
Contradiction.

4.3.6 IV.3.4 x g

3.4. Let X be the d-uple embedding (I, Ex. 2.12) of $\mathbf{P}^1$ in $\mathbf{P}^d$, for any $d \geqslant 1$. We call X the *rational normal curve of degree d* in $\mathbf{P}^d$.
(a) Show that X is projectively normal, and that its homogeneous ideal can be generated by forms of degree 2.

Since $\mathbb{P}^1$ is projectively normal, then d-uple is projectively normal.
If θ is the corresponding ring homomorphism, then the kernel is generated by quadrics.

4.3.7 b. x g

(b) If X is any curve of degree d in $\mathbf{P}^n$, with $d \leqslant n$, which is not contained in any $\mathbf{P}^{n-1}$, show that in fact $d = n$, $g(X) = 0$, and X differs from the rational normal curve of degree d only by an automorphism of $\mathbf{P}^d$. Cf. (II. 7.8.5).

We have $dim\,|D| = n$, $deg\,(D) = d$ and $l\,(D) = n + 1 \leq deg\,(D) + 1 = d + 1 \leq n + 1$.
Thus $n = d$. Since $deg\,(D) = dim\,(D)$, and $d \neq 0$, then by exc IV.1.5, $g = 0$.
Thus D corresponds to an $(n+1)$ dimensional subspace in $\Gamma\,(\mathbb{P}^1, \mathcal{O}\,(n))$

4.3.8 c. x

(c) In particular, any curve of degree 2 in any $\mathbf{P}^n$ is a conic in some $\mathbf{P}^2$.

By part (b).

4.3.9 d. x g

(d) A curve of degree 3 in any $\mathbf{P}^n$ must be either a plane cubic curve, or the twisted cubic curve in $\mathbf{P}^3$.

Ok well we know genus $\leq \frac{1}{2}\,(d-1)\,(d-2) = \frac{1}{2}\,(3-1)\,(3-2) = \frac{1}{2}2 = 1$. So it's either rational or elliptic. If genus is 1, then it's a plane cubic by IV.4.6. If genus is 0, then if $n = 4$, then it's a degree 2 in $\mathbb{P}^3$. then projecting down gives a negative genus, degree 1 in $\mathbb{P}^2$ so $g \leq \frac{1}{2}\,(d-1)\,(d-2)$ it's ≤ 0... so there are no nodes, or else genus is actually < 0 there. but then it's a line in $\mathbb{P}^2$, so contained in a plane. so it's a plane cubic. Now the only other choice is if it's a cubic in $\mathbb{P}^3$ of genus 0. Are all of these twisted? yes by definition.

4.3.10 IV.3.5 x g

3.5. Let X be a curve in $\mathbf{P}^3$, which is not contained in any plane.
(a) If $O \notin X$ is a point, such that the projection from O induces a birational morphism φ from X to its image in $\mathbf{P}^2$, show that $\varphi(X)$ *must* be singular. [*Hint*: Calculate dim $H^0(X, \mathcal{O}_X(1))$ two ways.]

Suppose to the contrary that $\varphi\,(X)$ is nonsingular.
Then φ is an isomorphism.
X is not contained in a hyperplane, so $H^0\,(\mathbb{P}^3, I_X\,(1)) = 0$.
Thus $H^0\,(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}\,(1)) \to H^0\,(X, \mathcal{O}_X\,(1))$ is injective.
Thus $dim\ H^0\,(X, \mathcal{O}_X\,(1)) \geq 4$.
As $\varphi\,(X)$ is a complex intersection, (it's a curve in $\mathbb{P}^2$).
Thus exc II.5.5.a gives that $dim\ H^0\left(\varphi\,(X)\,, \mathcal{O}_{\varphi(X)}\,(1)\right) \leq dim\ H^0\,(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}\,(1)) = 3$.
Contradiction since $\varphi\,(X) \approx X$.

4.3.11 b. x g

(b) If X has degree d and genus g, conclude that $g < \frac{1}{2}(d-1)(d-2)$. (Use (Ex. 1.8).)

Projection from a point preserves degree.

X is the normalization of this projection.
By exc IV.1.8, X has a lower degree than the normalization.
Once we project enough that we are in $\mathbb{P}^2$ use genus-degree formula.

4.3.12 c. x

(c) Now let $\{X_t\}$ be the flat family of curves induced by the projection (III, 9.8.3) whose fibre over $t = 1$ is X, and whose fibre X_0 over $t = 0$ is a scheme with support $\varphi(X)$. Show that X_0 always has nilpotent elements. Thus the example (III, 9.8.4) is typical.

X_0 is the curve given by projection as in (b). But this contradicts the fact that for a flat family the fibers have the same hilbert polynomial.

4.3.13 IV.3.6 x g Curves of Degree 4

3.6. *Curves of Degree* 4.
(a) If X is a curve of degree 4 in some $\mathbf{P}^n$, show that either

(1) $g = 0$, in which case X is either the rational normal quartic in $\mathbf{P}^4$ (Ex. 3.4) or the rational quartic curve in $\mathbf{P}^3$ (II, 7.8.6), or
(2) $X \subseteq \mathbf{P}^2$, in which case $g = 3$, or
(3) $X \subseteq \mathbf{P}^3$ and $g = 1$.

Suppose $X \subset \mathbb{P}^n$.
If $n \geq 4$, then by exc IV.3.4.b, X is rational normal.
If $n = 2$, then $g(X) = \frac{1}{2}(d-1)(d-2) = 3$.
If $n \subset \mathbb{P}^3 \backslash \mathbb{P}^2$, then by exc IV.3.5.b, $g < 3$.
If $g = 2$ by exc IV.3.1, any divisor of degree 3 is not very ample so X is not embedded in $\mathbb{P}^3$ contradiction.
If $g = 0$, then X is rational quartic.

4.3.14 b. x g

(b) In the case $g = 1$, show that X is a complete intersection of two irreducible quadric surfaces in $\mathbf{P}^3$ (I, Ex. 5.11). [*Hint*: Use the exact sequence $0 \to \mathscr{I}_X \to \mathscr{O}_{\mathbf{P}^3} \to \mathscr{O}_X \to 0$ to compute $\dim H^0(\mathbf{P}^3, \mathscr{I}_X(2))$, and thus conclude that X is contained in at least two irreducible quadric surfaces.]

Suppose $g = 1$. Consider the LES associated to
$0 \to \mathcal{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$.
As $dim\ H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \begin{pmatrix} 2+3 \\ 2 \end{pmatrix} = 1$, and $dim\ H^0(X, \mathcal{O}_X(2)) = h^0(2H) = 8 + 1 - 1 = 8$ by r.r., then $dim\ H^0(\mathbb{P}^3, \mathcal{I}_X(2)) \geq 2$.
As the intersection of 2 quadrics has degree 4 by bezout this intersection is all of X (i.e. not just a component).

4.3.15 IV.3.7 x

3.7. In view of (3.10), one might ask conversely, is every plane curve with nodes a projection of a nonsingular curve in $\mathbf{P}^3$? Show that the curve $xy + x^4 + y^4 = 0$ (assume char $k \neq 2$) gives a counterexample.

By using the jacobian, $xy + x^4 + y^4 = 0$.

Any curve projecting to this would have degree 4 and genus 2 by using the degree-genus formula for singular curves.

By exc IV.3.6, this is impossible.

4.3.16 IV.3.8 x

3.8. We say a (singular) integral curve in $\mathbf{P}^n$ is *strange* if there is a point which lies on all the tangent lines at nonsingular points of the curve.

(a) There are many singular strange curves, e.g., the curve given parametrically by $x = t$, $y = t^p$, $z = t^{2p}$ over a field of characteristic $p > 0$.

Computing the tangent line at the point (t, t^p, t^2p) gives a line pointing in the direction $(1, 0, 0)$ at each point since $char\ k = p$.

We can also parametrize in x, y, w coordinates as (t^{2p-1}, t^p, t^{2p}) .

The tangent at $(0, 0, 0)$ points again in $(1, 0, 0)$ direction.

Thus $(1 : 0 : 0 : 0)$ is a strangent point.

4.3.17 b. x g No strange curves in char 0!!!!

(b) Show, however, that if char $k = 0$, there aren't even any singular strange curves besides $\mathbf{P}^1$.

$char\ k = 0$ X has finitely many singular points.

Choosing a general point gives a projection to $\mathbb{P}^3$.

If P is a strange point we choose an affine cover where P is infinity on the x-axis as in thm 3.9.

As the morphism is unramified at finitely many points, the image is a point since the map is separable in char 0 .

Thus X is a line.

4.3.18 IV.3.9 x g

3.9. Prove the following lemma of Bertini: if X is a curve of degree d in $\mathbf{P}^3$, not contained in any plane, then for almost all planes $H \subseteq \mathbf{P}^3$ (meaning a Zariski open subset of the dual projective space $(\mathbf{P}^3)^*$), the intersection $X \cap H$ consists of exactly d distinct points, no three of which are collinear.

Recall the tangent variety is a subvariety of $\mathbb{P}^1 \times X$ and thus has $dim \leq 2$.

Using the trisecant lemma (not in Hartshorne) the dimension of multisecants is ≤ 1.

Thus the union of these spaces is a proper closed subset of $(\mathbb{P}^3)^\vee$.

Thus almost all hypeplanes intersect somewhere not tangent or multisecant.

Now recall tat a hyperplane intersects at d points iff not on a tangent line and three points are collinear iff they are not on a multisecant.

4.3.19 IV.3.10 x g

> **3.10.** Generalize the statement that "not every secant is a multisecant" as follows. If X is a curve in $\mathbf{P}^n$, not contained in any $\mathbf{P}^{n-1}$, and if char $k = 0$, show that for almost all choices of $n-1$ points $P_1, \ldots, P_{n-1}$ on X, the linear space L^{n-2} spanned by the P_i does not contain any further points of X.

Let $X \subset \mathbb{P}^n$ not contained in $\mathbb{P}^{n-1}$ having degree d.

Let H_0 a hyperplane which meets X in $p_1, ..., p_d$.

For H in a small neighborhood of H_0, then the intersections of H with X vary smoothly with H.

Thus for every multiindex $I = \{i_1, ..., i_n\} \subset \{1, ..., d\}$ there is a map

$\pi_I : U \to X^n = X \times X \times ... \times X$ where H maps to the points of intersection of X with H.

For any point $Q = (q_1, ..., q_n) \in X^n$ close to $\pi_1(H_0)$ we can find a hyperplane $H \in U$ containing Q such that the image of U contains a small opens subset.

Let $D \subset X^n$ the locus of points $(q_1, ..., q_n)$ such that $q_1, ..., q_n$ are linearly dependent.

Since X is not contained in $\mathbb{P}^{n-1}$, D is a proper subvariety of X^n .

Thus $\pi^{-1}(D)$ is a proper subvariety of U.

Thus for $H \in U - \cup_I \pi_I^{-1}(D)$, the points of $H \cap X$ satisfy no n of them are linearly dependent. Thus if we choose $n-1$ of them, the n^{th} intersection will not be dependent on the first $n-1$.

(From GH p 249)

4.3.20 IV.3.11 x g

> **3.11** (a) If X is a nonsingular variety of dimension r in $\mathbf{P}^n$, and if $n > 2r+1$, show that there is a point $O \notin X$, such that the projection from O induces a closed immersion of X into $\mathbf{P}^{n-1}$.

This is Shafarevich Algebraic Geometry 1, theorem 9, page 136

4.3.21 b. x g

> (b) If X is the Veronese surface in $\mathbf{P}^5$, which is the 2-uple embedding of $\mathbf{P}^2$ (I, Ex. 2.13), show that each point of every secant line of X lies on infinitely many secant lines. Therefore, the secant variety of X has dimension 4, and so in this case there is a projection which gives a closed immersion of X into $\mathbf{P}^4$ (II, Ex. 7.7). (A theorem of Severi [1] states that the Veronese surface is the only surface in $\mathbf{P}^5$ for which there is a projection giving a closed immersion into $\mathbf{P}^4$. Usually one obtains a finite number of double points with transversal tangent planes.)

If $r \in \mathbb{P}^5$ is a general point on a secant line $\overline{v(P)\,v(Q)}$ of X, then $\overline{pq}$ maps to a conic C in X and r then lies on the plane spanned by C. Any other line on the plane passing through r is also a secant line to C and thus to X. Thus a general point on a secant line to X lies on a one-dimensional family of secant lines to X. Since we are in $\mathbb{P}^5$ at any rate, then the secant variety has dimension at most 4. On the other hand, the secant variety clearly has dimension at least 4.

4.3.22 IV.3.12 x g (just explain the advanced method)

> **3.12.** For each value of $d = 2,3,4,5$ and r satisfying $0 \leqslant r \leqslant \frac{1}{2}(d-1)(d-2)$, show that there exists an irreducible plane curve of degree d with r nodes and no other singularities.

Basic Method:

1. We use $g = \frac{1}{2}(d-1)(d-2) - r$ to find curves of a certain degree with the right amount of singularities.
2. Now for each singularity, we substitute in so that the singularity is at $(0,0,1)$.
3. We write as poly in

$$z^{d-2}f_2 + z^{d-3}f_3 + \dots$$

in order for there to be a singularity there.

4. A double point is where f_2 is nonzero (Fulton, algebraic curves or Michael Artin's Plane algebraic curves lecture notes from MIT).

This is because it is quadratic, product of homogeneous linear poly, has two zeros

5. Specifically a node is a point where the discriminant of f_2 is nonzero.

This is when f is not the square of a linear polynomial.

6. Now we:

a. Check for genus in Sage easily

b. check for singularities on the affine patches with Maple once we have a candidate.

c....

d. Profit.

More advanced method

1. Use halpen's theorem

exa [0 nodes for d=2,3,4,5] Use the comment on page 314 about Bertini theorem.

Verbatim.

exa [The maximum amount of notes for $d = 3,4,5$] Using Comment on Page 314 again, Hartshorne says, For any d, we can embed $\mathbb{P}^1$ in $\mathbb{P}^d$ as a curve of degree d , and then project it into $\mathbb{P}^2$, by (3.5) and (3.10), to get a curve X of degree d in $\mathbb{P}^2$ having only nodes, and with $g\left(\tilde{X}\right) = 0$. This gives $r = \frac{1}{2}(d-1)(d-2)$.

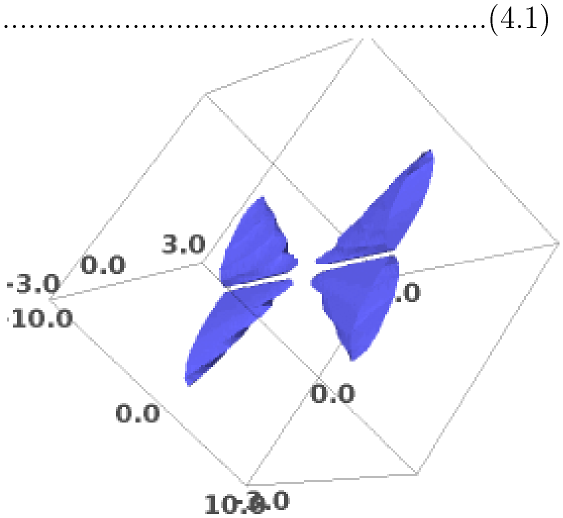
For any d, there are irreducible nonsingular curves of degree d in $\mathbb{P}^2$.

exa[1 node for d=4,5] compute partials and find singularities. Either draw the curve, or note that hessian must be invertible to see that we have nodes. Use the following polynomials:

degree 4: $xyz^2 + x^4 + y^4 = 0$ for char $\neq 2$ or $xyz^2 + x^3z + y^4 = 0$ for char 2

or

....(4.1)



Note we can also check these with Maple on the affine patches with the algcurves package

```
singularities(x*y*z+x^3 + y^3, x, y);
                 {[[0, 0, 1], 2, 1, 2]}
```

degree 5: $xyz^3 + x^5 + y^5 = 0$ for char $\neq 5$ or $xyz^3 + x^5 + y^5 + x^3y^2$ for char 5.

There is also a method using discriminants.

exa[2 nodes for $d = 4, 5$]

Now I want to do the same sort of thing for 2 nodes as 1 node, just the computations are harder, so I use a computer to help aid visualization.

x*y^3 + x^2*y*z+ x^2*z^2 + x*y*z^2 - y^2*z^2 ... for deg 4 - note genus is 1

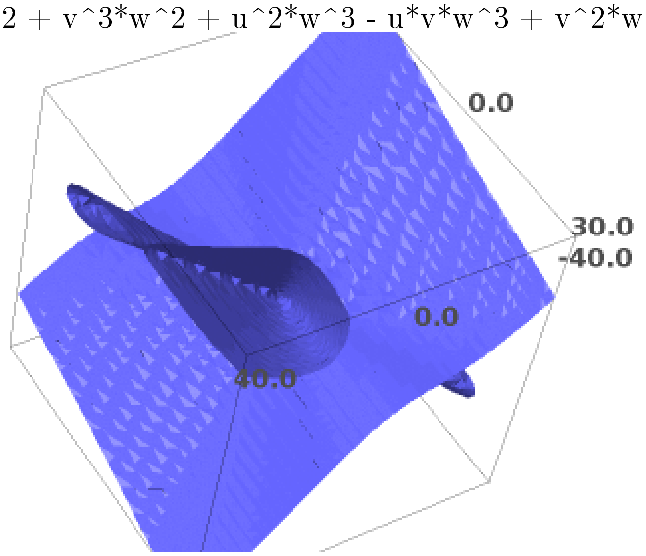
Maple checks this one out:

```
singularities(subs(z=1,x*y^3 + x^2*y*z+ x^2*z^2 + x*y*z^2 - y^2*z^2),x,y);
                 {[[0, 0, 1], 2, 1, 2], [[1, 0, 0], 2, 1, 2]}
```

We compute $g = \frac{1}{2}(4-1)(4-2) - 2 = 3 - 2 = 1$ so there can be no more singularities.

Next, I compute a genus 4 quintic with discriminant of the z^3 terms nonzero:

-u^4*v - u^3*v^2 - u^2*v^3 - u*v^4 - v^5 + u^4*w - u^3*v*w - u^2*v^2*w - v^4*w + u^3*w^2 + u^2*v*w^2 + v^3*w^2 + u^2*w^3 - u*v*w^3 + v^2*w^3



Maple likes this one too:

```
singularities(subs(w=1, f), u, v);
                 {[[0, -1, 1], 2, 1, 2], [[0, 0, 1], 2, 1, 2]}
```

theorem

Corollary [halphen corollary] There exists a curve X of degree d and genus g in $\mathbb{P}^3$, whose hyperplane section D is nonspecial, iff either

(1) $g = 0$ and $d \geq 1$,

(2) $g = 1$ and $g \geq 3$, or

(3) $g \geq 2$ and $d \geq g + 3$.

(pp366, page 350)

corollary [Any curve is birationally equivalent to...] a plane curve with at most nodes as singularities. (Hartshorne 3.11, pp331, page 314).

exa [degree 4 with 2 or 3 nodes, and degree 5 with 4, 5, or 6 nodes]

We have the following table:

degree d	Nodes r	Genus: $g = \frac{1}{2}(d-1)(d-2) - r$	Halphen Condition
5	4	$\frac{1}{2} \times 3 \times 4 - 4 = 2$	(3)
5	5	$\frac{1}{2} \times 3 \times 4 - 5 = 1$	(2)

By Halphen's Corollary, we have a curve of degree d and genus listed above in $\mathbb{P}^3$.

Now if we can argue as in the case for the top dimensional genus, then we should be all set.

Note genus is a birational invariant by Hartshorne I.8

exa [3, 4, and 5 nodes for a degree 5 curve]

So if we have three nodes, the two conditions are going to be we need genus of 3 since $\frac{1}{2}(5-1)(5-2)-3 = \frac{1}{2}(4)(3)-3 = 6-3 = 3$ is the number of singularities.

We also need discriminant of the z^3 term nonzero.

u^5 + u^3*v^2 - u*v^4 - v^5 + u^4*w + u^3*v*w - u^2*v^2*w - u^3*w^2 - u^2*v*w^2 + u*v^2*w^2 - v^3*w^2 + u^2*w^3 - u*v*w^3 + v^2*w^3

maple likes:

```
singularities(subs(w=1, f), u, v);
                                                  2
        {[[0, 0, 1], 2, 1, 2], [[RootOf(1 + _Z + _Z ), 1, 1], 2, 1, 2]}
```

```
(Projective Curve over Rational Field defined by u^5 + u^3*v^2 - u*v^4 - v^5 + u^4*w + u^3*v*w
 - u^2*v^2*w + u*v^3*w - v^4*w - u^3*w^2 + u^2*v*w^2 + u*v^2*w^2 + v^3*w^2 + u^2*w^3 - u*v*w^3
 + v^2*w^3, 3)
```

For 5 nodes, we will need genus 2.

For 5 nodes we will need genus 1.

exa [4 and 5 nodes for degree 5]

We will need genus 2 and 1 respectively.

u^4*v + u^3*v^2 + u*v^4 - u^4*w + u^3*v*w + u^2*v^2*w + v^4*w - u^2*v*w^2 + u*v^2*w^2 + u^2*w^3 - u*v*w^3 + v^2*w^3

v*w + u^2*v^2*w + v^4*w - u^2*v*w^2 + u*v^2*w^2 + u^2*w^3 - u*v*w^3 + v^2*w^3;

```
 singularities(subs(w=1, f), u, v);
                             2      3     4
[[[-1, -1 - RootOf(5 + 8 _Z + 8 _Z  + 4 _Z  + _Z ), 1], 2, 1, 2],

   [[0, 0, 1], 2, 1, 2]}
```

and at u=1

```
 singularities(subs(u=1, f), w, v);
emory used=7.6MB, alloc=4.1MB, time=1.00
                            2      3     4
[[-1, -1 - RootOf(5 + 8 _Z + 8 _Z  + 4 _Z  + _Z ), 1], 2, 1, 2],

  [[1, 0, 0], 2, 1, 2]}
```

and at v= 1

```
> singularities(subs(v=1, f), u, w);
{[[0, 1, 0], 2, 1, 2],

                   2       4                  2       4
   [[RootOf(1 + 2 _Z  + 2 _Z ), -RootOf(1 + 2 _Z  + 2 _Z ), 1], 2, 1, 2]}
```

4.4 IV.4 Elliptic Curves

Note: I will pretty much freely use and quote silverman's books for this chapter's solutions

4.4.1 IV.4.1 x g

4.1. Let X be an elliptic curve over k, with char $k \neq 2$, let $P \in X$ be a point, and let R be the graded ring $R = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nP))$. Show that for suitable choice of t, x, y,

$$R \cong k[t,x,y]/(y^2 - x(x-t^2)(x-\lambda t^2)),$$

as a graded ring, where $k[t,x,y]$ is graded by setting $\deg t = 1$, $\deg x = 2$, $\deg y = 3$.

Define $\varphi : k[x,y,t] \to R = \oplus_{n\geq 0} H^0(X, \mathcal{O}_X(nP))$, by $t \mapsto 1 \in H^0(X, \mathcal{O}_X(P))$, and define φ on x, y as in thm IV.4.6.

If $f \in k[x,y,t]$, and $(f) + nP \geq 0$ then f can have poles only at P.

Considering P to be the point at infinity, such an f lives in $\mathbb{A}^2$.

So for any section of the graded ring R, it satisfies $(f)+nP \geq 0$ some n, and we can find an $f \in \mathbb{A}^2$ mapping there under φ.

On the other hand, $y^2 - x(x-t^2)(x-\lambda t^2)$ is in the kernel so R is a quotient of the desired ring at least.

By r.r., $dim\ H^0(X, \mathcal{O}_X(nP)) = n$ and also dimension of the desired ring graded n part is n. So they must be equal.

4.4.2 IV.4.2 x

> **4.2.** If D is any divisor of degree $\geqslant 3$ on the elliptic curve X, and if we embed X in $\mathbf{P}^n$ by the complete linear system $|D|$, show that the image of X in $\mathbf{P}^n$ is projectively normal.
>
> *Note.* It is true more generally that if D is a divisor of degree $\geqslant 2g+1$ on a curve of genus g, then the embedding of X by $|D|$ is projectively normal (Mumford [4, p. 55]).

Denote by $\varphi_{|D|} : X \hookrightarrow \mathbb{P}^N$ the embedding. Suppose E is an effective divisor of degree $d-2$. Consider the s.e.s. $0 \to \mathscr{L}(-E) \to \mathcal{O}_X \to i_*\mathcal{O}_E \to 0$. This gives $0 \to \mathscr{L}(D) \otimes \mathscr{L}(-E) \to \mathscr{L}(D) \to i_*\mathcal{O}_E \to 0$, $supp\ i_*\mathcal{O}_D = Supp\ E$. Then $deg \mathscr{L}(D) \otimes \mathscr{L}(-E) = d - deg\ E = d - d + 2 = 2$. By serre duality, $H^1(\mathscr{L}(D-E)) = H^0(\mathscr{L}(K+E-D)) = 0$. This gives a commutative diagram:

$$\begin{tikzcd} \Gamma(-E+D) \otimes \Gamma(nD) \arrow[r] \arrow[d, "f"] & \Gamma(D) \otimes \Gamma(nD) \arrow[r] \arrow[d, "g"] & \Gamma(i_*\mathcal{O}_E) \otimes \Gamma(nD) \arrow[r] \arrow[d, "h"] & 0 \\ \Gamma(-E+D+nD) \arrow[r] & \Gamma(D+nD) \arrow[r] & \Gamma(i_*\mathcal{O}_E + nD) & \end{tikzcd}$$

Snake lemma gives us an s.e.s. $coker\ f \to coker\ g \to coker\ h$.

As $deg\ D \geq 3 > 2g-2 = 0$, $|D|$ is bpf and $Supp\ i_*\mathcal{O}_D$ is clearly 0 -dimensional. We can yse the base-point rfee pencil trick (Geom Alg Curves I, page 126) to get $coker\ h = 0$ so $coker\ f \twoheadrightarrow coker\ g$.

R.R. gives $h^0(D-E) = deg(D-E) + h^1(D-E) = d-d+2 = 2$. Therefore $H^0(D-E)$ is a basepoint free pencil, so again by BPFPT, $coker\ f = 0$ and g is surjective.

Since $|D|$ is complete, we have a surjection $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \to \Gamma(X, \mathcal{O}_X(1))$ by ex II.7.8.4. Suppose we have a similar surjection when twisted by n. We have a square

$$\begin{tikzcd} \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(1)) \otimes \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \arrow[r] \arrow[d] & \Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n+1)) \arrow[d] \\ \Gamma(X, \mathcal{O}_X(1)) \otimes \Gamma(X, \mathcal{O}_X(n)) \arrow[r] & \Gamma(X, \mathcal{O}_X(n+1)) \end{tikzcd}$$

The left is surjective clearly and since g is surjective then the bottom is surjective. Thus the right is surjective so by induction, X is projectively normal.

4.4.3 IV.4.3 x g

4.3. Let the elliptic curve X be embedded in $\mathbf{P}^2$ so as to have the equation $y^2 = x(x-1)(x-\lambda)$. Show that any automorphism of X leaving $P_0 = (0,1,0)$ fixed is induced by an automorphism of $\mathbf{P}^2$ coming from the automorphism of the affine (x,y)-plane given by

$$\begin{cases} x' = ax + b \\ y' = cy. \end{cases}$$

In each of the four cases of (4.7), describe these automorphisms of $\mathbf{P}^2$ explicitly, and hence determine the structure of the group $G = \operatorname{Aut}(X, P_0)$.

We can write E in weierstrass form as f so that if $k[E] = k[x,y]/(f)$ is the ring of regular functions. , a regular function can be written as $v(x) + y \cdot w(x)$. If E is defined by $y^2 = x^3 + ax^2 + bx + c$, then a rational function of x and y on E (i..e. in $frac\ (k[x,y]/(f))$) can be written as $a(x) + b(x)y$, with $a(x), b(x) \in K(x)$ (As in Algebra 1.46, NN.30, my summerstudychallenge2 notes). Thus we can write an isogeny as $\phi(x,y) = (a(x) + b(x)y, c(x) + d(x)y)$ for $a, b, c, d(x) \in K(x)$.

Since this is an isogeny then $\phi(P) + \phi(-P) = \phi(P - P) = O$ hence $(a(x) + b(x)y, c(x) + d(x)y) = (a(x) - b(x)y, -c(x) + d(x)y)$ so that $b(x), c(x)$ are 0 and thus $\phi(x,y) = (a(x), d(x)y)$. Now suppose that $[a(x)](0,1) = 0$ and $[d(x)y](0,1) = 1$. So clearly $d(x) \neq 0$. so it has a constant term. But since there are no y terms on rhs, there can be no x terms in $d(x)$. Thus $d(x)$ is just some constant. Now in order for the degrees to be correct, $a(x)$ must be linear.

4.4.4 IV.4.4 x

4.4. Let X be an elliptic curve in $\mathbf{P}^2$ given by an equation of the form

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

Show that the j-invariant is a rational function of the a_i, with coefficients in $\mathbf{Q}$. In particular, if the a_i are all in some field $k_0 \subseteq k$, then $j \in k_0$ also. Furthermore, for every $\alpha \in k_0$, there exists an elliptic curve defined over k_0 with j-invariant equal to α.

We'll write the j-invariant via the tate coefficients and show this equals Hartshorne's definition of the j-invariant. I will show my calculations and the maxima output even though it looks a tiny bit sloppy.

Write the tate coefficients as $b_2 = a_1^2 + 4a_2$, $b_4 = a_1a_3 + 2a_4$, $b_6 = a_3^2 + 4a_6$, $b_8 = b_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2$, $c_4 = b_2^2 - 24b_4$, $c_6 = -b_2^3 + 36b_2b_4 - 216b_6$, $\Delta = -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6$, $j = \frac{c_4^3}{\Delta}$.

```
f:y^2 + a1*x*y + a3*y - x^3 - a2*x^2 - a4*x - a6;
```

If char $\neq 2$, this means 2 is invertible so we can simplify by completing the square via $y \mapsto (y - a_1x - a_3)$.

```
g:expand(4*subst(1/2 * (y - a1*x - a3), y, f));
```

```
       2     3         2     2  2                                2
(%o4)  y  - 4 x  - 4 a2 x  - a1  x  - 4 a4 x - 2 a1 a3 x - 4 a6 - a3
```

This gives $E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6$,. Now replace (x,y) with $(x, 2y)$ and factor to get $y^2 = (x - e_1)(x - e_2)(x - e_3)$.

```
h:expand(1/4*subst(2*y, y, g));
i: -1*h + y^2;
```

```
                                      2  2                              2
              2    3       2   a1  x             a1 a3 x          a3
(%o10)        y  - x  - a2 x  - ------ - a4 x - ------- - a6 - ---
                                   4                 2             4
```

Now i is a cubic which will factor into
$(x - e_1)(x - e_2)(x - e_3)$ for some e_i in the algebraic closure so this gives an equation like $y^2 = x(x-1)(x-\lambda)$.
ok so that we get a j-invariant which agrees with Hartshorne's for algebraically closed.
Now we want to show that for the curve $y^2 = x(x-1)(x-\lambda)$, hartshorne's

$$j(E_\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda-1)^2}.$$

j-invariant of

$$j = c_4^3/\Delta,$$

agrees with our j-invariant of
We expand $y^2 = x(x-1)(x-\lambda)$

```
expand(x * (x - 1) * (x - L));
```

```
                             2          3    2
(%o14)               - x  L + x L + x  - x
```

So comparing to Weierstrass equation, $E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$ gives $a_1 = 0$, $a_3 = 0$, $a_2 = -\lambda - 1$, $a_4 = \lambda$ and $a_6 = 0$.

```
a1:0; a3:0; a2:-L-1;a4:L;a6:0;
```

We find the b_i's

```
b2:(a1)^2 + 4*a2; b4:2*(a4) + a1 * a3; b6:(a3)^2 + 4*a6;
```

$b_2 = 4\lambda$, $b_4 = 2\lambda$, and $b_6 = 0$.
Now we need c_4 and Δ.

```
b8:a1^2*a6 + 4*a2 * a6 - a1 * a3 * a4 + a2 * a3^2 - a4^2;
c4:b2^2 - 24 * b4;
c6:-b2^3 + 36 * b2 * b4 - 216 * b6;
Delta:-b2^2 * b8 - 8*b4^3 - 27 * b6^2 + 9*b2 * b4 * b6;
j:factor(c4^3 / Delta);
```

$b_8 = -\lambda^2$
$c_4 = 16(-\lambda - 1)^2 - 48\lambda$
$c_6 = 288(-\lambda - 1)\lambda - 64(-\lambda - 1)^3$
$\Delta = 16(-\lambda - 1)^2 \lambda^2 - 64\lambda^3$
$j = \frac{256(\lambda^2 - l + 1)^3}{(\lambda-1)^2\lambda^2}$.
so new x -coordinate is

```
        2         2        4
       L  - 2 x  L + x
  ----------------------------
               2               3
  4 x L + 4 x  (- L - 1) + 4 x
```

So we have shown our definitions agree. Note that $\Delta \neq 0$ is another definition for an elliptic curve (weierstrass equation, nonzero discriminant. Clearly defining j this way will give a fraction of coefficients of the weierstrass equation.

Now we may wish to show the second statement about finding j's.
We can use the following curves:
If $char\,(k) = 2$, $j_0 = 0$, then $y^2 + y = x^3$, $j_0 \neq 0$ $y^2 + xy + x^3 + x^2 + j_0^{-1}$.
If $char\,(k) = 3$, $j_0 = 0$, then $y^2 = x^3 + x$, $j_0 \neq 0$, then $y^2 = x^3 + x^2 - j_0^{-1}$.
If $char\,(k) \neq 2, 3$, $j_0 = 0$, then $y^2 = x^3 + 1$, $j_0 = 12^3$, then $y^2 = x^3 + x$, $j_0 \neq 0, 12^3$, then $y^2 = x^3 + 2\kappa X + 2\kappa$, $\kappa = \frac{j_0}{12^3 - j_0}$.

4.4.5 IV.4.5 x

4.5. Let X, P_0 be an elliptic curve having an endomorphism $f: X \to X$ of degree 2.

(a) If we represent X as a 2-1 covering of $\mathbf{P}^1$ by a morphism $\pi: X \to \mathbf{P}^1$ ramified at P_0, then as in (4.4), show that there is another morphism $\pi': X \to \mathbf{P}^1$ and a morphism $g: \mathbf{P}^1 \to \mathbf{P}^1$, also of degree 2, such that $\pi \circ f = g \circ \pi'$.

This goes almost exactly like 4.4.

4.4.6 b. x

(b) For suitable choices of coordinates in the two copies of $\mathbf{P}^1$, show that g can be taken to be the morphism $x \to x^2$.

So from part (a), we have
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\pi"] & X \arrow[d, "\pi'"] \\ \mathbb{P}^1 \arrow[r, "g"] & \mathbb{P}^1 \end{tikzcd}$$
and we know that f, g have degree 2.

By Silverman, exa III.4.5 we consider the two elliptic curves $E_1 : y^2 = x^2 + ax^2 + bx$, and $E_2 : Y^2 = X^2 - 2aX^2 + rX$.

We have isogenies of degree 2 connecting the curves:

$\phi : E_1 \to E_2$, $(x, y) \mapsto \left(\frac{y^2}{x^2}, \frac{y\left(b-x^2\right)}{x^2}\right)$, and $\hat{\phi} : E_2 \to E_1$, $(X, Y) \mapsto \left(\frac{Y^2}{4X^2}, \frac{Y\left(r-X^2\right)}{8X^2}\right)$.

Now if E has a degree 2 endomorphism has a kernel with two points, O and an order 2 point. Moving the order 2 point to $(0, 0)$ gives a weierstrass equation, $E : y^2 = x^3 + ax^2 + bx$ which looks like E_1 above and we know the equation for E_1's isogeny. Thus we just need to see (see part (d) for example) when E_1 and E_2 are isomorphic. This calculation is performed on page 110 Silverman, AEC II.

Now assuming we have such a curve with $E_1 \approx E_2$, the maps π and π' are given by projection of the x-coordinate. Clearly $g : x \mapsto \left(\frac{y^2}{x^2}\right)$ (if coordinates on the second $\mathbb{P}^1$ are given by $X := \left(\frac{y}{x}\right)$.

4.4.7 c. x

(c) Now show that g is branched over two of the branch points of π, and that g^{-1} of the other two branch points of π consists of the four branch points of π'. Deduce a relation involving the invariant λ of X.

Since it's a degree 2 morphism to $\mathbb{P}^1$, then by Riemann-Hurwitz, there are 4 branch points. Factoring $y^2 = x(x-1)(x-\lambda) = x^3 + (-1-\lambda)x^2 + \lambda x$ these branch points are values of x for which there are one value of y, as well as the point at infinite.

Note that $x \mapsto x^2 : \mathbb{P}^1 \to \mathbb{P}^1$ is degree 2 so by Riemann-Hurwitz, $-2 = 2 \cdot (-2) + R$ implies there are two branch points. (Point at infinity and 0). Taking $\pi : \infty \to \infty$ gives one of the branch points of g, and after taking our equation to the form $y^2 = x^3 + ax^2 + bx$ as in (b), we see there is another branch point at 0, corresponding to the root at 0. Clearly this is a branch point of $x \mapsto x^2$. To see the reverse direction, look at $\hat{\phi}$ given in (b).

Now to see a relation involving the invariant we look at (b) and see that we need, for $E_1 \approx E_2$ that the j-invariants be equal. Thus, by the forms of E_1, E_2 in (b) we need $\frac{256\left((-1-\lambda)^2-3\lambda\right)^3}{\lambda^2\left((-1-\lambda)^2-4(\lambda)\right)} = \frac{16\left((-1-\lambda)^2+12\lambda\right)^3}{\lambda\left((-1-\lambda)^2-4\lambda\right)^2}$.

4.4.8 d. x

> (d) Solving the above, show that there are just three values of j corresponding to elliptic curves with an endomorphism of degree 2, and find the corresponding values of λ and j. [*Answers:* $j = 2^6 \cdot 3^3$; $j = 2^6 \cdot 5^3$; $j = -3^3 \cdot 5^3$.]

Expanding the relation (I used maxima since lazy) given above gives the polynomial $256\lambda^{11} - 1808\lambda^{10} + 5504\lambda^9 - 21696\lambda^8 - 5760\lambda^7 + 47008\lambda^6 -$
$5760\lambda^5 - 21696\lambda^4 + 5504\lambda^3 - 1808\lambda^2 + 256\lambda$.

To find the roots, I use the numerical durandkerner algorithm. Below is an implementation I did for gma5th (my personal math library), though you would need some of my other code to run it ..

```
function durandkerner(eq)
        local i=1 local j=1
        local numterms = table.maxn(eq)
        local so = {} so[1] = {} so[1][1] = .4 so[1][2] = "" --initial point
        if verbose then
                print("Durand-Kerner")
                print"Roots of:"
                peq(eq)
                print""
        end

        so[2] = {}
        so[2][1] = .9 so[2][2]= "i"
        local ttab = {}
        local ntht = {}
        --finding max the first time
        local max = 0
        for i=1,table.maxn(eq) do --find max exp
                if numvars(eq[i][2],"x") > max then
                        max = numvars(eq[i][2],"x")
                end
        end
        --finding min exp to extract zero roots
        local min = max
        for i=1,table.maxn(eq) do --find max exp
                if numvars(eq[i][2],"x") < min then
                        min = numvars(eq[i][2],"x")
                end
        end
        --dividing by x if there are zero roots
        for i=1,min do
                for j=1,table.maxn(eq) do
                        eq[j][2] = string.sub(eq[j][2], 2)
                end
        end
        if min > 0 and verbose then
                print("Note that there is a root of multiplicity "..min.." at zero"
        end
        eq = ceq(eq)
        --peq(eq)
```

```
local max = 0
for i=1,table.maxn(eq) do --find max exp
	if numvars(eq[i][2],"x") > max then
		max = numvars(eq[i][2],"x")
	end
end
--then poly needs to be switched depending on max power
eq = seq(eq,(-1)^(max))

db("here1")
local nume = {}
local denom = {}
local rootab = {}
local v = verbose
verbose = false

for i=1,max do
	rootab[i] = exeq(so,i-1)
end
if verbose then
	print "Initial guess"
	for i=1,table.maxn(rootab) do
		peq(rootab[i])
	end
	print""
end

db("here2 ")
local n=1
--local last = 2341
repeat
	--last = rootab[1][1][1]
	db("here3")
	for i=1,table.maxn(rootab) do
		db("here 4")
		if rootab[i] == nil then rootab[i] = zeropoly() end
		nume = eveq(eq,rootab[i],"x")
		if type(nume) ~= "table" then print"error in durker" end
		denom = constantpoly()
		db("here 4")
		for j=1,table.maxn(rootab) do
			if i ~= j then
				--print("here2")
				if rootab[j] == nil then
					rootab[j] = zeropoly()
				end
				denom = ceq(meq(denom,aeq(rootab[i],seq(roo
			end
		end
		db("here5")
		--print("here1")
```

```
                    --printmat(rootab[i])
--                   print(table.maxn(nume))
                    --printmat(seq(conjdeq(nume,denom),-1))
                    db("here6")
                    --
                    if table.maxn(nume) > 0 and table.maxn(denom) > 0 then
                            rootab[i] = aeq(rootab[i],seq(conjdeq(nume,denom),-
                    end
                    dbprint(rootab[i][1][1])
                    for j=1,table.maxn(rootab[i]) do
                            if 1/rootab[i][j][1] > 1e+25 then
                                    rootab[i][j][1] = 0
                            end
                    end
                    db("here7")
            end

            n=n+1
            if v then
                    if n % 3 == 0 then
                            print("roots at iteration ",n)
                            for i=1,table.maxn(rootab) do
                                    peq(rootab[i])
                            end
                            print""
                    end
            end
    until n> 15
    verbose = v
    db("here end")
    if verbose then
            print " The roots of this polynomial are approximately: "
    end
            for i=1,table.maxn(rootab) do
                    peq(rootab[i])
        print("norm ", normeq(rootab[i]))
            end
    return rootab
end
```

Throwing away roots which give singular elliptic curves, and plugging into the j-invariant equation, we are left with $j = 1728$, $j = 8000$, $j = -3375$.

4.4.9 IV.4.6.a. x g

> **4.6.** (a) Let X be a curve of genus g embedded birationally in $\mathbf{P}^2$ as a curve of degree d with r nodes. Generalize the method of (Ex. 2.3) to show that X has $6(g-1)+3d$ inflection points. A node does not count as an inflection point. Assume char $k = 0$.

The arithmetic genus, $p_a(X) = \frac{1}{2}(d-1)(d-2) - r$ where r is the number of nodes.

Define φ to be the gauss map from the plane curve to a line with no nodes.

φ is rational so it gives a regular map $X \to \mathbb{P}^1$.

If P is a point on the plane curve which is not on a tangent that has an inflection or on a multiple tangent, and π gives projection from P, then π induces a map from X to $\mathbb{P}^1$ which has degree d.

By Hurwitz, this gives $d^2 - d - 2r$ tangents of the plane curve through P, which is therefore the degree of φ.

As in exc IV.2.3.a, ignore the ramification, then the plane curve has $3d^2 - 6d - 6r$ inflection points.

4.4.10 b. x osculating hyperplanes

> (b) Now let X be a curve of genus g embedded as a curve of degree d in $\mathbf{P}^n$, $n \geqslant 3$, not contained in any $\mathbf{P}^{n-1}$. For each point $P \in X$, there is a hyperplane H containing P, such that P counts at least n times in the intersection $H \cap X$. This is called an *osculating hyperplane* at P. It generalizes the notion of tangent line for curves in $\mathbf{P}^2$. If P counts at least $n+1$ times in $H \cap X$, we say H is a *hyperosculating hyperplane*, and that P is a *hyperosculation point*. Use Hurwitz's theorem as above, and induction on n, to show that X has $n(n+1)(g-1) + (n+1)d$ hyperosculation points.

Example 4 of http://www.math.lsa.umich.edu/~idolga/sol.pdf which are some notes from Dolgachaev.

4.4.11 c. x g

> (c) If X is an elliptic curve, for any $d \geqslant 3$, embed X as a curve of degree d in $\mathbf{P}^{d-1}$, and conclude that X has exactly d^2 points of order d in its group law.

By (b), X has d^2 hyperosculating points.

If X is embedded via $|dP_0|$, then P is a hyperosculating point when it is the divisor of a hyperplane is dP which happens when P has order dividing d in the group law (see also II.6 excercises).

4.4.12 IV.4.7 x g Dual of a morphism

> **4.7.** *The Dual of a Morphism.* Let X and X' be elliptic curves over k, with base points P_0, P'_0.
> (a) If $f: X \to X'$ is any morphism, use (4.11) to show that $f^*: \operatorname{Pic} X' \to \operatorname{Pic} X$ induces a homomorphism $\hat{f}: (X', P'_0) \to (X, P_0)$. We call this the *dual* of f.

Note that the by thm IV.4.11, the picard groups and jacobians coincide.

Now IV.4.10.6 gives that jacobian automatically has a group structure.

We know there is an induced homomorphism on picard groups.

So we just compose the correspondence between jacobian variety and picard group with the pullback on piard group.

4.4.13 b. x

> (b) If $f: X \to X'$ and $g: X' \to X''$ are two morphisms, then $(g \circ f)^{\wedge} = \hat{f} \circ \hat{g}$.

This is clear from (a).

4.4.14 c. x g

(c) Assume $f(P_0) = P'_0$, and let $n = \deg f$. Show that if $Q \in X$ is any point, and $f(Q) = Q'$, then $\hat{f}(Q') = n_X(Q)$. (Do the separable and purely inseparable cases separately, then combine.) Conclude that $f \circ \hat{f} = n_{X'}$ and $\hat{f} \circ f = n_X$.

Silverman, thm 6.1, around page 82

Case 1: f is separable. Since f has degree n, then $\sharp ker\ f = n$ so every element of $ker\ f$ has order dividing n. Thus $ker\ f \subset ker\ n$.

By Galois theory, there is an inclusion $[n]^* K(E) \subset f^* K(E) \subset K(E)$ so we can find a map λ satisfying $f^* \lambda^* K(E) = [n]^* K(E)$ so that $\lambda \circ f = [n]$. Clearly $\lambda = \hat{f}$.

---

*(d) If $f, g : X \to X'$ are two morphisms preserving the base points P_0, P'_0, then $(f + g)\hat{} = \hat{f} + \hat{g}$. [*Hints*: It is enough to show for any $\mathscr{L} \in \operatorname{Pic} X'$, that $(f + g)^* \mathscr{L} \cong f^* \mathscr{L} \otimes g^* \mathscr{L}$. For any f, let $\Gamma_f : X \to X \times X'$ be the graph morphism. Then it is enough to show (for $\mathscr{L}' = p_2^* \mathscr{L}$) that

$$\Gamma^*_{f+g}(\mathscr{L}') = \Gamma^*_f \mathscr{L}' \otimes \Gamma^*_g \mathscr{L}'.$$

Let $\sigma : X \to X \times X'$ be the section $x \to (x, P'_0)$. Define a subgroup of $\operatorname{Pic}(X \times X')$ as follows:

$\operatorname{Pic}_\sigma = \{\mathscr{L} \in \operatorname{Pic}(X \times X') | \mathscr{L}$ has degree 0 along each fibre of p_1, and $\sigma^* \mathscr{L} = 0$ in $\operatorname{Pic} X\}$.

Note that this subgroup is isomorphic to the group $\operatorname{Pic}^{\circ}(X'/X)$ used in the definition of the Jacobian variety. Hence there is a 1-1 correspondence between morphisms $f : X \to X'$ and elements $\mathscr{L}_f \in \operatorname{Pic}_\sigma$ (this defines $\mathscr{L}_f$). Now compute explicitly to show that $\Gamma^*_g(\mathscr{L}_f) = \Gamma^*_f(\mathscr{L}_g)$ for any f, g.

Use the fact that $\mathscr{L}_{f+g} = \mathscr{L}_f \otimes \mathscr{L}_g$, and the fact that for any $\mathscr{L}$ on X', $p_2^* \mathscr{L} \in \operatorname{Pic}^{\circ}_\sigma$ to prove the result.]

MISS - strred

4.4.15 e. x

(e) Using (d), show that for any $n \in \mathbf{Z}$, $\hat{n}_X = n_X$. Conclude that $\deg n_X = n^2$.

Silverman thm 6.2, page 83

4.4.16 f. x

(f) Show for any f that $\deg \hat{f} = \deg f$.

Silverman thm 6.2, page 83

4.4.17 IV.4.8 x Algebraic Fundamental Group

4.8. For any curve X, the *algebraic fundamental group* $\pi_1(X)$ is defined as $\varprojlim \mathrm{Gal}(K'/K)$, where K is the function field of X, and K' runs over all Galois extensions of K such that the corresponding curve X' is étale over X (III, Ex. 10.3). Thus, for example, $\pi_1(\mathbf{P}^1) = 1$ (2.5.3). Show that for an elliptic curve X,

$$\pi_1(X) = \prod_{l \text{ prime}} \mathbf{Z}_l \times \mathbf{Z}_l \qquad \text{if char } k = 0;$$

$$\pi_1(X) = \prod_{l \neq p} \mathbf{Z}_l \times \mathbf{Z}_l \qquad \text{if char } k = p \text{ and Hasse } X = 0;$$

$$\pi_1(X) = \mathbf{Z}_p \times \prod_{l \neq p} \mathbf{Z}_l \times \mathbf{Z}_l \quad \text{if char } k = p \text{ and Hasse } X \neq 0,$$

where $\mathbf{Z}_l = \varprojlim \mathbf{Z}/l^n$ is the l-adic integers.

[*Hints*: Any Galois étale cover X' of an elliptic curve is again an elliptic curve. If the degree of X' over X is relatively prime to p, then X' can be dominated by the cover $n_X : X \to X$ for some integer n with $(n,p) = 1$. The Galois group of the covering n_X is $\mathbf{Z}/n \times \mathbf{Z}/n$. Étale covers of degree divisible by p can occur only if the Hasse invariant of X is not zero.]

This is not something that incredibly excites me. Here are some notes that have the answer: http://math.berkeley

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4.4.18 IV.4.9 x g isogeny is equivalence relation.

4.9. We say two elliptic curves X, X' are *isogenous* if there is a finite morphism $f : X \to X'$.

(a) Show that isogeny is an equivalence relation.

Reflexivity is clear.
Symmetry is by exc IV.4.7.c. (dual isogeny)
Reflexivity is clear by composition of finites.

4.4.19 b. x g

(b) For any elliptic curve X, show that the set of elliptic curves X' isogenous to X, up to isomorphism, is countable. [*Hint*: X' is uniquely determined by X and $\ker f$.]

Every isogeny is a finite map of curves.
Thus we have an inclusion of function fields $K(X') \hookrightarrow K(X)$.
The degree of this inclusion is the degree of the field extension.
Since degree 1 would mean an isomorphism, and degrees come in nonnegative integer sizes, we are done.

4.4.20 IV.4.10 x picard of product on genus 1

4.10. If X is an elliptic curve, show that there is an exact sequence

$$0 \to p_1^* \operatorname{Pic} X \oplus p_2^* \operatorname{Pic} X \to \operatorname{Pic}(X \times X) \to R \to 0,$$

where $R = \mathrm{End}(X, P_0)$. In particular, we see that $\operatorname{Pic}(X \times X)$ is bigger than the sum of the Picard groups of the factors. Cf. (III, Ex. 12.6), (V, Ex. 1.6).

Following Mumford, Abelian Varieties, let $T_x : X \to X$ be the translation, $T_x(y) = x+y$. Let $m : X \times X \to X$ be addition. The theorem of the square (cor 4) gives us a homomorphism $\mathscr{L} \mapsto \phi_{\mathscr{L}} : Pic(X) \to R$, where $\phi_{\mathscr{L}}$ is defined by $\phi_{\mathscr{L}}(x)$ is the isomorphism class of $T_x^* \mathscr{L} \otimes \mathscr{L}^{-1}$ in $Pic(X)$. Then $Pic^0(X)$ is the set of line bundles $\mathscr{L}$ where $\phi_{\mathscr{L}}$ is identically 0. We therefore have an exact sequence $0 \to Pic^0(X) \to Pic(X) \to Hom(X, Pic^0(X)) \to 0$. Note moreover that $\mathscr{L} \in Pic^0(X) \iff T_x^* \mathscr{L} \approx \mathscr{L}$ for all $x \in X$ (by definition) $\iff m^* \mathscr{L} \approx p_1^* \mathscr{L} \otimes p_2^* \mathscr{L}$ on $X \times X$. This last equality follows since by the seesaw theorem (essentially the theorem of the cube) $m^* \mathscr{L} \otimes p_1^* \mathscr{L}^{-1} \otimes p_2^* \mathscr{L}^{-1}$ is trivial iff it is trivial on $X \times \{a\}$ and $\{0\} \times X$. Clearly it is always trivial on $\{0\} \times X$ and restricts to $T_a^* \mathscr{L} \otimes \mathscr{L}^{-1}$ on $X \times \{a\}$. Thus our exact sequence implies exactness of the desired sequence.

4.4.21 IV.4.11 x g

4.11. Let X be an elliptic curve over $\mathbf{C}$, defined by the elliptic functions with periods $1, \tau$. Let R be the ring of endomorphisms of X.
(a) If $f \in R$ is a nonzero endomorphism corresponding to complex multiplication by α, as in (4.18), show that $\deg f = |\alpha|^2$.

An elliptic curve corresponds to a lattice L. If $A(L)$ is the area, then $A(\alpha L) = |\alpha^2| A(L)$.
For a sublattice, the degree of the field extension is the inverse quotient of the areas.
Thus $deg\ f = deg(\alpha) = [L : \alpha L] = |\alpha|^2 \cdot A(L) / A(L)$.

4.4.22 b. x

(b) If $f \in R$ corresponds to $\alpha \in \mathbf{C}$ again, show that the dual $\hat{f}$ of (Ex. 4.7) corresponds to the complex conjugate $\bar{\alpha}$ of α.

By part (a), we have $deg\ f = |\alpha|^2$.
We know that $f \circ \hat{f} = \left[|\alpha|^2\right]$ by Silverman, thm 6.2, (page 52??).
Thus taking f^{-1} , we have $\hat{f} = \alpha^{-1} |\alpha|^2 = \overline{\alpha}$.

4.4.23 c. x

(c) If $\tau \in \mathbf{Q}(\sqrt{-d})$ happens to be integral over $\mathbf{Z}$, show that $R = \mathbf{Z}[\tau]$.

This is theorem VI.5.5 Silverman since integral means a finitely generated module contained in $\mathbb{Q}(\tau)$.

4.4.24 IV.4.12.a x

4.12. Again let X be an elliptic curve over $\mathbf{C}$ determined by the elliptic functions with periods $1, \tau$, and assume that τ lies in the region G of (4.15B).
(a) If X has any automorphisms leaving P_0 fixed other than ± 1, show that either $\tau = i$ or $\tau = \omega$, as in (4.20.1) and (4.20.2). This gives another proof of the fact (4.7) that there are only two curves, up to isomorphism, having automorphisms other than ± 1.

Write the curve as $y^2 = x^3 + Ax + B$.

Via Milne, Elliptic Curves, Theorem 2.1, automorphisms fixing the point have the form $x = u^2 x'$, $y = u^3 y'$ for $u \in \mathbb{C}^*$ and the substitution gives an automorphism of E iff $u^{-4} A = A$ and $u^{-6} B = B$. Checking possibilities, if $AB \neq$ then $u = \pm 1$. If $B = 0$ then writing A, B in terms of λ gives $j = 1728$. If $A = 0$ then $j = 0$. Now in part (b), we'll see these match the desired values of τ .

4.4.25 b. x

(b) Now show that there are exactly three values of τ for which X admits an endomorphism of degree 2. Can you match these with the three values of j determined in (Ex. 4.5)? [*Answers*: $\tau = i$; $\tau = \sqrt{-2}$; $\tau = \frac{1}{2}(-1 + \sqrt{-7})$.]

Note that we'll use exc 4.11 to connect τ from the lattice to the ring of integers, and then prove a supplementary lemma for the similar case. Basically the idea is we need to find quadratic extensions which have elements of norm 2.

Note for $x = a + \sqrt{m}b \in \mathbb{Q}(\sqrt{m})/\mathbb{Q}$ the norm is given by $N(x) = a^2 - mb^2$. So we need to solve $2 = a^2 - mb^2$ with m prime and negative. For example $2 = a^2 + 1b^2$ works for $a = b = 1$. so $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$ is one possibility For example $2 = a^2 + 2b^2$ works for $a = 0, b = 1$. So these are clearly the only solutions for $m \equiv 2, 3 \ mod\ 4$.

If $m \equiv 1 \ mod\ 4$, then $\mathcal{O}_K = \left\{a + b\left(\frac{1+\sqrt{d}}{2}\right), a, b \in \mathbb{Z}\right\}$ which we can write as $\left(a + \frac{b}{2}\right) + \frac{b}{2}\sqrt{d}$ to compute the norm. In later case we need $\left(a + \frac{b}{2}\right)^2 + m\left(\frac{b}{2}\right)^2 = 2$ so need $expand\left(\left(a + \frac{b}{2}\right)^2 + m \cdot \left(\frac{b}{2}\right)^2\right) = \frac{b^2 m}{4} + \frac{b^2}{4} + a\, b + a^2 = 2$. So if we let $m = 7$, $a = 0$, $b = 2$ then we get $\frac{1 \cdot 7}{4} + \frac{1}{4} + 0 + 0 = 2$.

If m is any larger, it is clear that the lhs will be too large, and thus these are the only rings of integers under quadratic extensions with elements of norm 2. So we are done.

Now we want to match j with τ. But the method of proof of exc IV.4.5.b accomplishes this since we are just using the fact of corresponding to degree 2.

4.4.26 IV.4.13 x

4.13. If $p = 13$, there is just one value of j for which the Hasse invariant of the corresponding curve is 0. Find it. [*Answer*: $j = 5 \pmod{13}$.]

This is $j = 5 \ mod\ 13$. The uniqueness is Silverman Theorem 4.1.c , page 149.

The curve is incidentally given by (y^2-x^3+165*x+110)^12.

So given a j-invariant, you can solve the j-invariant equation to get a curve.

You can check this is unique, and then use thm IV.4.21 (and some computer algebra software since it's a large polynomial) to see the Hasse invariant is 0.

4.4.27 IV.4.14 x Fermat Curve and Dirichlet's Theorem

4.14. The Fermat curve $X: x^3 + y^3 = z^3$ gives a nonsingular curve in characteristic p for every $p \neq 3$. Determine the set $\mathfrak{P} = \{p \neq 3 | X_{(p)}$ has Hasse invariant $0\}$, and observe (modulo Dirichlet's theorem) that it is a set of primes of density $\frac{1}{2}$.

Dirichlet's theorem gives that the Dirichlet density of primes in an arthmetic progression $a + nb$ for a, b coprime has dirichlet density $1/\varphi(b)$. The condition that the Hasse Invariant be 0 is that $(xyz)^{p-1}$ has coefficient 0 in the expansion of $(x^3 + y^3 - z^3)^{p-1}$. Taking a trinomial expansion,$(x^3 + y^3 - z^3)^{p-1} = \sum_{k_x+k_y+k_z=p-1} \begin{pmatrix} p-1 \\ k_x, k_y, k_z \end{pmatrix} (x^3)^{k_x} (y^3)^{k_y} (z^3)^{k_z}$. The term we want is $\frac{(p-1)!}{(((p-1)/3)!)^3} (xyz)^{p-1}$. So for what p is this coefficient 0? What if $p-1$ has no factor of 3? Ignore those primes since then automatically $(xyz)^{p-1}$ has coefficient 0 since it doesn't appear in the above summation. Thus all primes like $3k+2$ are automatically hasse invariant 0 for this. Modulo dirichlet theorem, this gives us a set of density $\frac{1}{\varphi(3)} = \frac{1}{2}$. Otherwise, this coefficient should not be zero mod p , since p is a larger prime! so we have the whole set.

4.4.28 IV.4.15 x

> **4.15.** Let X be an elliptic curve over a field k of characteristic p. Let $F' : X_p \to X$ be the k-linear Frobenius morphism (2.4.1). Use (4.10.7) to show that the dual morphism $\hat{F}' : X \to X_p$ is separable if and only if the Hasse invariant of X is 1. Now use (Ex. 4.7) to show that if the Hasse invariant is 1, then the subgroup of points of order p on X is isomorphic to $\mathbf{Z}/p$; if the Hasse invariant is 0, it is 0.

This is silverman V.3.1, page 144

4.4.29 IV.4.16 x

> **4.16.** Again let X be an elliptic curve over k of characteristic p, and suppose X is defined over the field $\mathbf{F}_q$ of $q = p^r$ elements, i.e., $X \subseteq \mathbf{P}^2$ can be defined by an equation with coefficients in $\mathbf{F}_q$. Assume also that X has a rational point over $\mathbf{F}_q$. Let $F' : X_q \to X$ be the k-linear Frobenius with respect to q.
> (a) Show that $X_q \cong X$ as schemes over k, and that under this identification, $F' : X \to X$ is the map obtained by the qth-power map on the coordinates of points of X, embedded in $\mathbf{P}^2$.

This is Qing Liu, 3.2.26a

4.4.30 b.x g kernel of frobenius

> (b) Show that $1_X - F'$ is a separable morphism and its kernel is just the set $X(\mathbf{F}_q)$ of points of X with coordinates in $\mathbf{F}_q$.

Separability is thm III.5.5 Silverman. Essentially we know that F is separable iff the pullback on sheaves of differentials is injective as in II.8. (The proof follows as in IV.2 in the proof of Hurwitz). Thus inseparable iff $\psi^*\omega = 0$ where ω is the invariant differential of the curve. Now compute the frobenius of invariant differential using the tate coefficients.

Now note that the fixed points of frobenius are the points in $\mathbb{F}_q$ since $a \in \mathbb{F}$ lies in $\mathbb{F}_q$ when $a^q = a$.

4.4.31 c. x

> (c) Using (Ex. 4.7), show that $F' + \hat{F}' = a_X$ for some integer a, and that $N = q - a + 1$, where $N = \# X(\mathbf{F}_q)$.

I give a proof on page 11, 12 of my Fall Break Number Theory Remix Notes from 2012 http://divisibility.files.word
alternatively

4.4.32 d. x Hasse's Riemann Hypothesis for Elliptic Curves

(d) Use the fact that $\deg(m + nF') > 0$ for all $m,n \in \mathbf{Z}$ to show that $|a| \leqslant 2\sqrt{q}$. This is Hasse's proof of the analogue of the Riemann hypothesis for elliptic curves (App. C, Ex. 5.6).

I give a proof on page 7,8 of my fall break number theory remix notes http://divisibility.files.wordpress.com/2012/
Alternatively, see my proof in exc V.1.10 and use $g = 1$.
alternatively, via granville

4.4.33 e. x

(e) Now assume $q = p$, and show that the Hasse invariant of X is 0 if and only if $a \equiv 0 \pmod p$. Conclude for $p \geqslant 5$ that X has Hasse invariant 0 if and only if $N = p + 1$.

The first assertion is proved in the first three paragraphs of Silverman theorem V.4.1.

Now, using the previous parts, if $p \geq 5$, have $a = p - N + 1 \equiv 0\,(mod\ p) \iff hasse = 0$. (since assume $q = p$) If $N = p + 1$, then clearly hasse $= 0$. Now if $p + 1 - N \equiv 0\,(mod\ p)$ then $pk = (p + 1) - N$ some k. Since $N \geq 0$, then we are done.

4.4.34 IV.4.17 a. x

4.17. Let X be the curve $y^2 + y = x^3 - x$ of (4.23.8).
(a) If $Q = (a,b)$ is a point on the curve, compute the coordinates of the point $P + Q$, where $P = (0,0)$, as a function of a,b. Use this formula to find the coordinates of nP, $n = 1,2,\ldots,10$. [*Check*: $6P = (6,14)$.]

Well here is a lua function I made for the group law:

- ../gma5th.lua -pv "wpeq(leq('g'))"
- where h is a file containing

```
andrew@andrew-HP-Folio-13-Notebook-PC:~$ ./gma5th.lua -pv "wpeq(leq('h'))"
Printing weierstrass a coeffs for equation
1yyy +1yy +-1xxx +1x
a1 =     0
a3 =     0
a2 =     0
a4 =     -1
a6 =     0
other form see hartshorne IV.4.4
y^2 -x^3 -0*x^2 -0*x^2 --1*x -0*x -0
j invariant      1728
Group law for weierstrass equation, following Silverman, pp76
may need some editing for + - combos or 1c coefficients before it runs

function GroupLaw(x1, y1, x2, y2)
  local x3 = 1
  local y3 = 1
  if x1 ~= x2 then
    lambda = (y2 - y1)/ (x2 - x1)
```

```
      nu = (y1 * x2 - y2*x1) / (x2 - x1)
    elseif x1 == x2 then
      lambda = (3*x1^2 + 2*(0)*x1 + -1 - (0)*y1) / (2 * y1 + (0)*x1 + (0) )
      nu = (-x1^3 + (-1)*x1 + 2*(0) - (0)*y1) / (2*y1 + (0)*x1 + (0))
    end
    local x3 = lambda^2 + (0)*lambda - (0) - x1 - x2
    local y3 = -1*(lambda + (0))*x3 - nu - (0)
    return x3, y3
  end
```

4.4.35 b. x

(b) This equation defines a nonsingular curve over $\mathbf{F}_p$ for all $p \neq 37$.

Such a curve is nonsingular iff the discriminant is nonzero.
Note the discriminant is defined via the Tate coefficients (definition 1.3 Schmitt)
so $a_3 = 1, a4 = -1$, so $b_2 = 0$, $b_4 = 2 \cdot a_4 = -2$, $b_6 = a_3^2 = 1^2$, $b_8 = -a_4^2 = -1$
so $\Delta = 0 - 8 \cdot 2^3 - 27 \cdot 1 + 9 \cdot 0 = 8 \cdot 8 - 27 = -64 - 27 = 37 \neq 0$.

4.4.36 IV.4.18 x

4.18. Let X be the curve $y^2 = x^3 - 7x + 10$. This curve has at least 26 points with integer coordinates. Find them (use a calculator), and verify that they are all contained in the subgroup (maybe equal to all of $X(\mathbf{Q})$?) generated by $P = (1,2)$ and $Q = (2,2)$.

- can probably find these with my calculator "sage", then use the additional law in silverman...

- ```
  sage: E = EllipticCurve(QQ,[0, 0, 0, -7, 10])
  sage: Q=E(2,2)
  sage: P=E(1,2)
  sage: E.integral_points(mw_base=[P,Q],both_signs=True)
  [(-3 : -2 : 1), (-3 : 2 : 1), (-2 : -4 : 1), (-2 : 4 : 1), (-1 : -4 : 1),
  (-1 : 4 : 1), (1 : -2 : 1), (1 : 2 : 1), (2 : -2 : 1), (2 : 2 : 1), (3 : -
  (3 : 4 : 1), (5 : -10 : 1), (5 : 10 : 1), (9 : -26 : 1), (9 : 26 : 1),
  (13 : -46 : 1), (13 : 46 : 1), (31 : -172 : 1), (31 : 172 : 1), (41 : -262
  (41 : 262 : 1), (67 : -548 : 1), (67 : 548 : 1), (302 : -5248 : 1), (302 :
  ```

  ```

4.4.37 IV.4.19 x

4.19. Let X, P_0 be an elliptic curve defined over $\mathbf{Q}$, represented as a curve in $\mathbf{P}^2$ defined by an equation with integer coefficients. Then X can be considered as the fibre over the generic point of a scheme $\bar{X}$ over Spec $\mathbf{Z}$. Let $T \subseteq$ Spec $\mathbf{Z}$ be the open subset consisting of all primes $p \neq 2$ such that the fibre $X_{(p)}$ of $\bar{X}$ over p is nonsingular. For any n, show that $n_X: X \to X$ is defined over T, and is a flat morphism. Show that the kernel of n_X is also flat over T. Conclude that for any $p \in T$, the natural map $X(\mathbf{Q}) \to X_{(p)}(\mathbf{F}_p)$ induced on the groups of rational points, maps the n-torsion points of $X(\mathbf{Q})$ *injectively* into the torsion subgroup of $X_{(p)}(\mathbf{F}_p)$, for any $(n,p) = 1$.

By this method one can show easily that the groups $X(\mathbf{Q})$ in (Ex. 4.17) and (Ex. 4.18) are torsion-free.

The fact that $n_X : X \to X$ is defined over T is theorem IV.5.3.c Silverman AEC II. Now let $t \in T$ and consider $[n_X]_t : X_t \to X_t$. Note that n_X is obtained by composing multiplication by prime factors p of n. By thm II.6.8, each such p_X is either constant or flat. If $[n_X]_t$ is constant, then one of the $[p_X]_t$ is constant. But this doesn't happen by the Criterion of Neron-Ogg-Shafarevich: thm VII.7.1 Silverman AEC I. Thus $[n_X]_t$ is finite flat by thm II.6.8 / thm IV.4.17. The last statement follows by Silverman AEC I, VII.3.1.b.

4.4.38 IV.4.20 x g

4.20. Let X be an elliptic curve over a field k of characteristic $p > 0$, and let $R = \operatorname{End}(X, P_0)$ be its ring of endomorphisms.

(a) Let X_p be the curve over k defined by changing the k-structure of X (2.4.1). Show that $j(X_p) = j(X)^{1/p}$. Thus $X \cong X_p$ over k if and only if $j \in \mathbf{F}_p$.

- Ok here's what we do. Assume for convenience the curve is in the form $y^2 = x\,(x-1)\,(x-\lambda) = expand\,(x \cdot (x -$
  $-x^2\,\lambda + x\,\lambda + x^3 - x^2$
- So $a_1 = 0, a_2 = -\lambda - 1,\ a_3 = 0,\ a_4 = \lambda,\ a_6 = 0.$ are the tate coefficients.
  - The j-invariant, a'la Ex, 4.4 is $\frac{256\left(\lambda^2-l+1\right)^3}{(\lambda-1)^2\lambda^2}$.
  - On the other hand, if we let $a_1' = 0, a_2' = (-\lambda-1)^p,\ a_3 = 0,\ a_4 = \lambda,\ a_6 = 0,$ be the tate coefficients of X^p then the j-invariant will be a'la Ex 4.4 $\frac{256\left(3\lambda^p-(-\lambda-1)^{2p}\right)^3}{\lambda^{2p}\left(4\lambda^p-(-\lambda-1)^{2p}\right)}$
  - Take p^{th} power of the first one, and factor mod p gives the second. (It helps to use a computer algebra system such as maxima to do the computations for you)

4.4.39 Slight issue?

(b) Show that p_X in R factors into a product $\pi\hat{\pi}$ of two elements of degree p if and only if $X \cong X_p$. In this case, the Hasse invariant of X is 0 if and only if π and $\hat{\pi}$ are associates in R (i.e., differ by a unit). (Use (2.5).)

- There is a slight error with this problem. Or there is an error on other sources. In characteristic p, then multiplication by p is never separable, so it always factors!

  - If it factors as frobenius -> separable, then both are size p.
  - if it factors as frobenius -> frobenius, then both are size p.

- So I think there is actually a slight error in this guy.. or it's trivial. Since I read Milne's Modular forms notes earlier this year...

  - Also it's in some notes from MIT

- By assumption of the problem, we are in characteristic p

- Thus the multiplication by p map is either purely inseparable (in which case, the multiplication by p map factors as $E \to E^{(p^2)} \approx E$ so in this case multiplication by p factors as two frobenius morphisms each of degree p.

- If the mult by p map is separable / inseparable, then it's separable / inseparable degrees are p, and it factors into two things of degree p.

- In characteristic p is it true that $X \approx X_p$?

- hmm...

- so if $E \approx E^p$ then $E \approx E^{p^2}$ then it factors as $E \to E^{(p^2)} \to E$.

4.4.40 c. x

(c) If Hasse $(X) = 0$ show in any case $j \in \mathbf{F}_{p^2}$.

Suppose $Hasse\,(X) = 0$.
By exc IV.4.15, the subgroup of points of order p on X is 0.
Now use Silverman, Theorem 3.1, page 144, 145.

4.4.41 d. x

(d) For any $f \in R$, there is an induced map $f^*: H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$. This must be multiplication by an element $\lambda_f \in k$. So we obtain a ring homomorphism $\varphi: R \to k$ by sending f to λ_f. Show that any $f \in R$ commutes with the (nonlinear) Frobenius morphism $F: X \to X$, and conclude that if Hasse $(X) \neq 0$, then the image of φ is $\mathbf{F}_p$. Therefore, R contains a prime ideal $\mathfrak{p}$ with $R/\mathfrak{p} \cong \mathbf{F}_p$.

Recall that R is the ring of endomorphisms of X fixing P_0. Recall that an isogeny is defined by polynomials with coefficients in k. Thus it is clear that f commutes with frobenius since all such polynomials do. Now how does hasse invariant relate to frobenius? So the frobenius F also gives a map $F^* : H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_X)$. Note that $\mathbb{F}_p \subset k$ is the fixed point set of frobenius, and if Hasse invariant is nonzero, $\mathbb{F}_p$ are the points of order p which will be potential images by what we will see next. So if $\lambda_f \notin \mathbb{F}_p$, then $\lambda_f \cdot F^*(x) \neq F^*(\lambda_f) \cdot F^*(x) = F^*(\lambda_f x)$. Since we know it does commute, then λ_f must be in $\mathbb{F}_p$. Then by group isomorphism theorem since $\varphi : R \to k$, then just take the kernel will be a prime ideal such that $R/\mathfrak{p} \approx \mathbb{F}_p$ (it's prime since $\mathbb{F}_p$ is a field).

4.4.42 IV.4.21 x skip - not algebraic geometry

4.21. Let O be the ring of integers in a quadratic number field $\mathbf{Q}(\sqrt{-d})$. Show that any subring $R \subseteq O$, $R \neq \mathbf{Z}$, is of the form $R = \mathbf{Z} + f \cdot O$, for a uniquely determined integer $f \geqslant 1$. This integer f is called the *conductor* of the ring R.

This isn't really algebraic geometry. See for instance, Dummit and Foote

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IV.4.22* (starred)

***4.22.** If $X \to \mathbf{A}^1_{\mathbf{C}}$ is a family of elliptic curves having a section, show that the family is trivial. [*Hints*: Use the section to fix the group structure on the fibres. Show that the points of order 2 on the fibres form an étale cover of $\mathbf{A}^1_{\mathbf{C}}$, which must be trivial, since $\mathbf{A}^1_{\mathbf{C}}$ is simply connected. This implies that λ can be defined on the family, so it gives a map $\mathbf{A}^1_{\mathbf{C}} \to \mathbf{A}^1_{\mathbf{C}} - \{0,1\}$. Any such map is constant, so λ is constant, so the family is trivial.]

MISS

4.5 IV.5 Canonical Embedding

4.5.1 IV.5.1 x g complete intersect is nonhyperelliptic

5.1. Show that a hyperelliptic curve can never be a complete intersection in any projective space. Cf. (Ex. 3.3).

3.3. gives us that the canonical bundle K is very ample if it's a complete intersection.
But 5.2 says $|K|$ is v.a. iff X is non-hyperelliptic.

4.5.2 IV.5.2 x g Aut X is finite.

5.2. If X is a curve of genus $\geqslant 2$ over a field of characteristic 0, show that the group Aut X of automorphisms of X is finite. [*Hint*: If X is hyperelliptic, use the unique g^1_2 and show that Aut X permutes the ramification points of the 2-fold covering $X \to \mathbf{P}^1$. If X is not hyperelliptic, show that Aut X permutes the hyperosculation points (Ex. 4.6) of the canonical embedding. Cf. (Ex. 2.5).]

Proof 1
See Dawei-Chen Notes MT845, proposition 4.8, and use Weierstrass points.

Proof 2
If X is hyperelliptic, then it has a g^1_2 so a degree 2 map $f : X \to \mathbb{P}^1$.
By hurwitz, it's ramified at $2g + 2$ points.
Any automorphism of X is determined by whether or not it permutes the ramification points.
By connectedness of X, any nontrivial automorphism has no fixed points.

Automorphisms of X are therefore determined by automorphisms of $\mathbb{P}^1$ permuting all the ramification points.

Automorphisms of $\mathbb{P}^1$ are determined by where the three points $\{0, 1, \infty\}$ are sent.

So we are permuting $2g+2$ points with $g \geq 2$, and thus since there are only finitely many ways to permute them, $Aut(X)$ is finite.

Now suppose X is non-hyperelliptic.

Then X has $(g-1)^2 g + gd$ hyperosculating points. by exc. IV.4.6.

An automorphism of $\mathbb{P}^{g-1}$ (where X is embedded) is determined by where it sends $g+1$ points not on the same hyperplane.

By comparing this number with d and $g-1$ we see that all hyperosculating points cannot lie on a hyperplane of degree d thus the finite number of hyperosculating points determine $Aut(X)$ so that it is finite.

4.5.3 IV.5.3 x g Moduli of Curves of Genus 4

5.3. *Moduli of Curves of Genus* 4. The hyperelliptic curves of genus 4 form an irreducible family of dimension 7. The nonhyperelliptic ones form an irreducible family of dimension 9. The subset of those having only one g_3^1 is an irreducible family of dimension 8. [*Hint*: Use (5.2.2) to count how many complete intersections $Q \cap F_3$ there are.]

Hyperelliptic.

These are classified by the hurwitz scheme (see my notes on Severi + Hurwitz scheme) which has dimension $2g-1$.

Nonhyperelliptic

So let E a projective bundle parametrizing complete intersections of cubics and quadrics in $\mathbb{P}^3$.

We have a surjection to the quadrics in $\mathbb{P}^3$, $\pi : E \to \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \mathbb{P}^9$.

Now we must add the dimension of a fiber (cubics intersecting each quadric) and then quotient by $PGL(3)$ action.

For the fiber over a point Q, we have an exact sequence:

$0 \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) \overset{Q}{\to} H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) \to H^0(\mathbb{P}^3, \mathcal{O}_Q(3)) = E_q \to 0$

And thus $dim\ E_q = H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(3)) - H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1)) - 1 =$

$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} - 1 = 20 - 4 - 1 = 15.$

Also $dim\ \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - 1 = \frac{5 \cdot 4}{2} - 1 = 9.$

So now we add $15+9 = 24$ and quotient by $PGL(3)$ which has dim $(3+1)^2 - 1 = 15$ gives us $24 - 15 = 9$.

Only one g_3^1 .

A curve with only one g_3^1 corresponds to sublocus where quadric is singular

So we counted the quadrics by $dim\ \mathbb{P}H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2)) = \begin{pmatrix} 5 \\ 3 \end{pmatrix} - 1 = 10 - 1.$

Note that 10 is also the number of parameters in the **symmetric** matrix below.

Note that a singular quadric is rank 3, so in the **symmetric** matrix, there are 9 parameters.

In this case we need Q to be singular, i.e. quadric cone.

So basically I just need to say that the dimension of the space of quadric cones is going to be 8.

Note that quadric forms are zero sets of the following matrix:

Consider $P(x, y, z) = ax^2 + by^2 + 2fxy + 2gyz + 2hzx + 2px + 2qy + 2rz + d$.

This is a matrix product $X^t \cdot A \cdot X = 0$, $X = (x, y, z, 1)^t$, and $A = \begin{pmatrix} a & f & h & p \\ f & b & g & q \\ h & g & c & r \\ p & q & r & d \end{pmatrix}$.

4.5.4 IV.5.4 x g

5.4. Another way of distinguishing curves of genus g is to ask, what is the least degree of a birational plane model with only nodes as singularities (3.11)? Let X be nonhyperelliptic of genus 4. Then:

(a) if X has two g_3^1's, it can be represented as a plane quintic with two nodes, and conversely;

Suppose X is nonhyperelliptic with two g_3^1's each giving degree 3 maps to $\mathbb{P}^1$. Let $\varphi : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the product morphism. If X_0 is the image its on a quadric so it has type (a, b) for some a, b. If φ has degree e, then $ea = 3$, $eb = 3$ since these are coming from g_3^1's so either $a = b = 1$ and $e = 3$ or $a = b = 3$ and $e = 1$. In the first case X_0 is smooth and rational and the projections from $X_0 \to \mathbb{P}^1$ are injective hence the two g_3^1's must coincide so that gives a contradiction.

Now we want to bound the number of singularities to 2. Consider the projection π from a point P_0 on X to $\mathbb{P}^2$. By bezout, if π maps P, Q to the same point, then the line through P_0, P, Q lies in Q. Since there are only two lines through any given point of Q the image of X under π can have only two singularities.

Now if more than two points are collapsed then 4 points are collinear on X. Since X is type $(3, 3)$ it is a complete intersection of a quadric and cubic. But then the cubic and the quadric both contain lines through 4 points which is a contradiction. Since we are on a quadric, geometrically it is clear that we cannot collapse a g_3^1 through a tangent line and similarly, there are no secants with coplanar tangent lines for the same reason. Computing the genus of normalization gives degree 5.

On the other hand if we have a plane quintic with two nodes then the normalization has genus 4, and genus is a birational invariant of a curve so X has genus 4. A line through one of the nodes meets X in 4 other points since it has multiplicity 2 there and using degree 5 and bezout. Thus we have a g_3^1 .

If X was hyperelliptic, then it has a g_2^1 , but also a g_3^1 taking the product gives a map to a quadric with $ea = 2$, $eb = 3$ as above. so $e = 1$ so the product morphism is birational to something with a different genus, contradiction.

4.5.5 x g

(b) if X has one g_3^1, then it can be represented as a plane quintic with a tacnode (I, Ex. 5.14d), but the least degree of a plane representation with only nodes is 6.

Note X is nonhyperelliptic, genus 4 by assumption.

By example IV.5.2.2, X lies on a unique irreducible quadric surface Q. By example IV.5.5.2, Q is singular. Also by example IV.5.2.2, X is the complete intersection of the quadric cone with a cubic surface F.

Projecting from a point P on X gives a morphism to $\mathbb{P}^2$. If P lies on a trisecant L the projection is $3-1$ at some points. By Bezout, as Q has degree 2 and L intersects Q in 3 points, then L must lie on Q, and as Q has a unique ruling through P, then the projection is birational from X to a plane curve which must have only the one singularity from the trisecant. If on the other hand there is a multisecant line L which meets X in more than 3 points, then by Bezout L must lie on both Q and F. But a quadric and a cubic forming a complete intersection don't share a line. This contradiction shows there must be at most trisecants.

Has one singular point. Then we have $4 = \frac{1}{2}(5-1)(5-2) - r = \frac{1}{2}4 \cdot 3 = 6 - r$ so $r = 2$. Now using the chart around page 506-508 which tells how much a one singularity will drop the genus, we see that the singular point corresponds to a tacnode. Geometrically, this is also fairly clear, we need to cut X with a hyperplane meeting X tangent at an inflection in one point P and tangent at a concave point Q and then project down from a point on the line $\overline{PQ}$ but not between P, Q .

Now suppose we have a plane quintic with degee less than 6 and only nodes. By degree genus formula for normalization, which was a previous excercise, we must have two nodes. Each node gives a g_3^1 as in ex IV.5.5.2. Thus we have two g_3^1 's which gives a contradiction.

4.5.6 IV.5.5 x g Curves of Genus 5

5.5. *Curves of Genus* 5. Assume X is not hyperelliptic.

(a) The curves of genus 5 whose canonical model in $\mathbf{P}^4$ is a complete intersection $F_2.F_2.F_2$ form a family of dimension 12.

There is probably a way to do this via Hartshorne, however, let $Hilb_r^{p(t)}$ denote the hilbert space corresponding to the hilbert polynomial $p(t)$.

By Arbarello, Cornalba, ... Geometry of Algebraic Curves II, I.5.11 we have

$h^0(X, N_{X/\mathbb{P}^r})$ is an upper bound for the dimension of $Hilb_r^{p(t)}$.

By the next theorem, 5.12, the dimension of irreducible components of $Hilb_r^{p(t)}$ at a point is at least $h^0(X, N_{X/\mathbb{P}^r}) - h^1(X, N_{X/\mathbb{P}^r})$.

So we need to compute cohomology of the normal bundle of X.

From Sernesi's book on moduli theory, (I forget the title)

$N_{X/\mathbb{P}^r} \approx \mathcal{O}_X(2) \oplus \mathcal{O}_X(2) \oplus \mathcal{O}_X(2)$.

So, just need to compute $h^0(X, \mathcal{O}(2)) \oplus h^0(X, \mathcal{O}(2)) \oplus h^0(X, \mathcal{O}(2))$ and $h^1(X, \mathcal{O}(2)) \oplus h^1(X, \mathcal{O}(2)) \oplus h^1(X, \mathcal{O}(2))$. Well, $h^0(X, \mathcal{O}(2)) - h^1(X, \mathcal{O}(2)) = d + 1 - g = 8 \cdot 2 + 1 - 5 = 17 - 5 = 12$ Now multiply that by 3 gives 36. Now modulo by $PGL(4) = (4+1)^2 - 1 = 24$, still get 12 since $36 - 24 = 12$. Note we should have an upper bound on the dimension, since these guys are in $\mathfrak{M}_5$ which has dimension $3g - 3 = 15 - 3 = 12$. does this actually work? Hopefully

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4.5.7 b. x g

(b) X has a g_3^1 if and only if it can be represented as a plane quintic with one node. These form an irreducible family of dimension 11. [*Hint*: If $D \in g_3^1$, use $K - D$ to map $X \to \mathbf{P}^2$.]

Has g13 implies plane quintic with one node

Suppose X has a divisor D which gives a degree 3 embedding to $\mathbb{P}^1$.

Thus $h^0(D) = 2$.

Then $deg(K_X - D) = 2 \cdot 5 - 2 - 3 = 5$.

By R.R., $\chi(D) = 3 + 1 - 5 = -1$ so $h^0(K_X - D) = h^0(D) + 1 = 3$.

Thus $K_X - D$ is a g_5^2 which maps X to $\mathbb{P}^2$.

By degree-genus, $5 = p_g(X) = \frac{1}{2}(5-1)(5-2) - \sharp\text{nodes}$.

Solving gives one node.

Plane quintic with one node implies g13

Let $f : X \to \mathbb{P}^2$, and $\mathcal{O}_X(E) = f^*\mathcal{O}_{\mathbb{P}^2}(1)$.

Then $deg(E) = 5$, so $deg(K_X - E) = 2 \cdot 5 - 2 - 5 = 3$. (2g-2 - 5)

By r.r., $h^0(E) - h^0(K_X - E) = 5 + 1 - 5 = 1$.

Necessarily, $h^0(f^*\mathcal{O}_{\mathbb{P}^2}(1)) \geq h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$ and so $h^0(K_X - E) \geq 2$.

Thus E is special, and hence by clifford, $dim|E| \leq \frac{1}{2}deg(E) = \frac{1}{2} \cdot 5$.

Thus $3 \leq h^0(E) = dim|E| + 1 \leq \frac{7}{2} = 3.5$.

Thus $h^0(E) = 3$, and thus $h^0(K_X - E) = 2$, so $K_X - E$ is a g_3^1 .

Irreducible family

See my notes on Severi Varieties and Hurwitz schemes: $dim\ V_{5,5} = 3d + g - 1 = 15 + 5 - 1 = 19$... Now subtract $dim\ PGL(2) = (2+1)^2 - 1 - 8$ and we're good.

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*(c) In that case, the conics through the node cut out the canonical system (not counting the fixed points at the node). Mapping $\mathbf{P}^2 \to \mathbf{P}^4$ by this linear system of conics, show that the canonical curve X is contained in a cubic surface $V \subseteq \mathbf{P}^4$, with V isomorphic to $\mathbf{P}^2$ with one point blown up (II, Ex. 7.7). Furthermore, V is the union of all the trisecants of X corresponding to the g_3^1 (5.5.3), so V is contained in the intersection of all the quadric hypersurfaces containing X. Thus V and the g_3^1 are unique.

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- . already showed g_3^1 is unique.

4.5.8 IV.5.6 x g

5.6. Show that a nonsingular plane curve of degree 5 has no g_3^1. Show that there are nonhyperelliptic curves of genus 6 which cannot be represented as a nonsingular plane quintic curve.

I feel slightly iffy about this one.
So a plane curve will have genus 6 by the degree genus formula if it is degree 5.
Suppose X has a divisor D which gives a degree 3 embedding to $\mathbb{P}^1$.
Thus $h^0(D) = 2$.
Then $deg(K_X - D) = 2 \cdot 6 - 2 - 3 = 7$.
By R.R., $\chi(D) = 3 + 1 - 6 = -2$ so $h^0(K_X - D) = h^0(D) + 2 = 4$.
Thus $K_X - D$ is a g_7^4 which maps X to $\mathbb{P}^2$.
By degree-genus, $6 = p_g(X) = \frac{1}{2}(7-1)(7-2) - \sharp\text{nodes}$.
Then $6 = \frac{1}{2} \cdot 6 \cdot 5 - \sharp nodes$ so $6 = 15 - \sharp nodes$.
So there are 7 nodes.

For the second part see Arbarello, Harris, Geometry of Algebraic Curves I.

4.5.9 IV.5.7.a x g

5.7. (a) Any automorphism of a curve of genus 3 is induced by an automorphism of $\mathbf{P}^2$ via the canonical embedding.

Suppose first that C is non-hyperelliptic.

Note that a morphism between nonsingular non-hyperelliptic curves then the pullback of regular differentials map to regular differentials. Thus such a morphism lifts to a morphism between projective spaces via the canonical embedding.

If C is hyperelliptic it has a unique degree two map f to $\mathbb{P}^1$ ramified at 8 points by Hurwitz theorem. Any automorphism preserves f but permutes the 8 points. Thus the morphism is determined by the action on the fibers. Since the complement of the 8 points on C is connected, then the action must be free. Such automorphisms are just given by automorphisms of $\mathbb{P}^1$ permuting 8 points. Now apply thm IV.5.3.

- .

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*(b) Assume char $k \neq 3$. If X is the curve given by

$$x^3y + y^3z + z^3x = 0,$$

the group Aut X is the simple group of order 168, whose order is the maximum $84(g-1)$ allowed by (Ex. 2.5). See Burnside [1, §232] or Klein [1].

solved:

*(c) Most curves of genus 3 have no automorphisms except the identity. [*Hint*: For each n, count the dimension of the family of curves with an automorphism T of order n. For example, if $n = 2$, then for suitable choice of coordinates, T can be written as $x \to -x$, $y \to y$, $z \to z$. Then there is an 8-dimensional family of curves fixed by T; changing coordinates there is a 4-dimensional family of such T, so the curves having an automorphism of degree 2 form a family of dimensional 12 inside the 14-dimensional family of all plane curves of degree 4.]

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4.6 IV.6 Curves In P3

4.6.1 IV.6.1 x g

6.1. A rational curve of degree 4 in $\mathbf{P}^3$ is contained in a unique quadric surface Q, and Q is necessarily nonsingular.

Denote by X a rational curve of degree 4 in $\mathbb{P}^3$. Consider the LES associated to $0 \to \mathscr{I}_X(2) \to \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_X(2) \to 0$. Let D the hyperplane section. $dim\,|2D| - dim\,|K-2D| = 8+1-0 = 9$. Also $deg\ K = 2g-2 = -2$ clearly $dim\,|K-2D| = 0$. Thus $h^0(\mathcal{O}_X(2)) = 9$ and since $h^0(\mathcal{O}_{\mathbb{P}^3}(2)) = \begin{pmatrix} 3+2 \\ 3 \end{pmatrix} = 10$, then by exactness, $h^0(\mathscr{I}_X(2)) \geq 1$. So at least X lies on a quadric. If X is contained in two quadrics, then by by Bezout, X is a complete intersection and by exc II.8.4 thus has genus $\frac{1}{2}4\cdot(2+2-4)+1 = 1$ contradiction. Thus Q is unique. Now note that by exc IV.5.6.b.3 and since X is, at any rate, nonsingular that X is the rational normal quartic which has $n+1$ linearly independent points in $\mathbb{P}^n$ and thus must be contained in a nondegenerate quadric. But nondegenerate quadrics are nonsingular by chapter I.

4.6.2 IV.6.2 x g

6.2. A rational curve of degree 5 in $\mathbf{P}^3$ is always contained in a cubic surface, but there are such curves which are not contained in any quadric surface.

Always in a cubic

To see it's always in a cubic, consider $h^0(\mathcal{O}_{\mathbb{P}^3}(3)) - h^0(\mathcal{O}_X(3)) = \frac{factorial(6)}{factorial(3)\cdot factorial(3)} - ... = 20 - ...$ To get the second term, note $d > 2g-2$ thus nonspecial (the degree of the divisor $3P$), so $h^0(\mathcal{O}_X(3)) = 3d+1-g = 15+1-0 = 16$ by R.R. Thus the ideal sheaf twisted by 3 has some global sections.

Exist ones not in any quadric

First note that the rational curve of degree 5 X cannot be contained in a quadric cone or else by exc V.2.9 (which doesn't use this result) $g(X) = 2$ which is a contradiction. Thus we can restrict our attention to finding a rational degree 5 curve not on a smooth quadric since at any rate such a curve won't be on a singular quadric. The remainder is Theorem 4 and Proposition 6 of Eisenbuds "On Normal Bundles of

Smooth Rational Space Curves." Alternatively, embed $\mathbb{P}^1$ via $(s:t)\mapsto(s^5:s^4t:st^4+as^3t^4:t^5)$ for $a\in k^*$. This is a degree 5 and we can check there are no degree 2 relations by checking each case (i.e. check $(s^5)^2-(s^4t)(st^4+as^3t^4)=0$ and such possibilities, there are finitely many and I'm too lazy to type them all). Now use thm II.7.3 to check it gives an embedding.

4.6.3 IV.6.3 x g

6.3. A curve of degree 5 and genus 2 in $\mathbf{P}^3$ is contained in a unique quadric surface Q. Show that for any abstract curve X of genus 2, there exist embeddings of degree 5 in $\mathbf{P}^3$ for which Q is nonsingular, and there exist other embeddings of degree 5 for which Q is singular.

Contained in a Quadric. Consider the LES associated to $0\to I_X(2)\to\mathcal{O}_{\mathbb{P}^3}(2)\to\mathcal{O}_X(2)\to 0$. Note that $h^0(\mathcal{O}_{\mathbb{P}^3}(2))=\begin{pmatrix}2+3\\3\end{pmatrix}=\frac{20}{2}=10$. By r.r., $h^0(\mathcal{O}_X(2))-h^1(\mathcal{O}_X(2))=5\cdot 2+1-2=9$. Note that $h^1(\mathcal{O}_X(2))=h^0(\mathcal{O}_X(K_X-2H))=0$ since $K_X-2\cdot H$ has negative degree. So if $h^0(I_X(2))=0$ then we have a contradiction to exactness.

Thus at least X is on a quadric Q. If that quadric is nonsingular, then by remark IV.6.4.1, checking the possible types gives X of type $(a,b)=(2,3)$. Then exc IV.5.6.b.3, gives X is projectively normal. Then exc II.5.14.d combined with the LES computation gives X that Q is the unique quadric on which X lies.

If the quadric is singular, then by the proof of exc V.2.9, X passes through the vertex of the cone. If it lies on another quadric, it must be a cone, and so since X is smooth the two vertices must coincide (think of the picture) so they are in fact the same cone.

As X has genus 2, then $deg(K_X)=2$ and K_X spans a line in $\mathbb{P}^3$. Let H be the hyperplane $\mathcal{O}_X(1)$ which has degree 5 since X does. Thus $H-2K_X$ has degree 1 but could either be effective or not effective depending on how we choose the points (both cases are possible).

For $H-2K_X$ not effective, then $|H-2K_X|=\emptyset$. Let $D=H-K_X$ which has degree 3. Then $h^0(\mathcal{O}_X(D))=2$ and $|H-D|=|K_X|$ has dimension 1. Thus the divisors of D are contained in two planes and therefore spans a line which is the intersection of the planes. This line is contained in Q since it meets X in 3 points by bezout since it has degree 3 and it's a line. Since $|H-2D|$ is empty, two different lines don't meet. But then we have rulings not meeting which gives a smooth quadric.

On the other hand if $H-2K\geq 0$, then by the dimension $|H-2K|$ is a point. So we have rulings meeting at a vertex giving a quadric cone.

4.6.4 IV.6.4 x g

6.4. There is no curve of degree 9 and genus 11 in $\mathbf{P}^3$. [*Hint*: Show that it would have to lie on a quadric surface, then use (6.4.1).]

Suppose X has degree 9 and genus 11 in $\mathbb{P}^3$. Consider the LES associated to $0\to\mathscr{I}_X\to\mathcal{O}_{\mathbb{P}^3}\to\mathcal{O}_X\to 0$. By riemann-roch, $h^0(\mathcal{O}_X(2D))-h^0(\mathcal{O}_X(K-2D))=18+1-11=8$. This curve should be special so $h^0(\mathcal{O}_X(2D))>8$. $h^0(\mathcal{O}_{\mathbb{P}^3}(2))=\begin{pmatrix}3+2\\3\end{pmatrix}=10$. If it's nonspecial, then it's clearly contained in a quadric. If $2D$ is special, then since $h^0(\mathcal{O}_X(2d))>0$, we can find an effective divisor linearly equivalent to $2D$. By Clifford, $dim|2D|\leq\frac{1}{2}deg\ 2D=9$. So in any case $h^0(\mathcal{O}_X(2D))<10$ and thus X lies on a quadric surface Q.

Now if Q is nonsingular, then by rmk IV.6.4.1, $9=a+b$, $11=ab-a-b+1$ and there are no possible solutions. If Q is the product of two hyperplanes, then it be a line and have genus 0 or it's on a plane so by degree genus for $\mathbb{P}^2$, the genus is $\frac{1}{2}(9-1)(9-2)\neq 11$. If Q is a quadric cone, then by exc V.2.9, $9=2\cdot a+1$ and $11=a^2-a$ so that $11=12$ which is a contradiction.

4.6.5 IV.6.5 x g complete intersection doesn't lie on small degree surface

6.5. If X is a complete intersection of surfaces of degrees a,b in $\mathbf{P}^3$, then X does not lie on any surface of degree $<\min(a,b)$.

We assume X is smooth I guess since we're in chapter 4. Via exc II.8.4 We get projectively normal. (smooth $\implies$ normal $\implies$ projectively normal) Thus $H^0(\mathcal{O}_{\mathbb{P}}(l)) \to H^0(\mathcal{O}_X(l))$ is surjective for all $l \geq 0$ via II.8.4. Now note if $m < min(a,b)$ we want to show that $h^0(\mathcal{I}_X(m)) = 0$. I can compute $h^0(\mathcal{O}_{\mathbb{P}^3}(m)) = \binom{m+3}{3}$

We know it's hilbert polynomial. so that's how we'll compute $h^0(\mathcal{O}_X(l))$. Since $\chi(l) = h^0(\mathcal{O}_X(l)) - h^1(\mathcal{O}_X(l)) + h^1(\mathcal{O}_X(l)) - ...$ but $h^1(\mathcal{O}_X(l)) = 0$ by projectively normal, and $h^2(\mathcal{O}_X(l)) = 0$ since X is dimension 1. Therefore, we have $h^0(I_X(m)) = h^0(\mathcal{O}_{\mathbb{P}^3}(m)) - h^0(\mathcal{O}_X(m)) =$

$$\binom{m+3}{3} - \binom{m+3}{3} + \binom{m+3-a}{3}$$
$$+\binom{m+3-b}{3} - \binom{m+3-a-b}{3}$$
$$= \binom{m+3-a}{3} + \binom{m+3-b}{3} - \binom{m+3-a-b}{3}.$$

So if $m < a$ and $m < b$ then $\binom{m+3-a}{3} = 0$ and $\binom{m+3-b}{3} = 0$ and $\binom{m+3-a-b}{3} = 0$ so we're good.

4.6.6 IV.6.6 x g Projectively normal curves not in a plane

6.6. Let X be a projectively normal curve in $\mathbf{P}^3$, not contained in any plane. If $d = 6$, then $g = 3$ or 4. If $d = 7$, then $g = 5$ or 6. Cf. (II, Ex. 8.4) and (III, Ex. 5.6).

- II.8.4
- ok here is almost everything for $d = 6$:
  - for $g = 0$, it's rational so in a plane
  - for $g = 1$, it's a plane cubic
  - for $g = 2$,??? (maybe something like canonical embedding)
    - * see below.
  - for $g = 3, 4$ ok
  - for $g \geq 5$, use castelnuovo's bound (6.4)
- For $d = 7$, then
  - for $g = 0, 1$ same reasoning as before.
  - for $g > 9$, we have castelnuovo's bound
  - For $g \geq 7$ we have castelnuovo again genus is bounded above by $\frac{1}{4} \cdot (7^2 - 1) - 7 + 1 = 6$.
  - Now we just have to check genus $2, 3, 4$.
- so for genus 2, and degree 6 in $\mathbb{P}^3$.

- if it were a complete intersection of surfaces of degree d, e
- then $2 = \frac{1}{2}de\,(d+e-4)+1$ and $d \cdot e = 6$ (by bezout) or $e = d/6$ so
- $2 = \frac{1}{2} \cdot 6 \cdot \left(d + \frac{d}{6} - 6\right) + 1 = 3\left(\frac{7d}{6} - 6\right) + 1$ so $\frac{1}{3} + 6 = \frac{7d}{6}$ or $\frac{6}{3} + 36 = 7d$ or $d = \frac{2.0+36}{7.0} =$ 5.428571428571429
- This is not an integer, so it's not a complete intersection.
- something's messed up here though, since consider degree 7, then $de = 7$, so it's contained in a degree 1 thing.
- so it's not a complete intersection (also $g = 2 \implies$ hyperelliptic $\implies$ not a complete intersection by IV.5.1), we still need to prove the thing.
- by halphen, it has a nonspecial, very ample divisor of degree $d \geq 5$.
- by 6.3, the hyperplane section D is nonspecial. -should allow us to compute $h^0(\mathcal{O}_X(1))$... yeah it's 5 so that confirms it's not in a plane. wait, but $h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$ so that's actually a contradiction, since we're supposed to have a surjection !

- So for degree 7, genus $2, 3, 4$,
  - if hyperplane section is special, then $g \geq \frac{1}{2}7 + 1 = 4.5$ by 6.3
  - so hyperplane section is nonspecial.
  - Now let's compute $h^0(\mathcal{O}_X(1))$ using Remark IV.1.3.1 so $h^0(\mathcal{O}_X(D)) = 7 + 1 - g$. if $g = 4$, then $h^0(\mathcal{O}_X(D)) = 4$ so that's still ok, however for $g = 2, 3$ we get a contradiction same as for $d = 6, g = 2$ case.

- for degree 7, genus 4. we'll take $2D$ it has degree $12 > 2g - 2$ so it's nonspecial.
  - Then $h^0(\mathcal{O}_X(2)) = 12 + 1 - 4 = 13 - 4 = 9$. Also $h^0(\mathbb{P}^3(2)) = \frac{factorial(5)}{factorial(3) \cdot factorial(2)} = 10$ so we see that our curve is contained in a quadric. (note for genus 5 this would not be the case)
  - So if it's on a quadric, it has a type. (a, b) where $a + b = 7$.
  - so $a = 1, b = 6$, $a = 2b = 5$, $a = 3, b = 4$.
  - Also $g = ab - 7 + 1$, so $g = 6 - 7 + 1 = 0$, or $g = 10 - 7 + 1 = 4$ or $g = 12 - 7 + 1$
  - so we know that it has type $(2, 5)$.
  - However, this contradicts III.5.6.b.c since $|a - b| \nleq 1$.

4.6.7 IV.6.7 x g

6.7. The line, the conic, the twisted cubic curve and the elliptic quartic curve in $\mathbf{P}^3$ have no multisecants. Every other curve in $\mathbf{P}^3$ has infinitely many multisecants. [*Hint*: Consider a projection from a point of the curve to $\mathbf{P}^2$.]

Suppose X is a curve with no multisecants.

Looking at the picture if X lies on a plane then it is a line or conic since any degree term higher than 3 would have at least one inflection and thus a trisecant.

If $X \subset \mathbb{P}^3 \backslash \mathbb{P}^2$ has degree d then the projection from a general point of X gives an isomorphism onto a smooth plane curve of degree $d - 1$ with genus $\frac{1}{2}(d-2)(d-3)$.

By Castelnuovo, this gives $\frac{1}{2}(d-2)(d-3) \leq \begin{cases} \frac{1}{4}d^2 - d + 1 & d \text{ even} \\ \frac{1}{4}(d^2-1) - d + 1 & d \text{ odd} \end{cases}$. Thus $\frac{1}{2}d^2 - \frac{5}{2}d + 3 \leq \frac{1}{4}d^2 - d + 1$, d even or $\frac{1}{2}d^2 - \frac{5}{2}d + 3 \leq \frac{1}{4}(d^2-1) - d + 1$ if d is odd. Thus $d^2 - 6d + 8 \leq 0$ so $2 \leq d \leq 4$. Thus we could have $d = 3$ so X is twisted cubic since it's degree 3 in $\mathbb{P}^3$ and genus 0. We could also have $d = 4$ so by the genus is $0, 1$ depending on nodes. Such an elliptic would be the complete intersection of two quadrics which has no trisecants. On the other hand, a rational quartic lives as a type $(1,3)$ on a smooth quadric so has trisecants which are lines on one of the rulings.

4.6.8 IV.6.8 x g

6.8. A curve X of genus g has a nonspecial divisor D of degree d such that $|D|$ has no base points if and only if $d \geqslant g + 1$.

Rephrase

Let K be the field of rational functions on X.

Define S to be the set of all $d \in \mathbb{N}$ such that there exists base point free g_d^1 on X.

Define d_0 to be the smallest $d \in S$ such that $m \geq d$ implies $m \in S$.

Note that if $d_0 \leq g+1$, and $d \geq g+1$ then there exists a base point free g_d^1 on X.

On the other hand if $d < d_0$, then there exists no base point free g_d^1 on X.

Thus it suffices to prove $d_0 \leq g+1$

d >= g+1 implies exists bpf ...

Halphen gives us that if $g \geq 2$, then X has a nonspecial very ample divisor iff $d \geq g+3$.

Thus if $d \geq g+3$, consider a nonspecial very ample g_d^{d-g} for any $d \geq g+3$, the general pencil of the very ample g_d^{d-g} gives a base point free so such $d \in S$.

Next we have the bpf, $g_{g+3}^3(-P)$ where $P \in C$ is general which gives $g+2 \in S$.

We may subtract a further point by $g_{g+3}^3(-P-Q) = g_{g+2}^2(-P) = g_{g+1}^1$ which is birationally very ample. Thus for any $d \geq g+1$, $d \in S$ and so $d_0 \leq g+1$.

Clearly these later linear systems are also nonspecial.

exists bpf implies d>= g+1

If there exists a base point free, nonspecial g_g^1, then we add two points not in g_g^1 to get a g_{g+2}^3 which is very ample by thm IV.3.1(b) contradicting Halphen.

4.6.9 IV.6.9 (starred)

***6.9.** Let X be an irreducible nonsingular curve in $\mathbf{P}^3$. Then for each $m >> 0$, there is a nonsingular surface F of degree m containing X. [*Hint*: Let $\pi : \tilde{\mathbf{P}} \to \mathbf{P}^3$ be the blowing-up of X and let $Y = \pi^{-1}(X)$. Apply Bertini's theorem to the projective embedding of $\tilde{\mathbf{P}}$ corresponding to $\mathcal{I}_Y \otimes \pi^*\mathcal{O}_{\mathbf{P}^3}(m)$.]

skip.

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5 V Surfaces

5.1 V.1 Geometry On A Surface

5.1.1 V.1.1 x g Intersection Via Euler Characteristic

1.1. Let C,D be any two divisors on a surface X, and let the corresponding invertible sheaves be $\mathscr{L},\mathscr{M}$. Show that

$$C.D = \chi(\mathcal{O}_X) - \chi(\mathscr{L}^{-1}) - \chi(\mathscr{M}^{-1}) + \chi(\mathscr{L}^{-1} \otimes \mathscr{M}^{-1}).$$

Proof 0. For full generality see mumford chapter 12.
Proof 1.
As in the proof of V.5.1,
write C and D as the difference of curves all meeting transversally.
Thus we can use V.1.1 TSADL: $\#(C \cap D) = C.D$
We have $0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$
$0 \to \mathcal{O}_C(-D) \to \mathcal{O}_C \to \mathcal{O}_{C\cap D} \to 0$ and
$0 \to \mathcal{O}_X(-D-C) \to \mathcal{O}_X(-D) \to \mathcal{O}_C(-D) \to 0$
Since euler characteristic is additive on exact sequences,
then χ of the middle is sum of χ outside.
Making the needed substitutions gives
$\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-C)) + \chi(\mathcal{O}_X(-C-D)) = \chi(\mathcal{O}_{C\cap D}) = h^0(C \cap D)$
Proof 2.
For simplicity, assume C, D are curves meeting transversely.
Consider the sequence
$0 \to \mathcal{O}_X(-C-D) \overset{(d,-c)}{\to} \mathcal{O}_X(-C) \oplus \mathcal{O}_X(-D) \overset{(c,d)}{\to} \mathcal{O}_X \to \mathcal{O}_{C\cap D} \to 0.$
Exactness can be checked at the stalks.
$0 \to \mathcal{O}_x \overset{(d,-c)}{\to} \mathcal{O}_x^2 \overset{(c,d)}{\to} \mathcal{O}_X \to \mathcal{O}_X/(c,d) \to 0.$
We must show that the kernel of (c,d) is the image of $(d,-c)$.
If $af + cd = 0$, then $af = -cd$.
Since the meetings of C, D are transverse, $\mathcal{O}_x$ is a UFD, since c, d are relatively prime.
Thus there is h such that $a = hd$ and $b = -hc$.

5.1.2 V.1.2 x g Degree via hypersurface

1.2. Let H be a very ample divisor on the surface X, corresponding to a projective embedding $X \subseteq \mathbf{P}^N$. If we write the Hilbert polynomial of X (III, Ex. 5.2) as

$$P(z) = \frac{1}{2}az^2 + bz + c,$$

show that $a = H^2$, $b = \frac{1}{2}H^2 + 1 - \pi$, where π is the genus of a nonsingular curve representing H, and $c = 1 + p_a$. Thus the *degree* of X in $\mathbf{P}^N$, as defined in (I, §7), is just H^2. Show also that if C is any curve in X, then the degree of C in $\mathbf{P}^N$ is just $C.H$.

By Riemann-Roch, $\chi(nH) = \frac{1}{2}(nH).(nH - K_X) + 1 + p_a$.
This is $P(n) = \frac{1}{2}n^2H^2 - \frac{1}{2}nH.K_X + 1 + p_a$.
Clearly $a = H^2$ and $c = 1 + p_a$.
We need that $\frac{1}{2}H.K_X = \frac{1}{2}H^2 + 1 - \pi$.
Note that $H.(H + K_X) = 2g - 2$ and so $H.K_X = 2g - 2 - H^2$

By I.7, the degree is $dim(X)!$ times the first coefficient, so it's H^2.

Now we have exact sequences
$0 \to \mathcal{O}_X(-C) \to \mathcal{O}_X \to \mathcal{O}_C \to 0$
$0 \to \mathcal{O}_X(-C-H) \to \mathcal{O}_X(-H) \to \mathcal{O}_C(-H|_C) \to 0.$
Plugging these into excercise V.1.2 gives
$C.H = \chi(C, \mathcal{O}_C) - \chi(C, -H|_C).$
But this is the definition of the degree of the line bundle associated to C.

5.1.3 V.1.3 x g:a,b adjunction computational formula

1.3. Recall that the *arithmetic genus* of a projective scheme D of dimension 1 is defined as $p_a = 1 - \chi(\mathcal{O}_D)$ (III, Ex. 5.3).
(a) If D is an effective divisor on the surface X, use (1.6) to show that $2p_a - 2 = D.(D + K)$.

Riemann-roch (for a surface) gives
$\chi(-D) = \frac{1}{2}(-D)(-D-K_X) + 1 + p_a.$
By the exact sequence $0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$,
we get $\chi(\mathcal{O}_D) = \frac{1}{2}D.(-D-K_X) - 1 - p_a + \chi(\mathcal{O}_X)$
(note the last term cancels out).

5.1.4 b. x g: with above

(b) $p_a(D)$ depends only on the linear equivalence class of D on X.

By part (a).

5.1.5 c. x g

(c) More generally, for *any* divisor D on X, we define the *virtual arithmetic genus* (which is equal to the ordinary arithmetic genus if D is effective) by the same formula: $2p_a - 2 = D.(D + K)$. Show that for any two divisors C, D we have

$$p_a(-D) = D^2 - p_a(D) + 2$$

and

$$p_a(C + D) = p_a(C) + p_a(D) + C.D - 1.$$

Note that $p_a(-D) = \frac{1}{2}(-D)(-D+K) + 1 = D^2 - \frac{1}{2}D.(D+K) + 1$
also $p_a(C+D) = \frac{1}{2}(C+D).(C+D+K) + 1 = \frac{1}{2}C.(C+K) + \frac{1}{2}D.(D+K) + C.D + 1$.

5.1.6 V.1.4 x g Self intersection of rational curve on surface

1.4. (a) If a surface X of degree d in $\mathbf{P}^3$ contains a straight line $C = \mathbf{P}^1$, show that $C^2 = 2 - d$.

First we compute K_X
We have $X \sim dH$ for a hypersurface
(since we're in $\mathbb{P}^3$, and dim 2).
Using adjunction, $K_X = (K_{\mathbb{P}^3} + dH)|_X$.

Recalling that $K_{\mathbb{P}^3} = -4H$, $K_X = (-4+d)\,H$.

Now we compute C^2.
By adjunction, $C.\,(C+K_X) = 2g-2 = 0-2$.
We can choose H generally enough to meet C at one point.
(i.e. $C.H = 1$)
Then $C^2 = -2 + (4-d)\,C.H = 2-d$.

5.1.7 b. x.

(b) Assume char $k = 0$, and show for every $d \geqslant 1$, there exists a nonsingular surface X of degree d in $\mathbf{P}^3$ containing the line $x = y = 0$.

The fermat hypersurface $x^d + y^d + z^d + w^d = 0$ contains the line $x -$ $\sqrt[n]{}$

5.1.8 V.1.5 x g Canonical for a surface in $\mathbb{P}^3$

1.5. (a) If X is a surface of degree d in $\mathbf{P}^3$, then $K^2 = d(d-4)^2$.

Let $X \sim dH$ for some hypersurface.
Then $K_X = (d-4)\,H$ by adjunction (since canonical in $\mathbb{P}^3$ is $-4H$)
Thus $K_X^2 = (d-4)^2\,H^2$.
Now $H^2 = d$ by V.1.2

5.1.9 b. x g

(b) If X is a product of two nonsingular curves C, C', of genus g, g' respectively, then $K^2 = 8(g-1)(g'-1)$. Cf. (II, Ex. 8.3).

Let p_1, p_2 the projections .
From exc II.8.3, $K_X = p_1^* K_C + p_2^* K_{C'}$.
Thus $K_X^2 = (p_1^* K_C)^2 + 2 p_1^* K_C . p_2^* K_{C'} + (p_2^* K_{C'})^2$.
Then middle term is $2 \cdot (2g-2) \cdot (2g'-2)$ and the outer terms disappear.

5.1.10 V.1.6 x g

1.6. (a) If C is a curve of genus g, show that the diagonal $\Delta \subseteq C \times C$ has self-intersection $\Delta^2 = 2 - 2g$. (Use the definition of $\Omega_{C/k}$ in (II, §8).)

Let p_1, p_2 the projections $C \times C \to C$.
Note the diagonal is isomorphic to C.
The intersection of the diagonal and a fiber is the point.
Thus $deg\ K_\Delta = (K_X + \Delta)\,.\Delta$ by adjunction.
This is $(p_1^* K_C + p_2^* K_C)\,.\Delta + \Delta.\Delta =$
$(2g-2) + (2g-2) + \Delta^2$.
Now $\Delta \approx C$ so $deg\ K_\Delta = deg\ K_C = 2g-2$.
Now solve for Δ^2.

5.1.11 b. x g

(b) Let $l = C \times \mathrm{pt}$ and $m = \mathrm{pt} \times C$. If $g \geqslant 1$, show that l, m, and Δ are linearly independent in $\mathrm{Num}(C \times C)$. Thus $\mathrm{Num}(C \times C)$ has rank $\geqslant 3$, and in particular, $\mathrm{Pic}(C \times C) \neq p_1^* \mathrm{Pic}\, C \oplus p_2^* \mathrm{Pic}\, C$. Cf. (III, Ex. 12.6), (IV, Ex. 4.10).

Note that $l^2 = 0$, $l.m = 1$, $l.\Delta = 1$, where Δ is again, the diagonal, and $m^2 = 0$, $\Delta^2 = 2g - 2$ by (a).
Suppose that $a \cdot l + b \cdot m + c \cdot \Delta = 0$ for some constants.
Then $l.(al + bm + c\Delta) = 0 \implies b + c = 0$,
$m.(al + bm + c\Delta) = 0 \implies a + c = 0$
$\Delta.(al + bm + c\Delta) = 0 \implies a + b + c(2g - 2) = 0$.
Thus $2(a + c - gc) = 0$.
Thus $a = b = c = 0$ by solving the system.

5.1.12 V.1.7 x Algebraic Equivalence of Divisors

1.7. *Algebraic Equivalence of Divisors.* Let X be a surface. Recall that we have defined an algebraic family of effective divisors on X, parametrized by a nonsingular curve T, to be an effective Cartier divisor D on $X \times T$, flat over T (III, 9.8.5). In this case, for any two closed points $0, 1 \in T$, we say the corresponding divisors D_0, D_1 on X are prealgebraically equivalent. Two arbitrary divisors are prealgebraically equivalent if they are differences of prealgebraically equivalent effective divisors. Two divisors D, D' are *algebraically equivalent* if there is a finite sequence $D = D_0, D_1, \ldots, D_n = D'$ with D_i and D_{i+1} prealgebraically equivalent for each i.

(a) Show that the divisors algebraically equivalent to 0 form a subgroup of $\mathrm{Div}\, X$.

Write $\equiv$ for algebraic equivalence.
Suppose D is prealgebraically equivalent to 0.
inverses
Write D as the difference of effective $D_1 - D_2$.
Then $-D = D_2 - D_1$ is prealgebraically equivalent to D is prealgebraically equivalent to $-D$.
Now suppose that $0 = D_0, ..., D_n = D$ is a sequence for D.
Then $0 = -D_0, -D_0, ..., -D_n$ is a sequence for $-D$ by above.
Thus $-D \equiv 0$ so it's closed under inverses.
Sums
Suppose $0 = D_0, ..., D_n = D$ and $0 = E_0, ..., E_m = E$.
Now D and $D + 0 = D + E_0$ are prealgebraically equivalent. Hence,
$0 = D_0, ..., D_n = D, D + E_0, ..., E + E_m = D + E$ is a sequence for $D + E$.

5.1.13 b. x

(b) Show that linearly equivalent divisors are algebraically equivalent. [*Hint*: If (f) is a principal divisor on X, consider the principal divisor $(tf - u)$ on $X \times \mathbf{P}^1$, where t, u are the homogeneous coordinates on $\mathbf{P}^1$.]

By II.9.8.5, an effective divisor on $X \times T$ is flat over T when the local equations of the divisor are nonzero when restricted to a fiber.

Since the difference of linearly equivalent divisors is principal, we just need to show that $(f) \equiv 0$ for $f \in K(X)$.

Note that $(tf-u)$ restricts to (f) over $(1,0)$ and to 0 over $(0,1)$.
Thus $(f)=0$.

5.1.14 c. x

(c) Show that algebraically equivalent divisors are numerically equivalent. [*Hint*: Use (III, 9.9) to show that for any very ample H, if D and D' are algebraically equivalent, then $D.H = D'.H$.]

By bertini we can consider differences of very ample divisors.
Thus we consider intersections with very ample divisors.
Thus we want to show $D.H = D'.H$ for prealgebraically equivalent effective D, D' and very ample H.
H induces an embedding $X \to \mathbb{P}_k^n$ which gives an embedding $X \times T \to \mathbb{P}_T^n$.
If $E \subset X \times T$ is a divisor with fibers $E_0 = D$, $E_1 = D'$, then E is flat over T, so by thm III.9.9 , the degrees of D and D' in $\mathbb{P}_k^n$ are equal.
But $D.H$ and $D'.H$ are exactly the degrees of D and D' in $\mathbb{P}_k^n$.
Thus $D.H = D'.H$.

5.1.15 V.1.8 x g cohomology class of a divisor

1.8. *Cohomology Class of a Divisor*. For any divisor D on the surface X, we define its cohomology class $c(D) \in H^1(X,\Omega_X)$ by using the isomorphism $\operatorname{Pic} X \cong H^1(X,\mathcal{O}_X^*)$ of (III, Ex. 4.5) and the sheaf homomorphism $d\log : \mathcal{O}^* \to \Omega_X$ (III, Ex. 7.4c). Thus we obtain a group homomorphism $c : \operatorname{Pic} X \to H^1(X,\Omega_X)$. On the other hand, $H^1(X,\Omega)$ is dual to itself by Serre duality (III, 7.13), so we have a nondegenerate bilinear map

$$\langle \ , \ \rangle : H^1(X,\Omega) \times H^1(X,\Omega) \to k.$$

(a) Prove that this is compatible with the intersection pairing, in the following sense: for any two divisors D,E on X, we have

$$\langle c(D), c(E) \rangle = (D.E) \cdot 1$$

in k. [*Hint*: Reduce to the case where D and E are nonsingular curves meeting transversally. Then consider the analogous map $c : \operatorname{Pic} D \to H^1(D,\Omega_D)$, and the fact (III, Ex. 7.4) that $c(\text{point})$ goes to 1 under the natural isomorphism of $H^1(D,\Omega_D)$ with k.]

By Bertini, write D as a difference of very ample divisors, thus wlog smooth curves.
Consider
$$\begin{tikzcd} Pic\ X \arrow[r, "c"] \arrow[d] & H^1(X,\Omega_X) \arrow[d, "f"] \\ Pic\ D \arrow[r, "c"] & H^1(D,\Omega_D) \approx k \end{tikzcd}$$
.
By the hint, the bottom map is degree.
Thus going down and right gives $\mathscr{L}(E) \mapsto \mathscr{L}(E) \otimes \mathcal{O}_D \mapsto deg_D \mathscr{L}(E) \otimes \mathcal{O}_D = D.E$ by 1.3.
Now going right then down gives $f(c(D)) = D.E$.

5.1.16 b. x g

(b) If char $k = 0$, use the fact that $H^1(X,\Omega_X)$ is a finite-dimensional vector space to show that Num X is a finitely generated free abelian group.

Via the exponential sequence, D determines $c_1(\mathcal{O}_X(D)) \in H^2(X,\mathbb{Z})$. (see page 446). This second group is finitely generated (note it sits between $H^1(X,\mathcal{O}_X^*)$ and $H^2(X,\mathcal{O}_X)$ in the sequence).

5.1.17 V.1.9 x g Hodge inequality

1.9. (a) If H is an ample divisor on the surface X, and if D is any divisor, show that

$$(D^2)(H^2) \leqslant (D.H)^2.$$

Let a divisor.
$D' = H^2D - (H.D)H$.
Then $D'.H = 0$. Thus by hodge index theorem $D'^2 \leq 0$.
Then
$(H^2D - (H.D)H)^2 \leq 0 \implies$
$H^4D^2 - 2H^2(H.D)D.H + (H.D)^2 H^2 \leq 0 \implies$
since $H^2 > 0$ (nak moish) we factor it out and have
$D^2H^2 - 2(H.D)^2 + (H.D)^2 \leq 0 \implies$
$D^2H^2 \leq (H.D)^2$.

5.1.18 b. x g

(b) Now let X be a product of two curves $X = C \times C'$. Let $l = C \times \mathrm{pt}$, and $m = \mathrm{pt} \times C'$. For any divisor D on X, let $a = D.l$, $b = D.m$. Then we say D has *type* (a,b). If D has type (a,b), with $a,b \in \mathbf{Z}$, show that

$$D^2 \leqslant 2ab,$$

and equality holds if and only if $D \equiv bl + am$. [*Hint*: Show that $H = l + m$ is ample, let $E = l - m$, let $D' = (H^2)(E^2)D - (E^2)(D.H)H - (H^2)(D.E)E$, and apply (1.9). This inequality is due to Castelnuovo and Severi. See Grothendieck [2].]

Define $H = l+m$ and $E = l-m$ so that $deg(D.H) = deg(D.l)+deg(D.m) = a+b$ and $deg(D.E) = a-b$.

Note as $l^2 = 0$, $l.m = 1$, $m^2 = 0$ then $deg(E^2) = -2$, $deg(H^2) = 2$, $deg(E.H) = 0$. By Nakai Moishezon, H is ample thus if $D' = -4D + 2(a+b)H - 2(a-b)E$ since $deg(D'.H) = 0$, then by Hodge index theorem if $D' \not\equiv 0$, then $0 > deg(D'^2) = 16(deg(D^2) - 2ab)$. Thus $2ab > deg(D^2)$. If $D' \equiv 0$ then $D \equiv bl + am$ and $deg(D^2) = deg\left((bl+am)^2\right) = 2ab$.

5.1.19 V.1.10 x g Weil Riemann Hypothesis for Curves

1.10. *Weil's Proof* [2] *of the Analogue of the Riemann Hypothesis for Curves.* Let C be a curve of genus g defined over the finite field $\mathbf{F}_q$, and let N be the number of points of C rational over $\mathbf{F}_q$. Then $N = 1 - a + q$, with $|a| \leqslant 2g\sqrt{q}$. To prove this, we consider C as a curve over the algebraic closure k of $\mathbf{F}_q$. Let $f:C \to C$ be the k-linear Frobenius morphism obtained by taking qth powers, which makes sense since C is defined over $\mathbf{F}_q$, so $X_q \cong X$ (IV, 2.4.1). Let $\Gamma \subseteq C \times C$ be the graph of f, and let $\Delta \subseteq C \times C$ be the diagonal. Show that $\Gamma^2 = q(2-2g)$, and $\Gamma.\Delta = N$. Then apply (Ex. 1.9) to $D = r\Gamma + s\Delta$ for all r and s to obtain the result. See (App. C, Ex. 5.7) for another interpretation of this result.

Note that Γ is the preimage of Δ under $(f,1) : C \times C \to C \times C$. Thus $\Gamma^2 = \Delta^2 \cdot deg(f) = (2-2g) \cdot q$ since q is the degree of frobenius and $\Delta^2 = 2-2g$ by exc V.1.6.a and $((f,1)_* (f,1)^* \Delta, \Delta) = deg(f,1) \cdot \Delta^2$. Now $\Gamma.\Delta = N$ clearly gives the number of fixed points and the fixed point set of frobenius are the points lying in $\mathbb{F}_q$ as in exc IV.4.16.b. Note that, using the notation of exc V.1.9, Γ meets $l = C \times pt$ at $f^*pt \times pt$, Γ meets $m = pt \times C$ at $pt \times f(pt)$. Since Frobenius has degree q, then $\Gamma.l = q$, $\Gamma.m = 1$. Similar logic gives $\Delta.l = 1$, $\Delta.m = 1$. Thus Γ has type $(q,1)$, Δ has type $(1,1)$.

Let $D = r\Gamma + s\Delta =$ as in the hint. Thus D has type $(rq+s, r+s)$ $D^2 = rq(2-2g) + s^2(2-2g) + 2rsN$. Now by exc V.1.9, $rq(2-2g) + s^2(2-2g) + 2rsN \leq 2(rq+s)(r+s)$. Rearranging gives $N \leq 1+q+\frac{r}{s}gq+\frac{s}{r}g$ for $rs > 0$ and $N \geq 1+q+\frac{r}{s}gq+\frac{s}{r}g$ for $rs < 0$. Since r, s can be arbitrary we have $|N-1-q| \leq sup_{r,s}\frac{r}{s}gq+\frac{s}{r}g$. Note that $g\left(qx+\frac{1}{x}\right)$ is maximized at the same place as $g(qx^2+1)$ is maximized which is at $x = \pm\frac{1}{\sqrt{q}}$ and thus we get $|N-1-q| \leq g\left(q\frac{1}{\sqrt{q}}+\sqrt{q}\right) = g(2\sqrt{q})$.

5.1.20 V.1.11 x g

1.11. In this problem, we assume that X is a surface for which Num X is finitely generated (i.e., any surface, if you accept the Néron Severi theorem (Ex. 1.7)).

(a) If H is an ample divisor on X, and $d \in \mathbf{Z}$, show that the set of effective divisors D with $D.H = d$, modulo numerical equivalence, is a finite set. [*Hint*: Use the adjunction formula, the fact that p_a of an irreducible curve is $\geqslant 0$, and the fact that the intersection pairing is negative definite on $H^\perp$ in Num X.]

WLOG assume that H is a very ample hyperplane of $deg(H) = 1$.

For $D.H = 0$ this follows from nakai moishezon.

Note that any such D can only have finitely many components not intersecting H since if $P_i.H = 0$, then $P_i.P_j < 0$ and with each additional such component, the genus, $2g-2 = C.C + C.K \geq -2$ will decrease.

On the other hand, if we can choose infinitely many components which each have different numerical equivalence class, then $Num\ X$ will not be finitely generated.

Thus we are choosing D from a finite set of a finite number of components.

5.1.21 b. x g

(b) Now let C be a curve of genus $g \geqslant 2$, and use (a) to show that the group of automorphisms of C is finite, as follows. Given an automorphism σ of C, let $\Gamma \subseteq X = C \times C$ be its graph. First show that if $\Gamma \equiv \Delta$, then $\Gamma = \Delta$, using the fact that $\Delta^2 < 0$, since $g \geqslant 2$ (Ex. 1.6). Then use (a). Cf. (IV, Ex. 2.5).

If $\sigma \in Aut(C)$, and let Γ be its graph.

Doing the hint by contrapositive, thm V.1.4, gives $\Gamma \neq \Gamma$ which implies $\Gamma.\Delta \geq 0$.
Then $\Gamma^2 < 0$ implies $\Gamma.\Delta \neq \Delta^2$ and thus $\Gamma \not\equiv \Delta$.
As in exc 6.1, $\Gamma^2 = \Delta^2 < 0$.
Thus two graphs of an automorphism are not numerically equivalent.
Let $H = l + m$ from exc V.1.9.b, then $\Gamma.H = 2$ for a graph of an automorphism Γ .
Now use (a).

5.1.22 V.1.12 x g Very Ample not numerically equiv

1.12. If D is an ample divisor on the surface X, and $D' \equiv D$, then D' is also ample. Give an example to show, however, that if D is very ample, D' need not be very ample.

First we examine a curve. Let C a curve with $p_a > 2$. Let D a divisor of degree $2g$. Recall from thm IV.3.4 that D is very ample iff for any two points, $h^0(D - P - Q) = h^0(D) - 2$. If D has degree $2g$, then r.r. gives
$h^0(D) - h^0(K_C - D) = 2g + 1 - g = g + 1.$
$h^0(D - P - Q) - h^0(K_C - D + P + Q) = (2g - 2) + 1 - g = g - 1.$
So this holds when $K_C - D + P + Q$ has nonpositive degree, in other words, when D is not of the form $K_C + P + Q$.
Since D's looking like $K_C + P + Q$ are parametrized by the set of P, Q which is a two dimensional proper subset of the possible divisors D, then degree is not determined by numerical equivalence on a curve.

Case of a Surface
Consider a decomposable ruled surface X over a curve with $p_a(C) > 2$, let $H = |C_0 + \mathfrak{b}f|$ a linear system on X. By Fuentes-Pedreira thm 3.9, $|H|$ is very ample iff $\mathfrak{b}$ and $\mathfrak{b} + \mathfrak{e}$ are very ample. By the above, we know that we may find $\mathfrak{b}$ of degree $2p_a$ which is very ample, and $\mathfrak{b}$ of degree $2p_a$ which is not very ample. But numerical equivalence is only determined by the coefficient on C_0 and the degree of the divisor by thm V.2.3.

5.2 V.2 Ruled Surfaces

5.2.1 V.2.1 x g

2.1. If X is a birationally ruled surface, show that the curve C, such that X is birationally equivalent to $C \times \mathbf{P}^1$, is unique (up to isomorphism).

Suppose that $C_0 \times \mathbb{P}^1 \cong X \cong C_1 \times \mathbb{P}^1$.
But two curves which are birational are isomorphic.
(since any isomorphism of open sets extends to an isomorphism of the whole curve by the valuative criterions).

5.2.2 V.2.2 x

2.2. Let X be the ruled surface $\mathbf{P}(\mathcal{E})$ over a curve C. Show that $\mathcal{E}$ is decomposable if and only if there exist two sections C', C'' of X such that $C' \cap C'' = \varnothing$.

Marumaye, Remark 1.20

5.2.3 V.2.3 x

2.3. (a) If $\mathscr{E}$ is a locally free sheaf of rank r on a (nonsingular) curve C, then there is a sequence

$$0 = \mathscr{E}_0 \subseteq \mathscr{E}_1 \subseteq \ldots \subseteq \mathscr{E}_r = \mathscr{E}$$

of subsheaves such that $\mathscr{E}_i/\mathscr{E}_{i-1}$ is an invertible sheaf for each $i = 1, \ldots, r$. We say that $\mathscr{E}$ is a *successive extension* of invertible sheaves. [*Hint*: Use (II, Ex. 8.2).]

Miyanishi, Algebraic Geometry Lemma 12.1

5.2.4 x g tangent sheaf not extension of invertibles

(b) Show that this is false for varieties of dimension $\geqslant 2$. In particular, the sheaf of differentials Ω on $\mathbf{P}^2$ is not an extension of invertible sheaves.

Suppose that the tangent bundle on $\mathbb{P}^2$ is an extension of line bundles.
We have $0 \to \mathcal{O}_X(d_1) \to \mathscr{T}_{\mathbb{P}^2} \to \mathcal{O}_X(d_2) \to 0$.
Let H a hyperplane.
Then $c(\mathscr{T}_{\mathbb{P}^2}) = (1 + d_1 H)(1 + d_2 H)$.
By the euler exact sequence
$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^3 \to \mathscr{T}_{\mathbb{P}^2} \to 0$
$c(\mathscr{T}_{\mathbb{P}^2}) = c\left(\mathcal{O}_{\mathbb{P}^2}(1)^3\right) = c(\mathcal{O}_{\mathbb{P}^2}(1))^3 =$
$1 + 3H + 3H^2$.
Since we cannot solve for d_1, d_2, this is a contradiction.

5.2.5 V.2.4 a x

2.4. Let C be a curve of genus g, and let X be the ruled surface $C \times \mathbf{P}^1$. We consider the question, for what integers $s \in \mathbf{Z}$ does there exist a section D of X with $D^2 = s$? First show that s is always an even integer, say $s = 2r$.
(a) Show that $r = 0$ and any $r \geqslant g + 1$ are always possible. Cf. (IV, Ex. 6.8).

Let $X = \mathbb{P}(\mathcal{E})$ the ruled surface.
So $D \sim C_0 + (\mathfrak{d} - \mathfrak{e}) f \in PicX$ corresponds to a surjection $\mathcal{E} \to \mathcal{L}(\mathfrak{d}) \to 0$
We can also write as $D \sim C_0 + (d - e) f \in NumX$ by 2.3 where $f^2 = 0$ and $C_0.f = 1$.
Let's compute D^2 via adjunction.
$2g - 2 = D.(D + K) = D^2 + D.K$
$D^2 = 2g - 2 - D.K$.
Now $K \overset{num}{\equiv} -2C_0 + (2g - 2 - e) f$ by 2.11.
Plugging in to $D.K$ is $(C_0 + (d - e) f).(-2C_0 + [2g - 2 - e].f)$.
By FOIL it's $-2e + 2g - 2 - e - 2(d - e) + 0$ and we are left with an e
In total: $D^2 = 2e + e + 2(d - e) = 2d + e$.
Now we note that $C \times \mathbb{P}^1$ is the ruled surface with the projection, by 2.0.1.
By 2.11.1, we have $e = 0$. And thus we have $D^2 = 2d$.
So we have that $D^2 = 2d$ by the above.
Then we need D with degree $d \geq g + 1$, on X and D with degree 0 on X.
(We can take the structure sheaf and the one given by Ex, IV.6.8)

5.2.6 b. x

(b) If $g = 3$, show that $r = 1$ is not possible, and just one of the two values $r = 2, 3$ is possible, depending on whether C is hyperelliptic or not.

If hyperelliptic, then has a g_2^1 i.e. closed immersion to $\mathbb{P}^1$ of degree 2.
Then $\mathcal{O}_C(-1)$ has $d = -2$ and so we're done..
If nonhyperelliptic, then by 5.5.2, it has a g_3^1 but not a g_2^1
And it is obtained by projecting from a point to $\mathbb{P}^1$ from it's deg 4, embedding in $\mathbb{P}^2$.
If it has an $r = 1$ then that would mean we have a degree 1 map from C to $\mathbb{P}^1$.
Thus it's an isomorphism. But the genii are different.

5.2.7 V.2.5 x g

2.5. *Values of e.* Let C be a curve of genus $g \geqslant 1$.
(a) Show that for each $0 \leqslant e \leqslant 2g - 2$ there is a ruled surface X over C with invariant e, corresponding to an indecomposable $\mathscr{E}$. Cf. (2.12).

I will do this proof in a way that solves this one and one of the exercises in section 5. Now consider X an arbitrary indecomposable with invariant e_0. Let C_0 with $C_0^2 = -e_0$ the minimum self-intersection curve.

Now an elementary transform X' of a ruled surface blows up a point x, and then blows down the strict transform, leaving the exceptional divisor as a fiber of the new surface (see ex V.5.7.1).

Using general rules of monoidal transformations in V.3, we find that if $C, D \in X$, are $nC_0 + af$, $mC_0 + bf$, and C', D' are their elementary transformed curves, then $C'.D' = C.D + nm + n \cdot mult_x(D) - m \cdot mult_x(C)$. Thus if $m, n = 1$, then for $x \in C \cap D$, $C'.D' = C.D - 1$, and for $x \notin C \cap D$ we have $C'.D' = C.D + 1$.

Note that C_0 the elementary transform of C_0 , C_0' is the new minimum self-intersection curve. For if $x \in C_0$, and D' is another one on X' then $D'^2 \geq D^2 - 1 \geq X_0^2 - 1 = X_0'^2$.

Thus if $x \in C_0$, then we can obtain a new ruled surface with $e_1 = e_0 + 1$. On the other hand, taking the reverse transformation, we can lower the invariant . Thus starting with e_0 I can arbitrarily lower or raise the invariant. Thus going high enough, I get to a decomposable ruled surface. By Fuentes, Pedreira 4.6, if $X = \mathbb{P}(\mathscr{E}_0)$, $\Lambda^2\mathscr{E}_0 \approx \mathcal{O}_X(\mathfrak{e})$, then the new surface correpsonds to $\mathscr{E}_0'$, $\Lambda^2\mathscr{E}_0' \approx \mathcal{O}_C(\mathfrak{e} - P)$ where x lives in the fiber over P. Now subtracting enough points we can get to $X \times \mathbb{P}^1$ corresponding to $\mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C)$ (example V.2.11.1). Thus we can go from any X to one with high invariant which is decomposable, and then back to $X \times \mathbb{P}^1$ using a finite number of elementary transformations. Thus gives the exercise in V.5 that I mentioned.

Finally, suppose X is arbitrary decomposable with two sections C_0, C_1 . Then if $x \notin X_0$, $x \notin X_1$ and P is a base point of $-\mathfrak{e}$, then X' is indecomposable with $e' = e - 1$. This will get us all invariants for this problem with the help of thm V.2.12. This is F,P thm 4.12.4.

5.2.8 b. x

(b) Let $e < 0$, let D be any divisor of degree $d = -e$, and let $\xi \in H^1(\mathscr{L}(-D))$ be a nonzero element defining an extension

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{L}(D) \to 0.$$

Let $H \subseteq |D + K|$ be the sublinear system of codimension 1 defined by ker ξ, where ξ is considered as a linear functional on $H^0(\mathscr{L}(D + K))$. For any effective divisor E of degree $d - 1$, let $L_E \subseteq |D + K|$ be the sublinear system $|D + K - E| + E$. Show that $\mathscr{E}$ is normalized if and only if for each E as above, $L_E \nsubseteq H$. Cf. proof of (2.15).

I Follow thm IV.2.15. If $\mathscr{E}$ is normalized, then $H^0(\mathscr{E} \otimes \mathscr{M}) = 0$ so the map $\gamma : H^0(\mathscr{L}(D+K-E)) \to H^1(\mathscr{L}(-E))$ must be injective. On the other hand, let $\xi \in H^1(\mathscr{L}(-D))$ be the element defining the extension $\mathscr{E}$. Then we have a commutative diagram, writing $\mathscr{L}(D+K-E)$ as $\mathscr{L}(S)$,

$$\begin{tikzcd} H^0(\mathcal{O}_C) \arrow[r, "\delta"] \arrow[d, "\alpha"] & H^1(\mathscr{L}(-D)) \arrow[d, "\beta"] \\ H^0(\mathscr{L}(s)) \arrow[r, "\alpha"] & H^1(\mathscr{L}(-E)) \end{tikzcd}$$

where $\delta(1) = \xi$, $\alpha(1) = t$, a nonzero section defining the divisor S, and β is induced from the map $\mathcal{O}_C \to \mathscr{L}(S)$ corresponding to t. Now β is dual to the map $\beta' : H^0(\mathscr{L}(E)) \to H^0(\mathscr{L}(D+K))$ also induced by t. The image of any nonzero element of $H^0(\mathscr{L}(E))$ by β' is a section of $H^0(\mathscr{L}(D+K))$ corresponding to the effective divisor $E+S \in |D+K|$. By varying E and S, we get every divisor in linear system $|D+K|$, therefore image of β' as E varies fills up whole $H^0(\mathscr{L}(D+K))$. So if $L_E \subset |D+K|$, then $|D+K-E| + E \subset |D+K|$

On the other hand, $E + S = (D+K-E) + E$.

So suppose to the contrary, that there exists E such that $L_E \subset H$, where H is the kernel. In this case, $\beta(\xi) = 0$ contradicting injectivity of γ, so $\mathscr{E}$ is not normalized. Thus we have shown the contrapositive of $\mathscr{E}$ is normalized $\implies$ for every such E, $L_E \not\subset H$. So in particular, we have shown that $\mathscr{E}$ normalized $\implies$ for every such E, then $L_E \not\subset H$.

On the other hand suppose $\mathscr{E}$ is not normalized. Thus there is $\mathscr{E}$ with $H^0(\mathscr{E} \otimes \mathscr{L}(-E)) \neq 0$. Thus the map $\gamma : H^0(\mathscr{L}(D+K-E)) \to H^1(\mathscr{L}(-E))$ is not injective and therefore we can find $L_E \subset ker(\xi)$ by using commutativity of the above diagram.

5.2.9 c. x

above, $L_E \not\subseteq H$. Cf. proof of (2.15).

(c) Now show that if $-g \leqslant e < 0$, there exists a ruled surface X over C with invariant e. [*Hint*: For any given D in (b), show that a suitable ξ exists, using an argument similar to the proof of (II, 8.18).]

By proof of (a), I only need to find one with invariant $-g$. Thus I need to find a ruled surface with $C_0^2 = g$. This is given by 3.13 in Maruyama.

5.2.10 d. x

(d) For $g = 2$, show that $e \geqslant -2$ is also necessary for the existence of X.
Note. It has been shown that $e \geqslant -g$ for any ruled surface (Nagata [8]).

By Theorem 1 in Nagata “Self-intersection number...”

5.2.11 V.2.6 x g Grothendieck’s Theorem

2.6. Show that every locally free sheaf of finite rank on $\mathbf{P}^1$ is isomorphic to a direct sum of invertible sheaves. [*Hint*: Choose a subinvertible sheaf of maximal degree, and use induction on the rank.]

(Following Potier) Given a base case of rank 1 so it’s already invertible, assume locally free sheaves of rank $r-1$ all split on $\mathbb{P}^1$. By thm III.8.8.c and Serre duality, we can find $n \gg 0$ such that $\mathscr{E}(-n)$ has no sections. If i is the largest integer where $\mathscr{E}(-i)$ admits a section, then after twisting, we get an s.e.s.

$0 \to \mathcal{O}(i) \to \mathscr{E} \to \mathscr{F} \to 0$, where $\mathscr{F}$ has rank $r-1$. Then $\mathscr{F} \approx \oplus_j \mathcal{O}(j)^{r_j}$ by induction, with $j \leq i$ for $r_j \neq 0$ or else $\mathscr{E}(-i-1)$ has a nonzero section. But then $H^1(X, \mathscr{H}om(\mathscr{F}, \mathcal{O}(i))) = 0$ so the sequence splits by Weibel 10.1 and exc III.6.1.

5.2.12 V.2.7 x

2.7. On the elliptic ruled surface X of (2.11.6), show that the sections C_0 with $C_0^2 = 1$ form a one-dimensional algebraic family, parametrized by the points of the base curve C, and that no two are linearly equivalent.

By 2.11.6, $e = -1$.

Thus by 2.12.a, $\mathcal{E}$ must be indecomposable. (by 2.11.6, locally free, rank 2)

By 2.15, there is exactly one ruled surface over $\times C$ for this value of e.

Now according to the defniitions 2.9, sections correspond to $\mathcal{E} \to \mathcal{L}(\mathfrak{d}) \to 0$ and by 2.15, $\mathfrak{d}$ has degree 1 and the exact sequence is $0 \to \mathcal{O}_C \to \mathcal{E} \to \mathcal{L}(P) \to 0$ for some point $P \in C$.

By 2.16, there is a natural 1-1 correspondence between set of isomorphism classes of indecomposable locally free sheaves of rank 2 and degree 1 on elliptic curve C and set of points of C.

Now suppose that $\mathcal{E} \not\approx \mathcal{E}'$. Then the two exact sequences correspond to different points $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ where $P \neq Q$.

Now for $\mathcal{E}$ we have $C_0^2 = -e = -(-1) = 1$ if we use the notation 2.8.1 on the exact sequence $\mathcal{E} \to \mathcal{L}(Q) \to 0$ and similarly, if we use the notation on the exact sequence $\mathcal{E} \to \mathcal{L}(Q) \to 0$ (maybe we can say it's C_0' in this case).

Suppose that $C_0 \sim C_0 + (P-Q)f$. Then $Pf \sim Qf$ so $\mathcal{L}(P) \approx \mathcal{L}(Q)$ but then $P \sim Q$.

Note that on an elliptic curve we have the following theorem:

"If E is an elliptic curve the map $E \to Div^0(E)/Prin(E)$ defined by $P \to [P-0]$ is a bijection" (brackets denote the equivalence class)

so this would mean P and Q are the same point.

5.2.13 V.2.8.a x g decomposable is never stable

2.8. A locally free sheaf $\mathscr{E}$ on a curve C is said to be *stable* if for every quotient locally free sheaf $\mathscr{E} \to \mathscr{F} \to 0$, $\mathscr{F} \neq \mathscr{E}$, $\mathscr{F} \neq 0$, we have

$$(\deg \mathscr{F})/\mathrm{rank}\, \mathscr{F} > (\deg \mathscr{E})/\mathrm{rank}\, \mathscr{E}.$$

Replacing $>$ by $\geqslant$ defines *semistable*.

(a) A decomposable $\mathscr{E}$ is never stable.

A decomposable is a direct sum of two invertible sheaves.

Claim: $\mathscr{E}$ stable $\implies$ every non-zero morphism of $\mathscr{E}$ to itself is an isomorphism.

Given this, then clearly $f : \mathscr{G} \oplus \mathscr{F} \to \mathscr{G} \oplus \mathscr{F} \to f = id_{\mathscr{G}} \oplus 0_{\mathscr{F}}$ is clearly not an iso.

Proof of claim.

Factor $\varphi : \mathscr{E} \to \mathscr{E}$ as $\varphi : \mathscr{E} \to im\ \varphi \to \mathscr{E}$.

If $im\ \varphi \neq \mathscr{E}$, then by stability, $\frac{deg(\mathscr{E})}{rk(\mathscr{E})} < \frac{deg(im\varphi)}{rk(im\varphi)} < \frac{deg(\mathscr{E})}{rk(\mathscr{E})}$ contradiction.

Thus $im\ \varphi = \varphi$ so the kernel is 0 since it's locally free.

5.2.14 b.x g

(b) If $\mathscr{E}$ has rank 2 and is normalized, then $\mathscr{E}$ is stable (respectively, semistable) if and only if $\deg \mathscr{E} > 0$ (respectively, $\geqslant 0$).

First note that if $\mathscr{E}$ is (semi) stable of (negative) non-positive degree , then $\mathscr{E}$ has no sections. For if $\mathscr{E}$ has a section, then $\mathcal{O}_C \subset \mathscr{E}$ so $deg\,(\mathscr{E})\,(\geq) > 0$, contradiction to (semi)-stability. Since $\mathscr{E}$ is normalized, it has global sections, and thus by contrapositive we have the if direction.

Now suppose $\mathscr{E}$ has positive degree and is rank 2 normalized. Since $\mathscr{E}$ is rank 2, we need for $\mathscr{F} \subset \mathscr{E}$, that $deg\,(\mathscr{F}) \leq \frac{1}{2} deg\,(\mathscr{E})$, since at any rate $rk\,(\mathscr{F}) < 2$. If $\mathscr{E}$ is decomposable, then by thm V.2.12(a), and the normalized assumption, and thm V.2.8 degree of $\mathscr{F}$ would be ≤ 0 so we have the inequality.

Assume $\mathscr{E}$ is indecomposable of rank r, degree d. By exc V.2.6, we may therefore assume $p_a\,(C) > 0$. Assume to the contrary that $\mathscr{E}$ is not stable. Thus there is $\mathscr{F} \subset \mathscr{E}$ with $deg\,(\mathscr{F}) > \frac{1}{2} deg\,(\mathscr{E}) > 0$. But then as there is a map $\mathscr{F} \to \mathscr{E}$, $\mathscr{E} \otimes \mathscr{F}^\vee$ has a section, and $\mathscr{F}$ is invertible and $deg\ \mathscr{F}^\vee < 0$ so by the normalized assumption $h^0\,(\mathscr{E} \otimes \mathscr{F}^\vee) = 0$ so actually it should have no sections.

To discern between semistable and stable in this direction, use 4.16, Teixido, vector bundles: An indecomposable of degree zero is semistable but not stable.

5.2.15 c. x g

(c) Show that the indecomposable locally free sheaves $\mathscr{E}$ of rank 2 that are not semistable are classified, up to isomorphism, by giving (1) an integer $0 < e \leq 2g - 2$, (2) an element $\mathscr{L} \in \operatorname{Pic} C$ of degree $-e$, and (3) a nonzero $\xi \in H^1(\mathscr{L}^\vee)$, ~~determined up to a nonzero scalar multiple.~~

Let $\mathscr{E}$ non-semistable.

Claim: The degrees of coherent subsheaves of $\mathscr{F}$ are bounded above.

On $\mathbb{P}^1$, this is clear by exc V.2.6. Otherwise, let $f : X \to \mathbb{P}_1$ with $f^*\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_X(1)$ be finite. The pushforward of any subsheaf $\mathscr{G}$ of $\mathscr{F}$ is a subsheaf of $f_*\mathscr{F}$. By Leray, exc III.8.1, $\chi\,(f_*\mathscr{G}) = \chi\,(\mathscr{G})$ and since the euler characteristic is additive in s.e.s., we find the $\chi\,(\mathscr{G})$ is bounded. Since degree is $deg\,(\mathscr{G}) = \chi\,(\mathscr{G}) - rk\,(\mathscr{G}) \cdot \chi\,(\mathcal{O}_C)$, then the degrees are bounded above.

Claim: $\mathscr{E}$ has an increasing filtration by $0 \subset \mathscr{F}_1 \subset \cdots \subset \mathscr{F}_k = \mathscr{F}$ where (a) $\mathscr{F}_i/\mathscr{F}_{i-1}$ is semi-stable and (b) $\mu\,(\mathscr{F}_i/\mathscr{F}_{i-1}) < \mu\,(\mathscr{F}_{i+1}/\mathscr{F}_i)$.

If $\mathscr{F}$ is semi-stable then this is clear. Else, by the first claim, we can find $\mathscr{F}_1 \subset \mathscr{F}$ with maximal rank among those of maximal slope. By maximality, $\mathscr{F}_1$ is semi-stable. Since the quotient is locally free we can repeat. Note that (b) follows since $\mu\,(\mathscr{G}) < \mu\,(\mathscr{F}_1)$ for $\mathscr{G} \subset \mathscr{F}/\mathscr{F}_1$. For the uniqueness of this filtration, see prop 5.4.2, Potier.

Now consider the maximal slope subsheaf $\mathscr{F}_1$ in the filtration, we have a nontrivial extension $0 \to \mathscr{F}_1 \to \mathscr{E} \to \mathscr{L} \to 0$. As in thm V.2.12, this corresponds to nonzero $\xi \in Ext^1\,(\mathscr{L}, \mathcal{O}_X) \approx H^1\,(C, \mathscr{L}^\vee)$ with $-deg\ \mathscr{L} \leq 2g - 2$. Since $\mathscr{E}$ is non-semistable and $\mathscr{F}_1$ is maximal slope, then $deg\ \mathscr{F}_1 > 0$. Since necessarily $deg\ \mathscr{L} + deg \mathscr{F}_1 = deg \mathscr{E}$, we must have $-deg\ \mathscr{L} > 0$. Since the filtration was unique we have this up to isomorphism.

5.2.16 V.2.9 x g Curves on Quadric Cone

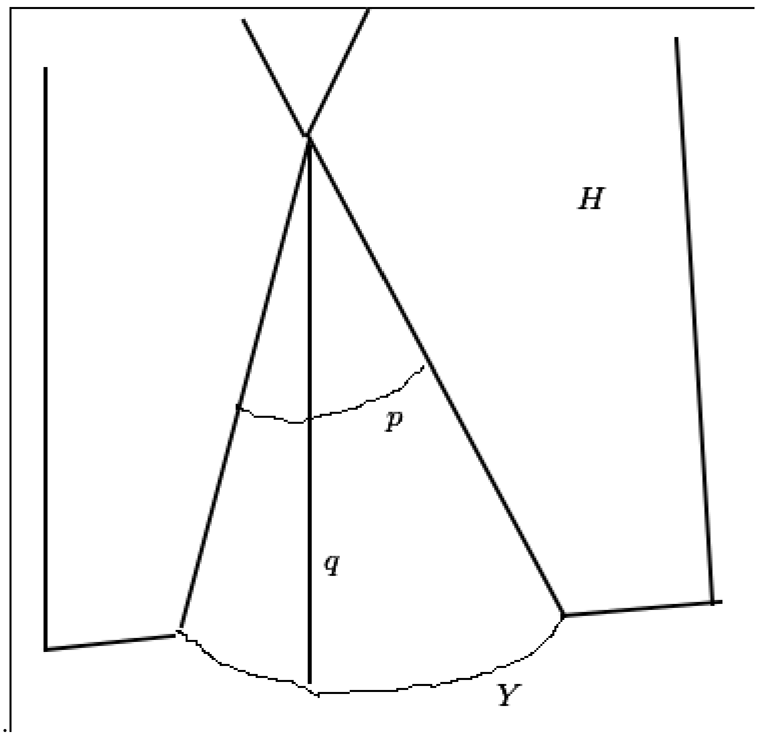
2.9. Let Y be a nonsingular curve on a quadric cone X_0 in $\mathbf{P}^3$. Show that either Y is a complete intersection of X_0 with a surface of degree $a \geqslant 1$, in which case $\deg Y = 2a$, $g(Y) = (a-1)^2$, *or*, $\deg Y$ is odd, say $2a+1$, and $g(Y) = a^2 - a$. Cf. (IV, 6.4.1). [*Hint*: Use (2.11.4).]

So if it's a complete intersection, then by exc II.8.4, $deg = 2a$, and $g = \frac{1}{2} 2 \cdot a\,(a + 2 - 4) + 1 = a\,(a - 2) + 1 = a^2 - 2a + 1 = (a-1)^2$.

On the other hand, we can split the two cases into curves intersecting the vertex and not. geometrically this corresponds to some multiplicity of the conic section together with the line. Note the line has multiplicity

1 since it's nonsingular and it's going through the vertex. (else too many tangent directions)

Using what we know of ruled surfaces, a divisor D on the cone will be some multiple of the section $p +$ some multiple of the ruling q. A hyperplane intersecting Y through the vertex will hit q once and p twice and using exc V.1.1, we see the degree is $2a + 1$ some a.

- 

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Geometrically considering the number of intersections of a plane passing through the vertex (this gives the line) and via exc V.1.1 we see that the degree is $2a + 1$ since this is the number of intersections.

To compute the genus, we blow up the point on the cone via V.2.11.4 achieving a ruled surface X over the conic, with $e = -d$. In particular, $K = -2C_0 - 4f$ where C_0 is the section of the conic. (this is via V.2.11). Thus the hyperplane section on the cone lifts to $C_1 = C_0 + 2f$. In particular,

$C_1^2 = C_0^2 + 4 = 2.$

$C_1.f = (C_0 + 2f).f = 1.$

$C_0.C_1 = C_0 (C_0 + 2f) = -2 + 2 = 0$

Our original curve which was in the form $aC + f$, therefore lifts to $aC_1 + f$. Now we attempt to compute genus using adjunction.

$expand \left(\frac{1}{2} \cdot (a \cdot C_1 + f) \cdot (a \cdot C_1 + f - 2 \cdot C_0 - 4 \cdot f) + 1\right) =$

$-\frac{3f^2}{2} - C_1 a f - C_0 f + \frac{C_1^2 a^2}{2} - C_0 C_1 a + 1$

This gives $-a - 1 + a^2 + 1 = a^2 - a$ which is what we wanted.

5.2.17 V.2.10 x

2.10. For any $n > e \geqslant 0$, let X be the rational scroll of degree $d = 2n - e$ in $\mathbf{P}^{d+1}$ given by (2.19). If $n \geqslant 2e - 2$, show that X contains a nonsingular curve Y of genus $g = d + 2$ which is a canonical curve in this embedding. Conclude that for every $g \geqslant 4$, there exists a nonhyperelliptic curve of genus g which has a g_3^1. Cf. (IV, §5).

See example 2.10 in Kollar's Complex Algebraic Geometry.

5.2.18 V.2.11 x

2.11. Let X be a ruled surface over the curve C, defined by a normalized bundle $\mathscr{E}$, and let $\mathfrak{e}$ be the divisor on C for which $\mathscr{L}(\mathfrak{e}) \cong \bigwedge^2 \mathscr{E}$ (2.8.1). Let $\mathfrak{b}$ be any divisor on C.

(a) If $|\mathfrak{b}|$ and $|\mathfrak{b}+\mathfrak{e}|$ have no base points, and if $\mathfrak{b}$ is nonspecial, then there is a section $D \sim C_0 + \mathfrak{b}f$, and $|D|$ has no base points.

Claim: If $\mathfrak{b}$ is nonspecial, then $h^i(\mathcal{O}_X(C_0+\mathfrak{b}f)) = h^i(\mathcal{O}_C(\mathfrak{b})) + h^i(\mathcal{O}_C(\mathfrak{b}+\mathfrak{e}))$.

Proof: Consider the LES associated to $0 \to \mathcal{O}_X(-C_0) \to \mathcal{O}_X \to \mathcal{O}_{C_0} \to 0$. From the picture, $h^i(\mathcal{O}_X(\mathfrak{b}f)) = h^i(\mathcal{O}_C(\mathfrak{b}))$ and $h^i(\mathcal{O}_{C_0}(C_0+\mathfrak{b}f)) = h^i(\mathcal{O}_X(\mathfrak{b}+\mathfrak{e}))$. Also h^2 on the curve vanishes and $h^1(\mathcal{O}_X(\mathfrak{b}f)) = 0$ by nonspecialness. Now use exactness.

Claim: If $P \in C$, then $H = |C_0+\mathfrak{b}f|$ is bpf on Pf(the fiber over P) iff $h^0(\mathcal{O}_X(H-Pf)) = h^0(\mathcal{O}_X(H)) - 2$. .

Proof: This is essentially rephrasing thm IV.3.1, using the same left exact sequence and noting that $Pf \approx \mathbb{P}^1$ since we're on a ruled surface.

Claim: $|D|$ has no basepoints

Proof:Since there are effective divisors linearly equivalent to $\mathfrak{b}$, $\mathfrak{b}+\mathfrak{e}$, any generic point P can't be a base point of both. Since $\mathfrak{b}$ is nonspecial, $\mathfrak{b}-P$ is nonspecial as P is not a base point of $\mathfrak{b}$, it being bpf. Thus by the first claim,

$h^0(\mathcal{O}_X(C_0+(\mathfrak{b}-P)f)) =$

$h^0(\mathcal{O}_C(\mathfrak{b}-P)) + h^0(\mathcal{O}_C(\mathfrak{b}+\mathfrak{e}-P)) =$ which by bpf is

$h^0(\mathcal{O}_C(\mathfrak{b})) - 1 + h^0(\mathcal{O}_C(\mathfrak{b}+\mathfrak{e}-P)) =$ if P is not a basepoint of $\mathfrak{b}+\mathfrak{e}$

$h^0(\mathcal{O}_C(\mathfrak{b})) - 1 + h^0(\mathcal{O}_C(\mathfrak{b}+\mathfrak{e})) - 1$ so we're done.

5.2.19 b. x

(b) If $\mathfrak{b}$ and $\mathfrak{b}+\mathfrak{e}$ are very ample on C, and for every point $P \in C$ we have $\mathfrak{b}-P$ and $\mathfrak{b}+\mathfrak{e}-P$ nonspecial, then $C_0+\mathfrak{b}f$ is very ample.

Note $\mathfrak{b}-P$ is bpf since $\mathfrak{b}$ is very ample and using thm IV.3.1. By the first claim in (a), for arbitrary P,Q on the curve,

$h^0(\mathcal{O}_X(C_0+\mathfrak{b}f-(P+Q)f)) =$

$h^0(\mathcal{O}_C(\mathfrak{b}-P-Q)) + h^0(\mathcal{O}_C(\mathfrak{b}+\mathfrak{e}-P-Q)) =$ by thm IV.3.1

$h^0(\mathcal{O}_C(\mathfrak{b})) + h^0(\mathcal{O}_C(\mathfrak{b}+\mathfrak{e})) - 4 =$ by the first claim in (a)

$h^0(\mathcal{O}_X(C_0+\mathfrak{b}f)) - 4$.

Now if $|C_0+\mathfrak{b}f|$ had a base point P, then $h^0(\mathcal{O}_X(C_0+\mathfrak{b}f-Pf)) \geq h^0(\mathcal{O}_X(C_0+\mathfrak{b}f)) - 1$, but then $h^0(\mathcal{O}_X(C_0+\mathfrak{b}f-(P+Q)f)) \geq h^0(\mathcal{O}_X(C_0+\mathfrak{b}f)) - 3$ using the first claim in (a) which is a contradiction.

Now let ϕ be the map determined by $|C_0+\mathfrak{b}f|$. Then ϕ is injective, for if x,y are two points on the same ruling then by the second claim in (a) and thm IV.3.1, $|C_0+\mathfrak{b}f|$ is very ample on the fiber and thus separates x,y .

Further, ϕ separates tangent vectors, since if x is in the fiber f over P and $t \in T_x(f)$, then $|C_0+\mathfrak{b}f|$ is very ample on f so there is a tangent vector through x meeting the fiber transversally. If $t \notin T_x(f)$, then you can just take a tangent along the fiber and containing x. Thus $|C_0+\mathfrak{b}f|$ separates tangent vectors.

5.2.20 V.2.12 x

2.12. Let X be a ruled surface with invariant e over an elliptic curve C, and let $\mathfrak{b}$ be a divisor on C.

(a) If $\deg \mathfrak{b} \geq e+2$, then there is a section $D \sim C_0 + \mathfrak{b}f$ such that $|D|$ has no base points.

This is Fuentes, Pedreira Prop 1.3.

5.2.21 b. x

(b) ~~The linear system $|C_0 + bf|$ is very ample if and o~~nly if $\deg b \geqslant e + 3$.
Note. The case $e = -1$ will require special attention.

Fuentes, Predeira prop 1.4

V.2.13 x

2.13. For every $e \geqslant -1$ and $n \geqslant e + 3$, there is an elliptic scroll of degree $d = 2n - e$ in $\mathbf{P}^{d-1}$. In particular, there is an elliptic scroll of degree 5 in $\mathbf{P}^4$.

(Follow thm V. 2.19 slightly)
So by thm V.2.15, $e = 0$ or $e = -1$ are possible for indecomposable
by thm V.2.12, a, $e \geq 0$ are possible for decomposable.
Let $D = C_0 + nf$. This is ample by thm V.2.20.b, V.2.21, b.
Then using thm V.2.3, that $C_0.f = 1$ and $f^2 = 0$, we have
$D.f = (C_0 + nf).f = C_0.f + nf^2 = 1$ so it's an elliptic scroll.
$D^2 = (C_0 + nf).(C_0 + nf) = 2n - e$ so image has degree $2n - e = d$.
so the first part is done... Now just need to find N.

We need $H^0(X, \mathcal{L}(D)) = H^0(C, \pi_* \mathcal{L}(D)) = ... = H^0(C, ...)$
where the last term has dimension $d = 2n - e$. This would give $N = d - 1$.
$H^0(X, \mathcal{L}(D)) = H^0(C, \pi_* \mathcal{L}(D)) = H^0(C, \mathcal{O}_C(n) \oplus \mathcal{O}_C(n - e))$ (c.f. thm IV.2.12.a, pp309, exa 3.3.3)
By riemann roche (c.f. thm pp 319, pp381), get $H^0(C, \mathcal{O}_C(n)) = n$ and so this dimension is $2n - 1 = d - 1$.

V.2.14 x

2.14. Let X be a ruled surface over a curve C of genus g, with invariant $e < 0$, and assume that char $k = p > 0$ and $g \geqslant 2$.
(a) If $Y \equiv aC_0 + bf$ is an irreducible curve $\neq C_0, f$, then either $a = 1, b \geqslant 0$, or $2 \leqslant a \leqslant p - 1, b \geqslant \frac{1}{2}ae$, or $a \geqslant p, b \geqslant \frac{1}{2}ae + 1 - g$.

C.f. thm V.2.21 assume genus is ≥ 2.

Let $\tilde{Y}$ the normalization of Y and consider the composition of natural map $\tilde{Y} \to Y$ with the projection $\pi : Y \to C$. If $char\ k = p$, then

This map is degree $a\,(mod\ p)$ so by (thm V.2.4, we have)
$2g\left(\tilde{Y}\right) - 2 \geq \alpha(2g - 2) + degR$ where R is (effective ram divisor).
On other hand, $p_a(Y) \geq g\left(\tilde{Y}\right)$ (thm IV.1.8) so
$2p_a(Y) - 2 \geq \alpha(2g - 2)$ by getting rid of $deg\ R$.
Futhermore, this last inequality is true in any char if $g = 0, 1$, since in any case, $p_a(Y) \geq g$.
By adjunction, we have
$2p_a(Y) - 2 = Y.(Y + K)$.

Substituting $Y =\equiv aC_0 + bf$ and $K \equiv -2C_0 + (2g - 2 - e) f$ from (thm V.2.11), and combining with inequality above we find that
$(aC_0 + bf).([aC_0 + bf] + -2C_0 + (2g - 2 - e) f) \geq \alpha(2g - 2)$

LHS is

$a^2C_0^2 + 2abC_0.f + b^2f^2 - 2aC_0^2 + aC_0.(2g-2-e)f - 2abC_0.f + b(2g-2-e).f^2$

we use that $C_0^2 = -e$, $C_0.f = 1$, and $f^2 = 0$ so that the above LHS becomes:

$a^2(-e) + 2ab - 2a(-e) + a(2g-2-e) - 2ab.$

In the end this becomes

$ae(1-a) + 2b(a-1) \geq (\alpha - a)(2g-2).$

Note that when $\alpha = a$, (as is the case when $2 \leq a \leq p-1$) then we retain the inequality from V.2.21, which is

$b(a-1) \geq \frac{1}{2}ae(a-1)$. So that if $p-1 \geq a \geq 2$, we have $b \geq \frac{1}{2}ae$ as required.

If $a \geq p$, then the term $(\alpha - a)(2g-2)$ will be $(-np)(2g-2) = np(1-g)$.

We will have $b(a-1) \geq \frac{1}{2}ae(a-1) + \frac{1}{2}(a - deg(\pi))(1-g)$

Now we don't know $deg\pi$ except it's less than a. Thus $(a - deg(\pi)) > 0$.

The most that $\frac{1}{2}(a - deg(\pi))(1-g)$ can be is when $deg(\pi) = 1$.

And thus we have $b \geq \frac{1}{2}ae + 1 - g$.

In the case $a = 1$, the same proof as thm V.2.21.a holds:

Y is a section, corresponding to surjective map $\mathcal{E} \to \mathcal{L} \to 0$.

Since $\mathcal{E}$ is normalized, $deg\mathcal{L} \geq deg\mathcal{E}$.

But $deg\mathcal{L} = C_0.Y$ by thm V.2.9, so $b - e \geq -e$ and $b \geq 0$.

5.2.22 b. x

(b) If $a > 0$ and $b > a(\frac{1}{2}e + (1/p)(g-1))$, then any divisor $D \equiv aC_0 + bf$ is ample. On the other hand, if D is ample, then $a > 0$ and $b > \frac{1}{2}ae$.

Using thm V.1.10, if D is ample, then since Nakai-Moishezon works in any characteristic, then $D.f > 0$ and $D^2 = (aC_0 + bf)^2 = a^2C_0^2 + 2abC_0.f = -a^2e + 2ab > 0$.

Thus $b > \frac{1}{2}ae$.

Suppose that $a > 0$ and $b > \frac{1}{2}a\left(e + \left(\frac{1}{p}\right)(g-1)\right) = \frac{1}{2}ae + \frac{1}{2}\frac{a}{p}(g-1)$.

Since $g > 2$, and $a > 0$, then $b > \frac{1}{2}ae$.

Then $D.f = (aC_0 + bf).f = a > 0$.

Also $D^2 = -a^2e + 2ab = 2a\left(b - \frac{1}{2}ae\right) > 0$. since $b - \frac{1}{2}ae > 0$ and $a > 0$.

Also $D.C_0 = (aC_0 + bf).C_0 = -ae + b > -ae + \frac{1}{2}ae = -\frac{1}{2}ae > 0$ (since a is positive, and $e < 0$ in the assumptions of the problem).

Let Y an irreducible curve $\neq C_0, f$. Then

$D.Y = (a.C_0 + bf).(a'C_0 + b'f) = aa'(-e) + ab'C_0.f + ba'C_0.f + 0 =$

$-aa'e + ab' + ba'$.

Suppose that $a' = 1$. Since Y is an irreducible curve, then by the first part of the excercise, we have that $b' \geq 0$ (in fact ≥ 1 since $Y \neq C_0$) so using: $a > 0$, $b \geq \frac{1}{2}ae$ and that $b' \geq 1$, we have $ab' > 0$ and thus

$D.Y = -ae + ab' + b > -ae + b \geq -ae + \frac{1}{2}ae = -\frac{1}{2}ae > 0$ since $e < 0$ by assumption.

Next suppose that $2 \leq a' \leq p-1$. By part (a) $b' \geq \frac{1}{2}a'e$ and by assumption, $b > \frac{1}{2}ae$.

Thus $D.Y = -aa'e + ab' + ba'$ has $ab' \geq \frac{1}{2}aa'e$ and $ba' > \frac{1}{2}aea'$.

In total, $D.Y > -aa'e + \frac{1}{2}aa'e + \frac{1}{2}aea' = 0$.

Finally, suppose that $a' \geq p$. By part (a), $b' \geq \frac{1}{2}a'e + 1 - g$. Note $g \geq 2$ so $(1-g) < 0$.

To recap, we also have $a > 0$, $b > a\left(\frac{1}{2}e + \left(\frac{1}{p}\right)(g-1)\right)$, $D.Y = -aa'e + ab' + ba'$.

Thus $ab' \geq \frac{1}{2}aa'e + a - ga$ and $a'b > a'a\frac{1}{2}e + \frac{a'}{p}(g-1)$.

In total, $D.Y > \left[-aa'e + \frac{1}{2}aa'e + \frac{1}{2}aa'e\right] + \left[a - ga + \frac{aa'}{p}(g-1)\right]$.

The first term is 0 and the second factors to $a(1-g)\left(1-\frac{a'}{p}\right) \geq 0$.

By Nakai-Moishezon, we are done...

V.2.15 x Funny behavior in char p

2.15. *Funny behavior in characteristic p.* Let C be the plane curve $x^3y + y^3z + z^3x = 0$ over a field k of characteristic 3 (IV, Ex. 2.4).

(a) Show that the action of the k-linear Frobenius morphism f on $H^1(C, \mathcal{O}_C)$ is identically 0 (Cf. (IV, 4.21)).

By degree genus in $\mathbb{P}^2$ is genus 3 quartic.
Let's try and follow thm IV.4.21
Calculate H^1 of quartic curve in $\mathbb{P}^2$.
The ideal sheaf is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(-4)$ so we have an exact sequence:
$0 \to \mathcal{O}_{\mathbb{P}}(-4) \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_X \to 0$.
Taking cohomology, we get $\to H^1(\mathcal{O}_{\mathbb{P}^2}) \to H^1(\mathcal{O}_X) \to H^2(\mathcal{O}_{\mathbb{P}^2}(-4)) \to H^2(\mathcal{O}_{\mathbb{P}^2})$
Via stacks,
we have $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(0)) = 0$ and $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(0 = -r-1-(-r-1))) \approx H^0(X, \mathcal{O}_X(-r-1)) = 0$ (since no global sections) so we obtain $H^1(\mathcal{O}_X) \approx H^2(\mathcal{O}_{\mathbb{P}^2}(-4))$.

Now $H^2(\mathcal{O}_{\mathbb{P}^2}(-4)) \approx H^0(\mathcal{O}_{\mathbb{P}^2}(1))$ of dim $\begin{pmatrix} 2+1 \\ 1 \end{pmatrix} = 3$ vector space...

We compute action of frobenius using this embedding.

If F_1 is Frobenius morphism on $\mathbb{P}^2$, then F_1^* takes $\mathcal{O}_X$ to $\mathcal{O}_{X^p}$, where X^p is subscheme of $\mathbb{P}^2$ defined by $f^p = 0$.

Thus:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}}(-4) \arrow[r] \arrow[d, "F_1^*"] & \mathcal{O}_{\mathbb{P}} \arrow[r] \arrow[d, "F_1^*"] & \mathcal{O}_X \arrow[r] \arrow[d, "F_1^*"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}}(-4p) \arrow[r] & \mathcal{O}_{\mathbb{P}} \arrow[r] & \mathcal{O}_{X^p} \arrow[r] & 0 \end{tikzcd}$$

Namely $x^9y^3 + y^9z^3 + z^9x^3 = 0$. (char 3) On other hand, X is closed subscheme of X^p , (since f is a factor of $(f)^3$) so we have a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}}(-4p) \arrow[r] \arrow[d, "f^{p-1}"] & \mathcal{O}_{\mathbb{P}} \arrow[r] \arrow[d] & \mathcal{O}_{X^p} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{O}_{\mathbb{P}}(-4) \arrow[r] & \mathcal{O}_{\mathbb{P}} \arrow[r] & \mathcal{O}_X \arrow[r] & 0 \end{tikzcd}$$

(we can multiply by the equation f^{p-1} to go from the scheme f to f^p and note that if we start in the thing twisted by $-4p$ (p is the characteristic - which is 3) then we will end in the thing twisted by -4 – compute a few examples if it seems mysterious))

Now combining the cohomologies of the above two diagrams gives:

$$\begin{tikzcd} H^1(X, \mathcal{O}_X) \arrow[r, "\approx"] \arrow[d, "F_1^*"] & H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(-4)) \arrow[d, "F_1^*"] \\ H^1(X^p, \mathcal{O}_{X^p}) \arrow[r, "\approx"] \arrow[d] & H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(-4p)) \arrow[d, "f^{p-1}"] \\ H^1(X, \mathcal{O}_X) \arrow[r, "\approx"] & H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(-4)) \end{tikzcd}$$

Now F^* is the map $H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ on the left hand column.

We can use 5.1 to get that $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(-4)) \approx H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(1))$ with dim 3.

Since a basis for $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(-3))$ is $(xyz)^{-1}$, then we must have a basis for $H^2(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}}(-4))$ is $(x^2yz)^{-1}$, $(xy^2z)^{-1}$, and $(xyz^2)^{-1}$ as a free $\mathbb{F}_3\left[\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right]$-module.

Then $F_1^*\left((x^2yz)^{-1}\right) = (x^2yz)^{-p}$, $F_1^*\left((xy^2z)^{-1}\right) = (xy^2z)^{-p}$, and $F_1^*\left((xyz^2)^{-1}\right) = (xyz^2)^{-p}$, and the image in $H^2(\mathcal{O}_{\mathbb{P}}(-4))$ (bottom right of above diagram) will be (for example) $f^{p-1} \cdot (x^2yz)^{-p} = f^{p-1}(x^{-6}y^{-3}z^{-3})$.

Now f^{p-1} is $(x^2z^6 + 2xy^3z^4 + 2x^4yz^3 + y^6z^2 + 2x^3y^4z + x^6y^2)$ and $(x^2yz)^{-3}$ is $(x^{-6}y^{-3}z^{-3})$.

We multiply these giving $\frac{z^3}{x^4y^3} + \frac{2y}{x^5} + \frac{y^3}{x^6z} + \frac{2y}{x^3z^2} + \frac{1}{yz^3} + \frac{2}{x^2y^2}$.

Similarly, $f^{p-1}(xy^2z)^{-3}$ is $\frac{z^3}{xy^6} + \frac{2z}{x^2y^3} + \frac{1}{x^3z} + \frac{2}{y^2z^2} + \frac{x^3}{y^4z^3} + \frac{2x}{y^5}$

and finally, $f^{p-1}(xyz^2)^{-3}$ is $\frac{2}{x^2z^2} + \frac{2x}{y^2z^3} + \frac{y^3}{x^3z^4} + \frac{2y}{z^5} + \frac{x^3}{yz^6} + \frac{1}{xy^3}$.

Now any monomial having a nonnegative exponent on x, y, or z is 0 , and thus each of the above expressions is 0. Thus F^* is identically 0.

5.2.23 b. x

> (b) Fix a point $P \in C$, and show that there is a nonzero $\xi \in H^1(\mathscr{L}(-P))$ such that $f^*\xi = 0$ in $H^1(\mathscr{L}(-3P))$.

Let H be hyperplane section. First make the bottom sequence: $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_H \to 0$ Now top sequence: $0 \to \mathcal{O}_X(-3) \to \mathcal{O}_X \to \mathcal{O}_{H^3} \to 0$ put them together:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-3) \arrow[r] \arrow[d, "f^{3-1}"] & \mathcal{O}_X \arrow[r] \arrow[d] & \mathcal{O}_{H^3} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_H \arrow[r] & 0
\end{tikzcd}$$

Now $F^*(\xi) = \xi^p$ and it's image in $H^1(C, \mathcal{O}_C(-1))$ will be $f^2 \cdot \xi^p$. I can look at f^2 in affine coords. Writing this out gives

$subst\left(1, z, expand\left((x^3 \cdot y + y^3 \cdot z + z^3 \cdot x)^2\right)\right) = y^6 + 2\,x^3\,y^4 + 2\,x\,y^3 + x^6\,y^2 + 2\,x^4\,y + x^2$. On the other hand, $H^1(\mathcal{O}_C(-1))$ has basis β , and any monomial having nonnegative exponent is 0. Thus the image is just β times coefficient of β. So I need to determine the basis β of $H^1(C, \mathcal{O}_C(-1))$. So since it says plane curve, we're in $\mathbb{P}^2$. In the exc III.4.7 I computed what these elements look like explicitly for a plane curve. Every element of $H^1(C, \mathcal{O}_C)$ can be represented by a polynomial like $\sum a_{ij}x^iy^j$ with $1 \le i < d$ and $-i < j < 0$. If we twist by negative 1, then $-4 < i \le -1$ and $i < -j < 0$ which, as in exc III.4.7 will give a zero image.

5.2.24 c. x

> (c) Now let $\mathscr{E}$ be defined by ξ as an extension
>
> $$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{L}(P) \to 0,$$
>
> and let X be the corresponding ruled surface over C. Show that X contains a nonsingular curve $Y \equiv 3C_0 - 3f$, such that $\pi: Y \to C$ is purely inseparable. Show that the divisor $D = 2C_0$ satisfies the hypotheses of (2.21b), but is not ample.

To see the curve $Y \equiv 3C_0 - 3f$ is inseparable is Miyanishi, Open Algebraic Surfaces, lemma 2.5.2.2. The significance of $\pi : Y \to C$ is that then they will be isomorphic as abstract schemes. So Y is some section of C lying on the surface.

Now since we have a genus 3 and we are not in characteristic 0, we have a chance of getting a counterexample to 2.21b since those hypothesis are required by 2.21b.

What I need to find are $a > 0, b > \frac{1}{2}ae$ where invariant e is invariant of ruled surface is < 0. Note that the invariant is the degree of the the sheaf $\mathcal{L}(P)$ (see for instance Theorem 2.15 or 2.12), so in particular, so it's -1 as $C_0^2 = -1$. If we let $a = 6, b = -6$, then $-6 > \frac{1}{2}(6) \cdot 6$. Clearly $2D$ will not be ample, since $C_0.(6C_0 - 6f) = -6 - 6 < 0$ and by Nakai moishezon.

5.2.25 V.2.16 x

2.16. Let C be a nonsingular affine curve. Show that two locally free sheaves $\mathscr{E}, \mathscr{E}'$ of the same rank are isomorphic if and only if their classes in the Grothendieck group $K(X)$ (II, Ex. 6.10) and (II, Ex. 6.11) are the same. This is false for a projective curve.

So suppose they are isomorphic. $\mathcal{E} \approx \mathcal{E}'$. Then have ses $0 \to 0 \to \mathcal{E} \to \mathcal{E}' \to 0$ so $\mathcal{E} - \mathcal{E}' - 0 = 0$ in $K(X)$. so classes in $K(X)$ are the same. On the other hand suppose class in groth group are the same. Since locally free, it must be $\widetilde{M}$ for some finitely generated projective $\mathcal{O}(X)$module M. Now by Rotman 7.77 gives that if C is a $\star$-category, then $A, B \in Obj(C)$ have the same classes in the grothendieck group iff there is $D \in Obj(C)$ with $A \star D \approx B \star D$. Now use the fact (Rotman 11.118) that stably isomorphic on dedekind domain (such as a smooth affine curve) is equivalent to isomorphic. Note that an affine variety is a smooth curve iff it's a dedekind domain (for this definition of a dedekind domain see Bruning coherent sheaves on an elliptic curve).

5.2.26 V.2.17* (starred)

***2.17.** (a) Let $\varphi: \mathbf{P}_k^1 \to \mathbf{P}_k^3$ be the 3-uple embedding (I, Ex. 2.12). Let $\mathscr{I}$ be the sheaf of ideals of the twisted cubic curve C which is the image of φ. Then $\mathscr{I}/\mathscr{I}^2$ is a locally free sheaf of rank 2 on C, so $\varphi^*(\mathscr{I}/\mathscr{I}^2)$ is a locally free sheaf of rank 2 on $\mathbf{P}^1$. By (2.14), therefore, $\varphi^*(\mathscr{I}/\mathscr{I}^2) \cong \mathcal{O}(l) \oplus \mathcal{O}(m)$ for some $l, m \in \mathbf{Z}$. Determine l and m.

(b) Repeat part (a) for the embedding $\varphi: \mathbf{P}^1 \to \mathbf{P}^3$ given by $x_0 = t^4$, $x_1 = t^3u$, $x_2 = tu^3$, $x_3 = u^4$, whose image is a nonsingular rational quartic curve. [*Answer*: If char $k \neq 2$, then $l = m = -7$; if char $k = 2$, then $l, m = -6, -8$.]

starred

5.3 V.3 Monoidal Transformations

5.3.1 V.3.1 x g

3.1. Let X be a nonsingular projective variety of any dimension, let Y be a nonsingular subvariety, and let $\pi: \tilde{X} \to X$ be obtained by blowing up Y. Show that $p_a(\tilde{X}) = p_a(X)$.

This follows from 3.4, which says that $H^i\left(\tilde{X}\right) \approx H^i(X)$ and the fact that p_a is calculated from the Euler characteristic.

5.3.2 V.3.2 x g

3.2. Let C and D be curves on a surface X, meeting at a point P. Let $\pi:\tilde{X} \to X$ be the monoidal transformation with center P. Show that $\tilde{C}.\tilde{D} = C.D - \mu_P(C)\cdot\mu_P(D)$. Conclude that $C.D = \sum\mu_P(C)\cdot\mu_P(D)$, where the sum is taken over all intersection points of C and D, including infinitely near intersection points.

We have the following rules of intersection theory, via 3.2:
$\pi^*D.\pi^*C = C.D$
$E.\pi^*C = 0$
and $E^2 = -1$.
Usnig thm V.3.6, $\pi^*C = \tilde{C} + rE$ so $\tilde{C} = \pi^*C - rE$ and $\tilde{D} = \pi^*D - r'E$.
Then $\tilde{C}.\tilde{D} = (\pi^*C - rE)\,.\,(\pi^*D - r'E) =$
$\pi^*C.\pi^*D - r'E.\pi^*C - (rE)\,.\,(\pi^*D) + rr'E.E =$
$C.D - rr'$. So that's the first part. The second part follows trivially.

5.3.3 V.3.3 x g

3.3. Let $\pi:\tilde{X} \to X$ be a monoidal transformation, and let D be a very ample divisor on X. Show that $2\pi^*D - E$ is ample on $\tilde{X}$. [*Hint*: Use a suitable generalization of (I, Ex. 7.5) to curves in $\mathbf{P}^n$.]

Ok well I.7.5, says that an irreducible curve Y of degree $d > 1$ in $\mathbb{P}^2$ cannot have a point of multiplicity $\geq d$. This generalizes simply to $\mathbb{P}^n$.
Now since D is very ample, then for any other curve C on X, then $D.C$ is the degree of C. (via V.1.2)
Then by V.3.2, anything in $Pic\tilde{X}$ may be written as $\pi^*C - rE$.
Now we compute
$(2\pi^*D - E)\,(\pi^*C - rE) = 2\pi^*D.C - 2\pi^*D.E - E.\pi^*C + rE^2$.
The middle terms drop out by V.3.2, and it becomes
$2deg\,(C) - r$ since $E^2 = -1$ by V.3.1.
But then we know that $deg\,(C) > r$ and so we have
$(2\pi^*D - E)\,(\pi^*C - rE) > 0$.

Further, note that
$(2\pi^*D - E)\,.\,(2\pi^*D - E) = 4\pi^*D^2 + E^2 > 0$ since $D^2 = 1, 2, ...$

5.3.4 V.3.4 x Multiplicity of local ring

3.4. *Multiplicity of a Local Ring.* (See Nagata [7, Ch III, §23] or Zariski–Samuel [1, vol 2, Ch VIII, §10].) Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$. For any $l > 0$, let $\psi(l) = \text{length}(A/\mathfrak{m}^l)$. We call ψ the *Hilbert–Samuel function* of A.
(a) Show that there is a polynomial $P_A(z) \in \mathbf{Q}[z]$ such that $P_A(l) = \psi(l)$ for all $l \gg 0$. This is the *Hilbert–Samuel polynomial* of A. [*Hint*: Consider the graded ring $\text{gr}_{\mathfrak{m}} A = \bigoplus_{d\geq 0} \mathfrak{m}^d/\mathfrak{m}^{d+1}$, and apply (I, 7.5).]

Note that $gr_m A = \oplus_{d\geq 0}\mathfrak{m}^d/\mathfrak{m}^{d+1}$ is a graded ring.
Now $\psi\,(l) = length\,\left(A/\mathfrak{m}^l\right)$ in the problem, and in I.7, we have
$\varphi_M\,(l) = dim_k M_l$ where M_l is the l^{th} graded part.
So I need to show that $M_l = A/\mathfrak{m}^l$ so note $M = \oplus_{d\in\mathbb{Z}} M_d$
so then in our case, $M_l = \mathfrak{m}^d/\mathfrak{m}^{d+1}$ so I wts that $length\,\left(A/\mathfrak{m}^l\right) = dim\,\left(\mathfrak{m}^{l-1}/\mathfrak{m}^l\right)$.

so ok we know that a sop $x_1, ..., x_n$ for $\mathfrak{m} = \sum Ax_i$ exists [3], so zariski samule, vol 2, vhap VIII, $10 gives us that

$gr_m A = \oplus_{d \geq 0} \mathfrak{m}^d/\mathfrak{m}^{d+1} \approx \frac{A}{\mathfrak{m}}[X_1, ..., X_n]$ in n variables. (note $A/\mathfrak{m}$ is a field)[4]

Thus in our case $\mathfrak{m}^{l-1}/\mathfrak{m}^l$ becomes $dim\ \left(\frac{A}{\mathfrak{m}}[X_1, ..., X_n]\right)/\mathfrak{m}^l$.

Since this is a vector space, length and dimension coincide.

Thus must show that $length\ \left(\frac{A}{\mathfrak{m}}[X_1, ..., X_n]\right)/\mathfrak{m}^l = length\left(A/\mathfrak{m}^l\right)$

But from the isomorphism, we have that $A \approx \frac{A}{\mathfrak{m}}[X_1, ..., X_n]$ and so

$length\left(\frac{A}{\mathfrak{m}}[X_1, ..., X_n]/\mathfrak{m}^l\right) = length\left(A/\mathfrak{m}^l\right)$.

Now applying thm I.7.5, [5] we have that there is $P_A(z) \in \mathbb{Q}[z]$ such that $P_A(l) = \varphi_M(l) = \psi(l)$ for $l \gg 0$.

5.3.5 b. x

(b) Show that deg P_A = dim A.

---

[3] **6.1.1 Definition**

Let R be a Noetherian local ring with maximal ideal $\mathcal{M}$, and let M be a finitely generated R-module of dimension n. A *system of parameters* for M is a set $\{a_1, \ldots, a_n\}$ of elements of $\mathcal{M}$ such that $M/(a_1, \ldots, a_n)M$ has finite length. The finiteness of the Chevalley dimension (see (5.3.2) and (5.3.3) guarantees the existence of such a system.

http://www.math.uiuc.edu/~r-ash/ComAlg/ComAlg6.pdf

[4] THEOREM 23. *Let A be a local ring, $\{x_1, \cdots, x_d\}$ a system of parameters of A, $\mathfrak{q}$ the ideal $\sum_{i=1}^{d} Ax_i$. Then $e(\mathfrak{q}) \leq \ell(A/\mathfrak{q})$. If $e(\mathfrak{q}) = \ell(A/\mathfrak{q})$, then the associated graded ring $G_\mathfrak{q}(A) = \sum_{n=0}^{\infty} \mathfrak{q}^n/\mathfrak{q}^{n+1}$ is isomorphic to the polynomial ring $B = (A/\mathfrak{q})[X_1, \cdots, X_d]$; and conversely.*

zariski samule, vol 2, vhap VIII, $10

[5] **Definition.** If $\mathfrak{p}$ is a minimal prime of a graded S-module M, we define the *multiplicity* of M at $\mathfrak{p}$, denoted $\mu_\mathfrak{p}(M)$, to be the length of $M_\mathfrak{p}$ over $S_\mathfrak{p}$.

Now we can define the Hilbert polynomial of a graded module M over the polynomial ring $S = k[x_0, \ldots, x_n]$. First, we define the *Hilbert function* φ_M of M, given by

$$\varphi_M(l) = \dim_k M_l$$

for each $l \in \mathbf{Z}$.

Theorem 7.5. (Hilbert–Serre). *Let M be a finitely generated graded $S = k[x_0, \ldots, x_n]$-module. Then there is a unique polynomial $P_M(z) \in \mathbf{Q}[z]$ such that $\varphi_M(l) = P_M(l)$ for all $l \gg 0$. Furthermore,* $\deg P_M(z) = \dim Z(\operatorname{Ann} M)$, *where Z denotes the zero set in $\mathbf{P}^n$ of a homogeneous ideal* (cf. §2).

This is given in Zariski samuel vol 2, ch VIII, \$10. [6]

alternatively, note by I.7.5, and part (a), we must have

$degP_A = dim\ Z\left(Ann\ \frac{A}{m}\left[X_1, ..., X_n\right]\right)$ where Z denotes the zero set in $\mathbb{P}^n$ of a homogeneous ideal.

Following chapter 9 Zariski-Samuel, we have that dimension of A is the length of the system of parameters, so it's n.

Since we are trying to annihilate a vector space, then the annihilator is just 0.

So $Z(0) = \mathbb{P}^n$ which will just have dimension n.

5.3.6 c. x

(c) Let $n = \dim A$. Then we define the *multiplicity* of A, denoted $\mu(A)$, to be $(n!)\cdot$ (leading coefficient of P_A). If P is a point on a noetherian scheme X, we define the *multiplicity* of P on X, $\mu_P(X)$, to be $\mu(\mathcal{O}_{P,X})$.

(he means $n! \times$ the leading coefficient of P_A).

he is not actually asking any question here...

5.3.7 d. x

(d) Show that for a point P on a curve C on a surface X, this definition of $\mu_P(C)$ coincides with the one in the text just before (3.5.2).

c.f. V.3.6

Assume that C has multiplicity r at P.

Let $\mathfrak{m}$ be sheaf of ideals of P on X.

x, y generate $\mathfrak{m}$ in some neighborhood U of P, we may assume affine, say $U = Spec\ A$.

Let $f(x, y)$ a local equation for C on U (shrinking U if necesarry).

By definition of multiplicity in chapter, $f \in \mathfrak{m}^r$, $f \notin \mathfrak{m}^{r+1}$.

Now note the local ring is being given by [7] $\mathcal{O}_{C,p} = \left\{\frac{h}{g} \in \frac{k[x,y]}{(f)} : g(p) \neq 0\right\}$.

Thus $length\left(\mathcal{O}_{C,p}/\mathfrak{m}\right) = length\left(\mathcal{O}_{C,p}/\left(x, y\right)\right)$ is small,

$length\left(\mathcal{O}_{C,p}/\mathfrak{m}^2\right) = length\left(\mathcal{O}_{C,p}/\left(x, y\right)^2\right)$ is a little bigger

...

$length\left(\mathcal{O}_{C,p}/\mathfrak{m}^r\right)$ is a little bigger but now note that since $f \notin \mathfrak{m}^{r+1}$, then further quotienting doesn't change the length.

In particular, $length\left(\mathcal{O}_{C,p}/\mathfrak{m}^r\right) = lim_{r\to\infty}\left(\mathcal{O}_{C,p}/\mathfrak{m}^r\right) = lim_{r\to\infty}P_{\mathcal{O}_{C,p}}(r)$

(Note that

Now we know the Hilbert-Samuel function looks like

$P_{\mathcal{O}_{C,p}}(l) = a_n l^n + a_{n-1} l^{n-1} + ... + a_0$.

:

§ **10. Theory of multiplicities.** Let A be a semi-local ring of dimension d, and $\mathfrak{q}$ an open ideal of A, admitting the intersection $\mathfrak{m}$ of the maximal ideals $\mathfrak{p}_j$ of A as radical. Then the characteristic polynomial $\bar{P}_{\mathfrak{q}}(n)$ is of degree d, by the definition of the dimension of A (§ 9). Its leading term has the form

$$e(\mathfrak{q})n^d/d!,$$

where $e(\mathfrak{q})$ is an integer (cf. VII, § 12). The integer $e(\mathfrak{q})$ is called the *multiplicity of the ideal* $\mathfrak{q}$. The integer $e(\mathfrak{m})$ is called the *multiplicity*

[6] *of the semi-local ring A.*

[7] (think gathmanns local ring def)

So the first coefficient is $a_n = lim_{l\to\infty}\frac{a_nl^n+...+a_0}{l^n} = lim_{l\to\infty}\frac{length\left(\mathcal{O}_{C,p}/\mathfrak{m}^l\right)}{l^n}$ where n is the dimension.

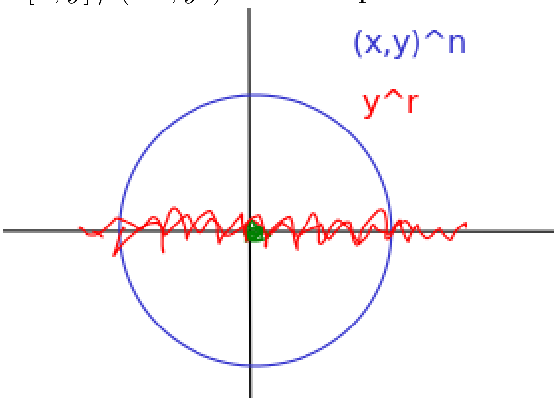
So for a curve ($n=1$), I want to show that $length\left(\mathcal{O}_{\mathcal{C},p}/\mathfrak{m}^l\right) \to l \times \frac{r}{n!} + constant.$

So we'll divide out the l, and then multiply through the $n!$ (for the dimension – it's $1! = 1$ in this case), and we just are left with showing that

$length\left(\mathcal{O}_{C,p}/\mathfrak{m}^l\right) \to l \times r + constant.$

To compute the length this is how I will argue:

$k\left[x,y\right]/\left(\mathfrak{m}^n, y^r\right)$ will be equidimensional. Here is the picture:



So we are localizing at the green point, and we are still in the fuzzy area with blue and red. Thus localizing won't change the length (This is called the equidimensional case).

Since localizing commutes with quotients, we have that length $length\left(\left[\frac{k[x,y]}{(f^r,(x,y)^n)}\right]_{\mathfrak{m}}\right) = length\left(\frac{k[x,y]_{\mathfrak{m}}}{(f^r,(x,y)^n)}\right)$.

Thus I just have to compute the length of $\frac{k[x,y]}{(f_r,(x,y)^n)}$.

we'll here's an example:

$M = k\left[x,y\right]/\left(y^7 - x^9\right)$ with $m = (x,y)$ has $dim\left(M\right) = 1$

$deg\left(M/m\right) = 1$,

$deg\left(M/m^2\right) = 3$, ...

up to it increases by 1 each time.

$deg\left(M/m^7\right) = 28$

Now it stabilizes, adding 7 each time

$deg\left(M/m^8\right) = 35$,

$deg\left(M/m^9\right) = 42$

Now note that $35 + 21 = 56$ which $/8$ is an integer

$42 + 21 = 63$ which $/9$ is an integer...

so $length\left(M/m^n\right) = a_1 \times n + a_0$

so assuming $a_0 = -21$, then $a_1 \times n$ must be $7 \times n$.

which makes sense, since it's the degree we want...

So I just have to show that the length stabilizes correctly

i.e. $length\left(M/\left[\left(y^r + ...\right) + (x,y)^n\right]\right) = r \times n + ...$

Ok, here's the idea: Note that $\mu_p\left(C\right)$ as defined by the problem (as the $n!$ times the coefficient of the leading term of samuel, hilbert thing can also be written as

$n!lim_{k\to\infty}\frac{length\left(M/q^kM\right)}{k^d}$.

To see this, let $c_nk^n + ... + c_0$ be the Hilbert Poly of M.

let's maybe try and figure out a_0 ?

$r = 1$, -1

$r = 2$, 0,

$r = 3$, 3

for $r = 4$ it's -6,

$r = 5$ it's -10

for $r = 6$ it's -15

for $r=7$ it's -21

so it goes up 1 each time... Note that arithmetic sum is $S_n = \frac{n(a_n - a_0)}{2}$ so the first term is -1, the last term we add is like $-(r-1)$ in total, In our case, we take $\frac{r(r-1)}{2}$ which coicidentally is $\begin{pmatrix} n \\ 2 \end{pmatrix}$.

so maybe we have $md + (n+1)$

So basically I need to prove that $length\left(\frac{M}{(y^r),(x,y)^n}\right) = rn - \begin{pmatrix} r \\ 2 \end{pmatrix}$ for $n > r$.

$M/\left(y^r, (x,y)^n\right)$

ideal on right is gen by $x^n, x^{n-1}y, ..., x^{n-(r-1)}y^{r-1}$ plus redundant terms.

Thus a basis: 1 and $y, y^2, y^3, ..., y^{r-1}$, and $x, x^2, ..., x^{n-1}$ and $yx, yx^2, .., yx^{n-2}$ and ... up to $y^{r-1}x, y^{r-1}x^2, ..., y^{r-1}x^{r}$

In total it's $1 + (r-1) + (n-1) + (n-2) + (n-3) + ... + (n-r)$.

This is $r + (n-1) + (n-2) + ... + (n-r)$

This is $(n-0) + (n-1) + (n-2) + ... + (n-(r-1))$

Thus it's $\sum_{j=0}^{r-1} n - \sum_{j=1}^{r-1} j$

This is $n \times r - \frac{(r-1)(r-1+1)}{2} = nr - \begin{pmatrix} r \\ 2 \end{pmatrix}$.

Since nr is the linear term, we are done.

5.3.8 e. x

(e) If Y is a variety of degree d in $\mathbf{P}^n$, show that the vertex of the cone over Y is a point of multiplicity d.

See Mumford, Algebraic Geometry 1, Proposition 5.11

5.3.9 V.3.5 x g hyperelliptic every genus

3.5. Let $a_1, \ldots, a_r$, $r \geqslant 5$, be distinct elements of k, and let C be the curve in $\mathbf{P}^2$ given by the (affine) equation $y^2 = \prod_{i=1}^{r}(x - a_i)$. Show that the point P at infinity on the y-axis is a singular point. Compute δ_P and $g(\tilde{Y})$, where $\tilde{Y}$ is the normalization of Y. Show in this way that one obtains hyperelliptic curves of every genus $g \geqslant 2$.

Singular at infinity on y -axis

So this is referring to the point $\infty = (0:1:0)$ on projective space $\mathbb{P}(x,y,z) = \mathbb{P}^2$.

(Since it's on y-axis, then $x = 0$ and infinity means $z = 0$)

The projective closure of C is $z^{r-2}y^2 = x^r + x^{r-1}z\sum a_i + ... + z^r \prod a_i$

The jacobian at ∞ is $\left(\begin{array}{ccc} rx^{r-1} + ... + z^{r-1} & 2yz^{r-2} & (r-2)\, z^{r-3}y^2 + \sum(x,y) \end{array} \right)|_\infty$

which is 0. Thus it's a singular point.

Compute δ_P

Now at $\infty = (0:1:0)$ we subs in 1 for y and the equation becomes:

$z^{r-2} = x^r + x^{r-1}z\sum a_i + ... + z^r\prod a_i$.

Then $r-2$ is the multiplicity since it's lowest order term (i.e. $f \in \mathfrak{m}^{r-2}$, not in $\mathfrak{m}^{r-1}$ where $\mathfrak{m}$ is the maximal ideal of the point ∞.

every genus ... + genus.

Now note that by projecting to the x-coordinate, then the equation gives a cover of $\mathbb{P}^1$ of degree 2.

By Riemann-Hurwitz, formula (IV.2) such a cover will have $2g+2$ branch points. The branch points are places where there is one value of x for the value of y, namely the roots x_i.

So choosing $n = 2g+2$, we obtain hyperelliptic curves of any genus we desire.

5.3.10 V.3.6 x

3.6. Show that analytically isomorphic curve singularities (I, 5.6.1) are equivalent in the sense of (3.9.4), but not conversely.

See Wall's, Singular Points of Plane Curves where he proves that Analytically isomorphic $\Longrightarrow$ same puiseux characteristic $\Longrightarrow$ equivalence in the sense of 3.9.4. Furthermore he shows that two curves are equisingular iff they have the same puiseux characteristic. However the example is then given of the two curves $C_1 : y^3 + x^7 = 0$ and $C_2 : y^3 + x^5 y + x^7 = 0$ which are equisingular but not analytically isomorphic.

5.3.11 V.3.7 x

3.7. For each of the following singularities at (0,0) in the plane, give an embedded resolution, compute δ_P, and decide which ones are equivalent.

(a) $x^3 + y^5 = 0$.

~~(b) $x^3 + x^4 + y^5 = 0$.~~

(c) $x^3 + y^4 + y^5 = 0$.

(d) $x^3 + y^5 + y^6 = 0$.

(e) $x^3 + xy^3 + y^5 = 0$.

The following code will compute an embedded resolution of singularities in Singular, find strict transforms, and exceptionals

```
LIB"resolve.lib";
LIB"reszeta.lib";
LIB"resgraph.lib";
//that loaded some libraries

ring R=0, (x,y), dp; //define the ring Q[x,y]
ideal I=x^2-y^3; //define a cusp for example
list L=resolve(I, 1);
list coll=collectDiv(L);
```

For example, let's do a resolution of $x^3 + x^5 = 0$.

```
+++++++++++++ Overview of Current Chart +++++++++++++++++++++++
Current number of final charts: 0
Total number of charts currently in chart-tree: 1
Index of the current chart in chart-tree: 1
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=y5+x3

==== Exceptional Divisors:
```

```
empty list

==== Images of variables of original ring:
_[1]=x
_[2]=y

-------------------------------- Upcoming Center --------------------------
_[1]=y
_[2]=x
---------------------------------------------------------------------------
++++++++++++++ Overview of Current Chart ++++++++++++++++++++++
Current number of final charts: 0
Total number of charts currently in chart-tree: 2
Index of the current chart in chart-tree: 2
+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=y(1)^3+x(2)^2

==== Exceptional Divisors:
[1]:
   _[1]=x(2)

==== Images of variables of original ring:
_[1]=x(2)*y(1)
_[2]=x(2)

-------------------------------- Upcoming Center --------------------------
_[1]=y(1)
_[2]=x(2)
---------------------------------------------------------------------------
++++++++++++++ Overview of Current Chart ++++++++++++++++++++++
Current number of final charts: 0
Total number of charts currently in chart-tree: 3
Index of the current chart in chart-tree: 3
+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=y(1)^2+x(2)

==== Exceptional Divisors:
[1]:
   _[1]=y(1)
[2]:
   _[1]=x(2)
```

```
==== Images of variables of original ring:
_[1]=x(2)^2*y(1)
_[2]=x(2)*y(1)

------------------------------ Upcoming Center ------------------------------
_[1]=y(1)
_[2]=x(2)
-----------------------------------------------------------------------------
++++++++++++++ Overview of Current Chart ++++++++++++++++++++++
Current number of final charts: 0
Total number of charts currently in chart-tree: 5
Index of the current chart in chart-tree: 4
+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=x(2)+y(1)

==== Exceptional Divisors:
[1]:
   _[1]=1
[2]:
   _[1]=y(1)
[3]:
   _[1]=x(2)

==== Images of variables of original ring:
_[1]=x(2)^3*y(1)^2
_[2]=x(2)^2*y(1)

------------------------------ Upcoming Center ------------------------------
_[1]=y(1)
_[2]=x(2)
-----------------------------------------------------------------------------
++++++++++++++ Overview of Current Chart ++++++++++++++++++++++
Current number of final charts: 0
Total number of charts currently in chart-tree: 7
Index of the current chart in chart-tree: 5
+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=x(1)*y(0)^2+1

==== Exceptional Divisors:
[1]:
```

```
   _[1]=y(0)
[2]:
   _[1]=1
[3]:
   _[1]=x(1)

==== Images of variables of original ring:
_[1]=x(1)^3*y(0)
_[2]=x(1)^2*y(0)

------------------------------------ Upcoming Center ------------------------------------
_[1]=x(1)*y(0)^2+1
-------------------------------------------------------------------------------------------
++++++++++++++ Overview of Current Chart ++++++++++++++++++++++
Current number of final charts: 0
Total number of charts currently in chart-tree: 7
Index of the current chart in chart-tree: 6
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0

==== Ideal of Variety:
_[1]=y(1)+1

==== Exceptional Divisors:
[1]:
   _[1]=1
[2]:
   _[1]=1
[3]:
   _[1]=y(1)
[4]:
   _[1]=x(2)

==== Images of variables of original ring:
_[1]=x(2)^5*y(1)^3
_[2]=x(2)^3*y(1)^2

------------------------------------ Upcoming Center ------------------------------------
_[1]=y(1)+1
-------------------------------------------------------------------------------------------
++++++++++++++ Overview of Current Chart ++++++++++++++++++++++
Current number of final charts: 1
Total number of charts currently in chart-tree: 7
Index of the current chart in chart-tree: 7
++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++

==== Ambient Space:
_[1]=0
```

```
==== Ideal of Variety:
_[1]=y(0)+1

==== Exceptional Divisors:
[1]:
   _[1]=1
[2]:
   _[1]=y(0)
[3]:
   _[1]=1
[4]:
   _[1]=x(1)

==== Images of variables of original ring:
_[1]=x(1)^5*y(0)^2
_[2]=x(1)^3*y(0)

------------------------------ Upcoming Center -------------------------
_[1]=y(0)+1
------------------------------------------------------------------------
============== result will be tested ===========

the number of charts obtained: 2
==============          result is o.k.      ===========
```

Now let's compare the charts for the exceptionals from (a)

```
0,0,0,0,
1,0,0,0,
1,2,0,0,
0,2,3,0,
1,0,3,0,
0,0,3,4,
0,2,0,4
```

(b)

```
0,0,0,0,
1,0,0,0,
1,2,0,0,
0,2,3,0,
1,0,3,0,
0,0,3,4,
0,2,0,4
```

(c)

```
0,0,0,0,
1,0,0,0,
1,2,0,0,
0,2,3,0,
1,0,3,0,
0,0,3,4,
1,0,0,4
```

(d)

```
0,0,0,0,
1,0,0,0,
1,2,0,0,
0,2,3,0,
1,0,3,0,
0,0,3,4,
0,2,0,4
```

(e)

```
0,0,0,
1,0,0,
1,2,0,
0,2,3,
1,0,3
```

Each row gives one chart, and each column tells which exceptional is appearing there. So for instance in (e) above, the 1 in the second row first column means E_1 is in the second chart. To see the multiplicity, you have to look at

So this would seem to say that (a), (b), (d) are equivalent and (c), (e) are not equivalent to any others. However, we should check the multiplicities of the exceptionals (a), (b), (d) to make sure they are actually equivalent resolutions. To do this we can use the additional singular commands:

```
> poly f=x^3+y^5;
> displayMultsequence(f);

> poly f=x^3+x^4+y^5;
> displayMultsequence(f);

> poly f=x^4+y^5+y^6;
> displayMultsequence(f);
```

This last set of commands should show that (d) is not equivalent.

5.3.12 V.3.8a,b x

3.8. Show that the following two singularities have the same multiplicity, and the same configuration of infinitely near singular points with the same multiplicities, hence the same δ_P, but are not equivalent.

(a) $x^4 - xy^4 = 0$.

(b) $x^4 - x^2y^3 - x^2y^5 + y^8 = 0$.

Clearly these two singularities have the same multiplicity by looking at the lowest term (it's 4).

The configuration of infinitely near singular points can be seen as in 3.7. The multiplicities are given as in the end of 3.7.

To see they are not equivalent,

Note to compute the multiplicity sequence, we use the list L=resolve(I, 1) command as in 3.7 which gives 3 charts for (a) yet (4) for (b).

5.4 V.4 Cubic Surface

5.4.1 V.4.1 x g P2 blown at 2 points

> **4.1.** The linear system of conics in $\mathbf{P}^2$ with two assigned base points P_1 and P_2 (4.1) determines a morphism ψ of X' (which is $\mathbf{P}^2$ with P_1 and P_2 blown up) to a nonsingular quadric surface Y in $\mathbf{P}^3$, and furthermore X' via ψ is isomorphic to Y with one point blown up.

Using the notation of the chapter, since $r \leq 4$, $dim\ \mathfrak{d} = 3$ by thm V.4.2.a.

$\mathfrak{d}'$ has no base points by thm V.4.1 and by thm II.7.8.1, it determines a morphism ψ of X' to $\mathbb{P}^3$.

Let the two blown up points be $P_1 = (0,0,1)$ and $P_2 = (0,1,0)$.

The vector space $V \subset H^0\left(\mathcal{O}_{\mathbb{P}^2}(2)\right)$ corresponding to $\mathfrak{d}$ is spanned by $x_0^2, x_1x_2, x_0x_2, x_0x_1$ which is the space of all conics passing through $(0,0,1)$ and $(0,1,0)$.

You can check this by looking at the defining equations of conics through some points

$$\left|\begin{pmatrix} x^2 & xy & y^2 & xz & yz & z^2 \\ p^2 & pq & q^2 & qr & .. & .. \\ .. & & & & & \\ .. & & & & & \\ .. & & & & & \\ .. & & & & & \end{pmatrix}\right| = 0 \text{ through points } (p,q,r) \text{ in the variables } x,y,z.$$

Now we define $\psi : X' \to \mathbb{P}^3$ by where it sends the basis elements.

Namely, $x_0^2 \mapsto y_0$, and $x_0x_1 \mapsto y_1$, and $x_0x_2 \mapsto y_2$, and $x_1x_2 \mapsto y_3$.

Note that for any point in X', the image satisfies $y_0y_3 = y_1y_2$ which is the equation defining the quadric surface Q $(xw = yz$).

Thus the image of ψ is contained in Y.

Now let $\pi : Q \to \mathbb{P}^2$ be projection from the point $p = (0,0,0,1) \in Q$ to the plane $y_3 = 0$, i.e. $\pi_p : (x,y,z,t) \mapsto (x,y,z)$.

Note that we have $\pi \circ \psi = Id_{\mathbb{P}^2}$ and $\psi \circ \pi = id_Q$.

Let $\Gamma \subset Q \times \mathbb{P}^2$ be the graph of π.

Let $p = (0,0,0,1) \in Q$ and $q = (0,0,1)$, $r = (0,1,0) \in \mathbb{P}^2$.

We have the following definitions for the blow up of a point and subvariety from Harris:

- Blowing up $\mathbb{P}^n$ at a point.
  - Let $\varphi : \mathbb{P}^n \to \mathbb{P}^{n-1}$ the rational map given by projection from a point $p \in \mathbb{P}^n$ and Γ_φ the graph. The map $\pi : \Gamma_\varphi \to \mathbb{P}^n$ is the blow up of $\mathbb{P}^n$ at p. The map π projects Γ_φ isomorphically to $\mathbb{P}^n$ away from p, while over p the fiber is isomorphic to $\mathbb{P}^{n-1}$.
- For the blowing up of Q at p
  - Let $X \subset \mathbb{P}^n$ a quasi-projective variety and $p \in X$ any point. Let $\tilde{X} = \Gamma_\varphi \subset X \times \mathbb{P}^{n-1}$ the graph of the projection map of X to $\mathbb{P}^{n-1}$ from p. The map $\pi : \tilde{X} \to X$ is then called the blow-up of X at p.
- For $\mathbb{P}^2$ with two points blown up:
  - If $X \subset \mathbb{P}^m$ is a projective variety and $Y \subset X$ is a subvariety, we define the blow-up of X along Y by taking a collection $F_0, ..., F_n$ of homogeneous polynomials of the same degree generating an ideal with saturation $I(Y)$ and letting $Bl_Y(X)$ the graph of the rational map $\varphi : X \to \mathbb{P}^n$ given by $[F_0, ..., F_n]$.

Then $\Gamma = X'$ is by definition the blow up of Q at p. Note that $\langle x_0^2, x_0x_1, x_0x_2, x_1x_2\rangle$ generate an ideal with saturation the homogeneous ideal of two points $I(\{q,r\})$. Thus the graph of the rational map $\psi : \mathbb{P}^2 \to \mathbb{P}^3$ given by $[x_0^2, x_0x_1, x_0x_2, x_1x_2]$ above is the blowing up of $\mathbb{P}^2$ at the two points q, r. This is again just Γ.

5.4.2 V.4.2 x g

4.2. Let φ be the quadratic transformation of (4.2.3), centered at P_1, P_2, P_3. If C is an irreducible curve of degree d in $\mathbf{P}^2$, with points of multiplicity r_1, r_2, r_3 at P_1, P_2, P_3, then the strict transform C' of C by φ has degree $d' = 2d - r_1 - r_2 - r_3$,

and has points of multiplicity $d - r_2 - r_3$ at Q_1, $d - r_1 - r_3$ at Q_2 and $d - r_1 - r_2$ at Q_3. The curve C may have arbitrary singularities. [*Hint*: Use (Ex. 3.2).]

Suppose C is defined by $f(x,y,z) = 0$ which is irreducible thus no x, y, z factors out. WLOG assume $P_1 = (0,0,1), P_2 = (0,1,0), P_3 = (1,0,0)$. The strict transform under the quadratic transformation of chapter 1 $(x,y,z) \to (yz, xz, xy)$, from the three points P_1, P_2, P_3 is then given by $g(x,y,z) := \frac{f(yz,xz,xy)}{x^{r_1}y^{r_2}z^{r_3}}$. Suppose P_1 has multiplicity r_1 on C. So the terms of lowest degree in x, y have degree r_1 in x, y. Then $f(x,y,z) = f_{r_1}(x,y) z^{d-r_1} + \cdots + f_{d-r_1}(x,y)$, where $f_i(x,y)$ are homogeneous of degree i. The transform satisfies $f(yz,xz,xy) = f_{r_1}(yz,xz)(xy)^{d-r_1} + \cdots + f_{d-r_1}(yz,xz)$ since z^{r_1} gets cancelled in g. Now g is just $f(yz,xz,xy)$ dividing out factors of x, y, z and doing this twice gets us back to f so since this agrees with the quadratic transform given in chapter 1, we see this is the strict transform, which clearly has degree $2d - r_1 - r_2 - r_3$. Now recall that multiplicity is the lowest degree term so looking at $g(x,y,z) = \sum_{i=0}^{d-r_i} f_{r_i+i}(y,x) x^{d-r_1-r_3} y^{d-r_1-r_2} z^{d-r_2-r_3}$. Looking at $(0,0,1)$, and since this is homogeneous, then $f_d(y,x) x^{-r_2} y^{-r_3}$ is the lowest degree term there of degree $d - r_2 - r_3$, similarly for the other points.

5.4.3 V.4.3 x

4.3. Let C be an irreducible curve in $\mathbf{P}^2$. Then there exists a finite sequence of quadratic transformations, centered at suitable triples of points, so that the strict transform C' of C has only *ordinary* singularities, i.e., multiple points with all distinct tangent directions (I, Ex. 5.14). Use (3.8).

See Algebraic Curves over Finite Fields by Hirschfel, Theorem 3.27

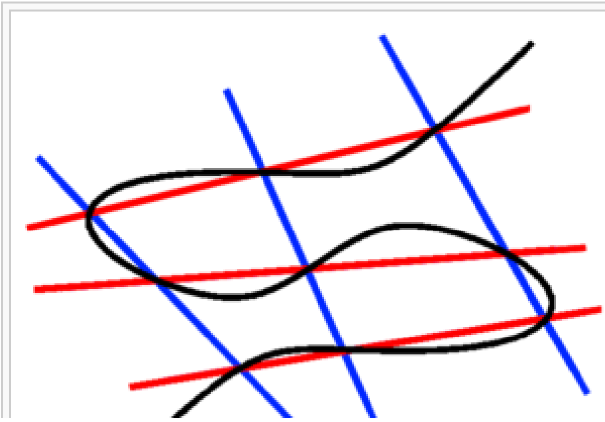
5.4.4 V.4.4 x g important?

4.4. (a) Use (4.5) to prove the following lemma on cubics: If C is an irreducible plane cubic curve, if L is a line meeting C in points P, Q, R, and L' is a line meeting C in points P', Q', R', let P'' be the third intersection of the line PP' with C, and define Q'', R'' similarly. Then P'', Q'', R'' are collinear.

Well, note that 9 points determine a cubic.

Also a reducible plane cubic is either 3 lines or 1 line and a conic.

Let the blue lines be the cubic given by the three lines PP' , QQ' and RR' in the following picture I stole from Wikipedia:



Now consider the cubic determined by the two lines $L\,(P \to Q \to R)$ and $L'\,(P' \to Q' \to R')$ and the third line $L''\,(P'' \to Q'')$.

Now use this Cayley-Bacharach theorem: If two cubics C_1 and C_2 meet in nine points, then every cubic that passes through eight of the nine also passes through the ninth.

This pretty much does it.

5.4.5 b. x

> (b) Let P_0 be an inflection point of C, and *define* the group operation on the set of regular points of C by the geometric recipe "let the line PQ meet C at R, and let P_0R meet C at T, then $P + Q = T$" as in (II, 6.10.2) and (II, 6.11.4). Use (a) to show that this operation is associative.

See Tate, Silverman: Rational Points. It's in the first chapter. They even give nice pictures

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5.4.6 V.4.5 x g Pascal's Theorem

> **4.5.** Prove Pascal's theorem: if A,B,C,A',B',C' are any six points on a conic, then the points $P = AB'.A'B$, $Q = AC'.A'C$, and $R = BC'.B'C$ are collinear (Fig. 22).

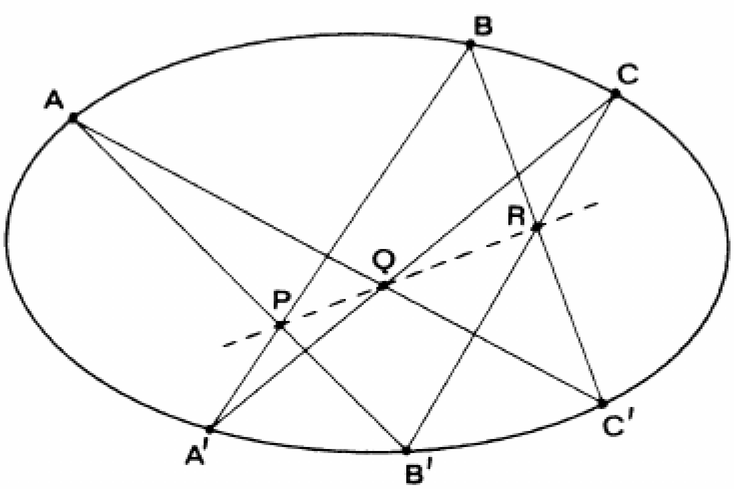

Figure 22. Pascal's theorem.

Consider the cubics $X = AB' + BC' + CA'$ and $Y = AC' + BA' + CB'$.

Then $X \cap Y$ meets on the outside along the conic in 6 points, and in the inside at P, Q, R. Thus in 9 points.

Now consider $E = conic + PQ$. which is a cubic. This meets the cubics at 8 of the 9 points, so by Cayley Bacharach, as in 4.4.a,

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5.4.7 V.4.6 x g

4.6. Generalize (4.5) as follows: given 13 points $P_1, \ldots, P_{13}$ in the plane, there are three additional determined points P_{14}, P_{15}, P_{16}, such that all quartic curves through $P_1, \ldots, P_{13}$ also pass through P_{14}, P_{15}, P_{16}. What hypotheses are necessary on $P_1, \ldots, P_{13}$ for this to be true?

In exc V.4.15, I prove that there is exactly one curve of degree 4 through $4 \cdot (4+3)/2 = 28/2$ points in general position, where this hypothesis means that the linear system of curves through the points has the smallest possible for any choice of that number of points. Thus 13 points determine a 1-dimensional pencil $C_1 + tC_2 = 0$ of quartics.

For another quartic C_3 through those points but not in the pencil then $C_1 + tC_2 + sC_3 = 0$ is a two dimensional family meeting the points. Thus if we fixed some additional two arbitrary points P, Q, then for some t, s there is a quartic through points 1 through 13 and P, Q. But 14 points determine a quartic so this is a contradiction.

Thus the pencil $C_1 + tC_2 = 0$ determines all quartics through the $P_1, ..., P_{13}$. By Bezout, $C_1 \cap C_2$ has 3 additional points through which clearly all other quartics in the pencil must pass.

5.4.8 V.4.7 x

4.7. If D is any divisor of degree d on the cubic surface (4.7.3), show that

$$p_a(D) \leqslant \begin{cases} \frac{1}{6}(d-1)(d-2) & \text{if } d \equiv 1,2 \pmod 3 \\ \frac{1}{6}(d-1)(d-2) + \frac{2}{3} & \text{if } d \equiv 0 \pmod 3). \end{cases}$$

Show furthermore that for every $d > 0$, this maximum is achieved by some irreducible nonsingular curve.

If it's not on a quadric, this is Harris / Eisenbud, Curves in Projective Spaces Theorem 3.13. Otherwise, proceed as in example IV.5.2.2.

Claim: for $d > 0$ and g with $\frac{1}{\sqrt{3}}d^{3/2} - d + 1 < g \leq \frac{1}{6}d(d-3) + 1$, there exists a smooth connected curve of degree d and genus g on the cubic surface.

Given this

$expand\left(\frac{1}{6} \cdot (d-1) \cdot (d-2)\right) = \frac{d^2}{6} - \frac{d}{2} + \frac{1}{3}$

$expand\left(\frac{1}{6} \cdot (d-1) \cdot (d-2) + \frac{2}{3}\right) = \frac{d^2}{6} - \frac{d}{2} + 1$

$expand\left(\frac{1}{6} \cdot d \cdot (d-3) + 1\right) = \frac{d^2}{6} - \frac{d}{2} + 1$

so we have achieved the maximum

Following Hartshorne, master of arithmetic and algebraic geometry. Consider, on the cubic, the irreducible nonsingular curve of degree $d = 3a - \sum b_i$ and genus $g = \frac{1}{2}(a^2 - \sum b_i^2 - d) + 1$ for $a \geq b_1 + b_2 + b_3$, $b_i \geq b_{i+1} \geq 0$.

Define for $(a, b_1, ..., b_6) \in Pic\ X$ $r = a - b_1$, and $\alpha_i = \frac{1}{2}r - b_i$ for $i = 2, ..., 6$ so that $r \in \mathbb{Z}$ and $\alpha_i \in \frac{1}{2}\mathbb{Z}$. Thus $a = \frac{1}{2}\left(d + \frac{3}{2}r - \sum \alpha_i\right)$, $b_1 = a - r$, $b_i = \frac{1}{2}r - \alpha_i$, $i = 2, ..., 6$. Note for $a, b_i \in \mathbb{Z}$ we need $\alpha_i \equiv \frac{1}{2}r \,(mod\ 1)$ and $d + \frac{3}{2}r - \sum \alpha_i \equiv 0 \,(mod\ 2)$. The inequalities at the end of the first paragraph become $|\alpha_2| \leq \alpha_3 \leq \cdots \leq \alpha_6 \leq \frac{1}{2}r$ and $-\alpha_2 + \alpha_3 + \cdots + \alpha_6 \leq d - \frac{3}{2}r$ and the genus becomes $g = \frac{1}{2}\left((r-1)\,d - \frac{3}{4}r^2 - \sum \alpha_i^2\right) + 1$.

Now define $F_d(r) = \frac{1}{2}\left((r-1)d - \frac{3}{4}r^2\right) + 1$, $g = F_d(r) - \frac{1}{2}\sum \alpha_i^2$. We want to find for arbitrary d, some $r \in \mathbb{Z}$ and $\alpha_i \in \frac{1}{2}\mathbb{Z}$ satisfying the required congruences and inequalities above for all $g = F_d(r) - \frac{1}{2}\sum \alpha_i^2$.

To compute the maximum value of $F_d(r)$, $\frac{-b}{2a}$ of the poly $expand\left(\frac{1}{2}\cdot\left((r-1)\cdot d - \frac{3}{4}\cdot r^2\right)+1\right) = -\frac{3r^2}{8} + \frac{dr}{2} - \frac{d}{2} + 1$ as a polynomial in r.

If $d \equiv 1 \; mod \; 3$, then $\left[\frac{2}{3}d\right] = \frac{2(d-1)}{3}$ so $F\left(\frac{2(d-1)}{2}\right) =$

$expand\left(subst\left(\frac{2}{3}\cdot(d-2), r, \frac{1}{2}\cdot\left((r-1)\cdot d - \frac{3}{4}\cdot r^2\right)+1\right)\right) = \frac{d^2}{6} - \frac{d}{2} + \frac{1}{3}$

So $F_d(r)$ has a max at $r = \frac{2}{3}d$, $F\left(\frac{2}{3}d\right) = \frac{1}{6}d(d-3)+1$. Thus for $d \equiv 0\,(mod\; 3)$, the maximum is attained by taking all $\alpha_i = 0$.

For other g, we use the sum of five squares theorem.

Claim: If $d > 0$ is an integer, $\frac{2}{\sqrt{3}}\sqrt{d} \le r \le \frac{2}{3}d$ and $F_d(r-1) < g \le F_d(r)$ then we can find $\alpha_i \in \frac{1}{2}\mathbb{Z}$, $i = 2, ..., 6$ satisfying the requirements so that $g = F_d(r) - \frac{1}{2}\sum \alpha_i^2$.

Proof: $F_d(r) - F_d(r-1) = \frac{1}{2}\left(d - \frac{3}{2}r + \frac{3}{4}\right)$ so that $F_d(r) - g < \frac{1}{2}\left(d - \frac{3}{2}r + \frac{3}{4}\right)$. Thus $F_d(r) - g < \frac{1}{2}\left(d - \frac{3}{2}r + \frac{3}{4}\right)$. We need $F_d(r) - g = \frac{1}{2}\sum \alpha_i^2$.

If r is even, then $F_d(r) \in \frac{1}{2}\mathbb{Z}$. Then $n = 2(F_d(r) - g)$ is an integer, and we need α_i with $|\alpha_i| \le \frac{1}{2}r$ such that $n = \sum \alpha_i^2$. By hypothesis, $n < d - \frac{3}{2}r + \frac{3}{4}$. On the other hand, since $\frac{2}{\sqrt{3}}\sqrt{d} - \frac{1}{3} \le r$, we have $n < \frac{3}{4}r^2 - r + \frac{5}{6}$. We need $\star$ integers $\alpha_i, i = 2, ..., 6$ with $|\alpha_i| \le \frac{1}{2}r$, $n = \sum \alpha_i^2$. Assuming we have $\star$, we must verify the congruences and inequalities from the first paragraph. Since g is an integer, $(r-1)d - \frac{3}{4}r^2 - \sum \alpha_i^2 \equiv 0\,(mod\; 2)$, so $d + \frac{3}{4}r - \sum \alpha_i \equiv 0\,(mod\; 2)$. After adjusting the order and signs of the α_i to satisfy the $|\alpha_2| \le \alpha_3 \le \cdots \le \alpha_6 \le \frac{1}{2}r$, since $\sum \alpha_i^2 = n \le d - \frac{3}{2}r + \frac{3}{4}$, then $-\alpha_2 + \alpha_3 + ... + \alpha_6 \le d - \frac{3}{2}r$.

Next assume r is odd. thus $F_d(r) \in \frac{1}{8}\mathbb{Z}$. For $n = 8(F_d(r) - g)$, we want to write $n = \sum_{i=2}^{6} x_i^2$ with $x_i = 2\alpha$, the sum of odd integers with $|x_i| \le r$. Thus $n \equiv 5\,(mod\; 8)$. By hypothesis, $n < 3r^2 - 4r + \frac{10}{3}$ as above. Then we need $\star\star$ integers x_i with $|x_i| \le r$ such that $n = \sum x_i^2$, $\alpha_i = \frac{1}{2}x_i$. As we hve assumed $|\alpha_2| \le \alpha_3 \le ... \le \alpha_6 \le \frac{1}{2}r$, the order and signs of the α_i are determined. The sign of α_2 is given by the assumption that $d + \frac{3}{2}r - \sum \alpha_i \equiv 0 \;\; (mod\; 2)$. We therefore need to check that $-\alpha_2 + \alpha_3 + .. + \alpha_6 \le d - \frac{3}{2}r$. Setting $-x_2 + x_3 + \cdots + x_6 \le 2d - 3r$, then since $n < 4d - 6r + 3$ and $n \equiv 5\,(mod\; 8)$, we have $n \le 4d - 6r - 1$. Since $n = \sum x_i^2$, we need $-x_2 + x_3 + \cdots + x_6 \le \frac{1}{2}\sum x_i^2 + \frac{1}{2}$ which is $(x_2+1)^2 + \sum_{i=3}^{6}(x_i - 1)^2 - 4 \ge 0$. For x_i odd, this is true unless $(x_2, ..., x_6) = (-1, 1, 1, 1, 1)$. Since $-x_2 + x_3 + \cdots + x_6 \le 2d - 3r$ unless $2d - 3r = 1$ or 3, and in the first case there is no $n \equiv 5$ with $n \le 4d - 6r - 1$, and in the second case, the congruence $-\alpha_2 + \alpha_3 + .. + \alpha_6 \le d - \frac{3}{2}r$ is clear, then we have shown the claim.

$\star$

The Sum of Five Squares

If $k \in \mathbb{Z}^+$ then any positive integer $n < 3k^2 - 2k + 3$ can be written as the sum of five squares $n = \sum_{i=1}^{5} x_i^2$ of integers x_i with $|x_i| \le k$.

Proof: Recall the sum of three squares theorem from Gauss: A positive integer n is the sum of 3 squares iff it is not of the form $4^a(8b-1)$,$a, b \in \mathbb{Z}$.

Now suppose that $n < (k+1)^2$. If $n = x_1^2 + x_2^2 + x_3^2$, then clearly $|x_i| \le k$ for all i so we are done. Else, write $n = 4^a m$, $m \equiv 7\,(mod\; 8)$. Then $m - 1$ is a sum of 3 squares, so n is the sum of 4 squares of integers $\le k$.

If $k^2 \le n < k^2 + (k+1)^2$, the same argument applies to $n - k^2$ so either n is the sum of 4 or of 5 squares of integers x_i with $|x_i| \le k$.

If on the other hand $n > 2k^2$, write $n = 2k + m$. If m is the sum of 3 squares and $m < (k+1)^2$, we are done as above. Else, $m = 4^a(8b-1)$, $m \equiv 0, 4, 7\,(mod\; 8)$. Write $n = 2(k-1)^2 + m'$, so $m' = m + 4k - 2$. Then $m' \equiv 1, 2, 5, 6\,(mod\; 8)$ so m' is the sum of 3 squares. If $m' < (k+1)^2$, then $n < 3k^2 - 2k + 3$ we have the result.

$\star\star$ Claim: Set $k > 0$ an odd integer. Then all positive integers $n \equiv 5 \; mod \; 8$ with $n < 3k^2 + 2k + 1$ are sums of 5 squares, $n = \sum_{i=1}^{5} x_i^2$, $|x_i| \le k$.

Proof: Any such n can be written as $1 + 1 + m$, $1 + k^2 + m$ or $k^2 + k^2 + m$, where $0 < m < (k+1)^2$ and $m \equiv (mod\; 8)$. Then m is the sum of 3 squares of integers $\le k$ which are odd.

-
- Existence

V.4.8*

***4.8.** Show that a divisor class D on the cubic surface contains an irreducible curve $\Leftrightarrow$ it contains an irreducible nonsingular curve $\Leftrightarrow$ it is either (a) one of the 27 lines, or (b) a conic (meaning a curve of degree 2) with $D^2 = 0$, or (c) $D.L \geqslant 0$ for every

line L, and $D^2 > 0$. [*Hint*: Generalize (4.11) to the surfaces obtained by blowing up 2, 3, 4, or 5 points of $\mathbf{P}^2$, and combine with our earlier results about curves on $\mathbf{P}^1 \times \mathbf{P}^1$ and the rational ruled surface X_1, (2.18).]

5.4.9 V.4.9 x genus bound for cubic surface.

4.9. If C is an irreducible non-singular curve of degree d on the cubic surface, and if the genus $g > 0$, then

$$g \geqslant \begin{cases} \frac{1}{2}(d-6) & \text{if } d \text{ is even, } d \geqslant 8, \\ \frac{1}{2}(d-5) & \text{if } d \text{ is odd, } d \geqslant 13, \end{cases}$$

and this minimum value of $g > 0$ is achieved for each d in the range given.

- Note $g = \frac{1}{2}\left(a^2 - \sum b_i^2 - d\right) + 1$ since it's a cubic surface.
- So $g > 0 \implies g - 1 \geq 0$ so $(a^2 - \sum b_i^2) \geq d$

The bound:

LHS I will work downwards from the genus:
We have $g = \frac{1}{2}(a-1)(a-2) - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i$
$= \frac{1}{2}(a^2 - 3a + 2) - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i$ lets get rid of $\frac{1}{2}$ since on both sides
$a^2 - 3a + 2 - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i$ get rid of 2 on each side
$a^2 - 3a - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i$ move a^2 to other side
$-3a - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i$ move $9a$ here
$6a - \frac{1}{2}\sum b_i^2 + \frac{1}{2}\sum b_i$ move $\frac{1}{2}\sum b_i$ down
$6a - \frac{1}{2}\sum b_i^2$ bring $6a\sum b_i$ here and move $\frac{1}{2}\sum b_i^2$ up
$6a + 6a\sum b_i$ hmm move $2.5\sum b_i$ here
$6a + 6a\sum b_i + 2.5\sum b_i$

Now, since $g = \frac{1}{2}\left(a^2 - \sum b_i^2 - 1\right) + 1 > 0, \implies \ldots \implies a^2 \geq 8 + \sum b_i^2 \geq \sum b_i$
Thus $6a\sum b_i \leq 6a^2$.
case 1
If $\sum_i b_i \geq 1$, then $a^2 \geq 9 \implies a \geq 3 \implies 2a^2 \geq 6a + 1$ so we are done.
case 2 if $\sum b_i = 0$, then we are done trivially.
case 3 if $\sum b_i < 0$, $a = 0$, then we are done
case 4 if $\sum b_i < 0$, $a < 0$, then we are still done, since we still have $6a\sum b_i \leq 6a^2$

$8a^2 + \left(\sum b_i\right)^2 + \frac{1}{2}\sum b_i^2$

$8a^2 + 2.5\sum b_i + \left(\sum b_i\right)^2 + \frac{1}{2}\sum b_i^2$ let's move $2.5\sum b_i$ down
$8a^2 + (2.5 - 6a)\sum b_i + \left(\sum b_i\right)^2$ let's move $6a\sum b_i$ up and bring $\frac{1}{2}\sum b_i^2$ here
$8a^2 - 6a\sum b_i + \left(\sum b_i\right)^2 + 3\sum b_i$ move $\frac{1}{2}\sum b_i$ here
$8a^2 - 6a\sum b_i + \left(\sum b_i\right)^2 - 9a + 3\sum b_i$ move $9a$ up
$9a^2 - 6a\sum b_i + \left(\sum b_i\right)^2 - 9a + 3\sum b_i$ move a^2 here
$9a^2 - 6a\sum b_i + \left(\sum b_i\right)^2 - 9a + 3\sum b_i + 2$ schwarz won't help. we can get rid of 2's
$\frac{1}{2}\left(d^2 - 3d + 2\right)$ let's evict $\frac{1}{2}$ for now
$\frac{1}{2}\left(d-1\right)\left(d-2\right), d = 3a - \sum b_i$
On the RHS I will work upwards from what we want.

Existence: Set $k > 0$ an integer. Then all positive integers $n < 3k^2 - 2k + 3$ is the sum of five squares $n = \sum_{i=1}^{5} x_i^2$ of integers x_i with $|x_i| \leq k$. (see ex 4.7)
we want to get min value $\frac{1}{2}\left(d-6\right) = \frac{1}{2}\left(a^2 - \sum b_i^2 - d\right) + 1$ the genus.
This is $a^2 - 2d + 8 - b_6^2 = \sum_{i=1}^{5} b_i^2$.
Choose $a \leq d$ such that $a^2 > d + 2$. (since $d \geq 8$, this is easy)
Let $b_6^2 = 3$.
Then $3a^2 - 2a + 3 =\geq a^2 - 2d + 3 + 5 - b_6^2 = a^2 - 2d + 3 - 4$.
Since $a^2 > d + 4$, then $2a^2 > 2d + 2$ and so $a^2 - 2d + 8 - b_6 > 0$.
Thus we can find $b_i, i = 1, ..., 5$ with $|b_i| \leq a$ such that $\sum b_i^2 = a^2 - 2d + 8 - b_6^2$.

•

5.4.10 V.4.10 x

4.10. A curious consequence of the implication (iv) ⇒ (iii) of (4.11) is the following numerical fact: Given integers $a, b_1, \ldots, b_6$ such that $b_i > 0$ for each i, $a - b_i - b_j > 0$ for each i,j and $2a - \sum_{i \neq j} b_i > 0$ for each j, we must necessarily have $a^2 - \sum b_i^2 > 0$. Prove this directly (for $a, b_1, \ldots, b_6 \in \mathbf{R}$) using methods of freshman calculus.

No algebraic geometry here. If your interested, see Nagata Rational Surfaces I, and proceed by cases

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5.4.11 V.4.11 x Weyl Groups

4.11. *The Weyl Groups.* Given any diagram consisting of points and line segments joining some of them, we define an abstract group, given by generators and relations, as follows: each point represents a generator x_i. The relations are $x_i^2 = 1$ for each i; $(x_i x_j)^2 = 1$ if i and j are not joined by a line segment, and $(x_i x_j)^3 = 1$ if i and j are joined by a line segment.

(a) The *Weyl group* $\mathbf{A}_n$ is defined using the diagram

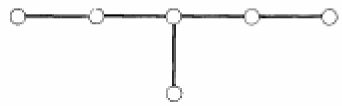
$$\circ\!-\!\circ\!-\!\circ \;\cdots\; -\!\circ$$

of $n - 1$ points, each joined to the next. Show that it is isomorphic to the symmetric group Σ_n as follows: map the generators of $\mathbf{A}_n$ to the elements $(12), (23), \ldots, (n-1, n)$ of Σ_n, to get a surjective homomorphism $\mathbf{A}_n \to \Sigma_n$. Then estimate the number of elements of $\mathbf{A}_n$ to show in fact it is an isomorphism.

I don't care about group theory.

5.4.12 b. x

(b) The *Weyl group* $\mathbf{E}_6$ is defined using the diagram



Call the generators $x_1, \ldots, x_5$ and y. Show that one obtains a surjective homomorphism $\mathbf{E}_6 \to G$, the group of automorphisms of the configuration of 27 lines (4.10.1), by sending $x_1, \ldots, x_5$ to the permutations (12),(23), . . . ,(56) of the E_i, respectively, and y to the element associated with the quadratic transformation based at P_1, P_2, P_3.

see above.

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*(c) Estimate the number of elements in $\mathbf{E}_6$, and thus conclude that $\mathbf{E}_6 \cong G$. *Note*: See Manin [3, §25,26] for more about Weyl groups, root systems, and exceptional curves.

5.4.13 V.4.12 x g kodaira vanishing for cubic surface

4.12. Use (4.11) to show that if D is any ample divisor on the cubic surface X, then $H^1(X, \mathcal{O}_X(-D)) = 0$. This is Kodaira's vanishing theorem for the cubic surface (III, 7.15).

Recall $X \approx \mathbb{P}^2$ which has known cohomology.

$H^0(X, \mathcal{O}_X) = k$, $H^1(X, \mathcal{O}_X) = 0$, $H^2(X, \mathcal{O}_X) = H^0(X, \mathcal{O}_X(-3)) = H^0(X, \omega_X) = p_g$, since $-3H$ is the canonical for $\mathbb{P}^2$. By ex II.8.20.1, the dimension of this space is the geometric genus is 0 for $\mathbb{P}^n$. So this is 0.

By chapter 1, $H^0(D, \mathcal{O}_D) = k$.

We have a LES

$0 \to H^0(\mathcal{O}_X(-D)) \to k \to H^0(D, \mathcal{O}_D) = k \to \cdots$

$H^1(X, \mathcal{O}_X(-D)) \to 0 \to H^1(D, \mathcal{O}_D) \to \cdots$

$H^2(X, \mathcal{O}_X(-D)) \to 0 \to 0.$

So if $H^0(\mathcal{O}_X(-D)) = 0$, then we are done as $k \hookrightarrow k$ thus the kernel of $k \to H^1(X, \mathcal{O}_X(-D))$ is k, and so by exactness.

Suppose there is an effective divisor linearly equivalent to $-D$.

Then by V.4.11, this effective divisor should meet at least one line with negative intersection which is silly. So we are done.

V.4.13 16 Lines x g

4.13. Let X be the Del Pezzo surface of degree 4 in $\mathbf{P}^4$ obtained by blowing up 5 points of $\mathbf{P}^2$ (4.7).

(a) Show that X contains 16 lines.

So first we really need to get the intersection theory on the new surface.

Using the methods of the chapter we find:

- $Pic\ X \approx \mathbb{Z}^6$ generated by $l, e_1, ..., e_6$
- intersection pairing on X given by $l^2 = 1$, $e_i^2 = -1$, $l.e_i = 0$, $e_i.e_{\neq i} = 0$.

- hyperplane is $3l - \sum e_i$
- canonical class is $-h = -3l - \sum e_i$
- If $D \sim al - \sum b_i e_i$, degree as a curve in $\mathbb{P}^3$ is $d = 3a - \sum b_i$
- $D^2 = a^2 - \sum b_i^2$,
- genus is $\frac{1}{2}(D^2 - d) + 1$ by adjunction.

Thus we claim:

- the del pezzo contains 16 lines, self-intersection -1, only irreducibles negative self-intersection
- exceptional curves E_i (5 of these)
- strict transforms F_{ij} of lines through P_i and P_j , (the number of lines through 2 points is $a_2 = 1$, number of lines through 3 points is $a_3 = a_2 + 2 = 3$, number of lines through 4 points is $a_4 = a_3 + 3 = 6$, and number of lines through 5 points is $a_5 = a_4 + 4 = 10$. So 10 of these.
- The strict transform of conic in $\mathbb{P}^2$ containing 5 of the P_i (one of these).

The first and second paragraphs of the proof of 27 lines hold exactly by looking at our intersection theory discovered above.

Now suppose C is irreducible curve on X, with deg $C = 1$, and $C^2 = -1$, then C is one of the 16 listed. If C is not one of the E_i, then $C \sim al - \sum b_i$, and since we are just doing monoidal transforms (3.7) cf 4.8.1, then $a > 0$, $b_i \geq 0$.

Also, $deg\ C = 3a - \sum b_i = 1$, $C^2 = a^2 - \sum b_i^2 = -1$.

$\implies \sum b_i = 3a - 1$, and $\sum b_i^2 = a^2 + 1$

We show that only $a, b_1, ..., b_5$ satisfying all these conditions are those corresponding to F_{ij} and G_j above.

Schwarz gives $\left(\sum b_i\right)^2 \leq 5\left(\sum b_i^2\right)$. Substituting, we get $(3 \cdot a - 1)^2 = 9a^2 - 6a + 1 \leq 5a^2 + 5$ or $4a^2 - 6a - 4 \leq 0$.

We need to solve the quadratic.

$allroots\,(4 \cdot a^2 - 6 \cdot a - 4, a) = [a = -0.5, a = 2.0]$ so $a = 1$ or $a = 2$.

Now if $a = 1$, then $\sum b_i = 3 - 1 = 2$ and $\sum b_i^2 = 1 + 1 = 2$ so two of the b_i are 1 , and rest are 0. This is one of the F_{ij}. If $a = 2$, then $\sum b_i = 3 \cdot 2 - 1 = 5$ and $\sum b_i^2 = 2^2 + 1 = 5$ so all the b_i are 1 and this is G_j.

5.4.14 b. x g

. Let X be the Del Pezzo surface of degree 4 in $\mathbf{P}^4$ obtained by blowing up 5 points of $\mathbf{P}^2$ (4.7).

(b) Show that X is a complete intersection of two quadric hypersurfaces in $\mathbf{P}^4$ (the converse follows from (4.7.1)).

Since X is del-pezzo, as in thm V.4.7, we have $\mathcal{O}_X(1) = -\omega_X$, and X is embedded in $\mathbb{P}^4$ via the linear system of cubics through the blow up points.

Then $\mathcal{O}_X(-2K_X) \approx \mathcal{O}_X(2)$ $\quad\star$ by tensoring the same thing on both sides.

Consider $0 \to H^0(\mathbb{P}^4, \mathcal{I}_X(2)) \to H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) \to H^0(X, \mathcal{O}_X(2)) \to 0$.

(This ends in 0 since $H^1(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(-2)) = 0$).

Now we count dimensions. Note $h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(2)) = 15$

Note that by Riemann-roch on a surface,

$dim\ H^0(X, \mathcal{O}_X(-2K_X)) - dim\ H^1(X, \mathcal{O}_X(-2K_X)) = \frac{1}{2}(-2K_X - K_X) . (-2K_X) + 1 = \frac{1}{2} \cdot 2(2+1) \cdot 4 + 1 = 13$

since $K_X^2 = 4$ by thm V.4.7.

Note that by Ramanujam vanishing and Serre duality, $H^1(X, \mathcal{O}_X(-2K_X)) = 0$.
Now use $\star$.

5.4.15 V.4.14 x

4.14. Using the method of (4.13.1), verify that there are nonsingular curves in $\mathbf{P}^3$ with $d = 8$, $g = 6,7$; $d = 9$, $g = 7,8,9$; $d = 10$, $g = 8,9,10,11$. Combining with (IV, §6), this completes the determination of all posible g for curves of degree $d \leqslant 10$ in $\mathbf{P}^3$.

From 4.7

- From method of V.4.7 we get $d = 8, g = 7$, and $d = 9, g = 8, 9$ and $d = 10, g = 10, 11$
- just need $d = 10, g = 8, 9$ and $d = 9, g = 7$, and $d = 8, g = 6$

For these remaining rest, I use the method of exc V.4.9 (Read that exercise solution first or this will be nonsense!)

We want to get $\frac{1}{2}(a^2 - \sum b_i^2 - d) + 1 = g$ for $d = 8,\ g = 6$
so $a^2 - \sum b_i^2 = 2g - 2 + d$
so $\sum_{i=1}^5 b_i^2 + b_6^2 = 2 - 2g + d + a^2$ —use this line
so $\sum b_i^2 = a^2 - 12 + 2 + 8 - b_6^2$
so $\sum b_i^2 = a^2 - 2 - b_6^2$. Solve for this using sum of 5 squares as in exc V.4.9 and the rest follow similarly.

5.4.16 V.4.15 x admissible transformation

4.15. Let $P_1, \ldots, P_r$ be a finite set of (ordinary) points of $\mathbf{P}^2$, no 3 collinear. We define an *admissible transformation* to be a quadratic transformation (4.2.3) centered at some three of the P_i (call them P_1, P_2, P_3). This gives a new $\mathbf{P}^2$, and a new set of r points, namely Q_1, Q_2, Q_3, and the images of $P_4, \ldots, P_r$. We say that $P_1, \ldots, P_r$ are *in general position* if no three are collinear, and furthermore after any finite sequence of admissible transformations, the new set of r points also has no three collinear.

(a) ~~A set of 6 points is in general position if and only if no three are~~ collinear and not all six lie on a conic.

I will show that there is exactly one curve of m^{th} order through $m(m+3)/2$ points.

This is equivalent to there being $\frac{1}{2}m(m+3)$ points with only one curve passing through those points for any m.

For $m = 1$ we can find a $1 \cdot 4/2 = 2$ points which only contain one curve clearly.

If the assertion is true for $m - 1$, then as $\frac{1}{2}m(m+3) = (m-1)(m-1+3)/2 + m + 1$, we choose $m + 1$ distinct points on a line L and the rest not on the line and in general position.

Suppose C is any curve passing through all the points. Now by Bezout, L has intersects any curve of m^{th} order passing through the first $m + 1$ points at least $m + 1$ times but $m \cdot 1$ for the degrees don't equal so L must be a component of such a curve, $C = L \cup C'$, with C' being of order $m - 1$. But then C' is determined by the choice of the remaining points by the induction hypothesis.

5.4.17 b. x

(b) If $P_1, \ldots, P_r$ are in general position, then the r points obtained by any finite sequence of admissible transformations are also in general position.

This is Nagata, Rational Surfaces II, corollary to proposition 9, and use (a) of this problem.

5.4.18 c. x g

(c) Assume the ground field k is uncountable. Then given $P_1, \ldots, P_r$ in general position, there is a dense subset $V \subseteq \mathbf{P}^2$ such that for any $P_{r+1} \in V, P_1, \ldots, P_{r+1}$ will be in general position. [*Hint*: Prove a lemma that when k is uncountable, a variety cannot be equal to the union of a countable family of proper closed subsets.]

Consider $(P_1, ..., P_r)$ as a point in $\mathbb{P}_2 \times ... \times \mathbb{P}_2$.

General position is equivalent to some determinants not vanishing.

Thus the tuples $(P_1, ..., P_r)$ in general position form the complement of an the vanishing of these determinants.

So unless the vanishing set is all of $\mathbb{P}_2 \times ... \times \mathbb{P}_2$, then it must be proper so we get a zariski open set which is open and dense for the general position points.

5.4.19 d. x

(d) Now take $P_1, \ldots, P_r \in \mathbf{P}^2$ in general position, and let X be the surface obtained by blowing up $P_1, \ldots, P_r$. If $r = 7$, show that X has exactly 56 irreducible nonsingular curves C with $g = 0$, $C^2 = -1$, and that these are the only irreducible curves with negative self-intersection. Ditto for $r = 8$, the number being 240.

at 7

By the logic of thm V.4.9, and proceeding as in exc V.4.13.a, we find there are 7 exceptional curves.

There are 21 lines through the points, since no 3 are collinear, and that gives 21 additional -1 curves, as in exc V.4.13.a.

There are 21 conics through 5 points, 7 choose 5.

Now note that cubics through 7 points have 1 degree of freedom (by Bezout for intsance). Thus if we take one of the points doubled (so make a base point), this gives 7 additional -1 curves.

Thus we have $14 + 42 = 56$

at 8

Using the same logic, we can have

exceptional curves (8)

Lines through points (8 choose 2 = 28)

conics through 5 (8 choose 5 = 56)

cubics through 7 with one double point (8 choose 7 = 8)

quartics through 8 with three double points (8 choose 3 = 56)

quintics though 8 with 6 double points (8 choose 6 = 28)

sextics through 8 with 7 double points and one triple point (the rest)

5.4.20 V.4.16 x Fermat Cubic

4.16. For the *Fermat cubic surface* $x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$, find the equations of the 27 lines explicitly, and verify their incidence relations. What is the group of automorphisms of this surface?

We can use Elkies parametrization: If (w, x, y, z) is a rational solution of $w^3 + x^3 + y^3 + z^3 = 0$, then there exist r, s, t such that (w, x, y, z) are proportional to

$-(s+r)t^2 + (s^2 + 2r^2)t - s^3 + rs^2 - 2r^2 s - r^3 + t^3$

$-(s+r)t^2 + (s^2 + 2r^2)t + rs^2 - 2r^2 s + r^3 - t^3 +$

$(s+r)t^2 - (s^2 + 2r^2)t + 2rs^2 - r^2 s + 2r^3 +$

$(s-2r)t^2 + (r^2 - s^2)t + s^3 - rs^2 + 2r^2 s - 2r^3$.

where r is proportional to $yz - wx$, s is propotional to $wy - wx + xz + w^2 - wz + z^2$, and t is proportional to $w + y, y, x$ unless $w + y$, $x + z$ are both zero, in which case r, s, t are proportional to $x + y$, y, x . and the blow down / points we blow up are

$x + az = x + ay = 0 \mapsto (-a : 1 : 1)$;

$w + a'z = x + a'y = 0 \mapsto (-a' : 1 : 1)$;

$x + az = y + aw = 0 \mapsto (0 : 1 : -a)$;

$x + a'z = y + a'w = 0 \mapsto (0 : 1 : -a')$;

$y + az = w + ax = 0 \mapsto (1 : -a' : -a)$;

$y + a'z = w + a'x = 0 \mapsto (1 : -a : -a')$.

By remark 4.10.1 we have:

So it's in $\mathbb{P}^3$, it's blow up of $\mathbb{P}^2$ at 6 points.

- E_i there are 6 exceptional lines, with self intersection -1,
- F_{jk} there are 15 strict transforms of lines containing P_i and P_j
- G_j there are 6 strict transforms of conics containing 5 of the P_i.

The are incidence relations

- so E_i doesn't meet E_j (obvious by description)
  - E_i meets F_{jk} iff $i = j$ or $i = k$ (obvious by description)
  - E_i meets G_j iff $i \neq j$ (obvious by description)
  - F_{ij} meets F_{kl} iff all i, j, k, l are distinct.. not quite obvious.
  - F_{ij} meets G_k iff $i = k$ or $j = k$ (obvious by description)
  - so as an example calculation> If we use the parametrization above, and we want to show E_i doesn't meet E_j, then they would meet if there is a point on the line where $x = x'$,$y = y'$, $z = z'$, $w = w'$ on the two curves $w + az = x + zy = 0$ and $w' + a'z' = x' + z'y = 0$ where a is a cubic root of unity. But then $a = a'$ which is false. The other's can be similarly verified.

Automorphism group? E_6. A proof is in Hirschfeld Finite projective spaces in three dimensions 20.3.1.

5.5 V.5 Birational Transformations

5.5.1 V.5.1 x g Resolving singularities of f

5.1. Let f be a rational function on the surface X. Show that it is possible to "resolve the singularities of f" in the following sense: there is a birational morphism $g: X' \to X$ so that f induces a morphism of X' to $\mathbf{P}^1$. [*Hints*: Write the divisor of f as $(f) = \sum n_i C_i$. Then apply embedded resolution (3.9) to the curve $Y = \bigcup C_i$. Then blow up further as necessary whenever a curve of zeros meets a curve of poles until the zeros and poles of f are disjoint.]

Let $Y = div\,(f)$

Take an embedded resolution using V.3.9, so $f^{-1}(Y)$ is normal crossings.

Now how will further blowing-up make the resulting curves disjoint?

Ok, further blowing up will make the curves disjoint because you have simle normal crossings to start. Now the blowing-up surface separates tangent directions at the node .

Now you have a bunch of copies of $\mathbb{P}^1$ so you can just define by projection.

5.5.2 V.5.2 x g Castelnuovo Lookalike

5.2. Let $Y \cong \mathbf{P}^1$ be a curve in a surface X, with $Y^2 < 0$. Show that Y is contractible (5.7.2) to a point on a projective variety X_0 (in general singular).

c.f.5.7

Let $-m = Y^2 < 0$

choose a very ample divisor H on X such that $H^1(X, \mathcal{L}(H)) = 0$ by III.5.2.

Let $k = H.Y$ and assume $k \geq 2$.

(H is ample so intersection > 0 and then take a multiple of that) – [8]

We will use the invertible sheaf $\mathcal{L}(mH + kY)$ to define a morphism of X to something.

First we prove that $H^1(X, \mathcal{L}(mH + iY)) = 0$ for $i = 0, 1, ..., k$.

For $i = 0$, it's true by assumption. Assume for $i - 1$.

Consider $0 \to L(mH + (i-1)T) \to L(mH + iY) \to \mathcal{O}_Y \otimes \mathcal{L}(mH + iY) \to 0$.

$Y \approx \mathbb{P}^1$ and $(mH + iY).Y = mk - im$, so

$\mathcal{O}_Y \otimes \mathcal{L}(mH + iY) \approx \mathcal{O}_{\mathbb{P}^1}(mk - im)$.

We get an exact cohomology sequence

$... \to H^1(X, \mathcal{L}(mH + (i-1)Y)) \to H^1(X, \mathcal{L}(mH + iY)) \to$

$\to H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(mk - im)) \to 0$

By the induction hypothesis, we know that $H^1(X, \mathcal{L}(H + (i-1)Y)) = 0$.

Also $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(mk - im)) = 0$ by III.5 and so we conclude $H^1(X, \mathcal{L}(mH + iY)) = 0$ for $i \leq k$.

step 2

Next we show that $\mathcal{M} = \mathcal{L}(mH + kY)$ is globally generated.

Since H is very ample, $|mH + kY|$ has no basepoints away from Y so $\mathcal{M}$ is generated by global sections away from Y.

On the other hand, the natural map $H^0(X, \mathcal{M}) \to H^0(Y, \mathcal{M} \otimes \mathcal{O}_Y)$ is surjective, because

$\mathcal{M} \otimes \mathcal{I}_Y \approx \mathcal{L}(mH + kY) \otimes \mathcal{L} - Y \approx \mathcal{L}(mH + (k-1)Y)$ and

$H^1(X, \mathcal{L}(mH + (k-1)Y)) = 0$ by step 1, and the LES of cohomology.

Next observe $(mH + kY).Y = mk - mk = 0$ and so $\mathcal{M} \otimes \mathcal{O}_Y \approx \mathcal{O}_{\mathbb{P}^1}$ which is generated by the global section 1. Lifting this section to $H^0(X, \mathcal{M})$, and using Nakayama lemma, we see that $\mathcal{M}$ is generated by global sections also at every point of Y.

Step 3.

[8] we can change k to the bigger

Therefore $\mathcal{M}$ determines a morphism $f_1 : X \to \mathbb{P}^N$ (II.7.1).

Let X_1 it's image.

Since $f_1^*\mathcal{O}(1) \approx \mathcal{M}$ (II.7), and since degree of $\mathcal{M} \otimes \mathcal{O}_Y$ is 0 (since its $\mathcal{O}_{\mathbb{P}^1}$), then f_1 must map Y to a point P_1.

On other hand, since H is very ample, the linear system $|mK + kY|$ separates points and tangent vectors away from Y, and also separates points of Y from points not on Y , so f_1 is an iso of $X - Y$ onto $X_1 - P_1$.

5.5.3 V.5.3 x g hodge numbers excercise

5.3. If $\pi:\tilde{X} \to X$ is a monoidal transformation with center P, show that $H^1(\tilde{X},\Omega_{\tilde{X}}) \cong H^1(X,\Omega_X) \oplus k$. This gives another proof of (5.8). [*Hints*: Use the projection formula (III, Ex. 8.3) and (III, Ex. 8.1) to show that $H^i(X,\Omega_X) \cong H^i(\tilde{X},\pi^*\Omega_X)$ for each i. Next use the exact sequence

$$0 \to \pi^*\Omega_X \to \Omega_{\tilde{X}} \to \Omega_{\tilde{X}/X} \to 0$$

and a local calculation with coordinates to show that there is a natural isomorphism $\Omega_{\tilde{X}/X} \cong \Omega_E$, where E is the exceptional curve. Now use the cohomology sequence of the above sequence (you will need every term) and Serre duality to get the result.]

Let $\mathscr{F} = \pi^*\Omega_X$. so probably since we are on smooth varieties, $H^i\left(\tilde{X}, \pi^*\Omega_X\right) \approx H^i(X, \pi_*\pi^*\Omega_X) \approx H^i(X, \Omega_X)$ by exc III.8.1.

Now $0 \to \pi^*\Omega_X \to \Omega_{\tilde{X}} \to \Omega_{\tilde{X}/X} \to 0$ is the relative cotangent sequence.

Now (cf IV.2.2) $\Omega_{\tilde{X}/X}$ has support equal to the set of ramified points, for at other points the first two sheaves are same dimensional so the quotient at the stalks is 0. Thus $\Omega_{\tilde{X}/X} \approx \Omega_E$.

Now consider

$0 \to H^0(X, \Omega_X) \to H^0\left(\tilde{X}, \Omega_{\tilde{X}}\right) \to H^0(E, \Omega_E) \to$

$H^1(X, \Omega_X) \to H^1\left(\tilde{X}, \Omega_{\tilde{X}}\right) \to H^1(E, \Omega_E) \to$

$H^2(X, \Omega_X) \to H^2\left(\tilde{X}, \Omega_{\tilde{X}}\right) \to 0$ (the last is 0 since Ω_E has support on a one dimensional space).

Note that $H^0(E, \Omega_E) \approx H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0$ since there are no monomials of negative degree in two variables and $H^1(E, \Omega_E) \approx H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \approx H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1})^\vee = k^\vee = k$ by serre duality.

If X is $\mathbb{P}^2$ at least, then we are done by exc III.7.3, and hence we are done for a rational surface.

Otherwise, note that the hodge numbers satisfy $h^{p,q} = h^{q,p}$ since $H^q(X, \Omega^p) = H^p(X, \Omega^q)^\vee$ by serre duality and further $h^{n-p,n-q} = h^{p,q}$ by poincare duality. As $h^{1,0}$ is a birational invariant since genus is, then we see $H^2(X, \Omega_X) \to H^2\left(\tilde{X}, \Omega_{\tilde{X}}\right)$ is an isomorphism.

5.5.4 V.5.4 x g hodge index theorem corollary x

5.4. Let $f:X \to X'$ be a birational morphism of nonsingular surfaces.

(a) If $Y \subseteq X$ is an irreducible curve such that $f(Y)$ is a point, then $Y \cong \mathbf{P}^1$ and $Y^2 < 0$.

Let H be a very ample divisor on X' .

Note that pullback of ample is ample.

So $f^*H.Y = 0$ and $(f^*H)^2 = 0$.

Thus by the hodge index theorem, $Y^2 \leq 0$.

5.5.5 b. x g Hodge Index negative definite

(b) (Mumford [6].) Let $P' \in X'$ be a fundamental point of f^{-1}, and let $Y_1, \ldots, Y_r$ be the irreducible components of $f^{-1}(P')$. Show that the matrix $\|Y_i . Y_j\|$ is negative definite.

That $Y_i^2 \leq 0$ is easy since if H is strict transform of a hyperplane not through P, then $H^2 > 0$, and $H.Y_i = 0$ so that $Y_i^2 \leq 0$ by Hodge index theorem.

On the other hand, let H_1 the strict transform of a hyperplane through P and H_2 the strict transform of a hyperplane not through P. Then on irreducible components we have $H_2 \equiv H_1 + \sum Y_i$, $m_i > 0$. Then for each j, $\sum_i \langle m_i E_i, m_j E_j \rangle = -\langle H_1, m_j E_j \rangle \leq 0$. Note further that $\langle Y_i, Y_j \rangle \geq 0$ for $i \neq j$. Since the matrix $\|Y_i.Y_j\|$ is symmetric, then by basic facts of symmetric matrices, we have $\|Y_i.Y_j\|$ is negative indefinite.

Now note that since H_1 passes through E_j then $\sum_i \|Y_i Y_j\| < 0$. By Zariski's Main theorem, we can't split the Y_i's into two groups which are not connected so the two groups always don't intersect each other. Since we have already proved indefiniteness, these two facts give negative definite.

5.5.6 V.5.5 x g

5.5. Let C be a curve, and let $\pi: X \to C$ and $\pi': X' \to C$ be two geometrically ruled surfaces over C. Show that there is a finite sequence of elementary transformations (5.7.1) which transform X into X'. [*Hints*: First show if $D \subseteq X$ is a section of π containing a point P, and if $\tilde{D}$ is the strict transform of D by elm_P, then $\tilde{D}^2 = D^2 \pm 1$ (Fig. 23). Next show that X can be transformed into a geometrically ruled surface X'' with invariant $e \gg 0$. Then use (2.12), and study how the ruled surface $\mathbf{P}(\mathscr{E})$ with $\mathscr{E}$ decomposable behaves under elm_P.]

Consequence of my proof of exc V.2.5.a

5.5.7 V.5.6 x

5.6. Let X be a surface with function field K. Show that every valuation ring R of K/k is one of the three kinds described in (II, Ex. 4.12). [*Hint*: In case (3), let $f \in R$. Use (Ex. 5.1) to show that for all $i \gg 0$, $f \in \mathcal{O}_{X_i}$, so in fact $f \in R_0$.]

If R is a valuation ring with valuation $\nu : R \to \Gamma$, then since X is projective, by the general assumptions in chapter 5, it is thus proper, and so the valuative criterion gives that $Spec\ K(X) \to X$ extends to $Spec\ R \to X$.

The image of the closed point of $Spec\ R$ corresponding to the maximal ideal is the center of the valuation ν. If $x \in X$ is the center then R dominates $\mathcal{O}_x$. If ν is nontrivial, then the dimension of ν , which is corresponds to the transcendence degree of $R/\mathfrak{m}_R$ which corresponds to the dimension of the center, and since we are on a surface this number must be 0 or 1.

If the center has codimension 1, then $\mathcal{O}_x$ and R must be discrete as R dominates $\mathcal{O}_x$ and at any rate a valuation group on a surface is either a one dimensional or two dimension $\mathbb{Z}$-module (see Vaqiue Valuations and local uniformization, remark 1.14). As both R and $\mathcal{O}_x$ are valuation rings, then $R = \mathcal{O}_x$. This corresponds to type (1) of exc II.4.12.

If the center x has codimension 2, then taking the monoidal transform with center x gives X_1 . The center of ν in X_1 is either the exceptional divisor or point contained in the exceptional divisor. In the former case we have type (2) of exc II.4.12, in the later case we repeat and get a type (3) of exc II.4.12.

5.5.8 V.5.7 x

5.7. Let Y be an irreducible curve on a surface X, and suppose there is a morphism $f: X \to X_0$ to a projective variety X_0 of dimension 2, such that $f(Y)$ is a point P and $f^{-1}(P) = Y$. Then show that $Y^2 < 0$. [*Hint*: Let $|H|$ be a very ample (Cartier) divisor class on X_0, let $H_0 \in |H|$ be a divisor containing P, and let $H_1 \in |H|$ be a divisor not containing P. Then consider f^*H_0, f^*H_1 and $\tilde{H}_0 = f^*(H_0 - P)^-$.]

The answer from 5.4 applies since at beginning of surfaces chapter, we assume that X is nonsingular (maybe X_0 isn't)

5.5.9 V.5.8 x A surface Singularity

5.8. *A surface singularity.* Let k be an algebraically closed field, and let X be the surface in $\mathbf{A}_k^3$ defined by the equation $x^2 + y^3 + z^5 = 0$. It has an isolated singularity at the origin $P = (0,0,0)$.

(a) Show that the affine ring $A = k[x,y,z]/(x^2 + y^3 + z^5)$ of X is a unique factorization domain, as follows. Let $t = z^{-1}$; $u = t^3x$, and $v = t^2y$. Show that z is irreducible in A; $t \in k[u,v]$, and $A[z^{-1}] = k[u,v,t^{-1}]$. Conclude that A is a UFD.

First we claim z is irreducible in A. So I wish to show $A/(z) \approx k[x,y]/(x^2+y^3)$ is an integral domain. Proceed as in Algebra.NN.30, my algebra/geometry/analysis solutions, summerstudychallenge2.pdf.

Next note that $x^2+y^3+z^5=0$ implies $-x^2/z^6 - t^3/z^6 = \frac{1}{z}$ so $t = -u^2 - v^3$ so $t \in k[u,v]$.

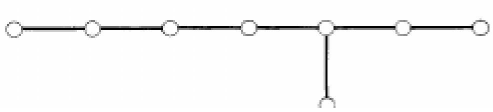
Now we have $x = (t^{-1})^3 t^3 x = (t^{-1})^3 u$, $y = (t^{-1})^2 t^2 y = (t^{-1})^2 v$, $z = t^{-1}$ so that $A \subset k[u,v,t^{-1}]$. Since $t \in k[u,v]$ then $A[z^{-1}] \subset k[u,v,t^{-1}]$. On the other hand, $u,v,t^{-1} = (z^{-1})^3 x, (z^{-1})^2, t^{-1} \in A[z^{-1}]$ so $A[z^{-1}] \supset k[u,v,t^{-1}]$. Thus $A[z^{-1}] = k[u,v,t^{-1}]$. These are both UFD.

Geometrically, $A[z^{-1}]$ is localizing at things not in z. So if f is irreducible not in (z), then f is irreducible in the localization. The converse also holds: If there is a nonzero irreducible element in the localization $A[z^{-1}]$ for $f \in A$, then $f = z^m g$ for an irreducible $g \in A$, $g \notin (z)$.

Now if $f \in A$ is nonzero, since $A[z^{-1}]$ is a UFD, then, denoting by $\frac{f}{1}$ the localization, we have $\frac{f}{1} = \frac{u}{z^m}\frac{f_1}{z^{m_1}}\cdots\frac{f_n}{z^{m_n}}$ for nonnegative $m, m_1, ..., m_n$, a unit u in A, and $f_1, ..., f_n$ in A , where f_i/z^{m_i} are irreducible in $A[z^{-1}]$. Since each z^{m_i} is a unit, we may write this as $\frac{f}{1} = \frac{u}{z^m}\frac{f_1}{1}\cdots\frac{f_n}{1}$, f_i irreducible. So $z^m f = u f_1 \cdots f_n$ in A. By above, $f_i = z^{r_i} g_i$ for an irreducible $g_i \in A$, $g_i \notin (z)$. Thus $z^m f = u z^{\sum r_i} \prod g_i$. Since $g_i \notin (z)$, then $m \geq \prod r_i$ so $f = u z^s \prod g_i$, $s = \sum r_i - m \geq 0$ so we have factored f. Uniqeness follows in the standard way.

5.5.10 V.5.8.b. x Surface singularity

(b) Show that the singularity at P can be resolved by eight successive blowings-up. If $\tilde{X}$ is the resulting nonsingular surface, then the inverse image of P is a union of eight projective lines, which intersect each other according to the Dynkin diagram $\mathbf{E}_8$:



Step 1: Start your computer.

Step 2: Enter the following commands into Singular

```
> ring R=0,(x,y,z),dp;
> ideal I=x2+y3+z5;
```

```
> list L=resolve(I,1s)

> list iD=intersectionDiv(L);
```

Alternatively, if you have a few hours to spare you can proceed as follows:

We first find sings of X.

Jacobian is $\begin{pmatrix} 2x \\ 3y^2 \\ 5z^4 \end{pmatrix}$

nonsingular when rank is $n - dim\ V$ where $n = 3$, dim of space, so when $rank$ is ≥ 1.

thus singular when x, y, z=0

total transform $S \subset \mathbb{A}^3 \times \mathbb{P}^2$ of blow-up at 0 is $S : (x^2 + y^3 + z^5, xZ = zX, yZ = zY, xY = yX)$

we compute locally.

If $X = 1$, $(x^2 + y^3 + z^5, xZ = z, yZ = zY, xY = y) \iff$
$(x^2 + x^3Y^3 + x^5Z^5, xZ = z, xY = y) \iff$
$(x^2, xZ = z, xY = y) \vee (1 + xY^3 + x^3Z^5, xZ = z, xY = y)$

The latter is strict transform.

The jacobian is $\begin{pmatrix} 3x^2z^5 + y^3 \\ 3xy^2 \\ 5x^3z^4 \end{pmatrix}$

note $x \neq 0$on the surface

if singular then Y, Z must both be zero, but that points not on surface either, thus nonsingular.

If $Y = 1$,
$(x^2 + y^3 + z^5, yZ = z, x = yX)$ strict is
$y^2X^2 + y^3 + y^5Z^5 = y^2 (X^2 + y + y^3Z^5)$
exceptional is y^2, strict $(X^2 + y + y^3Z^5)$

jacobian is $\begin{pmatrix} 2x \\ 3y^2z^5 + 1 \\ 5y^3z^4 \end{pmatrix}$

singular points need $x = 0$ for first col, either y or z is 0 for last col, but then middle col is nonzero, so this patch is nonsingular.

If $Z = 1$,
$(x^2 + y^3 + z^5, x = zX, y = zY) \iff$
$(z^2X^2 + z^3Y^3 + z^5, x = zX, y = zY) \iff$

$(z^2, x = zX, y = zY) \vee (X^2 + zY^3 + z^3, x = zX, y = zY)$ jacobian of RHS strict is $\begin{pmatrix} 2x \\ 3y^2z \\ y^3 + 3z^2 \end{pmatrix}$

so to be signular, first row must be 0 so $x = 0$, and second row also so y or $z = 0$, but then both y, z are zero by last row.

have one singularity at $(0, 0, 0)$

New blowup

$X_2 : (X^2 + zY^3 + z^3, bX = Ya, az = cX, Yc = bz)$on $a = 1$, $bX = Y$, $z = cX$, $\implies$
$X^2 + cXb^3X^3 + c^3X^3 =$
$X^2 (1 + cb^3X^2 + c^3X)$

jacobian of strict is

$$\begin{pmatrix} 3b^2cX^2 \\ b^3X^2 + 3c^2X \\ 2b^3cX + c^3 \end{pmatrix} = \begin{pmatrix} 3b^2cX^2 \\ X (b^3X + 3c^2) \\ c (2b^3X + c^2) \end{pmatrix}$$

from first row, one of c, b, X must be zero.

X cannot be, since that is not on the surface.

same for c.
If $b=0$, then by second row, still X or c must be zero.
Thus this patch is nonsingular.

Second patch:
$X_2 : (X^2 + zY^3 + z^3, bX = Ya, az = cX, Yc = bz)$
on $b = 1$, $X = Ya, Yc = z$ so
$Y^2a^2 + YcY^3 + Y^3c^3 =$
$Y^2 \left(a^2 + cY^2 + Yc^3\right)$
jacobian of strict is $\begin{pmatrix} 2a \\ 3c^2y + y^2 \\ c^3 + 2cy \end{pmatrix}$
clearly $a = 0$ at a sginularity
if $c \neq 0$, then $Y \neq 0$,
by second row of jacobian, Y is negative
we must solve $cY\left(Y + c^3\right) = 0$ to be on surface
so $c = -Y^3$
last row of jacobian: $-Y^9 - 2Y^4 = -Y^4\left(Y^5 + 2\right)$ so $Y = (-2)^{\frac{1}{5}}$
also by second row of jacobian
$-3Y^6 \cdot y + y^2 = 0$
so since this has different roots, $Y = 0$
only singularity is at $Y = 0$.

On the other patch, $c = 1$, we get it's nonsingular.

Let X_3 :$(a^2 + cY^2 + Yc^3, ae = cd, af = Yd, cf = Ye)$
on $d = 1$, $ae = c$, $af = Y$ so
$a^2\left(1 + ...\right)$ and we see it's nonsingular
on $e = 1$, $a = cd, cf = Y \implies$
$c^2\left(d^2 + cf^2 + cdc\right)$
on $f = 1$, $a = Yd, c = Ye \Longrightarrow$
$Y^2\left(d^2 + Ye + YYe^3\right)$
jacobian on $e = 1$ is
$\begin{pmatrix} 2cd + f^2 \\ c^2 + 2d \\ 2cf \end{pmatrix}$
so singularity at $(0, 0, 0)$
on $f = 1$, jacobian is
$\begin{pmatrix} 2d \\ 2ye^3 + e \\ 3y^2e^2 + y \end{pmatrix}$
so singularity at $(0, 0, 0)$
so $d = 0$
etc, etc.

5.6 V.6 Classification Of Surfaces

5.6.1 V.6.1 x g

6.1. Let X be a surface in $\mathbf{P}^n$, $n \geqslant 3$, defined as the complete intersection of hypersurfaces of degrees $d_1, \ldots, d_{n-2}$, with each $d_i \geqslant 2$. Show that for all but finitely many choices of $(n, d_1, \ldots, d_{n-2})$, the surface X is of general type. List the exceptional cases, and where they fit into the classification picture.

X is a surface, so in this chapter it's smooth, then $K_X = \mathcal{O}_X\left(\sum_{i=1}^{n-2} d_i - n - 1\right)$ by adjunction since $K_{\mathbb{P}^n} = (-n-1)\,H$.

If $n = 3$, then $-n-1 = -4$ so we could pick $d_1 = 2$ or $d_1 = 3$ to have K be negative multiple of ample so by 6.1, $|12K| = \emptyset \implies X$ rational or ruled. We could pick $n = 4$ to have $K_X \equiv 0$ (so by 6.3, X is $K3$.

If $n = 4$, then we could pick $d_1 = 2, d_2 = 3$ to get a $K3$.

If $n = 5$, we could pick $d_1, d_2, d_3 = 2$ to get a $K3$.

If $n = 6$ we need $d_1, d_2, d_3, d_4 = ...$ but since $d_i \geq 2$, and the canonical subtracts only 7, it will always be general type.

Same for larger n.

5.6.2 V.6.2 x g

6.2. Prove the following theorem of Chern and Griffiths. Let X be a nonsingular surface of degree d in $\mathbf{P}_{\mathbf{C}}^{n+1}$, which is not contained in any hyperplane. If $d < 2n$, then $p_g(X) = 0$. If $d = 2n$, then either $p_g(X) = 0$, or $p_g(X) = 1$ and X is a K3 surface. [*Hint*: Cut X with a hyperplane and use Clifford's theorem (IV, 5.4). For the last statement, use the Riemann–Roch theorem on X and the Kodaira vanishing theorem (III, 7.15).]

Let X span $\mathbb{P}^{n+1}$ of degree $d \leq 2n$ since it's not in a hyperplane. Let H be the hyperplane section of genus g. $|H|_H$ has degree d and projective dimension n. Using Clifford's theorem and Riemann-Roch either $h^1\left(\mathcal{O}_H(d)\right) > 0$ and $d \geq 2n,\ g-1 \geq n$ or $h^1\left(\mathcal{O}_H(H)\right) = 0$ and $n+1 \leq h^0\left(\mathcal{O}_H(d)\right) = 1 - g + d$.

Assuming F is not ruled, then by thm V.6.2, 6.3, there is $r > 0$, $D \geq 0$ with $D \sim rK_X$. By adjunction, $2g-2 = d + \frac{1}{r}H.D \geq d$ so the second case is impossible since then multiplying by 2 gives $2n+2 \leq 2-2g+2d$ rearranging contradicts $d \leq 2n$. Then $d = 2n$, $g = n+1$ so $d = 2(g-1) = 2g-2$, so by the genus formula, $n+1 = p_a(H) = 1 + \frac{1}{2}\left(H^2 + HK\right) = 1 + \frac{1}{2}(2n + HK)$ so $H.K = 0$. Since X is non-ruled then $|12K| \neq \emptyset$ by thm 6.2. But then $12K \sim 0$ so at least $\kappa = 0$. As $H - K$ is ample by Nakai Moishezon, then $h^1\left(\mathcal{O}_X(K-H)\right) = 0$ by Kodaira Vanishing so that $h^1\left(\mathcal{O}_X(H)\right) = 0$ by Serre duality. Using Serre duality and exc V.1.1, the degree of $K - H < 0$ so $l(K-H) = 0$ and now using thm V.1.6, Riemann-roch we get that $p_a(X) = 1$ so by thm V.6.3, X is $K3$. On the other hand, X is ruled.

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