1 Sheaves

Exercise 1.1. *Let A be an abelian group, and define the* constant presheaf *associated to A on the topological space X to be the presheaf $U \mapsto A$ for all $U \neq \varnothing$, with restriction maps the identity. Show that the constant sheaf $\mathcal{A}$ defined in the text is the sheaf associated to this presheaf.*

Solution. Let $\mathcal{A}^{pre}$ be the constant presheaf. There is an obvious morphism of sheaves $\mathcal{A}^{pre} \to \mathcal{A}$ which sends an element $a \in \mathcal{A}^{pre}(U) = A$ to the constant map $U \to A$. This induces a morphism from the sheafification of $\mathcal{A}^{pre}$ to $\mathcal{A}$ which we claim is an isomorphism. To see that it is an isomorphism we need only check the stalks, and since stalks are preserved under sheafification, we need only check that $\mathcal{A}^{pre} \to \mathcal{A}$ induces an isomorphism on the stalks. Clearly, the stalks of $\mathcal{A}^{pre}$ are A. Now consider a representative of the stalk of $\mathcal{A}$ at P. That is, an open set $U \ni P$ and section $s : U \to A$. The preimage $s^{-1}(s(P))$ of the value of s at P is an open subset of U on which the restriction of s is constant. Hence, every element of the stalk can be represented using a constant section and therefore $\mathcal{A}_P = A$.

Exercise 1.2. a *For any morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$ show that for each point P, $(\ker \phi)_P = \ker(\phi_P)$ and $(\operatorname{im} \phi)_P = \operatorname{im}(\phi_P)$.*

b *Show that ϕ is injective (respectively, surjective) if and only if the induced map on the stalks ϕ_P is injective (respectively, surjective) for all P.*

c *Show that a sequence $\cdots \to \mathscr{F}^{i-1} \to \mathscr{F}^i \to \mathscr{F}^{i+1} \to \dots$ of sheaves and morphisms is exact if and only if for each $P \in X$ the corresponding sequence of stalks is exact as a sequence of abelian groups.*

Solution. a Recall that filtered colimits commute with finite limits in the category of sets. Now kernel is a finite limit and stalk is a filtered colimit. Image is the kernel of $\mathscr{G} \to \operatorname{coker} \phi$.

b ϕ is injective if and only if its kernel is zero, and *phi* is surjective if and only if its cokernel is zero if and only if its image is $\mathscr{G}$. So these follow from part (a).

c Exactness can be stated as $\operatorname{im} \phi^i = \ker \phi^{i+1}$ and so it follows from part (a).

Exercise 1.3. a *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves on X. Show that ϕ s surjective if and only if the following condition holds: for every open set $U \subseteq X$, and for every $s \in \mathscr{G}(U)$, there is a covering $\{U_i\}$ of U, and there are elements $t_i \in \mathscr{F}(U_i)$, such that $\phi(t_i) = s|_{U_i}$ for all i.*

b *Give an example of a surjective morphism of sheaves $\phi : \mathscr{F} \to \mathscr{G}$, and an open set U such that $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is not surjective.*

Solution. a This is equivalent to saying that ϕ is surjective on each stalk.

b Consider the sheaf of holomorphic functions on $\mathbb{C} - \{0\}$ and the map $f \mapsto exp(f)$. For every holomorphic function defined on some open set of $\mathbb{C} - \{0\}$ we can write it locally as $f = \log g$ for some f so this morphism is surjective on stalks. Globally, we cannot.

Exercise 1.4. *a Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of presheaves such that $\phi(U) : \mathscr{F}(U) \to \mathscr{G}(U)$ is injective for each U. Show that the induced map $\phi^+ : \mathscr{F}^+ \to \mathscr{G}^+$ of associated sheaves is injective.*

b Use part (a) to show that if $\phi : \mathscr{F} \to \mathscr{G}$ is a morphism of sheaves, then $\operatorname{im} \phi$ can be naturally identified with a subsheaf of $\mathscr{G}$, as mentioned in the text.

Solution. a Sheafification preserves stalks, now use Exercise 1.2(a).

b The image is defined as the sheafification of the "presheaf image" which is certainly a subpresheaf of $\mathscr{G}$. Sheafification preserves injective morphisms and sheaves and so the image is a subsheaf of $\mathscr{G}$.

Exercise 1.5. *Show that a morphism of sheaves is an isomorphism if and only if it is both injective and surjective.*

Solution. Exercise 1.2(b) and Proposition 1.1.

Exercise 1.6. *a Let $\mathscr{F}'$ be a subsheaf of a sheaf $\mathscr{F}$. Show that the natural map of $\mathscr{F}$ to the quotient sheaf $\mathscr{F}/\mathscr{F}'$ is surjective, and has kernel $\mathscr{F}'$. Thus there is an exact sequence*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}/\mathscr{F}' \to 0$$

b Conversely, if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence show that $\mathscr{F}'$ is isomorphic to a subsheaf of $\mathscr{F}$, and that $\mathscr{F}''$ is isomorphic to the quotient of $\mathscr{F}$ by this subsheaf.

Solution. a Sheafification is a left adjoint and therefor preserves colimits (i.e. preserves surjections).

b The forgetful functor is the right adjoint to sheafification. Since it is a right adjoint it preserves kernels and so $0 \to \mathscr{F}' \to \mathscr{F}$ is exact as a sequence of presheaves. That is, $\mathscr{F}'$ is a subsheaf of $\mathscr{F}$. Then use part (a).

Exercise 1.7. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a morphism of sheaves.*

a Show that $\operatorname{im} \phi \cong \mathscr{F} / \ker \phi$.

b Show that $\operatorname{coker} \phi \cong \mathscr{G} / \operatorname{im} \phi$.

Solution. Follows from Exercises 1.6 and 1.4(b).

Exercise 1.8. *For any open subset $U \subseteq X$, show that the functor $\Gamma(U, -)$ from sheaves on X to abelian groups is a left exact functor.*

Solution. Since U is an open subset, there is a morphism of sites $i : X_{open} \to U_{open}$ with underlying functor the inclusion $U_{open} \to X_{open}$. We also have a morphism of sites defined by the continuous morphism $p : U \to \bullet$ of U to a point. Global sections of a sheaf $\mathscr{F}$ on U is then $p_* i_* \mathscr{F}$. Since both p_* and i_* are right adjoints, they are left exact.

Exercise 1.9. Direct sum. *Let $\mathscr{F}$ and $\mathscr{G}$ be sheaves on X. Show that the presheaf $U \mapsto \mathscr{F}(U) \oplus \mathscr{G}(U)$ is a sheaf.*

Solution. A consequence of the forgetful functor preserving limits.

Exercise 1.10. Direct Limit. *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limite to be the sheaf associated to the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$. Show that this is a diret limit in the category of sheaves on X.*

Solution. Sheafification is a left adjoint and so preserves colimits.

Exercise 1.11. *Let $\{\mathscr{F}_i\}$ be a direct system of sheaves on a noetherian topological space X. In this case show that the presheaf $U \mapsto \varinjlim \mathscr{F}_i(U)$ is already a sheaf. In particular, $\Gamma(X, \varinjlim \mathscr{F}_i) = \varinjlim \Gamma(X, \mathscr{F}_i)$.*

Solution. This is again a consequence of finite limits commuting with filtered colimits. Since each $\mathscr{F}_n$ is a sheaf, given an open set U and a cover $\{U_i \to U\}$ we can write $\mathscr{F}_n(U)$ as the limit $\varprojlim \mathscr{F}_n(U_{ij})$ where the limit is indexed by double intersections with inclusions as morphisms.

If the space is Noetherian, then we can choose the cover to be finite, hence, the limit is finite. So now for any open cover $\{U_i \to U\}$ (which we can choose to be finite)

$$\begin{aligned} \varprojlim_{ij}(\varinjlim_n \mathscr{F}_n)(U_{ij}) &= \varprojlim_{ij}(\varinjlim_n \mathscr{F}_n(U_{ij})) = \varinjlim_n(\varprojlim_{ij} \mathscr{F}_n(U_{ij})) \\ &= \varinjlim_n \mathscr{F}_n(U) = (\varinjlim_n \mathscr{F}_n)(U) \end{aligned}$$

Exercise 1.12. Inverse limit. *Let $\{\mathscr{F}_i\}$ be an inverse system of sheaves on X. Show that the section wise inverse limit is a sheaf.*

Solution. Same as the previous solution but since arbitrary limits commute, we don't need to assume the cover to be finite.

Exercise 1.13. Espace Étalé of a Presheaf. *Show that the sheaf $\mathscr{F}^+$ associated to a presheaf $\mathscr{F}$ can be described as follows: for any open sets $U \subseteq X$, $\mathscr{F}^+(U)$ is the set of continuous sections of Spé($\mathscr{F}$) over U.*

Solution. Let U be an open subset of X, and consider $s \in \mathscr{F}^+(U)$. We must show that $s : U \to$Spé($\mathscr{F}$) is continuous. Let $V \subseteq$Spé($\mathscr{F}$) be an open subset and consider the preimage $s^{-1}V$. Suppose $P \in X$ is in the preimage of V. Since $s(Q) \in \mathscr{F}_Q$ for each point $Q \in X$, we see that $P \in U$. This means that there is an open neighbourhood U' of P contained in P and a section $t \in \mathscr{F}(U')$ such that for all $Q \in U'$, the germ $t_{U'}$ of t at U' is equal to $s(U')$. That is, $s|_{U'} = t$.

So we have $s|_{U'}^{-1}(V) = t^{-1}(V)$, which is open since by definition of the topology on Spé($\mathscr{F}$), t is continuous. So there is an open neighbourhood $t^{-1}(V)$ of P that is contained in the preimage. The point P was arbitrary, and so we have shown that every point in the preimage $s^{-1}V$ has an open neighbourhood contained in the preimage $s^{-1}V$. Hence, it is the union of these open neighbourhoods, and therefore open itself. So s is continuous.

Now suppose that $s : U \to$Spé($\mathscr{F}$) is a continuous section. We want to show that s is a section of $\mathscr{F}^+(U)$. First we show that for any open V and any $t \in \mathscr{F}(V)$, the set $t(V) \subset$Spé($\mathscr{F}$) is open. To see this, recall that the topology on Spé($\mathscr{F}$) is defined as the strongest such that every morphism of this kind is continuous. If we have a topology $\mathscr{U}$ (where $\mathscr{U}$ is the collection of open sets) on Spé($\mathscr{F}$) such that each $t \in \mathscr{F}(U)$ is continuous, and $W \in$Spé($\mathscr{F}$) has the property that $t^{-1}W$ is open in X for any $t \in \mathscr{F}(V)$ and any open V, then the topology generated by $\mathscr{U} \cup \{W\}$ also has the property that each $t \in \mathscr{F}(U)$ is continuous. So since we are taking the strongest topology such that each $t \in \mathscr{F}(U)$ is continuous, if a subset $W \subset$Spé($\mathscr{F}$) has the property that $t^{-1}W$ is open in U for each $t \in \mathscr{F}(U)$, then W is open in Spé($\mathscr{F}$). Now fix one $s \in \mathscr{F}(U)$ and consider $t \in F(V)$. For a point $x \in t^{-1}s(U)$, it holds that $s(x) = t(x)$. That is, the germs of t and s are the same at x. This means that there is some open neighbourhood W of x contained in both U and V such that $s|_W = t|_W$, and hence $s = t$ for every $y \in W$ so $W \subset t^{-1}s(U)$. Since every point in $t^{-1}s(U)$ has an open neighbourhood in $t^{-1}s(U)$, we see that $t^{-1}s(U)$ is open and therefore by the reasoning just discussed we see that $s(U)$ is open in Spé($\mathscr{F}$).

Now let $s : U \to$Spé($\mathscr{F}$) be a continuous section. We want to show that s is a section of $\mathscr{F}^+(U)$. For every point $x \in U$, the image of x under s is some germ (t, W) in the stalk of $\mathscr{F}$ at x. That is, an open neighbouhood W of x (which we can assume is contained in U) and $t \in \mathscr{F}(W)$. Since s is continuous, and we have seen that $t(W)$ is open, it follows that $s^{-1}(t(W))$ is open in X. This means there is an open neighbourhood W' of x on which $t|_{W'} = s|_{W'}$. Since s is locally representable by sections of $\mathscr{F}$, it is a well defined section of $\mathscr{F}^+$.

Exercise 1.14. Support. *Let $\mathscr{F}$ be a sheaf on X, and let $s \in \mathscr{F}(U)$ be a section over an open set U. The* support *of s, denoted* Supp s*, is defined to be $\{P \in U | s_P \neq 0\}$, where s_P denotes the germ of s in the stalk $\mathscr{F}_P$. Show that* Supp s *is a closed subset of U. We define the* support *of $\mathscr{F}$,* Supp $\mathscr{F}$*, to be $\{P \in X | \mathscr{F}_P \neq 0\}$. It need not be a closed subset.*

Solution. We show that the complement of the support V is open. For every point $P \in V$, since P is not in the support the germ of the section s is zero. This means there is a neighourhood V_P of P on which s vanishes. Note that $V_P \cap \operatorname{Supp} s = \varnothing$ since an intersection would imply $s_Q = 0$ for all Q in the intersection. Now $V = \cup V_P$, a union of opens, therefore V is open.

An example of a sheaf whose support is not closed is $j_!\mathscr{F}$ from Exercise 1.19(b).

Exercise 1.15. Sheaf $\mathscr{H}om$. *Let $\mathscr{F}$, $\mathscr{G}$ be sheaves of abelian groups on X. For any open set $U \subseteq X$, show that the set* $\hom(\mathscr{F}|_U, \mathscr{G}|_U)$ *of morphisms of* *the restricted sheaves has a natural structure of abelian group. Show that the presheaf* $U \mapsto \hom(\mathscr{F}|_U, \mathscr{G}|_U)$ *is a sheaf.*

Solution. Suppose we have an open set U, a cover $\{U_i \to U\}$, and a set of natural transformations $\phi_i : \mathscr{F}|_{U_i} \to \mathscr{G}|_{U_i}$, that agree on restrictions to the U_{ij}. We define a natural transformation $\mathscr{F}|_U \to \mathscr{G}|_U$. Given an open subset $V \subset U$ we want a morphism $\mathscr{F}(V) \to \mathscr{G}(V)$. Since $\{V \cap U_i \to V\}$ is a cover of V, we can write $\mathscr{F}(V)$ and $\mathscr{F}(G)$ as a limit over $\{V \cap U_{ij}\}$ and we already have morphisms $\mathscr{F}(V \cap U_{ij}) \to \mathscr{G}(V \cap U_{ij})$ from our initial data. It doesn't matter which morphism we choose since the requirement that the ϕ_i agree on restrictions means they will be the same. So now we have a morphism $\mathscr{F}(V) \to \mathscr{G}(V)$ between the limits, it remains to show that these actually form a natural transformation, but this can be seen by drawing the appropriate diagram

$$\begin{tikzcd}
F(V) \arrow[rr] \arrow[dr] \arrow[dd] & & \prod F(V_i) \arrow[rr] \arrow[dr] \arrow[dd] & & \prod F(V_{ij}) \arrow[dr] \arrow[dd] & \\
& F(W) \arrow[rr] \arrow[dd] & & \prod F(W_i) \arrow[rr] \arrow[dd] & & \prod F(W_{ij}) \arrow[dd] \\
G(V) \arrow[rr] \arrow[dr] & & \prod G(V_i) \arrow[rr] \arrow[dr] & & \prod G(V_{ij}) \arrow[dr] & \\
& G(W) \arrow[rr] & & \prod G(W_i) \arrow[rr] & & \prod G(W_{ij})
\end{tikzcd}$$

Exercise 1.16. Flasque Sheaves. *A sheaf* $\mathscr{F}$ *on a topological space* X *is* flasque *if for every inclusion* $V \subseteq U$ *of open sets, the restriction map* $\mathscr{F}(U) \to \mathscr{F}(V)$ *is surjective.*

a Show that a constant sheaf on an irreducible topological space is flasque.

b If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathscr{F}'$ *is flasque, then for any open set* U, *the sequence* $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ *of abelian groups is also exact.*

c If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of sheaves, and if* $\mathscr{F}'$ *and* $\mathscr{F}$ *are flasque, then* $\mathscr{F}''$ *is flasque.*

d If $f : X \to Y$ *is a continuous map, and if* $\mathscr{F}$ *is a flasque sheaf on* X, *then* $f_*\mathscr{F}$ *is a flasque sheaf on* Y.

e Let $\mathscr{F}$ *be any sheaf on* X. *Show that the sheaf of discontinuous sections dis* $\mathscr{F}$ *is flasque and that there is a natural injective morphism of* $\mathscr{F}$ *into des* $\mathscr{F}$.

Solution. a If X is irreducible then every open set is connected and we have already seen that a constant sheaf takes every connected open subset to the same set/group. So all the restrictions are identity morphisms and hence, it is flasque.

b The only thing to prove is that $\mathscr{F}(U) \to \mathscr{F}''(U)$ is surjective. Let $f \in \mathscr{F}''(U)$. Since the morphism $\mathscr{F} \to \mathscr{F}''$ is a surjective morphism of sheaves there is a cover $\{U_i\}$ of U on which the restriction of f lifts to an element $\{f_i\}$ of $\mathscr{F}(U_i)$.

$$\begin{tikzcd}
\prod_{ij} \mathscr{F}'(U_{ij}) \arrow[r] & \prod_{ij} \mathscr{F}(U_{ij}) \arrow[r] & \prod_{ij} \mathscr{F}''(U_{ij}) \\
\prod_{i} \mathscr{F}'(U_{i}) \arrow[r] \arrow[u] & \prod_{i} \mathscr{F}(U_{i}) \arrow[r] \arrow[u] & \prod_{i} \mathscr{F}''(U_{i}) \arrow[u] \\
\mathscr{F}'(U) \arrow[r] \arrow[u] & \mathscr{F}(U) \arrow[r] \arrow[u] & \mathscr{F}''(U) \arrow[u]
\end{tikzcd}$$

Since f is a global section on U, the restriction to $\prod \mathscr{F}''(U_{ij})$ is zero and so the element $\{f_i|_{U_{ij}} - f_j|_{U_{ij}}\}$ gets sent to zero horizontally. Since sectionwise exactness in the middle is given automatically, this element pulls back horizontally to some $\{g_{ij}\}$. We have assumed $\mathscr{F}'$ to be flasque and so there is a $\{g_i\}$ in the preimage of $\{g_{ij}\}$.

$$\begin{tikzcd}
\{g_{ij}\} \arrow[r, dotted] & \{f_i|_{U_{ij}} - f_j|_{U_{ij}}\} \arrow[r, dotted] & 0 & \cdot & 0 & \cdot \\
\{g_i\} \arrow[u, dotted] & \{f_i\} \arrow[u, dotted] \arrow[r, dotted] & \{f|_{U_i}\} \arrow[u, dotted] & \cdot & \{f_i - g_i\} \arrow[u, dotted] \arrow[r, dotted] & \{f|_{U_i}\} \\
\cdot & \cdot & f \arrow[u, dotted] & \cdot & h \arrow[u, dotted] & \cdot
\end{tikzcd}$$

Now by commutivity of the diagram, $\{g_i - f_i\} \in \prod_i \mathscr{F}(U_i)$ is in the vertical kernel, and therefore lifts to some global section $h \in \mathscr{F}(U)$. Now if we push h up and to the right we get the restriction of f. So the image of h in $\mathscr{F}''(U)$ hasthe same restriction as f. Since $\mathscr{F}''$ is a sheaf this means that $h = f$ in $\mathscr{F}''(U)$. So we have found an element in the preimage of f.

c Let $V \subseteq U$ be open sets in the topological space X and consider the following diagram.

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathscr{F}'(V) \arrow[r] \arrow[u] & \mathscr{F}(V) \arrow[r] \arrow[u] & \mathscr{F}''(V) \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{F}'(U) \arrow[r] \arrow[u] & \mathscr{F}(U) \arrow[r] \arrow[u] & \mathscr{F}''(U) \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

We know that the rows are exact by the previous part, and the columns are exact by the assumption that $\mathscr{F}'$ and $\mathscr{F}$ are flasque. It is now a straightforward diagram chase to find a preimage in $\mathscr{F}''(U)$ to anything in $\mathscr{F}''(V)$.

d For opens $V \subseteq U$ of Y we have

$$\big(f_*\mathscr{F}(V) \to f_*\mathscr{F}(U)\big) = \big(\mathscr{F}(f^{-1}V) \to \mathscr{F}(g^{-1}U)\big)$$

which is surjective.

e In general, for a subset I of set J indexing objects in some category C, the morphism $\prod_J X_j \to \prod_I X_i$ induced by the inclusion $I \subset J$ is surjective. The restriction morphisms are special cases of this where the indexing sets are points in the opens. The natural injective morphism $\mathscr{F} \to dis\ \mathscr{F}$ is clear. It is injective since two sections of a sheaf are the same if and only if they agree on stalks. If $\mathscr{F}$ were to be a nonseparated presheaf the morphism wouldn't be injective.

Exercise 1.17. Skyscraper Sheaves. *Let $i_P(A)$ be the skyscraper sheaf of a space X at a point P for an abelian group A. Verify that the stalk of $i_P(A)$ is A at every point $Q \in \overline{\{P\}}$ and 0 elsewhere. Show that this sheaf can be described as $i_*(A)$, the pushforward of the constant sheaf A on $\overline{\{P\}}$.*

Solution. Let Q be a point. If $Q \notin \overline{\{P\}}$ then there is a neighbourhood $U \ni Q$ which doesn't contain P. Every element in the stalk can be represented by (s, W) where $W \subseteq U$ and there for $s = 0$. So the stalk is zero. Conversely, if $Q \in \overline{P}$, then every open neighbourhood of Q contains P, and so every group in the limit defining the stalk is A, with transition morphisms identities. Therefore the stalk is A.

Let $\mathcal{A}$ be the constant sheaf of $\overline{\{P\}}$. If $U \subset X$ doesn't contain P then $i^{-1}U = \varnothing$ and so $i_*(A)(U) = \mathcal{A}(i^{-1}U) = \mathcal{A}(\varnothing) = 0$. If U does contain P, then $i^{-1}U = \overline{\{P\}}$ and so $i_*(A)(U) = \mathcal{A}(i^{-1}U) = \mathcal{A}(\overline{\{P\}}) = \Gamma(\mathcal{A}, \overline{\{P\}})$. It remains only to show that $\overline{\{P\}}$ is connected so that $\Gamma(\mathcal{A}, \overline{\{P\}}) = A$ but this follows from it having a unique generic point.

Exercise 1.18. Adjoint Property of f^{-1}. *Show that f^{-1} is the left adjoint to f_*.*

Solution. If we denote f^{-1}_{pre} the functor that sends a sheaf $\mathscr{F}$ to the PREsheaf $U \mapsto \varinjlim_{V \supset f(U)} \mathscr{F}(V)$ then we have

$$\hom_{Sh(Y)}(af^{-1}_{pre}\mathscr{F}, \mathscr{G}) \cong \hom_{PreSh(Y)}(f^{-1}_{pre}\mathscr{F}, \mathscr{G}) \cong \hom_{Sh(X)}(\mathscr{F}, f_*\mathscr{G})$$

The theory of Kan extensions shows that f^{-1}_{pre} is the left adjoint to f_* and we already know that sheafification a is the left adjoint to the forgetful functor. So the composition $f^{-1} = af^{-1}_{pre}$ is left adjoint to the "composition" f_*.

Exercise 1.19. Extending a Sheaf by Zero. *Let X be a topological space, let Z be a closed subset, let $i : Z \to X$ be the inclusion, let $U = X - Z$ be the complementary open subset, and let $j : U \to X$ be its inclusion.*

a *Let $\mathscr{F}$ be a sheaf on Z. Show that the stalk $(i_*\mathscr{F})_P$ of the direct image sheaf on X is $\mathscr{F}_P$ if $P \in Z$, 0 if $P \notin Z$.*

b *Let $\mathscr{F}$ be a sheaf on U. Show that the stalk $(j_!(\mathscr{F}))_P$ is equal to $\mathscr{F}_P$ if $P \in U$, 0 if $P \notin U$, and show that $j_!\mathscr{F}$ is the only sheaf on X which has this property, and whose restriction to U is $\mathscr{F}$.*

c *For $\mathscr{F} \in Sh(X)$ show that there is an exact sequence of sheaves on X*

$$0 \to j_!(\mathscr{F}|_U) \to \mathscr{F} \to i_*(\mathscr{F}|_Z) \to 0$$

Solution. a If $P \notin Z$ and $(U, s) \in (i_*\mathscr{F})_P$ then there is an open subset V of U containing P but not intersecting Z. Since V doesn't intersect Z, $(i_*\mathscr{F})(V) = 0$ and so $s|_V = 0$, hence $(U, s) = 0$. Now suppose that $P \in Z$ via $k : P \to Z$. The stalk $(i_*\mathscr{F})_P$ is the group of global sections of $(ik)^*(i_*\mathscr{F}) = k^*i^*i_*\mathscr{F} = k^*\mathscr{F}$ which is the stalk $\mathscr{F}_P$.

b If $P \notin U$ then every open set containing P is not contained in U and so every group in the diagram that defines the limit $(j_!(\mathscr{F}))_P$ is zero. Hence $(j_!(\mathscr{F}))_P$ is zero. Alternatively, if $P \in U$ then for every open set $P \in V$ there is an open set $P \in V' \subseteq U$ and so every element (V, s) of the stalk is equivalent to an element $(V', s|_{V'})$ of the stalk $\mathscr{F}_P$.

c We just need to show that the sequence is exact on each of the stalks. From the previous two parts of the exercise however, depending on whether P is in U or Z we either get an isomorphism followed by a zero object, or the zero object followed by an isomorphism. So the sequence is exact on the stalks.

Exercise 1.20. Subsheaf with Supports. *Let Z be a closed subset of X, and let $\mathscr{F}$ be a sheaf on X. We define $\Gamma_Z(X, \mathscr{F})$ to be the subgroup of $\Gamma(X, \mathscr{F})$ consisting of all sections whose support is contained in Z.*

a *Show that the presheaf $V \mapsto \Gamma_{Z\cap V}(V, \mathscr{F}|_V)$ is a sheaf. It is denoted $\mathscr{H}^0_Z(\mathscr{F})$.*

b *Let $U = X - Z$, and let $j : U \to X$ be the inclusion. Show that there is an exact sequence of sheaves on X*

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*(\mathscr{F}|_U)$$

Furthermore, if $\mathscr{F}$ is flasque, the map $\mathscr{F} \to j_*(\mathscr{F}|_U)$ is surjective.

Solution. a Since it is a subpresheaf of a sheaf we know that it is separated. Let U be an open subset of X and $\{U_i\}$ a cover of U. Suppose that s_i is a section of $\Gamma_{Z\cap U_i}(U_i, \mathscr{F}|_{U_i})$ for each i and that the restrictions of s_i and s_j to U_{ij} agree. Since all this takes place in a subpresheaf of a sheaf there is some $s \in \mathscr{F}(U)$ whose restriction to the U_i are s_i. The only thing to check is that s has support inside Z. Suppose that $P \in U\backslash Z$. Since the

U_i cover U, the point P is in one of the U_i. Since s_i is the restriction of s to U_i, the germ s_P agrees with $(s_i)_P$ which is zero. Hence, $s_P = 0$ for all $P \in U \backslash Z$. So $s \in \Gamma_{Z \cap U}(U, \mathscr{F}|_U)$ and hence, $\mathscr{H}^0_Z(\mathscr{F})$ is a sheaf.

b First note that if $U \cap Z = \varnothing$ then $\mathscr{H}^0_Z(\mathscr{F})(U) = 0$ since $\mathscr{H}^0_Z(\mathscr{F})(U)$ is the group of sections whose support is contained in Z, but $Z \cap U = \varnothing$ and so $\mathscr{H}^0_Z(\mathscr{F})(U)$ is the group of sections with empty support. Since $\mathscr{F}$ is a sheaf, any section whose germ is zero at every point is trivial itself, and so $\mathscr{H}^0_Z(\mathscr{F})(U) = 0$.

Now for a point $P \notin Z$, any section of the stalk (V, s) can be represented by $(V', s|_{V'})$ with $V' \cap Z = \varnothing$ (take $V' = V \cap Z^c$). But this means that $s|_{V'} = 0$ and so the stalk of $\mathscr{H}^0_Z(\mathscr{F})$ ar $P \notin Z$ is zero. As has been noted in the previous exercise, for $P \in U$ the stalks $\mathscr{F}_P$ and $(j_*(\mathscr{F}|_U))_P$ are the same, and so the sequence is exact on stalks at these points.

Now consider a point $P \in Z$ and an element $(V, s) \in \mathscr{F}_P$. If this element gets sent to zero it means that there is some open subset $V' \subseteq V$ such that $s|_{V'}$ is zero in $j_*(\mathscr{F}|_U)(V') = \mathscr{F}(U \cap V')$. If $U \cap V' \neq \varnothing$ we have $(V, s) \sim (U \cap V', s|_{U \cap V'})$ and so our original germ was trivial. If $U \cap V' = \varnothing$ then $V' \subseteq Z$ and so $(V', s|_{V'})$ is an element of $(\mathscr{H}^0_Z(\mathscr{F}))_P$.

Now consider an element (V, s) of $(\mathscr{H}^0_Z(\mathscr{F}))_P$, with $P \in Z$ still, and its image in $(j_*(\mathscr{F}|_U))_P$. This is (V, t) where t is the image of s under the map $\mathscr{H}^0_Z(\mathscr{F})(V) \to j_*(\mathscr{F}|_U)(V) = \mathscr{F}(U \cap V)$ which is essentially restriction of s to $U \cap V$. Since the support of S is contained in Z, restricting it to something contained in $U = Z^c$ will give zero. Hence, $(V, t) = 0$. So the sequence is exact in the middle term.

As has just been noted, for any open set V the morphism $\mathscr{F}(V) \to j_*(\mathscr{F}|_U) = F(U \cap V)$ is restriction, and so if $\mathscr{F}$ is flasque, the rightmost arrow is surjective as a morphism of presheaves. This means that it is also surjective as a morphism of sheaves.

Exercise 1.21. Some Examples of Sheaves on Varieties. *Let X be a variety over an algebraically closed field k. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.*

a *Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathscr{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathscr{I}_Y(U)$ is a sheaf.*

b *If Y is a subvariety, then the quotient sheaf $\mathcal{O}_Y/\mathscr{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$ where $i : Y \to X$ is the inclusion and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.*

c *Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $P, Q \in X$. Then there is an exact sequence of sheaves on X, where $\mathscr{F} = i_*\mathcal{O}_P \oplus i_*\mathcal{O}_Q$*

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathscr{F} \to 0$$

Show however that the induced map on global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{F})$ is not surjective.

d *Again let $X = \mathbb{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathscr{K}$ be the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathscr{K}$. Show that the quotient sheaf $\mathscr{K}/\mathcal{O}$ is isomorphic to the direct sum of sheaves $\sum_{P \in X} i_P(I_P)$ where I_p is the group $K/\mathcal{O}_P$, and $i_P(I_P)$ denotes the skyscraper sheaf given by I_P at the point P.*

e *Finally show that in the case of (d) the sequence*

$$0 \to \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}_X) \to 0$$

is exact.

Solution. a Let $\{U_i\}$ be an open cover of an open subset U, and suppose we are given sections $f_i \in \mathscr{I}_Y(U_i)$ that agree on their restrictions to the intersections $U_i \cap U_j$. Since $\mathscr{I}_Y$ is a subpresheaf of a sheaf, we know that we can find a section $f \in \mathcal{O}_X(U)$ whose restrictions to U_i are the f_i, we just need to check that it is indeed in $\mathscr{I}_Y(U)$. That is, that the function $f : U \to k$ vanishes at all points of $Y \cap U$. If P is a point of $Y \cap U$ then since $\{U_i\}$ is a cover, P is contained in some U_i. The restriction of f to U_i is f_i, so $f(P) = f_i(P)$ which is zero since $f_i \in \mathscr{I}_Y(U_i)$. Hence, f vanishes at all points of $U \cap Y$ and is therefore in $\mathscr{I}_Y(U)$.

The fact that $\mathscr{I}_Y$ is a separated presheaf comes from the fact that every presheaf of a separated presheaf is separated, and every sheaf is separated.

b Let U be an open subset of X. If $f \in \mathcal{O}_X(U)$ is a regular function on U, then it is a function $U \to k$ that is locally representable as a quotient of polynomials. Restricting to $Y \cap U$ gives a section of $\mathcal{O}_Y(U \cap Y) = i_*(U \cap Y)$ and so we obtain a morphism $\mathcal{O}_X \to i_*\mathcal{O}_Y$. We want to see that the sequence

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0$$

is exact. It follows from the definition of $\mathscr{I}_Y$ that the sequence is exact at $\mathscr{I}_Y$ and at $\mathcal{O}_X$. To see exactness at $i_*\mathcal{O}_Y$, consider a point $P \in X$ and the morphism of stalks $(\mathcal{O}_X)_P \to (i_*\mathcal{O}_Y)_P$. If $P \notin Y$ then $(i_*\mathcal{O}_Y)_P$ is zero, as there is an open neighbourhood of P not intersecting Y, so the morphism of stalks is surjective. Suppose $P \in Y$. An element of $(i_*\mathcal{O}_Y)_P$ is represented by a rational function on the ambient affine or projective space, which doesn't have a pole at P. This ambient space also includes X, and so this rational function also represents an element of $(\mathcal{O}_X)_P$. Hence, $(\mathcal{O}_X)_P \to (i_*\mathcal{O}_Y)_P$ is surjective, and so the sequence of sheaves is exact.

c Recall that $\mathbb{P}^1$ is the set of linear subspaces of k^2. Since the projective general linear group is transitive on pairs of distinct points, we can assume that $P = (0,1)$, and $Q = (1,1)$ and therefore, the sequence will be exact if and only if it is exact on its restriction to $\mathbb{A}^1 = \{(0,a) | a \in k\}$ where $P = 0$ and $Q = 1$.

Now the sequence on the stalk at a point R falls into three cases: either $R = P$, $R = Q$ or $R \neq P, Q$. In case $R \neq P, Q$, there is an open set U containing R which does not contain P and Q: the complement of the closed subset defined by $x(x-1) \in k[x]$. On this open set we have $\mathscr{F}(U) = 0$, by definition the skyscraper sheaves, and $\mathscr{I}_Y(U) = \mathcal{O}_X(U)$, by definition of $\mathscr{I}_Y$. Hence, the sequence is exact for any point in U.

If $R = P, Q$ then the sequence is the same so suppose that $R = Q$. The stalk of $\mathcal{O}_X$ at Q is the ring of rational functions whose denominator doesn't vanish at Q. That is, $(\mathcal{O}_X)_Q = \{\frac{f}{g} | g(1) \neq 0\}$. The ideal $(\mathscr{I}_Y)_Q$ is the subset of functions whose numerator does vanish at Q, that is $(\mathscr{I}_Y)_Q = \{\frac{f}{g} | g(1) \neq 0, f(1) = 0\}$. The quotient is isomorphic to k via evaluation at 1, which is the stalk of $\mathscr{F}$ at Q. So the sequence is exact at Q, and by symmetry, also at P.

On global sections however, the sequence is

$$0 \to 0 \to k \to k \oplus k \to 0$$

which cannot be exact.

d By definition, a regular function on U is a function that is represented locally by a rational function, that is, a section of $\mathscr{K}(U)$. More explicitely, a regular function on U is a function $f : U \to k$, such that there is an open cover $\{U_i\}$ of U on which $f|_{U_i}$ is a rational function with no poles in U_i. Since the f_i are restrictions of f as functions, they agree as functions on the intersections U_{ij}, and therefore define a section of $\mathscr{K}(U)$, the sheafification of $U \mapsto K$.

The morphism $\mathscr{K} \to \sum_{P \in X} i_P(I_P)$ should be clear. To show exactness it is enough to show exactness on the stalks. The sequence on a stalk takes the form

$$0 \to \mathcal{O}_P \to \mathscr{K}_P \to (\sum_{Q \in X} i_Q(I_Q))_P \to 0$$

Since $\mathscr{K}$ is a constant sheaf, it takes the value K at every stalk. On the right, we have a sum of stalks of skyscraper sheaves, all of which vanish except $Q = P$ which by definition is $K/\mathcal{O}_P$. Hence, the sequence is

$$0 \to \mathcal{O}_P \to K \to K/\mathcal{O}_P \to 0$$

which is exact.

e The global sections functor is left exact so we only need to show that $\Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}_X)$ is surjective. Using the description of $\mathscr{K}/\mathcal{O}$ from the previous part as $\sum i_P(I_P)$, our task is the following: given a rational function $f \in K$ and a point P, find another rational function $f' \in K$ such that $f' \in \mathcal{O}_Q$ for every $Q \neq P$ and $f' - f \in \mathcal{O}_P$.

Using the isomorphism $K \cong k(x)$, we can write $f = \frac{\alpha(x)}{\beta(x)} = \frac{\prod_{i=1}^{n}(x-a_i)}{\prod_{i=1}^{m}(x-b_i)}$ and then the points in $\mathbb{A}^1 \subset \mathbb{P}^1$ for which $f \notin \mathcal{O}_Q$ are those corresponding to b_i, and $f \notin \mathcal{O}_\infty$ if $m < n$. Infact, write f as $f = x^{-\nu}\frac{\alpha}{\beta'}$ with $x \not| \alpha, \beta$. Since $PGL(1)$ is transitive on points, without loss of generality we can assume that our point P is $0 \in \mathbb{A}^1$. If $\nu \leq 0$ then choosing $f' = 1$ satisfies the required conditions. If $\nu > 0$, then choose $f' = \frac{\sum_{i=0}^{\nu} c_i}{x^\nu}$ with c_i defined iteratively via $c_0 = \frac{\alpha_0}{\beta_0}$ and $c_i = \beta_0^{-1}(a_i - \sum_{j=0}^{i-1} c_j \beta_{i-j})$ where α_i, β_i are the coefficients for $\alpha = \sum \alpha_i x^i$ and $\beta' = \sum \beta_i x^i$ respectively. Our thus chosen f' satisfies the requirement that $f' \in \mathcal{O}_Q$ for all $Q \neq P$ and so consider $f - f'$. We have $f - f' = \frac{\alpha}{x^\nu \beta'} - \frac{\sum_{i=0}^{\nu} c_i}{x^\nu} = \frac{\alpha - \beta' \sum_{i=0}^{\nu} c_i}{x^\nu \beta'}$. The ith coefficient of the numerator for $i \leq \nu$ is $\alpha_i - \sum_{j=0}^{i} c_j \beta_{i-j}$ which is zero due to our careful choice of the c_i. So the x^ν in the denominator vanishes and we see that $f - f' \in \mathcal{O}_P$ since $x \not| \beta'$.

Exercise 1.22. Glueing Sheaves. *Let X be a topological space, let $\{U_i\}$ be an open cover of X, and suppose we are given for each i a sheaf $\mathscr{F}_i$ on U_i, and for each i, j an isomorphism $\phi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \cong \mathscr{F}_j|_{U_i \cap U_j}$ such that (1) for each i we have $\phi_{ii} = id$, and (2) for each i, j, k we have $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$ on $U_i \cap U_j \cap U_k$. Then there exists a unique sheaf $\mathscr{F}$ on X, together with isomorphisms $\psi_i : \mathscr{F}|_{U_i} \cong \mathscr{F}_i$ such that for each i, j we ahve $\psi_j = \phi_{ij} \circ \psi_i$ on $U_i \cap U_j$.*

Solution. Let $\iota_i : U_i \to X$ and $\iota_{ij} : U_i \cap U_j \to X$ be the inclusions of the open sets and define $\mathscr{G}_i = \iota_{i*}\mathscr{F}_i$ and $\mathscr{G}_{ij} = \iota_{ij*}(\mathscr{F}_i|_{U_{ij}})$. Restriction induces morphisms $\mathscr{G}_i \to \mathscr{G}_{ij}$ and restriction composed with ϕ_{ji} gives morphisms $\mathscr{G}_j \to \mathscr{G}_{ij}$. Define $\mathscr{F} = \varprojlim \mathscr{G}$ to be the inverse limit of the system of $\mathscr{G}_i$'s and $\mathscr{G}_{ij}$'s. This comes naturally with morphisms $\mathscr{F} \to \mathscr{G}_i$. By considering stalks we see that the morphisms $\mathscr{F}|_{U_i} \to \mathscr{G}_i|_{U_i} = \mathscr{F}_i$ are isomorphisms, since on stalks, every morphism of the system that we took the limit over becomes either zero or an isomorphism, and the isomorphisms are compatible due to the cocycle condition. If there were to be another sheaf $\mathscr{F}'$ with isomorphisms as stated in the question, this would define a cone of the system. So there would be a morphism $\mathscr{F}' \to \mathscr{F}$ and considering stalks shows that this would be an isomorphism.

2 Schemes

Exercise 2.1. *Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.*

Solution. Let $\phi : A \to A_f$ be the obvious ring homomorphism. This evidently defines a scheme morphism $\operatorname{Spec} A_f \to \operatorname{Spec} A$.

The ideal generated by the image $\phi\mathfrak{p}$ of a prime ideal $\mathfrak{p}$ of A is the set $A_f\mathfrak{p} = \{\frac{a}{f^n} : n \in \mathbb{N}, a \in \mathfrak{p}\}$ which is prime in A_f, and its inverse image $\phi^{-1}(A_f\mathfrak{p})$ is $\mathfrak{p}$, unless $f \in \mathfrak{p}$ in which case $A_f\phi\mathfrak{p} = A_f$. So the morphism $\operatorname{Spec} A_f \to \operatorname{Spec} A$ is surjective onto the underlying space of $D(f)$.

Let $\mathfrak{p}, \mathfrak{q}$ be two points of $\operatorname{Spec} A_f$ that get sent to the same image in $\operatorname{Spec} A$. This means that all of their elements of the form $\frac{a}{1}$ are the same. Now $\frac{a}{f^n} \in \mathfrak{p}$ if and only iff $f^n \frac{a}{f^n} = a \in \mathfrak{p}$ if and only if $a \in \mathfrak{q}$ if and only if $\frac{1}{f^n} a = \frac{a}{f^n} \in \mathfrak{q}$ and so $\mathfrak{p} = \mathfrak{q}$. Hence, the morphism is injective on the underlying space of $\operatorname{Spec} A_f$.

This bijection of sets is continuous automatically since if comes from a ring homomorphism. To see that it is a homeomorphism we need to show that it is open. Let $\mathfrak{a} \subset A_f$ be an ideal and $\mathfrak{b} = (f) \cap \phi^{-1}\mathfrak{a} \subset A$. A prime ideal $\mathfrak{p} \in \operatorname{Spec} A_f$ is in the open complement of $V(\mathfrak{a})$ if and only if $\mathfrak{p} \not\supset \mathfrak{a}$ if and only if $\phi^{-1}\mathfrak{p} \not\supset \mathfrak{b}$ and conversely, $\mathfrak{q} \in \operatorname{Spec} A$ is in the open complement of $V(\mathfrak{b})$ if and only if $\mathfrak{q} \not\supset \mathfrak{b}$ if and only if $A_f\mathfrak{q} \not\supset \mathfrak{a}$ and so we have a homeomorphism.

It remains to show that the morphism of structure sheaves $\mathcal{O}_{\operatorname{Spec} A|_{D(f)}} \to F_*\mathcal{O}_{\operatorname{Spec} A_f}$ (where F is the scheme morphism) is an isomorphism. It is enough to check this on the stalks. Let $\mathfrak{p} \in D(f)$. The stalk of $\mathcal{O}_{\operatorname{Spec} A}$ is $A_\mathfrak{p}$ and the stalk of $F_*\mathcal{O}_{\operatorname{Spec} A_f}$ is $(A_f)_\mathfrak{p}$ where we confuse $\mathfrak{p}$ with its preimage in $\operatorname{Spec} A_f$. Since $f \neq \mathfrak{p}$ the morphism $A_\mathfrak{p} \to (A_f)_\mathfrak{p}$ induced by ϕ is clearly an isomorphism.

Exercise 2.2. *Let $(X, \mathcal{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathcal{O}_X|_U)$ is a scheme.*

Solution. Let $\operatorname{Spec} A_i$ be an affine open cover for X. The intersection of each $\operatorname{Spec} A_i$ with U is an oen subset of $\operatorname{Spec} A_i$ which is therefore covered by basic open affines $D(f_{ij})$. Hence, we obtain an open affine cover $\operatorname{Spec}(A_i)_{f_ij}$ for U.

The resoning behind the glueing here needs to be checked and/or made more explicit.

Exercise 2.3. Reduced Schemes.

a *Show that $(X, \mathcal{O}_X)$ is reduced if and only if for every $P \in X$, the local ring $\mathcal{O}_{X,P}$ has no nilpotent elements.*

b *Let $(X, \mathcal{O}_X)$ be a scheme. Show that $X_{red} \overset{red}{=} (X, (\mathcal{O}_X)_{red})$ is a scheme. Show that there is a morphism of schemes $X_{red} \to X$, which is a homeomorphism on the underlying topological spaces.*

c *Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced Show that there is a unique morphism $g : X \to Y_{red}$ such that f is obtained by composing g with the natural map $Y_{red} \to Y$.*

Solution. a Suppose that $(X, \mathcal{O}_X)$ is reduced. So $\mathcal{O}_X(U)$ has no nilpotent elements for each U. Let $P \in X$ be a point and consider a representative (U, s) of an element of the stalk. If this element is nilpotent, then there is some subneighbourhood $V \ni P$ of P on which s^n vanishes, but $\mathcal{O}_X(V)$ has no nilpotents, so s vanishes on V an therefore $(V, s|_V) = (U, s)$ is zero. Hence, the stalk has no nilpotents.

Suppose conversely, that each stalk has no nilpotents, and suppose that $s \in \mathcal{O}_X(U)$ is nilpotent, say $s^n = 0$. Then the germ of s^n is zero at each point in U. Since the stalks have no nilpotents, this means that the stalk of s vanishes at each point of U. But this means that $s = 0$ since a sheaf is a separated presheaf. So $\mathcal{O}_X(U)$ has no nilpotents.

b Suppose that $X = \operatorname{Spec} A$ is affine and denote by A_{red} the quotient $A/\mathfrak{N}$ where $\mathfrak{N} = \mathfrak{N}(A)$ is the nilradical of A. Since every prime ideal of A contains $\mathfrak{N}$, as topological spaces, sp $\operatorname{Spec} A = sp$ $\operatorname{Spec} A_{red}$. Now for a basic open affine $D(f)$ we have $\mathcal{O}_{\operatorname{Spec}(A_{red})}(D(f)) \cong (A/\mathfrak{N})_f \cong A_f/(\mathfrak{N}(A_f))$. That is, on a basic open affine U we have $\mathcal{O}_{\operatorname{Spec}(A_{red})}|_U \cong \mathcal{O}_{(\operatorname{Spec} A)_{red}}|_U$. Since the basic opens cover X this shows that $\operatorname{Spec}(A_{red}) \cong (X, (\mathcal{O}_X)_{red})$.

Now for a general scheme X, a cover of X with open affines $\operatorname{Spec} A_i$ gives a coer $\operatorname{Spec}(A_i)_{red}$ for $(X, (\mathcal{O}_X)_{red})$. Hence, the latter is a scheme.

The homomorphism $X \to X_{red}$ is induced on an affine cover $\operatorname{Spec} A_i \subset X$ by the ring morphisms $A_i \to A_i/\mathfrak{N}(A_i)$. We have already seen that it is a homeomorphism on the underlying topological spaces.

c Let $V_i = \operatorname{Spec} B_i$ be an open affine cover for Y, and let $U_{ij} = \operatorname{Spec} A_{ij}$ be an open affine cover of $f^{-1}V_i$. As in the previous part $V_i^{red} = \operatorname{Spec} B_i^{red}$ is an open affine cover for Y_{red} and the morphism $Y_{red} \to Y$ is induced by the ring homomorphisms $B_i \to B_i^{red}$. Now since each A_{ij} is reduced, $\mathfrak{N}(B_i)$ is in the kernel of each of the ring homomorphisms $B_i \to A_{ij}$ and so these factor uniquely as $B_i \to B_i^{red} \to A_{ij}$. So the morphisms $U_{ij} \to V_i$ factor uniquely as $U_{ij} \to V_i^{red} \to V_i$. The same is true of each intersection of te U_{ij}'s and so this gives rise to a unique factorization $f^{-1}V_i \to V_i^{red} \to V_i$. These patch to give a unique factorization $X \to Y_{red} \to Y$.

Exercise 2.4. *Let A be a ring and let $(X, \mathcal{O})X$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$ we have an associated map on sheaves $f^\# : \mathcal{O}_{\operatorname{Spec} A} \to f_*\mathcal{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathcal{O}_X)$. Thus there is a natural map*

$$\alpha : \hom_{\mathfrak{Sch}}(X, \operatorname{Spec} A) \to \hom_{\mathfrak{Ring}}(A, \Gamma(X, \mathcal{O}_X))$$

Show that α is bijective.

Solution. We will show that $\operatorname{Spec}(-) : \mathfrak{Ring} \to \mathfrak{Sch}$ is a right adjoint to $\Gamma(-, \mathcal{O}_-) : \mathfrak{Sch} \to \mathfrak{Ring}$. One way to show this is to provide two natural transformations

$$\eta : id_{Sch} \to \operatorname{Spec} \circ \Gamma \qquad \varepsilon : id_{Rings} \to \Gamma \circ \operatorname{Spec}$$

such that for all $A \in \mathfrak{Rings}$ and $X \in \mathfrak{Sch}$ we have

$$\Gamma\eta_X \circ \varepsilon_{\Gamma X} \cong id_{\Gamma X} \qquad \text{and} \qquad \operatorname{Spec}\varepsilon_A \circ \eta_{\operatorname{Spec} A} \cong id_{\operatorname{Spec} A}$$

The obvious choice for ε is the isomorphism of Proposition 2.2 (c). For a scheme X we define the natural transformation η as follows. Let $U_i = \operatorname{Spec} A_i$ be an affine cover of the scheme. Each restriction $\Gamma X \to A_i$ gives a morphism $\operatorname{Spec} A_i \to \operatorname{Spec}\Gamma X$ and since the restriction $\Gamma X \to \mathcal{O}_X(U_i) \to \mathcal{O}_X(U_{ij})$ is the same as $\Gamma X \to \mathcal{O}_X(U_j) \to \mathcal{O}_X(U_{ij})$ these morphisms glue to give a morphism $X \to \operatorname{Spec}\Gamma X$.

So now since ε is an isomorphism, we just have to show that for any scheme X and A a ring, $\Gamma\eta_X$ is an isomorphism, and $\eta_{\operatorname{Spec} A}$ is an isomorphism. For a scheme consider the morphism $\eta : X \to \operatorname{Spec}\Gamma X$ just defined. It comes with a sheaf morphism $\mathcal{O}_{\operatorname{Spec}\Gamma X} \to \eta_*\mathcal{O}_X$ whose global sections we want to be an isomorphism. This is indeed the case. Now for a ring A, we have a sheaf morphism $\operatorname{Spec} A \to \operatorname{Spec}\Gamma\operatorname{Spec} A$ and since $\Gamma\operatorname{Spec} A \cong A$ this too is an isomorphism. Hence, the functors are adjoints and so α is a bijection.

We can more explicitely describe the bijection now as sending $\phi : A \to \Gamma(X, \mathcal{O}_X)$ to the composition

$$X \stackrel{\eta_X}{\to} \operatorname{Spec}\Gamma(X, \mathcal{O}_X) \stackrel{\operatorname{Spec}\phi}{\to} \operatorname{Spec} A$$

and sending $(f, f^\#) : X \to \operatorname{Spec} A$ to

$$A \stackrel{\varepsilon}{\cong} \Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \stackrel{f^\#(\operatorname{Spec} A)}{\to} \Gamma(X, \mathcal{O}_X)$$

Exercise 2.5. *Describe* $\operatorname{Spec}\mathbb{Z}$ *and show that it is a final object for the category of schemes.*

Solution. Description. There is one closed point (p) for every prime number p and one generic point (0). As the ideals of $\mathbb{Z}$ are (n), the closed subsets are finite sets of primes (the prime divisors of n) and the open sets are their complements, together with the empty set. As a consequence, every closed subset is of the form $D(n)$ for some integer n and so the structure sheaf takes an open set $D(n)$ to $\mathbb{Z}$ localized at the prime divisors of n that is, $\mathcal{O}_{\operatorname{Spec}\mathbb{Z}}(D(n)) = \{\frac{a}{b} : p \not| b \ \forall\ p|n\}$. The value of the structure sheaf on the whole space is $\mathbb{Z}$ (since it is affine).

Final object. We have seen by the adjunction between Spec and Γ that the morphisms from a scheme X to an affine scheme $\operatorname{Spec} A$ are in one to one correspondence with the morphisms $A \to \Gamma(X, \mathcal{O}_X)$. Since we consider only identity preserving ring homomorphisms, there is a unique one of these if $A = \mathbb{Z}$.

Exercise 2.6. *Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes.*

Solution. The zero ring as no points as there are no proper prime ideals. The structure sheaf takes the usual value on the empty open set. As the point set is empty, there is a unique morphism of topological spaces from *sp* $\operatorname{Spec} 0$ to any topological space X. If X is a scheme, then the structure sheaf pulls back to the structure sheaf of $\operatorname{Spec} 0$, which has a unique morphism to itself - the identity morphism. Hence, $\operatorname{Spec} 0$ is initial.

Exercise 2.7. *Let X be a scheme and K a field. Show that to give a morphism of* $\operatorname{Spec} K$ *to* X *it is equivalent to give a point $x \in X$ and an inclusion map* $k(x) \to K$.

Solution. Since $\operatorname{Spec} K$ has a unique point, unique nonempty open set, and global sections K, given a point x we obtain immediately a continuous morphism of topological spaces $i : \operatorname{Spec} K \to X$. To define the sheaf morphism $i^{\#} : \mathcal{O}_X \to i_*\mathcal{O}_{\operatorname{Spec} K}$ note that $i_*\mathcal{O}_{\operatorname{Spec} K}$ is the skyscraper sheaf with ring of sections K so for every open set $U \ni x$ we need to give a ring homomorphism $\mathcal{O}_X(U) \to K$ in a way natural in U. We define these morphisms as the composition

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k(x) \to K$$

they are natural in U by definition of $\mathcal{O}_{X,x}$.

Conversly, given a scheme morphism $(i, i^{\#}) : \operatorname{Spec} K \to X$ we obtain a point $x = i((0))$, the image of the unique point of $\operatorname{Spec} K$. For the inclusion map, consider an affine open $\operatorname{Spec} A$ containing x. In $\operatorname{Spec} A$, the point x is a prime ideal $\mathfrak{p}$ and so if $\phi : A \to K$ is the corresponding ring homomorphism, $\mathfrak{p}$ is the kernel of ϕ and so we get an induced inclusion $k(x) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}} \to K$.

Exercise 2.8. *Let X be a scheme over a field k. Show that to give a k-morphism of* $\operatorname{Spec} k[\epsilon]/\epsilon^2$ *to X is equivalent to giving a point $x \in X$, rational over k and an element of* $\hom_{k-Vec}(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$.

Solution. Let $T = \operatorname{Spec} k[\epsilon]/\epsilon^2$.

Supose that we have a morphism of schemes $(\tau, \tau^{\#}) : T \to X$. We get a point x by taking the image of the unique point in T. To see that it is k-rational note that we have an inclusion of fields $k(x) \to k$ induced by τ but since the morphism τ is a k-morphism, this has to be compatible with the structural morphism to k. So we have inclusions $k \subset k(x) \subset k$ and therefore $k(x) = k$. Now consider $\tau^{\#}_x : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$, the stalk of $\tau^{\#}$. Taking an open affine $\operatorname{Spec} A$ containing x we can write this as $A_{\mathfrak{p}} \to k[\epsilon]/\epsilon^2$ where $\mathfrak{p}$ is the prime ideal correpsonding to x. This morphism is induced a ring homomorphism $\phi : A \to k[\epsilon]/\epsilon^2$ whose scheme morphism $T \to \operatorname{Spec} A$ sends (ϵ), the only point of T, to $\mathfrak{p}$. So $\phi^{-1}((\epsilon)) = \mathfrak{p}$ and therefore every element in $(\mathfrak{p}A_{\mathfrak{p}})^2 = \mathfrak{m}_x^2$ gets sent to (ϵ^2) which is zero. Hence, the composition

$$\mathfrak{m}_x \subset \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2 \to k$$

passes to a k-homomorphism $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k$.

Now suppose that we are given a point $x \in X$, rational over k, and an element $\phi \in \hom_{k-Vec}(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$. The morphism $\tau : T \to X$ of topological spaces is easily defined by sending the unique point of T to x. To define a mophism of sheaves $\tau^{\#} : \mathcal{O}_X \to \tau_*\mathcal{O}_T$ we need to give a morphism $\mathcal{O}_X(U) \to k[\epsilon]/\epsilon^2$ for every open subset $U \ni x$ containing x. We will give a morphism $\mathcal{O}_{X,x} \to k[e]/\epsilon^2$ and then define $\mathcal{O}_X(U)$ as the composition

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k[e]/\epsilon^2$$

Let α denote the morphism $\mathcal{O}_{X,x} \to k(x) = k$. Then we claim that

$$\begin{aligned} \mathcal{O}_{X,x} &\to k[e]/\epsilon^2 \\ f &\mapsto \alpha(f) + \phi(f - \alpha(f))\epsilon \end{aligned}$$

is a ring homomorphism. Assuming it is well-defined, it is immediate that it is k-linear, and so we just need to notice that $f - \alpha(f)$ really is in $\mathfrak{m}$ the maximal ideal [1] and that for $f, g \in \mathcal{O}_{X,x}$ the relation

$$\phi(fg - \alpha(fg)) = \phi(f - \alpha(f))\alpha(g) + \alpha(f)\phi(g - \alpha(g))$$

holds. [2]

Exercise 2.9. *If X is a scheme show that every (non)empty irreducible closed subset has a unique generic point.*

Solution. Claim 1: If η is a generic point of Z then η is in $Z \cap U$ for all open sets U that have nontrivial intersection with Z. Suppose that $\eta \neq Z \cap U$. Then η is in its complement $Z^c \cup U^c$. We know that $\eta \in Z$ and so $\eta \neq Z^c$ and therefore $\eta \in U^c$. Since U^c is closed and contains η, it must contain Z, the closure of η. Hence, $Z \cap U = \varnothing$.

Using the claim we have just proven, we can reduced to the affine case by chosing an open affine, say $\operatorname{Spec} A$, that has nontrivial intersection with Z.

Claim 2: If a closed subset $V(I) \subseteq \operatorname{Spec} A$ is irreducible then $\sqrt{I}$ is prime. Let $fg \in \sqrt{I}$ and consider the closed subsets $Z_1 = V(I_1)$ and $Z_2 = V(I_2)$ where $I_1 = (f) + \sqrt{I}$ and $I_2 = (g) + \sqrt{I}$. If $h \in I_1 \cap I_2$ then we can write $h = af + i = bg + j$ for some $a, b \in A$ and $i, j \in \sqrt{I}$. Then $h^2 = abfg + ij + ibg + afj$ and so all these terms are in $\sqrt{I}$, so is h^2 and therefore so is h. So $I_1 + I_2 = \sqrt{I}$, but since $V(I)$ is irreducible this means either $I_1 = \sqrt{I}$ or $I_2 = \sqrt{I}$. Hence, either f or g are in $\sqrt{I}$ and so $\sqrt{I}$ is prime.

It is now straightforward to see that $\sqrt{I}$ is the unique generic point of $V(I)$.

Exercise 2.10. *Describe* $\operatorname{Spec} \mathbb{R}[x]$. *How does its topological space compare to the set $\mathbb{R}$? To $\mathbb{C}$?*

Solution. $\operatorname{Spec} \mathbb{R}[x]$ has one point for every irreducible polynomial, together with the generic point (0). There is one closed point for every real number $(x - a)$ and one for every nonreal complex number $(x + \alpha)(x + \overline{\alpha})$ where $\alpha \in \mathbb{C}$. The residue field at the real numbers is $\mathbb{R}$ and at the "complex numbers" is $\mathbb{C}$. The closed sets are finite collections of points and the open sets their complements.

Exercise 2.11. *Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe* $\operatorname{Spec} k[x]$. *What are the residue field of its points? How many points are there with a given residue field?*

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[1] This is a consequence of the composition $k \to \mathcal{O}_{X,x} \xrightarrow{\alpha} k(x) = k$ being the identity.

[2] This is more involved. By k-linearity of ϕ we just need to show that $fg - \alpha(fg)$ and $f\alpha(g) - \alpha(f)\alpha(g) + \alpha(f)g - \alpha(f)\alpha(g)$ get sent to the same place by ϕ. This will hapen if their difference is in $\mathfrak{m}_x^2$, and this can be seen by expanding $(f - \alpha(f)(g - \alpha(g)) \in \mathfrak{m}_x^2$.

Solution. $\operatorname{Spec} k[x]$ has the generic point and one point for every (monic) irreducible polynomial. The residue field of a point corresponding to a polynomial of degree n is the finite field with p^n elements. To count how many irreducible polynomials there are of degree n, consider the field $\mathbb{F}_{p^n}$. Every irreducible polynomial $f(x)$ of degree n gives an element of $\mathbb{F}_{p^n}$ via the isomorphism $\mathbb{F}_p[x]/(f(x)) \to \mathbb{F}_{p^n}$ and every element α of $\mathbb{F}_{p^n}$ that is not contained in any subfields gives an irreducible polynomial of degree n by taking its minimal polynomial $\prod_{i=0}^{n-1}(x - \alpha^{p^i})$. These processes are inverses of each other and so we want to count the number of elements of $\mathbb{F}_{p^n}$ not contained in any subfields. This quantity is

$$\sum_{d|n} \mu(d) p^d$$

where

$$\mu(d) = \begin{cases} 0 & \text{if } d \text{ has repeated prime divisors} \\ (-1)^{(\#\text{ prime divisors}-1)} & \text{otherwise} \end{cases}$$

Exercise 2.12. Glueing Lemma. *Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an opent subset $U_{ij} \subseteq X_i$ and let if have the induced scheme structure. Suppose also given for each $i \neq j$ an isomorphism of schemes $\phi_{ij} : U_{ij} \to U_{ji}$ such that (1) for each $i, j, \phi_{ji} = \phi_{ij}^{-1}$, and (2) for each i, j, k $\phi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\phi_{ik} = \phi_{jk}\phi_{ij}$ on $U_{ij} \cap U_{ik}$.*

The show that there is a scheme X, together with morphisms $\pi_i : X_i \to X$ for each i, such that (1) ψ_i is an isomorphism of X_i onto an open subscheme of X, (2) the $\psi_i(X_i)$ cover X, (3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$ and (4) $\psi_i = \psi_j \circ \phi_{ij}$ on U_{ij}.

Solution. First define a topological space X as the quotient of $\coprod X_i$ by the equivalence relation $x \sim y$ if $x = y$, or if there are i, j such that $x \in U_{ij} \subset X_i$, $y \in U_{ji} \subset X_j$, and $\phi_{ij} x = y$. This relation is reflexive by definition, symmetric since $\phi_{ji} = \phi_{ij}^{-1}$, and transitive since $\phi_{ik} = \phi_{jk} \circ \phi_{ij}$, hence is really an equivalence relation. We take the quotient topology on $X = \coprod X_i / \sim$ (a set is open in X if and only if its preimage under $\coprod X_i \to X$ is open; in particular the image $\psi_i(X_i)$ of X_i is open for each i since its preimage is $X_i \coprod(\coprod_j U_{ji})$). Now for each i we have a sheaf $\psi_* \mathcal{O}_{X_i}$ on the image of X_i by pushing forward the structure sheaf of X_i, and on the intersections, we have the pushforward of the isomorphisms $\phi_{ij}^{\#}$, and these satisfy the required relation to use Exercise I.1.22 to glue the sheaves together obtaining a sheaf $\mathcal{O}_X$ together with isomorphisms $\psi_i : \mathcal{O}_X|_{\psi X_i} \xrightarrow{\sim} \psi_* \mathcal{O}_{X_i}$. So now we have a locally ringed space $(X, \mathcal{O}_X)$ and we immediately see that $\psi_i : X_i \to X$ is an isomorphism of locally ringed spaces onto an open locally ringed subspace of X. Hence, X is a scheme that satisfies (1). Tohat (2) is satisfied follows from our definition of the uderlying space of X as a quotient of $\coprod X_i$. To see (3) let x be a point in $\psi_i(U_{ij})$. Then the preimage of x in $\coprod X_i$ is certainly contained in X_j as well so $\psi(U_{ij}) \subseteq \psi_i(X_i) \cap \psi_j(X_j)$. Conversely, if $x \in \psi_i(X_i) \cap \psi_j(X_j)$ then there are $x_i \in X_i$ and $x_j \in X_j$ that are equivalent under $\sim$. Hence, $x_i \in U_{ij}$, $x_j \in U_{ji}$, and $\psi_{ij}(x_i) = x_j$ so $x \in \psi_i U_{ij}$ and therefore $\psi(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$. (4) is fairly clear as well.

Exercise 2.13. *A topological space is* quasi-compact *if every open cover has a finite subcover.*

a *Show that a topological space is noetherian if and only if every open subset is quasi-compact.*

b *If X is an affine scheme show that $sp(X)$ is quasi-compact, but not in general noetherian.*

c *If A is a neotherian ring, show that $sp(\operatorname{Spec} A)$ is a noetherian toplogical space.*

d *Give an example to show that $sp(\operatorname{Spec} A)$ can be noetherian even when A is not.*

Solution. a Let X be noetherian, U an open subset and $\{U_i\}$ a cover of U. Define an increasing sequence of open subsets by $V_0 = \varnothing$ and $V_{i+1} = V_i \cup U_i$ where U_i is an element of the cover not contained in V_i. If we can always find such a U_i then we obtain a strictly increasing sequence of open subsets of X, which contradicts X being noetherian. Hence, there is some n for which $\cup_{i=1}^{n} U_i = U$ and therefore $\{U_i\}$ has a finite subcover.

Suppose every open subset of X is quasi-compact. Suppose $U_1 \subset U_2 \subset \dots$ is an increasing sequence of open subsets of X. Then $\{U_i\}$ is a cover for $\cup U_i$. Since this has a finite subcover, there must be some n for which $U_n = U_{n+1}$, hence, the sequence stabilizes and X is noetherian.

b Let $\{U_i\}$ be an open cover for $sp(X)$. The complements of U_i are closed and therefore determined by ideals I_i in $A = \Gamma(\mathcal{O}_X, X)$. Since $\cup U_i = X$ the I_i generate the unit ideal and hence $1 = \sum_{j=1}^{n} f_j g_{i_j}$ for some f_j where $g_{i_j} \in I_{i_j}$. Then $\{I_{i_1}, \dots, I_{i_n}\}$ also generate the unit ideal and therefore we have a finite subcover $\{U_{i_1}, \dots, U_{i_n}\}$.

An example of a non noetherian affine scheme is $\operatorname{Spec} k[x_1, x_2, \dots]$ which has a decreasing chain of closed subsets $V(x_1) \supset V(x_1, x_2) \supset V(x_1, x_2, x_3) \supset \dots$.

c A decreasing sequence of close subsets $Z_1 \supset Z_2 \supset \dots$ corresponds to an increasing sequence $I_1 \subset I_2 \subset \dots$ of ideals of A. Since A is noetherian this stabilizes at some point and therefore, so does the sequence of closed subsets.

d If A is the ring of p-adic integers, then there is one prime idea so the space is noetherian, but there is an increasing chain of ideals $(0) \subset (p) \subset (p^2) \subset \dots$.

Exercise 2.14. a *Let S be a graded ring. Show that $\operatorname{Proj} S = \varnothing$ if and only if every element of S_+ is nilpotent.*

b *Let $\phi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \operatorname{Proj} T | \mathfrak{p} \not\supseteq \phi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$ and show that ϕ determines a natural morphism $f : U \to \operatorname{Proj} S$.*

c *The morphism f can be an isomorphism even when ϕ is not. For example, suppose that $\phi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.*

d *Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong \operatorname{Proj} S$.*

Solution. a Since every prime ideal contains every nilpotent, if every element of S_+ is nilpotent then $\mathfrak{p} \supset S_+$ for every homogeneous prime ideal $\mathfrak{p}$. Hence, $\operatorname{Proj} S$ is empty.

Conversely, suppose $\operatorname{Proj} S$ is empty and consider $s \in S_+$. Let $\mathfrak{p} \subset S$ be a prime ideal, and consider the homogeneous prime ideal $\mathfrak{q} = \sum_{d \geq 0} \mathfrak{p} \cap S_d \subset \mathfrak{p}$ (check that it is prime!). Since $\operatorname{Proj} S$ is empty, $D_+(s)$ is empty, so every homogeneous prime ideal contains s. Hence If $\mathfrak{p}$ contains s. Since every prime ideal contains s, it is nilpotent, so every element of S_+ is nilpotent.

b Let $\mathfrak{p} \in U$. Then $\phi(S_+) \not\subseteq \mathfrak{p}$ and so unless $S_+ = 0$, there is some $f \in S_+$ such that $\phi f \notin \mathfrak{p}$. If $\phi f_i \in \mathfrak{p}$ for every homogeneous component f_i of f then $\phi f \in \mathfrak{p}$, so there is some homogeneous component f_i of f such that $\phi f_i \notin \mathfrak{p}$. Hence, we have found a basic open $D_+(\phi f_i)$ that contains $\mathfrak{p}$. Moreover, $D_+(\phi f_i)$ is contained in U since every prime in $D_+(\phi f_i)$ doesn't contain ϕf_i and so doesn't contain $\phi(S_+)$. The basic opens of this kind cover U and therefore it is open since it is union of open sets.

For $\mathfrak{p} \in U$ define $f(\mathfrak{p}) = \phi^{-1}\mathfrak{p}$. Since $\mathfrak{p} \not\supseteq \phi(S_+)$ we have $\phi^{-1}\mathfrak{p} \not\supseteq S_+$ so the morphism is well defined. As in the affine case, this morphism preserves all ideals, and therefore closed subsets so it is a continuous morphism of topological spaces. As in the affine case, the morphism of sheaves $f^\#$ is induced by $S_{(\phi^{-1}\mathfrak{p})} \to T_{(\mathfrak{p})}$ for $\mathfrak{p} \in U$.

c *The open U is* $\operatorname{Proj} T$. Let $\mathfrak{p} \in \operatorname{Proj} T$, suppose that $\mathfrak{p} \supseteq \phi(S_+)$ and let $t \in T_e$ with $i > 0$. Since ϕ_d is an isomorphism for $d \geq d_0$, there is some $s \in S_{ed_0}$ such that $\phi_{ed_0} s = t^{d_0}$. Since $\mathfrak{p} \supseteq \phi(S_+)$ this means that $\phi_{ed_0} s = t^{d_0} \in \mathfrak{p}$ and since $\mathfrak{p}$ is prime, $t \in \mathfrak{p}$. But $\mathfrak{p} \subseteq T_+$ contradicts the assumption that $\mathfrak{p} \in \operatorname{Proj} T$, so $\mathfrak{p} \not\subseteq \phi(S_+)$. Since $\mathfrak{p}$ was arbitrary, this shows that $U = \operatorname{Proj} T$.

Surjectivity. Let $\mathfrak{p} \in \operatorname{Proj} S$. Define $\mathfrak{q} = \sqrt{\langle \phi\mathfrak{p} \rangle}$ to be the radical of the *homogeneous* ideal generated by $\phi\mathfrak{p}$ the image of ϕ (note that radicals of homogeneous ideals are homogeneous). We will show that (i) $\phi^{-1}\mathfrak{q} = \mathfrak{p}$, and (ii) $\mathfrak{q}$ is prime. We start with (i). The inclusion $\phi^{-1}\mathfrak{q} \supseteq \mathfrak{p}$ is clear, so suppose we have $a \in \phi^{-1}\mathfrak{q}$. Then $\phi a^n \in \langle \phi\mathfrak{p} \rangle$ for some integer n. This means that $\phi a^n = \sum b_i \phi s_i$ for some $b_i \in T$ and $s_i \in \mathfrak{p}$. If we take a high enough m, then the every monomial in the b_i will be in $T_{\geq d_0}$, and since we have isomorphisms $T_d \cong S_d$ for $d \geq d_0$ this means that these monomials correspond to some $c_j \in S$. The element $(\sum b_i \phi s_i)^m$ is a polynomial in the ϕs_i whose coefficients are monomials of degree m in the b_i, and this corresponds in S to a polynomial in the s_i with coefficients in the c_j, which is in $\mathfrak{p}$, as all the s_i are. Hence, $\phi a^{nm} \in \phi\mathfrak{p}$ and so $a^{nm} \in \mathfrak{p}$ and therefore $a \in \mathfrak{p}$. So $\phi^{-1}\mathfrak{q} \subseteq \mathfrak{p}$ and combining this with the other inclusion shows that $\phi^{-1}\mathfrak{q} = \mathfrak{p}$. (ii) Suppose that $ab \in \mathfrak{q}$ for some $a, b \in T$. Then using the same reasoning as for (i) we see that $(ab)^{nm} \in \phi\mathfrak{p}$ for some n, m such that $(ab)^{nm} \in T_{\geq d_0}$. If necassary, take higher power so that $a^{nmk}, b^{nmk} \in T_{\geq d_0}$ as well. Using the isomorphism $T_{\geq d_0} \cong S_{\geq d_0}$ this means that a^{nmk}, b^{nmk} correspond to elements of S and we see that their product is in $\mathfrak{p}$. Hence, one of a^{nmk} or b^{nmk} are in $\mathfrak{p}$, say a^{nmk}. Then $a^{nmk} \in \phi\mathfrak{p}$ and so $a \in \mathfrak{q}$. So $\mathfrak{q}$ is prime.

Injectivity. Suppose that $\mathfrak{p}, \mathfrak{q} \in \operatorname{Proj} T$ have the same image under $f : \operatorname{Proj} T \to \operatorname{Proj} S$. Then $\phi^{-1}\mathfrak{p} = \phi^{-1}\mathfrak{q}$. Consider $t \in \mathfrak{p}$. Since $t \in \mathfrak{p}$ we have $t^{d_0} \in \mathfrak{p}$ and since ϕ_d is an isomorphism for $d \geq d_0$ it follows that there is a unique $s \in S$ with $\phi s = t^{d_0}$. The element s is in $\phi^{-1}\mathfrak{p}$ and so since $\phi^{-1}\mathfrak{p} = \phi^{-1}\mathfrak{q}$ this implies that $s \in \phi^{-1}\mathfrak{q}$. So $\phi s = t^{d_0} \in \mathfrak{q}$. Now $\mathfrak{q}$ is prime and so $t \in \mathfrak{q}$. Hence $\mathfrak{p} \subseteq \mathfrak{q}$. By symmetry $\mathfrak{q} \subseteq \mathfrak{p}$ as well and therefore $\mathfrak{p} = \mathfrak{q}$.

Isomorphism of structure sheaves. Since $\operatorname{Proj} S$ is covered by open affines of the form $D_+(s)$ for some homogeneous element of S, it is enough to check the isomorphism on these. Note that $D_+(s) = D_+(s^i)$ so we can assume that the degree of s is $\geq d_0$. With this assumption it can be seen that $f^{-1}D_+(s) = D_+(t) \subseteq \operatorname{Proj} T$ where t is the element of T corresponding to s under the isomorphism $S_{\deg s} \to T_{\deg s}$ since a homogeneous prime ideal $\mathfrak{q} \subset T$ gets sent to $D_+(s)$ if and only if s is not in its preimage, if and only if t is not in $\mathfrak{q}$. So our task is to show that the morphism $S_{(s)} \to T_{(t)}$ is an isomorphism. If $\frac{f}{s^n}$ gets sent to zero then $0 = t^m\phi f = \phi(s^m)\phi f$ for some m (choose $m > 0$ so that we don't have to handle the case $\deg f = 0$ separately), and so $s^m f \in \ker\phi$. Taking a high enough power of $s^m f$ puts it in one of the S_d for which $S_d \to T_d$ is an isomorphism and so $s^m f = 0$ and therefore $\frac{f}{s^n} = 0$ so our morphism is injective. Now suppose that $\frac{f}{t^n} \in T_{(t)}$. This is equal in $T_{(t)}$ to $\frac{t^{d_0} f}{t^{n+d_0}}$ and now $t^{d_0} f$ has degree high enough to have a preimage in S. So our morphism is surjective.

Exercise 2.15. *a Let V be a variety over the algebraically closed field k. Show that a point $P \in t(V)$ is a closed point if and only if its residue field is k.*

b If $f : X \to Y$ is a morphism of schemes over k, and if $P \in X$ is a point with residue field k, then $f(P) \in Y$ also has residue field k.

c Now show that if V, W are any two varieties over k, then the natural map

$$\hom_{\mathfrak{Var}}(V, W) \to \hom_{\mathfrak{Sch}/k}(t(V), t(W))$$

is bijective.

Solution. a Every point of $t(V)$ is by definition, an irreducible closed subset of V. If P is not a closed point its corresponding irreducible closed subset Z is not a point, but a subvariety of V of dimension greater than zero. Then by Theorem 1.8A the transcendence degree of its residue field over k is greater than zero. Hence, a residue field of k implies P is closed. Conversely, a closed point comes gives a residue field of transcendence degree zero, and since k is algebraically closed this means that $k(P) = k$.

b The morphism of structure sheaves $\mathcal{O}_Y \to f_* \mathcal{O}_X$ induces a morphism of residue field $k(f(P)) \to k(P)$. Since X and Y are schemes over k, these residue fields are both extensions of k. So if $k(P) = k$ then we have a tower $k \hookrightarrow k(f(P)) \hookrightarrow k$, and so $k(f(P)) \cong k$.

Exercise 2.16. *Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of x at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.*

a If $U = \operatorname{Spec} B$ is an open affine subscheme of X and if $\overline{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.

b Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.

c Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0, f^n b$ is the restriction of an element of A.

d With the hypothesis of (c), conclude that $\Gamma(X_f, \mathcal{O}_{X_f}) \cong A_f$.

Solution. a A point x is in $U \cap X_f$ if and only if it is in U and the stalk f_x of f is not in the maximal ideal at x. Since U is affine we can take x to be a prime $\mathfrak{p} \in \operatorname{Spec} B$ and so the maximal ideal of the local ring is $\mathfrak{m} = \mathfrak{p} B_{\mathfrak{p}}$. The element $\overline{f}$ is in $\mathfrak{m}$ if and only if $\overline{f} \in \mathfrak{p}$ and so $U \cap X_f = D(\overline{f})$. Since a subset of a topological space is open if and only if it is open in every element of an open cover, we conclude that X_f is open in X.

b Let $U_i = \operatorname{Spec} A_i$ be an affine cover of X, finite since X is quasi-compact. The restriction of a to $U_i \cap X_f = \operatorname{Spec}(A_i)_f$ is zero for each i and so $f^{n_i} a = 0$ in A_i for some n_i. Choose an n bigger than all the n_i. Then $f^n a = 0$ in each $\operatorname{Spec} A_i$, and so since the $\operatorname{Spec} A_i$ cover X and $\mathcal{O}_X$ is a sheaf (in particular since it is a separated presheaf), this implies that $f^n a = 0$.

c Let $U_i = \operatorname{Spec} A_i$ (different from the previous part!!!). The restriction of b to each intersection $X_f \cap U_i$ can be written in the form $\frac{b_i}{f^{n_i}}$ for some $n_i \in \mathbb{N}$. Since there are finitely many affines, we can choose the expression so that all the n_is are the same, say n. In other words, we have found $b_i \in A_i$ such that $f^n b|_{U_i \cap X_f} = b_i$. Now consider $b_i - b_j$ on $U_i \cap U_j$. Since $U_i \cap U_j$ is quasi-compact and the restriction of $b_i - b_j$ to $U_i \cap U_j \cap X_f = (U_i \cap U_j)_f$ vanishes, we can apply the previous part to find m_{ij} such that $f^{m_{ij}}(b_i - b_j) = 0$ on $U_i \cap U_j$. Again, we choose m bigger than all the m_{ij} so that they are all the same. So the situation now is that we have sections $f^m b_i$ on each U_i that agree on the intersections. Hence, they lift to some global section $c \in \Gamma(X, \mathcal{O}_X)$. Now consider $c - f^{n+m} b$ on X_f. Its restriction to each $U_i \cap X_f$ is $f^m b_i - f^m b_i = 0$ and so $c = f^{n+m} b$ on X_f. Hence, $f^{n+m} b$ is the restriction of the global section c.

d Consider the morphism $A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$. If an element $\frac{a}{f^n}$ is in the kernel then $a|_{X_f} = 0$ and so by part (b) we have $f^m a = 0$ as global sections for some m. Hence, $\frac{a}{f^n}$ is zero and the morphism is injective. Now suppose we have a section b on X_f. By part (c) there is an m such that $f^m b$ is the restriction of some global section, say c. Hence, we have found $\frac{c}{f^m} \in A_f$ that gets sent to b so the morphism is surjective.

Exercise 2.17. A Criterion for Affineness.

a *Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.*

b *A scheme X is affine if and only if there is a finite set of elements $f_1, \dots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ such that the open subsets X_{f_i} are affine, and $f_1, \dots, f_r$ generate the unit ideal in A.*

Solution. a Take a cover of Y by open affines V_j and then cover each intersection $V_j \cap U_i$ by basic open affines of V_j. So we end up with a cover of Y composed of affines W_k, such that each one is an subset of some U_i. Since $f^{-1}(U_i) \to U_i$ is an isomorphism, its restriction to $f^{-1}(W_k) \to W_k$ will be for any $W_k \subset U$, so X is now covered by the same set of affines as Y. It can be checked that the gluing morphisms are the same and so Y and X are both isomorphic to the scheme obtained by gluing together the W_k and these isomorphisms are compatible with f.

b If A is affine we can take $f_1 = 1$.

Suppose then that we have elements $f_1, \dots, f_r \in A$, that each $X_{f_i} = \operatorname{Spec} A_i$, and that the f_i generate A. We always have a morphism $f : X \to \operatorname{Spec} A$ and we wish to show that this is an isomorphism. Since the f_i generate A the basic opens $D(f_i) = \operatorname{Spec} A_{f_i}$ cover $\operatorname{Spec} A$. It is immediate that their preimages are X_{f_i} which we have already assumed are affine $X_{f_i} \cong \operatorname{Spec} A_i$. So our morphism f restricts to a morphism $\operatorname{Spec} A_i \to \operatorname{Spec} A_{f_i}$ which comes from a ring homomorphism $\phi_i : A_{f_i} \to A_i$. If we can show that the ϕ_i are isomorphisms then the result will follow from the previous part of this exercise.

Stated more clearly, we want to show that

$$\phi_i : \Gamma(X, \mathcal{O}_X)_{f_i} \to \Gamma(X_{f_i}, \mathcal{O}_X)$$

is an isomorphism for each i.

Injectivity. Let $\frac{a}{f_i^n} \in A_{f_i}$ and suppose that $\phi_i \frac{a}{f_i^n} = 0$. This means that it also vanishes in each of the intersections $X_{f_i} \cap X_{f_j} = \operatorname{Spec}(A_j)_{f_i}$ so for each j there is some n_j such that $f_i^{n_j} a = 0$ in A_j. Choose an m bigger than all the n_j. Now the restriction of $f_i^m a$ to each open set in a cover vanishes, therefore $f_i^m a = 0$. So $\frac{a}{f_i^n} = 0$ in A_{f_i}.

Surjectivity. Let $a \in A_i$. For each $j \neq i$ we have $\mathcal{O}_X(X_{f_i f_j}) \cong (A_j)_{f_i}$ so $a|_{X_{f_i f_j}}$ can be written as $\frac{b_j}{f_i^{n_j}}$ for some $b_j \in A_j$ and $n_j \in \mathbb{N}$. That is, we have elements $b_j \in A_j$ whose restriction to $X_{f_i f_j}$ is $f_i^{n_j} a$. Since there are finitely many, we can choose them so that all the n_i are the same, say n.

Now on the triple intersections $X_{f_i f_j f_k} = \operatorname{Spec}(A_j)_{f_i f_k} = \operatorname{Spec}(A_k)_{f_i f_j}$ we have $b_j - b_k = f_i^n a - f_i^n a = 0$ and so we can find $m_{jk} \in \mathbb{N}$ so that $f_i^{m_{jk}}(b_j - b_k) = 0$ on $X_{f_j f_k}$. Replacing each m_{jk} by m larger than all of them, the relation $f_i^m(b_j - b_k)$ still holds. So now we have a section $f_i^m b_j$ for each X_{f_j} $j \neq i$ together with a section $f_i^{n+m} a$ on X_{f_i} and these sections agree on all the intersections. This gives us a global section d whose restriction to X_{f_i} is $f_i^{n+m} a$ and so $\frac{d}{f_i^{n+m}}$ gets mapped to a by ϕ_i.

Exercise 2.18. a *Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.*

b *Let $\phi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that ϕ is injective if and only if the map of sheaves $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant.*

c *With the same notation, show that if ϕ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.*

d *Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\# : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective, then ϕ is surjective.*

Lemma 1. *Let $(f, f^\#) : \operatorname{Spec} B \to \operatorname{Spec} A$ be a scheme morphims of affine schemes with corresponding ring homomorphism $\phi : A \to B$. Then for a point $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $(f_*\mathcal{O}_{\operatorname{Spec} B})_\mathfrak{p}$ is $S^{-1}B = B \otimes_A A_\mathfrak{p}$ where $S = \phi(A \backslash \mathfrak{p})$.*

Proof. Since we can shrink every open subset U containing $\mathfrak{p}$ to one of the form $D(a)$ with $a \in A$, we can compute the stalk by taking the colimit over these. Notice that the preimage of $D(a)$ is $D(\phi a) \subseteq \operatorname{Spec} B$. [3] So $(f_*\mathcal{O}_{\operatorname{Spec} B})_\mathfrak{p}$ is then

[3] If a prime $\mathfrak{q} \in \operatorname{Spec} B$ is in the preimage of $D(a)$ then $\phi^{-1}\mathfrak{q} \in D(a)$ and so $a \notin \phi^{-1}\mathfrak{q}$ and therefore $\phi a \notin \mathfrak{q}$. Conversely, if a prime $\mathfrak{q}$ is in $D(\phi a)$ then $\phi a \notin \mathfrak{q}$ and so $a \notin \phi^{-1}\mathfrak{q}$ so $\phi^{-1}\mathfrak{q} \in D(a)$.

the colimit of $\mathcal{O}_{\operatorname{Spec} B}$ evaluated at opens $D(a)$ with $a \notin \mathfrak{p}$, that is, the colimit of $B_{\phi a}$ for $a \notin \mathfrak{p}$. This is $S^{-1}B$. To see that it is the same as the tensor product, use the universal property of tensor products. $\square$

Solution. a If f is nilpotent then $f^n = 0$ for some $n \in \mathbb{N}$ and so $f^n \in \mathfrak{p}$ for every prime ideal $\mathfrak{p}$. Hence, $f \in \mathfrak{p}$ for every prime ideal and therefore, $\mathfrak{p} \notin D(f)$ for every prime ideal $\mathfrak{p}$.

b If the map of sheaves is injective then in particular, taking global sections, we see that $\Gamma(X, \mathcal{O}_X) \to \Gamma(X, f_*\mathcal{O}_Y)$ is injective. That is, $A \to B$ is injective. Conversely, suppose $A \to B$ is injective, pick a prime $\mathfrak{p} \in \operatorname{Spec} A$, and consider the stalk $A_\mathfrak{p} \to S^{-1}B$ of the morphism $f^\#$ at $\mathfrak{p}$ where $S = A \backslash \mathfrak{p}$ (see Lemma 1). That this is injective follows immediately from $A \to B$ being injective.

To see that it is dominant consider the complement of the closure of the image. That is, the biggest open set that doesn't intersect the image. This is covered by open affines of the form $D(f)$ where $f \in \phi^{-1}\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} B$. For such an f, we have $\phi f \in \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} B$ and so ϕf is in the nilradical, so ϕf is nilpotent. Since ϕ is injective, this means, f is nilpotent, so $D(f)$ is empty. So the closure of the image is the entire space.

c We immediately have a bijection between primes of A containing I and primes of $A/I \cong B$ where I is the kernel of ϕ. We already know the morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ is continuous so we just need to see that it is open to find that it is a homeomorphism. Note that for $f + I \in A/I$ the preimage of $D(f) \subset \operatorname{Spec} A$ is $D(f + I) \subset \operatorname{Spec}(A/I)$, so basic opens of $\operatorname{Spec}(A/I)$ are open in the image (with the induced topology). Since arbitrary unions of open sets are open, and the basic opens are a base for the topology, the image of every open set is open. The stalk $A_\mathfrak{p} \to B \otimes_A A_\mathfrak{p}$ of the sheaf morphism at $\mathfrak{p} \in \operatorname{Spec} A$ is clearly surjective.

d If $f^\#$ is surjective then it is surjective on each stalk. So for an element $b \in B$, for each point $\mathfrak{p}_i \in \operatorname{Spec} A$ there is an open neighbourhood which we can take to be a basic open $D(f_i)$ of $\operatorname{Spec} A$ such that the germ of b is the image of some $\frac{a_i}{f_i^{n_i}} \in A_{f_i}$. That is, $f_i^{m_i}(a_i - f_i^{n_i} b) = 0$ in B. Since all affine schemes are quasi-compact, we can find a finite set of the $D(f_i)$ that cover $\operatorname{Spec} A$, which means we can assume all the n_i and m_i are the same, say n and m. Since $D(f_i)$ is a cover, the f_i generated A and therefore so do the f_i^{n+m}, so we can write $1 = sum g_i f_i^{n+m}$ for some $g_i \in A$. We now have

$$b = \sum g_i f_i^{n+m} b = \sum g_i f_i^m a_i \in \operatorname{im} \phi$$

So ϕ is surjective.

Exercise 2.19. *Let A be a ring. Show that the following conditions are equivalent:*

a Spec A *is disconnected;*

b there exist nonzero elements $e_1, e_2 \in A$ *such that* $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1 + e_2 = 1$.

c A *is isomorphic to a direct product* $A_1 \times A_2$ *of two nonzero rings.*

Solution. ($1 \Rightarrow 3$) If Spec A is disconnected then it is the udisjoint union of two open sets, say as $\operatorname{Spec} A = U \coprod V$. In particular, this means that U and V are also both closed sets, and therefor correspond to ideals, say I and J. That is, $U = \operatorname{Spec} A/I$ and $V = \operatorname{Spec} A/J$. It follows that $\operatorname{Spec} A = \operatorname{Spec}(A/I) \times (A/J)$ and therefore $A = A_1 \times A_2$ where $A_1 = A/I$ and $A_2 = A/J$.

($3 \Rightarrow 2$) Choose $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

($2 \Rightarrow 1$) Since $e_1 e_2 = 0$, for every prime, either $e_1 \in \mathfrak{p}$ or $e_2 \in \mathfrak{p}$. The closed sets $V((e_1)), V((e_2))$ cover Spec A. Now if a prime $\mathfrak{p}$ is in both these closed sets then $e_1, e_2 \in \mathfrak{p}$ and therefore $1 = e_1 + e_2 \in \mathfrak{p}$ and so $\mathfrak{p} = A$. So the closed sets $V((e_1)), V((e_2))$ are disjoint. Since we have a cover of Spec A by disjoint closed sets, Spec A is disconnected.

3 First Properties of Schemes

Lemma 1. *a If B is a finitely generated A_f-algebra, then it is a finitely generated A-algebra.*

b Let $F : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes with associated ring homomorphism $\phi : B \to A$. Then the preimage of $\operatorname{Spec} B_f$ is $A_{\phi f}$.

c If B is a finitely generated A-algebra via $\phi : A \to B$, then for any element $f \in A$, the ring $B_{\phi f}$ is a finitely generated A_f algebra.

d Let $f_1, \dots, f_n \in A$ be elements which generate a B-algebra A. If A_{f_i} is a finitely generated B-algebra for every i, then A is a finitely generated B-algebra.

e Let $f_1, \dots, f_n \in B$ be elements which generate the unit ideal, and let A be a B-module. If A_{f_i} is a finitely generated B_{f_i}-module for every i, then A is a finitely generated B-module.

Proof. a Let $\{b_k\}$ be a finite set of elements of B such that $B = A_f[b_1, \dots, b_n]$. Then $B = A[b_1, \dots, b_n, \frac{1}{f}]$.

b Let $F : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of affine schemes with associated ring homomorphism $\phi : B \to A$. Then the preimage of $\operatorname{Spec} B_f$ is $A_{\phi f}$.

c Obvious.

d If $\{\frac{a_{i1}}{f_i^{k_{i1}}}, \dots, \frac{a_{in}}{f_i^{k_{ni}}}\}$ is a generating set for A_{f_i} over B then so is $\{a_{i1}, \dots, a_{in}, \frac{1}{f_i}\}$ so we can assume that the generating sets are of this form. Let $S = \{a_{ij}, f_i\}$. We claim that $A = B[S]$. For an element $a \in A$, for each i we can write $a \in A_{f_i}$ as $a = \frac{p_i}{f_i^{k_i}}$ for some $k_i \in \mathbb{N}$ and $p_i \in B[a_{i1}, \dots, a_{in}, f_i]$. Replacing p_i by $f_i^{\nu_i} p_i$ for suitable ν_i we can assume that all the k_i are the same, say $k \in \mathbb{N}$. Now by definition of the localization, writing a in this form means that for each i we have $f^{\ell_i}(f_i^k a - p_i) = 0$ for some ℓ_i. Again, we can replace p_i and k so that we have $(f_i^m a - p_i) = 0$ for each i. Now since the f_i generate A, the same is true of their Nth power for any N. So choosing $N = m$ we find that $f_1^m, \dots, f_n^m$ generate A, and so we can write the unit as $1 = \sum_{i=1}^n g_i f_i^m$ for some $g_i \in A$. Coming back to our expressions with the p_i we now see that

$$0 = \sum_{i=1}^n g_i(f_i^m a - p_i) = \sum_{i=1}^n g_i f_i^m a - \sum_{i=1}^n g_i p_i = a - \sum_{i=1}^n g_i p_i$$

and so we have found an expression for a as $\sum_{i=1}^n g_i p_i \in B[S]$.

e This is essentially the same idea as in the previous part.

If $\{\frac{a_{i1}}{f_i^{k_{i1}}}, \dots, \frac{a_{in}}{f_i^{k_{ni}}}\}$ is a generating set for A_{f_i} over B_{f_i} then so is $\{a_{i1}, \dots, a_{in}\}$ so we can assume that the generating sets are of this form. Let $S = \{a_{ij}\}$.

We claim that $A = B[S]$. For an element $a \in A$, for each i we can write $a \in A_{f_i}$ as $a = \frac{\sum_{j=1}^n b_{ij} a_{ij}}{f_i^{k_i}}$ for some $k_i \in \mathbb{N}$ and $b_{ij} \in B$. As before, we can assume that all the k_i are the same, say $k \in \mathbb{N}$. Now by definition of the localization, writing a in this form means that for each i we have $f^{\ell_i}(f_i^k a - \sum_j b_{ij} a_{ij}) = 0$ for some ℓ_i. Again, we can replace the b_{ij} and k so that we have $(f_i^m a - \sum_j b_{ij} a_{ij}) = 0$ for each i. Now since the f_i generate B, the same is true of their Nth power for any N. So choosing $N = m$ we find that $f_1^m, \dots, f_n^m$ generate B, and so we can write the unit as $1 = \sum_{i=1}^n g_i f_i^m$ for some $g_i \in B$. Coming back to our expressions with the a_{ij} we now see that

$$0 = \sum_{i=1}^n g_i(f_i^m a - \sum_j b_{ij} a_{ij}) = \sum_{i=1}^n g_i f_i^m a - \sum_{i=1}^n g_i \sum_j b_{ij} a_{ij} = a - \sum_{i=1}^n g_i \sum_j b_{ij} a_{ij}$$

and so we have found an expression for a as $\sum_{i=1}^n g_i \sum_j b_{ij} a_{ij} \in B[S]$. □

Lemma 2. *Let* $\operatorname{Spec} A, \operatorname{Spec} B$ *be two open affine subsets of a scheme* X*. Then for every point* $\mathfrak{p} \in \operatorname{Spec} A \cap \operatorname{Spec} B$ *there exists an open subset* U *with* $\mathfrak{p} \in U \subset \operatorname{Spec} A \cap \operatorname{Spec} B$ *such that* $U \cong \operatorname{Spec} A_f \cong \operatorname{Spec} B_g$ *for some* $f \in A$, $g \in B$.

Proof. The basic open affines form a basis for affine schemes and so since $\operatorname{Spec} A \cap \operatorname{Spec} B$ is open in $\operatorname{Spec} A$ it is a union of basic opens, one of which, say $\operatorname{Spec} A_{f'}$, contains $\mathfrak{p}$. This open will also be open in $\operatorname{Spec} B$ as well and so for the same reason there is some $g \in B$ such that $\mathfrak{p} \in \operatorname{Spec} B_g \subseteq \operatorname{Spec} A_{f'}$. In particular, the inclusion $\operatorname{Spec} A_{f'} \subseteq \operatorname{Spec} B$ gives us a ring homomorphism $B \xrightarrow{\phi} A_{f'}$. Now it can be checked that $A_f \cong B_g$ for some f and so we are done. □

Exercise 3.1. *Show that a morphism* $f : X \to Y$ *is locally of finite type if and only if for every open affine subset* $V = \operatorname{Spec} B$ *of* Y*,* $f^{-1}(V)$ *can be covered by open affine subsets* $U_j = \operatorname{Spec} A_j$*, where each* A_j *is a finitely generated* B*-algebra.*

Solution. ($\Leftarrow$) It is immediate from the definitions.

($\Rightarrow$) We use $F : X \to Y$ to denote the morphism of schemes. Let $V_i = \operatorname{Spec} B_i$ be a covering of Y by open affine subschemes such that $F^{-1}V_i$ is covered by open affines $\operatorname{Spec} A_{ij}$ where each A_{ij} is a finitely generated B_i-algebra. Each intersection $V_i \cap V$ is open in V_i and so is a union of basic open sets $\operatorname{Spec}(B_i)_{f_{ik}}$ of V_i since they form a base for the topology of $\operatorname{Spec} B_i$. Considering f_{ik} as an element of A_{ij} under the morphisms $B_i \to A_{ij}$, the preimage of $\operatorname{Spec}(B_i)_{f_{ik}}$ is $\operatorname{Spec}(A_{ij})_{f_{ik}}$, and the induced ring morphisms make each $(A_{ij})_{f_{ik}}$ a finitely generated $(B_i)_{f_{ik}}$-algebra.

So we can cover $\operatorname{Spec} B$ with open affines $\operatorname{Spec} C_i$ whose preimages are covered with open affines $\operatorname{Spec} D_{ij}$ such that each D_{ij} is a finitely generated C_i-algebra. Now given a point $\mathfrak{p}$ of $\operatorname{Spec} B$, it is contained in some $\operatorname{Spec} C_i$, but since these are open, there is a basic open affine $\operatorname{Spec} B_{g_\mathfrak{p}} \subseteq \operatorname{Spec} C_i$ that contains $\mathfrak{p}$. Associating $g_\mathfrak{p}$ with its image under the induced ring homomorphisms $B \to C_i$ and then $C_i \to D_{ij}$, it can be seen that $\operatorname{Spec}(C_i)_{g_\mathfrak{p}} \cong \operatorname{Spec} B_{g_\mathfrak{p}}$, the preimage of this is $\operatorname{Spec}(D_{ij})_{g_\mathfrak{p}}$, and $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated $B_{g_\mathfrak{p}}$-algebra. The $\operatorname{Spec}(D_{ij})_{g_\mathfrak{p}}$ together cover the preimage of $\operatorname{Spec} B$, and since $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated $B_{g_\mathfrak{p}}$-algebra, it follows that $(D_{ij})_{g_\mathfrak{p}}$ is a finitely generated B-algebra (add $g_\mathfrak{p}$ to the generating set). Hence, the preimage of $\operatorname{Spec} B$ can be covered by open affines $\operatorname{Spec} A_i$ such that each A_i is a finitely generated B algebra.

Exercise 3.2. *A morphism $f : X \to Y$ of schemes is* quasi-compact *if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.*

Lemma 3. *If a topological space has a finite cover consisting of quasi-compact open sets, it is quasi-compact.*

Proof. Suppose X is the topological space and $\{U_i\}_{i=1}^n$ the open cover with each U_i quasi-compact. Let $\mathcal{V} = \{V_j\}_{j\in J}$ be a cover for X. Then $\{V_j \cap U_i\}_{j\in J}$ is an open cover of U_i which has a subcover $\{V_j \cap U_i\}_{j\in J_i}$ where J_i is finite, since U_i is quasi-compact. Then $\cup_{i=1}^n \{V_j\}_{j\in J_i}$ is a finite subcover of $\mathcal{V}$. □

Solution. Let $\{\operatorname{Spec} B_i\}_{i\in I}$ be an open affine cover of Y such that the preimage $f^{-1} \operatorname{Spec} B_i$ of each $\operatorname{Spec} B_i$ is quasi-compact. Let $\operatorname{Spec} C \subseteq Y$ be an arbitrary open affine subset. Each intersection $\operatorname{Spec} B_i \cap \operatorname{Spec} C$ can be covered by opens that are basic in $\operatorname{Spec} B_i$ and since the $\operatorname{Spec} B_i$ form a cover for X, these opens, basic in the various $\operatorname{Spec} B_i$, cover $\operatorname{Spec} C$. Since $\operatorname{Spec} C$ is quasi-compact (Exercise II.2.13(b)), we can find a finite subcover $\{D(b_k)\}_{k=1}^n$ where for each k, $b_k \in B_{i_k}$ for some i_k. Now we cover each $f^{-1} \operatorname{Spec} B_i$ with open affine subschemes $\{\operatorname{Spec} A_{ij}\}_{j\in J_i}$. Since $f^{-1} \operatorname{Spec} B_i$ is quasi-compact, we can choose these in such a way that J_i is finite. The preimage of $D(b_k)$ in $\operatorname{Spec} A_{i_k j}$ is $\operatorname{Spec}(A_{i_k j})_{b_k}$, so we now have a finite cover $\cup_{k=1}^n \{\operatorname{Spec}(A_{i_k j})_{b_k}\}_{j\in J_{i_k}}$ of $f^{-1} \operatorname{Spec} C$ by open affines. Each open affine is quasi-compact (Exercise II.2.13(b)) and so applying the Lemma 3 we see that $f^{-1} \operatorname{Spec} C$ is quasi-compact.

Exercise 3.3. *a Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.*

b Conclude from this that f is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_i = \operatorname{Spec} A_i$ where each A_i is a finitely generated B-algebra.

c Show also if f is of finite type, then for every open affine subset $V = \operatorname{Spec} B \subseteq Y$ and for every open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Solution. a We need only show that if f is of finite type then it is quasi-compact, the others follow immediately from the definitions. Since f is of finite type there is a cover of Y by open affines $\operatorname{Spec} B_i$ whose preimages are covered by finitely many open affines $\operatorname{Spec} A_{ij}$. We know from Exercise 2.13(b) that each $\operatorname{Spec} A_{ij}$ is quasi-compact. In general if a space can be covered by finitely many quasi-compact opens then it itself is quasi-compact [1], so we have found an open affine cover of Y whose preimages are quasi-compact. Hence, f is quasi-compact.

b Follows directly from Exercise 3.1, 3.2, and 3.3(a).

c Cover $f^{-1}(V)$ by affines $U_i = \operatorname{Spec} A_i$ such that each A_i is a finitely generated B-algebra. We can cover each of the intersections $U_i \cap U$ with opens that are basic in both U and U_i by Lemma 2. Let $\operatorname{Spec} A_{f_i} = \operatorname{Spec}(A_i)_{g_i}$ be a cover of U by these basic opens, which we can choose to be finite since the morphism is quasi-compact. Since each A_i is a finitely generated B-algebra, $(A_i)_{g_i} = A_{f_i}$ is a finitely generated B algebra (Lemma 1), and therefore, since the $\operatorname{Spec} A_{f_i}$ form a finite cover of U, the ring A is a finitely generated B-algebra (Lemma 1).

Exercise 3.4. *Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to* $\operatorname{Spec} A$*, where A is a finite B-module.*

Solution. As usual, let $V_i = \operatorname{Spec} B_i$ be an affine cover of Y such that each preimage $f^{-1}V_i$ is affine $U_i = \operatorname{Spec} A_i$ and each A_i is a finitely generated B_i-module. We cover each intersection $U \cap U_i$ with opens $D(f_{ij}) = (B_i)_{f_{ij}}$ of U_i that are basic in both U and U_i and note that the preimage of $D(f_{ij})$ is $\operatorname{Spec}(A_i)_{f_{ij}}$ where f_{ij} is associated with its image in A_i. Since A_i is a finitely generated B_i-module, it follows that $(A_i)_{f_{ij}}$ is a finitely generated $(B_i)_{f_{ij}}$-module.

So now we have a cover of $V = \operatorname{Spec} B$ by opens $\operatorname{Spec} B_{g_i}$ that are basic in V and each of the preimages is affine $\operatorname{Spec} C_i$ and each C_i is a finitely generated B_{g_i}-module. We now use the affineness criterion from Exercise 2.17 as follows. Since $\operatorname{Spec} B$ is affine it is quasi-compact (Exercise 2.13(b)) there is a finite subcover $\operatorname{Spec} B_{g_1}, \dots, \operatorname{Spec} B_{g_n}$. Since this is a cover, the $g_1, \dots, g_n$ generated the unit ideal. This means their image in $\Gamma(U, \mathcal{O}_U)$ where $U = f^{-1} \operatorname{Spec} B$ also generate the unit ideal. Furthermore, the preimage of each $\operatorname{Spec} B_{g_i}$ is in fact U_{g_i} where we associated g_i with its image in $\Gamma(U, \mathcal{O}_U)$. So we can apply the criterion of Exercise 2.17(b) and find that U is affine.

Let $U = \operatorname{Spec} A$. To see that A is a finitely generated B-module we use Lemma 1.

Exercise 3.5. *Let $f : X \to Y$ be a morphism of schemes.*

[1] Let X be the space and U_i the finite cover. For any cover $\{V_i\}$ of X we get a cover $\{V_i \cap U_j\}$ for each U_j, which has a finite subcover by the assumption that the U_j are quasi-compact. The union of the V_i appearing in these finite subcovers will cover X since it covers a cover, and by construction it is finite.

a *Show that a finite morphism is quasi-finite.*

b *Show that a finite morphism is closed.*

c *Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.*

Solution. a Let $\mathfrak{p} \in Y$ be a point. By assumption of the morphism being finite there is an open affine scheme $\operatorname{Spec} B$ containing $\mathfrak{p}$ such that the preimage $f^{-1} \operatorname{Spec} B$ is affine, say $\operatorname{Spec} A$, and A is a finitely generated B-module, so we immediately reduce to the case where $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. To show that the preimage of $\mathfrak{p}$ is finite is the same as showing that the fiber $\operatorname{Spec} A \otimes_B k(\mathfrak{p})$ has finitely many primes (Exercise II.3.10). Since A is a finitely generated B-module, it follows that $A \otimes_B k(\mathfrak{p})$ is a finitely generated $k(\mathfrak{p})$-module, that is, a vector space of finite rank. Hence, there are a finite number of prime ideals.

b Note that a subset of a topological space is closed if and only if it is closed in every element of an open cover so we can reduce to the case where $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, and A is a finitely generated B-module, via say $\phi : B \to A$. So now we want to show that for every ideal $I \subset A$ there is an ideal $J \subset B$ such that $V(J) \subseteq \operatorname{Spec} B$ is the image of $V(I) \subseteq \operatorname{Spec} A$. We immediately have a candidate: $\phi^{-1} I$ so let $J = \phi^{-1} I$. For a point $\mathfrak{p} \in \operatorname{Spec} A$ we have $\mathfrak{p} \supseteq I \Rightarrow \phi^{-1}\mathfrak{p} \supseteq \phi^{-1} I$ so $fV(I) \subseteq V(J)$ and it remains to show that $V(I)$ is mapped surjectively onto $V(J)$. Replacing A and B by A/I and B/J, we just want to show that f is surjective. Given a point $\mathfrak{p} \in \operatorname{Spec} B$ the Going Up Theorem provides us with a point $\mathfrak{q} \in \operatorname{Spec} A$ that maps to $\mathfrak{p}$, and so we are done.

c
$$\operatorname{Spec} k[t, t^{-1}] \oplus k[t, (t-1)^{-1}] \to \operatorname{Spec} k[t]$$

Exercise 3.6. *Let X be an integral scheme. Show that the local ring $\mathcal{O}_\xi$ of the generic point ξ of X is a field. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.*

Solution. Let $U = \operatorname{Spec} A$ be an open affine subset of X. By definition, A is an integral domain and so (0) is a prime ideal. A closed subset $V(I)$ contains (0) if and only if (0) contains I and so we see that the closure of (0) is $V((0))$, i.e. all U. Hence, (0) is the generic point ξ of X. $\mathcal{O}_X(U)_{(0)} = \mathcal{O}_\xi$ is the fraction field of $\mathcal{O}_X(U)$.

Exercise 3.7. *Let $f : X \to Y$ be a dominant, generically finite morphism of finite type of integral schemes. Show that there is an open dense subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.*

Solution. Step 1: $k(X)$ is a finite field extension of $k(Y)$. Choose an open affine $\operatorname{Spec} B = V \subset Y$ and an open affine in its preimage $\operatorname{Spec} A = U \subset f^{-1}V$ such that A is a finitely generated B-algebra (by the finite type hypothesis). From the hypothesis that X is irreducible, it follows that U is irreducible, implying that A is integral.

Now A is finitely generated over B and therefore so is $k(B) \otimes_B A \cong B^{-1}A$. So by Noether's normalization lemma, there is an integer n and a morphism $k(B)[t_1, \ldots, t_n] \to B^{-1}A$ for which $B^{-1}A$ is integral over $k(B)[t_1, \ldots, t_n]$. Since it is integral, the induced morphism of affine schemes is surjective. But $\operatorname{Spec} B^{-1}A$ has the same underlying space as $f^{-1}(\eta) \cap U$ where η is the generic point of Y, and by assumption this is finite. Hence, since affine space always has infinitely many points and the Going-Up Theorem tells us that the morphism $\operatorname{Spec} B^{-1}A \to \operatorname{Spec} k(B)[t_1, \ldots, t_n]$ is surjective ($B^{-1}A$ is integral and integral over $k(B)[t_1, \ldots, t_n]$) we see that $n = 0$ and moreover, we have found that $B^{-1}A$ is integral over $k(B)$. Since it is also of finite type, this implies that it is finite over $k(B)$. With some work clearing denominators from elements of A, This implies that $k(B^{-1}A) = k(A)$ is finite over $k(B)$. That is, it is a finite field extension of $k(B)$.

Step 2: The case where X and Y are affine. Let $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ and consider a set of generators $\{a_i\}$ for A over B. Considered as an element of $k(A)$, each one satisfies some polynomial in $k(B)$ since it is a finite field extension. Clearing denominators we get a set of polynomials with coefficients in B. Let b be the product of the leading coefficients in these polynomials. Replacing B and A by B_b and A_b, all these leading coefficients become units, and so multiplying by their inverses, we can assume that the polynomials are monic. That is, A_b is finitely generated over B_b and there is a set of generators that all satisfy monic polynomials with coefficients in B_b. Hence, A_b is integral over B_b and therefore a finitely generated B_b-module.

Step 3: The general case. Now we return to the case where X and Y are not necassarily affine. Take an affine subset $V = \operatorname{Spec} B$ of X and cover $f^{-1}V$ with finitely many affine subsets $U_i = \operatorname{Spec} A_i$. By Step 2, for each i there is a dense open subset of V for which the restriction of f is finite. Taking the intersection of all of these gives a dense open subset V' of V such that $f^{-1}V' \cap U_i \to V'$ is finite for all i. Furthermore, a look at the previous step shows that V' is in fact a distinguished open of V. We want to shrink V' further so that $f^{-1}V'$ is affine. To start with, replace V with V' and similarly, replace U_i with $U_i \cap f^{-1}V'$. Since V' is a distinguished open of V, we still have an open affine subset of Y and the $U_i \cap f^{-1}V'$ (now written as U_i) form an affine cover of $f^{-1}V'$.

Let $U' \subseteq \cap U_i$ be an open subset that is a distinguished open in each of the U_i. So there are elements $a_i \in A_i$ such that $U' = \operatorname{Spec}(A_i)_{a_i}$ for each i. Since each A_i is finite over B, there are monic polynomials g_i with coefficients in B that the a_i satisfy. Take g_i of smallest possible degree so that the constant terms b_i are nonzero and define $b = \prod b_i$. Now the preimage of $\operatorname{Spec} B_b$ is $\operatorname{Spec}((A_i)_{a_i})_b$ (any i gives the same open) and $((A_i)_{a_i})_b$ is a finitely generated B_b module. so we are done.

Exercise 3.8. Normalization. *Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \operatorname{Spec} \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a* *normal integral scheme $\widetilde{X}$, and that there is a morphism $\widetilde{X} \to X$ having the following universal property: for every normal integral scheme Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism.*

To be done.

Solution.

Exercise 3.9. The Topological Space of a Product.

a *Let k be a field, and let $\mathbb{A}^1_k = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbb{A}^1_k \times_{\operatorname{Spec} k} \mathbb{A}^1_k \cong \mathbb{A}^2_k$ and show that the underlying point set of the product is not the product of the underlying pointsets of the factors (even if k is algebraically closed).*

b *Let k be a field, let x and t be indeterminates over k. Then $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$.*

Solution. a The stated product is the affine scheme of the ring $k[x] \otimes_k k[x]$ which is clearly isomorphic to $k[x, y]$ via (for example)

$$x \otimes 1 \mapsto x, \qquad 1 \otimes x \mapsto y$$

To see that the underlying point set of the product is not the product of the underlying point sets of the factors consider the point $(x-y) \subset k[x, y]$ (or equivalently $(x \otimes 1 - 1 \otimes x) \subset k[x] \otimes_k k[x]$). Each pair of points $((f), (g)) \subset sp \ \operatorname{Spec} k[x] \times sp \ \operatorname{Spec} k[y]$ (where f and g are irreducible or zero) gives a point $(f, g) \in \operatorname{Spec} k[x, y]$ which gets sent back to (f) and (g) via the projections. However, $(x-y)$ gets sent to (0) via both projections, yet $(0) \neq (x - y)$.

b Using greatest common denominators, every element of $k(s) \otimes_k k(t)$ can be written as

$$\frac{1}{c(s) \otimes d(t)} \left(\sum a_i(s) \otimes b_i(t)\right)$$

for some $a_i, c \in k[s], b_i, d \in k[t]$. So if

$$S = \{c(s)d(t) | c \in k[s], d \in k[t]\} \qquad \subset k[s, t]$$

then we can associate $k(s) \otimes_k k(t)$ with $S^{-1}k[s, t]$, the "holomorphic functions whose poles form horizontal and vertical lines in the plane". To see what this looks like geometrically, note that $S^{-1}k[s, t] = \varinjlim_{f \in S} k[s, t]_f$ and so $\operatorname{Spec} S^{-1}k[s, t]$ is the intersection (Spec is contravariant) of basic opens of $\mathbb{A}^2_k$ which are "complements of horizontal or vertical lines". More concretely, the points of $\operatorname{Spec} k(s) \otimes_k k(t)$ are the points of $\operatorname{Spec} k[s, t]$ that aren't in the preimage of one of the projections (excluding the generic point (0)). The topology and structure sheaf are the induced ones.

Exercise 3.10. Fibres of a Morphism.

a *If $f : X \to Y$ is a morphism, $y \in Y$ a point, show that $sp(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.*

b *Let $X = \operatorname{Spec} k[s,t]/(s - t^2)$ let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \mapsto s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed).*

Solution. a In the affine case we want to show that for a morphism $f : \operatorname{Spec} A \to \operatorname{Spec} B$ induced by $\phi : B \to A$, and a point $\mathfrak{p} \in \operatorname{Spec} B$, the preimage of $\mathfrak{p}$ is topologically homeomorphic to the space of $\operatorname{Spec} A \otimes_B (B_\mathfrak{p}/(\mathfrak{p}B_\mathfrak{p}))$. Consider the commutative diagram

$$\begin{tikzcd} X_y \arrow[d] & A \otimes_B k(\mathfrak{p}) & k(\mathfrak{p}) \arrow[l] \\ X & A \arrow[u, "\pi"] & B \arrow[l, "\phi"] \arrow[u] \end{tikzcd}$$

and a prime $\mathfrak{q}' \subset A \otimes_B k(\mathfrak{p})$. Along the top $\mathfrak{q}'$ gets pulled back to (0) and along the right, (0) gets pulled back to $\mathfrak{p}$ so along the left, $\mathfrak{q}'$ gets pulled back to $\mathfrak{q} \in f^{-1}\mathfrak{p}$ and so the morphism of topological spaces $\operatorname{Spec}(A \otimes_B k(\mathfrak{p})) \to \operatorname{Spec} A$ factors through $f^{-1}(y)$.

Injectivity. Suppose that $\mathfrak{q}, \mathfrak{q}' \in \operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$ get sent via π^{-1} to the same prime in A and consider an element of $\mathfrak{q}$. It can be represented by a sum $\sum_{i=1}^n a_i \otimes \frac{b_i}{c_i}$ where $a_i \in A, b_i, c_i \in B, c_i \notin \mathfrak{p}$. Since the tensor is over B, replacing a_i by $a_i b_i$ we can assume that $b_i = 1$. We have $\sum a_i \otimes \frac{1}{c_i} \in \mathfrak{q} \Rightarrow (\prod 1 \otimes c_j)\left(\sum a_i \otimes \frac{1}{c_i}\right) = \sum (a_i \prod_{j \neq i} c_j) \otimes 1) \in \mathfrak{q}$ and so $(a_i \prod_{j\neq i} c_j) \in \pi^{-1}\mathfrak{q} = \pi^{-1}\mathfrak{q}'$ which implies that $\sum (a_i \prod_{j \neq i} c_j) \otimes 1) \in \mathfrak{q}'$. Now multiplying by $1 \otimes \frac{1}{\prod c_i}$ we see that our original element $\sum a_i \otimes \frac{1}{c_i}$ is in $\mathfrak{q}'$. Therefore $\mathfrak{q} \subseteq \mathfrak{q}'$. By symmetry we also have $\mathfrak{q}' \subseteq \mathfrak{q}$ and so the morphism $X_y \to f^{-1}y$ is injective.

Surjectivity. Let $\mathfrak{q} \in \operatorname{Spec} A$ be in the preimage of $\mathfrak{p}$ under f and consider the subset $\mathfrak{q}' = \{a \otimes \frac{1}{b} | a \in \mathfrak{q}, b \in B \backslash \mathfrak{p}\}$ of $A \otimes_B k(\mathfrak{p})$. With some elementary work it can be seen that this is an ideal, which is in fact prime, and that the preimage $\pi^{-1}\mathfrak{q}'$ is $\mathfrak{q}$.

Closedness. Let $I \subset A \otimes_B k(\mathfrak{p})$ be the radical ideal associated to a closed subset of $\operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$. Let $I' = \pi^{-1} I \subset A$, an ideal of A. For any prime ideal $\mathfrak{q} \in \operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$, if $\mathfrak{q} \supseteq I$ then $\pi^{-1}\mathfrak{p} \supseteq \pi^{-1} I = I'$. Conversely, consider $\mathfrak{q} \in V(I') \cap f^{-1}\mathfrak{p}$ and its preimage $\mathfrak{q}' = \{a \otimes \frac{1}{b} | a \in \mathfrak{q}, b \in B \backslash \mathfrak{p}\} \in \operatorname{Spec}(A \otimes_B k(\mathfrak{p}))$. For any $a \otimes \frac{1}{b} \in I$ the element $(1 \otimes b)(a \otimes \frac{1}{b}) = a \otimes 1$ is also in I, and so $a \in I'$. Since $\mathfrak{q} \supseteq I'$ this implies that $a \in \mathfrak{a}$ and so $a \otimes \frac{1}{b} \in \mathfrak{q}'$. Hence $\mathfrak{q}' \supseteq I$. What we have shown is that a prime $\mathfrak{q}$ is in $V(I)$ if and only if $\pi^{-1}\mathfrak{q}$ is in $V(I') \cap f^{-1}\mathfrak{p}$. So the morphism $\operatorname{Spec}(A \otimes_B k(\mathfrak{p})) \to f^{-1}\mathfrak{p}$ is closed. Since it is also a continuous bijection, this proves that it is a homeomorphism.

b The ring of the fibre is the tensor $(k[s,t]/(s-t^2)) \otimes_k (k[s]/(s-a))$ which is isomorphic to the ring $k[s,t]/(s-t^2, s-a)$. Since $s = t^2 = a$ in this ring, every class can be represented uniquely by a polynomial of the form $a_0 + a_1 t$. Recalling that $t^2 = a$ it can be checked that if $a \neq 0$, a ring isomorphism $k[s,t]/(s-t^2, s-a) \cong k \oplus k$ is given by

$$(1,0) \leftrightarrow \frac{1}{2\sqrt{a}}t + \frac{1}{2} \qquad (0,1) \leftrightarrow -\frac{1}{2\sqrt{a}}t + \frac{1}{2}$$

and so the fibre has two points, both with residue field k. If $a = 0$, then $k[s,t]/(s-t^2, s-a) \cong k[t]/(t^2)$, which is a nonreduced one-point scheme. For the generic point, the ring of the fibre is $(k[s,t]/(s-t^2)) \otimes_k k(s) \cong k(t)[s]/(s-t^2)$ which is an extension of degree zero.

Exercise 3.11. Closed Subschemes.

a *Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.*

b *If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$.*

c *Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'.*

d *Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.*

Solution. a *Step 1: X and X' are affine.* In this case by part (b) of this exercise is also affine and is infact $\operatorname{Spec} A/I$ for some suitable ideal of $A = \Gamma(X, \mathcal{O}_X)$. Then if $\Gamma(X', \mathcal{O}_{X'}) = B$ the morphism $Y \times_X X' \to X'$ is $\operatorname{Spec}(B \otimes_A (A/I)) \to \operatorname{Spec} B$ and since $B \otimes_A (A/I) \cong B/J$ where $J = \langle \phi I \rangle$ the ideal generated by the image of I, we see that the morphism $Y \times_X X' \to X'$ is a closed immersion.

Step 2: X is affine. Let $x \in X'$ be a point of X' and $\operatorname{Spec} A = U$ an open affine neighbourhood of X. As we have just seen, $Y \times_X U \to U$ is a closed immersion, and so since $Y \times_X U = (f')^{-1}U$ it follows that the morphism of stalks $(\mathcal{O}_{X'})_x \to (f'_* \mathcal{O}_{Y \times_X X'})_x$ is surjective. Furthermore, it shows that locally, f' is a homeomorphism onto a closed subset of X'. This is enough to conclude that f' is globally a homeomorphism onto a closed subset of X'. [2]

Step 3: X and X' general. Take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X and let $f^{-1}U_i = Y_i$ and $g^{-1}U_i = X'_i$ where $g : X' \to X$. From the previous step we know that the morphisms $Y_i \times_{U_i} X'_i \to X'_i$ are closed immersions. But these are the same as the morphisms $X'_i \times_X Y \to X'_i$ and so we have found an open cover of X' on which f' is a closed immersion. This is enough to conclude that $X' \times_X Y \to X'$ is a closed immersion (see footnote). check this

To be done.

b Let $V_i = \operatorname{Spec} B_i$ be an open affine cover of Y. By definition of the induced topology, if the V_i are open in Y then there is some open $U_i \subseteq \operatorname{Spec} A$ such that $U_i \cap Y = V_i$. Since $\operatorname{Spec} A$ is affine the U_i are covered by basic open affines $D(a_{ij})$. Consider the composition $V_i = \operatorname{Spec} B_i \to Y \to \operatorname{Spec} A$. There is an induced ring homomorphism $\phi_i : A \to B_i$ and the preimage of $D(a_{ij})$ is $D(\phi_i a_{ij}) = \operatorname{Spec}(B_i)_{\phi_i a_{ij}} \subseteq \operatorname{Spec} B_i$. The complement $Y^c \subseteq \operatorname{Spec} A$ is open and therefore covered by basic open affines $D(g_i)$. Putting these two sets of basic opens together we get a cover of $\operatorname{Spec} A$ since every point in Y is covered as well as every point not in Y. Using the quasi-compactness of $\operatorname{Spec} A$ (since it is affine) we find a finite subcover $\{D(h_i)\}$ where $h_i = f_j$ or g_k for some j, k. As this is a cover, the h_i generate the unit ideal in A. That is, there are $k_i \in A$ such that $1 = \sum h_i k_i$. Under the ring homomorphism $A \to \Gamma(Y, \mathcal{O}_Y)$ unity is preserved and so the images of the h_i generate the unit ideal there also. But recall that $D(h_i) \subseteq Y$ were all affine, and so the criteria of Exercise I.2.17(b) is satisfied and we see that Y is affine. Now we use Exercise II.2.18(d) to find that $\phi : A \to B = \Gamma(Y, \mathcal{O}_Y)$ is surjective ($f^{\#}$ is surjective since $Y \to \operatorname{Spec} A$ is a closed embedding). Hence, $B \cong A/\ker\phi$ and Y is determined by the ideal $\ker\phi$.

finitely many?

check the structure sheaf isomorphism on stalks

c First assume that $X = \operatorname{Spec} A$ is affine, so $Y = \operatorname{Spec} A/I$ for some radical ideal I. As we have seen in the previous part, this implies that $Y' = \operatorname{Spec} B'$ is affine and is determined by an ideal $I' \subseteq A$. That is, $B' \cong A/I'$. Since Y' and Y share the same underlying closed set, $\sqrt{I} = \sqrt{I'}$. But I is already reduced and so $I = \sqrt{I'}$. Hence, the morphism $A \to A/I$ factors as $A \to A/I' \to A/I$. In fact, it factors uniquely. If X is not affine, we can take an open affine cover $\{\operatorname{Spec} A_i\}$. If $\{\operatorname{Spec} B_i\}$ and $\{\operatorname{Spec} B'_i\}$ are the retrictions of this cover to Y and Y' respectively, then as we have just seen, we obtain morphisms $g_i : \operatorname{Spec} B_i \to \operatorname{Spec} B'_i$ which factor

[2] Let Z be the image of f' in X'. To see that Z is closed it is enough to note that the closure of a set is the union of its closures on an open cover $\{U_i\}$ since $\overline{Z} \subseteq \bigcup \overline{U_i \cap Z} \subseteq \bigcup (U_i \cap \overline{Z}) = \overline{Z}$. To see that it is mapped homeomorphically we need just to show that it is closed (since we already know that it is bijective and continuous). But this follows from the same reasoning. That is, for a closed subset $Z \subseteq Y \times_X X'$ its image $f'(Z)$ is closed globally if and only if it is closed locally on an open cover. But this we have seen since locally $Y \times_X X'$ is mapped homeomorphically onto is image.

$\operatorname{Spec} B_i \to \operatorname{Spec} A_i$. If $\mathfrak{p} \in \operatorname{Spec} A$ is a point in an intersection $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$, we take a basic open affine eighbourhood $D(f)$ of $\mathfrak{p}$ (contained in $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ (say with $f \in A_i$) and then we are still in the affine world, so we get a factoring $\operatorname{Spec}(B_i)_f \to \operatorname{Spec}(B_i')_f \to \operatorname{Spec}(A_i)_f$. Since this factoring was unique, this shows that the restriction of the g_i to the intersections agrees, and so the g_i give a well defined morphism $Y \to Y'$ which factors $Y \to X$. check the gluing

d

To be done.

Exercise 3.12. Closed Subschemes of $\operatorname{Proj} S$.

a *Let $\phi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of Exercise II.2.14 is equal to* $\operatorname{Proj} T$*, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.*

b *If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as the mage of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geq d_0} I_d$. Show that I and I' determine the same closed subscheme.*

To be done.

Solution. a

Exercise 3.13. Properties of Morphisms of Finite Type.

a *A closed immersion is a morphism of finite type.*

b *A quasi-compact open immersion is of finite type.*

c *A composition of two morphisms of finite type is of finite type.*

d *Morphisms of finite type are stable under base extension.*

e *If X and Y are schemes of finite type over S, then $X \times_S Y$ is of finite type over S.*

f *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and it f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.*

g *If $f : X \to Y$ is a morphism of finite type, and if Y is noetherian, then X is noetherian.*

Solution. a Take an open affine cover $U_i = \operatorname{Spec} A_i$ of the target scheme X. The restriction of $f : Y \to X$ to $f^{-1}U_i \to U_i$ is still a closed immersion and so it follows from Exercise II.3.11(b) that $f^{-1}U_i$ is affine, say $f^{-1}U_I = \operatorname{Spec} B_i$. Since the morphism of structure sheaves is surjective, the morphism $(A_i)_{\mathfrak{p}} \to (B_i)_{\phi^{-1}\mathfrak{p}}$ is surjective at every prime $\mathfrak{p} \in \operatorname{Spec} A_i$ and from this it follows that $A_i \to B_i$ is surjective. Hence, each B_i is a finitely generated A_i module.

b Let $i : U \to X$ be a quasi-compact open immersion. Since we already know that i quasi-compact, by Exercise II.3.3(a) we only need to show that it is locally of finite type. Let $\operatorname{Spec} A_i$ be an open affine cover of X. Then i restricts to open immersions $U_i \to \operatorname{Spec} A$. Each U_i is covered by basic open sets $D(f_{ij}) \cong \operatorname{Spec}(A_i)_{f_{ij}}$. Clearly, each $(A_i)_{f_{ij}}$ is a finitely generated A_i-algebra (generated by 1 and $\frac{1}{f_{ij}}$ for example) and so we have shown that i is locally of finite type.

c Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a pair of composable morphisms of finite type. Let $W = \operatorname{Spec} C$ be an open affine subscheme of Z. By Exercise I.3.1 the preimage $g^{-1}W$ can be covered by finitely many open affine subschemes $\operatorname{Spec} B_i$ such that each B_i is a finitely generated C-algebra. Again by Exercise I.3.1, the preimage $f^{-1} \operatorname{Spec} B_i$ of each $\operatorname{Spec} B_i$ can be covered by finitely many open affine subschemes $\operatorname{Spec} A_{ij}$ such that each A_{ij} is a finitely generated B_i-algebra. Hence, the preimage $(g \circ f)^{-1}W$ of W can be covered by finitely many open affine subschemes $\operatorname{Spec} A_{ij}$ such that each A_{ij} is a finitely generated C-algebra, so by Exercise I.3.1 $g \circ f$ is a morphism of finite type.

d Consider a pullback square with f a morphism of finite type:

$$\begin{tikzcd} X' \times_X Y \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"] & X \end{tikzcd}$$

We want to show that $X' \times_X Y \to X'$ is a morphism of finite type. If X', X and Y are affine this is certainly true since $A \otimes_C B$ is a finitely generated B-algebra if A is a finitely generated C-algebra. If X' and X are both affine then it is true since a finite open affine cover $U_i \subseteq Y$ leads to a finite open affine cover $U_i \times_X X'$ of $Y \times_X X'$ and as we just noted, if the U_i are of finite type over X then the $U_i \times_X X'$ are of finite type over X'. Now suppose that just X is affine and let V_i be an open affine cover of Y'. Then each $V_i \times_X Y$ is of finite type over V_i and so since they cover X' and $V_i \times_X Y$ is the preimage of V_i we see that $X' \times_X Y$ is of finite type over X'.

So the only case left is when X is not affine. In this case, take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X and let $X'_i = g^{-1}U_i$ and $Y_i = f^{-1}U_i$. From the above work we see that $X'_i \times_{U_i} Y_i$ is of finite type over X'_i. But this is the same morphism as $X'_i \times_X Y \to X'_i$ and so we have found an open cover of X' on which f' is of finite type. This is enough to conclude that f' is of finite type.

e Let $\{\operatorname{Spec} C_i\}$ be an open cover of S. Since $X \xrightarrow{f} S$ (resp. $Y \xrightarrow{g} S$) is a scheme of finite type over S, the preimages $f^{-1} \operatorname{Spec} C_i$ (resp. $g^{-1} \operatorname{Spec} C_i$) can be covered by finitely many open affines, say $\{\operatorname{Spec} A_{ij}\}$ (resp. $\{\operatorname{Spec} B_{ik}\}$)

such that each A_{ij} (resp. B_{ik}) is a finitely generated C_i algebra. It can be seen from the construction of $X \times_S Y$ given in the text that $X \times_S Y$ is covered by the open affines $\operatorname{Spec}(A_{ij} \otimes_{C_i} B_{ik})$ for various i, j, k. Notice that for fixed i there are finitely many of these. Since A_{ij} and B_{ik} are finitely generated C_i algebras, it follows that $A_{ij} \otimes_{C_i} B_{ik}$ is a finitely generated C_i algebra (if $\{\alpha_\ell\} \subseteq A_{ij}$ and $\{\beta_m\} \subseteq B_{ik}$ are finitely generating sets then take $\{\alpha_\ell \otimes \beta_m\}$). So we have found an affine cover of S such that each of the preimages in $X \times_S Y$ satisfies the required property. Hence, $X \times_S Y \to S$ is a morphism of finite type.

f Since we are given that f is quasi-compact, by Exercise II.3.3(a) we just need to show that it is locally of finite type. Let $\mathcal{C} = \{\operatorname{Spec} C_i\}$ be an open affine cover of Z. Since gf is of finite type, the preimages $(gf)^{-1} \operatorname{Spec} C_i$ are covered by finitely many open affines $\operatorname{Spec} A_{ij}$ such that each A_{ij} is a fintely generated C_i-algebra. Let $\{\operatorname{Spec} B_{ik}\}$ be an open affine cover of $g^{-1} \operatorname{Spec} C_i$ in Y. Then the preimage of each $\operatorname{Spec} B_{ik}$ is contained in $\cup_j \operatorname{Spec} A_{ij}$ so we can cover it with basic open affines coming from the $\operatorname{Spec} A_{ij}$. Stated differently, for each ik, the preimage $f^{-1} \operatorname{Spec} B_{ik}$ can be covered with affine schemes of the form $\operatorname{Spec}(A_{ij})_{a_{ik\ell}}$ for some j and some $a_{ik\ell} \in A_{ij}$. We then have a sequence of ring homomorphisms $C_i \to B_{ik} \to (A_{ij})_{a_{ik\ell}}$. The composition makes $(A_{ij})_{a_{ik\ell}}$ a finitely generated C_i-algebra (since A_{ij} is a finitely generated C_i algebra we can choose the generators for A_{ij} together with $\frac{1}{a_{ik\ell}}$) and hence, $(A_{ij})_{a_{ik\ell}}$ is a finitely generated B_{ik}-algebra (we can take the same generators as for over C_i as everything in C_i goes through B_{ik} anyway).

g Let $V_i = \operatorname{Spec} B_i$ be a finite affine cover of Y and $U_{ij} = \operatorname{Spec} A_{ij}$ be a finite affine cover of V_i such that each A_{ij} is a finitely generated B_i-algebra. Since each A_{ij} is a finitely generated B_i-algebra, and each B_i is noetherian (since Y is noetherian) it follows from Hilbert's Basis Theorem that the A_{ij} are noetherian. Hence, Y is locally noetherian.

To see that Y is quasi-compact, consider a finite open affine cover $\{U_i\}$ of X. Since f is of finite type it is quasi-compact (Exercise I.3.3(a)), and so the preimage $f^{-1}U$ of each U_i is quasi-compact (Exercise I.3.2). Now let $\{V_j\}_{j \in J}$ be an open cover of Y indexed by a set J. This gives an open cover of $f^{-1}U_i$ for each i, and since each of these is quasi-compact, there is a finite subcover, indexed by a finite set, say J_i. Now $\cup J_i$ is finite and the subcover indexed by it is still a cover so we have found a finite subcover of an arbitrary cover. Hence, Y is noetherian.

Exercise 3.14. *If X is a scheme of finite type over a field, show that the closed points of X are dense. Give an example to show that this is not true for arbitrary schemes.*

Solution. We immediately reduce to the affine case, for if V is a closed subset containing every closed point, then for each U_i in an open affine cover, $V \cap U_i$ is a closed subset containing every closed point of U_i. So we can't have a proper closed subset containing every closed point globally, unless we can have them locally on affines.

So let $X = \operatorname{Spec} A$ be an affine scheme of finite type over a field. If we can show that the Jacobson radical is the same as the nilradical of A we are done, since the Jacobson radical corresponds to the smallest closed subset containing all the maximal points, and the nilradical corresponds to the closed set which is the whole space. But this is a statement of the Nullstellensatz.

An example of a scheme for which this is not true is $\operatorname{Spec} R$ for any local ring R of dimension greater than 0. The maximal ideal is unique and therefor equal to its own closure. Since the dimension is positive however, this is not the whole space.

Exercise 3.15. *Let X be a scheme of finite type over a field k (not necessarily algebraically closed).*

a *Show that the following three conditions are equivalent.*

  (a) $X \times_k \overline{k}$ is irreducible, where $\overline{k}$ denotes the algebraic closure of k.

  (b) $X \times_k k_s$ is irreducible, where k_s denotes the separable closure of k.

  (c) $X \times_k K$ is irreducible for every extension field K of k.

b *Show that the following three conditions are equivalent.*

  (a) $X \times_k \overline{k}$ is reduced.

  (b) $X \times_k k_p$ is reduced, where k_p denotes the perfect closure of k.

  (c) $X \times_k K$ is reduced for all extension fields K of k.

c *Give examples of integral schemes which are neither geometrically irreducible nor geometrically reduced.*

To be done.

Solution. a

To be done.

b

c Consider
$$\operatorname{Spec} \mathbb{Q}[x, y]/(x^2 + y^2)$$
Since $x^2 + y^2$ is an irreducible polynomial in $\mathbb{Q}[x, y]$ the ideal $(x^2 + y^2)$ is prime and therefore the affine scheme is integral. However, after tensoring with $\mathbb{C}$ we get
$$\operatorname{Spec} \mathbb{C}[x, y]/((x - iy)(x + iy))$$
which is certainly not irreducible.

Now consider
$$\operatorname{Spec} \mathbb{Z}[x]/(x^2 - p)$$
for some prime p.In the integers since p is prime there is not solution to $x^2 = p$ and so $x^2 - p$ is irreducible implying that the affine scheme is integral. However after tensoring with $\mathbb{Z}/p$ we get
$$\operatorname{Spec}(\mathbb{Z}/p)[x]/(x^2)$$

which is certainly not reduced.

For an example over a field consider $\mathbb{F}_p(t)$ the function field over the field with p elements. We take our example to be

$$\operatorname{Spec} \mathbb{F}_p(t)[x]/(x^p - t)$$

which is integral as $x^p - t$ has no solutions in $\mathbb{F}_p(t)$. Tensoring with $\mathbb{F}_p(t^{\frac{1}{p}})$ however, our scheme becomes

$$\operatorname{Spec} \mathbb{F}_p(t^{\frac{1}{p}})[x]/(x - t^{\frac{1}{p}})^p$$

Exercise 3.16. Noetherian Induction. *Let X be a noetherian topological space, and let $\mathscr{P}$ be a property of closed subsets of X. Assume that for any closed subset Y of X, if $\mathscr{P}$ holds for every proper closed subset of Y, then $\mathscr{P}$ holds for Y (in particular, $\mathscr{P}$ holds for the empty set). Then $\mathscr{P}$ holds for X.*

Solution. Let NP be the collection of closed subsets of X for which $\mathscr{P}$ does not hold. If NP is not empty, then since X is noetherian, it has a least element Z. If every proper closed subset of Z satisfies $\mathscr{P}$ then so does Z, however if there is a proper closed subset of Z that doesn't satisfy $\mathscr{P}$ then Z is not minimal. Hence, we have a contradiction and NP must be empty. So X satisfies $\mathscr{P}$.

Definition. A topological space X is a Zariski space if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point.

Exercise 3.17. Zariski spaces.

a *Show that if X is a noetherian scheme, then $sp(X)$ is a Zariski space.*

b *Show that any minimal nonempty closed subset of a Zariski space consists of one point. These are called closed points.*

c *Show that a Zariski space X satisfies T_0: given any two distinct points of X, there is an open set containing one but not the other.*

d *If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset of X.*

e *Show that the minimal points, for the partial ordering determined by $x_1 > x_0$ if $x_0 \in \overline{\{x_1\}}$, are the closed points, and the maximal points are the generic points of the irreducible components of X. Show also that a closed subset contains every specialization of any of its points.*

f *Let t be the functor on topological spaces introduced in the proof of (2.6): the points of $t(X)$ are the irreducible closed subsets of X and the closed subsets are the sets of the form $t(Y)$ where Y is a closed subset of X.*

If X is a noetherian topological space, show that $t(X)$ is a Zariski space. Furthermore, X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

Solution. a We have already seen in the text (Caution 3.1.1) that $sp(X)$ is a noetherian topological space so we just need to show that each closed irreducible subset has a unique generic point. Note that for a closed irreducible subset Z of any topological space and an open subset U, either U contains the generic points of Z, or $U \cap Z = \varnothing$ (since if $\eta \notin U$ then U^c is a closed subset containing η and so $\overline{\{\eta\}} \subseteq U^c$ and therefore $U \cap Z = \varnothing$). So we can reduce to the affine case.

Suppose that X is affine. Then the irreducible closed subsets correspond to ideals I with the property that $\sqrt{I} = \sqrt{JK} \Rightarrow \sqrt{I} = \sqrt{J}$ or $\sqrt{K}$. We claim that ideals with this property are prime. To see this, suppose that $fg \in \sqrt{I}$. Then $\sqrt{I} = ((f)+\sqrt{I})((g)+\sqrt{I})$ and so either $\sqrt{I} = (f)+\sqrt{I}$ or $\sqrt{I} = g + \sqrt{I}$. Hence, either $f \in \sqrt{I}$ or $g \in \sqrt{I}$. It is straightforward that $\mathfrak{p}$ is a generic point for $V(\mathfrak{p})$ so we just need to show uniqueness. Suppose that $\mathfrak{p}, \mathfrak{q}$ are two generic points for a closed subset determind by an ideal I. Then $\mathfrak{p} = \sqrt{\mathfrak{p}} = \sqrt{I} = \sqrt{\mathfrak{q}} = \mathfrak{q}$.

b Let Z be a minimal nonempty closed subset. Since Z is minimal it is irreducible and therefore, by the previous part has a unique generic point η. For any point $x \in Z$, again since Z is minimal, we have $Z = \overline{\{x\}}$ and so $x = \eta$.

c Let x, y be the two distinct points and let $U = \overline{\{x\}}^c$. If $y \in U$ we are done. If not, then $y \in \overline{\{x\}}$. If $x \in \overline{\{y\}}$ then x and y are both generic points for the same closed irreducible subset, which contradicts the assuption that they were distinct. Hence, $x \in \overline{\{y\}}^c$.

d If $\eta \notin U$ then $\eta \in U^c$, a closed subset, and so $X = \overline{\{\eta\}} \subseteq U^c$. Therefore $U = \varnothing$.

e Let $X = \cup Z_i$ be the expression of X as the union of its irreducible closed subsets. In particular, the Z_i are the maximal irreducible closed subsets. Let η be the generic point of Z_i and x a point such that $\eta \in \overline{\{x\}}$. This implies that $Z_i \subseteq \overline{\{x\}}$ and so since the Z_i are maximal, $Z_i = \overline{\{x\}}$. Since the generic points of irreducible closed subsets are unique, this implies that $\eta = x$. So η is maximal. Conversely, suppose that η is maximal. η is in Z_i for some i. If η' is the unique generic point of Z_i then $\eta \in \overline{\{\eta'\}}$ and so since η is maximal, $\eta = \eta'$.

Let Z be a closed subset and $z \in Z$. Since $\overline{\{z\}}$ is the smallest closed subset containing z, and Z contains z, we have $\overline{\{z\}} \subseteq Z$.

f Since the lattice of closed subsets of $t(X)$ is the same as the lattice of closed subsets of X, we immediately have that $t(X)$ is noetherian. Now consider η, a closed irreducible subset of X, and its closure $\overline{\{\eta\}}$ in $t(X)$. This is the smallest closed subset of X containing η. Since η is itself a closed subset of X, we see that this is η. So if η' is a generic point for $\overline{\{\eta\}} \subseteq t(X)$ then $\overline{\{\eta\}} = \overline{\{\eta'\}}$, and so $\eta = \eta'$. Hence, each closed irreducible subset has a unique generic point.

If X is itself a Zariski space then there is a one-to-one correspondence between points and irreducible closed subsets. Hence, α isa bijection on the underlying sets. It is straightforward to see that its inverse is also continuous.

Exercise 3.18. Constructible sets. *Let X be a Zarisiki topological space. A constructible ubset of X is a subset which belongs to the smallest family $\mathfrak{F}$ of subsets such that (1) every **open subset** is in $\mathfrak{F}$, (2) a **finite intersection** of elements of $\mathfrak{F}$ is in $\mathfrak{F}$, and (3) the **complement** of an element of $\mathfrak{F}$ is in $\mathfrak{F}$.*

a *A subset of X is locally closed if it is the intersection of an open subset with a closed subset. Show that a subset of X is constructible if and only if it can be written as a finite disjoint union of locally closed subsets.*

b *Show that a constructible subset of an irreducible Zariski space X is dense if and only if it contains the generic point. Furthermore, in that case it contains a nonempty open subset.*

c *A subset S of X is closed if and only if it is constructible and stable under specialization. Similarly, a subset T of X is open if and only if it is constructible and stable under generization.*

d *If $f : X \to Y$ is a continuous map of Zariski spaces, then the inverse image of any constructible subset of Y is a constructible subset of X.*

Solution. a Consider $\coprod_{i=1}^n Z_i \cap U_i \subseteq X$ where Z_i are closed subsets of X and U_i are open subsets of X. Note that (1) + (3) implies that all closed subsets of X are in $\mathfrak{F}$ and (2) + (3) implies that finite unions of elements of $\mathfrak{F}$ are in $\mathfrak{F}$. Hence, as long as the $Z_i \cap U_i$ are disjoint, $\coprod_{i=1}^n Z_i \cap U_i = \cup_{i=1}^n Z_i \cap U_i \in \mathfrak{F}$.

Let $\mathfrak{F}'$ be the collection of subsets of X that can be written as a finite disjoint union of locally closed subsets. We have just shown that $\mathfrak{F}' \subset \mathfrak{F}$, so by definition, if $\mathfrak{F}'$ satisfies (1), (2), and (3) then $\mathfrak{F}' = \mathfrak{F}$. We immediately have that (1) is satisfied since $U \cap X = U$ and X is closed. If $\coprod_{i=1}^n Z_i \cap U_i$ and $\coprod_{i=1}^n Z_i' \cap U_i'$ are two elements of $\mathfrak{F}'$ then their intersection is

$$\left(\coprod_{i=1}^n Z_i \cap U_i\right) \cap \left(\coprod_{i=1}^n Z_i' \cap U_i'\right) = \coprod_{i,j=1}^n (Z_i \cap Z_j') \cap (U_i \cap U_j')$$

which is in $\mathfrak{F}'$ so (2) is satisfied. We show (3) by induction on n. Let $\mathfrak{F}_n' \subset \mathfrak{F}$ be the collection of subsets of X that can be written as a finite disjoint union of n locally closed ubsets. Note that $\cup_n \mathfrak{F}_n' = \mathfrak{F}'$ and that we have already shown that, an intersection an element of an element of $\mathfrak{F}_n'$ and an element of $\mathfrak{F}_m'$ is in $\mathfrak{F}'$. Let $S \in \mathfrak{F}_1'$. So $S = U \cap Z$. Then its complement is

$$S^c = (U \cap Z)^c = U^c \cup Z^c = U^c \coprod (Z^c \cap U)$$

which is in $\mathfrak{F}'$. Now let $S \in \mathfrak{F}'_n$ and suppose that for all $i < n$, complements of memebers of $\mathfrak{F}'_i$ are in $\mathfrak{F}'$. We can write S as $S = S_{n-1} \coprod S_1$ for some $S_{n-1} \in \mathfrak{F}'_{n-1}$ and $S_1 \in \mathfrak{F}'_1$. The complement of S is then $S^c_{n-1} \cap S^c_1$. We know that S^c_{n-1} and S^c_1 are in $\mathfrak{F}'$ by inductive hypothesis and we know that their intersection is in $\mathfrak{F}'$ by (2) which we proved above. Hence, S^c is in $\mathfrak{F}'$ and we are done.

b Let $S \in \mathfrak{F}$. If if the generic point η is in S then $\overline{S} \supseteq \overline{\{\eta\}} = X$ so S is dense.

For the converse, we use the fact that for an irreducible Zariski space, every nonempty open subset contains the generic point (Exercise 3.17(d)). Suppose $S = \coprod_{i=1}^n Z_i \cap U_i$ is dense, that is, its closure is X. The closure $\overline{S}$ is the smallest closed subset that contains S so any closed subets, for example $\cup_i Z_i$ that contains S, contains the closure. Hence, $\cup_i Z_i \supseteq \overline{S} = X$. But X is irreducible and so $Z_i = X$ for some i. So up to reindexing, $S = U_n \coprod \left(\coprod_{i=1}^{n-1} Z_i \cap U_i\right)$. Since every nonempty open subset contains the generic point, S contains the generic point.

c It is immediate the closed (resp. open) subsets are constructible and stable under specialization (resp. generization). Suppose that $S = \coprod_{i=1}^n Z_i \cap U_i$ is a constructible set stable under specialization and let x be the generic point of an irreducible component of Z_i that intersects U_i nontrivially. Since S is closed under specialization, S contains every point in the closure of $\{x\}$. So S contains every point of every irreducible component of each Z_i. That is, $S \supseteq \cup Z_i$. Now consider a point $x \in S$. It is contained in so Z_i and so $S \subseteq \cup Z_i$. Hence $S = \cup Z_i$ is closed.

Now suppose S is a constructible set, stable under generization. Then S^c is a closed set, stable under specialization and therefore closed, so S is open.

d
$$f^{-1}\left(\coprod_{i=1}^n Z_i \cap U_i\right) = \coprod_{i=1}^n f^{-1}(Z_i \cap U_i) = \coprod_{i=1}^n f^{-1}(Z_i) \cap f^{-1}(U_i)$$

Since f is continuous, $f^{-1}Z_i$ is closed and $f^{-1}U_i$ is open, hence, the primage of a constructible set is constructible.

Exercise 3.19. *Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Then the image of any constructible subset of X is a constructible subset of Y. In particuler, $f(X)$, which need not be either open or closed, is a constructible subset of Y.*

a *Reduce to showing that $f(X)$ itself is constructible, in the case where X and Y are affine, integral, noetherian schemes, and f is a dominant morphism.*

b *In that case, show that $f(X)$ contains a nonempty open subset of Y by using the following result from commutative algebra: let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\phi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\phi(a) \neq 0$, then ϕ extends to a homomorphism ϕ' of B into K, such that $\phi'(b) \neq 0$.*

c *Use noetherian induction to complete the proof.*

d *Give some examples of morphisms $f : X \to Y$ of varieties over an algebraically closed field k, to show that $f(X)$ need not be open of closed.*

Solution. a If $S \subseteq X$ is a constructible set then we can restrict the morphism to $f|_S : S \to Y$. So it is enough to show that $f(X)$ itself is constructible. If $\{V_i\}$ is an affine cover of Y and $\{U_{ij}\}$ is an affine cover for each $f^{-1}(V_i)$ then if $f(U_{ij})$ is constructible for each i, j then $f(X) = \cup f(U_{ij})$ is constructible, so we can assume that X and Y are affine. Similarly, if $\{V_i\}$ are the irreducible components of Y and $\{U_{ij}\}$ the irreducible components of $f^{-1}(V_i)$, then if $f(U_{ij})$ is constructible for each i, j then $f(X) = \cup f(U_{ij})$ is constructible, so we can assume that X and Y are irreducible. Reducing a scheme (or ring) doesn't change the topology, so we can assume that X and Y are reduced. Putting these last two together, we can assume that X and Y are integral.

The last thing is to show that we can assume f is dominant. Suppose that $f(X)$ is constructible for every dominant morphism. We have an induced morphism $f' : X \to \overline{f(X)} = C$ from X into the closure of its image C. Then f' is certainly dominant, so $f'(X)$ is constructible in C. This means it can be written as $\coprod U_i \cap Z_i$ a disjoint union of locally closed subsets. Since C is closed in Y, each Z_i is still closed in Y. The subsets U_i on the otherhand, can be obtained as $U_i = V_i \cap C$ for some open subsets V_i of Y, by the definition of the induced topology on C. We now have, $f(X) = \coprod U_i \cap Z_i = \coprod V_i \cap (C \cap Z_i)$, which is constructible.

b If $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$ are affine integral noetherian schemes, and f is a dominant morphism, then $f : X \to Y$ is induced by a morphism $\phi : A \to B$. Since A is integral it has a generic point $\eta = (0)$ and since f is dominant, η is in the image of f. That is, there is some $\mathfrak{p} \subseteq B$ such that $\phi^{-1}\mathfrak{p} = (0)$. Since every element of B is contained in a prime ideal, in particular this means that ϕ is injective. By the assumption that f is finite type, we have that B is a finitely generated A-algebra. We now use the following lemma, with $b = 1$ to find an element a of A with the stated properties. We claim that $D(a) \subseteq f(\operatorname{Spec} B)$. To see this, suppose that $\mathfrak{p} \in D(a)$. So $a \neq \mathfrak{p}$ and the image of a under the composition $\phi : A \to A/\mathfrak{p} \to Frac(A/\mathfrak{p}) \to \overline{Frac(A/\mathfrak{p})}$ is nonzero. This means that we can lift ϕ to a homomorphism $\phi' : B \to K$ in which 1 is not zero. This means the kernel of ϕ' is a proper prime ideal $\mathfrak{q}$ of B. We now have

$A \cap \mathfrak{q} = A \cap \ker \phi' = \ker \phi = \mathfrak{p}$ and so $\mathfrak{q}$ gets sent to $\mathfrak{p}$ under $X \to Y$. Hence, $D(f)$ is contained in the image of f.

Lemma 4. Let $A \subseteq B$ be an inclusion of noetherian integral domains, such that B is a finitely generated A-algebra. Then given a nonzero element $b \in B$, there is a nonzero element $a \in A$ with the following property: if $\phi : A \to K$ is any homomorphism of A to an algebraically closed field K, such that $\phi(a) \neq 0$, then ϕ extends to a homomorphism ϕ' of B into K, such that $\phi'(b) \neq 0$.

Proof. First suppose that B is generated over A by one element. Then either $B \cong A[x]$ or $B \cong A[x]/(f(x))$ where $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is an irreducible polynomial of degree, say n (irreducible since B is integral).

In the first case, given $b = g(x) = b_0 + b_1x + \cdots + b_dx^d$ we choose $a = b_d$. Then for a homomorphism $\phi : A \to K$ into an algebraically closed field, we get a nonzero (since $\phi(b_d) \neq 0$) polynomial $\phi(g)(x) \in K[x]$ which has d roots. Since K is algebraically closed, we choose an element α that is not a root, and define $\phi' : B \to K$ by sending x to α.

Now suppose that $B \cong A[x]/(f(x))$. Let $b \in B$ and let $g(x) = b_0 + b_1x + \cdots + b_mx^m \in A[x]$ be a representative for b with $m < n$. Choose $a = b_m \in A$. Now given a morphim $\phi : A \to K$, we obtain polynomials $\phi(f)(x), \phi(g)(x) \in K[x]$ which since K is algebraically closed can be written as $a_n \prod_{i=1}^n (x - \alpha_i)$ and $b_m \prod_{i=1}^m (x - \beta_j)$ for some $\alpha_i, \beta_j \in K$. Note that the α_i are all distinct. Choosing an $\alpha_i \notin \{\beta_1, \ldots, \beta_m\}$ we get a morphism $\phi' : B \to K$ defined by $x \mapsto \alpha_i$ which extends ϕ. Now the image of b is $b_m \prod_{i=1}^m (\alpha_i - \beta_j)$ which is nonzero by our choice of α_i and the fact that $\phi(a) = \phi(b_m) \neq 0$. justify this

Now suppose that we have a B generated by n elements over A. So $B \cong A[x_1, \ldots, x_n]/\mathfrak{p}$ for some prime idea $\mathfrak{p}$. Let $\psi : A[x_1, \ldots, x_{n-1}] \to A[x_1, \ldots, x_n]$ denote the inclusion. It can be shown that $A' \overset{def}{=} A[x_1, \ldots, x_{n-1}]/\psi^{-1}\mathfrak{p}$ is a noetherian integral domain and $A' \subset B$ satisfies the assumptions of the lemma. So we have reduced to the case where B is generated by one element, which we have already proven. □ justify this

c We need to show that given a closed subset $Z \subseteq Y$ of Y, if $Z' \cap f(X)$ is constructible for every proper closed subset of Z, then $Z \cap f(X)$ is constructible. The result that $f(X)$ is constructible will then follow by Noetherian induction.

So suppose that $Z' \cap f(X)$ is constructible for every closed proper subset of a closed subset $Z \subseteq Y$. To be done.

d Consider the morphism $\operatorname{Spec} k[t, t^{-1}, (t-1)^{-1}] \to \operatorname{Spec} k[x, y]$ determined by $y \mapsto 0, x \mapsto t$. It is a composition

$$\mathbb{A}^1_k - \{0, 1\} \to \mathbb{A}^1_k - \{0\} \to \mathbb{A}^2_k$$

where the image of the second morphism is the hyperbola $xy = 1$, a closed subset of $\mathbb{A}^2_k$. The closure of the image of the composition is $xy = 1$ but the image is missing the point $(1, 1)$.

Exercise 3.20. Dimension. *Let X be an integral scheme of finite type over a field k (not necassarily algebraically closed). Use appropriate results from Section I.1 to prove the following:*

a *For any closed point $P \in X$, $\dim X = \dim \mathcal{O}_P$.*

b *Let $K(X)$ be the function field of X. Then $\dim X = tr.d.\ K(X)/k$.*

c *If Y is a closed subset of X, then $\operatorname{codim}(Y, X) = \inf\{\dim \mathcal{O}_{P,X} : P \in Y\}$.*

d *If Y is a closed subset of X, then $\dim Y + \operatorname{codim}(Y, X) = \dim X$.*

e *If U is a nonempty open subset of X, then $\dim U = \dim X$.*

f *If $k \subseteq k'$ is a field extension, then every irreducible component of $X' = X \times_k k'$ has dimension $= \dim X$.*

Lemma 5. *Let P be a point of X. Then there is an inclusion reversing bijection between irreducible subsets of X containing P and prime ideals of $\mathcal{O}_{X,P}$.*

Proof. Let $U = \operatorname{Spec} B$ be an open affine subset of X containing P and let $\mathfrak{p}$ be the prime ideal of B corresponding to P. So we have an isomorphism $\mathcal{O}_{X,P} \cong B_\mathfrak{p}$. We will use bijections

$$\begin{aligned}\{Z \subseteq X : Z \text{ cl. irr. and } P \in Z\} &\leftrightarrow \{Z \subseteq U : Z \text{ cl. irr. and } P \in Z\} \\ &\leftrightarrow \{I \subseteq B : I \text{ prime and } \mathfrak{p} \supseteq I\} \\ &\leftrightarrow \{I \subseteq B_\mathfrak{p} : I \text{ prime }\}\end{aligned}$$

The only one of these that is not immediately a bijection is the first one.

If Z is a closed irreducible subset of X containing P, then $U \cap Z$ is a nonempty closed subset of U. We can write $Z = (U^c \cap Z) \cup (\overline{U \cap Z})$. Since we assumed that Z is irreducible and intersects U we have $Z = \overline{U \cap Z}$. To see that $Z \cap U$ is irreducible, suppose we write it as $U \cap Z = Z_1 \cup Z_2$ for closed subsets $Z_1, Z_2 \subseteq U$. Since the Z_i are closed in U we have $Z_i = U \cap \overline{Z_i}$ where $\overline{Z_i}$ is their closure in X. Now $Z = \overline{U \cap Z} = \overline{Z_1 \cup Z_2} = \overline{Z_1} \cup \overline{Z_2}$ and so $Z = \overline{Z_i}$ for $i = 1$ or 2. Say 1. Then $Z \cap U = Z_1$ and so $Z \cap U$ is irreducible.

Conversely, if Z is a closed irreducible subset of U, then consider $\overline{Z}$. Writing it as $\overline{Z} = Z_1 \cup Z_2$ we get $Z = U \cap \overline{Z} = (U \cap Z_1) \cup (U \cap Z_2)$ and so either $Z = U \cap Z_1$ or $Z = U \cap Z_2$. Say $Z = U \cap Z_1$. Then $\overline{Z} \subseteq Z_1$ and so since $\overline{Z} = Z_1 \cup Z_2$ we find that $Z = Z_1$. So Z is irreducible. □

Solution. a Via the lemma, any chain of distinct prime ideals of $\mathcal{O}_P$ gives a chain of distinct closed irreducibles of X containing P, so $\dim \mathcal{O}_P \leq \dim X$. In particular, since any maximal chain of distinct closed irreducible subset of X ends in a closed point, say Q, we have equality for at least one point Q.

Now for P again an arbitrary point, the fraction field of $\mathcal{O}_P$ is the same as the function field of X (since it is integral) and so by Theorem I.1.8A(a), we have

$$\begin{aligned}\dim \mathcal{O}_P &= tr.d.\ K(\mathcal{O}_P)/k \\ &= tr.d.K(X)/k = tr.d.\ K(\mathcal{O}_Q)/k = \dim \mathcal{O}_Q = \dim X\end{aligned}$$

b This is contained in the proof of the previous part.

c By definition, $\operatorname{codim}(Y, X)$ is the infimum of the codimension of the closed irreducible subsets of Y and so we can assume that Y is irreducible. In the case where Y is irreducible, it has a unique generic point η, and $\mathcal{O}_{\eta,X} \supseteq \mathcal{O}_{P,X}$ for any $P \in Y$. This implies that $\dim \mathcal{O}_{\eta,X} \leq \dim \mathcal{O}_{P,X}$ for any P and hence, $\inf\{\dim \mathcal{O}_{P,X} : P \in Y\} = \dim \mathcal{O}_{\eta,X}$. Now the result follows from the lemma.

d Suppose that X, Y are affine and Y is irreducible. Then Y corresponds to a prime ideal $\mathfrak{p}$ in $B = \Gamma(X, \mathcal{O}_X)$. We have the following immediate equalities: $\dim Y =$ height $\mathfrak{p}$, $\dim B/\mathfrak{p} = \operatorname{codim}(Y, X), \dim X = \dim B$. From these and Theorem I.1.8A(b) the result follows.

Now suppose X and Y are not necassarily affine, but Y is still irreducible. Then it has a generic point η, and choosing an affine neighbourhood $U = \operatorname{Spec} A$ of η we get a new pair, U and $Y' = U \cap Y$ which are affine, and therefore satisfy

$$\dim Y' + \operatorname{codim}(Y', U) = \dim U$$

To be done.

e They have the same function fields, and so the equality follows from the second part of this exercise.

To be done.

f

Exercise 3.21. *Let R be a discrete valuation ring containing its residue field k. Let $X = \operatorname{Spec} R[t]$ be the affine line over $\operatorname{Spec} R$. Show that statements (a), (d), (e) of Exercise 3.20 are false for X.*

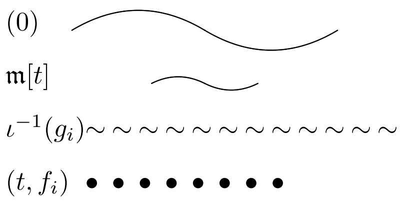
Solution. First we describe X. To list the points, we separate them into two groups by considering the preimages of the closed and generic points under $R[t] \to R$. Topologically, these are isomorphic to $\operatorname{Spec} k[t]$ and $\operatorname{Spec} K[t]$ and so the points of $\operatorname{Spec} R[t]$ are of four kinds. To describe these we name the morphisms $\pi : R[t] \to k[t]$ and $\iota : R[t] \to K[t]$. Then a point of $\operatorname{Spec} R[t]$ is one of

a $\iota^{-1}(0) = (0)$,

b $\iota^{-1}(f) = (f)$ for a polynomial $f \in R[t]$ irreducible in $K[t]$, and therefore also in $R[t]$ (note that by clearing denominators of coefficients, every polynomial in $K[t]$ can be written as a product of a unit in K and a polynomial in $R[t]$).

c $\pi^{-1}(0) = \mathfrak{m}[t]$,

d $\pi^{-1}(\overline{f}) = \mathfrak{m}[t] + (f)$ for a polynomial $f \in R[t]$ which is irreducible module $\mathfrak{m}[t]$, and therefore also irreducible $R[t]$.



To be done.

(a) A closed point of the form $\iota^{-1}\mathfrak{p}$. To be done.

(d) To be done.

(e)

Exercise 3.22. Dimension of the Fibres of a Morphism. To be done.

Solution.

Exercise 3.23. *If V, W are two varieties over an algebraically closed field k, and if $V \times W$ is their product, as defined in Exercises I.3.15 and I.3.16, and it t is the functor of II.2.6 then $t(V \times W) = t(V) \times_k t(W)$.* To be done.

Solution.

4 Separated and Proper Morphisms

Exercise 4.1. *Show that a finite morphism is proper.*

Solution. Let $f : X \to Y$ be the finite morphism. Finite implies finite type so we only need to show that f is universally closed and separated.

f is separated. We want to show that $X \to X \times_Y X$ is a homeomorphism onto a closed subset of $X \times_Y X$. It is enough to show this locally so take an open affine cover $\{V_i = \operatorname{Spec} B_i\}$ of Y. Since f is finite, the preimages of the V_i are also affine, say $U_i = \operatorname{Spec} A_i$. Now $U_i \times_{V_i} U_i$ are open affine subsets of $X \times_Y X$ which cover the image of the diagonal and so it is enough to show that each $\Delta^{-1} U_i \times_{V_i} U_i \to U_i \times_{V_i} U_i$ is a closed immersion. Now the preimages are $\Delta^{-1} U_i \times_{V_i} U_i = U_i$ so we want to show that the scheme morphism induced by $A_i \otimes_{B_i} A_i \to A_i$ is a closed immersion. Since this ring homomorphism is surjective, the result follows from Exercise II.2.18(c).

f is universally closed. The proof of Exercise II.3.13(d) goes through to show that finite morphisms are stable under base change (in fact, the proof becomes easier). Secondly, we know that finite morphisms are closed (Exercise II.3.5) and therefore finite morphisms are universally closed.

Exercise 4.2. *Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced , or (b) Y is nonseparated.*

Solution. Let U be the dense open subset of X on which f and g agree. Consider the pullback square(s):

$$\begin{tikzcd} U \arrow[r, equal] \arrow[d] & U \arrow[d] \\ Z \arrow[r, "\Delta'"] \arrow[d] & X \arrow[d, "f,g"] \\ Y \arrow[r, "\Delta"] & Y \times_S Y \end{tikzcd}$$

Since Y is separated, the lower horizontal morphism is a closed immersion. Closed immersions are stable under base extension (Exercise II.3.11) and so $Z \to X$ is also a closed immersion. Now since f and g agree on U, the image of U in $Y \times_S Y$ is contained in the diagonal and so the pullback is, again U (at least topologically. But this means that $U \to X$ factors through Z, whose image is a closed subset of X. Since U is dense, this means that $sp\ Z = sp\ X$. Since $Z \to X$ is a closed immersion, the morphism of sheaves $\mathcal{O}_X \to \mathcal{O}_Z$ is surjective. Consider an open affine $V = \operatorname{Spec} A$ of X. Restricted to V, the morphism $Z \cap V \to V$ continues to be a closed immersion and so $Z \cap V$ is an affine scheme, homeomorphic to V, determined by an ideal $I \subseteq A$. Since $\operatorname{Spec} A/I \to \operatorname{Spec} A$ is a homeomorphism, I is contained in the nilradical. But A is reduced and so $I = 0$. Hence, $Z \cap V = Z$ and therefore $Z = X$.

a Consider the case where $X = Y = \operatorname{Spec} k[x, y]/(x^2, xy)$, the affine line with nilpotents at the origin, and consider the two morphisms $f, g : X \to Y$, one the identity and the other defined by $x \mapsto 0$, i.e. killing the nilpotents at the origin. These agree on the complement of the origin which is a dense open subset but the sheaf morphism disagrees at the origin.

b Consider the affine line with two origins, and let f and g be the two open inclusions of the regular affine line. They agree on the complement of the origin but send the origin two different places.

Exercise 4.3. *Let X be a separated scheme over an affine scheme S. Let U and V be open affine subsets of X. Then $U \cap V$ is also affine. Give a example to show that this fails if X is not separated.*

Solution. Consider the pullback square

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \times_S V \arrow[d] \\ X \arrow[r, "\Delta"] & X \times_S X \end{tikzcd}$$

Since X is separated over S the diagonal is a closed immersion. Closed immersions are stable under change of base (Exercise II.3.11(a)) and so $U \cap V \to U \times_S V$ is a closed immersion. But $U \times_S V$ is affine since all of U, V, S are. So $U \cap V \to U \times_S V$ is a closed immersion into an affine scheme and so $U \cap V$ itself is affine (Exercise II.3.11(b)).

For an example when X is not separated consider the affine plane with two origins X and the two copies U, V of the usually affine plane inside it as open affines. The intersection of U and V is $\mathbb{A}^2 - \{0\}$ which is not affine.

Exercise 4.4. *Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure is proper over S.*

Solution. Since $Z \to S$ is proper and $Y \to S$ separated it follows from Corollary II.4.8e that $Z \to Y$ is proper. Proper morphisms are closed and so $f(Z)$ is closed.

$f(Z) \to S$ is finite type. This follows from it being a closed subscheme of a scheme Y of finite type over S (Exercise II.3.13(a) and (c)).

$f(Z) \to S$ is separated. This follows from the change of base square and the fact that closed immersions are preserved under change of base.

$$\begin{tikzcd} f(Z) \arrow[r] \arrow[d, "\Delta"] & Y \arrow[d, "\Delta"] \\ f(Z) \times_S f(Z) \arrow[r] & Y \times_S Y \end{tikzcd}$$

$f(Z) \to S$ is universally closed. Let $T \to S$ be some other morphism and consider the following diagram

$$\begin{tikzcd}
T \times_S Z \arrow[r] \arrow[d, "f'"] & Z \arrow[d, "f"] \\
T \times_S f(Z) \arrow[r] \arrow[d, "s'"] & f(Z) \arrow[d, "s"] \\
T \arrow[r] & S
\end{tikzcd}$$

Our first task will be to show that $T \times_S Z \to T \times_S f(Z)$ is surjective. Suppose $x \in T \times_S f(Z)$ is a point with residue field $k(x)$. Following it horizontally we obtain a point $x' \in f(Z)$ with residue field $k(x') \subset k(x)$ and this lifts to a point $x'' \in Z$ with residue field $k(x'') \supset k(x')$. Let k be a field containing both $k(x)$ and $k(x'')$. The inclusions $k(x''), k(x) \subset k$ give morphisms $\operatorname{Spec} k \to T \times_S f(Z)$ and $\operatorname{Spec} k \to Z$ which agree on $f(Z)$ and therefore lift to a morphism $\operatorname{Spec} k \to T \times_S, Z$ giving a point in the preimage of x. So $T \times_S Z \to T \times_S f(Z)$ is sujective.

Now suppose that $W \subseteq T \times_S f(Z)$ is a closd subset of $T \times_S f(Z)$. Its vertical preimage $(f')^{-1}W$ is a closed subset of $T \times_S Z$ and since $Z \to S$ is universally closed the image $s' \circ f'((f')^{-1}(W))$ in T is closed. As f' is surjective, $f'((f')^{-1}(W)) = W$ and so $s' \circ f'((f')^{-1}(W)) = s'(W)$. Hence, $T \times_S f(Z)$ is closed in T.

Exercise 4.5. *Let X be an integral scheme of finite type over a field k, having function field K. We say that a valuation of K/k has center x on X if its valuation ring R dominates the local ring $\mathcal{O}_{x,X}$.*

a *If X is separated over k, then the center of any valuation of K/k on X (if it exists) is unique.*

b *If X is proper over k, then every valuation of K/k has a unique center on X.*

c *Prove the converses of (a) and (b).*

d *If X is proper over k, and if k is algebraically closed, show that $\Gamma(X, \mathcal{O}_X) = k$.*

Solution. a Let R be the valuation ring of a valuation on K. Having center on some point $x \in X$ is equivalent to an inclusion $\mathcal{O}_{x,X} \subseteq R \subseteq K$ (such that $\mathfrak{m}_R \cap \mathcal{O}_{x,X} = \mathfrak{m}_x$) which is equivalent to a diagonal morphism in the diagram

$$\begin{tikzcd}
\operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec} R \arrow[r] \arrow[ur] & \operatorname{Spec} k
\end{tikzcd}$$

But by the valuative criterion for separability this diagonal morphism (if it exists) is unique. Therefore, the center, if it exists, is unique.

b Same argument as the previous part.

c

d Suppose that there is some $a \in \Gamma(X, \mathcal{O}_X)$ such that $a \notin k$. Consider the image $a \in K$. Since k is algebraically closed, a is transcendental over k and so $k[a^{-1}]$ is a polynomial ring. Consider the localization $k[a^{-1}]_{(a^{-1})}$. This is a local ring contained in K and therefore there is a valuation ring $R \subset K$ that dominates it. Since $\mathfrak{m}_R \cap k[a^{-1}]_{(a^{-1})} = (a^{-1})$ we see that $a^{-1} \in \mathfrak{m}_R$.

Now since X is proper, there exists a unique dashed morphism in the diagram on the left.

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k \end{tikzcd} \qquad \begin{tikzcd} K & \Gamma(X, \mathcal{O}_X) \arrow[l] \arrow[dl, dashed] \\ R \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

Taking global sections gives the diagram on the right which implies that $a \in R$ and so $v_R(a) \geq 0$. But $a^{-1} \in \mathfrak{m}_R$ and so $v_R(a^{-1}) > 0$ This gives a contradiction since $0 = v_R(1) = v_R(\frac{a}{a}) = v_R(a) + v_R(\frac{1}{a}) > 0$.

Exercise 4.6. *Let $f : X \to Y$ be a proper morphisms of affine varieties over k. Then f is a finite morphism.*

Solution. Since X and Y are affine varieties, by definition they are integral and so f comes from a ring homomorphism $B \to A$ where A and B are integral. Let $K = k(A)$. Then for valuation ring R of K that contains $\phi(B)$ we have a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

Since f is proper, the dashed arrow exists (uniquely, but we don't need this). From Theorem II.4.11A the integral closure of $\Phi(B)$ in K is the intersection of all valuation rings of K which contain $\phi(B)$. As the dashed morphism exists for any valuation ring K containing $\phi(B)$ so it follows that A is contained in the integral closure of $\phi(B)$ in K. Hence every element of A is integral over B, and this together with the hypothesis that f is of finite type implies that f is finite.

Exercise 4.7. Schemes over $\mathbb{R}$.

a Let X be a separated scheme of finite type over $\mathbb{C}$, let σ be a semilinear involution on X, and assume that for any two points $x_1, x_2 \in X$ there is an open affine subset containing both of them. Show that there is a unique separated scheme X_0 of finite type over $\mathbb{R}$ such that $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$, and such that this isomorphism identifies the conjugation involution of X with the one on $X_0 \times_{\mathbb{R}} \mathbb{C}$.

For the following statements, X_0 will denote a separated scheme of finite type over $\mathbb{R}$, and X, σ will denote the corresponding scheme with oncolution over $\mathbb{C}$.

b *Show that X_0 is affine if and only if X is.*

c *If X_0, Y_0 are two such schemes over $\mathbb{R}$, then to give amorphism $f_0 : X_0 \to Y_0$ is equivalent to giving a morphism $f : X \to Y$ which commutes with the involutions.*

d *If $X \cong \mathbb{A}^1_{\mathbb{C}}$ then $X_0 \cong \mathbb{A}^1_{\mathbb{R}}$.*

e *If $X \cong \mathbb{P}^1_{\mathbb{C}}$ then either $X_0 \cong \mathbb{P}^1_{\mathbb{R}}$ or X_0 is isomorhic to the conic in $\mathbb{P}^2_{\mathbb{R}}$ given by the homogeneous equation $x_0^2 + x_1^2 + x_2^1 = 0$.*

Solution. a

b Since $X_0 \times_{\mathbb{R}} \mathbb{C} \cong X$ if X_0 is affine then certainly X is. Conversely, if $X = \operatorname{Spec} A$ is affine then as above, $X_0 = \operatorname{Spec}(A^\sigma)$.

c Certainly, given f_0 we get an f that commutes with the involution. Conversely, suppose that we are given f that commutes with σ. In the case where Y and X are affine $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ we get and induced morphism on σ invarants $A^\sigma \to B^\sigma$ and this gives us the morphism $X_0 \to Y_0$. If X and Y are not affine then take a cover of X by σ preserved open affines $\{U_i\}$ and for each i take a cover $\{V_{ij}\}$ of $f^{-1}U_i$ with each V_{ij} a σ preserved open affine of Y. Let $\pi : Y \to Y_0$ is the projection and recall that it is affine (part (b)). By the affine case, we get $\pi V_{ij} \to \pi U_i$ and by the way these are defined it can be seen that they glue together to give a morphism $Y_0 \to X_0$.

d See Case II of part (e).

e *Case I: σ has no closed fixed points.* Let $x \in X \cong \mathbb{P}^1_{\mathbb{C}}$ be a closed point and consider the space $U = X \backslash \{x, \sigma x\}$. Since σ has no fixed points and $PGL_{\mathbb{C}}(1)$ is transitive on pairs of distinct points we can find a $\mathbb{C}$-automorphism f that sends $(x, \sigma x)$ to $(0, \infty)$ and therefore assume that x and σx are 0 and ∞ and so $U \cong \operatorname{Spec} \mathbb{C}[t, t^{-1}]$. Note that the lift of σ is still $\mathbb{C}$-semilinear by the commutativity of the following diagram.

$$\begin{array}{ccccccc}
X & \xrightarrow{f} & X & \xrightarrow{\sigma} & X & \xrightarrow{f^{-1}} & X \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{C} & \xrightarrow{id} & \mathbb{C} & \xrightarrow{\alpha} & \mathbb{C} & \xrightarrow{id} & \mathbb{C}
\end{array}$$

Now σ induces an invertible semilinear $\mathbb{C}$-algebra homomorphism on $\mathbb{C}[t, t^{-1}]$. We will show that σ acts via $t \mapsto -t^{-1}$. The element t must get sent to something invertible and therefore gets sent to something of the form

at^k for some k an integer. [1] Since $\sigma^2 = id$ it follows that $k = \pm 1$. Furthermore, by considering σ on the function field $\mathbb{C}(t)$ it can be seen that $k = -1$ since otherwise the valution ring $\mathbb{C}[t]_{(t)} \subset \mathbb{C}(t)$ would be fixed, implying that σ has a fixed point. Now $t\sigma t = a$ is fixed by σ and σ acts by conjugation on constants, we see that $a \in \mathbb{R}$. If a is positive, the ideal $(t - \sqrt{a})$ is preserved contradicting the assumption of no fixed points, so $a \in \mathbb{R}_{\leq 0}$. Now we make a change of coordinates by replacing t with $\frac{1}{\sqrt{-a}}$. This amounts to choosing a slightly different element of $PGL(1)$ at the beginning when we were sending x and σx to 0 and ∞. With this new t our involution is $t \mapsto -t^{-1}$.

Now we rewrite $\mathbb{C}[t, t^{-1}]$ as $\frac{\mathbb{C}[\frac{X}{Z}, \frac{Y}{Z}]}{(1+\frac{XY}{Z^2})}$ via

$$\left\{ \begin{array}{c} \frac{X}{Z} = t^{-1} \\ \frac{Y}{Z} = -t \end{array} \right\}$$

so the involution acts by switching $\frac{X}{Z}$ and $\frac{Y}{Z}$ (and conjugation on scalars). Now consider the two subrings $\mathbb{C}[-t]$ and $\mathbb{C}[t^{-1}]$ of the function field $\mathbb{C}(t)$. We have isomorphisms

$$\frac{\mathbb{C}[\frac{Y}{X}, \frac{Z}{X}]}{(\frac{Y}{X} + (\frac{Z}{X})^2)} \cong \mathbb{C}[-t] \qquad t = \frac{Z}{X}$$

$$\frac{\mathbb{C}[\frac{X}{Y}, \frac{Z}{Y}]}{(\frac{X}{Y} + (\frac{Z}{Y})^2)} \cong \mathbb{C}[t^{-1}] \quad -t^{-1} = \frac{Z}{Y}$$

and σ acts by swapping these two rings (and conjugation on scalars). These three open affines patch together in a way compatible with σ to form an isomorphism

$$\operatorname{Proj} \frac{\mathbb{C}[X, Y, Z]}{(XY + Z^2)} \cong \mathbb{P}^1_{\mathbb{C}}$$

where σ acts on the quadric by swapping X and Y, and conjugation on scalars. Making a last change of coordinates

$$U = \frac{1}{2}(X + Y) \qquad V = \frac{i}{2}(Y - X)$$

we finally get the isomorphism

$$\mathcal{Q} = \operatorname{Proj} \frac{\mathbb{C}[X, Y, Z]}{(U^2 + V^2 + Z^2)} \cong \operatorname{Proj} \frac{\mathbb{C}[X, Y, Z]}{(XY + Z^2)} \cong \mathbb{P}^1_{\mathbb{C}} = X$$

[1] *The group of units in* $\mathbb{C}[t, t^{-1}]$ *is* $\{at^k : a \neq 0, k \in \mathbb{Z}\}$. Suppose that $(\sum_{i=m'}^m a_i)(\sum_{i=n'}^n b_i) = 1$ the term of highest order in the product is $a_m b_n t^{n+m} = 1$ and so $n = 1 - m$. Similarly, the term of lowest order is $a_{m'} b_{n'} t^{n'+m'} = 1$ and so $n' = 1 - m'$. Now $n' \leq n = 1 - m \leq 1 - m' = n'$ and so $n = n'$. The same argument shows that $m = m'$. Hence, both elements of the product are of the form at^k for some k.

where σ acts on $\mathscr{Q}$ by conjugation of scalars alone. Hence

$$X_0 \cong \mathscr{Q}_0 = \operatorname{Proj} \frac{\mathbb{R}[X,Y,Z]}{(U^2 + V^2 + Z^2)}$$

Case II: σ has at least one fixed point. Now suppose that σ fixes a closed point x. This means that σ restricts to a semilinear automorphism of the complement of the fixed point $\operatorname{Spec} \mathbb{C}[t] \subset \mathbb{P}^1_{\mathbb{C}}$. Since σ is invertible, t gets sent to something of the form $at + b$. There exists a change of coordinates $s = ct + d$ such that $\sigma s = s$ and so in these new coordinates we get a σ invariant isomorphism $X \cong \mathbb{P}^1_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$.

Exercise 4.8. *Let $\mathscr{P}$ be a property of morphisms of schemes such that:*

a *a closed immersion has $\mathscr{P}$;*

b *a composition of two morphisms having $\mathscr{P}$ has $\mathscr{P}$;*

c *$\mathscr{P}$ is stable under base extension.*

Then show that

d *a product of morphisms having $\mathscr{P}$ has $\mathscr{P}$;*

e *if $f : X \to Y$ and $g : Y \to Z$ are two morphisms, and if $g \circ f$ has $\mathscr{P}$ and g is separated, then f has $\mathscr{P}$;*

f *If $f : X \to Y$ has $\mathscr{P}$, then $f_{red} : X_{red} \to Y_{red}$ has $\mathscr{P}$.*

Solution. d Let $X \xrightarrow{f} Y$ and $X' \xrightarrow{f'} Y'$ be the morphisms. The morphism $f \times f'$ is a composition of base changes of f and f' as follows:

$$\begin{tikzcd}
 & X \arrow[dd] \\
X \times X' \arrow[ur] \arrow[dd] & \\
 & Y \\
Y \times X' \arrow[ur] \arrow[dr] \arrow[dd] & \\
 & X' \arrow[dd] \\
Y \times Y' \arrow[dr] & \\
 & Y'
\end{tikzcd}$$

Therefore $f \times f'$ has property $\mathscr{P}$.

e Same argument as above but we should also note that since g is separated the diagonal morphism $Y \to Y \times_Z Y$ is a closed embedding and therefore satsifies $\mathscr{P}$.

$$\begin{tikzcd}
 & Y \arrow[d] \\
X \arrow[ur] \arrow[d] & Y \times_Z Y \\
X \times_Z Y \arrow[ur] \arrow[dr] \arrow[d] & \\
 & X \arrow[d] \\
Y \arrow[r] & Z
\end{tikzcd}$$

f Consider the factorization

$$\begin{tikzcd}
X_{red} \arrow[dr, "\Gamma_{f_{red}}"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "f_{red}"'] & & \\
 & Y_{red} \times_Y X_{red} \arrow[r] \arrow[d] & X_{red} \arrow[d] \\
 & Y_{red} \arrow[r] & Y
\end{tikzcd}$$

The morphism $X_{red} \to X \to Y$ is a composition of a closed immersion and a morphism with property scP and therefore it has property $\mathscr{P}$. Therefore the vertical morphism out of the fibre product is a base change of a morphism with property $\mathscr{P}$ and therefore, itself has property $\mathscr{P}$. To se that f_{red} has property $\mathscr{P}$ it therefore remains only to see that the graph $\Gamma_{f_{red}}$ has property $\mathscr{P}$ for then f_{red} will be a composition of morphisms with property $\mathscr{P}$. To see this, recall that the graph is following base change

$$\begin{tikzcd}
X_{red} \arrow[r] \arrow[d, "\Gamma"] & Y_{red} \arrow[d, "\Delta"] \\
X_{red} \times_Y Y_{red} \arrow[r] & Y_{red} \times_Y Y_{red}
\end{tikzcd}$$

But $Y_{red} \times_Y Y_{red} = Y_{red}$ and $\Delta = id_{Y_{red}}$ and so Δ is a closed immersion. Hence, Γ is a base change of a morphism with property $\mathscr{P}$.

Exercise 4.9. *Show that a composition of projective morphisms is projective. Conclude that projective morphisms have properties (a)-(f) of Exercise II.4.8 above.*

Solution. Let $X \overset{f}{\to} Y \overset{g}{\to} Z$ be two projective morphisms. This gives rise to a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[dr, "f"'] & \mathbb{P}^r \times Y \arrow[r, "id\times g'"] \arrow[d] & \mathbb{P}^r \times \mathbb{P}^s \times Z \arrow[d] \\
& Y \arrow[r, "g'"] \arrow[dr, "g"'] & \mathbb{P}^s \times Z \arrow[d] \\
& & Z
\end{tikzcd}$$

where f' and g' (and therefore $id \times g'$) are closed immersions. Now using the Segre embedding the projection $\mathbb{P}^r \times \mathbb{P}^s \times Z \to Z$ factors as

$$\mathbb{P}^r \times \mathbb{P}^s \times Z \to \mathbb{P}^{rs+r+s} \times Z \to Z$$

So since the Segre embedding is a closed immersion then we are done since we have found a closed immersion $X \to \mathbb{P}_Z^{rs+r+s}$ which factors $g \circ f$.

Exercise 4.10. Chow's Lemma. *Let X be proper over a noetherian scheme S. Then there is a scheme X' and a morphism $g : X' \to X$ such that X' is projective over S, and there is an open dense subset $U \subseteq X$ such that g induces an isomorphism of $g^{-1}(U)$ to U. Prove this result in the following steps.*

a *Reduce to the case X irreducible.*

b *Show that X can be covered by a finite number of open subsets $U_i, i = 1, \ldots, n$, each of which is quasi-projective over S. Let $U_i \to P_i$ be an open immersion of U_i into a scheme P_i which is projective over S.*

c *Let $U = \bigcap U_i$ and consider the map*

$$U \to X \times_S P_1 \times_s \cdots \times_S P_n$$

deduced from the give maps $U \to X$ and $U \to P_i$. Let X' be the closed image subscheme structure. Let $g : X' \to X$ be the projection onto the first factor, and let $h : X' \to P = P_1 \times_S \cdots \times_S P_n$ be the projection onto the product of the remaining factors. Show that h is a closed immersion, hence X' is projective over S.

d *Show that $g^{-1}(U) \to U$ is an isomorphism, thus completing the proof.*

Exercise 4.11. Valuative criteria using discrete valuation rings.

a *If $\mathcal{O}, \mathfrak{m}$ is a noetherian local domain with quotient field K, and if L is a finitely generated field extension of K, then there exists a discrete valuation ring R of L dominating $\mathcal{O}$.*

b *Let $f : X \to Y$ be a morphism of finite type of noetherian schemes. Show that f is separated (resp. proper) if and only if the criterion of 4.3 (resp. 4.7) holds for all discrete valuation rings.*

Exercise 4.12. Examples of Valuation Rings. *Let k be an algebraically closed field.*

a *If K is a function field of dimension 1 over k, then every valuation ring of K/k (except for K itself) is discrete.*

b *If K/k is a function field of dimension two, there are several different kinds of valuations. Suppose that X is a complete nonsingular surface with function field K.*

(a) *If Y is an irreducible curve on X, with generic point x_1, then the local ring $R = \mathcal{O}_{x_1,X}$ is a discrete valuation ring of K/k with center at the (nonclosed) point x_1.*

(b) *If $f : X' \to X$ is a birational morphism, and if Y' is an irreducible curve in X' whose image in X is a single closed point x_0, then the local ring R of the generic point of Y' on X' is a discrete valuation rign of K/k with center at the closed point x_0 on X.*

(c) *Let $x_0 \in X$ be a closed point. Let $f : X_1 \to X$ be the blowing up of x_0 and let $E_1 = f^{-1}x_0$ be the exceptional curve. Choose a closed point $x_1 \in E_1$, let $f_2 : X_2 \to X_1$ be the blowing-up of x_1, and let $E_2 = f_2^{-1}x_1$ be the exceptional curve. Repeat. In this manner we obtain a sequence of varieties X_i with closed points x_i chosen on them, and for each i, the local ring $\mathcal{O}_{X_{i+1},x_{i+1}}$ dominates $\mathcal{O}_{X_i,x_i}$. Let $R_0 = \cup_{i=0}^{\infty}\mathcal{O}_{X_i,x_i}$. Then R_0 is a local ring, so it is dominated by some valuation ring R of K/k. Show that R is a valuation ring of K/k and that it has center x_0 on X. When is R a discrete valuation ring?*

Solution. a Let $R \subset K$ be a valuation ring of K. We will show that $\mathfrak{m}_R$ is principal, which will imply that R is discrete. Let $t \in \mathfrak{m}_R$. If $(t) = \mathfrak{m}_R$ then we are done. If not choose some $s \in \mathfrak{m}_R \backslash (t)$. Note that t is transcendental over k. To see this, suppose that it satisfies some polynomial $\sum_{i=0}^{n} a_i t^i = 0$ chosen so that $a_0 \neq 0$. Then $a_0 = t \sum a_i t^{i-1}$ and so $a_0 \in (t)$. But a_0 is a unit and so we get a contradiction, hence there is no such polynomial. Now since K has dimension 1 and t is transcendental, K is a finite algebraic extension of $k(t)$. The element $s \notin (t)$ and so it is algebraic over k. Hence, it satisfies some polynomial with coefficients in $k(t)$. Let $\sum_{i=0}^{n} a_i s^i = 0$ be such a polynomial, chosen so that $a_0 \neq 0$. Again, this implies that $a_0 = s \sum a_i s^{i-1}$. Write $a_0 = \frac{f(t)}{g(t)}$. Then we have $\frac{f(t)}{g(t)} = s \sum a_i s^{i-1}$ and so $f(t) = g(t) s \sum a_i s^{i-1}$ implying that $f(t) \in (s) \subseteq \mathfrak{m}_R$. Since $tin\mathfrak{m}_R$, the polynomial $f(t)$ can't have any constant term (otherwise this term would be in $\mathfrak{m}_R$ contradicting the fact that it is a proper ideal) and so $t \in (s)$ and hence $(s) \supset (t)$. If $(s) = \mathfrak{m}_R$ we are done. If not, repeat the process to obtain increasing chain of ideals $(t) \subset (s) \subset (s_1) \subset \ldots$ all contained in $\mathfrak{m}_R$. Since R is noetherian, this chain must terminate and we find so s_i such that $(s_i) = \mathfrak{m}_R$. Hence, $\mathfrak{m}_R$ is principal, and therefore by Theorem I.6.2A the valuation ring R is discrete.

5 Sheaves of Modules

Exercise 5.1. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathscr{E}$ be a locally free $\mathcal{O}_X$-module of finite rank. We define the dual of $\mathscr{E}$ denoted $\check{\mathscr{E}}$ to be the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathcal{O}_X)$.*

a Show that $(\check{\mathscr{E}})^\vee \cong \mathscr{E}$.

b For any $\mathcal{O}_X$-module $\mathscr{F}$, we have $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{F}) \cong \check{\mathscr{E}} \otimes \mathscr{F}$.

c For any $\mathcal{O}_X$-modules $\mathscr{F}, \mathscr{G}$, we have $\hom_{\mathcal{O}_X}(\mathscr{E} \otimes \mathscr{F}, \mathscr{G}) \cong \hom_{\mathcal{O}_X}(\mathscr{F}, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{E}, \mathscr{G}))$.

d Projection Formula. *If $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces, if $\mathscr{F}$ is an $\mathcal{O}_X$-module, and if $\mathscr{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then there is a natural isomorphism $f_*(\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{E}) \cong f_*(\mathscr{F}) \otimes_{\mathcal{O}_Y} \mathscr{E}$.*

Solution. a Even without any conditions on $\mathscr{E}$ there is a canonical morphism $\mathscr{E} \to \mathscr{H}om(\mathscr{H}om(\mathscr{E}, \mathcal{O}_X), \mathcal{O}_X)$ defined by evaluation. Given an open subset U we want to define for every section, $s \in \mathscr{E}(U)$ a natural transformation $\hom(\mathscr{E}, \mathcal{O}_X)|_U \to \mathcal{O}_X|_U$. For every open subset $V \subseteq U$ we define an element of $\hom_{\mathcal{O}_X}(\mathscr{E}(V), \mathcal{O}_X(V)) \to \mathcal{O}_X(V)$ by evaluating at $s|_V$.

In the case where $\mathscr{E}$ is locally free, it can be seen that this canonical morphism is an isomorphism by looking at the stalks. On the stalks, this morphism is the canonical morphism of $\mathcal{O}_{X,x}$-modules,

$$\mathscr{E}_x \to \hom_{\mathcal{O}_{X,x}}(\hom_{\mathcal{O}_{X,x}}(\mathscr{E}_x, \mathcal{O}_{X,x}))$$

Since $\mathscr{E}_x$ is free, this morphism is an isomorphism.

b Again, we have a canonical isomorphism

$$\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes_{\mathcal{O}_X} \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{F})$$

To define this consider the presheaf $U \mapsto \mathscr{H}om(\mathcal{O}_X, \mathscr{E})(U) \otimes_{\mathcal{O}_X(U)} \mathscr{F}(U)$. Denote this presheaf by $\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes^{pre}_{\mathcal{O}_X} \mathscr{F}$. Its sheafification is $\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes_{\mathcal{O}_X} \mathscr{F}$ so we need just define a morphism

$$\mathscr{H}om(\mathcal{O}_X, \mathscr{E}) \otimes^{pre}_{\mathcal{O}_X} \mathscr{F} \to \mathscr{H}om(\mathscr{E}, \mathscr{F})$$

and then sheafification will give a morphism of the kind we require. Define the morphism of presheaves section wise by noticing that every section $s \in \mathscr{F}(U)$ gives a natural transformation $\mathcal{O}_X|_U \to \mathscr{F}|_U$ by multiplying by the restriction of s. Then we define

$$\begin{aligned} \hom(\mathcal{O}_X|_U, \mathscr{E}|_U) \otimes_{\mathcal{O}_X|_U} \mathscr{F}(U) &\to \hom(\mathscr{E}|_U, \mathscr{F}|_U) \\ \phi \otimes s &\mapsto \left(\mathscr{E}|_U \xrightarrow{\phi} \mathcal{O}_X|_U \xrightarrow{s} \mathscr{F}|_U \right) \end{aligned}$$

The stalk of the morphism we have just defined is the obvious canonical morphism of $\mathcal{O}_{X,x}$-modules. When $\mathscr{E}$ is locally free, $\mathscr{E}_x$ is free and so this is an isomorphism.

Exercise 5.2. *Let R be a discrete valuation ring with quotient field K, and let $X = \operatorname{Spec} R$.*

a *To give an $\mathcal{O}_X$-module is equivalent to giving an R-module M, a K-vector space L, and a homomorphism $\rho : M \otimes_R KtoL$.*

b *That $\mathcal{O}_X$-module is quasi-coherent if and only if ρ is an isomorphism.*

Solution. a Since R is a discrete valuation ring, there are two nontrivial open subsets of $\operatorname{Spec} R$: the total space and $U = \{\eta\}$ the set containing only the generic point. So by definition, an $\mathcal{O}_X$-module consists of an $\mathcal{O}_X(X) = R$ module M and an $\mathcal{O}_X(U) = K$ module L, together with an R-module homomorphism $M \to L_R$ where L_R is L considered as an R-module. Since restriction and extension of scalars are adjoint the R-module homomorphism is that same as giving an K-module homomorpism $M \otimes_R K \to L$.

b Let $\mathscr{F}$ be the $\mathcal{O}_X$-module. If $\mathscr{F}$ is quasi-coherent then every point has a neighbourhood on which $\mathscr{F}$ has the form $\widetilde{M}$. The only neighbourhood of the unique closed point of $\operatorname{Spec} R$ is the whole space, so $\mathscr{F}$ is of the form $\widetilde{M}$ and therefore $L = \mathscr{F}(U) = M_{(0)} = M \otimes_R K$. Conversely, suppose $M \otimes_R K \to L$ is an isomorphism. $M \otimes_R K \to L$ is the adjunct morphism of $M \to L_R$ so we get a factorization $M \to M \otimes_R K \to L_R$ where the first morphism is the unit of the adjunction. This factorization means gives a morphism of sheaves $\widetilde{M} \to \mathscr{F}$ and since it is an isomorphism, the morphism of sheaves is. So $\mathscr{F}$ is quasi-coherent.

Exercise 5.3. *Let $X = \operatorname{Spec} A$ be an affine scheme. Show that the functors $^\sim$ and Γ are adjoint.*

Solution. We begin by defining a morphism of sheaves $\eta : \Gamma(X, \mathscr{F})^\sim \to \mathscr{F}$. On a distinguished affine open, $D(a)$ of $\operatorname{Spec} A$ we have $\Gamma(X, \mathscr{F})^\sim(D(a)) \cong \Gamma(X, \mathscr{F})_a$ and so restriction $\mathscr{F}(X) \to \mathscr{F}(D(a))$ induces a morphism $\Gamma(X, \mathscr{F})^\sim(D(a)) \to \mathscr{F}(D(a))$. If we have two distinguished open subsets $D(a), D(b)$ of X, it can be seen that the restriction of the morphisms agree on the intersection, and so we have defined a morphism of sheaves. Furthermore, since $X = D(1)$, on global sections we have the identity $\Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$.

Now for a morphism $\phi \in \hom_A(M, \Gamma(X, \mathscr{F}))$ we define a morphism in $\hom_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F})$ by $\eta \circ \widetilde{\phi}$. By the observation that $\widetilde{\phi}_X : \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F})$ is the identity, we see that $\Gamma : \hom_{\mathcal{O}_X}(\widetilde{M}, \mathscr{F}) \to \hom_A(M, \Gamma(X, \mathscr{F}))$ is an inverse to this assignment. Hence we have a bijection.

Exercise 5.4. *Show that a sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$ on a scheme X is quasi-coherent if ad only if every point of X has a neighbourhood U, such that $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves on U. If X is noetherian, then $\mathscr{F}$ is coherent if and only if it is locally a cokernel of a morphism of free sheaves of finite rank.*

Solution. First suppose that $\mathscr{F}$ is quasi-coherent. Then every point has a neighbourhood U on which $\mathscr{F}|_U \cong \widetilde{\mathscr{F}(U)}$. Since every module is a cokernel of a morphism between free modules, [1] $\mathscr{F}(U)$ is the cokernel of a morphism $F_1 \to F_0$ of free $\mathcal{O}_X(U)$-modules. Since the functor $\sim$ is a left adjoint it is right exact and therefore preserves cokernels. So $\mathscr{F}|_U$ is isomorphic to the cokernel of $\widetilde{F_1} \to \widetilde{F_0}$. The functor $\sim$ also preserves arbitrary products and so $\widetilde{F_1}, \widetilde{F_0}$ are free $\mathcal{O}_X$-modules.

Conversely, suppose that locally $\mathscr{F}$ is isomorphic to a cokernel of a morphism of free sheaves. Take an affine neighbourhood $U = \operatorname{Spec} A$ of a point x on which $\mathscr{F}|_U$ is isomorphic to a cokernel of a morphism of free sheaves $\mathscr{F}_1 \to \mathscr{F}_0$. Since the $\mathscr{F}_i$ are free, the adjunction morphisms $\widetilde{\mathscr{F}_i(U)} \to \mathscr{F}_i$ are isomorphisms. So we have a diagram

$$\begin{tikzcd}
\mathscr{F}_1 \arrow[r] \arrow[d, equal] & \mathscr{F}_0 \arrow[r] \arrow[d, equal] & \widetilde{\mathscr{F}(U)} \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\
\mathscr{F}_1 \arrow[r] & \mathscr{F}_0 \arrow[r] & \mathscr{F} \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact. So it follows from the five lemma that the adjunction morphism $\widetilde{\mathscr{F}(U)} \to \mathscr{F}$ is an isomorphism. Hence, $\mathscr{F}$ is quasi-coherent.

The proof for the coherent case is the same. To get a cokernel of finite rank free modules we do the following. M is finitely generated so there is a surjective morphism $R^n \to M$ that sends each standard basis element to a generator. Its kernel is not finitely generated a priori but we have assumed that R is noetherian, hence R^n is a noetherian module, hence every submodule is finitely generated. So we can find a morphism $R^m \to R^n$ that is surjective onto the kernel of $R^n \to M$. Hence, M is a cokernel of finite rank free R-modules.

Exercise 5.5. *Let $f : X \to Y$ be a morphism of schemes.*

a *Show by example that if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ need not be coherent on Y, even if X and Y are varieties over a field k.*

b *Show that a closed immersion is a finite morphism.*

c *If f is a finite morphism of noetherian schemes, and if $\mathscr{F}$ is coherent on X, then $f_*\mathscr{F}$ is coherent on Y.*

Solution. a Consider the pushforward of the structure sheaf under the morphism $\operatorname{Spec} k(t) \to \operatorname{Spec} k$ for a field k. Certainly, $\mathcal{O}_{\operatorname{Spec} k(t)}$ is coherent on $k(t)$ but since $k(t)$ is not a finitely generated k-module, its pushforward is not coherent.

[1] Take F_0 to be the free R-module with basis the underlying set of M equipped with the obviousmorphism $F_0 \to M$. Now let F_1 be the free R-module with underlying set the elements of the kernel of $F_0 \to M$. This comes with a morphism $F_1 \to F_0$ and the cokernel of this morphism is, of course, M.

b Let $i : Z \to X$ be a closed immersion of schemes. Let $\{U_j = \operatorname{Spec} A_j\}$ be an open affine cover of X. The restrictions $i^{-1}U_j \to U_j$ are also closed immersions and so by Exercise II.3.11(b) the are of the form $\operatorname{Spec}(A_j/I_j) \to \operatorname{Spec} A_j$ for suitable ideals $I_j \subset A_j$. Since each A_j/I_j is a finitely generated A_j-module, this shows that i is finite.

c Let $\{\operatorname{Spec} B_i\}$ be an open affine cover of Y. Since f is finite, each $f^{-1} \operatorname{Spec} B_i$ is affine (say $\operatorname{Spec} A_i$) and since $\mathscr{F}$ is coherent and X noetherian, the sheaf $\mathscr{F}$ is of the form $\widetilde{M_i}$ on each $\operatorname{Spec} A_i$ (Proposition II.5.4) where the A_i are finitely generated B_i-modules and the M_i are finitely generated A_i-modules. On $\operatorname{Spec} B_i$ we have $f_*\mathscr{F}|_{\operatorname{Spec} B_i} \cong ({}_{B_i}M_i)^{\sim}$ by Proposition II.5.2(d). Since A_i is a finitely generated B_i-module and M_i is a finitely generated A_i-module it follows that ${}_{B_i}M_i$ is a finitely generated B_i-module. Hence, $f_*\mathscr{F}$ is coherent.

Exercise 5.6. Support.

a *Let A be a ring, let M be an A-module, let $X = \operatorname{Spec} A$, and let $\mathscr{F} = \widetilde{M}$. For any $m \in M$ show that $\operatorname{Supp} m = V(\operatorname{Ann} m)$.*

b *Now suppose that A is noetherian, and M finitely generated. Show that $\operatorname{Supp} \mathscr{F} = V(\operatorname{Ann} M)$.*

c *The support of a coherent sheaf on a noetherian scheme is closed.*

d *For any ideal $\mathfrak{a} \subseteq A$, we define a submodule $\Gamma_{\mathfrak{a}}(M)$ of M by $\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$. Assume that A is noetherian, and M any A-module. Show that $\Gamma_{\mathfrak{a}}(M)^{\sim} \cong \mathscr{H}^0_Z(\mathscr{F})$, where $Z = V(\mathfrak{a})$ and $\mathscr{F} = \widetilde{M}$.*

e *Let X be a noetherian scheme, and let Z be a closed subset. If $\mathscr{F}$ is a quasi-coherent (respectively coherent) $\mathcal{O}_X$-module, then $\mathscr{H}^0_Z(\mathscr{F})$ is also quasi-coherent (respectively coherent).*

Solution. a By definition $\operatorname{Supp} m$ is the set of points $\mathfrak{p} \in \operatorname{Spec} A$ such that $m_{\mathfrak{p}} \neq 0$. This condition is equivalent to asking that $sm \neq 0$ for all $s \notin \mathfrak{p}$. If $\mathfrak{p} \in V(\operatorname{Ann} m)$ then $\mathfrak{p} \supseteq \operatorname{Ann} m$ and so $sm \neq 0$ for all $s \notin \mathfrak{p}$ and therefore $\mathfrak{p} \in \operatorname{Supp} m$. Conversely, if $\mathfrak{p} \in \operatorname{Supp} m$ then there is some nonzero $s \notin \mathfrak{p}$ such that $sm = 0$ and therefore $\mathfrak{p} \not\supseteq \operatorname{Ann} m$ so $p \notin V(\operatorname{Ann} m)$.

b By definition $\operatorname{Supp} \mathscr{F}$ is the set of primes $\mathfrak{p} \in \operatorname{Spec} A$ such that $M_{\mathfrak{p}} \neq 0$. Suppose $\mathfrak{p} \in \operatorname{Supp} \mathscr{F}$. If $\mathfrak{p} \notin V(\operatorname{Ann} M)$ then there is some $s \notin \mathfrak{p}$ such that $sm = 0$ for all m. This means that $M_{\mathfrak{p}} = 0$ which contradicts the assumption that $\mathfrak{p} \in \operatorname{Supp} \mathscr{F}$. Hence, $\operatorname{Supp} \mathscr{F} \subseteq V(\operatorname{Ann} M)$. Conversely, suppose that $\mathfrak{p}$ is not in the support of M, so $M_{\mathfrak{p}} = 0$. Then for each element $m \in M$ there is some $s \in A \backslash \mathfrak{p}$ such that $sm = 0$. In particular, if $\{m_i\}$ is a finite set of generators for M then there are $s_i \in A \backslash \mathfrak{p}$ such that $s_i m_i = 0$. This means that $s = \prod s_i \in A \backslash \mathfrak{p}$ is in $\operatorname{Ann} M$. Hence, $\mathfrak{p} \not\supseteq \operatorname{Ann} M$.

c The support of a sheaf is the union of the supports of the sheaf resricted to each element of an open cover. Take an open affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathscr{F}|_{U_i} = \widetilde{M_i}$ for some finitely generated A_i-modules. Then by the previous part, for each i, the support intersected with U_i is a closed subset of U_i. This implies that the support is closed in X. [2]

d Let $U = X - Z$ and $j : U \to X$ the inclusion. Exercise I.1.20(b) gives us an exact sequence:

$$0 \to \mathscr{H}^0_Z(\mathscr{F}) \to \mathscr{F} \to j_*\mathscr{F}$$

Since j is quasi-compact and separated, we can apply Proposition II.5.8(c) to find that $j_*\mathscr{F}$ is quasi-coherent. Using a similar diagram to that in the proof of Exercise II.6.4 we see that $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. So we only need to show that the module of global sections are isomorphic. That is, we want an isomorphism between the following two modules

$$\Gamma_{\mathfrak{a}}(M) = \{m \in M | \mathfrak{a}^n m = 0 \text{ for some } n > 0\}$$

$$\Gamma_Z(\mathscr{F}) = \{m \in M | \operatorname{Supp} m \subseteq Z\}$$

First suppose that $m \in \Gamma_{\mathfrak{a}}(M)$. Then $\mathfrak{a}^n \subseteq \operatorname{Ann} m$ for some $n > Z$ so $V(\mathfrak{a}^n) \supseteq V(\operatorname{Ann} m)$. But $V(\mathfrak{a}^n) = V(\mathfrak{a})$ (Lemma II.2.1(a)) and by the first part of this exercise $V(\operatorname{Ann} m) = \operatorname{Supp} m$. Furthermore, by definition $Z = V(\mathfrak{a})$. So our inclusion $V(\mathfrak{a}^n) \supseteq V(\operatorname{Ann} m)$ becomes $Z \supseteq \operatorname{Supp} m$. Hence, $m \in \Gamma_Z(\mathscr{F})$.

Conversely, suppose that $m \in \Gamma_Z(\mathscr{F})$, so $\operatorname{Supp} m \subseteq Z$. By what we have just written we immediately see that this implies that $V(\operatorname{Ann} m) \subseteq V(\mathfrak{a})$. By Lemma II.2.1(c) this implies that $\sqrt{\operatorname{Ann} m} \supseteq \sqrt{\mathfrak{a}}$ and so $\sqrt{\operatorname{Ann} m} \supseteq \mathfrak{a}$. Now since A is noetherian, $\mathfrak{a}$ is finitely generated by, say n elements $\{a_i\}$. Since $\mathfrak{a} \subseteq \sqrt{\operatorname{Ann} M}$ there is some j_i such that for eacj i we have $a_i^{j_i} \in \operatorname{Ann} M$. Let $N = \frac{\max\{j_i\}}{n}$. Now every element of $\mathfrak{a}$ can be written as a polynomial in the a_i with no constant term, and so every element of $\mathfrak{a}^N$ can be written as a polynomial in the a_i where the degree of the smallest homogeneous part is N. For a polynomial of this form, every mononial contains a factor of the form a_i^k where $k \geq \max\{j_i\} \geq j_i$ by definition of N. Hence, every element of $\mathfrak{a}^N$ is in $\operatorname{Ann} M$ and so $\mathfrak{a}^N m = 0$. Hence, $m \in \Gamma_{\mathfrak{a}}(M)$.

e Let $\{U_i\}$ be an affine cover on which $\mathscr{F}$ is locally of the form $\widetilde{M_i}$. Since X is noetherian we can apply the previous part of this question to find that $\mathscr{H}^0_Z(\mathscr{F})|_{U_i} \cong \Gamma_{\mathfrak{a}_i}(M_i)^\sim$ where $\mathfrak{a}_i$ is the ideal of $Z \cap U_i$ (see Exercise II.3.11(b)). Hence, $\mathscr{H}^0_Z(\mathscr{F})$ is quasi-coherent. The same proof works for the coherent case.

[2] Let Z denote the support of X and Z^c its complement. For each i since $Z \cap U_i$ is closed in U_i we see that $Z^c \cap U_i$ is open in U_i. Since U_i is open in X this implies that $Z^c \cap U_i$ is open in X as well. So $Z^c = \cup(Z^c \cap U_i)$ is a union of open sets, and therefore open itself. Hence, Z is closed.

Exercise 5.7. *Let X be a noetherian scheme, and let $\mathscr{F}$ be a coherent sheaf.*

a *If the stalk $\mathscr{F}_x$ is a free $\mathcal{O}_x$-module for some point $x \in X$, then there is a neighbourhood U of x such that $\mathscr{F}|_U$ is free.*

b *$\mathscr{F}$ is ocally free if and only if its stalks $\mathscr{F}_x$ are free $\mathcal{O}_x$-modules for all $x \in X$.*

c *$\mathscr{F}$ is invertible (i.e. locally free of rank 1) if and only if there is a cogerent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$.*

Solution. a Consider a neighbourhood of x on which $\mathscr{F}$ has the form $\widetilde{M}$ (where M is a finitely generated A-module where A is the ring of global sections of said neighbourhood), so that $\mathscr{F}_x \cong M_\mathfrak{p}$ for some prime $\mathfrak{p} \in \operatorname{Spec} A$. Hence, we have an isomorphism $M_\mathfrak{p} \cong A_\mathfrak{p}^{\oplus n}$. Let e_i be the images in $M_\mathfrak{p}$ of the standard basis elements (note that we can choose the isomorphism so that $e_i \in M$). Let $\{m_i\}$ be a finite set of elements that generate M and $(\frac{a_{i1}}{s_{i1}}, \dots, \frac{a_{in}}{s_{in}})$ their images in $A_\mathfrak{p}^{\oplus n}$. Let $s = \prod_{ij} s_{ij}$ and consider the open affine subset $D(s)$ of $\operatorname{Spec} A$. As s is invertible in M_s so are all the s_{ij} and so we have the relation $m_i = \sum \frac{a_{ij}}{s_{ij}} e_j$. This shows that we have a surjective morphism $\phi : A_s^{\oplus n} \to M_s$. We wish it to be injective as well. Consider the kernel. Since A is noetherian, every submodule of $A_s^{\oplus n}$ is finitely generated and so there is a morphism $A_s^{\oplus m} \to A_s^{\oplus n}$ whose image is the kernel of ϕ. This can be represented by a matrix with entries in A_s and multiplying the basis of $A_s^{\oplus m}$ by a suitable invertible element of A_s (the inverse of the product of the denominators of the entries of the matrix of ϕ would do nicely) we can assume the entries of the matrix b_{ij} are in A. Now when we restrict (= tensor with) to $A_\mathfrak{p}$, the kernel vanishes as $id_{A_\mathfrak{p}} \times \phi$ is our original isomorphism so this means that every element b_{ij} is zero in $A_\mathfrak{p}$. This means that there is some t_{ij} for each b_{ij} such that $t_{ij} b_{ij} = 0$ in M. Let $t = s \prod t_{ij}$ and consider $D(t) \subseteq D(s)$. Tensoring our exact sequence with A_t now kills ϕ, by our choice of t and so we obtain an isomorphism $A_t^{\oplus n} \cong M_t$.

b If $\mathscr{F}$ is locally free then by definition the stalks are free $\mathcal{O}_x$-modules for all $x \in X$. Conversely, if the stalks are all free $\mathcal{O}_x$-modules then by part (a) each point has a neighbourhood U on which $\mathscr{F}|_U$ is a free $\mathcal{O}_X|_U$-module, hence $\mathscr{F}$ is locally free.

c If $\mathscr{F}$ is invertible then consider $\mathscr{G} = \mathscr{H}om(\mathscr{F}, \mathcal{O}_X)$. There is a canonical morphism $\mathscr{F} \otimes \operatorname{hom}(\mathscr{F}, \mathcal{O}_X) \to \mathcal{O}_X$ defined by evaluation and on the stalks, this is an isomorphism since $\mathscr{F}$ is locally free of rank 1.

Conversely, suppose that there is a coherent sheaf $\mathscr{G}$ such that $\mathscr{F} \otimes \mathscr{G} \cong \mathcal{O}_X$. Let x be a point of X. The vector space $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} \mathscr{G}_x) \otimes_{\mathcal{O}_{X,x}} k(x)$ is isomorphic to $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)) \otimes_{k(x)} (\mathscr{G}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ as well as $k(x)$. Hence, the vector space $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ is of dimension one, and similarly for $\mathscr{G}$. Consider an affine neighbourhood of x on which $\mathscr{F}$ has the form

$\widetilde{M}$ and $\mathscr{G}$ the form $\widetilde{N}$ and let $\mathfrak{p} \in A$ be the prime corresponding to x. Since $\mathscr{F}$ and $\mathscr{G}$ are coherent, M and N are finitely generated and so by Nakayama's lemma, a set of generators for $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x)) = M_\mathfrak{p} \otimes k(\mathfrak{p})$ lifts to a set of generators for $M_\mathfrak{p}$. We have seen that $(\mathscr{F}_x \otimes_{\mathcal{O}_{X,x}} k(x))$ is a vector space of dimension one and so it follows that $M_\mathfrak{p}$ is generated by a single element, say $m \in M$, as an $A_\mathfrak{p}$-module, and similarly, $N_\mathfrak{p}$ is generated by a single element, say n, as an $A_\mathfrak{p}$-module. Hence, $M_\mathfrak{p} \otimes N_\mathfrak{p}$ is generated by $m \otimes n$. Recall that it is also isomorphic to $A_\mathfrak{p}$. We define three morphisms:

$$\begin{array}{ccc} A_\mathfrak{p} \to M_\mathfrak{p} & M_\mathfrak{p} \to M_\mathfrak{p} \otimes N_\mathfrak{p} & M_\mathfrak{p} \otimes N_\mathfrak{p} \to A_\mathfrak{p} \\ \frac{a}{s} \mapsto \frac{a}{s} m & \frac{m'}{s} \mapsto \frac{m'}{s} \otimes n & \frac{a}{s}(m \otimes n) \mapsto \frac{a}{s} \end{array}$$

By recalling that the first morphism is surjective we see that the composition of the second two is an inverse to the first. Hence $\mathscr{F}_x \cong \mathcal{O}_{X,x}$ and so $\mathscr{F}$ is locally free of rank one.

Exercise 5.8. *Again let X be a noetherian scheme, and $\mathscr{F}$ a coherent sheaf on X. We will consider the function*

$$\phi(x) = \dim_{k(x)} \mathscr{F}_x \otimes_{\mathcal{O}_x} k(x)$$

where $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ is the residue field at the point x. Use Nakayama's lemma to prove the following:

a *The function ϕ is upper semi-continuous. That is, for any $n \in \mathbb{Z}$ the set*

$$\Phi(n) = \{x \in X | \phi(x) \geq n\}$$

is closed.

b *If $\mathscr{F}$ is locally free, and X is connected, then ϕ is a constant function.*

c *Conversely, if X is reduced, and ϕ is constant, then $\mathscr{F}$ is locally free.*

Solution. a Since a set is closed only if it is closed on each element of an open cover, we need only prove the result for the case when X is affine. We will show that the set $\Phi(n)^c$ is open by showing that every point in it has a neighbourhood contained in it. Let $x \in \Phi(n)^c$. So $\dim_{k(\mathfrak{p})}(M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}) < n$ where $\mathfrak{p} \in \operatorname{Spec} A = X$ is the prime corresponding to x and $M = \Gamma(X, \mathscr{F})$. In fact, let m be this dimension. By Nakayama's lemma, a basis of the vector space $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ lifts to a set of generators $\{m_i\}$ for the $A_\mathfrak{p}$-module $M_\mathfrak{p}$, so $M_\mathfrak{p}$ is generated by $m < n$ elements [3]. Note that we can assume $m_i \in M$. Now let $\{n_i\}$ be a generating set for the A-module M. In $M_\mathfrak{p}$ we can write each n_i as $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$. So setting $s = \prod s_{ij}$, we can write sn_i as $\sum a'_{ij} m_j$ for suitable a'_{ij}. None of the s_{ij} are in $\mathfrak{p}$ and so $s \notin \mathfrak{p}$, hence

[3] Let $\{\overline{v_i}\}$ be a set of generators for $M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ where $v_i \in M_\mathfrak{p}$ represents the class of $\overline{v_i}$. Let N be the submodule of $M_\mathfrak{p}$ generated by the v_i. Then $M_\mathfrak{p} = \mathfrak{p}M_\mathfrak{p} + N$ since $N \to M_\mathfrak{p}/\mathfrak{p}M_\mathfrak{p}$ is surjective. So Nakayama's lemma says that $M = N$.

$\mathfrak{p} \in D(s)$. Consider another prime $\mathfrak{q} \in D(s)$. Since $s \notin \mathfrak{q}$, the element s is invertible in $A_{\mathfrak{q}}$ and so recalling our expressions $sn_i = \sum a'_{ij} m_j$ we can write n_i as an $A_{\mathfrak{q}}$-linear combination of the m_j. Since n_i generate M, they also generate $M_{\mathfrak{q}}$ and since the m_j generate the n_i in $M_{\mathfrak{q}}$ we see that the m_j generate $M_{\mathfrak{q}}$. Hence, $M_{\mathfrak{q}}$ is generated by $m < n$ elements and therefore, so is $M_{\mathfrak{q}}/\mathfrak{q}M_{\mathfrak{q}}$. Hence $\mathfrak{q} \in \Phi(n)^c$. Since $\mathfrak{q}$ was arbitrarily chosen, this shows that $D(s) \subseteq \Phi(n)^c$. So every point in $\Phi(n)^c$ has an open neighbourhood contained in $\Phi(n)^c$, hence, $\Phi(n)^c$ is a union of open sets and therefore open itself. It follows that $\Phi(n)$ is closed.

b If $\mathscr{F}$ is locally free then for each point x there is an open nighbourhood U on which ϕ is constant. So $\Phi(n)$ is a union of open sets, and therefore open itself. In part (a) we have shown that it is also a closed set and so if X is connected, the set is either empty or the whole space. Since $\mathscr{F}$ is locally free it is locally of finite rank and so there is some n for which $\Phi(n)$ empty and therefore $\min_i\{i|\Phi(i) \neq \varnothing\}$ is a finite integer, say m. The set of points such that $\phi(x) = m$ is $\Phi(m)\backslash\Phi(m+1)$ and by definition of m this is the whole space. Hence, ϕ is constant.

c Let x be a point. We will find an open neighbourhood U of x such that $\mathscr{F}|_U$ is free of finite rank. Let $\operatorname{Spec} A$ be an affine neighbourhood of x, let $\mathfrak{p}$ be the prime of A corresponding to X, and let M be the finitely generated A-module corresponding to $\mathscr{F}|_{\operatorname{Spec} A}$. Since X is reduced, A has no nilpotents and similarly for all $A_{\mathfrak{q}}$ and A_f for $f \in A$, $\mathfrak{q} \in \operatorname{Spec} A$. Let $n = \dim M_{\mathfrak{p}} \otimes k(\mathfrak{p})$, choose a basis for this vector space and lift it to a set of generators $m_1, \ldots, m_n$ for $M_{\mathfrak{p}}$ (using Nakayama's lemma as in the first part). Let $\{n_i\}$ be a finite set of generators for M. In $M_{\mathfrak{p}}$ these can each be written as $n_i = \sum \frac{a_{ij}}{s_{ij}} m_j$. Setting $s = \prod s_{ij}$, these expressions hold also in A_s so we get a short exact sequence

$$0 \to \ker\phi \to A_s^{\oplus n} \xrightarrow{\phi} M_s \to 0$$

This sequence holds in each $A_{\mathfrak{q}}$ for $\mathfrak{q} \in D(s)$ but since ϕ is constant, each $M_{\mathfrak{q}} \otimes k(\mathfrak{q})$ has dimension n and so ϕ tensored with $k(\mathfrak{q})$ is an isomorphism. That is, $k(\mathfrak{q}) \otimes \ker\phi = 0$ for all $\mathfrak{q} \in D(s)$. This implies that for every element of $\ker\phi$, the components of the tuples are in $\mathfrak{q}A_s$ for all $\mathfrak{q} \in D(s)$. This implies that they are in the nilradical of A_s. But A_s is reduced, since X is reduced, so the nilradical is zero. Hence, $\ker\phi = 0$, and M_s is free.

Exercise 5.9. *Let S be a graded ring, generated by S_1 as an S_0-algebra, let M be a graded S-module, and let $X = \operatorname{Proj} S$.*

a *Show that there is a natural homomorphism $\alpha : M \to \Gamma_*(\widetilde{M})$.*

b *Assume now that $S_0 = A$ is a finitely generated k-algebra for some field k, that S_1 is a finitely generated A-module, and that M is a finitely generated S-module. Show that the map α is an isomorphism in all large enough degrees.*

c *With the same hypothesis, we define an approxalence relation $\approx$ on graded S-modules by saying that $M \approx M'$ if there is an integer d such that $M_{\geq d} \cong M'_{\geq d}$. We will sa that a graded S-module M is quasi-finitely generated if it is approxalent to a finitely generated module. Now show that the functors $\sim$ and Γ_* induce an approxalence of categories between the category of quasi-finitely generated graded S-modules modulo the approxalence relation $\approx$, and the category of coherent $\mathcal{O}_X$-modules.*

Solution. a Since S_1 generates S, there is a cover of distinguished open affines $D_+(f)$ with $f \in S_1$. Now to give a global section of $M(n)^\sim$ is the same as giving a section on each $D_+(f)$ such that the intersections agree. For $m \in M_d$, the element m defines a section on each $D_+(f)$, as it has degree zero in $M(d)_{(f)} = \Gamma(D_+(f), M(n)^\sim)$, and these sections agree on the intersections where they are, again, $m \in M(d)_{(fg)} = \Gamma(D_+(fg), M(n)^\sim)$. Hence, they define a global section and we obtain a morphism of abelian groups $\alpha : M \to \Gamma_*(\widetilde{M})$.

If $s \in S_e$ and $m \in M_d$ then $s\alpha(m) \in \Gamma_*(\widetilde{M})$ is defined as the image of $m \otimes s$ in $\Gamma(X, M(d)^\sim \otimes \mathcal{O}_X(e))$ under the isomorphism $M(d)^\sim \otimes \mathcal{O}_X(e) \cong M(d+e)^\sim$. In our case (that is, where $\mathscr{F}$ from the definition on page 118 is of the form $\widetilde{M}$) the isomorphism is the one induced by $M(d) \otimes_S S(e) \cong M(d+e)$ and so $s\alpha(m) = \alpha(sm)$ and therefor α is a morphism of graded modules.

b

c Part (b) of this exercise shows that M is equivalent to $\Gamma_*(\widetilde{M})$ if M is finitely generated, and Proposition II.5.15 says that $\Gamma_*(\mathscr{F})^\sim$ is isomorphic to $\mathscr{F}$ for any quasi-coherent sheaf $\mathscr{F}$. So if Γ and $\sim$ have images in the appropriate subcategories we are done. That is, we want to show that for a quasi-finitely generated graded S-module M, the sheaf $\widetilde{M}$ is coherent, and for a coherent sheaf $\mathscr{F}$ that $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Suppose that M is a quasi-finitely generated graded S-module. Then there is a finitely generated graded S-module M' such that $M_{\geq d} \cong M'_{\geq d}$ for some d. This implies that for every element $f \in S_1$ we have $M_(f) \cong M'_{(f)}$ since $\frac{m}{f^n} = \frac{mf^d}{f^{n+d}}$. Since M' is finitely generated, $M'_{(f)}$ is finitely generated. S is generated by S_1 as an S_0 algebra so open subsets of the form $M_{(f)}$ cover $X = \operatorname{Proj} S$ and so there is a cover of X on which $\widetilde{M}$ is locally equivlanent to a coherent sheaf. Hence $\widetilde{M}$ is coherent.

Now consider a coherent $\mathcal{O}_X$-module $\mathscr{F}$. Then by Theorem II.5.17 $\mathscr{F}(n)$ is generated by a finite number of global sections for sufficiently large n. Let M' be the submodule of $\Gamma_*(\mathscr{F})$ generated by these sections. We have an inclusion $M' \hookrightarrow \Gamma_*(\mathscr{F})$ which induces an inclusion of sheaves $\widetilde{M'} \hookrightarrow \widetilde{\Gamma_*(\mathscr{F})} \cong \mathscr{F}$ where the latter isomorphism comes from Proposition II.5.15. Tensoring with $\mathcal{O}(n)$ we have an inclusion $\widetilde{M(n)}' \hookrightarrow \mathscr{F}(n)$ that is actually an isomorphism since $\mathscr{F}(n)$ is generated by global sections in

M'. Tensoring again with $\mathcal{O}(-n)$ we then find that $\widetilde{M'}$ is isomorphic to $\mathscr{F}$. Now M' is finitely generated and so by part (b) there is a d_0 such that for all $d > d_0$ we have $M_d \cong \Gamma(X, \widetilde{M'}(d)) \cong \Gamma(X, \mathscr{F}(d)) = \Gamma_*(\mathscr{F})_d$. Hence, $M_{\geq d_0} \cong \Gamma_*(\mathscr{F})_{\geq d_0}$ and so $\Gamma_*(\mathscr{F})$ is quasi-finitely generated.

Exercise 5.10. *Let A be a ring, let $S = A[x_0, \ldots, x_r]$ and let $X = \operatorname{Proj} S$.*

a *For any homogeneous ideal $I \subseteq S$, we define the saturation $\overline{I}$ of I to be $\{s \in S |$ for each $i = 0, \ldots, r$ there is an n such that $x_i^n s \in I\}$. Show that $\overline{I}$ is a homogeneous ideal.*

b *Two homogeneous ideals I_1 and I_2 of S define the same closed subscheme of X if and only if they have the same saturation.*

c *If Y is any closed subscheme of X, then the ideal $\Gamma_*(\mathscr{I}_Y)$ is saturated. Hence it is the largest homogeneous ideal defining the subscheme Y.*

d *There is a 1-1 correspondence between saturated ideals of S and closed subschemes of X.*

Solution. a Suppose that $s, t \in \overline{I}$ then for each i there is an n and an m such that $x_i^n s, x_i^m t \in I$. So $x_i^{n+m} st \in I$, $x_i^{n+m}(s + t) \in I$, and for any $a \in S$ we have $a x_i^n s \in I$. So $\overline{I}$ is certainly an ideal. Now write $s = s_0 + \cdots + s_k$ with each s_i homogeneous. Since x_i is homogeneous of degree 1, each $x_i^n s_k$ is homogeneous of degree $n + k$. Since I is a homogeneous ideal and $x_i^n(s_0 + \cdots + s_k) \in I$ it follows that $x_i^n s_k \in I$. Hence, $s_k \in \overline{I}$, so $\overline{I}$ is a homogeneous ideal.

b Suppose that two homogeneous ideals I_1 and I_2 define the same closed subscheme of X. Then by Proposition II.5.9 they define the same quasi-coherent sheaf of ideals $\mathscr{I}$ on X. Suppose s is a homogeneous element of I_1 of degree d. Then for each i, the element $\frac{s}{x_i^d}$ is a section of $\mathscr{I}(D_+(x_i))$. Since the sheaf of ideals of I_1 is the same as that of I_2, for each i there is some $t_i \in I_2$, homogeneous of degree d such that $\frac{s}{x_i^d} = \frac{t_i}{x_i^d}$, which implies that $x_i^{n_i}(s - t_i) = 0$ for some n_i. Since $t_i \in I_2$ so is $x_i^{n_i} t_i = x_i^{n_i} s$ and so s is in the saturation of I_2, hence $I_1 \subseteq \overline{I}_2$. By symmetry $I_2 \subseteq \overline{I}_1$ and since the operation of saturation is an idempotent we see that $\overline{I}_2 = \overline{I}_1$.

c Suppose $s \in S$ is a homogeneous element of degree d, in the saturation of $\Gamma_*(\mathscr{I}_Y)$. That is, for each i there is some n such that $x_i^n s \in \Gamma_*(\mathscr{I}_Y)$. There are only finitely many i and so we can assume it is the same n for all of them. Since $\mathscr{I}_Y$ is a subsheaf of $\mathcal{O}_X$, to show that $s \in \Gamma_*(\mathscr{I}_Y)_d = \Gamma(X, \mathscr{I}_Y(d))$ it will be enough to show that its restriction to each open $U_i = D_+(x_i)$ is in $\Gamma(U_i, \mathscr{I}_Y(d))$.

We know that $x_i^n s \in \Gamma(X, \mathscr{I}_Y(d + n))$ and so, $x_i^{-n} \otimes x_i^n s$ is a section in $\Gamma(U_i, \mathscr{I}_Y(d+n) \otimes \mathcal{O}(-n))$. But $\mathscr{I}_Y(d+n) \otimes \mathcal{O}(-n) \cong \mathscr{I}_Y(d)$ and under this isomorphism, $x_i^{-n} \otimes x_i^n s$ corresponds to $x_i^{-n} x_i^n s = s$. So $s \in \Gamma(U_i, \mathscr{I}_Y(d))$ for all i, hence $s \in \Gamma(X, \mathscr{I}_Y(d)) \subset \Gamma_*(\mathscr{I}_Y)$. So $\Gamma_*(\mathscr{I}_Y)$ is saturated.

d We have the following three sets and maps between them:

$$\left\{ \begin{array}{c} \text{homogeneous} \\ \text{ideals of } S \end{array} \right\} \underset{\sim}{\overset{\Gamma_*(-)}{\leftrightarrows}} \left\{ \begin{array}{c} \text{quasi-coherent} \\ \text{sheaves of ideals} \end{array} \right\} \underset{\text{Prop II.5.9}}{\overset{\mathscr{I}_-}{\leftrightarrows}} \left\{ \begin{array}{c} \text{closed} \\ \text{subschemes of} X \end{array} \right\}$$

Proposition II.5.9 says that the maps between the two rightmost sets are bijective, and Proposition II.5.15 says that the composition left, then right from the middle is an isomorphism. Keeping in mind the bijection between the two rightmost sets, part (b) of this exercise says that two homogeneous ideals determine the same quasi-coherent sheaf of ideals if and only if they have the same saturation. Since we already know that $\sim$ is surjective, and we now know that each preimage has a unique saturated homogeneous ideal in it, we see that $\sim$ defines a bijection between the saturated homogeneous ideals of S and quasi-coherent sheaves of ideals. Part (c) of this question says that $\Gamma_*(-)$ is its inverse.

Exercise 5.11. *Let S and T be two graded rings with $S_0 = T_0 = A$. We define the Cartesian product $S \times_A T$ to be the graded ring $\oplus_{d \geq 0} S_d \otimes_A T_d$. If $X = \operatorname{Proj} S$ and $Y = \operatorname{Proj} T$, show that $\operatorname{Proj}(S \times_A T) \cong X \times_A Y$ and show that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}(S \times_A T)$ is isomorphic to the sheaf $p_1^*(\mathcal{O}_X(1)) \otimes p_2^*(\mathcal{O}_Y(1))$ on $X \times Y$.*

Solution. Let $f \in S_d$ and $g \in T_d$. We have a ring isomorphism $S_{(f)} \otimes_A T_{(g)} \to (S \times T)_{(f \otimes g)}$ defined by $\frac{s}{f^n} \otimes \frac{t}{g^m} \mapsto \frac{f^m s \otimes g^n t}{(f \otimes g)^{nm}}$ with inverse given by $\frac{s \otimes t}{(f \otimes g)^n} \mapsto \frac{s}{f^n} \otimes \frac{t}{g^n}$. Hence $D_+(f) \times D_+(g) \cong D_+(f \otimes g)$ and so composing with the inclusion $D_+(f) \times D_+(g) \to \operatorname{Proj} S \times \operatorname{Proj} T$ we get morphisms $D_+(f \otimes g) \to \operatorname{Proj} S \times \operatorname{Proj} T$ that are isomorphic onto their images.we get morphisms. Now consider $f' \otimes g' \in (S \times_A T)_{d'}$ and the restriction of the two morphisms $D_+(f \otimes g) \to \operatorname{Proj} S \times \operatorname{Proj} T$ and $D_+(f' \otimes g') \to \operatorname{Proj} S \times \operatorname{Proj} T$ to their intersection $D_+(f'f \otimes g'g) = D_+(f \otimes g) \cap D_+(f' \otimes g')$. We have a diagram

$$\begin{tikzcd}
D_+(f \otimes g) \arrow[r] & D_+(f) \times D_+(g) \arrow[dr] & \\
D_+(f'f \otimes g'g) \arrow[r] \arrow[u] \arrow[d] & D_+(f'f) \times D_+(g'g) \arrow[r] \arrow[u] \arrow[d] & \operatorname{Proj} S \times \operatorname{Proj} T \\
D_+(f' \otimes g') \arrow[r] & D_+(f') \times D_+(g') \arrow[ur] &
\end{tikzcd}$$

with corresponding ring homomorphisms

$$\begin{array}{ccc}
(S\times T)_{(f\otimes g)} & \longleftarrow & S_{(f)}\otimes_A T_{(g)} \\
\downarrow & & \downarrow \\
(S\times T)_{(f'f\otimes g'g)} & \longleftarrow & S_{(f'f)}\otimes_A T_{(g'g)} \\
\uparrow & & \uparrow \\
(S\times T)_{(f'\otimes g')} & \longleftarrow & S_{(f')}\otimes_A T_{(g')}
\end{array}$$

Following an element through the upper square gives

$$\begin{array}{ccc}
\frac{f^m s\otimes g^n t}{(f\otimes g)^{nm}} & \leftsquigarrow & \frac{s}{f^n}\otimes\frac{t}{g^m} \\
\wr & & \wr \\
\frac{(f')^{nm}f^m s\otimes(g')^{nm}g^n t}{(f'f\otimes g'g)^{nm}} & \leftsquigarrow & \frac{(f')^n s}{(f'f)^n}\otimes\frac{(g')^m t}{(g'g)^m}
\end{array}$$

and so we see that the two squares commute. Hence, the restriction of the morphisms to intersections agree. Therefore, the morphisms patch together to give a global morphism $\operatorname{Proj} S\times T\to\operatorname{Proj} S\times\operatorname{Proj} T$ which we can see by the way we have defined it is an isomorphism.

To show that $\mathcal{O}(1)$ on $\operatorname{Proj}(S\times_A T)$ is isomorphic to $p_1^*(\mathcal{O}_X(1))\otimes p_2^*(\mathcal{O}_Y(1))$ on $X\times Y$ we use a similar method. That is, we provide an isomorphism on each of the distinguished opens of the form we have ben using and show that they agree on the intersections.

Exercise 5.12. *a Let X be a scheme over a scheme Y, and let $\mathscr{L},\mathscr{M}$ be two very ample invertible sheaves on X. Show that $\mathscr{L}\otimes\mathscr{M}$ is also very ample.*

b Let $f:X\to Y$ and $g:Y\to Z$ be two morphisms of schemes. Let $\mathscr{L}$ be a very ample invertible sheaf on X relative to Y, and let $\mathscr{M}$ be a very ample invertible sheaf on Y relative to Z. Show that $\mathscr{L}\otimes f^*\mathscr{M}$ is a very ample invertible sheaf on X relative to Z.

Exercise 5.13. *Let S be a graded ring, generated by S_1 as an S_0-algebra. For any integer $d>0$, let $S^{(d)}$ be the graded ring $\oplus_{n\geq 0}S_n^{(d)}$ where $S_n^{(d)}=S_{nd}$. Let $X=\operatorname{Proj} S$. Show that $\operatorname{Proj} S^{(d)}\cong X$ and that the sheaf $\mathcal{O}(1)$ on $\operatorname{Proj} S^{(d)}$ corresponds via this isomorphism to $\mathcal{O}_X(d)$.*

Exercise 5.14. *Assume that k is an algebraically closed field, and that X is a connected, normal closed subscheme of $\mathbb{P}_k^r$. Show that for some $d>0$, the d-uple embedding of X is projectively normal, as follows.*

a Let $S=k[x_0,\dots,x_r]/\Gamma_*(\mathscr{I}_X)$ be the homogeneous coordinate ring of X, and let $S'=\bigoplus_{n\geq 0}\Gamma(X,\mathcal{O}_X(n))$. Show that S is a domain, and that S' is its integral closure.

b *Use Exercise II.5.9 to show that $S_d = S'_d$ for all sufficiently large d.*

c *Show that $S^{(d)}$ is integrally closed for sufficiently large d, and hence conclude that the d-uple embedding of X is projectively normal.*

d *As a corollary of (a), show that a closed subscheme $X \subseteq \mathbb{P}^r_A$ is projectively normal if and only if it is normal, and for every $n \geq$) the natural map $\Gamma(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \to \Gamma(X, \mathcal{O}_X(n))$.*

Exercise 5.15. Extension of coherent sheaves.

a *On a noetherian affine scheme, every quasi-coherent sheaf is the union of its coherent subsheaves.*

b *Let X be an affine noetherian scheme, U an open subset, and $\mathscr{F}$ coherent on U. Then there exists a coherent sheaf $\mathscr{F}'$ on X with $\mathscr{F}'|_U \cong \mathscr{F}$.*

c *With $X, U, \mathscr{F}$ as in (b) suppose furthermore we are given a quasi-coherent sheaf $\mathscr{G}$ on X such that $\mathscr{F} \subseteq \mathscr{G}|_U$. Show that we can find $\mathscr{F}'$ a coherent subsheaf of $\mathscr{G}$, with $\mathscr{F}'|_U \cong \mathscr{F}$.*

d

e

Solution. a Since the scheme is affine, a quasi-coherent sheaf corresponds to a module and a coherent sheaf a finitely generated module. So if $X = \operatorname{Spec} A$ showing that every A-module is a union of its finitely generated submodules is sufficient. But this is clear since for an A-module M, every element $m \in M$ is contained in a finitely generated submodule (take the submodule generated by m).

b Consider the pushforward $i_*\mathscr{F}$ where $i : U \to X$ is the inclusion. By Proposition 5.8(c) we know that it is at least quasi-coherent. Then by the previous part of this question it is the union of its coherent subsheaves, that is, $i_*\mathscr{F} = \cup_{\mathscr{G} \text{ coh.}} \mathscr{G}$. Restricting this union gives a system of subsheaves of $\mathscr{F}$ whose union is $\mathscr{F}$. But $\mathscr{F}$ is coherent on a Noetherian affine scheme, so the corresponding module is Noetherian. This means that the system of submodules corresponding to sheaves of the form $i^*\mathscr{G}$ (for $\mathscr{G}$ a coherent subsheaf of $i_*\mathscr{F}$) has a maximal element. But this system is directed and so the maximal element is the union. If $i^*\mathscr{F}'$ is the sheaf corresponding to this maximal element, then we have found a coherent subsheaf $\mathscr{F}'$ of $i_*\mathscr{F}$ such that $\mathscr{F}'|_U = \mathscr{F}$.

c We have a natural morphism $\mathscr{G} \to i_*i^*\mathscr{G}$ and so we can consider the subsheaf $\mathscr{G}'$ of $\mathscr{G}$ which is the preimage of $i_*\mathscr{F} \subseteq i_*(i^*\mathscr{G})$. On open sets V contained in U the morphism $\mathscr{G}(V) \to i_*i^*\mathscr{G}(V)$ is an isomorphism and so $\mathscr{G}'|_U = \mathscr{F}$. Consider the directed system of coherent subsheaves of $\mathscr{G}$ that are contained in $\mathscr{G}'$. Notice that by the following pullback diagram and the fact that the horizontal morphisms are injective, these are in one-to-one correspondence with coherent subsheaves of $i_*\mathscr{F}$, so their union is $\mathscr{G}'$.

$$\begin{tikzcd}
\mathscr{G}' \arrow[r] \arrow[d] & \mathscr{G} \arrow[d] \\
i_*\mathscr{F} \arrow[r] & i_*i^*\mathscr{G}
\end{tikzcd}$$

Now the argument of the previous part goes through. Since $\mathscr{G}'$ is the union of our directed system, and the restriction of this union to U is $\mathscr{F}$, there is a maximal element $\mathscr{F}'$ whose restriction to U is $\mathscr{F}$. So we have found a coherent subsheaf of $\mathscr{G}$ whose restriction to U is $\mathscr{F}$

d Let $\{U_i\}$ be an affine cover of X. Since X is noetherian, we can assume the the cover is finite. Restricting to U_1 and $U \cap U_1$, the hyotheses of the previous part are satisfied and so we can find a coherent subsheaf $\mathscr{F}_1$ of $\mathscr{G}|_{U_1}$ such that the restriction to $U_1 \cap U$ is isomorphic to $\mathscr{F}|_{U_1}$. Now consider $\mathscr{G}|_{U_1 \cup U_2}$ (note the union, not intersection!). Setting $X' = U_2$ and $U' = U_2 \cap (U \cup U_1)$ we have a quasi-coherent sheaf $\mathscr{G}|_{U_2}$ on $X' = U_2$ and a coherent subsheaf $\mathscr{F}_1|_{U'}$ on U'. The conditions of the previous part are satisfied and so we can find a coherent subsheaf $\mathscr{F}_2$ of $\mathscr{G}|_{U_2}$ whose restriction to U' is isomorphic to $\mathscr{F}|_{U'}$. In particular, the restriction to $U_1 \cap U_2$ is the same as that of F_1 so their "union" is a coherent subsheaf of $\mathscr{G}|_{U_1 \cap U_2}$ whose restriction to $U \cap (U_1 \cup U_2)$ is isomorphic to $\mathscr{F}|_{U \cap (U_1 \cup U_2)}$. Continuing in this was we eventually run out of U_i and end up with a coherent subsheaf $\mathscr{F}'$ of $\mathscr{G}$ such that the restriction to U is isomorphic to $\mathscr{F}$. In general, for the iterative step we will have $X' = U_i$ and $U' = U_i \cap (U \cup U_1 \cup \cdots \cup U_{i-1})$.

e If s is a section of $\mathscr{F}$ over an open set U, we apply (d) to the subsheaf of $\mathscr{F}|_U$ generated by s. In this way, for every open subset U and every section $s \in \mathscr{F}(U)$ there is a coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ such that $s \in \mathscr{F}'(U)$. Hence, $\mathscr{F}$ is the union of all of these.

Exercise 5.16. Tensor Operations on Sheaves.

a *Suppose that $\mathscr{F}$ is locally free of rank n. Then $T^r(\mathscr{F})$, $S^r(\mathscr{F})$, and $\bigwedge^r(\mathscr{F})$ are also locally free, of ranks n^r, $\binom{n+r-1}{n-1}$, and $\binom{n}{r}$ respectively.*

b *Again let $\mathscr{F}$ be locally free of rank n. Then the multiplication map $\wedge^r\mathscr{F} \otimes \wedge^{n-r}\mathscr{F} \to \wedge^n\mathscr{F}$ is a perfect pairing for ane r, i.e., it induces an isomorphism of $\wedge^r\mathscr{F}$ with $(\wedge^{n-r}\mathscr{F})^\vee \otimes \wedge^n\mathscr{F}$.*

c *Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of locally free sheaves. Then for any r there is a finite filtration of $S^r(\mathscr{F})$,*

$$S^r(\mathscr{F}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p-1} \cong S^p(\mathscr{F}') \otimes S^{r-p}(\mathscr{F}'')$$

for each p.

d *Same statement as (c), with exterior powers instead of symmetric powers.*

e *Let $f : X \to Y$ be a morphism of ringed spaces, and let $\mathscr{F}$ be an $\mathcal{O}_Y$-module. Then f^* commutes with all the tensor operations on $\mathscr{F}$.*

Solution. a Suppose that $\mathscr{F}$ is a free sheaf with basis global sections $e_1, \ldots, e_n$. That is, the e_i are global sections and for each open set U, we have $\mathscr{F}(U) \cong \mathcal{O}_X(U)e_1|_U \oplus \cdots \oplus \mathcal{O}_X(U)e_n|_U$, and these isomorphisms respect the restriction homorphisms. Then the presheaf $U \mapsto \Phi(\mathscr{F}(U))$ (where Φ is one of $T^r, S^r, \bigwedge^r$) is free with basis $\{e_{i_1} \otimes \cdots \otimes e_{i_r} | 1 \leq i_1, \ldots, i_r \leq n\}, \{e_{i_1} e_{i_2} \ldots e_{i_r} | 1 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq n\}, \{e_{i_1} \wedge \cdots \wedge e_{i_r} | 0 < i_1 < i_2 < \cdots < i_r < n+1\}$ respectively. As this presheaf of $\mathcal{O}_X$-modules is free, it is a sheaf. Now if $\mathscr{F}$ is an arbitrary locally free sheaf, we take a cover $\{U_i\}$ of X on which each $\mathscr{F}|_{U_i}$ is free. Then $\Phi(\mathscr{F})|_{U_i} = \Phi(\mathscr{F}|_{U_i})$, and so $\Phi(\mathscr{F})$ is locally free.

The ranks of $T^r(\mathscr{F})$ and $\bigwedge^r(\mathscr{F})$ are straightforward from the description of the basis: for T^r we have n choices for each of the i_j, of which there are r, and so there are n^r; for $\bigwedge^r$ the basis global sections are in one-to-one correspondence with subsets of $\{1, \ldots, n\}$ of size r. For the rank of $S^r(\mathscr{F})$ we want to count how many tuples $(i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$ there are such that $i_j \leq i_{j+1}$. Tuples of this form are in one-to-one correspondence with subsets of $\{1, \ldots, n+r-1\}$ of size $n-1$. To see this, choose a subset $\{k_1, \ldots, k_{n-1}\}$ and suppose that the indexing is chosen so that $k_i < k_{i+1}$ for all i. Now define $k_i - k_{i-1} - 1$ to be the number of times that i appears in the tuple $(i_1, \ldots, i_r)$. That is, our basis global section is $e_1^{k_1-1} e_n^{n+r-1-k_{n-1}} \prod_{i=2}^{n-1} e_i^{k_i - k_{i-1} - 1}$. Conversely, given such a tuple $(i_1, \ldots, i_r)$ we define $k_i = \sum_{j=1}^{i}(1 + \#\{i_\ell | i_\ell = j\})$. It can be seen that these are inverse operations.

b Suppose that $\mathscr{F}$ is free of rank n with basis of global sections $e_1, \ldots, e_n$. Then the pairing is defined by $\omega \otimes \lambda \mapsto \omega \wedge \lambda$. Since $\mathscr{F}$ is free of rank n we have an isomorphism $\mathcal{O}_X \to \bigwedge^n \mathscr{F}$ given by $f \mapsto f(e_1 \wedge \cdots \wedge e_n)$. Every global section λ of $\bigwedge^{n-r} \mathscr{F}$ defines a morphism $\bigwedge^r \mathscr{F} \to \bigwedge^n \mathscr{F} \cong \mathcal{O}_X$ via $\omega \mapsto \omega \wedge \lambda$. Alternatively, given a morphism of $\mathcal{O}_X$-modules $\bigwedge^r \mathscr{F} \to \bigwedge^n \mathscr{F} \cong \mathcal{O}_X$ we have a morphism of global sections $\phi : \bigwedge^r \mathscr{F}(X) \to \bigwedge^n \mathscr{F}(X) \cong \mathcal{O}_X(X)$ and so we can define a global section of $\bigwedge^{n-r} \mathscr{F}$ by $\sum (-1)^{\kappa_I} \phi(e_{i_1} \wedge \cdots \wedge e_{i_r}) e_{j_1} \wedge \cdots \wedge e_{j_{n-r}}$ where the j_k are the elements of $\{1, \ldots, n\}$ that don't appear as i_ℓ for some ℓ and κ_I is an appropriately chosen integer depending on the i_ℓ. It can be shown that these operations are inverses using the fact that if λ, μ are two basis global sections of $\bigwedge^r \mathscr{F}$ and $\bigwedge^{n-r} \mathscr{F}$ then $\lambda \wedge \mu$ is zero unless μ has all the complement elements to λ, in which case it is $\pm e_1 \wedge \cdots \wedge e_n$ (this is how we choose κ_I).

If $\mathscr{F}$ is not free, but still locally free, we can define such isomorphisms $\wedge^r \mathscr{F}|_{U_i} \cong (\wedge^{n-r}\mathscr{F})^\vee \otimes \wedge^n \mathscr{F}|_{U_i}$ locally on an open cover $\{U_i\}$ on which $\mathscr{F}$ is free. Then we need to check that the isomorphisms agree on their restrictions to $U_i \cap U_j$ for each i, j. But notice that when we defined the morphism $\bigwedge^r \mathscr{F} \to (\bigwedge^{n-r} \mathscr{F})^\vee$ we didn't explicitely use the basis. So since the inverse exists locally, it exists globally, by virtue of the fact that it is the inverse to an isomorphism of sheaves.

Exercise 5.17. Affine Morphisms.

a *Show that $f : X \to Y$ is an affine morphism if and only if for every open affine $V \subseteq Y$, the open subscheme $f^{-1}V$ of X is affine.*

b *An affine morphism is quasi-compact and separated. Any finite morphism is affine.*

c *Let Y be a scheme, and let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. Show that there is a unique scheme X, and a morphism $f : X \to Y$, such that for every open affine $V \subseteq Y$, $f^{-1}(V) \cong \operatorname{Spec} \mathcal{A}(V)$, and for every inclusion $U \subset V$ of open affines of Y, the morphism $f^{-1}(U) \to f^{-1}(V)$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(U)$.*

d *If $\mathcal{A}$ is a quasi-coherent $\mathcal{O}_Y$-algebra, then $f : X \to \boldsymbol{Spec}\, \mathcal{A} \to Y$ is an affine morphism, and $\mathcal{A} \cong f_*\mathcal{O}_X$. Conversely, if $f : X \to Y$ is an affine morphism then $\mathcal{A} = f_*\mathcal{O}_X$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-algebras, and $X \cong \boldsymbol{Spec}\, \mathcal{A}$.*

e *Let $f : X \to Y$ be an affine morphism, and let $\mathcal{A} = f_*\mathcal{O}_X$. Show that f_* induces an equivlance of categories from the category of quasi-coherent $\mathcal{O}_X$-modules to the category of quasi-coherent $\mathcal{A}$-modules.*

Solution. a Let $\{V_i\}$ be an open affine cover of Y such that $f^{-1}V_i$ is affine for all i. Given another open affine subset $V \subseteq Y$ we can consider the intersections $V \cap V_i$. These are open subsets of the affines V_i and so are covered by distinguished open affines $D(f_{ij})$ of the V_i. Let $A_i = \Gamma(V_i, \mathcal{O}_Y)$ and $B_i = \Gamma(f^{-1}V_i, \mathcal{O}_X)$. Then since both V_i and $f^{-1}V_i$ are affine, the morphism $f|_{f^{-1}V_i} : f^{-1}V_i \to V_i$ is induced by a ring homomorphism $\phi_i : A_i \to B_i$ and the preimage of $D(f_{ij})$ is $D(\phi f_{ij})$, also affine. So we have found an open cover (the $D(f_{ij})$) of V for which the preimage of every element in the cover is affine. Hence, the restricted morphism $f|_{f^{-1}V_i} : f^{-1}V_i \to V_i$ is affine. Now if the result holds for Y affine, then it will hold for this restricted morphism and in particular, the preimage of the whole space V_i will be affine. Hence, we just need to show that the result holds for Y affine.

So suppose that $Y = \operatorname{Spec} B$ is affine, and that the morphism f is affine. So there is an open cover $\{\operatorname{Spec} B_i\}$ of Y such that each of the preimages $f^{-1} \operatorname{Spec} B_i$ is an affine subscheme of X. We will show first that X is affine using the criterion of Exercise II.2.17. First refine the open affine cover

$\{\operatorname{Spec} B_i\}$ to one which consists of distinguished open subsets $D(f_i)$ of Y. The preimages will still be affine as the preimage of a distinguished open subset under a morphism of affine schemes is still a distinguished open affine (as we just saw above). Now since Y is affine, it is quasi-compact, so we can find a subcover of the $\{D(f_i)\}$ which is finite. So now we have a finite set of elements $\{f_i\}$ of B, which generate the unit ideal, and the preimage of each $D(f_i)$ under $f : X \to Y$ is an open affine subscheme of X. The sheaf morphism $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ induces a morphism of global sections, and since unity in B is a finite linear combination of the f_i, unity in $\Gamma(X, \mathcal{O}_X)$ is a finite linear combination of their images g_i under this morphism of global sections. It remains only to see that $X_{g_i} = f^{-1}D(f_i)$, and restricting to an open affine cover of X shows this. [4]

So now we know that X is affine. Open immersions are preserved by base change and so are morphisms between affine schemes. So the preimage $f^{-1}U = U \times_X Y$ of under of an open affine subset $U \subseteq X$ of X is affine.

b Let $f : X \to Y$ be an affine morphism. Take an open affine cover $\{V_i\}$ of Y. Since f is affine, each $U_i = f^{-1}V_i$ is affine, and since every affine scheme is quasi-compact, we have found a cover of Y for which each of the preimges is quasi-compact. Hence, f is quasi-compact.

Now consider the diagonal morphism $\Delta : X \to X \times_Y X$. This factors through the open subscheme $\bigcup U_i \times_{V_i} U_i \hookrightarrow X \times_Y X$ and so if $X \to \bigcup U_i \times_{V_i} U_i$ is a closed immersion, then so is Δ and f will be separated. The preimage of each $U_i \times_{V_i} U_i$ is U_i and so we just want to see that $U_i \to U_i \times_{V_i} U_i$ is a closed immersion. But this is a morphism of affine schemes whose corresponding ring homomorphism of global sections is surjective, hence, it is a closed immersion. So f is separated.

If f is finite then it follows from the definition that f is affine.

c

d That f is affine follows from the definition of $\mathbf{Spec}\,\mathcal{A}$. If $U \subseteq Y$ is an open subset, then by definition $(f_*\mathcal{O}_X)(U) = \mathcal{O}_X(f^{-1}U) = \mathcal{O}_X(\operatorname{Spec} \mathcal{A}(U)) = \mathcal{A}(U)$. So $f_*\mathcal{O}_X = \mathcal{A}$.

Conversely, suppose $f : X \to Y$ is an affine morphism. Let $\{V_i\}$ be an open affine cover of Y. Since f is affine $f^{-1}V_i$ is affine for each i, say $f^{-1}V_i = \operatorname{Spec} A_i$. We have $(f_*\mathcal{O}_X)|_{V_i} = f_*\mathcal{O}_{U_i}$ which is $f_*(\widetilde{A_i})$. By Proposition 5.2(d) this is $({}_{B_i}A_i)^{\sim}$ where $B_i = \mathcal{O}_Y(V_i)$, hence $f_*\mathcal{O}_X$ is a

[4] Explicitely, let $\operatorname{Spec} A$ be an open affine subset of X and $\rho : \Gamma(X, \mathcal{O}_X) \to A$ be the restriction morphism. Then $X_{g_i} \cap \operatorname{Spec} A = D(\rho g_i)$. But $f^{-1}D(f_i) \cap \operatorname{Spec} A$ is the preimage of the composition $\operatorname{Spec} A \to X \to \operatorname{Spec} B$. This composition gives an induced morphism $\operatorname{Spec} A \to \operatorname{Spec} B$ and the morphism $B \to A$ of global sections of this restricted morphism factors into the morphism of global sections follows by restriction $B \to \Gamma(X, \mathcal{O}_X) \to B$. The preimage of $D(f_i)$ under the morphism $specA \to \operatorname{Spec} B$ is the distinguised open corresponding to the image of f_i in A, that is, $D(\rho g_i)$. So $f^{-1}D(f_i) \cap \operatorname{Spec} A = X_{g_i} \cap \operatorname{Spec} A$. This works for any open affine and so taking a cover of them, we see that the intersection of the two subsets $f^{-1}D(f_i), X_{g_i}$ with every element in an open cover is the same, therefore they are the same.

quasi-coherent sheaf of $\mathcal{O}_Y$-algebras. To see that $\operatorname{Spec} \mathcal{A} \cong X$ we need to check that (i) for every open affine $V = \operatorname{Spec} B$ of Y we have $f^{-1}(V) = \operatorname{Spec} \mathcal{A}(V)$, and (ii) that for every inclusion of open affines $V' \subseteq V$ of Y the morphism $f^{-1}V' \subseteq f^{-1}V$ corresponds to the restriction homomorphism $\mathcal{A}(V) \to \mathcal{A}(V')$.

For (i) since f is affine know that $f^{-1}(V)$ as affine and therefore $f^{-1}(V) = \operatorname{Spec} \mathcal{O}_X(f^{-1}(V)) = \operatorname{Spec}(f_* \mathcal{O}_X)(V)$. For (ii), again since f is affine we know that $f^{-1}(V)$ and $f^{-1}(V')$ are affine and so $f^{-1}V' \hookrightarrow f^{-1}V$ corresponds to the ring homomorphism $\mathcal{O}_X(f^{-1}V) \to \mathcal{O}_X(f^{-1}V')$ which is none other than $f_*\mathcal{O}_X(V) \to f_*\mathcal{O}_X(V')$. That is, $\mathcal{A}(V) \to \mathcal{A}(V')$.

e

Exercise 5.18. Vector Bundles.

6 Divisors

Exercise 6.1. *Let X be a scheme satisfying (*). Then $X \times \mathbb{P}^n$ also satisfies (*) and* $\mathrm{Cl}(X \times \mathbb{P}^n) \cong \mathrm{Cl}(X) \times \mathbb{P}^n$.

Solution. As in the proof of Proposition II.6.6 we see immediately that $X \times \mathbb{P}^1$ is noetherian, integral, and separated. To see that it is regular in codimension one, note that it can be covered by (two) open affines of the form $X \times \mathbb{A}^1$. Each of these is shown to be regular in codimension one in the proof of II.6.6 and so $X \times \mathbb{P}^1$ is regular in codimension one.

After Proposition II.6.5 and II.6.5 we have an exact sequence

$$\mathbb{Z} \xrightarrow{i} \mathrm{Cl}(X \times \mathbb{P}^1) \xrightarrow{j} \mathrm{Cl}\, X \to 0$$

The first map sends n to nZ where Z is the closed subscheme $\pi_2^{-1}\infty \subset X \times \mathbb{P}^1$ (where $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$ is the second projection), and the second is the composition of $\mathrm{Cl}(X \times \mathbb{P}^1) \to \mathrm{Cl}(X \times \mathbb{A}^1) \xleftarrow{\sim} \mathrm{Cl}\, X$. Consider the map $\mathrm{Cl}\, X \to \mathrm{Cl}(X \times \mathbb{P}^1)$ that sends $\sum n_i Z_i$ to $\sum n_i \pi_1^{-1} Z_i$. The composition $\mathrm{Cl}(X) \to \mathrm{Cl}(X \times \mathbb{P}^1) \to \mathrm{Cl}(X \times \mathbb{A}^1) \xleftarrow{\sim} \mathrm{Cl}(X)$ sends a prime divisor Z to $\pi_1^{-1}Z$, then $(X \times \mathbb{A}^1) \cap \pi_1^{-1} Z$, and then back to Z since $(X \times \mathbb{A}^1) \cap \pi_1^{-1} Z$ is the preimage of Z under the projection $X \times \mathbb{A}^1 \to X$. Hence, the epimorphism in the exact sequence above is split.

We now show that the morphism $\mathbb{Z} \to \mathrm{Cl}(X \times \mathbb{P}^1)$ is split as well, by defining a morphism $\mathrm{Cl}(X \times \mathbb{P}^1) \to \mathbb{Z}$ which splits i. Let $k : \mathrm{Cl}\, X \to \mathrm{Cl}(X \times \mathbb{P}^1)$ denote the morphism we used to split j. Then we send a divisor ξ to $\xi - kj\xi$. This is in the kernel of j (since $jk = id$) and therefore in the image of i. So it remains only to see that i is injective.

Suppose that $nZ \sim 0$ for some integer n. Taking the "other" $X \times \mathbb{A}^1$ we have Z as $\pi_2^{-1}0$ under the projection $\pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1$. In the open subset $X \times \mathbb{A}^1$ we have Z as X embedded at the origin. So the local ring of Z in the function field $K(t)$ (where K is the function field of X) is $K[t]_{(t)}$. Since $nZ \sim 0$ there is a function $f \in K(t)$ such that $v_Z(f) = n$ and $v_Y(f) = 0$ for every other prime divisor Y. So f is of the form $t^n \frac{g(t)}{h(t)}$ where $g, h \in K[t]$ and $t \not| g(t), h(t)$. If the degree of g and h is 0 then changing coordinates back $t \mapsto t^{-1}$ we see that $v_Y(f) = -n$ where Y is another copy of X embedded at the origin, or infinity, depending on which coordinates we are using; the one opposite to Z at any rate. If one of g or h has degree higher than zero then, it will have an irreducible factor in $K[t]$, which will correspond to a prime divisor of the form $\pi_2^{-1}x$ for some $x \in \mathbb{P}^1$, and the value of f will not be zero at this prime divisor. Hence, there is no rational function with $(f) = nZ$ and so i is injective. Hence $\mathrm{Cl}(X \times \mathbb{P}^1) \cong \mathrm{Cl}(X) \times \mathbb{Z}$.

Exercise 6.2.

Exercise 6.3.

Exercise 6.4. *Let k be a field of characteristic $\neq 2$. Let $f \in k[x_1, \ldots, x_n]$ be a square free nonconstant polynomial, i.e., in the unique factorization of f into ir-*

reducible polynomials, there are no repeated factors. Let $A = k[x_1, \dots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring.

Solution. Let $B = k[x_1, \dots, x_n], L = \operatorname{Frac} B$ and consider the quotient field K of A. In this field we have $\frac{1}{g+zh}\frac{g-zh}{g-zh} = \frac{g-zh}{g^2-fh^2}$ since $z^2 = f$ in A, and so every element can be written in the form $g' + zh'$ where $g', h' \in L$. Hence, $K = L[z]/(z^2 - f)$. This is a degree 2 extension of L with automorphism $\sigma : z \mapsto -z$ and is therefore Galois. So we have the situation of Problem 5.14 from Atiyah-Macdonald (with badly chosen notation). Let A^c be the integral closure of A in K. We will show that $A = A^c$ by showing that for $\alpha = f + zg \in K$ (with $g, h \in L$) we have $\alpha \in A^c$ if and only if $f, g \in B$.

The minimal polynomial of α is $X^2 - 2gX + (g^2 - h^2 f)$. So if $g, h \in B$ then $\alpha \in A^c$. Conversely, suppose that $\alpha \in A^c$. Then $\alpha + \sigma\alpha = 2f$ and $\alpha - \sigma\alpha = 2g$ are both σ invariant and in A^c and are therefore in B, by the Atiyah-Macdonald exercise.

Exercise 6.5. Quadric Hypersurfaces. *Let char $k \neq 2$, and let X be the affine quadric hypersurface* $\operatorname{Spec} k[x_0, \dots, x_n]/(x_0^2 + x_1^2 + \dots + x_r^2)$.

a *Show that X is normal if $r \geq 2$.*

b *Show by a suitable linear change of coordinates that the equation of X could be written as $x_0 x_1 = x_2^2 + \dots + x_r^2$. Now imitate the method of (6.5.2) to show that:*

   (a) If $r = 2$ then $\operatorname{Cl} X \cong \mathbb{Z}/2\mathbb{Z}$;

   (b) If $r = 3$ then $\operatorname{Cl} X \cong \mathbb{Z}$;

   (c) If $r \geq 4$ then $\operatorname{Cl} X = 0$.

c *Now let Q be the projective quadric hypersurface in $\mathbb{P}^n$ defined by the same equation. Show that:*

   (a) If $r = 2$, $\operatorname{Cl} Q \cong \mathbb{Z}$, *and the class of a hyperplane section $Q.H$ is twice the generator;*

   (b) If $r = 3$, $\operatorname{Cl} Q \cong \mathbb{Z} \oplus \mathbb{Z}$;

   (c) If $r \geq 4$, $\operatorname{Cl} Q \cong \mathbb{Z}$, *generated by $Q.H$.*

d *Prove Klein's theorem, which says that if $r \geq 4$, and if Y is an irreducible subvariety of codimension 1 on Q, then there is an irreducible hypersurface $V \subseteq \mathbb{P}^n$ such that $Y \cap Q = Y$, with multiplicity one. In other words, Y is a complete intersection.*

Solution. a Let $A = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0^2 + x_1^2 + \dots + x_r^2)$. By taking $f = x_1^2 + \dots + x_r^2$, if we can show that f is square free, then we will have the situation of Exericise II.6.4 and so A will be integrally closed, implying that X is normal. But the polynomial f has degree 2 and so it is a product of at most 2 other nonconstant polynomials, which by degree, must be linear. Suppose $\sum a_i x_i$ is a linear polynomials such that $(\sum a_i x_i)^2 = f$.

Then $a_i^2 = 1$ for all $i = 0, \dots, r$, and $2a_ia_j = 0$ for $i \neq j \in \{0, \dots, r\}$. But this implies that $2 = 2a_i^2a_j^2 = 0$ and we have assumed that k doesn't have characteristic 2. Hence f is square free.

b We assume -1 has a square root i in k, otherwise there isn't a suitable change of coordinates. Take the change of coordinates $x_0 \mapsto \frac{y_0+y_1}{2}$ and $x_1 \mapsto \frac{y_0-y_1}{i2}$. Then $x_0^2 + x_1^2 = y_0y_1$.

Let $A = \operatorname{Spec} k[x_0, \dots, x_n]/(x_0x_1+x_2^2+\dots+x_r^2)$. Now we imitate Example II.6.5.2. We take the closed subscheme of $\mathbb{A}^{n+1}$ with ideal $\langle x_1, x_2^2+\dots+x_r^2\rangle$. This is a subscheme of X and is in fact $V(x_1)$ considering $x_1 \in A$. We have an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}(X) \to \mathrm{Cl}(X - Z) \to 0$$

Now since $V(x_1) \cap X = X - Z$ the coordinate ring of $X - Z$ is

$$k[x_0, x_1, x_1^{-1}, x_2, \dots, x_n]/(x_0x_1 + x_2^2 + \dots + x_r^2)$$

As in Example II.6.5.2 since $x_0 = -x_1^{-1}(x_2^2 + \dots + x_r^2)$ in this ring we can eliminate x_0 and since every element of the ideal $(x_0x_1 + x_2^2 + \dots + x_r^2)$ has an x_0 term, we have an isomorphism between the coordinate ring of $X - Z$ and $k[x_1, x_1^{-1}, x_2, \dots, x_n]$. This is a unique factorization domain so by Proposition II.6.2 $\mathrm{Cl}(X - Z) = 0$. So we have a surjection $\mathbb{Z} \to \mathrm{Cl}(X)$ which sends n to $n \cdot Z$.

$r = 2$ In this case the same reasoning as in Example II.6.5.2 works. Let $\mathfrak{p} \subset A$ be the prime associated to the generic point of Z. Then $\mathfrak{m}_\mathfrak{p}$ is generated by x_2 and $x_1 = x_0^{-1}x_2^2$ so $v_Z(x_1) = 2$. Since Z is cut out by x_1 there can be no other prime divisors Y with $v_Y(x_1) \neq 0$. It remains to see that Z is not a principle divisor. If it were then $\mathrm{Cl}(X)$ would be zero and by Proposition II.6.2 this would imply that A is a unique factorization domain (since A is normal by the first part of this exercise) which would imply that every hieght one prime ideal is principle. Consider the prime ideal $\langle x_1, x_2\rangle$ of A which defines Z. Let $\mathfrak{m} = (x_0, x_1, \dots, x_n)$. we have $\mathfrak{m}/\mathfrak{m}^2$ is a vector space of dimension n over k with basis $\{\overline{x}_i\}$. The ideal $\mathfrak{m}$ contains $\mathfrak{p}$ and its image in $\mathfrak{m}/\mathfrak{m}^2$ is a subspace of dimension at least 2. Hence, $\mathfrak{p}$ cannot be principle.

$r = 3$ We use Example II.6.6.1 and Exercise II.6.3(b). Using a similar change of coordinates as the beginning of this part of this exercise, we see that X is the affine cone of the projective quadric of Example II.6.6.1. This, by Exercise II.6.3(b) we have an exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to \mathrm{Cl}(X) \to 0$. We already know that $\mathrm{Cl}(X)$ is $\mathbb{Z}, \mathbb{Z}/n$ or 0. Tensoring with $\mathbb{Q}$ gives an exact sequence $\mathbb{Q} \to \mathbb{Q}^2 \to \mathrm{Cl}(X) \otimes \mathbb{Q} \to 0$ of $\mathbb{Q}$ vector spaces. Hence, $\mathrm{Cl}(X) = \mathbb{Z}$, as the other two cases contradict the exactness of the sequence of $\mathbb{Q}$-vector spaces.

$r \geq 4$ In this case we claim that Z is principle. Consider the ideal (x_1) in A. Its corresponding closed subset is Z and so if we can show that (x_1) is prime, then Z will be the principle divisor associated to the rational function x_1. Showing that (x_1) is prime is the same as showing that $A/(x_1)$ is integral, which is the same as showing that $\frac{k[x_0,\dots,x_n]}{(x_1,x_2^2+\dots+x_r^2)}$ is integral since $(x_1, x_0x_1 + x_2^2 + \dots + x_r^2) = (x_1, x_2^2 + \dots + x_r^2)$. This is the same as showing that $\frac{k[x_0,x_2,\dots,x_n]}{(x_2^2+\dots+x_r^2)}$ is integral (where the variable x_1 is missing on the top) which is the same as showing that $f = x_2^2 + \dots + x_r^2$ is irreducible. Suppose f is a product of more than one nonconstant polynomial. Since it has degree two, it is the product of at most two linear polynomials, say $a_0x_0+a_2x_2+\dots+a_nx_n$ and $b_0x_0 + b_2x_2 + \dots + b_nx_n$. Expanding the product of these two linear polynomials and comparing coefficients with f we find that (I) $a_ib_i = 1$ for $2 \leq i \leq r$, and (II) $a_ib_j + a_jb_i = 0$ for $2 \leq i,j \leq r$ and $i \neq j$. Without loss of generality we can assume that $a_2 = 1$. The relation (I) implies that $b_2 = 1$, and in general, $a_i = b_i^{-1}$ for $2 \leq i \leq r$. Putting this in the second relation gives (III) $a_i^2 + a_j^2 = 0$ for $2 \leq i \neq j \leq r$ and this together with the assumption that $a_2 = 1$ implies that (IV) $a_j^2 = -1$ for each $2 < j \leq r$. But if $r \geq 4$ then we have from (III) that $a_3^2 + a_4^2 = 0$ which contradicts (IV). Hence $x_2^2 + \dots + x_r^2$ is irreducible, so $\frac{k[x_0,x_2,\dots,x_n]}{(x_2^2+\dots+x_r^2)}$ is integral, so $A/(x_1)$ is integral, so (x_1) is prime and hence Z is the principle divisor corresponding to x_1. So $\mathrm{Cl}(X) = 0$.

c For each of these we use the exact sequence of Exercise II.6.3(b).

$r = 2$ We have an exact sequence $0 \to \mathbb{Z} \to \mathrm{Cl}(Q) \to \mathbb{Z}/2 \to 0$ where the first mophism sends 1 to the class of $H \cdot Q$ a hyperplane section. Tensoring with $\mathbb{Q}$ we get an exact sequence $\mathbb{Q} \xrightarrow{2} \mathrm{Cl}(Q) \otimes \mathbb{Q} \to 0 \to 0$ and so since $\mathrm{Cl}(Q)$ is an abelian group we see that it is $\mathbb{Z} \oplus T$ where T is some torsion group. Tensoring with $\mathbb{Z}/p$ for a prime p we get either $\mathbb{Z}/2 \xrightarrow{0} Cl(Q) \otimes (\mathbb{Z}/2) \to \mathbb{Z}/2 \to 0$ if $p = 2$ or $\mathbb{Z}/p \xrightarrow{2} \mathrm{Cl}(Q) \otimes (\mathbb{Z}/p) \to 0 \to 0$ if $p \neq 2$. Hence, $T = 0$, and so $\mathrm{Cl}(Q) \cong \mathbb{Z}$ and the class of a hyperplane section is twice the generator.

$r = 3$ This is Example II.6.6.1.

$r \geq 4$ We have an exact sequence $0 \to \mathbb{Z} \to \mathrm{Cl}(Q) \to 0 \to 0$, hence, $\mathrm{Cl}(Q) = \mathbb{Z}$ and it is generated by $Q \cdot H$.

Exercise 6.6. *Let X be the nonsingular plane cubic curve $y^2z = x^3 - xz^2$ of (6.10.2).*

a *Show that three points P, Q, R of X are collinear if and only if $P+Q+R = 0$ in the group law on X. (Note that the point $P_0 = (0, 1, 0)$ is the zero element in the group structure on X).*

b *A point $P \in X$ has order 2 in the group law on X if and only if the tangent line at P passes through P_0.*

c *A point $P \in X$ has order 3 in the group law on X if and only if P is an inflection point (an inflection point of a plane curve is a nonsingular point P of the curve, whose tangent line (Exercise I.7.3) has intersection multiplicity ≥ 3 with the curve at P.)*

d *Let $k = \mathbb{C}$. Show that the points of X with coordinates in $\mathbb{Q}$ form a subgroup of the group X. Can you determine the structure of this subgroup explicitly?*

Solution. a Suppose that P, Q, R are collinear. Then there is a line L on which they all lie and since every line meets X in exactly three points (counting multiplicities) P, Q, R are the only points where L meets X. In $\mathbb{P}^2$ any line is equivlant to z and so $P + Q + R \sim 3P_0$ as divisors, hence $(P - P_0) + (Q - P_0) + (R - P_0) \sim (P_0 - P_0)$ as divisors, and therefore $P + Q + R = 0$ in the group law on X.

Conversely, suppose that $P+Q+R = 0$ in the group law on X. If P, Q, R are not all distinct, then they are collinear in $\mathbb{P}^2$ since any two points are collinear in $\mathbb{P}^2$. Suppose they are distinct and consider the unique line L on which P and Q lie. This line intersects X in a unique third point T and we have $P + Q + T \sim 3P_0$. Hence, $P + Q + T = 0$ in the group law on X and therefore $R = -P - Q = T$. So P, Q, R are collinear.

b Recall that the tangent line to P is the unique line $T_P(X)$ whose intersection multiplicity with X at P is > 1 (Exercise I.7.3).

If $P = P_0$ then certainly the tangent line passes through P_0. Suppose that $P \neq P_0$ has order 2 and consider the tangent line $T_P(X)$ to P. This line intersects X in three points (counting multiplicities) and since it hits P with multiplicity greater than one, these three points are P, P and R for some other point R (which is possibly also P). Now P, P and R being collinear means that $P + P + R + 0$ in the group law on X. But P has order 2 and so we see that $R = 0 = P_0$. Hence, the tangent line $T_P(X)$ passes through P_0.

Conversely, suppose that the tangent line $T_P(X)$ passes through P_0. Since P_0 is the identity, it has order 2 so suppose that $P \neq P_0$. Again, $T_P(X)$ hits X in three points (counting multiplicities) of which at least two are P, and since we have assumed that $P_0 \neq P$ these three points are P, P and P_0. Hence, $P + P + P_0 = 0$ and since $P_0 = 0$ we see that P has order 2.

c If P is an inflection point then the intersection multiplicity of $T_P(X)$ and X at P is ≥ 3. Since X has degree three it can't be more than three and so we see that it is exactly three. So the three points of X that $T_P(X)$ hits, counting multiplicites, are all P, and so $P + P + P = 0$ in the group law. Hence, P has order three.

Conversely, if P has order three then $P+P+P=0$ then the three points P, P, P are collinear. That is, there is a line L such that L intersects X in the unique point P with intersection multiplicity three. Since there is a unique line of $\mathbb{P}^2$ that intersects X at P with multiplicity greater than one–the tangent line–we see that the tangent line intersects X at P with multiplicity three, and therefore P is an inflection point.

d If the base field is $\mathbb{C}$ then the elliptic curve is isomorphic as an abelian variety to the quotient of the complex plane by a lattice $\mathbb{Z}^2$.

Exercise 6.7. *Let X be the nodal cubic curve $y^2z = x^3 + x^2z$ in $\mathbb{P}^2$. Imitate (6.11.4) and show that the group of Cartier divisors of degree 0,* $\operatorname{CaCl}^0 X$*, is naturally isomorphic to the multiplicative group $\mathbb{G}_m$.*

Exercise 6.8. *a Let $f : X \to Y$ be a morphism of schemes. Show that $\mathscr{L} \mapsto f^*\mathscr{L}$ induces a homomorphim of Picard groups, $f^* : \operatorname{Pic} Y \to \operatorname{Pic} X$.*

b If f is a finite morphism of nonsingular curves, show that this homomorphism corresponds to the homomorphism $f^* : \operatorname{Cl} Y \to \operatorname{Cl} X$ defined in the text, via the isomorphism of (6.16).

c If X is a locally factorial integral closed subscheme of $\mathbb{P}^n_k$, and if $f : X \to \mathbb{P}^n$ is the inclusion map, then f^* on Pic *agrees with the homomorphism on divisor class groups defined in (Ex. 6.2) via the isomorphisms of (6.16).*

Exercise 6.9. Singular curves.

Exercise 6.10. The Grothendieck Group $K(X)$. *Let X be a noetherian scheme. We define $K(X)$ to be the quotient of the free abelian group genertaed byt all the coherent sheaves on X, by the subgroup generated by all expressions $\mathscr{F} - \mathscr{F}' - \mathscr{F}''$, whenever there is an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves on X. If $\mathscr{F}$ is a coherent sheaf, we denote by $\gamma(\mathscr{F})$ its image in $K(X)$.*

a If $X = \mathbb{A}^1_k$, then $K(X) \cong \mathbb{Z}$.

b If X is any integral scheme, and $\mathscr{F}$ a coherent sheaf, we define the rank of $\mathscr{F}$ to be $\dim_k \mathscr{F}_\xi$ where ξ is the generic point of X, and $K = \mathcal{O}_\xi$ is the function field of X. Show that the rank function defines a surjective homomorphism $\operatorname{rank} : K(X) \to \mathbb{Z}$.

c If Y is a closed subscheme of X, there is an exact sequence

$$K(Y) \to K(X) \to K(X - Y) \to 0$$

where the first map is extension by zero, and the second map is restriction.

Solution. a Let $\mathscr{F}$ be a coherent sheaf on X. Then $\mathscr{F}$ corresponds to a finitely generated $k[t]$-module M. We take a presentation $k[t]^{\oplus n} \to k[t]^{\oplus m} \to M \to 0$ of M and since $k[t]$ is a principle ideal domain, we can choose the first morphism to be injective.[1] Hence, we arrive at an exact sequence $0 \to \mathcal{O}_X^{\oplus n} \to \mathcal{O}_X^{\oplus m} \to \mathscr{F} \to 0$ so in the Grothendieck group we have $\gamma(\mathscr{F}) = (m-n)\gamma(\mathcal{O}_X)$. So the morphism $\mathbb{Z} \to K(X)$ sending n to $n\gamma(\mathcal{O}_X)$ is surjective. To see that this morphism is injective, we use the rank homomorphism from the next part of this exercise to split it.

b First we show that it defines a homomorphism. Let $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ be an exact sequence of coherent sheaves on X. Since this sequence is exact, it is exact at every stalk. In particular, it is exact at the stalk at the generic point ξ. So we have an exact sequence of finitely generated $\mathcal{O}_\xi$-modules $0 \to \mathscr{F}'_\xi \to \mathscr{F}_\xi \to \mathscr{F}''_\xi \to 0$. Hence, $\dim_K \mathscr{F}_\xi = \dim_K \mathscr{F}''_\xi + \dim_K \mathscr{F}'_\xi$. So rank is a well-defined homomorphism.

To see that it is surjective, notice that $\gamma(\mathcal{O}_X) \mapsto 1$, and so $n \cdot \gamma(\mathcal{O}_X) \mapsto n$.

c *Surjectivity on the right.* Every coherent sheaf $\mathscr{F}$ on $X - Y$ can be extended to a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_{X-Y} = \mathscr{F}$ by Exercise II.5.15. So the morphism on the right is surjective.

Exactness in the middle. Suppose that $\mathscr{F}$ is a coherent sheaf on X with support in Y. We will show (below) that there is a finite filtration $\mathscr{F} = \mathscr{F}_0 \supseteq \mathscr{F}_1 \supseteq \cdots \supseteq \mathscr{F}_n = 0$ such that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y. Assuming we have such a finite filtration, we have $\gamma(\mathscr{F}_i) = \gamma(\mathscr{F}_{i+1}) + \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$ in $K(X)$ and so $\gamma(\mathscr{F}) = \sum_{i=0}^{n-1} \gamma(\mathscr{F}_i/\mathscr{F}_{i+1})$. Hence, the class represented by $\mathscr{F}$ is in the image of $K(Y) \to K(X)$. Now if $\sum n_i \gamma(\mathscr{F}_i)$ is in the kernel of $K(X) \to K(X-Y)$ the *Proof of claim.* Let $i : Y \to X$ be the closed embedding of Y into X and consider the two functors $i_* : Coh(X) \to Coh(Y)$ (Exercise II.5.5) and $i^* : Coh(Y) \to Coh(X)$. These functors are adjoint (page 110) and so we have a natural morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ for any coherent sheaf $\mathscr{F}$ on X. Let $\operatorname{Spec} A$ be an open affine subscheme of X on which $\mathscr{F}$ has the form $\widetilde{M}$. Closed subschemes of affine schemes correspond to ideals bijectively and so $\operatorname{Spec} A \cap Y = \operatorname{Spec} A/I$ for some ideal $I \subset A$ and the morphism $\eta : \mathscr{F} \to i_* i^* \mathscr{F}$ restricted to $\operatorname{Spec} A$ has the form $M \to M/IM$. Thus we see that η is surjective. Let $\mathscr{F}_0 = \mathscr{F}$ and define $\mathscr{F}_j$ inductively as $\mathscr{F}_j = \ker(\mathscr{F}_{j-1} \to i_* i^* \mathscr{F}_j)$. It follows from our

[1] *If N is a submodule of a free A-module M of rank n where A is an integral PID then N is free.* Induction on n. If $n = 1$ then a submodule is an ideal and since A is a PID the ideal is of the form (a) for some $a \in A$. Since A is integral the map $b \mapsto ab$ is an isomorphism of modules. Now suppose $M = A^n$. Consider the submodule A^{n-1} of elements whose last component is zero. Then by the inductive hypothesis $N' = A^{n-1} \cap N$ is free; let $m_1, \dots, m_r$ be a basis for N' as a free A-module. If $\pi : A^n \to A$ is projection onto the last component then its image is an ideal I of A. If $I = 0$ then $N' = N$ and we are done. If not, choose an elemen $n \in N$ such that $\pi n = a$ where $(a) = I$. Then we claim that $N = N' \oplus An$. Certainly, $N' + An \subseteq N$. If $m \in N$ then $m = (m - (\pi m)n) + (\pi m)n$ is a decomposition into an element of N' and of An so $N' + An \supseteq N$ and therefore $N' + An = N$, so it remains to see that $N' \oplus An \to N' + An$ is injective. Suppose (x, bn) is in the kernel. Then $x + bn = 0$ and so $\pi(x + bn) = 0$. But $\pi(x + bn) = ba$ and since A is integral this implies that $b = 0$. Hence, $x + 0n = 0$ and so $x = 0$. So $N' \oplus An \to N' + An = N$ is an isomorphism.

definition that each $\mathscr{F}_i/\mathscr{F}_{i+1}$ is the extension by zero of a coherent sheaf on Y so we just need to show that the filtration $\mathscr{F} \supseteq \mathscr{F}_1 \supseteq \dots$ is finite.

On our open affine we have $\mathscr{F}_j|_{\operatorname{Spec} A} = I^j M$. Now the support of $\widetilde{M}$ contained in the closed subscheme $\operatorname{Spec} A/I = V(I)$ so by Exercise II.5.6(b) we have $\sqrt{\operatorname{Ann} M} \supseteq \sqrt{I} \supseteq I$. Since A is noetherian, every ideal is finitely generated. In particular, I is finitely generated. So there exists some N such that $\operatorname{Ann} M \supseteq I^N$ (see the proof of Exercise II.5.6(d) for details). Hence, $0 = I^N M$ and so the filtration is finite when restricted to an open affine. Since X is noetherian, there is a cover by finitely many affine opens $\{U_i\}$ and so if n_i is the point at which $\mathscr{F}_i|_{U_i} = 0$ then $\mathscr{F}_{\max\{n_i\}} = 0$. So the filtration is finite.

Exercise 6.11. The Grothendieck Group of a Nonsingular Curve. *Let X be a nonsingular curve over an algebraically closed field k.*

a *For any divisor $D = \sum n_i P_i$, let $\psi(D) = \sum n_i[k(P_i)] \in K(X)$ where $k(P_i)$ is the skyscraper sheaf k at P_i and 0 elsewhere. If D is an effective divisor, let $\mathcal{O}_D$ be the stucture sheaf of the associated subscheme of codimension 1, and show that $\psi(D) = [\mathcal{O}_D]$. Then use (6.18) to show that for any $D, \psi(D)$ depends only on the linear equivalence class of D, so ψ defines a homomorphism $\psi : \operatorname{Cl} X \to K(X)$.*

b *For any coherent sheaf $\mathscr{F}$ on X, show that there exists locally free sheaves $\mathscr{E}_0$ and $\mathscr{E}_1$ and an exact sequence $0 \to \mathscr{E}_1 \to \mathscr{E}_0 to \mathscr{F} \to 0$. Let $r_0 = \operatorname{rank} \mathscr{E}_0$, $r_1 = \operatorname{rank} \mathscr{E}_1$, and define $\det \mathscr{F} = (\wedge^{r_0} \mathscr{E}_0) \otimes (\wedge^{r_1} \mathscr{E}_1)^{-1} \in \operatorname{Pic} X$. Show that $\det \mathscr{F}$ is independent of the resolution chosen, and that it goves a homomorphism $\det : K(X) \to \operatorname{Pic} X$. Finally show that if D is a divsor, then $\det(\psi(D)) = \mathscr{L}(D)$.*

c *If $\mathscr{F}$ is any coherent sheaf of rank r, show that there is a divisor D on X and an exact sequence $0 ø \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$, where $\mathscr{T}$ is a torsion sheaf. Conclude that if $\mathscr{F}$ is a sheaf of rank f, then $[\mathscr{F}] - r[\mathcal{O}_X] \in \operatorname{im} \psi$.*

d *Using the maps $\psi, \det, \operatorname{rank}$, and $1 \mapsto [\mathcal{O}_X]$ from $\mathbb{Z} \to K(X)$, show that $K(X) \cong \operatorname{Pic} X \oplus \mathbb{Z}$.*

Solution. a We denote the associated subscheme of D also by D. So its sheaf of ideals is $\mathscr{I}_D$. For each closed point $P \in X$ let $\mathscr{F}_P$ be the skyscraper sheaf $\operatorname{coker}((\mathscr{I}_D)_P \to \mathcal{O}_P)$ at P and zero elsewhere. There are surjections $\mathcal{O}_X \to \mathscr{F}_P$ for each P and so we have an exact sequence

$$0 \to \mathscr{I}_D \to \mathcal{O}_X \to \bigoplus_{P \in X} \mathscr{F}_P \to 0$$

Hence, $\mathcal{O}_D \cong \oplus \mathscr{F}_P$ and so $\gamma(\mathcal{O}_D) = \sum \gamma(\mathscr{F}_P)$. Now consider $\mathscr{F}_P$ for some $P \in X$ with $\mathscr{F}_P$ nonzero (there are only finitely many as there are only finitely many points in D). Choose a representation $\{(U_i, f_i)\}$ of the Cartier divisor corresponding to the Weil divisor D. Since D is effective, this can be chosen so that $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$ for each i, and in this case the sheaf of ideals $\mathscr{I}_D$ is locally generated by f_i (by the definition on page 145). If U_i is an open that contains P then $v_P(f_i) = n$, where n is the coefficient of P in the sum D. So in the local ring $\mathcal{O}_P$ we have $f_i = t^n$ where t is a generator of $\mathfrak{m}_P$. The stalk of $\mathscr{F}_P$ at P is by our definition above $\operatorname{coker}((\mathscr{I}_D)_P \to \mathcal{O}_P)$ which we can now see to be $\mathcal{O}_P/\mathfrak{m}_P^n$. For each i we have an exact sequence of $\mathcal{O}_P$ modules $0 \to \mathfrak{m}_P^i/\mathfrak{m}_P^{i+1} \to \mathcal{O}_P/\mathfrak{m}_P^{i+1} \to \mathcal{O}_P/\mathfrak{m}_P^i \to 0$ and we have isomorphisms of $\mathcal{O}_P$-modules $\mathfrak{m}_P^i/\mathfrak{m}_P^{i+1} \cong \mathfrak{m}_P/\mathfrak{m}_P^2 \cong k$ so it follows that $\gamma(\mathscr{F}_P) = n\gamma(k(P))$. Combining this with the equality $\gamma(\mathcal{O}_D) = \sum \gamma(\mathscr{F}_P)$ shows that $\psi(D) = \gamma(\mathcal{O}_D)$.

If D' is some other effective divisor in the same linear equivalence class as D then we have

$$\begin{aligned}\psi(D) &= \gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_D) \\ &\overset{6.18}{=} \gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(-D)) \overset{6.13}{=} \gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(-D')) \\ &\overset{6.18}{=} \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_{D'})) = \gamma(\mathcal{O}_{D'}) = \psi(D')\end{aligned}$$

So ψ defines a homomorphism (for an arbitrary divisor D, write it as a difference of two effective divisors $D = D_+ - D_-$ and then we have $\psi(D) = \gamma(\mathcal{O}_{D_+}) - \gamma(\mathcal{O}_{D_-})$).

b *Existence of the exact sequence.* By Corollary II.5.18 we can write $\mathscr{F}$ as the quotient of a finite direct sum $\mathscr{E}_0 = \oplus\mathcal{O}(n_i)$ of twisted structure sheaves $\mathcal{O}(n_i)$ for various n_i. Let $\mathscr{E}_1$ be the kernel of the map $\mathscr{E}_0 \to \mathscr{F}$. At each closed point we then have an exact sequence

$$0 \to (\mathscr{E}_1)_x \to \mathcal{O}_x^{\oplus n} \to \mathscr{F}_x \to 0$$

That is, $(\mathscr{E}_1)_x$ is a submodule of $\mathcal{O}_x^{\oplus n}$. But each $\mathcal{O}_x$ is a reduced regular local ring of dimension one, and therefore a principle ideal domain (the only two ideals are zero since it is reduced, and $\mathfrak{m}$ which is principle since $\mathcal{O}_x$ is regular) and every submodule of a free module over a principle ideal domain is free. Hence $(\mathscr{E}_1)_x$ is free for every closed point x. Then by Exercise II.5.7 $\mathscr{E}_1$ is locally free.

Independence of $\mathscr{E}_1$ and $\mathscr{E}_0$. Suppose that we choose another locally free resolution $0 \to \mathscr{E}_1' \to \mathscr{E}_0' \to \mathscr{F} \to 0$. Consider the sequence $0 \to \mathscr{G} \to$

$\mathscr{E}_0 \oplus \mathscr{E}_0' \to \mathscr{F} \to 0$. We have a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E}_0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{E}_0 \oplus \mathscr{E}_0' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E}_0' \arrow[r, equal] \arrow[d] & \mathscr{E}_0' \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and we know that the rows and the two right columns are exact. Hence, the left column is exact as well by the nine lemma. We also get a similar diagram using $\mathscr{E}_1', \mathscr{E}_0'$ in the top row which gives an exact sequence $0 \to \mathscr{E}_1' \to \mathscr{G} \to \mathscr{E}_0 \to 0$ as the left column. We know that $\mathscr{G}$ is a locally free sheaf by the same argument we used to show that existence of the exact sequence and so using the isomorphism of exercise II.5.16(d) we see that

$$\begin{aligned}
(\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{E}_1)^{-1} &\cong (\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{E}_1)^{-1} \otimes (\wedge\mathscr{E}_0')^{-1} \otimes (\wedge\mathscr{E}_0') \\
&\cong (\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{G})^{-1} \otimes (\wedge\mathscr{E}_0') \\
&\cong (\wedge\mathscr{E}_0) \otimes (\wedge\mathscr{E}_1')^{-1} \otimes (\wedge\mathscr{E}_0)^{-1} \otimes (\wedge\mathscr{E}_0') \\
&\cong (\wedge\mathscr{E}_0') \otimes (\wedge\mathscr{E}_1')^{-1}
\end{aligned}$$

So the determinant is independent of the resolution chosen.

The map det *defines a homomorphism* $K(X) \to \operatorname{Pic}(X)$. We need to show that whenever we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves, it holds that $\det \mathscr{F} \cong (\det \mathscr{F}') \otimes (\det \mathscr{F}'')$. Let $0 \to \mathscr{E}_1' \to \mathscr{E}_0' \to \mathscr{F}' \to 0$ be an exact sequence, and $\mathscr{E}_0 \to \mathscr{F}$ a surjective morphism with $\mathscr{E}_0, \mathscr{E}_0', \mathscr{E}_1'$ all locally free. We define $\mathscr{G} = \ker(\mathscr{E}_0 \oplus \mathscr{E}_0' \to \mathscr{F})$

and $\mathscr{H} = \ker \mathscr{E}_0 \to \mathscr{F}''$ to obtain a diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathscr{E}_1' \arrow[r] \arrow[d] & \mathscr{G} \arrow[r] \arrow[d] & \mathscr{H} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{E}_0' \arrow[r] \arrow[d] & \mathscr{E}_0 \oplus \mathscr{E}_0' \arrow[r] \arrow[d] & \mathscr{E}_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

whose columns and the lower two rows are exact by construction. Hence, by the nine lemma the top row is exact. $\mathscr{G}$ and $\mathscr{H}$ are locally free sheaves by the same argument we used to show the existence of the exact sequence above. using the isomorphism of exercise II.5.16(d) we see that

$$\begin{aligned}
\det \mathscr{F} &\cong (\wedge \mathscr{E}_0) \otimes (\wedge \mathscr{E}_0') \otimes (\wedge \mathscr{G})^{-1} \\
&\cong (\wedge \mathscr{E}_0) \otimes (\wedge \mathscr{E}_0') \otimes (\wedge \mathscr{H})^{-1} \otimes (\wedge \mathscr{E}_1')^{-1} \\
&\cong \det \mathscr{F}' \otimes \det \mathscr{F}''
\end{aligned}$$

Hence, $\det : K(X) \to \operatorname{Pic}(X)$ is a well defined homomorphism.

For a divisor D, $\det(\psi(D)) = \mathscr{L}(D)$. Suppose D is an effective divisor. Then we have an exact sequence $0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ where Y is the corresponding closed subscheme. Since both $\mathcal{O}_X$ and $\mathscr{I}_Y$ are locally free of one, and by definition $\psi(D) = \gamma(\mathcal{O}_D)$ we have $\det(\psi(D)) = \mathcal{O}_X \otimes \mathscr{I}_Y^{-1} = \mathscr{I}_Y^{-1}$. Then using Proposition II.6.18 this is equal to $\mathscr{L}(-D)^{-1}$ and then by Proposition II.6.13 this is isomorphic to $\mathscr{L}(D)$. If D is not effective, write it as a difference of effective divisors and use the fact that det and ψ are both group homomorphisms together with Proposition II.6.13.

c To construct the injective morphism, the idea is to take a basis for the $K(X)$-vector space $\mathscr{F}_\xi$, and find a suitable $\mathscr{L}(D)$ such that this basis gives global sections of $\mathscr{L}(D) \otimes \mathscr{F}$. This defines a morphism $\mathcal{O}_X^{\oplus n} \to \mathscr{L}(D) \otimes \mathscr{F}$ which we show to be injective, and then tensor everything with $\mathscr{L}(D)^{-1}$.

Cover X with finitely many open affines $\{U_i = \operatorname{Spec} A_i\}_{i=1}^n$. On each of these, the restriction of $\mathscr{F}$ has the form $\widetilde{M_i}$ for some A_i-module M_i. Now consider the stalk $\mathscr{F}_\xi$ of $\mathscr{F}$ at the generic point. Since X is integral each A_i is integral and so the generic point appears as (0) in each U_i, so we have isomorphisms $\mathscr{F}_\xi \cong (\operatorname{Frac} A_i) \otimes_{A_i} M_i$ for each i. If $e_1, \dots, e_n$ is a basis for $\mathscr{F}_\xi$ as a $K(X)$-vector space, then these isomorphisms gives each e_j as $\frac{m_{ij}}{a_i}$ for some $m_{ij} \in M_i$ and $a_i \in A_i$ (if for some i the denominators of each $\frac{m_{ij}}{a_{ij}}$ were not the same, multiply by $\frac{\prod_{k \neq j} a_{ik}}{\prod_{k \neq j} a_{ik}}$ to get $\frac{m'_{ij}}{\prod a_{ij}}$). Now we want to use the a_i to define a Cartier divisor but $\frac{a_i}{a_j}$ might not be in $\mathcal{O}_X(U_i \cap U_j)$. We rectify this by shrinking the U_i as follows. First define $U'_i = U_i \backslash V(a_i)$ for each i. If $\cup U'_i \neq X$ then its complement is a finite set of points (since X is a curve), each one of which is contained in $V(a_i)$ for some i (since $\{U_i\}$ was a cover). For each of these points x, choose a $V(a_i)$ that it is in, and put it back in U_i. So if Z_i is the set of points in $V(a_i)$ that we have decided to leave in U_i, we have $U'_i = U_i \backslash (V(a_i) \backslash Z_i)$. The end result is that for $i \neq j$, if x is a point in $V(a_i) \cup V(a_j)$ then $x \notin U'_i \cap U'_j$. So $V(a_i) \cap (U'_i \cap U'_j)$ and $V(a_j) \cap (U'_i \cap U'_j)$ are both empty. It follows that a_i and a_j are both invertible in $\mathcal{O}_X(U'_i \cap U'_j)$. [2]

So we can define a Cartier divisor $D' = \{(U'_i, a_i\}$ whose associated sheaf is locally generated by $\frac{1}{a_i}$ on U'_i. The point is that our basis vectors e_j from $\mathscr{F}_\xi$ are now sections $\frac{1}{a_i} \otimes m_{ij}$ of $\Gamma(U'_i, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$. Futhermore, these sections agree on the intersections and so we have global sections $e_i \in \Gamma(X, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$ and this we obtain a morphism $\mathcal{O}_X^{\oplus n} \to \mathscr{L}(D') \otimes \mathscr{F}$. We claim that this is injective. To see this, it will be enough to show that the $\frac{1}{a_i} \otimes m_{ij}$ generate a free submodule of $\Gamma(U'_i, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$. To see this let $M = \Gamma(U'_i, \mathscr{L}(D') \otimes_{\mathcal{O}_X} \mathscr{F})$ and consider the morphism $M \to M \otimes K(X)$. Let $A = \mathcal{O}_X(U'_i)$ and let $A^n \to M$ be the morphism defined by sending $(a_1, \dots, a_n)$ to $\sum_j a_j \frac{1}{a_i} \otimes m_{ij}$. If $A^n \to M$ were to have a kernel, say N, then we would have an exact sequence

$$N \otimes K \to A^n \otimes K \to M \otimes K$$

but the second morphism is an isomorphism and so $N \otimes K$ is zero. Hence the composition $N \to N \otimes K \to A^n \otimes K$ is zero. But this is the same as the composition $N \to A^n \to A^n \otimes K$, and both of these maps are injective. Hence, $N = 0$.

So we have an injective morphism of sheaves $\mathcal{O}_X \to \mathscr{L}(D') \otimes \mathscr{F}$. Now we need just tensor with $\mathscr{L}(D')^{-1} = \mathscr{L}(D)$ and we obtain an exact sequence $0 \to \mathscr{L}(D)^{\oplus n} \to \mathscr{F} \to \mathscr{T} \to 0$ where $\mathscr{T}$ is the cokernel of $\mathscr{L}(D)^{\oplus n} \to \mathscr{F}$. To see that $\mathscr{T}$ is torsion, consider the stalk of this exact sequence at the generic point. We get an exact sequence of $K(X)$-vector spaces $0 \to V' \to V \to V'' \to 0$ and the ranks of V' and V are the same. Hence $\mathscr{T}_\xi = 0$ and $\mathscr{T}$ is torsion.

To show that $[\mathscr{F}] - r[\mathcal{O}_X]$ is in the image of ψ we first use the exact sequence $0 \to \mathscr{L}(D)^{\oplus r} \to \mathscr{F} \to \mathscr{T} \to 0$ to see that $[\mathscr{F}] - r[\mathcal{O}_X] = r[\mathscr{L}(D)] + [\mathscr{T}] - r[\mathcal{O}_X]$. So if $[\mathscr{T}]$ and $[\mathscr{L}(D)] - [\mathcal{O}_X]$ are in the image of ψ then we are done.

[2] For any affine scheme $\operatorname{Spec} A$, if a is not invertible, then (a) is a proper ideal of A, and therefore contained in some maximal idea (Zorn's Lemma) $\mathfrak{m}$ which implies that $a \in \mathfrak{m}$ and so $\mathfrak{m} \in V(a)$. Therefore, if $V(a) = \varnothing$ then a is invertible.

(i) $[\mathscr{L}(D)] - [\mathcal{O}_X]$ *is in the image of* ψ. As we saw in part (a) of this exercise, for effective divisors D there is an exact sequence $0 \to \mathscr{L}(D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$ (c.f. Proposition II.6.18) so in $K(X)$ we have $[\mathcal{O}_D] = [\mathcal{O}_X] - [\mathscr{L}(D)]$. Now if D is not necassarily effective, then it can be written as a difference $D = D_+ - D_-$ of effective divisors. Then we have $\psi(D) = [\mathcal{O}_{D_+}] - [\mathcal{O}_{D_-}] = [\mathcal{O}_X] - [\mathscr{L}(D_+)] - [\mathcal{O}_X] + [\mathscr{L}(D_-)] = [\mathscr{L}(D_-)] - [\mathscr{L}(D_+)]$. Now since $\mathscr{L}(D_-)^{-1}$ is locally free, tensoring with it preserves exact sequences, so $\Phi : [\mathscr{F}] \mapsto [\mathscr{F} \otimes \mathscr{L}(D_-)^{-1}]$ is a well defined (set) function on $K(X)$. So $\Phi(\psi(D)) = \Phi\big([\mathscr{L}(D_-)] - [\mathscr{L}(D_+)]\big) = [\mathcal{O}_X] - [\mathscr{L}(D)]$. But $\psi(D) = \sum n_i[k(P_i)]$ where $D = \sum n_i P_i$. and so $\psi(D)$ is unchanged by Φ. Hence $[\mathcal{O}_X] - [\mathscr{L}(D)]$ is in the image of ψ.

(ii) $[\mathscr{T}]$ *is in the image of* ψ. By Exercise II.5.6 the support of $\mathscr{T}$ is a closed subset of X. Since X is a curve, this is a finite set of points, so $\mathscr{T} = \oplus \mathscr{T}_{P_i}$ is a finite sum of skyscraper sheaves. If we can show that $[\mathscr{T}_P]$ is in the image of ψ for every coherent skyscraper sheaf $\mathscr{T}_P$ then we are done. As we are not assuming X complete, it is enough to do this in the affine case. So suppose that $X = \operatorname{Spec} A$ and that $\widetilde{M}$ is a coherent skyscraper sheaf, concentrated at the maximal prime $\mathfrak{p} \in \operatorname{Spec} A$. For each i we have an exact sequence $0 \to \mathfrak{p}^{i+1}M \to \mathfrak{p}^i M \to \mathfrak{p}^i M/\mathfrak{p}^{i+1}M \to 0$. The A-module $\mathfrak{p}^i M/\mathfrak{p}^{i+1}M$ is a finite rank $A/\mathfrak{p}$-module; that is, a finite dimensional vector space. Hence, $\mathfrak{p}^i M/\mathfrak{p}^{i+1}M \cong (A/\mathfrak{p})^{\oplus n_i}$ for some n_i. The associated sheaf to $A/\mathfrak{p}$ is the skyscraper sheaf $k(P)$ and so by induction, we have $[\widetilde{M}] = \sum_{i \geq 0} n_i[k(P)]$, if this sum is finite. As the support of $\widetilde{M}$ is $\mathfrak{p}$, Exercise II.5.6(b) shows that $\sqrt{\operatorname{Ann} M} = \mathfrak{p}$. The ring A is noetherian and so $\mathfrak{p}^N \subseteq \operatorname{Ann} M$ for some N. [3] This means that $\mathfrak{p}^N M = 0$. Hence, $n_j = 0$ for each $j > N$ and so the sum $[\widetilde{M}] = \sum_{i \geq 0} n_i[k(P)]$ is finite. Therefore, $[\mathscr{T}]$ is in the image of ψ.

d The diagram is

$$\begin{tikzcd} \operatorname{Pic} X \arrow[r, shift right, "\psi"'] & K(X) \arrow[l, shift right, "\det"'] \arrow[r, shift right, "\text{rank}"'] & \mathbb{Z} \arrow[l, shift right, "n\gamma(\mathcal{O}_X)"'] \end{tikzcd}$$

It is fairly evident that $\text{rank}(n\gamma(\mathcal{O}_X)) = n$ and $\det(n\gamma(\mathcal{O}_X)) = \mathcal{O}_X^{\otimes n} = \mathcal{O}_X = 1 \in \operatorname{Pic}(X)$. Furthermore, since ψ takes a divisor to a sum of skyscraper sheaves, and the rank of a skyscraper sheaf is zero, we have $\text{rank} \circ \psi = 0$. So we just need to show that $\det \circ \psi = id_{\operatorname{Pic} X}$.

Suppose that D is an effective divisor and $\mathscr{L}(D)$ the corresponding invertible sheaf. Then by part (a) ψ sends D to $\gamma(\mathcal{O}_D) = \gamma(\mathcal{O}_X) - \gamma(\mathscr{I}_D)$. By Proposition II.6.18 this is equal to $\gamma(\mathcal{O}_X) - \gamma(\mathscr{L}(-D))$. The homomorphism det then takes this to $\mathcal{O}_X \otimes (\mathscr{L}(-D))^\vee \cong (\mathscr{L}(-D))^\vee \cong \mathscr{L}(D)$. Hence $\det \circ \psi = id_{\operatorname{Pic} X}$.

[3]Since A is noetherian, $\mathfrak{p}$ is finitely generated. Let $a_1, \ldots, a_n$ be generators. For each i there is some n_i such that $a_i^{n_i} \in \operatorname{Ann} M$. Taking N high enough, every monomial of degree N in the a_i will contain at least one term of the form a_i^m with $m > n_i$. Hence, $\mathfrak{p}^N \subseteq \operatorname{Ann} M$.

Exercise 6.12. *Let X be a complete nonsingular curve. Show that there is a unique way to define the degree of any coherent sheaf on X, $\deg \mathscr{F} \in \mathbb{Z}$, such that:*

a *If D is a divisor, $\deg \mathscr{L}(D) = \deg D$;*

b *If $\mathscr{F}$ is a torsion sheaf then $\deg \mathscr{F} = \sum_{P \in X} \operatorname{length}(\mathscr{F}_P)$; and*

c *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, then $\deg \mathscr{F} = \deg \mathscr{F}' + \deg \mathscr{F}''$.*

7 Projective Morphisms

Exercise 7.1. *Let $(X, \mathcal{O}_X)$ be a locally ringed space, and let $f : \mathscr{L} \to \mathscr{M}$ be a surjective map of invertible sheaves on X. Show that f is an isomorphism.*

Solution. The morphism $\mathscr{L} \to \mathscr{M}$ of sheaves is surjective (resp. isomorphic) if and only if it is surjective (resp. isomorphic) on stalks (Exercise I.1.2). Furthermore, $\mathscr{L}$ and $\mathscr{M}$ being invertible means that they are locally free of rank one. So we are reduced to the question, given a local ring $(A, \mathfrak{m})$ and a surjective morphism $\phi : A \to A$ of A-modules, show that ϕ is an isomorphism. Since $\phi(a) = \phi(a \cdot 1) = a\phi(1)$ the morphism ϕ is determined by $b \mapsto b\phi(1)$. Since ϕ is surjective, there is some element $c \in A$ that gets mapped to 1, so $c\phi(1) = 1$ and therefore $\phi(1)$ is invertible. Then we can define $\psi : A \to A$ by $a \mapsto ac$ and this gives an inverse to ϕ. So ϕ is an isomorphism.

Exercise 7.2. *Let X be a scheme over a field k. Let $\mathscr{L}$ be an invertible sheaf on X, and let $\{s_0, \ldots, s_n\}$ and $\{t_0, \ldots, t_m\}$ be two sets of sections of $\mathscr{L}$, which generate the same subspace $V \subseteq \Gamma(X, \mathscr{L})$, and which generate the sheaf $\mathscr{L}$ at every point. Suppose $n \leq m$. Show that the corresponding morphisms $\phi : X \to \mathbb{P}^n_k$ and $\psi : X \to \mathbb{P}^m_k$ differ by a suitable linear projection $\mathbb{P}^m - L \to \mathbb{P}^n$ and an automorphism of $\mathbb{P}^n$, where L is a linear subspace of $\mathbb{P}^m$ of dimension $m - n - 1$.*

Solution. Now since the s_i and t_i generate the same subspace of $\Gamma(X, \mathscr{L})$ each s_i can be written (possibly non-uniquely) as a k-linear combination $s_i = \sum a_{ij} t_j$ of the t_j. We choose the a_{ij} so that the corresponding $(n+1) \times (m+1)$ matrix has linearly independent rows. [1] The coefficients a_{ij} determine $n + 1$ global sections $u_i = \sum a_{ij} x_j$ of $\mathcal{O}(1)$ on $\mathbb{P}^m$ and we have $\phi^* u_i = \phi^* \sum a_{ij} x_j = \sum a_{ij} \phi^* x_j = \sum a_{ij} t_j = s_i$. So the morphism $\rho : \mathbb{P}^m - L \to \mathbb{P}^n$ determined by the u_i satisfies $\rho \circ \phi = \psi$ by the uniqueness in Theorem II.7.1. It remains to see that ρ is a linear projection, which Hartshorne fails to define. We define it to be a morphism $\mathbb{P}^m - L \to \mathbb{P}^n$ defined by $n + 1$ linearly independent global sections of $\mathcal{O}(1)$ where L is the closed subvariety determined by the global sections considered as homogeneous elements of degree 1 of the homogeneous coordinate ring. The since the global sections are linearly independent and of degree 1, L will be a linear subspace of $\mathbb{P}^n$ of projective dimension $m - n - 1$. We don't need the automorphism because we have probably defined linear projection in a more general way tha Hartshorne has in mind.

Exercise 7.3. *Let $\phi : \mathbb{P}^n \to \mathbb{P}^m$ be a morphism. Then:*

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[1] Let $\dim V = r + 1$ and notice that we suppose $n \leq m$. Now notice that we can find a subset of $\{s_i\}$ that are linearly independent (inductively, choose s_{i_j} not in the span of $s_{i_1}, s_{i_2}, \ldots, s_{i_{j-1}}$) and similarly for the t_i. Without loss of generality, we can assume that these linearly independent subsets are $\{s_0, \ldots, s_r\}$ and $\{t_0, \ldots, t_r\}$. Now for each $i = 0, \ldots, r$ we can express s_i uniquely as a linear combination of the t_j for $j = 0, \ldots, r$, thus obtaining an $(r + 1) \times (r + 1)$ matrix that is invertible. For each $i > r$, express s_i as $s_i = t_i + \sum_{j=0}^{r} a_{ij} t_j$. Then the full $(n + 1) \times (m + 1)$ matrix $[a_{ij}]$ consists of an upper left square $(r + 1) \times (r + 1)$ square which is invertible, and for each $i > r$ we have a nonzero entry in the ith column and zeros in the jth columns for $j > i$. So the rows are linearly independent.

a *either* $\phi(\mathbb{P}^n) = pt$ *or* $m \geq n$ *and* $\dim \phi(\mathbb{P}^n) = n$*;*

b *in the second case,* ϕ *can be obtained as the composition of (1) a d-uple embedding* $\mathbb{P}^n \to \mathbb{P}^N$ *for a uniquely determined* $d \geq 1$*, (2) a linear projection* $\mathbb{P}^N - L \to \mathbb{P}^m$*, and (3) an automorphism of* $\mathbb{P}^m$*. Also,* ϕ *has finite fibres.*

Solution. a A morphism from $\mathbb{P}^n$ to $\mathbb{P}^m$ is equivalant to giving a line bundle $\mathscr{L}$ on $\mathbb{P}^n$ and $m+1$ global sections $s_0, \dots, s_m$ that generate $\mathscr{L}$ at every point of $\mathbb{P}^n$. Consider the subsets $Z_i = \{P \in \mathbb{P}^n | (s_j)_P \notin \mathfrak{m}_P \mathscr{L}_P, j = 0, 1, \dots, i\}$. These are closed subset of $\mathbb{P}^n$ and $Z_i \supseteq Z_{i+1}$. Since the s_i generate $\mathscr{L}$ at every point $Z_m = \varnothing$. Now $\mathbb{P}^n$ has dimension n so either $Z_i = \varnothing$ for every i, or $m \geq n$. In the first case, the global sections are all of degree zero in the homogeneous coordinate ring of $\mathbb{P}^n$ so $d = 0$ and its image in $\mathbb{P}^m$ is a point. In the second case, we show that $\dim \phi(\mathbb{P}^n) = n$ by induction on m.

We have already seen that if $m < n$ then the image of ϕ is a point. Consider $n \leq m$ and ϕ is surjective then $\dim \phi(\mathbb{P}^n) = \dim \mathbb{P}^m = m$ and so $m = n$. If ϕ is not surjective then there is a point P not in the image, and so we can compose $\mathbb{P}^n \to \mathbb{P}^m - P$ with projection from the point $\mathbb{P}^m - P \to \mathbb{P}^{m-1}$. By the inductive hypothesis on $\phi' : \mathbb{P}^n \to \mathbb{P}^{m-1}$ either $\dim \phi'(\mathbb{P}^n) = n$ in which case $\dim \phi(\mathbb{P}^n) \geq n$ and is therefore n, or $\phi'(\mathbb{P}^n)$ is a point. If $\phi'(\mathbb{P}^n)$ is a point then $\phi(\mathbb{P}^n)$ is contained in the preimage of this point under the projection. But this preimage is isomorphic to $\mathbb{A}^1$. So we have a morphism $\mathbb{P}^n \to \mathbb{A}^1$. Since $\mathbb{P}^n$ is proper and connected, its image is proper (Exericse II.4.4) and connected, and the only proper connected subschemes of $\mathbb{A}^1$ singleton points. Hence, the image of $\mathbb{P}^n$ is a point.

Exercise 7.4. a *Use (7.6) to show that if* X *is a scheme of finite type over a noetherian ring* A*, and if* X *admits an ample invertible sheaf, then* X *is separated.*

b *Let* X *be the affine line over a field* k *with the origin doubled. Calculate* $\operatorname{Pic} X$*, determine which invertible sheaves are generated by global sections, and then show directly (without using (a)) that there is no ample invertible sheaf on* X*.*

Solution. a If X admits an ample invertible sheaf $\mathscr{L}$ then Theorem II.7.6 tells use that $\mathscr{L}^n$ is very ample for some $n > 0$ and so X admits an imbedding in projective space. So there is a morphism $X \to \mathbb{P}^n$ for some n that factors as an open imbedding followed by a closed imbedding. Projective space is separated and so the structural morphism $\mathbb{P}^n \to \operatorname{Spec} A$ is separated. But then $X \to \operatorname{Spec} A$ is a composition of an open immersion, a closed immersion, and $\mathbb{P}^n \to \operatorname{Spec} A$, all of which are separated. Hence $X \to \operatorname{Spec} A$ is separated.

b An invertible sheaf $\mathscr{L}$ on X restricts to invertible sheaves on U_0, U_1, the two copies of the affine line that we have constructed X out of. Using Proposition II.6.2 and Corollary II.6.16 we see that $\operatorname{Pic} U_i = 0$ so every invertible sheaf is isomorphic to the structure sheaf. So $\mathscr{L}$ is determined by the isomorphism $\mathcal{O}_{U_0}|_{U_1 \cap U_0} \tilde{\to} \mathscr{L}|_{U_1 \cap U_0} \tilde{\to} \mathcal{O}_{U_1}|_{U_1 \cap U_0}$. Using II.6.2 and II.6.16 again we see that $\operatorname{Pic} U_0 \cap U_1 = 0$ so $\mathscr{L}|_{U_0 \cap U_1} \cong \mathcal{O}_{U_0 \cap U_1}$ and therefore the isomorphism is an automorphism of $k[x, x^{-1}]$ as a module over itself. Automorphisms of this form are determined by a unit in the ring, and the units of $k[x, x^{-1}]$ are the polynomials of the form ax^n for $a \in k^*$ and $n \in \mathbb{Z}$. So every element of $\operatorname{Pic} X$ is determined by a polynomial of the form ax^n. Following our construction, it can be seen that the corresponding Cartier divisor is $\{(U_0, 1), (U_1, ax^n)\}$. In this form it can be seen that ax^n and by^m define the same invertible sheaf if and only if $n = m$, so $\operatorname{Pic} X \cong \mathbb{Z}$. Denote by $\mathscr{L}_n$ the invertible sheaf corresponding to $n \in \mathbb{Z}$.

Given a Cartier divisor $\{(U_0, 1), (U_1, x^n)\}$, the corresponding invertible sheaf $\mathscr{L}_n$ is the subsheaf of $\mathscr{K}$ generated locally on U_0 by 1 and on U_1 by x^{-n}. A global section of $\mathscr{L}_n$ is a section on U_0 and a section on U_1 that agree on the intersection. That is, an element of $k[x]$ and an element of $x^{-n}k[x]$ that agree when restricted to $U_0 \cap U_1$. So the element of $x^{-n}k[x]$ must have homogeneous components of nonnegative degree, and so if $n > 0$ the local ring at the origin of U_1 cannot be generated by a global section. So each of the invertible sheaves $\mathscr{L}_n$ for $n > 0$ aren't generated by global sections.

Now suppose that $\mathscr{L}$ is an ample invertible sheaf, say $\mathscr{L} = \mathscr{L}_n$. Then by Theorem II.7.6 $\mathscr{L}^m = \mathscr{L}_{nm}$ is very ample over $\operatorname{Spec} k$ for some $m > 0$. This means there is a morphism to some projective space $\phi : X \to \mathbb{P}^n_k$ such that $\mathscr{L}_{nm} \cong \phi^*\mathcal{O}(1)$. But since $\mathbb{P}^n$ is separated, the two origins get sent to the same point, and so the morphism factors through $X \xrightarrow{f} \mathbb{A}^1 \xrightarrow{g} \mathbb{P}^n$. Since $\operatorname{Pic} \mathbb{A}^1 = 0$ we have $g^*\mathcal{O}(1) \cong \mathcal{O}_{\mathbb{A}^1}$ and so $\phi^*\mathcal{O}(1) = f^*g^*\mathcal{O}(1) = f^*\mathcal{O}_{\mathbb{A}^1} = \mathcal{O}_X$. So $n = 0$. Now consider the coherent sheaf $\mathscr{L}_n$ for some $n > 0$. If $\mathcal{O}_X$ really were ample then there would be some i_0 such that for $i > i_0$ the sheaf $\mathscr{L}_n \otimes \mathcal{O}_X^{\otimes i}$ was generated by its global sections. But we have seen that this is not the case. So not even $\mathcal{O}_X$ is ample, and therefore there are no ample invertible sheaves on X.

Exercise 7.5. *Establish the following properties of ample and very ample invertible sheaves on a noetherian scheme X. $\mathscr{L}, \mathscr{M}$ will denote invertible sheaves, and for (d), (e) we assume furthermore that X is of finite type over a noetherian ring A.*

a *If $\mathscr{L}$ is ample and $\mathscr{M}$ is generated by global sectrions, then $\mathscr{L} \otimes \mathscr{M}$ is ample.*

b *If $\mathscr{L}$ is ample, and $\mathscr{M}$ is arbitrary, then $\mathscr{M} \otimes \mathscr{L}^n$ is ample for sufficiently large n.*

Solution. a Note that if $\mathscr{F}$ and $\mathscr{G}$ are two sheaves of $\mathcal{O}_X$-modules that are generated by global sections $\{f_1, \dots, f_n\}$ and $\{g_1, \dots, g_m\}$ then the tensor product, is generated by the global sections $\{f_i \otimes g_j\}$. Now consider some coherent sheaf of $\mathcal{O}_X$-modules $\mathscr{F}$. Since $\mathscr{L}$ is ample, there is some n_0 such that for all $n > n_0$ the sheaf $\mathscr{F} \otimes \mathscr{L}^n$ is generated by global sections. By the remark we just made, this implies that $(\mathscr{F} \otimes \mathscr{L}^n) \otimes \mathscr{M}^d \cong \mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L})^n$ is generated by global sections. Hence, $\mathscr{L} \otimes \mathscr{M}$ is ample.

b $\mathscr{M}$ is coherent and so for sufficiently large i the sheaf $\mathscr{L}^i \otimes \mathscr{M}$ is generated by global sections. By the remark we made in part (a) this means that $(\mathscr{L}^i \otimes \mathscr{M})^d$ is generated by global sections for all positive d. Take $n = i+1$. For another arbitrary coherent sheaf $\mathscr{F}$, there is some d_0 such that for all $d > d_0$ the sheaf $\mathscr{F} \otimes \mathscr{L}^d$ is generated by global sections. It follows that $(\mathscr{F} \otimes \mathscr{L}^d) \otimes (\mathscr{L}^i \otimes \mathscr{M})^d \cong \mathscr{F} \otimes (\mathscr{L}^n \otimes \mathscr{M})^d$ is generated by global sections for all $d > d_0$. Hence, $\mathscr{L}^n \otimes \mathscr{M}$ is ample for sufficiently large n.

c $\mathcal{O}_X$ is a coherent sheaf so there is some d_0 such that for all $d > d_0$ $\mathcal{O}_X \otimes \mathscr{M}^d$ is generated by global sections. For an arbitrary coherent sheaf $\mathscr{F}$, there is some e_0 such that for all $e > e_0$ the sheaf $\mathscr{F} \otimes \mathscr{L}^e$ is generated by global sections. Choose n_0 bigger than e_0 and d_0. Then for all $n > n_0$ we have $\mathscr{F} \otimes (\mathscr{M} \otimes \mathscr{L})^n \cong (\mathscr{F} \otimes \mathscr{L}^n) \otimes (\mathcal{O}_X \otimes \mathscr{M}^n)$ is generated by global sections. So $\mathscr{L} \otimes \mathscr{M}$ is ample.

d

e From Theorem II.7.6 we see that there is some $n > 0$ for which $\mathscr{L}^n$ is very ample. Using $\mathscr{F} = \mathscr{L}$ in the definition of ample shows that there is some d_0 for which $\mathscr{L}^d$ is generated by global sections for all $d > d_0$. Then by the previous part $\mathscr{L}^d \otimes \mathscr{L}^n = \mathscr{L}^{d+n}$ is very ample for all $d + n > d_0 + n$.

Exercise 7.6. The Riemann-Roch Problem.

a Show that if D is very ample, and if $X \hookrightarrow \mathbb{P}^n$ is the corresponding embedding in projective space, then for all n sufficiently large, $\dim |nD| = P_X(n) - 1$*, where P_X is the Hilbert polynomial of X.*

b If D corresponds to a torsion element of $\operatorname{Pic} X$*, of order r, then* $\dim |nD| = 0$ *if $r|n$ and* $\dim |nD| = -1$ *otherwise. In this case the function is periodic of period r.*

Solution. a Recall that the Hilbert polynomial is the numerical polynomial associated to the Hilbert function $\phi : n \mapsto \dim_k S_n$ where S is the homogeneous coordinate ring of X. Via the embedding we can assocoiate $\mathscr{L}$ with $S(1)^\sim$ and then using Exercise II.5.9(b) we see that $S_n \to \Gamma(X, S(n)^\sim) = \Gamma(X, \mathscr{L}^n)$ is an isomorphism for all large enough n. So for all dn large enough have $\dim |nD| = \dim \Gamma(X, \mathscr{L}^n) - 1 = \dim S_n - 1 = \phi(n) - 1$. For n large enough, by definition $\phi(n) = P_X(n)$ and so for n large enough we get $\dim |nD| = P_X(n) - 1$.

b If D is a torsion element of degree r then rD is trivial and so its corresponding line bundle is the structure sheaf, whose vector space of global sections has dimension one. So $\dim |rD| = \dim \Gamma(X, \mathcal{O}_X) - 1 = 1 - 1 = 0$. Similarly, if $n = rk$ for some integer k, then $\dim |rkD| = \dim \Gamma(X, \mathcal{O}_X^k) - 1 = \dim \Gamma(X, \mathcal{O}_X) - 1 = 1 - 1 = 0$.

For the case $r \not| n$ we will first show that $\dim \Gamma(X, \mathscr{L}) = 0$. Consider a global section $s \in \Gamma(X, \mathscr{L})$ and let $Z_i = \{P \in X | s_P^{\otimes i} \in \mathfrak{m}_P \mathscr{L}_P^{\otimes i}\}$. If we take an open affine subset U on which we have an isomorphism $\mathscr{L}|_U \cong \mathcal{O}_U$ then s gives a section $t \in \mathcal{O}_U(U)$ and the set $Z_i \cap U$ is $\{P \in X | t_P^i \in \mathfrak{m}_P\}$ and so we see that from this that $Z_i = Z_1$ for all $i \geq 1$. Furthermore, since $\mathscr{L}^r = \mathcal{O}_X$, we see that $Z_{ri} = \varnothing$ or X since the only global sections of $\mathcal{O}_X$ are constants. Hence, $Z_1 = \varnothing$ or X. If $Z_1 = \varnothing$ then recalling the construction of D from $\mathscr{L}$ we see that $D = 0$ and so $r = 1$, and so $r | n$ for all n and we have the previous case. If $Z_1 = X$ then our original global section s was zero and so there are no nonzero global sections of $\mathscr{L}$.

Now for any $i = 1, \ldots, r-1$, the sheaf $\mathscr{L}^i$ is again a torsion sheaf of rank dividing r and so we see that $\mathscr{L}^i$ has no global sections for each of these i. Now $\Gamma(X, \mathscr{L}^n) = \Gamma(X, \mathscr{L}^{kr+i}) = \Gamma(X, \mathscr{L}^i)$ for some $i = 1, \ldots, r-1$ and so for any n that is not a multiple of r, there are no nonzero global sections of $\mathscr{L}^n$. Hence $\dim |nD| = \Gamma(X, \mathscr{L}^n) - 1 = 0 - 1 = -1$.

Exercise 7.7. Some Rational Surfaces. *Let $X = \mathbb{P}_k^2$, and let $|D|$ be the complete linear system of all divisors of degree 2 on X (conics). D corresponds to the invertible sheaf $\mathcal{O}(2)$, whose space of global sections has a basis $x^2, y^2, z^2, xy, xz, yz$, where x, y, z are the homogeneous coordinates of X.*

a *The complete linear system $|D|$ gives an embedding of $\mathbb{P}^2$ in $\mathbb{P}^5$, whose image is the Veronese surface.*

b *Show that the subsystem defined by $x_0^2, x_1^2, x_2^2, x_1(x_0 - x_2), (x_0 - x_1)x_2$ gives a closed immersion of X into $\mathbb{P}^4$.*

c *Let $\mathfrak{d} \subseteq |D|$ be the linear system of all conics passing though a fixed point P. Then $\mathfrak{d}$ gives an immersion of $U = X - P$ into $\mathbb{P}^4$. Furthermore, if we blow up P, to get a surface $\widetilde{X}$, then this map extends to give a closed immersion of $\widetilde{X}$ in $\mathbb{P}^4$. Show that $\widetilde{X}$ is a surface of degree 3 in $\mathbb{P}^4$, and that the lines in X through P are transformed into straight lines in $\widetilde{X}$ which do not meet.*

Solution. a Recall that the Veronese surface is the 2-uple embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$. That is, the embedding that corresponds the the ring homomorphism

$$(y_0, y_1, y_2, y_3, y_4, y_5) \mapsto (x_0^2, x_1^2, x_2^2, x_0x_1, x_0x_2, x_1x_2)$$

from $k[y_i]$ to $k[x_i]$. Consider the morphism $\phi : \mathbb{P}^2 \to \mathbb{P}^5$ corresponding to the linear system $|D|$. In the proof of Theorem II.7.1 the morphism ϕ is defined via $\mathbb{P}_{s_i}^2 \to D_+(y_i)$ where s_i is the $(i+1)$th basis vector of $|D|$.

Take $s_0 = x_0^2$. Then the morphism is $\operatorname{Spec}[\frac{x_1}{x_0}, \frac{x_2}{x_0}] \to \operatorname{Spec}[\frac{y_1}{y_0}, \ldots, \frac{y_5}{y_0}]$ and is defined via the ring homomorphism

$$\left(\frac{y_1}{y_0}, \ldots, \frac{y_5}{y_0}\right) \mapsto \left(\frac{x_2^2}{x_1^2}, \frac{x_0x_1}{x_1^2}, \frac{x_0x_2}{x_1^2}, \frac{x_1x_2}{x_1^2}\right)$$

Clearly, this agrees with the Veronese embedding described above. Now we can do the same thing for the other s_i or evoke Exercise II.4.2 to see that the two morphisms agree.

b We use the cirteria from Proposition 7.3. Since $D_+(x_i^2) = D_+(x_i)$, the five open sets corresponding to the chosen global sections cover $\mathbb{P}^2$, hence (1) is satisfied. Now we want to show that for every closed point $P \in \mathbb{P}^2$, the global sections whose germ are in $\mathfrak{m}_P\mathscr{L}_P$ generate $\mathfrak{m}_P\mathscr{L}_P/\mathfrak{m}_P^2\mathscr{L}_P$ as a $k(P)$-vector space.

Each closed point appears in one of the open affines $D_+(x_0), D_+(x_1), D_+(x_2)$. The system is symmetric under $x_1 \leftrightarrow x_2$ so show (2) is satisfied for all closed points in $D_+(x_1)$ will imply it for $D_+(x_2)$ and then that will leave the one remaining closed point $(1,0,0)$ that is not in $D_+(x_1) \cup D_+(x_2)$.

We start with $P = (1,0,0)$ which is the origin in $\mathbb{A}^2 \cong D_+(x_0)$. Choose coordinates $u = \frac{x_1}{x_0}, v = \frac{x_2}{x_0}$. We use the isomorphism $\mathcal{O}(2)|_{D_+(x_0)} \cong \mathcal{O}_X|_{D_+(x_0)}$ so for sections of $\mathcal{O}(2)$ we have $x_0^2 = 1$, $x_0^2\frac{x_1}{x_0} = u$, and $x_0^2\frac{x_2}{x_0} = v$. Then our global sections are $1, u^2, v^2, u(1-v), (1-u)v$. What we want to show is that $\mathfrak{m}_P/\mathfrak{m}_P^2 = (u,v)\mathcal{O}_P/(u,v)^2\mathcal{O}_P$ is generated by the linear combinations of the given global sections. Note that the images of u and v in this vector space are basis vectors. We need only the global sections $u(1-v), (1-u)v$ for in $(u,v)\mathcal{O}_P/(u,v)^2\mathcal{O}_P$ we have $uv = 0$ since $uv \in (u,v)^2$. So everything is fine.

While we are in $D_+(x_0)$ we do the point $(u+1, v+1)$ as well; we will see why later. We have the global section $u(1-v)+2 = u+1-(u+1)(v+1)$ which is $u+1$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$ and $v(1-u)+2 = v+1-(u+1)(v+1)$ which is $v+1$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$. So our global sections generate the vector space $(u+1, v+1)\mathcal{O}_P/(u+1,v+1)^2\mathcal{O}_P$.

Now consider the closed points in $D_+(x_1)$. Choose coordinates $u = \frac{x_0}{x_1}, v = \frac{x_2}{x_1}$. We use the isomorphism $\mathcal{O}(2)|_{D_+(x_1)} \cong \mathcal{O}_X|_{D_+(x_1)}$ so for sections of $\mathcal{O}(2)$ we have $x_1^2 = 1$, $x_1^2\frac{x_0}{x_1} = u$, and $x_1^2\frac{x_2}{x_1} = v$. Then our global sections are $u^2, 1, v^2, (u-v), -uv$. What we want to show is that $\mathfrak{m}_P/\mathfrak{m}_P^2 = (u-a, v-b)\mathcal{O}_P/(u-a,v-b)^2\mathcal{O}_P$ is generated by the linear combinations of the given global sections. If $a+b \neq 0$ then consider

$$\begin{aligned} uv - ab &= (u-a)(v+b) - b(u-a) + a(v-b) \\ (u-v) + (b-a) &= (u-a) + (v-b) \end{aligned}$$

written on the left as a linear combination of our global sections, and on the right as elements of $\mathfrak{m}_P = (u-a, v-b)\mathcal{O}_P$. We have

$$uv - ab = b(u-a) + a(v-b) + \Big((v-b)(u-a)\Big)$$
$$(u-v) + (b-a) = (u-a) - (v-b)$$

and so modulo $\mathfrak{m}_P^2$ these generate $\mathfrak{m}_P$ as long as $a+b \neq 0$ (recall that the images of $u-a$ and $v-b$ in $\mathfrak{m}_P/\mathfrak{m}_P^2$ are basis vectors).

So we have seen that (2) holds for all points not in the hypersurface $V(x_0 + x_2)$. Actually, everything that we did for $D_+(x_1)$ holds for $D_+(x_2)$ as well, with 1 switched with 2 so we actually see that (2) holds for all points not in $V(x_0+x_2)$ or $V(x_0+x_1)$. That is, all points except $(1,-1,-1)$. But we saw that it holds for $(1,-1,-1)$ earlier in $D_+(x_0)$. Hence, (2) holds for all closed points, and $X \to \mathbb{P}^4$ is a closed immersion.

c If we use coordinates $y_0, \dots, y_4$ for $\mathbb{P}^4$ and x_0, x_1, x_2 for $\mathbb{P}^2$ and take our point to be $(0,0,1) = \langle x_0, x_1 \rangle$, it can be seen by looking at the basic opens $D_+(x_0), D_+(x_1)$ and $D_+(x_2) - P$ that the linear system $\mathfrak{d}$ with basis vectors $x_0^2, x_1^2, x_0x_1, x_1x_2, x_0x_2$ maps U homeomorphically onto an open subset of the closed subvariety $V = V(y_2y_3 - y_0y_4, y_1y_3 - y_2y_4)$. Since the image of $\widetilde{X}$ is a closed subset and $U \cong \pi^{-1}$ is dense in $\widetilde{X}$, the closure V of the image of U must be the image of $\widetilde{X}$. Now picking a global section y_0 of $\mathcal{O}(1)$ it can be seen to correspond to the divisor $V(y_0, y_1, y_2) + V(y_0, y_2, y_3) + V(y_0, y_3, y_4)$ and so has degree 3.

The image of the line $ax_0 + bx_1 = 0$ (minus P) of $U \subset \mathbb{P}^2$ in V has as its closure the line $V(ay_0 + by_2, ay_1 + by_2, ay_4 + by_3)$ and it follows from some linear algebra that if the ratio $a : b$ is different to $a' : b'$ then the two corresponding lines in $V \subset \mathbb{P}^4$ do not share a point.

Exercise 7.8. *Let X be a noetherian scheme, let $\mathscr{E}$ be a coherent locally free sheaf on X, and let $\pi : \mathbb{P}(\mathscr{E}) \to X$ be the corresponding projective space bundle. Show that there is a natural 1-1 corresondence between sections of π and quotient invertible sheaves $\mathscr{E} \to \mathscr{L} \to 0$ of $\mathscr{E}$.*

Solution. By Proposition 7.12 to give a morphism $X \to \mathbb{P}(\mathscr{E})$ over X (that is, a section) it is equivalent to give an invertible sheaf $\mathscr{L}$ on X and a surjective map of sheaves $\mathscr{E} \to \mathscr{L}$. So we are done.

Exercise 7.9. *Let X be a regular noetherian scheme, and $\mathscr{E}$ a locally free coherent sheaf of rank ≥ 2 on X.*

a *Show that* $\operatorname{Pic} \mathbb{P}(\mathscr{E}) \cong \operatorname{Pic} X \times \mathbb{Z}$.

b *If $\mathscr{E}'$ is another locally free coherent sheaf on X, show that $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$ (over X) if and only if there is an invertible sheaf $\mathscr{L}$ on X such that $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{L}$.*

Solution. a There is a natural morphism $\alpha : \operatorname{Pic} X \times \mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$ defined by $(\mathscr{L}, n) \mapsto (\pi^* \mathscr{L}) \otimes \mathcal{O}(n)$. We claim that this gives the desired isomorphism. Let r be the rank of $\mathscr{E}$. Pick a point $\iota : x \hookrightarrow X$ and an open affine neighbourhood U of it on which $\mathscr{E}$ is free and let $k(x)$ be the residue field. On U we have $\pi^{-1}U = \mathbb{P}^{r-1}_U$ and so we obtain an embedding $\mathbb{P}^{r-1}_{k(x)} \to \mathbb{P}^{r-1}_U \to \mathbb{P}(\mathscr{E})$. Clearly, $\mathcal{O}_{\mathbb{P}(\mathscr{E})}(n)|_U \cong \mathcal{O}_U(n)$ and we know that $\operatorname{Pic} \mathbb{P}^{r-1}_{k(x)} = \mathbb{Z}$ so we have obtain a left inverse to $\mathbb{Z} \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$. So it remains to show that α is surjective, and that $\operatorname{Pic} X \to \operatorname{Pic} \mathbb{P}(\mathscr{E})$ is injective.

Injectivity of α. Suppose that $\pi^* \mathscr{L} \otimes \mathcal{O}(n) \cong \mathcal{O}_{\mathbb{P}(\mathscr{E})}$. Then by Proposition II.7.11 we see that $\pi_*(\pi^* \mathscr{L} \otimes \mathcal{O}(n)) \cong \mathcal{O}_X$ and by the Projection Formula (Exercise II.5.1(d)) we have $\mathscr{L} \otimes \pi_* \mathcal{O}(n) \cong \mathcal{O}_X$. Again by Proposition II.7.11 we know that $\pi_* \mathcal{O}(n)$ is the degree n part of the symmetric algebra on $\mathscr{E}$ and since rank $\mathscr{E} \geq 2$ this implies that $n = 0$ and $\mathscr{L} \cong \mathcal{O}_X$. Hence α is injective.

Surjectivity of α. Let $\{U_i\}$ be an open cover of X for which $\mathscr{E}$ is locally trivial, and such that each U_i is integral and separated. We can find such a cover since every affine scheme is separated, and X is regular implies that the local rings are reduced. The subschemes $V_i \overset{def}{=} \mathbb{P}(\mathscr{E}|_{U_i}) \cong U_i \times \mathbb{P}^{r-1}$ form an open cover of $\mathbb{P}(\mathscr{E})$ and since X is regular, each U_i is regular, and in particular, regular in codimension one, and hence satisfies (*), so we can apply Exercise II.6.1 to find that $\operatorname{Pic} V_i \cong \operatorname{Pic} U_i \times \mathbb{Z}$.

Now if $\mathscr{L} \in \operatorname{Pic} \mathbb{P}(\mathscr{E})$ then for each i, by restricting we get an element $\mathcal{O}_i(n_i) \otimes \pi_i^* \mathscr{L}_i \in \operatorname{Pic} V_i \cong \operatorname{Pic} U_i \times \mathbb{Z}$ together with transition isomorphisms

$$\alpha_{ij} : (\mathcal{O}_i(n_i) \otimes \pi_i^* \mathscr{L}_i)|_{V_{ij}} \to (\mathcal{O}_j(n_j) \otimes \pi_j^* \mathscr{L}_j)|_{V_{ji}}$$

that satisfy the cocycle condition. These isomorphisms pushforward to give isomorphisms

$$\alpha_{ij} : \pi_*(\mathcal{O}_i(n_i)|_{V_{ij}}) \otimes \mathscr{L}_i \to \pi_*(\mathcal{O}_j(n_j)|_{V_{ji}}) \otimes \mathscr{L}_j$$

via the projection formula. A quick look at Proposition II.7.11 and considering ranks, we see that $n_i = n_j$. Furthermore, it can be seen from the definition of $\mathbb{P}(\mathscr{E})$ that $\mathcal{O}_j(n)|_{V_{ij}} = \mathcal{O}_{ij}(n)$ and so our isomorphism α_{ij} is $\mathcal{O}_{ij}(n) \otimes \pi_i^* \mathscr{L}_i|_{V_{ij}} \to \mathcal{O}_{ij}(n) \otimes \pi_j^* \mathscr{L}_j|_{V_{ij}}$. Tensoring this with $\mathcal{O}_{ij}(-n)$ we get isomorphisms $\mathcal{O}_{ij} \otimes \pi_i^* \mathscr{L}_i|_{V_{ij}} \to \mathcal{O}_{ij} \otimes \pi_j^* \mathscr{L}_j$ and the projection formula together with II.7.11 again then tells us that we have isomorphisms $\beta_{ij} : \mathscr{L}_i|_{U_{ij}} \cong \mathscr{L}_j|_{U_{ij}}$, and it can be shown that these satisfy the cocycle condition as a cosequence of the α_{ij} satisfying it. Hence, we can glue the $\mathscr{L}_i$ together to obtain a sheaf $\mathscr{M}$ on X such that $\pi^* \mathscr{M} \otimes \mathcal{O}(n)$ is isomorphic to $\mathscr{L}$ on each connected component of X (where n depends on the component).

b One direction follows immediately from Lemma II.7.9 but we choose to do it more explicitely, using Yoneda's Lemma.

Suppose we have $Z \xrightarrow{f} Y \xrightarrow{g} X$ for arbitrary schemes Y and Z and morphisms f, g. Proposition II.7.12 says that we have an isomorphism $\hom_X(Y, \mathbb{P}(\mathscr{E})) \xrightarrow{\sim} \{$ quotient invertible sheaves $g^*\mathscr{E} \to \mathscr{L}\}$ and that this is given by $(Y \xrightarrow{u} \mathbb{P}(\mathscr{E})) \mapsto (u^*\pi^*\mathscr{E} \to u^*\mathcal{O}(1))$. It is straightforward that the following square commutes

$$\begin{tikzcd}
\hom_X(Y, \mathbb{P}(\mathscr{E})) \arrow[r] \arrow[d, "- \circ g"] & \{ \text{ quotient invertible sheaves of } g^*\mathscr{E}\} \arrow[d, "g^*"] \\
\hom_X(Z, \mathbb{P}(\mathscr{E})) \arrow[r] & \{ \text{ quotient invertible sheaves of } f^*\mathscr{E}\}
\end{tikzcd}$$

since $(ab)^* \cong b^*a^*$ and so we actually have an isomorphism of functors between $\hom_X(-, \mathbb{P}(\mathscr{E}))$ and the functor $F_{\mathscr{E}}$ that sends a scheme $g : Y \to X$ over X to the set of quotient invertible sheaves of $g^*\mathscr{E}$.

Now Yoneda's Lemma says that if two representable functors are isomorphic then their representatives are isomorphic. If we have an isomorphism $\mathscr{E} \cong \mathscr{E}' \otimes \mathscr{L}$ we get an induced isomorphism $F_{\mathscr{E}} \cong F_{\mathscr{E}'}$ by sending a quotient invertible sheaf $g^*\mathscr{E} \to \mathscr{M}$ to $g^*(\mathscr{E}' \otimes \mathscr{L}) \to \mathscr{M}$ and then $g^*\mathscr{E}' \to \mathscr{M} \otimes (g^*\mathscr{L})^{-1}$. It can be checked that this is functorial using Exercise II.6.8(a), and so we obtain via Yondea, an isomorphism $\mathbb{P}(\mathscr{E}) \cong \mathbb{P}(\mathscr{E}')$.

Suppose that we have an isomorphism $\alpha : \mathbb{P}(\mathscr{E}) \to \mathbb{P}(\mathscr{E}')$ with inverse β. Since α_* and α^* are adjoints, we obtain for every quasi-coherent sheaf $\mathscr{F}$ on $\mathbb{P}(\mathscr{E})$ a morphism $\alpha^*\alpha_*\mathscr{F} \to \mathscr{F}$. If we choose an open affine subset $U = \operatorname{Spec} A$ of $\mathbb{P}(\mathscr{E})$, this morphism on U takes the form $(({}_BM) \otimes_B A)^{\sim} \to M$ where $\operatorname{Spec} B = \alpha(U)$, $A \cong B$ is induced by α and M is an A-module. This is an isomorphism and so $\alpha^*\alpha_*\mathscr{F} \to \mathscr{F}$ is an isomorphism. Now take $\mathscr{F} = \mathcal{O}(1)$. Then we have $\alpha^*\alpha_*\mathcal{O}(1) \cong \mathcal{O}(1)$ and so $\alpha_*\mathcal{O}(1) \cong \beta^*\mathcal{O}(1)$ since β is the inverse to α. We know that $\beta^*\mathcal{O}(1)$ is in the Picard group of $\mathbb{P}(\mathscr{E}')$ and so by part (a) it has the form $((\pi')^*\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n)$ for some invertible sheaf $\mathscr{L}$ on X and some integer n. Pushing the isomorphism $\alpha_*\mathcal{O}(1) \cong ((\pi')^*\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n)$ forward through π' and using the Projection formula (Exercise II.5.1d) and Proposition II.7.11 gives

$$\begin{aligned}
\mathscr{E} \cong \pi_*\mathcal{O}(1) \cong (\pi')_*\alpha_*\mathcal{O}(1) \cong (\pi')_* \Bigg(((\pi')^*\mathscr{L}) \otimes \mathcal{O}_{\mathbb{P}(\mathscr{E}')}(n) \Bigg) \\
\cong \mathscr{L} \otimes (\pi')_*\mathbb{P}_{\mathbb{P}(\mathscr{E}')}(n) \cong \mathscr{L} \otimes \mathscr{S}^n(\mathscr{E}')
\end{aligned}$$

Now since the rank of $\mathscr{E}'$ is $r \geq 2$, the rth degree of the symmetric algebra on $\mathscr{E}'$ has rank $\binom{n+r}{r}$ and so $n = 1$ and we have an isomorphism $\mathscr{E} \cong \mathscr{L} \otimes \mathscr{E}'$ for some line bundle $\mathscr{L}$.

Exercise 7.10. $\mathbb{P}^n$-Bundles Over a Scheme. *Let X be a noetherian scheme.*

a *By anaogy with Exercise II.5.18, define the notion of a projective bundle over X.*

b *If $\mathscr{E}$ is a locally free sheaf of rank $n+1$ on X, then $\mathbb{P}(\mathscr{E})$ is a $\mathbb{P}^n$-bundle over X.*

c *Assume that X is regular, and show that every $\mathbb{P}^n$-bundle P over X is isomorphic to $\mathbb{P}(\mathscr{E})$ for some locally free sheaf $\mathscr{E}$ on X. Can you weaken the hypothesis "X regular"?*

d *Conclude (in the case X regular) that there is a one-to-one correspondence between $\mathbb{P}^n$-bundlesover X, and equivalence classes of locally free sheaves $\mathscr{E}$ of rank $n+1$ under the equivlanece $\mathscr{E}' \sim \mathscr{E}$ if and only if $\mathscr{E}' \cong \mathscr{E} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$ on X.*

Solution. a A projective bundle of rank n over X is a scheme P and a morphism $f : P \to X$, together with additional data consisting of an open covering $\{U_j\}$ of X, and isomorphisms $\psi_i : f^{-1}(U_i) \to \mathbb{P}^n_{U_i}$, such that for any i, j and for any open affine subset $V = \operatorname{Spec} A \subseteq U_i \cap U_j$, the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{P}^n_V = \operatorname{Proj} A[x_0, x_1, \ldots, x_n]$ is given by a linear automorphism θ of $A[x_0, x_1, \ldots, x_n]$.

b We take an affine cover $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathscr{E}$ is free on U_i. So we have isomorphims $\mathscr{E}|_{U_i} \cong \mathcal{O}_{U_i}^{\oplus(n+1)}$. By definition of $\pi : \mathbb{P}(\mathscr{E}) \to X$ we have $\pi^{-1}U_i = \operatorname{Proj} \mathscr{S}(\mathscr{E})(U_i) \cong \operatorname{Proj} \mathscr{S}(\mathcal{O}_{U_i}^{\oplus(n+1)})(U_i) = \operatorname{Proj} A_i[x_0, \ldots, x_n] = \mathbb{P}^n_{U_i}$ where $\mathscr{S}(\mathscr{F})$ is the symmetric algebra associated to a locally free sheaf $\mathscr{F}$. So we have our automorphisms ψ_i. Now for any open affine subscheme $V = \operatorname{Spec} A$ of $U_i \cap U_j$, again from the definition of $\mathbb{P}(\mathscr{E})$ we have an isomorphism $\pi^{-1}V \cong \mathbb{P}^n_V$ and the automorphism $\psi = \psi_j \circ \psi_i^{-1}$ of $\mathbb{P}^n_V$ is defined via the automorphism $\mathcal{O}_{U_i}^{n+1}|_V \cong \mathcal{O}_{U_j}^{n+1}|_V$ coming from the restriction morphisms $\mathscr{E}(U_i) \to \mathscr{E}(V) \leftarrow \mathscr{E}(U_j)$. Clearly this is of the desired form.

c

d Given a locally free sheaf of rank $n+1$ we obtain a projective bundle $\mathbb{P}(\mathscr{E})$ by part (b) of this question, so $\mathbb{P}(-) : \mathscr{L}oc_{n+1}(X) \to \mathscr{P}\mathscr{B}_n(X)$ is a map from locally free sheaves of rank $n+1$ to projective bundles of rank n. Conversely, given a projective bundle P, by part (c) we obtain a locally free sheaf $\mathscr{E} = \mathscr{E}_P$ of rank $n+1$ and an isomorphism $\mathbb{P}(\mathscr{E}) \cong P$, so we have a map $\mathscr{E}_- : \mathscr{P}\mathscr{B}_n(X) \to \mathscr{L}oc_{n+1}(X)$ which is a right inverse to $\mathbb{P}(-)$. The only thing left to see is that $\mathscr{E}_-$ is a left inverse to $\mathbb{P}(-)$ as well. So suppose that we have a localy free sheaf $\mathscr{F}$ or rank $n+1$ on X. Then we have seen that $\mathbb{P}(\mathscr{E}_{\mathbb{P}(\mathscr{F})}) \cong \mathbb{P}(\mathscr{F})$. But by Exercise II.7.9(b) this implies that $\mathscr{E}_{\mathbb{P}(\mathscr{F})} \cong \mathscr{F} \otimes \mathscr{M}$ for some invertible sheaf $\mathscr{M}$. So we have the desired one-to-one correspondence after we note that $\mathbb{P}$ is still well defined on $\mathscr{L}oc_{n+1}(X)$ modulo the equivalence relation (again by Exercise II.7.9(b)).

Exercise 7.11. *a If $\mathscr{I}$ is any coherent sheaf of ideals on X, show that blowing up $\mathscr{I}^d$ for any $d \geq 1$ gives a scheme isomorphic to the blowing up of $\mathscr{I}$.*

b *If $\mathscr{I}$ is any coherent sheaf of ideals, and if $\mathscr{J}$ is an invertible sheaf of ideals, then $\mathscr{I} \cdot \mathscr{J}$ give isomorphic blowings-up.*

c *If X is regular, show that (7.17) can be strengthened as follows. Let $U \subseteq X$ be the largest open set such that $f : f^{-1}U \to U$ is an isomorphism. Then $\mathscr{I}$ can be chosen such that the corresponding closed subscheme Y has support equal to $X - U$.*

Solution. a By definition, the blowing up of $\mathscr{I}$ is $\mathbf{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^n)$, and the blowing up of $\mathscr{I}^d$ is $\mathbf{Proj}(\bigoplus_{n\geq 0} \mathscr{I}^{nd})$. Locally–that is on an affine subscheme U of X, these blowing ups are $\operatorname{Proj} \bigoplus_{n\geq 0} \mathscr{I}(U)^n$ and $\operatorname{Proj} \bigoplus_{n\geq 0} \mathscr{I}(U)^{dn}$. By Exercise II.5.13 we know that these are isomorphic, and so if we can show that the isomorphism from Exercise II.5.13 is natural we are done, since these local isomorphisms will then agree on the pairwise intersections $U_i \cap U_j$ of two open affine subschemes. That is, we want to show that for a morphism of graded rings $T \to S$, the square commutes

$$\begin{array}{ccc} \operatorname{Proj} S & \longrightarrow & \operatorname{Proj} T \\ \downarrow & & \downarrow \\ \operatorname{Proj} S^{(d)} & \longrightarrow & \operatorname{Proj} T^{(d)} \end{array}$$

But in the proof of Exercise II.5.13, the horizontal morphisms come from inclusions $S^{(d)} \to S$ and $T^{(d)} \to T$ and so this square commutes. So we are done.

b This follows from Lemma II.7.9 or we can use Yoneda's Lemma as follows.

Proposition II.7.14 says that $\widetilde{X}$ represents the functor that sends Z to the set of morphisms $f : Z \to X$ such that $f^{-1}\mathscr{I} \cdot \mathcal{O}_Z$ is an invertible sheaf of ideals on Z. Now since $\mathscr{J}$ is invertible, $f^*\mathscr{J}$ is invertible, and so if we can show that $f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z \cong (f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes f^*\mathscr{J}$, then $f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z$ will be invertible if and only if $f^{-1}\mathscr{I} \cdot \mathcal{O}_Z$ is invertible and so the two functors represented by the blowings-up of $\mathscr{I}$ and $\mathscr{I} \cdot \mathscr{J}$ will be isomorphic, implying that the blowing-ups themselves are isomorphic.

The sheaf $f^{-1}\mathscr{J} \cdot \mathcal{O}_Z$ is the image of $f^*\mathscr{J} \to \mathcal{O}_Z$, so we have natural maps

$$\begin{aligned}(f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes f^*\mathscr{J} \to (f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes (f^{-1}\mathscr{J} \cdot \mathcal{O}_Z) \to & f^{-1}\mathscr{I} \cdot f^{-1}\mathscr{J} \cdot \mathcal{O}_Z \\ = & f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z\end{aligned}$$

Since $\mathscr{J}$ is invertible, it is locally isomorphic to $\mathcal{O}_X$, and so $f^*\mathscr{J}$ is locally isomorphic to $\mathcal{O}_Z$. Let U be an open subset of U on which we have an isomorphism $f^*\mathscr{J}|_U \cong \mathcal{O}_Z|_U$. Then $f^*\mathscr{J}|_U$ is of the form $(\mathcal{O}_Z|_U)s$ for some section $s \in \mathcal{O}_Z(U)$ (that is, s generates $f^*\mathscr{J}|_U$ as a free $\mathcal{O}_Z|_U$ module). Assuming that $U = Z$ so that we can stop writing $|_U$ everywhere, our morphisms above become

$$(f^{-1}\mathscr{I} \cdot \mathcal{O}_Z) \otimes s \xrightarrow{\sim} f^{-1}\mathscr{I} \cdot \mathcal{O}_Z \; s = f^{-1}(\mathscr{I} \cdot \mathscr{J}) \cdot \mathcal{O}_Z$$

Since we only need to check isomorphisms of sheaves locally, we are done.

Exercise 7.12. *Let X be a noetherian scheme and let Y, Z be two closed subschemes, neither one containing the other. Let $\widetilde{X}$ be obtained by blowing up $Y \cap Z$ (defined by the ideal sheaf $\mathscr{I}_Y + \mathscr{I}_Z$). Show that the strict transform $\widetilde{Y}$ and $\widetilde{Z}$ of Y and Z in $\widetilde{X}$ do not meet.*

Solution. Suppose that they do meet at some point $P \in \widetilde{X}$. The image of this point πP in X is contained in some open affine scheme $U = \operatorname{Spec} A$ and the preimage of this open is $\pi^{-1}U = \operatorname{Proj} \bigoplus_{d \geq 0}(I_Y + I_Z)^d$ where $I_Y = \mathscr{I}_Y(U), I_Z = \mathscr{I}_Z(U)$. The intersections of Y and Z with U are $Y \cap U = \operatorname{Spec}(A/I_Y)$, and $Z \cap U = \operatorname{Spec}(A/I_Z)$ and the preimage of these opens of Y and Z are $\pi^{-1}(U \cap Y) = \operatorname{Proj} \bigoplus_{d \geq 0}((I_Y + I_Z)(A/I_Y))^d \subset \widetilde{Y}$ and similarly for Z. The closed imbedding $\pi^{-1}(U \cap Y) \to \pi^{-1}(U)$ is given by a homomorphism of homogeneous rings $\bigoplus_{d \geq 0}(I_Y + I_Z)^d \to \bigoplus_{d \geq 0}((I_Y + I_Z)(A/I_Y))^d$ and similarly for Z. Clearly the kernel of this ring homomorphism is the homogeneous ideal $\bigoplus_{d \geq 0} I_Y^d$ and similarly for Z. Now if the two closed subschemes intersect as we have supposed then there is a homogeneous prime ideal of $\bigoplus_{d \geq 0}(I_Y + I_Z)^d$ that contains both of these homogeneous ideals. But $\bigoplus_{d \geq 0} I_Y^d$ and $\bigoplus_{d \geq 0} I_Z^d$ generate $\bigoplus_{d \geq 0}(I_Y + I_Z)^d$ so there can be no proper homogeneous prime ideal containing them both. Hence, the intersection is trivial.

8 Differentials

Exercise 8.1. *a Generalize (8.7) as follows. Let B be a local ring containing a field k, and assume that the residue field $k(B) = M/\mathfrak{m}$ of B is a separable generated extension of k. Then the exact sequence of (8.4A),*

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

is exact on the left also.

b Generalize (8.8) as follows. With B, k as above, assume furthermore that k is perfect, and that B is a localization of an algebra of finite type over k. Then show that B is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $= \dim B + \operatorname{tr.d.} k(B)/k$.

c Strengthen (8.15) as follows. Let X be an irreducible scheme of finite type over a perfect field k, and let $\dim X = n$. For any point $x \in X$, not necassarily closed, show that the local ring $\mathcal{O}_x$ is a regular local ring if and only if the stalk $(\Omega_{X/k})_x$ of the sheaf of differentials at x is free of rank n.

d Strengthen (8.16) as follows. If X is a variety over an algebraically closed field k, then $U = \{x \in X | \mathcal{O}_x$ is a regular local ring$\}$ is an open dense subset of X.

Solution. a To show that δ is injective is equivalent to showing that the morphism of vector spaces

$$\delta^* : \hom_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \to \hom_{k(B)}(\mathfrak{m}/\mathfrak{m}^2, k(B))$$

is surjective. Note the isomorphisms:

$$\hom_{k(B)}(\Omega_{B/k} \otimes k(B), k(B)) \cong \hom_B(\Omega_{B/k}, k(B)) \cong \mathrm{Der}_k(B, k(B))$$

So given a $k(B)$-linear homomorphism $h : \mathfrak{m}/\mathfrak{m}^2 \to k(B)$ we want to find a k-derivation $d' : B \to k(B)$ such that pushing it through the isomorphisms and then δ^* gives the original h. First we describe the image of a derivation d' through the isomorphisms and then δ^*. The deriviation $d' : B \to k(B)$ first becomes the B-homomophism described by $db \mapsto d'b$ (use the expression of Ω as a free module generated by the db modulo the suitable relations). This then becomes the $k(B)$-homomorphism $db \otimes c \mapsto cd'b$ and then applying δ^* gives $b \mapsto d'b$. So a derivation d' just gets mapped to its restriction to $\mathfrak{m}$ (note that if $b \in \mathfrak{m}^2$ then $b = \sum a_i c_i$ for some $a_i, c_i \in \mathfrak{m}$ and so $d'b = \sum a_i d' c_i + c_i d' a_i = 0$ in $k(B) = B/\mathfrak{m}$).

Now given a $k(B)$-linear homomorphism $h : \mathfrak{m}/\mathfrak{m}^2 \to k(B)$ we describe a k-derivation $d' : B \to k(B)$. For $b \in B$, write $b = c + \lambda$ with $\lambda \in k(B), c \in \mathfrak{m}$ in the unique way using the section $\begin{tikzcd} k(B) \arrow[r] \arrow[rr, bend left, "id"] & B \arrow[r] & k(B) \end{tikzcd}$ from Theorem 8.25A. Then define $d'(b) = h(c)$.

b Suppose that $\Omega_{B/k}$ is free of the given rank. Then we have the exact sequence from part (a):

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \xrightarrow{\delta} \Omega_{B/k} \otimes k(B) \to \Omega_{k(B)/k} \to 0$$

We know that the dimension of $\Omega_{B/k} \otimes k(B)$ is $\dim B + \operatorname{tr.d.} k(B)/k$ by assumption and that $\dim \Omega_{k(B)/k} = \operatorname{tr.d.} k(B)/k$ by Theorem II.8.6A ($k(B)$ is separably generated over k since we have assumed k perfect, see Theorem I.4.8A). Hence, the dimension of $\mathfrak{m}/\mathfrak{m}^2$ is $\dim B$ and so by definition B is regular.

Now suppose that B is regular. By the argument just described, we know that $\dim_{k(B)} \Omega_{B/k} \otimes k(B)$ is $\dim B + \operatorname{tr.d.} k(B)/k$. If we can show also that $\dim_K \Omega_{B/k} \otimes K$ is $\dim B + \operatorname{tr.d.} k(B)/k$ (where K is the quotient field of B) then we will be done by Lemma II.8.9 for the same reasons that it works in the proof of Theorem II.8.8. As in that proof we have $\Omega_{B/k} \otimes_B K = \Omega_{K/k}$ by Proposition II.8.2A and since k is perfect, K is a separably generated extension of k (Theorem I.4.8A) so $\dim_K \Omega_{K/k} = \operatorname{tr.d.} K/k$ by Theorem 8.6A. Hence $\dim_K \Omega_{B/k} \otimes K = \operatorname{tr.d.} K/k$. Now we have assumed that B is the localization of an algebra A of finite type over k, so $B = A_\mathfrak{p}$ for some prime $\mathfrak{p} \in \operatorname{Spec} A$. This means that we have $\operatorname{Frac} A = \operatorname{Frac} B$ and height $\mathfrak{p} = \dim B$. So by Theorem I.1.8A we have $\operatorname{tr.d.} K/k = \dim A = \text{height } \mathfrak{p} + \dim A/\mathfrak{p} = \dim B + \dim A/\mathfrak{p} = \dim B + \operatorname{tr.d.} \operatorname{Frac}(A/\mathfrak{p})/k = \dim B + \operatorname{tr.d.} k(B)$. So we have shown that $\dim_K \Omega_{B/k} \otimes K$ is $\dim B + \operatorname{tr.d.} k(B)/k$ and now we can happily apply Lemma II.8.9 to get the desired result.

c Take an affine neighbourhood $\operatorname{Spec} A$ of x in which x corresponds to the prime ideal $\mathfrak{p}$. Define $B = A_\mathfrak{p}$ and we have the hypotheses of part (b) satisfied so we see that $\mathcal{O}_x = B$ is a regular local ring if and only if $\Omega_{B/k}$ is free of rank $\dim B + \operatorname{tr.d.} k(B)/k = \dim A = \dim X$ (see the proof of the previous part for the former equality). The stalk $(\Omega_{X/k})_x$ is $\Omega_{A/k} \otimes_A B$ and we have an isomorphism $\Omega_{B/k} \cong \Omega_{S^{-1}A/k} \cong S^{-1}\Omega_{A/k} \cong \Omega_{A/k} \otimes_A B$ by Proposition II.8.2A where S is the multiplicative set of elements not in $\mathfrak{p}$, so $\mathcal{O}_x = B$ is a regular local ring if and only if $\Omega_{B/k} \cong \Omega_{A/k} \cong (\Omega_{X/k})_x$ is free of rank $\dim B + \operatorname{tr.d.} k(B)/k = \dim A = \dim X$.

d By (8.16) we know that there exists some open dense subset V of X which is nonsingular, hence U is dense since it contains any such V. At every point x of U, the coherent sheaf $\Omega_{X/k}$ is locally free by part (c) and so by Exercise II.5.7(a) there is an open neighbourhood W of x on which $\Omega_{X/k}|_W$ is free of rank n. This implies that at every point w of W, the stalks $(\Omega_{X/k})_w$ are free of rank n and therefore, again by part (c), $w \in U$. So every point of U has an open neighbourhood contained in U, and therefore U is open.

Exercise 8.2. *Let X be a variety of dimension n over k. Let $\mathscr{E}$ be a locally free sheaf of rank $> n$ on X, and let $V \subseteq \Gamma(X, \mathscr{E})$ be a vector space of global sections* *which generate $\mathscr{E}$. Then show that there is an element $s \in V$, such that for each $x \in X$, we have $s_x \notin \mathfrak{m}_x \mathscr{E}_x$. Conclude that there is a morphism $\mathcal{O}_X \to \mathscr{E}$ giving rise to an exact sequence*

$$0 \to \mathcal{O}_X \to \mathscr{E} \to \mathscr{E}' \to 0$$

where $\mathscr{E}'$ is locally free.

Solution. Consider the scheme $X \times V$ and the subset of points $Z = \{(x, s) | s_x \in \mathfrak{m}_x \mathscr{E}_x\}$. We label the projections by $\pi_1 : Z \subset X \times V \to X$ and $\pi_2 : Z \subset X \times V \to V$. Now for any point $x \in X$, the preimage $\pi^{-1}x$ in Z is the set of global sections s in V that $s_x \in \mathfrak{m}_x \mathscr{E}_x$. Otherwise said, it is the kernel of the $k(x)$-vector space morphism $V \otimes_k k(x) \to \mathscr{E}_x \otimes_{\mathcal{O}_x} k(x)$. Since $\mathscr{E}$ is generated by global sections this morphism is always surjective and since $\mathscr{E}$ is locally free of rank r this kernel will then have rank $m - r$ where $m = \dim V$. Hence, the dimension of Z as a closed subset of $X \times V$ is $n + m - r$. By assumption $r > n$ and so $n + m - r < m$. Hence, the second projection $\pi_2 : Z \to V$ cannot be surjective. Any point not in the image will be a global section with the required property.

Using this global section s we define a morphism $\mathcal{O}_X \to \mathscr{E}$ by sending $1 \mapsto s$, and define $\mathscr{E}'$ as the cokernel of $\mathcal{O}_X \to \mathscr{E}$. To see that we have an exact sequence as desired, consider the stalk at $x \in X$. We want to show that $\mathcal{O}_x \to \mathcal{O}_x^{\oplus r}$ is injective where we are using the isomorphism $\mathcal{O}_x^{\oplus r} \cong \mathscr{E}_x$; let $s_x = (a_1, \dots, a_r)$. This morphism sends $a \mapsto a(a_1, \dots, a_r)$ and so if $aa_i = 0$ for all i then $a = 0$ or $a_i = 0$ for all i (since X is integral the local rings have no zero divizors) but we have chosen s so that $s_x \notin \mathfrak{m}_x \mathscr{E}_x$ and so $a_i \notin \mathfrak{m}_x$ for some i, and therefore $a = 0$.

Now we must show that $\mathscr{E}'_x = \mathcal{O}_x^{\oplus r}/\mathcal{O}_x$ is free, then the local free-ness of $\mathscr{E}'$ will follow from Exercise II.5.7(b). We do this by explicitely constructing an isomorphism $\mathcal{O}_x^{\oplus(r-1)}$. We have assumed that one of a_i is not in $\mathfrak{m}_x$. Without loss of generality we can assume that it is a_r. Writing $\mathcal{O}_x$ as $A_\mathfrak{p}$ for some affine $\operatorname{Spec} A$ containing $\mathfrak{p}$ we see that a_r is invertible since it is not in $\mathfrak{p}A_\mathfrak{p} = \mathfrak{m}_x$. Now consider the composition $A_\mathfrak{p}^{r-1} \to \mathcal{A}_\mathfrak{p}^r \to \mathcal{A}_\mathfrak{p}^r / sA_\mathfrak{p}$ where the first morphism sends $(b_1, \dots, b_{r-1})$ to $(b_1, \dots, b_{r-1}, 0)$. Clearly the composition is injective for $(b_1, \dots, b_{r-1}, 0) \in sA_\mathfrak{p}$ contradicts the assumption that $a_r \notin \mathfrak{m}_x$. For surjectivity, let $b = (b_1, \dots, b_r)$ represent an element of $\mathcal{A}_\mathfrak{p}^r / sA_\mathfrak{p}$. Then $b - a_r^{-1} b_r s \in A_\mathfrak{p}^{r-1}$ and $(b - a_r^{-1} b_r s) - b \in sA_\mathfrak{p}$. So we are done.

Exercise 8.3. Product Schemes.

a *Let X and Y be schemes over another scheme S. Use (8.10) and (8.11) to show that $\Omega_{X \times Y/S} \cong p_1^* \Omega_{X/S} \oplus p_2^* \Omega_{Y/S}$.*

b *If X and Y are nonsingular varieties over a field k, show that $\omega_{X \times Y} \cong p_1^* \omega_X \otimes p_2^* \omega_Y$.*

c *Let Y be a nonsingular plane cubic curve, and let X be the surface $Y \times Y$. Show that $p_g(X) = 1$ but $p_a(X) = -1$ (I. Ex. 7.2).*

Solution. a From (8.10) it follows that $\Omega_{X\times Y/X} \cong p_2^*(\Omega_{Y/S})$ and $\Omega_{X\times Y/Y} \cong p_1^*(\Omega_{X/S})$. Combining these with Proposition (8.11) gives exact sequences

$$\Omega_{X\times Y/X} \to \Omega_{X\times Y/S} \to \Omega_{X\times Y/Y} \to 0$$

$$\Omega_{X\times Y/Y} \to \Omega_{X\times Y/S} \to \Omega_{X\times Y/X} \to 0$$

To see that the relavent morphisms actually do decompose $\Omega_{X\times Y/S}$ into $p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$ we go to Matsumura to find the definitions of these morphisms. It is enough to consider the affine case, so let A and B be rings over C. We want to know if the composition

$$\Omega_{A\otimes_C B/A} \xleftarrow{\sim} \Omega_{B/C} \otimes_B (B \otimes_C A) \to \Omega_{A\otimes_C B/C} \to \Omega_{A\otimes_C B/A}$$

is the identity. The first module is generated by elements of the form dx where $x \in A \otimes_C B$. Since d is a morphism of abelian groups and $d(a \otimes b) = d(a \otimes 1) + d(1 \otimes b) = d(1 \otimes b)$ it is enough to consider elements of the form $d(1 \otimes b)$. The first map takes such an element to $(1 \otimes 1) \otimes db$. This then gets taken to $d(1 \otimes b)$ which gets taken back to $d(1 \otimes b)$ so the composition is the identity.

b Suppose that the dimensions of X and Y are n and m respectively. Then we have

$$\begin{aligned}
\omega_{X\times Y} &= \wedge^{nm}\Omega_{X\times Y} && \text{(by definition)}\\
&\cong \wedge^{nm}(p_1^*(\Omega_X) \oplus p_2^*(\Omega_Y)) && \text{(part (a))}\\
&\cong (\wedge^n p_1^*(\Omega_X)) \otimes (\wedge^m p_2^*(\Omega_Y)) && \text{(Exercise I.5.16(d))}\\
&\cong (p_1^*(\wedge^n\Omega_X)) \otimes (p_2^*(\wedge^m\Omega_Y)) && \text{(Exercise I.5.16(e))}\\
&\cong p_1^*(\omega_X) \otimes p_2^*(\omega_Y) && \text{(by definition)}
\end{aligned}$$

c In Example 8.20.3 we see that $\omega_Y \cong \mathcal{O}_Y$ and so by part (b) we have $\omega_{Y\times Y} \cong p_1^*\omega_Y \otimes p_2^*\omega_Y \cong p_1^*\mathcal{O}_Y \otimes p_2^*\mathcal{O}_Y \cong \mathcal{O}_{Y\times Y}$. By Exercise II.4.5(d) the vector space of global sections of the structure sheaf of $Y \times Y$ has dimension one.

In Exercise I.7.2 we calculate the arithmetic genus of a plane cubic curve to be 1 in part (b) and then the arithmetic genus of $Y \times Y$ is calculated in part (e) as $1 - 1 - 1 = -1$.

Exercise 8.4. Complete Intersections in $\mathbb{P}^n$.

Exercise 8.5. Blowing up a Nonsingular Subvariety. *As in (8.24), let X be a nonsingular variety, let Y be a nonsingular subvariety of codimension $r \geq 2$, let $\pi : \widetilde{X} \to X$ be the blowing up of X along Y, and let $Y' = \pi^{-1}(Y)$.*

a *Show that the maps $\pi^* : \operatorname{Pic} X \to \operatorname{Pic} \widetilde{X}$, and $\mathbb{Z} \to \operatorname{Pic} \widetilde{X}$ defined by $n \mapsto$ class of nY', give rise to an isomorphism $\operatorname{Pic} \widetilde{X} \cong \operatorname{Pic} X \oplus \mathbb{Z}$.*

b *Show that $\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.*

Solution. a Since X is nonsingular we can associate each each invertible sheaf to a class of divisors (Remark II.6.11.1A). Then from Proposition II.6.5 we have the exact sequence and isomorphism:

$$\mathbb{Z} \to \operatorname{Cl} \widetilde{X} \to \operatorname{Cl} U \to 0 \qquad \operatorname{Cl} U \cong \operatorname{Cl} X$$

where $U = X - Y$. The composition $\operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} U$ is the same as the composition $\operatorname{Pic} X \xrightarrow{\sim} \operatorname{Pic} U$ and so $\operatorname{Pic} X \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} X$ is the identity. Furthermore, the composition $\mathbb{Z} \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} X$ is zero as a direct consequence of the exact sequence. So it remains only to find a splitting for $\mathbb{Z} \to \operatorname{Pic} \widetilde{X}$. Consider the embedding $j : Y' \to \widetilde{X}$. This provides a morphism $\operatorname{Pic} \widetilde{X} \to \operatorname{Pic} Y'$. We know by Theorem II.8.24(b) that Y' is a projective bundle over Y and then from Exercise II.7.9 that $\operatorname{Pic} Y' \cong \operatorname{Pic} Y \oplus \mathbb{Z}$. We follow 1 through the composition $\mathbb{Z} \to \operatorname{Pic} \widetilde{X} \to \operatorname{Pic} Y' \to \operatorname{Pic} Y \oplus \mathbb{Z} \to \mathbb{Z}$. We have 1 gets sent to $\mathscr{L}(Y') \in \operatorname{Pic} \widetilde{X}$ which by Proposition II.6.18 is isomorphic to $\mathscr{I}_{Y'}^{-1}$ which we know is $\mathcal{O}_{\widetilde{X}}(-1)$ (from the proof of (7.13) for example). This then becomes $\mathcal{O}_{Y'}(-1)$ which is then sent to -1. So our composition is not the identity, but is an isomorphism, and we only wanted to find a splitting for $\mathbb{Z} \to \operatorname{Pic} \widetilde{X}$ so compose with $1 \mapsto -1$ and we obtain our desired splitting.

b By (a) we can write $\omega_{\widetilde{X}}$ as $f^*\mathscr{M} \otimes \mathscr{L}(qY')$ for some invertible sheaf $\mathscr{M} \in \operatorname{Pic} X$ and some integer q. We have an isomorphisms $X - Y \cong \widetilde{X} - Y'$ (Proposition II.7.13) and so $\omega_{\widetilde{X}}|_{\widetilde{X}-Y'} \cong \omega_U \cong \omega_X|_{X-Y}$. We also have an isomorphism $\operatorname{Pic} X \cong \operatorname{Pic} U$ (Proposition II.6.5) and so if $\mathscr{M}|_{X-Y} \cong \omega_X|_{X-Y}$, which it is, then $\mathscr{M} \cong \omega_X$. Now by Proposition II.8.20 we have $\omega_{Y'} \cong \omega_{\widetilde{X}} \otimes \mathscr{L}(Y') \otimes \mathcal{O}_{Y'} \cong f^*\omega_X \otimes \mathscr{L}((q+1)Y') \otimes \mathcal{O}_{Y'}$. Then by Proposition II.6.18 $\mathscr{L}((q+1)Y') \cong \mathscr{I}_{Y'}^{-q-1}$ and we know that $\mathscr{I}_{Y'} = \mathcal{O}_{\widetilde{X}}(1)$ (from the proof of (7.13) for example). Putting all this together we get $\omega_{Y'} \cong f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)$. Now we take a closed point $y \in Y$ and let Z be the fibre of Y' over y; that is, $Z = y \times_Y Y'$. We can use Exercise II.8.3(b) to find that $\omega_Z \cong \pi_1^*\omega_y \otimes \pi_2^*\omega_{Y'} \cong \pi_2^*(f^*\omega_X \otimes \mathcal{O}_{Y'}(-q-1)) \cong \mathcal{O}_Z(-q-1)$ since $\omega_y = \mathcal{O}_y$ and pulling ω_X back to Z can be done via y on which it becomes the structure sheaf. Now Z is just projective space of dimension $r - 1$ (Theorem II.8.24) and so $\omega_Z \cong \mathcal{O}_Z(-r)$ (Example II.8.20.1) so $q = r - 1$. Hence $\omega_{\widetilde{X}} \cong f^*\omega_X \otimes \mathscr{L}((r-1)Y')$.

Exercise 8.6. Infinitesimal Lifting Property. *Let k be an algebraically losed field, let A be a finitely generated k-algebra such that* $\operatorname{Spec} A$ *is a nonsingular variety over k. Let $0 \to I \to B' \to B \to 0$ be an exact seqeuence, where B' is a k-algebra, and I is an ideal with $I^2 = 0$. Finally suppose given a k-algebra homomorphism $f : A \to B$. Then there exists a k-algebra homomorphism $g : A \to B'$ lifting f.*

a *First suppose that $g : A \to B'$ is a given homomorphism lifting f. If $g' : A \to B'$ is another such homomorphism, show that $\theta = g - g'$ is a k-derivation of A into I, which we can consider as an element of*

$\hom_A(\Omega_{A/k}, I)$. *Conversely, for any* $\theta \in \hom_A(\Omega_{A/k}, I)$, *show that* $g' = g + \theta$ *is another homomorphism lifting* f.

b *Now let* $P = k[x_1, \ldots, x_n]$ *be a polynomial ring over* k *of which* A *is a quotient, and let* J *be the kernel. Show that there does exist a homomorphism* $j : P \to B'$ *making a commutative diagram*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
J \arrow[d] & I \arrow[d] \\
P \arrow[r, "h"] \arrow[d] & B' \arrow[d] \\
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
0 & 0
\end{tikzcd}$$

and show that h *induces an* A*-linear map* $\overline{h} : J/J^2 \to I$.

c *Conclude by finding the desired morphism* $g : A \to B'$ *(Hartshorne essentially walks the reader through the proof of this part in his statement of the exercise).*

Solution. a Since g and g' both lift f, the difference $g - g'$ is a lift of zero, and therefore, the image lands in the submodule I of B'. The homomorphisms g and g' are algebra homomorphisms and so they both send 1 to 1, hence the difference sends 1 to 0 and so for any $c \in k$ we have $\theta(k) = k\theta(1) = 0$. For the Leibniz rule we have

$$\begin{aligned}
\theta(ab) &= g(ab) - g'(ab) \\
&= g(a)g(b) - g'(a)g'(b) \\
&= g(a)g(b) - g'(a)g'(b) + (g'(a)g(b) - g'(a)g(b)) \\
&= g(b)\theta(a) + g'(a)\theta(b)
\end{aligned}$$

We can consider it as an element of $\hom_A(\Omega_{A/k}, I)$ by the universal property of the module of relative differentials.

Conversely, for any $\theta \in \hom_A(\Omega_{A/k}, I)$ we obtain a derivation $\theta \circ d : A \to I$ which we can compose with the inclusion $I \to B'$ to get a k-linear morphism from A into B' Since the sequence is exact, this θ vanishes on composition with $B' \to B$ and so $g + \theta$ is another k-linear homomorphism lifting f and we just need to show that it is actually a morphism of k-algebras; that is, that it preserves multiplication.

$$\begin{aligned}
g(ab) + \theta(ab) &= g(ab) + \theta(a)g(b) + g(a)\theta(b) \\
&= g(ab) + \theta(a)g(b) + g(a)\theta(b) + \theta(a)\theta(b) \qquad \text{since } I^2 = 0 \text{ and } \theta(a), \theta(b) \in I \\
&= (g(a) + \theta(a))(g(b) + \theta(b))
\end{aligned}$$

b A k-homomorphism out of P is uniquely determined by the images of the x_i, which can be anything. So for each i choose a lift b_i of $f(x_i)$ in B' and we obtain a morphism h by sending x_i to b_i and extending to a k-algebra homomorphism. If $a \in P$ is in J then by commutivity, the image of $h(a)$ in B will be zero, implying that $h(a) \in I$ so we have at least a k-linear map $J \to I$. If $a \in J^2$ then $h(a) \in I^2 = 0$ so this map descends to $\overline{h} : J/J^2 \to I$. The last thing to check is that the map $\overline{h}$ is A-linear, and this follows from h preserving multiplication.

c Applying the global sections functor to the exact sequence of (8.17) with $X = \operatorname{Spec} P$, $Y = \operatorname{Spec} A$ gives an exact sequence

$$0 \to J/J^2 \to \Omega_{P/k} \otimes A \to \Omega_{A/k} \to 0$$

which is exact on the right as well by (8.3A). Now since A is nonsingular, $\Omega_{A/k}$ is locally free and and therefore projective so $\operatorname{Ext}^i(\Omega_{A/k}, I) = 0$ for all $i > 0$. So the exact sequence

$$0 \to \hom_A(\Omega_{A/k}, I) \to \hom_A(\Omega_{P/k} \otimes A, I) \to \hom_A(J/J^2, I) \to \operatorname{Ext}^1_A(\Omega_{A,k}, I) \to \dots$$

shows that $\hom(\Omega_{P/k} \otimes A, I) \to \hom(J/J^2, I)$ is surjective. So we can find a P-morphism $\theta : \Omega_{P/k} \to I$ whose image is $\overline{h}$ from part (b). We then define θ' as the composition $P \xrightarrow{d} \Omega_{P/k} \to I \to B'$ to obtain a k-derivation $P \to B'$. Let $h' = h - \theta$. For any element $b \in J$ we have $h'(b) = h(b) - \theta(b) = \overline{h}(b) - \overline{h}(b) = 0$ so h' descends to a morphism $g : A \to B'$ which lifts f.

Exercise 8.7. *If X is affine and nonsingular, then show that any extension of X by a coherent sheaf $\mathscr{F}$ is isomorphic to the trivial one.*

Solution. Since everything is affine, the problem restated is this: given a ring A', an ideal $I \subset A'$ such that I^2, and an isomorphism $A'/I \cong A$, such that $I \cong M$ as an A-module (where M is the finitely generated A-module corresponding to $\mathscr{F}$), show that $A' \cong A \oplus M$ as an abelian group, with multiplication defined by $(a, m)(a', m') = (aa', am' + a'm)$.

Using the infinitesimal lifting property we obtain a morphism $A \to A'$ that lifts the given isomorphism $A'/I \cong A$. This together with the given data provides the isomorphism $A \oplus M \cong A'$ of abelian groups where we use the isomorphism $M \cong I$ to associate M with I as an A-module. If $a \in A$ then $(a, 0)(a', m') = (aa', am')$ using the A-module structure on A and $M \cong I$. If $m \in M \cong I$ then $(0, m)(a', m') = (0, a'm)$ since $mm' \in I^2$. So we have the required isomorphism.

Exercise 8.8. *Using the method of (8.19), show that $P_n = \dim_k \Gamma(X, \omega_X^{\otimes n})$ and $h^{q,0} = \dim_k \Gamma(X, \wedge^q \Omega_{X/k})$ are birational invariants of X, a projective nonsingular variety over k.*

Solution. The proof of (8.19) translates almost verbatim.

Suppose that we have another nonsingular, projective variety X', birationally equivalent to X. Consider a birationally invertible map $X \to X'$ and let $V \subset X$ be the largest open subset of X on which it is representable, and $f : V \to X'$ a representative morphism. We obtain a morphism of sheaves $f^*\Omega_{X'} \to \Omega_V$ via Proposition II.8.11. These are locally free sheaves of rank $n = \dim X$ and so we obtain morphisms $f^*\omega_{X'}^{\otimes n} \to \omega_V^{\otimes n}$ and $f^*\Omega_{X'}^q \to \Omega_V^q$ both of which induce morphisms of global sections. By (I, 4.5) there is an open subset U of V that is mapped isomorphically onto its image in X' by f. This $\Omega_V|_U \cong \Omega'_X|_{f(U)}$ via f. We have a commutative square

$$\begin{tikzcd}
\Gamma(\omega_{X'}^{\otimes n}, X') \arrow[r] \arrow[d] & \Gamma(\omega_V^{\otimes n}, V) \arrow[d] \\
\Gamma(\omega_{f(U)}^{\otimes n}, f(U)) \arrow[r] & \Gamma(\omega_U^{\otimes n}, U)
\end{tikzcd}$$

and a similar one for $f^*\Omega_{X'}^q \to \Omega_V^q$. Since $f(U)$ is dense and open in X', and a nonzero global section cannot vanish on a dense open subset, we see that the morphisms

$$\Gamma(\omega_{X'}^{\otimes n}, X') \to \Gamma(\omega_V^{\otimes n}, V) \qquad \Gamma(\Omega_{X'}^q, X') \to \Gamma(\Omega_V^q, V)$$

are both injective.

Now we compare $\Gamma(V, -)$ to $\Gamma(X, -)$. First we claim that $X - V$ has codimension > 1 in X. This follows from the valuative criterion of properness (4.7). If $P \in X$ is a point of codimension 1 then $\mathcal{O}_{X,P}$ is a discrete valuation ring because X is nonsingular. The map from the generic point η_X of X to that of X' fits into a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K(X) \arrow[r] \arrow[d] & X' \arrow[d] \\
\operatorname{Spec} \mathcal{O}_{X,P} \arrow[r] \arrow[ur, dashed] & \operatorname{Spec} k
\end{tikzcd}$$

and so we can extend V to include P and so by the definition of V, it already includes P.

To show that $\Gamma(V, \mathscr{F}) \cong \Gamma(X, \mathscr{F})$ for the sheaves $\mathscr{F}$ that we are interested in, it suffices to show that $\Gamma(V \cap U, \mathscr{F}|_{U\cap V}) \cong \Gamma(U, \mathscr{F}|_U)$ for each open U in a cover of X (use the sequences $0 \to \Gamma(X, -) \to \oplus\Gamma(U_i, -) \to \oplus\Gamma(U_{ij}, -)$). Choose the open cover $\{U_i\}$ such that on each U_i the sheaf $\mathscr{F}$ ($= \Omega_X^q$ or $\omega_X^{\otimes n}$) is free, and each U_i is affine. Then what we need to show is that for each of these U_i, the morphism $\Gamma(U_i, \mathcal{O}_{U_i}) \to \Gamma(U \cap V, \mathcal{O}_{U_i\cap V})$ is bijective. Since X is nonsingular, and therefore normal, and since $U_i - U_i \cap V$ has codimension > 1 in U_i, this is a consequence of (6.3A).

So the culmination is that we have an injective morphism $\Gamma(X', \mathscr{F}_{X'}) \to \Gamma(V, \mathscr{F}_X|_V)$ and a bijective morphism $\Gamma(X, \mathscr{F}_X) \to \Gamma(V, \mathscr{F}_X|_V)$ (where $\mathscr{F}_- =$

Ω^q_- or $\omega_-^{\otimes n}$). Hence, $P_n(X') \leq P_n(X)$ and $h^{q,0}(X') \leq h^{q,0}(X)$. By symmetry we get inequalities in the other direction and so these inequalities are actually equalities.

Contents

2 Cohomology of Sheaves.

Exercise 2.1. *a Let $X = \mathbb{A}^1$ be the affine line over an infinite field k. Let P, Q be distinct closed points of X, and let $U = X - \{P, Q\}$. Show that $H^1(X, \mathbb{Z}_U) \neq 0$.*

b More generally, let $Y \subseteq X = \mathbb{A}^n$ be the union of $n+1$ hyperplanes in suitable general position, and let $U = X - Y$. Show that $H^n(X, \mathbb{Z}_U) \neq 0$, thus the result of (2.7) is the beset possible.

Solution. a The sheaf $\mathbb{Z}_U$ is a subsheaf of $\mathbb{Z}_X$ and so we get an exact sequence $0 \to \mathbb{Z}_U \to \mathbb{Z} \to i_{P*}\mathbb{Z} \oplus i_{Q*}\mathbb{Z} \to 0$ where $i_{P*}\mathbb{Z}$ and $i_{Q*}\mathbb{Z}$ are the skyscraper sheaves at P and Q with value $\mathbb{Z}$. Taking cohomology gives a long exact sequence, one piece of which is $\dots \to H^0(X, \mathbb{Z}_X) \to H^0(X, i_{P*}\mathbb{Z} \oplus i_{Q*}\mathbb{Z}) \to H^1(X, \mathbb{Z}_U) \to \dots$, so if $H^1(X, \mathbb{Z}_U) = 0$, then $H^0(X, \mathbb{Z}_X) \to H^0(X, i_{P*}\mathbb{Z} \oplus i_{Q*}\mathbb{Z})$ is surjective. But this is $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ which cannot be surjective.

Exercise 2.2. *Let $X = \mathbb{P}^1$ be the projective line over an algebraically closed field k. Show that the exact sequence $0 \to \mathcal{O} \to \mathscr{K} \to \mathscr{K}/\mathcal{O} \to 0$ of (II, Ex. 1.21d) is a flasque resolution of $\mathcal{O}$. Conclude from (II, Ex. 1.21e) that $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.*

Solution. Since every pair of open subsets of X intersect nontrivially, every open subset is connected. So the constant sheaf $\mathscr{K}$ is actually the constant presheaf $\mathscr{K}$, and therefore flasque. To see that $\mathscr{K}/\mathcal{O}$ is flasque, write it as $\oplus_{P \in X} i_P(I_P)$ (Exercise II.1.21(d)). Exercise II.1.21(e) then tells us that applying the global sections functor we get an exact sequence, so $\Gamma(X, \mathscr{K}) \to \Gamma(X, \mathscr{K}/\mathcal{O}) \to 0 \to \dots$ is exact, and since we can use this to calculate the cohomology, $H^i(X, \mathcal{O}) = 0$ for all $i > 0$.

Exercise 2.3. Cohomology with Supports. *Let X be a topological space, let Y be a closed subset, and let $\mathscr{F}$ be a sheaf of abelian groups. Let $\Gamma_Y(X, \mathscr{F})$ denote the group of sections of $\mathscr{F}$ with support in Y.*

a *Show that $\Gamma_Y(X, \cdot)$ is a left exact functor from $\mathscr{A}b(X)$ to $\mathscr{A}b$.*

b *If $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence of sheaves, with $\mathscr{F}'$ flasque, show that*

$$0 \to \Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'') \to 0$$

is exact.

c *Show that if $\mathscr{F}$ is flasque, then $H^i_Y(X, \mathscr{F}) = 0$ for all $i > 0$.*

d *If $\mathscr{F}$ is flasque, show that the sequence*

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F}) \to 0$$

is exact.

e *Let $U = X - Y$. Show that for ny $\mathscr{F}$, there is a long exact sequence of cohomology groups*

$$\begin{aligned} 0 &\to H^0_Y(X, \mathscr{F}) \to H^0(X, \mathscr{F}) \to H^0(U, \mathscr{F}|_U) \to \\ &\to H^1_Y(X, \mathscr{F}) \to H^1(X, \mathscr{F}) \to H^1(U, \mathscr{F}|_U) \to \\ &\to H^2_Y(X, \mathscr{F}) \to \dots \end{aligned}$$

f Excision. *Let V be an open subset of X containing Y. Then there are natural functorial isomorphisms, for all i and $\mathscr{F}$,*

$$H^i_Y(X, \mathscr{F}) \cong H^i_Y(V, \mathscr{F}|_V)$$

Solution. a Let $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ be an exact sequence of sheaves of abelian groups on X. If $\Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F})$ is injective as a consequence of $\Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F})$ being injective. Similarly, the composition $\Gamma_Y(X, \mathscr{F}') \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$ is zero as a consequence of $\Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F}) \to \Gamma(X, \mathscr{F}'')$ being zero. Consider a section $s \in \Gamma_Y(X, \mathscr{F})$ and suppose that it gets sent to zero in $\Gamma_Y(X, \mathscr{F}'')$. This implies that as an element of $\Gamma(X, \mathscr{F})$, the section s gets sent to zero in $\Gamma(X, \mathscr{F}'')$ and so is the image of some section $t \in \Gamma(X, \mathscr{F}')$. We just need to check that $t_x = 0$ for every $x \notin Y$. Let $x \in X - Y$ be such a point. Since $\mathscr{F}' \to \mathscr{F} \to \mathscr{F}''$ is exact, we have an exact sequence of stalks $0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$. The stalk of s_x is zero since $s \in \Gamma_Y(X, \mathscr{F})$ and therefore $t_x = 0$. Hence $t \in \Gamma_Y(X, \mathscr{F}')$.

b By part (a) we know that $\Gamma_Y(X, \cdot)$ is left exact so we just need to show that $\Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{F}'')$ is surjective. Suppose that we have a section $s \in \Gamma_Y(X, \mathscr{F}'')$. This is a section of $\Gamma(X, \mathscr{F}'')$ and since $\mathscr{F}'$ is flasque, there is a section $t \in \Gamma(X, \mathscr{F})$ in its preimage (Exercise II.1.16(b)). This section does not necassarily have support in Y however. For every point $x \in X - Y$ consider the exact sequence of stalks $0 \to \mathscr{F}'_x \to \mathscr{F}_x \to \mathscr{F}''_x \to 0$. The germ t_x gets sent to $s_x \in \mathscr{F}''_x$ which is zero since s has support in Y. So there is a germ $u_x \in \mathscr{F}'_x$ which gets sent to t_x. This means there is a neighbourhood U_i of x (which we can assume doesn't intersect Y) and a section u_i which gets sent to $t_i|_{U_i}$. In this way we get an open cover $\{U_i\}$ of $X - Y$ and for each i, a section u_i which gets sent to $t|_{U_i}$. Consider the intersections of the u_i. The sections $u_i|_{U_i \cap U_j} - u_j|_{U_i \cap U_j}$ get sent to $t|_{U_i \cap U_j} - t|_{U_i \cap U_j} = 0$ and since $\mathscr{F}' \to \mathscr{F}$ is injective, this means that $u_i|_{U_i \cap U_j} - u_j|_{U_i \cap U_j} = 0$ and so the u_i glue together to give a section $u' \in \mathscr{F}'(U)$ which gets sent to $t|_U$. Since $\mathscr{F}'$ is flasque, this lifts to a global section $u \in \Gamma(X, \mathscr{F}')$. Now consider $t - u \in \Gamma(X, \mathscr{F})$ this gets sent to $s \in \Gamma(X, \mathscr{F}'')$ since u came from $\mathscr{F}'$ and t got sent to s. Furthermore, for any point $x \in X - Y$, the germs of t and u agree since $t|_{U_i} = u|_{U_i} = u_i$ for every i in our cover above. Hence, we have found a global section $t - u \in \Gamma_Y(X, \mathscr{F})$ that gets sent to s.

c The proof from Proposition III.2.5 works. Embed $\mathscr{F}$ in an injective object $\mathscr{I}$ and let $\mathscr{G}$ be the quotient $\mathscr{F}/\mathscr{I}$. The sheaf $\mathscr{F}$ is flasque by hypothesis, and $\mathscr{I}$ is flasque by (2.4) so $\mathscr{G}$ is flasque by Exercise II.I.16(c). Since $\mathscr{F}$ is flasque, we have an exact sequence

$$0 \to \Gamma_Y(X, \mathscr{F}) \to \Gamma_Y(X, \mathscr{I}) \to \Gamma_Y(X, \mathscr{G}) \to 0$$

from part (b). On the other hand, $\mathscr{I}$ is injective and so $H^i_Y(X, \mathscr{I}) = 0$ for all $i > 0$. Thus, from the long exact sequence of cohomology, we get $H^1_Y(X, \mathscr{F}) = 0$ and $H^i_Y(X, \mathscr{F}) \cong H^{i-1}_Y(X, \mathscr{G})$ for each $i \geq 2$. But $\mathscr{G}$ is also flasque, and so by induction on i we get the result.

d This sequence is what you get if you apply the global sections functor to the sequence of Exercise II.1.20(b) so we just need to show that $\Gamma(X, \mathscr{F}) \to \Gamma(X - Y, \mathscr{F})$ is surjective. But this is true, since $\mathscr{F}$ is flasque.

e To compute the cohomology of $\mathscr{F}$ we choose at the beginning an injective resolution $\mathscr{I}^i$ for $\mathscr{F}$. The functor $-|_U$ preserves injectives so we can use $\mathscr{I}^i|_U$ as an injective resolution to calculate the cohomology on U of $\mathscr{F}|_U$. Now injective sheaves are flasque by Lemma III.2.4 so for each i we have an exact sequence

$$0 \to \Gamma_Y(X, \mathscr{I}) \to \Gamma(X, \mathscr{I}) \to \Gamma(U, \mathscr{I}|_U) \to 0$$

since $\Gamma(U, \mathscr{I}|_U) = \mathscr{I}(U)$. Now the long exact sequence is a consequence of the snake lemma.

f We use the espace étale of Exercse II.1.13 to show that there is an isomorphism of functors $\Gamma_Y(X, -) \to \Gamma_Y(V, -|_V)$. Given a sheaf $\mathscr{F}$ and an open subset $U \subset X$, using the espace étale we can consider $\mathscr{F}(U)$ as a set of continuous morphisms $U \to \text{Spé}\mathscr{F}$. Any section of $\Gamma_Y(X,\mathscr{F})$ takes the value $0 \in \mathscr{F}_x \subset \text{Spé}\mathscr{F}$ on any point x not in Y. So since $Y \subset V$, if two sections of $\Gamma_Y(X,\mathscr{F})$ agree on their restrictions to V, then they agree in $\Gamma_Y(X,\mathscr{F})$ so $\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(U,\mathscr{F})$ is injective. On the other hand, if we have a section $s \in \Gamma_Y(U,\mathscr{F})$ we extend it to a section t in $\Gamma_Y(X,\mathscr{F})$ by sending $x \mapsto 0 \in \mathscr{F}_x$ for any point $x \in X - V$. This defines a function $X \to \text{Spé}\mathscr{F}$ which is a section but is not necassarily continuous. To see that it is continuous, consider the restriction to an open cover $\{U_i\}$ where for each i, either $U_i \subset V$ or $U_i \cap Y = \varnothing$ (or both). Since t came from a section s, for the i with $U_i \subset V$ we have $t|_{U_i} = s|_{U_i}$ and so these are continuous. For the i with $U_i \cap Y = \varnothing$, we have $t|_{U_i} = 0$, which is continuous by definition of the espace étale since these morphisms come from sections of $\mathscr{F}(U_i)$. So the restrictions of t to every element of an open cover of X are continuous, and therefore t is continuous, hence $\Gamma_Y(X,\mathscr{F}) \to \Gamma_Y(V,\mathscr{F}|_V)$ is surjective.

Now as we mentioned in the previous part, if $\mathscr{I}^i$ is an injective resolution for $\mathscr{F}$, then $\mathscr{I}^i|_V$ is an injective resolution for $\mathscr{F}|_V$ and so the isomorphism $\Gamma_Y(X,-) \cong \Gamma_Y(V,-|_V)$ leads to the isomorphism of cohomology groups.

Exercise 2.4. Mayer-Vietoris Sequence. *Let Y_1, Y_2 be two closed subsets of X. Then there is a long exact sequence of cohomology with supports*

$$\cdots \to H^i_{Y_1\cap Y_2}(X,\mathscr{F}) \to H^i_{Y_1}(X,\mathscr{F}) \oplus H^i_{Y_2}(X,\mathscr{F}) \to H^i_{Y_1\cup Y_2}(X,\mathscr{F}) \to H^{i+1}_{Y_1\cap Y_2} \to \dots$$

Solution. Define $Y_{12} = Y_1 \cap Y_2$, $Y = Y_1 \cup Y_2$, $U_{12} = X - Y_{12}$, $U_i = X - Z_i$, $U = X - Y$ and consider the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \Gamma_{Y_{12}}(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma_{Y_1}(X,\mathscr{I}) \oplus \Gamma_{Y_2}(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma_Y(X,\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X,\mathscr{I}) \oplus \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(X,\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(U_{12},\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(U_1,\mathscr{I}) \oplus \Gamma(U_2,\mathscr{I}) \arrow[r] \arrow[d] & \Gamma(U,\mathscr{I}) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

If the sheaf $\mathscr{I}$ is flasque, then the columns are exact. The lower two rows are exact (the lower one being exact as consequence of $\mathscr{I}$ being a sheaf) and so we can apply the Nine Lemma to find that the top row is exact. So if $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ is an injective resolution of $\mathscr{F}$, then we get an exact sequence of complexes

$$0 \to \Gamma_{Y_{12}}(X,\mathscr{I}^\bullet) \to \Gamma_{Y_1}(X,\mathscr{I}^\bullet) \oplus \Gamma_{Y_2}(X,\mathscr{I}^\bullet) \to \Gamma_Y(X,\mathscr{I}^\bullet) \to 0$$

The Snake Lemma applied to this exact sequence of complexes gives the desired long exact sequence.

Exercise 2.5. *Let X be a Zariski space. Show that for all i, $\mathscr{F}$, we have*

$$H^i_P(X, \mathscr{F}) = H^i_P(X_P, \mathscr{F}_P)$$

(see Hartshorne's statement of the exercise for notation).

Solution. We show a natural isomorphism $\Gamma_P(X, \mathscr{G}) \cong \Gamma_P(X_P, \mathscr{G}_P)$. By definition, $\Gamma(X_P, \mathscr{G}_P) = \varinjlim_{U \ni P} \mathscr{G}(U) = \mathscr{G}_P$ since $P \in U$ if and only if $U \supset X_P$, so there is a natural morphism $\Gamma(X, \mathscr{G}) \to \Gamma(X_P, \mathscr{G}_P)$ which induces a morphism $\Gamma_P(X, \mathscr{G}) \to \Gamma_P(X_P, \mathscr{G}_P)$. Injectivity: let s and t be two global sections with support on P. If they get sent to the same element in $\Gamma_P(X_P, \mathscr{G}_P)$ then the germs $s_P = t_P$ agree. But s and t have support in P so they are identically zero in every other stalk. Therefore they agree on every stalk and hence, $s = t$. Surjectivity: let $s \in \Gamma_P(X_P, \mathscr{G}_P) = \mathscr{G}_P$. Then there is an open neighbourhood of P and $s_U \in \mathscr{G}(U)$ which represents s. Since s has support in P we can choose U small enough so that $(s_U)_Q = 0$ for every point $Q \neq P$. Now consider $V = X - P$ and the zero section in $\mathscr{G}(U)$. Since the germ of s_U is zero on all points that aren't P, we have $s_U|_{U \cap V} = 0$ and so s_U and 0 glue together to give a global section with support in P. So the map is surjective.

Exercise 2.6. *Let X be a noetherian topological space, and let $\{\mathscr{I}_\alpha\}_{\alpha \in A}$ be a direct system of injective sheaves of abelian groups on X. Then $\varinjlim \mathscr{I}_\alpha$ is also injective.*

Solution. For an open subset $U \subset X$ we define $\mathbb{Z}_U = i_! \mathbb{Z}_U$ where $\mathbb{Z}_U$ is the constant sheaf associated to the group $\mathbb{Z}$ and $i : U \to X$ is the inclusion.

Step 1. First we show that a sheaf $\mathscr{I}$ is injective if and only if for every open set $U \subseteq X$, and subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$, and every map $f : \mathscr{R} \to \mathscr{I}$, there is an extension of f to a map of $\mathbb{Z}_U \to \mathscr{I}$.

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{R} \arrow[r] \arrow[dr] & \mathbb{Z}_U \arrow[d, dashed] \\ & & \mathscr{I} \end{tikzcd}$$

The direction ($\Rightarrow$) follows from the definition of an injective object. For the direction ($\Leftarrow$) we adapt the proof from the proof of Baer's Criterion (Theorem 2.3.1) in Weibel. Let $\mathscr{F} \subset \mathscr{G}$ be an injective morphism of sheaves, and suppose we have a morphism $\phi : \mathscr{F} \to \mathscr{I}$. Consider the poset of extensions of ϕ to a subsheaf $\mathscr{F}'$ of $\mathscr{G}$ containing $\mathscr{F}$, where the order is $\alpha \leq \alpha'$ if α' extends α. By Zorn's lemma this poset has a maximal element $\psi : \mathscr{F}' \to \mathscr{I}$ and so we just need to show that $\mathscr{F}' = \mathscr{G}$.

$$\begin{tikzcd} \mathscr{F} \arrow[r, hook] \arrow[dr] & \mathscr{F}' \arrow[r, hook] \arrow[d] & \mathscr{G} \\ & \mathscr{I} & \end{tikzcd}$$

Suppose that there is an open set U and a section $s \in \mathscr{G}(U)$ that is not in $\mathscr{F}'(U)$. This defines a morphism $\mathbb{Z}_U \to \mathscr{G}$ and the inclusion $\mathscr{F}' \to \mathscr{G}$ defines a subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$. Let $\mathscr{F}''$ be the subsheaf of $\mathscr{G}$ generated by $\mathscr{F}'$ and s. Then we can extend ψ to $\mathscr{F}''$ and so $\mathscr{F}' = \mathscr{G}$. Hence $\mathscr{I}$ is injective.

Step 2. Secondly, we show that any such subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$ is finitely generated. Let $U = \coprod U_i$ be a decomposition of U into its connected components U_i. Since X is noetherian, the ascending chain $U_1 \subset U_1 \cup U_2 \subset \dots$ stabilizes (Exercise I.1.7(a)), say at n. So U is a finite union of connected open subsets. For each i we have subgroups $\mathscr{R}(U_i) \subset \mathbb{Z}_U(U_i) = \mathbb{Z}$, say that these are generated by $s_i \in \mathbb{Z}_U(U_i)$. Then these finitely many s_i generate $\mathscr{R}$.

Step 3. Let $s_i \in \mathscr{R}(U_i)$ be generating elements of $\mathscr{R}$ where $i = 0, \dots, n$. For any map $\mathscr{R} \to \varinjlim \mathscr{I}_\alpha$, the image of s_i is represented by some $t_i \in \mathscr{I}_{\alpha_i}(U_i)$ for some α_i. Due to the system being direct, there is an index β so that the image of s_i can be represented by $t_i' \in \mathscr{I}_\beta(U_i)$. Hence, the morphism factors as $\mathscr{R} \to \mathscr{I}_\beta \to \varinjlim \mathscr{I}_\alpha$. Now use the first part. For every open subset $U \subseteq X$, and subsheaf $\mathscr{R} \subseteq \mathbb{Z}_U$, and map $f : \mathscr{R} \to \varinjlim \mathscr{I}_\alpha$ the map f factors through some $f_\beta : \mathscr{R} \to \mathscr{I}_\beta$. Since $\mathscr{I}_\beta$ is injective, f_β extends to a map $\mathbb{Z}_U \to \mathscr{I}_\beta$ and so we get an extension $\mathbb{Z}_U \to \mathscr{I}_\beta \to \varinjlim \mathscr{I}_\alpha$ of f. Hence, by Step 1, $\varinjlim \mathscr{I}_\alpha$ is injective.

Exercise 2.7. *Let S^1 be the circle (with its usual topology), and let $\mathbb{Z}$ be the constant sheaf $\mathbb{Z}$.*

a Show that $H^1(S^1, \mathbb{Z}) \cong \mathbb{Z}$, using sheaf cohomology.

b Now let $\mathscr{R}$ be the sheaf of germs of continuous real-valued functions on S^1. Show that $H^1(S^1, \mathscr{R}) = 0$.

3 Cohomology of a Noetherian Affine Scheme

Exercise 3.1. *Show that a noetherian scheme X is affine if and only if X_{red} is affine.*

Solution. If X is affine then $X_{red} = \operatorname{Spec}(A/N)$ where $A = \Gamma(X, \mathcal{O}_X)$ and N is the nilradical of A.

Conversely, suppose that X_{red} is affine. We want to show that X is affine by using Theorem 3.7 and induction on the dimension of X. If X has dimension 0 then affineness follows from the noetherian hypothesis since it must have finitely many points and each of these is contained in an affine neighbourhood. So suppose that result is true for noetherian schemes of dimension $< n$, and that $\dim X = n$. Let $\mathscr{N}$ be the sheaf of nilpotents on X and consider a coherent sheaf $\mathscr{F}$. For every integer i we have a short exact sequence

$$0 \to \mathscr{N}^{d+1} \cdot \mathscr{F} \to \mathscr{N}^d \cdot \mathscr{F} \to \mathscr{G}_d \to 0$$

where $\mathscr{G}_d$ is the appropriate quotient. This short exact sequence gives rise to a long exact sequence in cohomology:

$$\dots \to H^0(X, \mathscr{G}_d) \to H^1(X, \mathscr{N}^{d+1} \cdot \mathscr{F}) \to H^1(X, \mathscr{N}^d \cdot \mathscr{F}) \to H^1(X, \mathscr{G}_d) \to \dots$$

Since X is noetherian, there is some m for which $\mathscr{N}^d = 0$ for all $d \geq m$, so if we can show that $H^1(X, \mathscr{G}_d)$ is zero for each d, then the statement $H^1(X, \mathscr{F}) = 0$ will follow by induction and the long exact sequence above.

So the sheaf $\mathscr{G}_d = \mathscr{N}^d \cdot \mathscr{F} / \mathscr{N}^{d+1} \cdot \mathscr{F}$ on X. Recall that X_{red} has the same underlying topological space as X, but with sheaf of rings $\mathcal{O}_{X_{red}} = \mathcal{O}_X / \mathscr{N}$. So $\mathscr{G}_d$ is also a sheaf of $\mathcal{O}_{X_{red}}$-modules. Since cohomology is defined as cohomology of sheaves of abelian groups we have $H^1(X, \mathscr{G}_d) = H^1(X_{red}, \mathscr{G}_d)$ and so it follows from Theorem 3.7 that $H^1(X, \mathscr{G}_d) = 0$.

Exercise 3.2. *Let X be a reduced noetherian scheme. Show that X is affine if and only if each irreducible component is affine.*

Solution. If X is affine then every closed subscheme is affine (Exercise II.3.11(b)) and so every irreducible component is affine.

Conversely, suppose that each irreducible component is affine. Let Y_1, Y_2 be two closed subschemes of X and consider the coherent sheaves of ideals $\mathscr{I}_{Y_1}, \mathscr{I}_{Y_1 \cup Y_2}$. We have an exact sequence

$$0 \to \mathscr{I}_{Y_1 \cup Y_2} \to \mathscr{I}_{Y_1} \to \mathscr{F} \to 0$$

and it can be seen (reduced to the affine case) that $\mathscr{F} = i_* \mathscr{I}_{Y_1 \cap Y_2}$ where $i : Y_2 \to X$ is the closed imbedding. Let $Y = Y_1$ be an arbitrary closed subscheme. If $Z = Y_2$ is one of the irreducible components, it is affine and so $H^1(X, i_* \mathscr{I}_{Y \cap Z}) = H^1(Z, \mathscr{I}_{Y \cap Z}) = 0$. Hence, from the long exact sequence associated to the cohomology of the short exact sequence above, we see that $H^1(X, \mathscr{I}_{Y \cup Z}) \to H^1(X, \mathscr{I}_Y)$ is surjective.

Now let $Z_1, \ldots, Z_n$ be the irreducible components of X. By induction we see that $H^1(X, \mathscr{I}_{Y \cup Z_1 \cup \cdots \cup Z_n}) \to H^1(X, \mathscr{I}_Y)$ is surjectve. But $Y \cup Z_1 \cup \cdots \cup Z_n = X$ and $\mathscr{I}_X = 0$ since X is reduced. Hence, $H^1(X, \mathscr{I}_Y)$ is zero and so it follows from Theorem III.3.7 that X is affine.

Exercise 3.3. *Let A be a noetherian ring and let $\mathfrak{a}$ be an ideal of A.*

a *Show that $\Gamma_{\mathfrak{a}}(\cdot)$ is a left-exact functor from the category of A-modules to itself.*

b *Now let $X = \operatorname{Spec} A$, $Y = V(\mathfrak{a})$. Show that for any A-module M,*

$$H^i_{\mathfrak{a}}(M) = H^i_Y(X, \widetilde{M})$$

c *For any i, show that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) = H^i_{\mathfrak{a}}(M)$.*

Solution. a Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of A-modules. Since $\Gamma_a(N) \subset N$ for any A-module, we know that

$$0 \to \Gamma_{\mathfrak{a}}(M') \to \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(M'')$$

is exact on the left, and the composition of the two rightmost morphisms is zero. So the only thing to show is that if m is in the kernel of $\Gamma_{\mathfrak{a}}(M) \to$

$\Gamma_{\mathfrak{a}}(M'')$, then it is in the image of $\Gamma_{\mathfrak{a}}(M') \to \Gamma_{\mathfrak{a}}(M)$. By exactness of the original short exact sequence we know that there is a unique $n \in M'$ which gets sent to m. Since $m \in \Gamma_{\mathfrak{a}}(M)$ there is some i for which $\mathfrak{a}^i m = 0$. But $M' \to M$ is injective, and so $\mathfrak{a}^i n = 0$ and so $n \in \Gamma_{\mathfrak{a}}(M')$.

b Let $0 \to M \to I^\bullet$ be an injective resolution for M in the category of A-modules. Then we have an exact sequence of sheaves $0 \to \widetilde{M} \to \widetilde{I^\bullet}$ on X. Each $\widetilde{I^i}$ is flasque by (3.4) so we can use this resolution of $\widetilde{M}$ to calculate $H^i_Y(X, \widetilde{M})$ (Exercise III.2.3(c) and Proposition III.1.2A). The only thing left to show is that $\Gamma_{\mathfrak{a}}(\cdot) = \Gamma_Y(X, \widetilde{\cdot})$.

Consider $m \in \Gamma_{\mathfrak{a}}(M)$ for some arbitrary A-module M. Then by definition there is some n for which $\mathfrak{a}^n m = 0$. Let $\mathfrak{p}$ be a point of X not contained in Y. So $\mathfrak{p}$ doesn't contain $\mathfrak{a}$ and there is some $a \in \mathfrak{a}$ which is not in $\mathfrak{p}$. Then a^n is also not in $\mathfrak{p}$. Since $\mathfrak{a}^n m = 0$ we see that $a^n m = 0$ and so $m = 0$ in the localized module $M_{\mathfrak{p}}$. Hence, m is a global section of $\widetilde{M}$ with support in $Y = V(\mathfrak{a})$.

Conversely, let m be a global section of $\widetilde{M}$ with support in $Y = V(\mathfrak{a})$. So *Supp* $m \subseteq V(\mathfrak{a})$. By Exercise II.5.6(a) we have *Supp* $m = V(Ann\ m)$ and so $\sqrt{Ann\ m} \supseteq \sqrt{\mathfrak{a}}$ (Lemma II.2.1(c)). Since A is noetherian, $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (f_1, \dots, f_n)$. For each i, there is n_i such that $f_i^{n_i} \in \sqrt{Ann\ m}$, and so there is some j_i such that $f_i^{n_i j_i} \in Ann\ m$. Set $N = \prod n_i j_i$ so that $f_i^N \in Ann\ m$ for all i. Then, there is some $N'(= nN)$ such that $\sum_{i=1}^n k_i \geq N' \Rightarrow k_i \geq N$ for some i where $k_i \geq 0$. So every element of $\mathfrak{a}^{N'}$ is a sum of elements which are divisible by f_i^N for some i. Hence, $\mathfrak{a}^{N'} \subseteq Ann\ m$, so $m \in \Gamma_{\mathfrak{a}}(M)$.

c By definition we know that $\Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M)) \subseteq H^i_{\mathfrak{a}}(M)$. Consider $m \in H^i_{\mathfrak{a}}(M)$. So we have taken an injective resolution $0 \to M \to I^\bullet$ of M, we have $\dots \to \Gamma_{\mathfrak{a}}(I^{i-1}) \xrightarrow{d^i} \Gamma_{\mathfrak{a}}(I^i) \xrightarrow{d^{i+1}} \Gamma_{\mathfrak{a}}(I^{i+1}) \to \dots$ and m is an element of $\frac{\ker d^{i+1}}{\operatorname{im} d^i}$. In particular, it is represented by an element of $\ker d^{i+1} \subset \Gamma_{\mathfrak{a}}(I^i)$ and so there is some n for which $\mathfrak{a}^n m = 0$. Hence $H^i_{\mathfrak{a}}(M) \subseteq \Gamma_{\mathfrak{a}}(H^i_{\mathfrak{a}}(M))$.

Exercise 3.4. *a Assume that A is noetherian, Show that if* $\operatorname{depth}_{\mathfrak{a}} M \geq 1$, *then $\Gamma_{\mathfrak{a}}(M) = 0$, and the converse is true if M is finitely generated.*

b Show inductively, for M finitely generated, that for any $n \geq 0$, the following conditions are equivalent:

(a) $\operatorname{depth}_{\mathfrak{a}} M \geq n$;

(b) $H^i_{\mathfrak{a}}(M) = 0$ *for all $i < n$.*

For the converse to part (a) we use some Commutative Algebra results that can be found in Section 3.1 of Eisenbud's "Commutative Algebra".

Solution. a If $\operatorname{depth}_{\mathfrak{a}} M \geq 1$ then there is some $x \in \mathfrak{a}$ which is not a zero divisor of M. Let $m \in \Gamma_{\mathfrak{a}}(M)$. Then there is some n for which $\mathfrak{a}^n m = 0$, and so $x^n m = 0$. But x is not a zero divisor and so $m = 0$.

Conversely, suppose that M is finitely generated and that $\Gamma_{\mathfrak{a}}(M) = 0$. So for any (nonzero) $m \in M$ and $n \geq 0$ there is an $x \in \mathfrak{a}^n$ such that $xm \neq 0$. This means that $\mathfrak{a} \not\subseteq \mathfrak{p}$ for any associated prime $\mathfrak{p}$ of M (i.e. primes $\mathfrak{p}$ such that $\mathfrak{p} = Ann(m)$ for some $m \in M$). So $\mathfrak{a} \not\subseteq \cup_{\mathfrak{p} \in Ass(M)} \mathfrak{p}$ [Eisenbud, Lemma 3.3, Theorem 3.1(a)]. The latter set is the set of zero divisors of M (including zero) [Eisenbud, Theorem 3.1(b)] and so we find that there is an element $x \in \mathfrak{a}$ that is not a zero divisor in M. Hence $\operatorname{depth}_{\mathfrak{a}} M \geq 1$.

b

Exercise 3.5. *Let X be a noetherian scheme, and let P be a closed point of X. Show that the following conditions are equivalent:*

a *depth $\mathcal{O}_P \geq 2$;*

b *If U is any open neighbourhood of P, then every section of $\mathcal{O}_X$ over $U - P$ extends uniquely to a section of $\mathcal{O}_X$ over U.*

Solution. First note that we can assume U is affine, since given a point P and an open subscheme containing it, there is an open affine subscheme V of U containing P, and a section of $\mathcal{O}_X(U)$ is the same as giving a section of $\mathcal{O}_X(V)$ and a section of $\mathcal{O}_X(U - P)$ which agree on $V - P$ since $\mathcal{O}_X$ is a sheaf. So suppose that $U = X = \operatorname{Spec} A$ is an affine noetherian scheme.

Secondly, note that we have the long exact sequence of Exercise III.2.3(e):

$$\cdots \to H^0_P(X, \mathcal{O}_X) \to H^0(U, \mathcal{O}_X) \to H^0(U - P, \mathcal{O}_X) \to H^1_P(X, \mathcal{O}_X) \to \ldots$$

So the second statement in the problem is equivalent to showing that $H^i_P(X, \mathcal{O}_X) = 0$ for $i = 0, 1$. By Exercise III.3.3 this is the same as showing that $H^i_{\mathfrak{p}}(A) = 0$ for $i = 0, 1$ where $\mathfrak{p}$ is the prime ideal of A corresponding to the point P. Furthermore, by Exercise III.3.4(b) this is the same as showing that $\operatorname{depth}_{\mathfrak{p}} A \geq 2$ since A noetherian implies that A is finitely generated.

So we have reduced the problem to showing that $\operatorname{depth}_{\mathfrak{p}} A \geq 2$ if and only if $\operatorname{depth} A_{\mathfrak{p}} \geq 2$. If $\operatorname{depth}_{\mathfrak{p}} A \geq 2$ then there are $x_1, x_2 \in \mathfrak{p}$ such that x_1 is not a zero divisor of A and x_2 is not a zero divisor of A/x_2. We can consider the x_i as elements of $A_{\mathfrak{p}}$ and so we get a regular sequence of length 2 of $A_{\mathfrak{p}}$. Conversely, if $\operatorname{depth} A_{\mathfrak{p}} \geq 2$ then there is a regular sequence $\frac{x_1}{s_1}, \frac{x_2}{s_2} \in \mathfrak{p}A_{\mathfrak{p}}$ of $A_{\mathfrak{p}}$ where $x_1, x_2 \in \mathfrak{p}$ and $s_1, s_2 \in A \backslash \mathfrak{p}$. It can be seen that x_1, x_2 is then a regular sequence for A and so $\operatorname{depth}_{\mathfrak{p}} A \geq 2$.

Exercise 3.6. *Let X be a noetherian scheme.*

a *Show that the sheaf $\mathscr{G}$ constructed in the proof of (3.6) is an injective object in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X. Thus $\mathfrak{Qco}(X)$ has enough injectives.*

b *Show that any injective object of $\mathfrak{Qco}(X)$ is flasque.*

c *Conclude that one can compute cohomology as the derived functors of $\Gamma(X, \cdot)$, considered as a functor from $\mathfrak{Qco}(X)$ to $\mathfrak{Ab}$.*

Solution. a Recall that the sheaf $\mathscr{G}$ is constructed as follows. Cover X with a finite number of open affines $U_i = \operatorname{Spec} A_i$, and let $\mathscr{F}|_{U_i} = \widetilde{M_i}$. Embed M_i in an injective A_i-module I_i. For each i let $f_i : U_i \to X$ be the inclusion, and let $\mathscr{G} = \oplus f_{i*}(\widetilde{I_i})$. Now suppose we have an inclusion of quasi-coherent sheaves $0 \to \mathscr{F}' \to \mathscr{F}$ and a morphism $\mathscr{F}' \to \mathscr{G}$. We want to show that this lifts to a morphism $\mathscr{F} \to \mathscr{G}$.

First notice that for each i, a morphism $\mathscr{F}' \to f_{i*}\widetilde{I}_i$ corresponds to a morphism $\mathscr{F}'|_{U_i} \to \widetilde{I}_i$ which lifts to $\mathscr{F}|_{U_i} \to \widetilde{I}_i$ (since I_i is injective), and this corresponds to a morphism $\mathscr{F} \to f_{i*}\widetilde{I}_i$. So each $f_{i*}\widetilde{I}_i$ is injective. Now notice that a direct sum $\oplus_{i=1}^n \mathscr{G}'_i$ of arbitrary injective objects $\mathscr{G}'_i$ is injective, since a morphism to $\oplus_{i=1}^n \mathscr{G}'_i$ is the same as an n-tuple of one morphism into each G'_i. Hence, $\mathscr{G}$ is injective, since it is a direct sum of injectives.

b By definition,the derived functors are calculated using injective resolutions. We have seen that the cohomology of a sheaf of abelian groups as it was defined in the text can be calculated using flasque resolutions. Hence, the derived functors of $\Gamma(X, \cdot)$ are the same as the cohomology groups $H^i(X, -)$.

Exercise 3.7. *Let A be a noetherian ring, let $X = \operatorname{Spec} A$, let $\mathfrak{a} \subseteq A$ be an ideal, and let $U \subseteq X$ be the open set $X - V(\mathfrak{a})$.*

a For any A-module M, establish the following formula of Deligne:

$$\Gamma(U, \widetilde{M}) \cong \varinjlim_n \hom_A(\mathfrak{a}^n, M),$$

b Apply this in the case of an injective A-module I, to give another proof of (3.4).

Remark. A more general version of this is proved using a similar method in EGA I 6.9.17.

Solution. a First we define a morphism $\varinjlim_n \hom_A(\mathfrak{a}, M) \to \Gamma(U, \widetilde{M})$. Since A is noetherian, $\mathfrak{a}$ is finitely generated, say $\mathfrak{a} = (f_1, \dots, f_n)$. Furthermore, the basic opens $D(f_i)$ form a cover of U. This means that every section of $\Gamma(U, \widetilde{M})$ can be written as an element of $\oplus M_{f_i}$, and conversely, every element of $\oplus M_{f_i}$ which is in the kernel of $\oplus M_{f_i} \to \oplus M_{f_i f_j}$ defines a section of $\Gamma(U, \widetilde{M})$. So given a morphism $\phi : \mathfrak{a}^r \to M$ define a section by $\left(\frac{\phi(f_1^r)}{f_1^r}, \dots, \frac{\phi(f_n^r)}{f_n^r}\right)$. It can be checked fairly readily that this tuple actually does define a section (since $f_j^r \phi(f_i^r) - f_i^r \phi(f_j^r) = \phi(f_i^r f_j^r) - \phi(f_i^r f_j^r) = 0$) and furthermore, two representatives $\phi : \mathfrak{a}^r \to M$ and $\phi' : \mathfrak{a}^{r'} \to M$ of the same element of $\varinjlim_n \hom_A(\mathfrak{a}^r, M)$ give rise to the same section (since $\frac{\phi(f_i^{r+s})}{f_i^{r+s}} = \frac{f^s \phi(f_i^r)}{f_i^s f_i^r} = \frac{\phi(f_i^r)}{f_i^r}$). So we have a well-defined morphism

$$\varinjlim_n \hom_A(\mathfrak{a}^r, M) \to \Gamma(U, \widetilde{M})$$

This morphism is injective: if an element represented by $\phi : \mathfrak{a}^r \to M$ gets sent to zero, then $\frac{\phi(f_i^r)}{f_i^r} = 0 \in M_{f_i}$ for each i and so $f_i^{s_i}\phi(f_i^r) = 0 \in M$ for some s_i. Since there are finitely many f_i choose some $s > s_i$ so that we have $f_i^s\phi(f_i^r) = 0 \in M$ for all i. We then consider R big enough so that $\mathfrak{a}^R$ is generated by f_i^{s+r} (for example, $R > n(s+r)$) and the induced morphism $\phi : \mathfrak{a}^R \to M$ is consequently zero, since $\phi(f_i^{s+r}) = f_i^s\phi(f_i^r) = 0$.

Now to see that the morphism is surjective. Choose a section $f \in \Gamma(U, \widetilde{M})$. As already mentioned, this section gives rise to a tuple $(\frac{m_1}{f_1^{r_1}}, \dots, \frac{m_n}{f_n^{r_n}})$. By replacing $\frac{m_i}{f_i^{r_i}}$ with $\frac{f_i^{r-r_i}m_i}{f_i^r}$ where $r = \max r_i$ we can assume that all the r_i are the same. Since the tuple $(\frac{m_1}{f_1^{r_1}}, \dots, \frac{m_n}{f_n^{r_n}})$ came from a section $f \in \Gamma(U, \widetilde{M})$, for each i, j we have $(f_if_j)^{s_{ij}}(f_im_j - f_jm_i) = 0 \in M$ for some s_{ij}. Again, we can choose s big enough so that we can assume $s_{ij} = s$ for all i, j. Now define $m_i' = f_i^s m_i$. So we have $(f_i^{r+s}f - m_i')|_{D(f_j)} = (f_i^{r+s}\frac{m_j}{f_j^r} - f_i^s m_i) = \frac{1}{f_j^r}(f_i^{r+s}m_j - f_i^s f_j^r m_i) = \frac{1}{f_j^{r+s}}(f_if_j)^s(f_i^r m_j - f_j^r m_i) = 0$. The point of this is that since the $D(f_j)$ cover U, and we have $(f_i^{r+s}f - m_i')|_{D(f_j)} = 0$ for each j, we now have the relation

$$f_i^{r+s}f = m_i'$$

on U for each i.

Now choose R big enough so that $\mathfrak{a}^R$ is generated by the f_i^{r+s} (for example $R > n(r+s)$) and define a morphism $\phi : \mathfrak{a}^R \to M$ by sending $\sum a_i f_i^{r+s}$ to $(\sum a_i m_i')|_U$ (note that $a_i \in A$ are global sections of $\Gamma(X, \mathcal{O}_X)$). We need to check that this is a well defined homorphism. Suppose that $\sum a_i f_i^{r+s} = 0$. Then we need $(\sum a_i m_i')|_U$ to be zero also. But we have $(\sum a_i m_i')|_U = \sum(a_i f_i^{r+s} f) = (\sum a_i f_i^{r+s})f = 0$ and so we really do have a well defined morphism. Moreover, the image of the morphism ϕ in $\Gamma(U, \widetilde{M})$ is $(\frac{m_1'}{f_1^{r+s}}, \dots, \frac{m_n'}{f_n^{r+s}}) = (\frac{f_1^s m_1}{f_1^{r+s}}, \dots, \frac{f_n^s m_n}{f_n^{r+s}}) = (\frac{m_1}{f_1^r}, \dots, \frac{m_n}{f_n^r}) = f$, the section we started with. So we have lifted $f \in \Gamma(U, \widetilde{M})$ to an element of $\varinjlim_n \hom_A(\mathfrak{a}, M)$ and consequently, the morphism $\varinjlim_n \hom_A(\mathfrak{a}, M) \to \Gamma(U, \widetilde{M})$ is surjective.

b Suppose I is an injective A-module. We want to show that for any two open subsets $V \subseteq U$, the restriction morphism $\Gamma(U, \widetilde{I}) \to \Gamma(V, \widetilde{I})$ is surjective. Using Deligne's formula, we can write the restriction as

$$\varinjlim_n \hom_A(\mathfrak{a}^n, I) \to \varinjlim_n \hom_A(\mathfrak{b}, I)$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are the (radical) ideals of the closed complements of U and V respectively. Since $V \subseteq U$, we have $V(\mathfrak{b}) \supseteq V(\mathfrak{a})$ and since we assumed $\mathfrak{a}$ and $\mathfrak{b}$ to be radical this implies $\mathfrak{b} \subseteq \mathfrak{a}$. The point is that this is an inclusion of A-modules, and so given a representative $\phi : \mathfrak{b}^n \to I$ is an element of $\varinjlim_n \hom_A(\mathfrak{b}, I)$, the fact that I is injective implies that there is a lifting to $\mathfrak{a}^n \to I$ Since $0 \to \mathfrak{b}^n \to \mathfrak{a}^n$ is an exact sequence of A-modules. Hence, the restriction homomorphism is surjective and so $\widetilde{I}$ is flasque.

Exercise 3.8. *Let $A = k[x_0, x_1, x_2, \dots]$ with the relations $x_0^n x_n = 0$ for $n = 1, 2, \dots$. Let I be an injective A-module containing A. Show that $I \to I_{x_0}$ is not surjective.*

Solution. Suppose that $I \to I_{x_0}$ is surjective. Then there is some $m \in I$ which gets sent to $\frac{1}{x_0}$. That is, $x_0^n(x_0 m - 1) = 0$ in I for some n. Multiplying by x_{n+1} and using the relation $x_{n+1}x_0^{n+1} = 0$ gives $x_0^n x_{n+1} = 0$ in A. But this is not true, and so we have a contradiction. Hence $I \to I_{x_0}$ is not surjective.

4 Čech Cohomology

Exercise 4.1. *Let $f : X \to Y$ be an affine morphism of noetherian separated schemes. Show that for any quasi-coherent sheaf $\mathscr{F}$ on X, there are natural isomorphisms for all $i \geq 0$*

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

Solution. Let $\{V_i\}$ be an open affine cover of Y. Since f is an affine morphism the set of preimages $\{U_i = f^{-1}V_i\}$ form an open affine cover of X. Furthermore, since X and Y are separated, the intersections $V_{i_0,\dots,i_p}$ and $U_{i_0,\dots,i_p} = f^{-1}V_{i_0,\dots,i_p}$ are also affine. Let $U_{i_0,\dots,i_p} = \operatorname{Spec} A_{i_0,\dots,i_p}$ and $V_{i_0,\dots,i_p} = \operatorname{Spec} B_{i_0,\dots,i_p}$. As $\mathscr{F}$ is quasi-coherent its restrictions to each $U_{i_0,\dots,i_p}$ are of the form $\mathscr{F}|_{U_{i_0,\dots,i_p}} \cong \widetilde{M}_{i_0,\dots,i_p}$ where $M_{i_0,\dots,i_p}$ is an $A_{i_0,\dots,i_p}$-module. Proposition II.5.2(d) says that $f_*\mathscr{F}|_{V_{i_0,\dots,i_p}} = ({}_{B_{i_0,\dots,i_p}}M_{i_0,\dots,i_p})^\sim$.

Now consider the appropriate Čech complexes. In degree p we have

$$C^p(\mathfrak{U}, \mathscr{F}) = \prod_{i_0 < \dots < i_p} M_{i_0,\dots,i_p} \qquad \text{and} \qquad C^p(\mathfrak{V}, f_*\mathscr{F}) = \prod_{i_0 < \dots < i_p} {}_{B_{i_0,\dots,i_p}}M_{i_0,\dots,i_p}$$

As complexes of abelian groups, these are identical and so their cohomology groups are the same. Since we can use Čech complexes of affine covers to compute the cohomology of quasi-coherent sheaves (Theorem III.4.5) we find the natural isomorphisms required.

Exercise 4.2. *Prove Chevalley's theorem: Let $f : X \to Y$ be a finite surjective morphism of noetherian separated schemes, with X affine. Then Y is affine.*

a *Let $f : X \to Y$ be a finite surjective morphism of integral noetherian schemes. Show that there is a coherent sheaf $\mathscr{M}$ on X, and a morphism of sheaves $\alpha : \mathcal{O}_Y^r \to f_*\mathscr{M}$ for some $r > 0$, such that α is an isomrophism at the generic point of Y.*

b *For any coherent sheaf $\mathscr{F}$ on Y, show that there is a coherent sheaf $\mathscr{G}$ on X, and a morphism $\beta : f_*\mathscr{G} \to \mathscr{F}^r$ which is an isomorphism at the generic point of Y.*

c *Now prove Chevalley's theorem. First use Exercise III.3.1 and Exercise III.3.2 to reduce to the case X and Y integral. Then use Theorem 3.7, Exercise 4.1, consider* $\ker\beta$ *and* $\operatorname{coker}\beta$*, and use noetherian induction on* Y.

Solution. a If we apply $\mathscr{H}om(\cdot, \mathscr{F})$ to α we get a morphism $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \to \mathscr{H}om(\mathcal{O}_Y^r, \mathscr{F})$ which is an isomorphism at the generic point (to see this consider an affine neighbourhood of the generic point). We have an isomorphism $\mathscr{H}om(\mathcal{O}_Y^r, \mathscr{F}) \cong \mathscr{F}$ and by Exercise II.5.17, since $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F})$ is a quasi-coherent $f_*\mathcal{O}_X$-module, there is a quasi-coherent $\mathcal{O}_X$-module $\mathscr{G}$ such that $\mathscr{H}om(f_*\mathscr{M}, \mathscr{F}) \cong f_*\mathscr{G}$.

b The morphism $f : X \to Y$ induces a morphism $f_{red} : X_{red} \to Y_{red}$ which is still surjective (since the underlying topological spaces are the same) and still finite (since if a B-algebra A is finitely generated as a B-module then B_{red} is finitely generated as a A_{red}-module). Exercise III.3.1 says that Y_{red} is affine if and only if Y is and so we can assume that X and Y are reduced.

Now for each connected component Y' of Y, the induced morphism $f^{-1}Y' \to Y'$ is still surjective, and finite because the other three morphisms in the commutative square

$$\begin{array}{ccc} f^{-1}Y' & \rightarrowtail & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

are finite (closed immersions are finite). Since it is surjective, we can consider an irreducible component X' of $f^{-1}Y'$ that contains a point in the preimage of the generic point of Y'. The induced morphism $X' \to Y'$ is finite, since it is a composition of finite morphisms $X' \to f^{-1}Y' \to Y'$ (closed immersions are finite). Now since X is affine, each irreducible component is (since a closed subscheme of an affine scheme is affine (Exercise II.3.11(b)) and Exercise III.3.2 says that Y is affine if and only if each irreducible component is. So we can assume X and Y irreducible.

So we can assume X, Y integral. Now we use Theorem III.3.7 to show that Y is affine. The goal is to show that for any coherent sheaf of ideals $\mathscr{I}$ we have $H^1(Y, \mathscr{I}) = 0$. So let $\mathscr{I}$ be a coherent sheaf of ideals on Y. Then by part (b) we have a coherent sheaf $\mathscr{G}$ on X and a morphism $\beta : f_*\mathscr{G} \to \mathscr{I}^r$ which is an isomorphism at the generic point. This gives an exact sequence $0 \to \ker\beta \to f_*\mathscr{G} \to \mathscr{I}^r \to \operatorname{coker}\beta \to 0$ which we break up into to short exact sequences

$$0 \to \ker\beta \to f_*\mathscr{G} \to \operatorname{im}\beta \to 0 \qquad 0 \to \operatorname{im}\beta \to \mathscr{I}^r \to \operatorname{coker}\beta \to 0$$

which give rise to long exact sequences on cohomology. Since $H^i(Y, f_*\mathscr{G}) = H^i(X, \mathscr{G})$ (Exercise III.4.1) and X is affine, we have $H^i(Y, f_*\mathscr{G}) = 0$ for all $i > 0$ (Theorem III.3.7) and so $H^i(Y, \operatorname{im}\beta) \cong H^{i+1}(Y, \ker\beta)$ for $i > 0$.

On the other hand, since β is an isomorphism at the generic point, both $\ker\beta$ and $\operatorname{coker}\beta$ are zero at the generic point, and therefore have support in some closed subscheme, necassarily of smaller dimension than Y. That is, we have $\ker\beta = i_* i^* \ker\beta$ where $i : Z \to X$ is the closed immersion of the support and similarly for $\operatorname{coker}\beta$. By the inductive hyopothesis and Exercise III.4.1, we then have that $H^i(Y, \ker\beta) = H^i(X, i^* \ker\beta) = 0$ for $i > 0$ and similarly, for $\operatorname{coker}\beta$. Putting this together with the isomorphism $H^i(Y, \operatorname{im}\beta) \cong H^{i+1}(Y, \ker\beta)$ described above, we see that $H^i(Y, \operatorname{im}\beta) = 0$ for $i > 0$ as well and so putting these into the long exact sequence associated to the short exact sequence $0 \to \operatorname{im}\beta \to \mathscr{I}^r \to \operatorname{coker}\beta \to 0$ we obtain finally $H^i(Y, \mathscr{I}^r) = H^i(Y, \mathscr{I})^r = 0$ for $i > 0$. Hence, Y is affine by Theorem III.3.7.

Exercise 4.3. *Let $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$, and let $U = X - \{(0, 0)\}$. Using a suitable cover of U by open affine subsets, show that $H^1(U, \mathcal{O}_U)$ is isomorphic to the k-vector space spanned by $\{x^i y^j | i, j < 0\}$. In particular, it is infinite dimensional.*

Solution. Take the open cover $\{U_x = \operatorname{Spec} k[x, y, x^{-1}], U_y = \operatorname{Spec} k[x, y, y^{-1}]\}$. The intersection is $U_{xy} = U_x \cap U_y = \operatorname{Spec} k[x, y, x^{-1}, y^{-1}]$ and so the Čech complex of this cover is

$$0 \to k[x, y, x^{-1}] \oplus k[x, y, y^{-1}] \to k[x, y, x^{-1}, y^{-1}] \to 0 \to \dots$$

The first cohomology group of this complex is $k[x, y, x^{-1}, y^{-1}]$ over the image of the boundary morphism. This image consists of all polynomials which are linear combinations of monomials $x^i y^j$ where at least one of i or j are not negative. Hence, the first cohomology group consists of linear combinations of monomials $x^i y^j$ with $i, j < 0$.

Exercise 4.4. a *Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open covering of the topological space X. If $\mathfrak{V}$ is a refinement of $\mathfrak{U}$ (that is, a covering $\mathfrak{V} = \{V_j\}_{j \in J}$ together with a map $\lambda : J \to I$ of index sets, such that for each $j \in J$, $V_j \subseteq U_{\lambda(j)}$), show that there is a natural induced map on the Čech cohomology, for any abelian sheaf $\mathscr{F}$, and for each i,*

$$\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$$

b *For any abelian sheaf $\mathscr{F}$ on X, show that the natural maps (4.4) for each covering $\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ are compatible with the refinement maps above.*

c *Now prove the following theorem. Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups. Then the natural map*

$$\varinjlim_{\mathfrak{U}} \check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$$

is an isomorphism.

Solution. For each $p \geq 0$ and each tuple $(j_0, \dots, j_p) \in J^{p+1}$ we have a morphism induced by the restriction morphisms $\mathscr{F}(U_{\lambda(j_0)\dots\lambda(j_p)}) \to \mathscr{F}(V_{j_0\dots j_p})$ which induces a morphism $C^p(\mathfrak{U}, \mathscr{F}) \to C^p(\mathfrak{V}, \mathscr{F})$. Since

$$
\begin{aligned}
(\lambda^{i+1} d\alpha)_{j_0\dots j_{p+1}} &= (d\alpha)_{\lambda(j_0)\dots\lambda(j_{p+1})}|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\dots\lambda(\hat{j}_k)\dots\lambda(j_{p+1})}|_{U_{\lambda(j_0)\dots\lambda(j_{p+1})}}\Big|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\dots\lambda(\hat{j}_k)\dots\lambda(j_{p+1})}|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k \alpha_{\lambda(j_0)\dots\lambda(\hat{j}_k)\dots\lambda(j_{p+1})}|_{V_{j_0\cdots\hat{j}_k\cdots j_{p+1}}}|_{V_{j_0\cdots j_{p+1}}} \\
&= \sum_{k=0}^{p+1} (-1)^k (\lambda^i \alpha)_{j_0\dots\hat{j}_k\dots j_{p+1}}|_{V_{j_0\cdots j_{p+1}}} \\
&= (d\lambda^i \alpha)_{j_0\dots j_{p+1}}
\end{aligned}
$$

we have commutative squares

$$
\begin{tikzcd}
C^p(\mathfrak{U}, \mathscr{F}) \arrow[r, "d"] \arrow[d, "\lambda^i"] & C^{p+1}(\mathfrak{U}, \mathscr{F}) \arrow[d, "\lambda^{i+1}"] \\
C^p(\mathfrak{V}, \mathscr{F}) \arrow[r, "d"] & C^{p+1}(\mathfrak{V}, \mathscr{F})
\end{tikzcd}
$$

and so we have a morphism of the Čech complexes $C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{V}, \mathscr{F})$ induced by the restriction morphisms and λ. This induces a morphism on the Čech cohomology.

The maps $\check{H}^i(\mathfrak{U}, \mathscr{F}) \to H^i(X, \mathscr{F})$ come from choosing an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of chain complexes and obtaining a map of chain complexes $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ unique up to homotopy. Our maps $\lambda^i : \check{H}^i(\mathfrak{U}, \mathscr{F}) \to \check{H}^i(\mathfrak{V}, \mathscr{F})$ from part (a) were induced by maps of chain complexes. Since the map $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ is unique up to homotopy, the map obtained as the composition $C^\bullet(\mathfrak{U}, \mathscr{F}) \to C^\bullet(\mathfrak{V}, \mathscr{F}) \to \mathscr{I}^\bullet$ is homotopic to $C^\bullet(\mathfrak{U}, \mathscr{F}) \to \mathscr{I}^\bullet$ and therefore induces the same maps on cohomology. Therefore we have a commutative triangle

$$
\begin{tikzcd}
\check{H}^i(\mathfrak{U}, \mathscr{F}) \arrow[r] \arrow[rr, bend left] & \check{H}^i(\mathfrak{V}, \mathscr{F}) \arrow[r] & H^i(X, \mathscr{F})
\end{tikzcd}
$$

Exercise 4.5. *Show that* $\operatorname{Pic} X \cong H^1(X, \mathcal{O}_X^*)$ *for any ringed space* $(X, \mathcal{O}_X)$.

Solution. The map $\operatorname{Pic} X \to H^1(X, \mathcal{O}_X^*)$. Let $\mathscr{L}$ be an invertible sheaf on X. That is, a sheaf that is locally frem of rank one. By definition there is an open cover $\{U_i\}$ of X for which we have isomorphisms $\phi_i : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$.

Restricting to the pairwise intersections we get isomorphisms $\phi_{ij} = \phi_j^{-1} \circ \phi_i : \mathcal{O}_{U_{ij}} \to \mathscr{L}|_{U_{ij}} \to \mathcal{O}_{U_{ij}}$ and on the triple intersections U_{ijk} the restriction of these isomorphisms satisfy the cocycle condition $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ where ϕ_{ii} is the identity. Each of the isomorphisms ϕ_{ij} is determined by an element $\alpha_{ij} \in \mathcal{O}_X(U_{ij})$ (the image of the identity) which is a unit by consequence of the ϕ_{ij}'s being isomorphisms. The cocycle condition amounts to the relation $\alpha_{ij}\alpha_{jk} = \alpha_{ik}$ and $\alpha_{ii} = 1$. So the elements $\{\alpha_{ij}\}$ determine an element of $C^1(\{U_i\}, \mathcal{O}_X^*)$ which is a cocycle as a consequence of the cocycle conditions, as

$$(d\alpha)_{ijk} = \alpha_{jk}\alpha_{ik}^{-1}\alpha_{ij} = 1$$

So we have defined a map, of sets at least, from $\operatorname{Pic} X$ to $H^1(X, \mathcal{O}_X^*)$ via the morphisms $\check{H}^i(\mathfrak{U}, \mathcal{O}_X^*) \to H^i(X, \mathcal{O}_X^*)$.

Independance with respect to the ϕ_i. If we have chosen different isomorphisms $\phi_i' : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathscr{L}|_{U_i}$ then we obtain isomorphisms $\psi_i = \phi_i^{-1} \circ \phi_i' : \mathcal{O}_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}$ which correspond to elements $\gamma_i \in \mathcal{O}_X(U_i)^*$ (the image of the identity global section) as above. if $\alpha' = \{\alpha'_{ij}\}$ is the cocycle associated with the isomorphisms ϕ_i' then we have the relations $\alpha_{ij}^{-1}\alpha'_{ij} = \gamma_j^{-1}\gamma_i$ by the commutivity of the following diagram:

$$\begin{tikzcd} \mathcal{O}_{U_{ij}} \arrow[r, "\phi_i'"] \arrow[d, "\psi_i"] & \mathscr{L}|_{U_i} \arrow[r, "\phi_j'^{-1}"] \arrow[d, equal] & \mathcal{O}_{U_{ij}} \arrow[d, "\psi_j"] \\ \mathcal{O}_{U_{ij}} \arrow[r, "\phi_i"] & \mathscr{L}|_{U_i} \arrow[r, "\phi_j^{-1}"] & \mathcal{O}_{U_{ij}} \end{tikzcd}$$

Hence, $\alpha^{-1}\alpha'$ is a coboundary and so α and α' determine the same element in $\check{H}^1(\mathfrak{U}, \mathcal{O}_X^*)$.

Compatibility with restriction (independence with respect to the cover). If $\mathscr{L}$, $\mathfrak{U}$ are as above and if $(\mathfrak{V}, \lambda)$ is a refinement of $\mathfrak{U}$ as in Exercise III.4.4, then by restricting a choice of isomorphism ϕ_i for $\mathfrak{U}$, we get isomorphisms for $\mathfrak{V}$ for which the corresponding cocycle $\beta = \{\beta_{k\ell}\}$ in $C^1(\mathfrak{V}, \mathcal{O}_X^*)$ is precisely the image of the cocycle $\alpha = \{\alpha_{ij}\}$ obtained from the ϕ_i under the morphism $C^1(\mathfrak{U}, \mathcal{O}_X^*) \to C^1(\mathfrak{V}, \mathcal{O}_X^*)$ described in Exercise III.4.4(a). So via Exercise III.4.4(b) we see that the image of $\mathscr{L}$ in $H^1(X, \mathcal{O}_X^*)$ is independent of the cover chosen.

Compatibilty with the group structure. If we have two invertible sheaves $\mathscr{L}$ and $\mathscr{M}$ then choose a cover $\mathfrak{U} = \{U_i\}$ on which both sheaves are trivial. Then $\mathscr{L} \otimes \mathscr{M}$ is trivial on this cover as well, and we can take the isomorphisms $\phi_i : \mathcal{O}_{U_i} \cong \mathscr{L} \otimes \mathscr{M}|_{U_i}$ to be $\phi_{i,\mathscr{L}} \otimes \phi_{i,\mathscr{L}}$ where $\phi_{i,\mathscr{L}}$ and $\phi_{i,\mathscr{L}}$ are isomorphisms for $\mathscr{L}$ and $\mathscr{M}$ respectively. It is now straightforward to see that the cocycle for $\mathscr{L} \otimes \mathscr{M}$ and is the product of that for $\mathscr{L}$ and that for $\mathscr{M}$, so the map $\operatorname{Pic} X \to H^1(X, \mathcal{O}_X)$ is actually a group homomorphism.

The map is an isomorphism. To see that the map defined is an isomorphism we construct an inverse via the isomorphism $\varinjlim_{\mathfrak{U}} \check{H}^1(\mathfrak{U}, \mathscr{F}) \to H^1(X, \mathscr{F})$ discussed in Exercise III.4.4(c). This isomorphism implies that *every* element of $H^1(X, \mathscr{F})$ can be realized as an element of $\check{H}^1(\mathfrak{U}, \mathscr{F})$ for some cover $\mathfrak{U}$. So given an element of $H^1(X, \mathcal{O}_X^*)$ there is a cover $\mathfrak{U}$ for which the element is represented by a cocycle $\{\alpha_{ij}\} \in C^1(\mathfrak{U}, \mathcal{O}_X^*)$. By virtue of the fact that $\{\alpha_{ij}\}$ is a cocycle, these α_{ij} define isomorphisms $\mathcal{O}_{U_i}|_{U_{ij}} \to \mathcal{O}_{U_j}|_{U_{ij}}$ which satisfy the necassary condition for us to be able to glue the $\mathcal{O}_{U_i}$ together into an invertible sheaf (Exercise II.1.22). By construction it can be seen that this provides an inverse.

Exercise 4.6. *Let $(X, \mathcal{O}_X)$ be a ringed space, let $\mathscr{I}$ be a sheaf of ideals with $\mathscr{I}^2 = 0$, and let X_0 be the ringed space $(X, \mathcal{O}_X/\mathscr{I})$. Show that there is an exact sequence of sheaves of abelian groups on X,*

$$\dots \to H^1(X, \mathscr{I}) \to \operatorname{Pic} X \to \operatorname{Pic} X_0 \to H^2(X, \mathscr{I}) \to \dots$$

Solution. Checking that the sequence is exact on stalks is fairly straightforward. As a consequence we have an exact sequence

$$\dots \to H^1(X, \mathscr{I}) \to H^1(X, \mathcal{O}_X^*) \to H^1(X, \mathcal{O}_{X_0}^*) \to H^2(X, \mathscr{I}) \to \dots$$

and the required exact sequence then follows from Exercise III.4.5 above.

Exercise 4.7. *Let X be a subscheme of $\mathbb{P}^2_k$ defined by a single homogeneous equation $f(x_0, x_1, x_2) = 0$ of degree d (without assuming that f is irreducible). Assume that $(1, 0, 0)$ is not on X. Then show that X can be covered by the two open affine subsets $U = X \cap \{x_1 \neq 0\}$ and $V = X \cap \{x_2 \neq 0\}$. Now calculate the Čech complex*

$$\Gamma(U, \mathcal{O}_X) \oplus \Gamma(V, \mathcal{O}_X) \to \Gamma(U \cap V, \mathcal{O}_X)$$

explicitely, and thus show that

$$\dim H^0(X, \mathcal{O}_X) = 1$$

$$\dim H^1(X, \mathcal{O}_X) = \frac{1}{2}(d-1)(d-2).$$

Solution. There is a standard cover of $\mathbb{P}^2$ consisting of the opens $U_0 = \{x_0 \neq 0\}, U_1 = \{x_1 \neq 0\}, U_2 = \{x_2 \neq 0\}$ and so $\{U_0 \cap X, U_1 \cap X, U_2 \cap X\}$ is an open cover of X. Since closed subschemes of affine schemes are affine, this is an affine cover. The only point of $\mathbb{P}^2$ not in U_1 or U_2 is $(1, 0, 0)$ and since this is not in X, the open affine $U_0 \cap X$ can be removed from the set and it will still be an open affine cover.

The Čech complex is then

$$\begin{array}{ccc} \dfrac{k[\frac{x_0}{x_1}, \frac{x_2}{x_1}]}{f(\frac{x_0}{x_1}, 1, \frac{x_2}{x_1})} \oplus \dfrac{k[\frac{x_0}{x_2}, \frac{x_1}{x_2}]}{f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)} & \to & \dfrac{k[\frac{x_0}{x_2}, \frac{x_1}{x_2}, \frac{x_2}{x_1}]}{f(\frac{x_0}{x_2}, \frac{x_1}{x_2}, 1)} \\ (\overline{g}(\dfrac{x_0}{x_1}, \dfrac{x_2}{x_1}), \overline{h}(\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2})) & \mapsto & \overline{g}(\dfrac{x_0}{x_2}\dfrac{x_2}{x_1}, \dfrac{x_2}{x_1}) - \overline{h}(\dfrac{x_0}{x_2}, \dfrac{x_1}{x_2}) \end{array}$$

which is written more legibly as

$$\frac{k[u, v]}{f(u, 1, v)} \oplus \frac{k[x, y]}{f(x, y, 1)} \quad \to \quad \frac{k[x, y, y^{-1}]}{f(x, y, 1)}$$

$$(\overline{g}(u,v), \overline{h}(x,y)) \quad \mapsto \quad \overline{g}(xy^{-1}, y^{-1}) - \overline{h}(x,y)$$

If $(\overline{g}, \overline{h})$ is in the kernel of this morphism then $g - h$ is in the ideal generated by $f(x,y,1)$. So $g - h = f'f$ for some $f' \in k[x,y,y^{-1}]$. Now the assumption that $(1,0,0)$ is not a point implies $f(x_0,x_1,x_2) = \widetilde{f} + a_0 x_0^d$ for some $\widetilde{f}$ and some nonzero a_0. Since scaling by units doesn't change the variety, we can assume that $a_0 = 1$. So we have $f(x,y,1) = \sum_{0 \leq i \leq d, 0 \leq j \leq d} a_{ij} x^i y^j$ with $a_{0d} = 1$. The polynomial f' in the expression $g - h = f'f$ is a linear combination of monomials. Write it as $f' = f_0 + f_1 + f_2$ where the f_k are linear combinations of monomials $x^i y^j$ with $i \leq -d - j$ for f_0, with $j \geq 0$ for f_1, and with $j < 0$ and $i > -d - j$ for f_2. The point is that $f_0 f$ is in the image of $\frac{k[u,v]}{f(u,1,v)}$ and $f_1 f$ is in the image of $\frac{k[x,y]}{f(x,y,1)}$ and the monomials spanning these images overlap only on the constant term. So if we can show that f_2 is necassarily zero, then we necassarily have $g = f_0 f + g_0$ and $h = -f_1 f + h_0$ where g_0 and h_0 are constants. So it will imply that $(\overline{g}, \overline{h})$ represents the same element as one where g and h are constant, and therefore equal.

To see that f_2 is necassarily zero consider a summand of it $a_{ij} x^i, y^j$ with i maximal and j minimal. Then $a_{ij} x^{i+d} y^j$ is a summand of $f_2 f$. but $f_2 f$ is in the image of the boundary map of the Čech complex so either $i + d \leq -j$ or $j \geq 0$, both of which contradict our assumptions on f_2. Hence, $(\overline{g}, \overline{h})$ represents the same element as one where g and h are constants, and therefore equal, so the kernel is (a,a) with $a \in k$ and therefore $\dim H^0(X, \mathcal{O}_X) = 1$.

Consider now the cokernel. Each element of the cokernel can be represented by a polynomial in $k[x,y,y^{-1}]$. Write it as a linear combination of monomials $\sum_{i \geq 0, j \in \mathbb{Z}} a_{ij} x^i y^j$. Any mononial with $j \geq 0$ represents zero in the cokernel as it is the image of $(0, x^i y^j))$. Similarly, any monomial with $j \geq i$ is the image of $(u^i v^{j-i}, 0)$. So we can represent an element of the cokernel with a polynomial $\sum_{j<0, j<i} a_{ij} x^i y^j$. Now the assumption that $(1,0,0)$ is not a point implies $f(x_0,x_1,x_2) = \widetilde{f} + a_0 x_0^d$ for some $\widetilde{f}$ and some nonzero a_0. Since scaling by units doesn't change the variety, we can assume that $a_0 = 1$. Hence, in the ring $\frac{k[x,y,y^{-1}]}{f(x,y,1)}$ we have the relation $x^d = -\widetilde{f}(x,y,1)$ where $\widetilde{f}(x,y,1)$ linear combination of monomials $x^i y^j$ with $0 \leq i < d$ and $0 \leq j$. Coming back to the cokernel, this means that every element of the cokernel can be represented by a polynomial of the form $\sum a_{ij} x^i y^j$ where $1 \leq i < d$ and $-i < j < 0$. So $\dim H^1(X, \mathcal{O}_X) \leq \frac{1}{2}(d-1)(d-2)$. To show that equality holds we need to show that polynomials of this form don't represent zero elements of the cokernel. Clearly, they are not in the image of the boundary map, by the argument already given, so we just need to show that they are not in the ideal generated by $f(x,y,1)$. But since $f(x,y,1) = x^d + \widetilde{f}(x,y,1)$ if they were, there would be a fact or of x with power $\geq d$. So we have equality.

Exercise 4.8. Cohomological Dimension. *Let X be a noetherian separated scheme.*

- a *In the definition of $cd(X)$ show that it is sufficient to consider only coherent sheaves on X.*

b *If X is quasi-projective over a field k, then it is even sufficient to consider only locally free coherent sheaves on X.*

c *Suppose X has a covering by $r+1$ open affine subsets. Use Čech cohomology to show that* $\mathrm{cd}(X) \leq r$.

d *If X is a quasi-projective scheme of dimension r over a field k, then X can be covered by $r+1$ open affine subsets. Conclude (independently of (2.7)) that* $\mathrm{cd}(X) \leq \dim X$.

e *Let Y be a set theoretic complete intersection of codimension r in $X = \mathbb{P}^n_k$. Show that* $\mathrm{cd}(X - Y) \leq r - 1$.

Solution. a Suppose that $H^i(X, \mathscr{F}) = 0$ for all $i > n$ and all coherent sheaves $\mathscr{F}$. If $\mathscr{F}$ is a quasi-coherent sheaf then it is the union of its coherent subsheaves (Exercise II.5.15(a)), that is, $\mathscr{F} = \varinjlim \mathscr{F}_\alpha$ where $\mathscr{F}_\alpha$ are the coherent subsheaves. Then by Proposition 2.9 for $i > n$ we have $H^i(X, \mathscr{F}) = H^i(X, \varinjlim \mathscr{F}_\alpha) \cong \varinjlim H^i(X, \mathscr{F}_\alpha) = \varinjlim 0 = 0$.

b By Proposition II.5.18 every coherent sheaf $\mathscr{F}$ can be written as a quotient of a finite rank locally free sheaf $\mathscr{E}$ so we have a short exact sequence $0 \to \mathscr{G} \to \mathscr{E} \to \mathscr{F} \to 0$ which gives rise to an exact sequence

$$\cdots \to H^i(X, \mathscr{E}) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{G}) \to H^{i+1}(X, \mathscr{E}) \to \dots$$

So if $H^i(X, \mathscr{E})$ for all locally free sheaves $\mathscr{E}$ and all $i > n$ then $H^i(X, \mathscr{F}) \cong H^{i+1}(X, \mathscr{G})$ for all $i > n$. Grothendieck's Theorem says that $H^i(X, \mathscr{G}) = 0$ for $i > \dim X$ so by induction, $H^i(X, \mathscr{F}) = 0$ for $i > n$.

c Since X is separated, we can use the Čech cohomology of an affine cover to calculate the cohomology of X. If there are only $r+1$ elements in the cover $\mathfrak{U}$ then for $p > r$ there are no p-tuples of indicies $(i_0, \dots, i_p)$ with $i_0 < \cdots < i_p$ and so $C^p(\mathfrak{U}, \mathscr{F}) = 0$ and hence $H^i(X, \mathscr{F}) = 0$ for $p > r$ and therefore $cd(X) \leq r$.

d By definition if Y is a set-theoretic complete intersection of codimension r then it is the intersection of r hypersurfaces. The complement of each of these hypersurfaces is an affine variety (Proposition II.2.5) and so these r complements form an affine cover of $X - Y$ which is separated by virtue of it being projective (Theorem 4.9). So it follows from part (c) of this exercise that $cd(X - Y) \leq r - 1$.

Exercise 4.9. *Let $X = \operatorname{Spec} k[x_1, x_2, x_3, x_4]$ be affine four-space over a field k, Let Y_1 be the plane $x_1 = x_2 = 0$ and let Y_2 be the plane $x_3 = x_4 = 0$. Show that $Y = Y_1 \cup Y_2$ is not a set-theoretic complete intersection in X. Therefore the projective closure $\overline{Y}$ in $\mathbb{P}^4_k$ is also not a set-theoretic complete intersection.*

Solution. If Y is a set theoretic complete intersection then $cd(X - Y) \leq 1$ (the same proof as for Exercise III.4.8(e) works). So to show that Y is not a complete intersection then we just need to show that $H^2(X-Y,\mathscr{F})$ for some quasi-coherent sheaf $\mathscr{F}$. Consider $\mathcal{O}_X$. We have the exact sequence from Exercise III.2.3:

$$\cdots \to H^2(X,\mathcal{O}_X) \to H^2(X-Y,\mathcal{O}_X) \to H^3_Y(X,\mathcal{O}_X) \to H^3(X,\mathcal{O}_X) \to \dots \quad (1)$$

Since X is affine, we have $H^i(X,\mathcal{O}_X)$ for $i>0$ and so $H^2(X-Y,\mathcal{O}_X) \to H^3_Y(X,\mathcal{O}_X)$ is an isomorphism. So our task is reduced to showing that $H^3_Y(X,\mathcal{O}_X) \neq 0$. Now consider the following exact sequence from Exercise III.2.4:

$$\begin{aligned} \dots H^3_P(X,\mathcal{O}_X) &\to H^3_{Y_1}(X,\mathcal{O}_X) \oplus H^3_{Y_2}(X,\mathcal{O}_X) \\ &\to H^3_Y(X,\mathcal{O}_X) \to H^4_P(X,\mathcal{O}_X) \to H^4_{Y_1}(X,\mathcal{O}_X) \oplus H^4_{Y_2}(X,\mathcal{O}_X) \to \dots \end{aligned} \quad (2)$$

Using a similar exact sequence to 1 we see that $H^i_{Y_j}(X,\mathcal{O}_X) \cong H^{i-1}(X-Y_j,\mathcal{O}_X)$ for $i=3,4$ and the later is zero since $X-Y_j$ is covered by the two open affines $x_{2j-1} \neq 0$ and $x_{2j} \neq 0$, and so the Čech complex is zero in these degrees (Exercise III.4.8(c)). Hence we have an isomorphism $H^3_Y(X,\mathcal{O}_X) \xrightarrow{\sim} H^4_P(X,\mathcal{O}_X)$ and so we want to show that $H^4_P(X,\mathcal{O}_X) \neq 0$.

Consider the exact sequence:

$$\cdots \to H^3(X,\mathcal{O}_X) \to H^3(X-P,\mathcal{O}_X) \to H^4_P(X,\mathcal{O}_X) \to H^4(X,\mathcal{O}_X) \to \dots \quad (3)$$

Since X is affine, we have $H^i(X,\mathcal{O}_X)$ for $i>0$ and so $H^3(X-P,\mathcal{O}_X) \to H^4_P(X,\mathcal{O}_X)$ is an isomorphism. Now we can calculate $H^3(X-P,\mathcal{O}_X)$ explicitely using the Čech complex of the cover $\mathfrak{U}$ consisting of the U_i with $x_i \neq 0$. We have $C^4(\mathfrak{U},\mathcal{O}_X)=0$ because there are four elements in the cover, so the cohomology group in question is the cokernel of $C^2(\mathfrak{U},\mathcal{O}_X) \to C^3(\mathfrak{U},\mathcal{O}_X)$. This morphism is

$$\bigoplus_{i=1}^4 A_i \to k[x_1,x_2,x_3,x_4,x_1^{-1},x_2^{-1},x_3^{-1},x_4^{-1}]$$

where A_i is $k[x_1,x_2,x_3,x_4]$ with x_j inverted for all $i \neq j$. The image of this morphism is spanned by all the monomials $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}$ such that at least one i_j is not negative. So the cokernel (and hence the cohomology) is spanned by monomials $x_1^{i_1}x_2^{i_2}x_3^{i_3}x_4^{i_4}$ with all $i_j<0$. In particular, it is not zero.

So $H^3(X-P,\mathcal{O}_X) \neq 0$ and therefore $H^4_P(X,\mathcal{O}_X) \neq 0$ by 3 and so $H^3_Y(X,\mathcal{O}_X) \neq 0$ by 2 and therefore $H^2(X-Y,\mathcal{O}_X) \neq 0$ by 1. Hence, $cd(X-Y)>1$ and so Y is not a set theoretic complete intersection.

Now if $\overline{Y}$ was a set theoretic complete intersection then we could restrict the two relavlent hypersurfaces to $\mathbb{A}^4$ and find that Y is a set theoretic complete intersection. But we have just proven that Y isn't, and so therefore, $\overline{Y}$ isn't either.

Exercise 4.10. *Let X be a nonsingular variety over an algebraically closed field k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a one-to-one correspondence between the set of infinitesimal extensions of X by $\mathscr{F}$ up to isomorphism, and the group $H^1(X,\mathscr{F}\otimes\mathscr{T})$, where $\mathscr{T}$ is the tangent sheaf of X.*

Exercise 4.11. *Let X be a topological space, $\mathscr{F}$ a sheaf of abelian groups, and $\mathfrak{U} = \{U_i\}$ an open cover. Assume for any finite intersection $V = U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets of the covering, and for any $k > 0$ that $H^k(V, \mathscr{F}|_V) = 0$. Then prove that for all $p \geq 0$, the natural maps*

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) \to H^p(X, \mathscr{F})$$

of (4.4) are isomorphisms. Show also that one can recover (4.5) as a corollary of this more generaly result.

Solution. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Consider the double complex $E_0^{p,q} = \prod_{i_0 < \cdots < i_p} \mathscr{I}^p(U_{i_0,\ldots,i_q})$. There are two spectral sequences associated to this double complex, one coming from the filtration of the total complex by columns and the other by rows.

Since for any open subset U and any i the sheaf $\mathscr{I}^i|_U$ is injective as a sheaf of abelian groups on U, the restriction $0 \to \mathscr{F}|_U \to \mathscr{I}^\bullet|_U$ is an injective resolution of $\mathscr{F}|_U$. So the "horizontal" cohomology groups $E_1^{p,q} \stackrel{def}{=} H^p(E_0^{\bullet,q})$ of this complex calculate the cohomology of $\mathscr{F}|_U$. By assumption, we then have

$$E_1^{p,q} = \begin{cases} C^q(\mathscr{F}, \mathfrak{U}) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

The "vertical" differentials $E_0^{0,q} \to E_0^{0,q+1}$ induce the usual differentials on the complex $C^q(\mathscr{F}, \mathfrak{U})$ and so the "vertical" cohomology groups of E_1 are

$$E_2^{p,q} \stackrel{def}{=} H^q(E_1^{p,\bullet}) = \begin{cases} \check{H}^q(\mathscr{F}, \mathfrak{U}) & \text{if } p = 0 \\ 0 & \text{otherwise} \end{cases}$$

Now suppose we start with the vertical differentials first. So we define ${}'E_1^{p,q} \stackrel{def}{=} H^q(E_0^{p,\bullet})$. These calculate the Čech cohology of the sheaves $\mathscr{I}^p$. Since the $\mathscr{I}$ are flasque (Lemma III.2.4), their Čech cohomology vanishes in nonzero degree and so we have

$${}'E_1^{p,q} = \begin{cases} \Gamma(X, \mathscr{I}^p) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

As above, the horizontal differentials induce the usual morphisms on the complex $\Gamma(X, \mathscr{I}^\bullet)$ and so we have

$${}'E_2^{p,q} \stackrel{def}{=} H^p({}'E_1^{\bullet}, q) = \begin{cases} H^q(X, \mathscr{F}) & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

So the cohomology of the total complex is isomorphic to both $H^\bullet(X, \mathscr{F})$ and $\check{H}^\bullet(\mathfrak{U}, \mathscr{F})$, hence, they are isomorphic.

5 The Cohomology of Projective Space

Exercise 5.1. *Let X be a projective scheme over a field k, and let $\mathscr{F}$ be a coherent sheaf on X. if*

$$0 \to \mathscr{F}' \xrightarrow{\phi} \mathscr{F} \xrightarrow{\psi} \mathscr{F}'' \to 0$$

is a short exact sequence of coherent sheaves on X, show that $\chi(\mathscr{F}) = \chi(\mathscr{F}') + \chi(\mathscr{F}'')$.

Solution. Consder the long exact sequence of cohomology

$$\dots \xrightarrow{\delta^{i-1}} H^i(X, \mathscr{F}') \xrightarrow{\phi^i} H^i(X, \mathscr{F}) \xrightarrow{\psi^i} H^i(X, \mathscr{F}'') \xrightarrow{\delta^i} H^{i+1}(X, \mathscr{F}') \to \dots$$

Since it is exact, we have (for example) $\dim H^i(X, \mathscr{F}) = \dim \ker \delta^i + \dim \ker \psi^i$. Now noting that $H^i(X, -)$ is zero for $i > \dim X = n$ (Grothendieck's Theorem) we can write

$$\begin{aligned}
\chi(\mathscr{F}) &= \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}) \\
&= \sum_{i=0}^{n} (-1)^i (\dim \ker \delta^i + \dim \ker \psi^i) \\
&= \sum_{i=0}^{n} (-1)^i \Big(\dim \ker \delta^i + \dim \ker \psi^i + \dim \ker \phi^i - \dim \ker \phi^i \Big) \\
&= \sum_{i=0}^{n} (-1)^i (\dim \ker \phi^i + \dim \ker \psi^i) \\
&\qquad\qquad + \sum_{i=0}^{n} (-1)^i (\dim \ker \delta^i - \dim \ker \phi^i) \\
&= \sum_{i=0}^{n} (-1)^i (\dim \ker \phi^i + \dim \ker \psi^i) \\
&\qquad\qquad + \sum_{i=0}^{n} (-1)^i (\dim \ker \delta^i + \dim \ker \phi^{i+1}) \\
&\qquad\qquad - \dim \ker \phi^{n+1} - \dim \ker \phi^0 \\
&= \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}') + \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{F}'') - 0 - 0 \\
&= \chi(\mathscr{F}') + \chi(\mathscr{F}'')
\end{aligned}$$

We have that $\dim \ker \phi^{n+1}$ is zero from Grothendiecks Theorem (since $H^{n+1}(X, -) = 0$) and $\dim \ker \phi^0$ is zero since $\phi^0 : \Gamma(X, \mathscr{F}') \to \Gamma(X, \mathscr{F})$ is injective.

Exercise 5.2. *a Let X be a projective scheme over a field k, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over k, and let $\mathscr{F}$ be a coherent sheaf on X. Show that there is a polynomial $P(z) \in \mathbb{Q}[z]$, such that $\chi(\mathscr{F}(n)) = P(n)$ for all $n \in \mathbb{Z}$.*

b Now let $X = \mathbb{P}^r_k$, and let $M = \Gamma_*(\mathscr{F})$, considered as a graded $S = k[x_0, \dots, x_r]$-module. Use (5.2) to show that the Hilbert polynomial of $\mathscr{F}$ just defined is the same as the Hilbert polynomial of M defined in (Chapter I, Section 7).

Solution. a

b As a consequence of Theorem III.5.2(b), for each $n \geq n_0$ (where n_0 is the one from the statement of the theorem that depends on $\mathscr{F}$) and $i > 0$ we have $H^i(X, \mathscr{F}(n)) = 0$ and so $\chi(\mathscr{F}(n)) = \dim H^0(X, \mathscr{F}(n))$. That is, $P(n) = \dim M_n$, which is exactly the definition of the Hilbert function for M. Since this equality holds for $n \gg 0$, and $P(z)$ and $P_M(z)$ are both polynomials, it follows that $P(z) = P_M(z)$.

Exercise 5.3. Arithmetic Genus.

a *If X is integral, and k algebrically closed, show that $H^0(X, \mathcal{O}_X) \cong k$, so that*

$$p_a(X) = \sum_{i=0}^{r-1} (-1)^i \dim_k H^{r-i}(X, \mathcal{O}_X)$$

In particular, we have

$$p_a(X) = \dim_k H^1(X, \mathcal{O}_X)$$

b *If X is a closed subvariety of $\mathbb{P}^r_k$, show that this $p_a(X)$ coincides with the one defined in (I, Ex 7.2), which apparently depended on the projective embedding.*

c *If X is a nonsingular projective curve over an algebraically closed field k, show that $p_a(X)$ is in fact a birational invariant. Conclude that a nonsingular plane curve of degree $d \geq 3$ is not rational.*

Solution. a As X is integral, it is isomorphic to a variety (Proposition II.4.10). So we can use Theorem I.3.4(a) to see that $H^0(X, \mathcal{O}_X) = k$. The desired result then follows from the definitions.

Exercise 5.4. a *Let X be a projective scheme over a field k, and let $\mathcal{O}_X(1)$ be a very amply invertible sheaf on X. Show that there is a unique additive homomorphism*

$$P : K(X) \to \mathbb{Q}[z]$$

such that for each coherent sheaf $\mathscr{F}$ on X, $P(\gamma(\mathscr{F}))$ is the Hilbert polynomial of $\mathscr{F}$.

b *Now let $X = \mathbb{P}^r_k$. For each $i = 0, \ldots, r$, let L_i be a linear space of dimension i in X. Then show that*

(a) *$K(X)$ is the free abelian group generated by $\{\gamma(\mathcal{O}_{L_i}) | i = 0, \ldots, r\}$, and*

(b) *the map $P : K(X) \to \mathbb{Q}[z]$ is injective.*

Solution. a Since we have a map defined from the set of coherent sheaves (the free generators of $K(X)$) to $\mathbb{Q}[z]$ we just need to show that the map is compatible with the relations. That is, for every short exact sequence of coherent sheaves $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ we want to show that $P(\gamma(\mathscr{F})) = P(\gamma(\mathscr{F}')) + P(\gamma(\mathscr{F}''))$. This follows immediately from the definition of the Hilbert polynomial and Exercise III.5.1.

b First suppose that (a) is indeed true and consider $P(\gamma(\mathcal{O}_{L_i}))$. We have $\mathcal{O}_{L_i} = i_*\mathcal{O}_{\mathbb{P}^i}$ for an appropriate linear embedding $i : \mathbb{P}^i \to \mathbb{P}^r$. We know the Hilbert polynomial of $\mathcal{O}_{\mathbb{P}^i}$ from the explicite calculations of Theorem 5.1 to be $\binom{i+z}{i}$. So an element $\sum a_i\gamma(\mathcal{O}_{L_i})$ of $K(X)$ gets sent to the polynomial $\sum a_i\binom{i+z}{i}$. If this is zero then by induction on the highest nonzero coefficient we see that each a_i is zero and so (b) is true.

Now having seen that (a) $\Rightarrow$ (b) we prove (a) and (b) together. The case $r = 0$ is trivially true so suppose that (a) and (b) are true for $\mathbb{P}^{r-1}$. By Exercise II.6.10 we have an exact sequence

$$K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to K(\mathbb{P}^r - \mathbb{P}^{r-1}) \to 0$$

where the first map is extension by zero. Suppose at the beginning we choose L_i such that $L_i \subseteq L_{r-1}$ for all $i < r$. The map P clearly factors through the first map of the exact sequence and so since the composition $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r) \to \mathbb{Q}[z]$ is injective, we see that $K(\mathbb{P}^{r-1}) \to K(\mathbb{P}^r)$ is injective. So $K(\mathbb{P}^r)$ has a subgroup $\mathbb{Z}^r$ with basis $\mathcal{O}_{L_i}$ for $i = 0, \dots, r-1$ and this subgroup is the kernel of the surjective morphism $K(\mathbb{P}^r) \to K(\mathbb{P}^r - \mathbb{P}^{r-1})$. The scheme $\mathbb{P}^r - \mathbb{P}^{r-1}$ is isomorphic to $\mathbb{A}^r$ and since $k[x_1, \dots, x_n]$ is a principle ideal domain $K(\mathbb{A}^r) = \mathbb{Z}$ generated by $\gamma(\mathcal{O}_{\mathbb{A}^r})$, which is in the image of $\gamma(\mathcal{O}_{\mathbb{P}^r})$ (see the proof of Exercise II.6.10). So $K(\mathbb{P}^r)$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}^r$. Since $\mathbb{Z}$ is projective, $Ext^1(\mathbb{Z}, \mathbb{Z}) = 0$ and so there are no nontrivial extensions and therefore, $K(\mathbb{P}^r) = \mathbb{Z}^{r+1}$, generated by $\gamma(\mathcal{O}_{L_i})$ for $i = 0, \dots, r$. We have already seen that (a) implies (b) and so (a) and (b) are both true for $\mathbb{P}^r$, completing the induction step.

Exercise 5.5. *Let k be a field, let $X = \mathbb{P}^r_k$, and let Y be a closed subscheme of dimension $q \geq 1$, which is a complete intersection. Then:*

a for all $n \in \mathbb{Z}$, the natural map

$$H^0(X, \mathcal{O}_X(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

is surjective.

b Y is connected;

c $H^i(Y, \mathcal{O}_Y(n)) = 0$ for $0 < i < q$ and all $n \in \mathbb{Z}$;

d $p_a(Y) = \dim_k H^q(Y, \mathcal{O}_Y)$.

Solution. a

b We know from Theore III.5.1 that $H^0(X, \mathcal{O}_X) \cong k$ and from Theorem 5.2 that $H^0(Y, \mathcal{O}_Y)$ is a finitely generated k-algebra. From part (a) of this exercise, $H^0(X, \mathcal{O}_X) \to H^0(Y, \mathcal{O}_Y)$ is surjective and so $H^0(Y, \mathcal{O}_Y) \cong k$. If Y were to have more than one connected component then $\Gamma(Y, \mathcal{O}_Y)$ would have zero divisors. This is not the case and so Y is connected.

Exercise 5.6.

Exercise 5.7. *Let X (resp. Y) be proper schemes over a noetherian ring A. We denote by $\mathscr{L}$ an invertible sheaf.*

a If $\mathscr{L}$ is ample on X, and Y is any closed subscheme of X, then $i^*\mathscr{L}$ is ample on Y, where $i : Y \to X$ is the inclusion.

b $\mathscr{L}$ is ample on X if and only if $\mathscr{L}_{red} = \mathscr{L} \otimes \mathcal{O}_{X_{red}}$ is ample on X_{red}.

c Suppose X is reduced. Then $\mathscr{L}$ is ample on X if and only if $\mathscr{L} \otimes \mathcal{O}_{X_i}$ is ample on X_i, for each irreducible component X_i of X.

d Let $f : X \to Y$ be a finite surjective morphism, and let $\mathscr{L}$ be an invertible sheaf on Y. Then $\mathscr{L}$ is ample on Y if and only if $f^*\mathscr{L}$ is ample on X.

Solution. a Let $\mathscr{F}$ be a coherent sheaf on Y. Then by Proposition III.5.3 there is some n_0 such that for each $n > n_0$ we have $H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n) = 0$ for $i > 0$. Using the projection formula (Exercise II.5.1(d)) and Lemma III.2.10 we have $H^i(X, i_*\mathscr{F} \otimes \mathscr{L}^n) = H^i(X, i_*(\mathscr{F} \otimes i^*\mathscr{L}^n)) = H^i(Y, \mathscr{F} \otimes i^*\mathscr{L}^n)$. Since $\mathscr{F}$ was arbitrary, it follows now from Proposition III.5.3 that $i^*\mathscr{L}$ is ample.

b Let $i : X_{red} \to X$ be the canonical closed immersion. Then we have $\mathscr{L} \otimes \mathcal{O}_{X_{red}} = i^*\mathscr{L}$ and so if $\mathscr{L}$ is ample, the ampleness of $i^*\mathscr{L}$ follows from part (a) of this exercise. Conversely, suppose that $i^*\mathscr{L}$ is ample. We use a similar strategy to Exercise III.3.1. We have a finite descending sequence

$$\mathscr{F} \supseteq \mathscr{N} \cdot \mathscr{F} \supseteq \mathscr{N}^2 \cdot \mathscr{F} \supseteq \cdots \supseteq 0$$

where $\mathscr{N}$ is the sheaf of nilpotents on X (finite since X is proper, and therefore finite type, over a noetherian base). At each d and n we have a short exact sequence $0 \to \mathscr{N}^{d+1} \cdot \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{N}^d \cdot \mathscr{F} \otimes \mathscr{L}^n \to \mathscr{G}_d \otimes \mathscr{L}^n \to 0$. giving rise to long exact sequences

$$\cdots \to H^i(X, \mathscr{G}_d \otimes \mathscr{L}^n) \to H^{i+1}(X, \mathscr{N}^{d+1} \cdot \mathscr{F} \otimes \mathscr{L}^n) \to H^{i+1}(X, \mathscr{N}^d \cdot \mathscr{F} \otimes \mathscr{L}^n) \to H^{i+1}(X, \mathscr{G}_d \otimes \mathscr{L}^n) \to$$

So if we can show that $H^i(X, \mathscr{G}_d \otimes \mathscr{L}^n) = 0$ for all i and all $n > n_0$ for some n_0, then it will follow by induction on d that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n > n_0$ and $i > 0$.

Since $i^*\mathscr{L}$ is ample, and $\mathscr{G}_d$ are coherent sheaves on X_{red}, by Proposition III.5.3 and the finiteness of the filtration, there is some n_0 such that for all $n > n_0$ and $i > 0$ we have $H^i(X_{red}, i^*\mathscr{L}^n \otimes \mathscr{G}_d) = 0$. Since the $\mathscr{G}_d$

are already $\mathscr{O}_{X_{red}}$-modules, cohomology is defined via sheaves of abelian groups, and the fact that X and X_{red} share the same underlying topological space, we have $H^i(X, \mathscr{L}^n \otimes \mathscr{G}_d) = H^i(X_{red}, i^*\mathscr{L}^n \otimes \mathscr{G}_d) = 0$ for all $n > n_0$ and $i > 0$. Via the above mentioned long exact sequences, this shows that $H^i(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n > n_0$ and $i > 1$, and that there is a sequence of surjections

$$\dots \to H^1(X, \mathscr{N}^{d+1} \cdot \mathscr{F} \otimes \mathscr{L}^n) \to H^1(X, \mathscr{N}^d \cdot \mathscr{F} \otimes \mathscr{L}^n) \to \dots$$

Since $\mathscr{N}^d = 0$ for some d big enough, this is enough to show also that $H^1(X, \mathscr{F} \otimes \mathscr{L}^n) = 0$ for all $n > n_0$. Hence, it follows from Proposition III.5.3 that $\mathscr{L}$ is ample.

Exercise 5.8.

Exercise 5.9.

Exercise 5.10. *Let X be a projective scheme over a noetherian ring A, and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \dots \mathscr{F}^r$ be an exact sequence of coherent sheaves on X. Show that there is an integer n_0 such that for all $n \geq n_0$, the sequence of global sections*

$$\Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \dots \to \Gamma(X, \mathscr{F}^r(n))$$

is exact.

Solution. Proof by induction. In the case $r < 3$ there is nothing to prove. In the case $r = 3$, we have a short exact sequence $0 \to \mathscr{F}^1 \to \mathscr{F}^2 \to \mathscr{F}^3 \to 0$. By Proposition III.5.2 there is an integer n_0 for $\mathscr{F}^1$ such that for each $i > 0$ and each $n \geq n_0$, $H^i(X, \mathscr{F}^1(n)) = 0$. So considering the long exact sequence

$$0 \to \Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \Gamma(X, \mathscr{F}^3(n)) \to H^1(X, \mathscr{F}^1(n)) \to \dots$$

we see that for $n > n_0$ the sequence $0 \to \Gamma(X, \mathscr{F}^1(n)) \to \Gamma(X, \mathscr{F}^2(n)) \to \Gamma(X, \mathscr{F}^3(n)) \to 0$ is exact.

Now suppose the result is true for $r - 1$. Given an exact sequence of sheaves $0 \to \mathscr{F}^1 \to \dots \to \mathscr{F}^{n-2} \xrightarrow{f} \mathscr{F}^{n-1} \to \mathscr{F}^n \to 0$ we obtain two exact sequences

$$0 \to \mathscr{F}^1 \to \dots \to \mathscr{F}^{n-2} \xrightarrow{f} \operatorname{coker} f \to 0$$

$$0 \to \operatorname{coker} f \to \mathscr{F}^{n-1} \to \mathscr{F}^n \to 0$$

Choose n_0 bigger than the two n_0 provided for both of these exact sequences by the induction hypothesis. Then for each $n > n_0$ we have exact sequences

$$0 \to \Gamma(X, \mathscr{F}^1(n)) \to \dots \to \Gamma(X, \mathscr{F}^{n-2}(n)) \xrightarrow{f} \Gamma(X, \operatorname{coker} f(n)) \to 0$$

$$0 \to \Gamma(X, \operatorname{coker} f(n)) \to \Gamma(X, \mathscr{F}^{n-1}(n)) \to \Gamma(X, \mathscr{F}^n(n)) \to 0$$

which we can stick back together to get a long exact sequence

$$0 \to \Gamma(X, \mathscr{F}^1(n)) \to \dots \to \Gamma(X, \mathscr{F}^{n-2}(n)) \to \Gamma(X, \mathscr{F}^{n-1}(n)) \to \Gamma(X, \mathscr{F}^n(n)) \to 0$$

6 Ext Groups and Sheaves

Exercise 6.1. *Show that there is a one-to-one correspondence between isomorphism classes of extensions of $\mathscr{F}''$ by $\mathscr{F}'$, and element of the group* $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$.

Solution. In Hartshorne's statement of the exercise we are given a map $E = \{\text{extensions of } \mathscr{F}'' \text{ by } \mathscr{F}'\} \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$; we construct an inverse. Let

$$0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{G} \to 0 \tag{4}$$

be an embedding of into an injective sheaf and $\mathscr{G}$ the cokernel. From this short exact sequence we get a long exact sequence and from this, a surjection $\hom(\mathscr{F}'', \mathscr{G}) \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to 0$ since $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{I}) = 0$ as a consequence of $\mathscr{I}$ being injective. So we can lift our element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ to a morphism in $\hom(\mathscr{F}'', \mathscr{G})$. We then define $\mathscr{F} = \mathscr{I} \times_{\mathscr{G}} \mathscr{F}''$ (pullbacks exist in $\mathfrak{Mod}(X)$ since kernels and products do, and in fact are defined component wise in the sense that $(\mathscr{I} \times_{\mathscr{G}} \mathscr{F}'')(U) = \mathscr{I}(U) \times_{\mathscr{G}(U)} \mathscr{F}''(U)$). The two morphisms $\mathscr{F}' \to \mathscr{I}$ and $\mathscr{F}' \xrightarrow{0} \mathscr{F}''$ define a morphism $\mathscr{F}' \to \mathscr{I} \times_{\mathscr{G}} \mathscr{F}''$ and so we get a sequence

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0 \tag{5}$$

which turns out to be exact (it is exact even as a sequence of presheaves; this is straight forward to check since $\mathscr{F} = \mathscr{I} \times_{\mathscr{G}} \mathscr{F}''$ is defined component-wise). So now we have two morphisms of sets $E \leftrightarrows \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ and we just need to check that they are actually inverses to each other.

Suppose we start with an element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$, construct an extension as above, and then look at what element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ this gives us. There is a morphism from sequence 5 to sequence 4 and therefore a morphism between the long exact sequences obtained through $\operatorname{Ext}^i(\mathscr{F}'', -)$. One square in this is

$$\begin{tikzcd}
\cdots \arrow[r] & \hom(\mathscr{F}'', \mathscr{F}'') \arrow[r] \arrow[d] & \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \hom(\mathscr{F}'', \mathscr{G}) \arrow[r] & \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \arrow[r] & \cdots
\end{tikzcd}$$

The image of the identity morphism $\mathscr{F}'' \xrightarrow{id} \mathscr{F}''$ in $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ is what we are concerned with. Following $\mathscr{F}'' \xrightarrow{id} \mathscr{F}''$ down and to the right gives us back the element of $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ that we started with. Since the right vertical morphism is the identity, this shows that the composition $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to E \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ is the identity.

Now we will show that $\operatorname{Ext}^1(\mathscr{F}', \mathscr{F}') \to E$ is surjective, and this together with $\operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}') \to E \to \operatorname{Ext}^1(\mathscr{F}'', \mathscr{F}')$ being the identity shows that the two maps in question are inverses to each other. Embed $\mathscr{F}'$ in an injective $\mathscr{I}$ and let $\mathscr{G}$ be the cokernel so that we have an exact sequence $0 \to \mathscr{F}' \to \mathscr{I} \to \mathscr{G} \to 0$. Then by construction, every short exact sequence in the image of $\operatorname{Ext}^1(\mathscr{F}', \mathscr{F}') \to E$ is of the form $0 \to \mathscr{F}' \to \mathscr{I} \times_{\psi, \mathscr{G}, \phi} \mathscr{F}'' \to \mathscr{F}'' \to 0$ for some morphism $\phi : \mathscr{F}'' \to \mathscr{G}$. So given a short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ we have to show it is of this form. Since $\mathscr{I}$ is injective and $\mathscr{F}' \to \mathscr{F}$ injective, the identity $\mathscr{F}' = \mathscr{F}'$ lifts to a morphism $\mathscr{F} \to \mathscr{I}$, and then since $\hom(-, \mathscr{G})$ is right exact we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d, "id"] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d, "\phi"] & 0 \\ 0 \arrow[r] & \mathscr{F}' \arrow[r] & \mathscr{I} \arrow[r, "\psi"] & \mathscr{G} \arrow[r] & 0 \end{tikzcd}$$

So we have a sequence $0 \to \mathscr{F} \to \mathscr{F}'' \oplus \mathscr{I} \stackrel{\phi-\psi}{\to} \mathscr{G}$ and if this sequence is exact, then $\mathscr{F} \cong \mathscr{F}'' \times_{\mathscr{G}} \mathscr{I}$ and so we are done. Consider the stalks at a point x, so we obtain diagrams of $\mathcal{O}_x$-modules. In the world of modules, we can chase elements around diagrams, and in this way prove that for every morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r, "g"] & C' \arrow[r] & 0 \end{tikzcd}$$

results in an isomorphism $B \cong B' \times_{g,C',f} C$. So for every point x, the sequence $0 \to \mathscr{F}_x \to \mathscr{F}''_x \oplus \mathscr{I}_x \stackrel{\phi_x-\psi_x}{\to} \mathscr{G}_x$ is exact. This implies that it is an exact sequence of sheaves. So we have our isomomorphism $\mathscr{F} \cong \mathscr{F}'' \times_{\mathscr{G}} \mathscr{I}$ and therefore $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is in the image of $\operatorname{Ext}^1(\mathscr{F}', \mathscr{F}') \to E$.

Exercise 6.2. *Let $X = \mathbb{P}^1_k$ with k an infinite field.*

a *Show that there does not exist a projective object $\mathscr{P} \in \mathfrak{Mod}(X)$, together with a surjective map $\mathscr{P} \to \mathcal{O}_X \to 0$.*

b *Show that there does not exist a projective object $\mathscr{P}$ in either $\mathfrak{Qco}(X)$ or $\mathfrak{Coh}(X)$ together with a surjection $\mathscr{P} \to \mathcal{O}_X \to 0$.*

Solution. a Suppose that we have such a projective object, with such a surjection. Let $x \in U$ be a point in U and let $V \subset U$ be a neighbourhood of x strictly smaller than U, so $U \neq V$, and consider the surjection $\mathcal{O}_V \to k(x) \to 0$ where $\mathcal{O}_V = j_!(\mathcal{O}_X|_V)$, $j : V \to X$ is the inclusion, and $k(x)$ is the skyscraper sheaf at x with value the stalk $\mathcal{O}_x$ of $\mathcal{O}_X$ at x. The composition $\mathscr{P} \to \mathcal{O}_X \to k(x)$ gives a surjection $\mathscr{P} \to k(x)$ which then lifts to $\mathscr{P} \to \mathcal{O}_V$ by the assumption that $\mathscr{P}$ is projective, so we have a commutative square

$$\begin{tikzcd} \mathscr{P} \arrow[r] \arrow[d] & \mathcal{O}_X \arrow[r] \arrow[d] & 0 \\ \mathcal{O}_V \arrow[r] & k(x) \arrow[r] & 0 \end{tikzcd}$$

Evalutating at U, we see that $\mathscr{P}(U) \to k(x) = k(x)(U)$ factors through zero since $\mathcal{O}_V(U) = 0$, so for every section in $\mathscr{P}(U)$ the stalk at x is zero. Since U and x were arbitrarily chosen, we see that every section in $\mathscr{P}(U)$ for every open U is zero at every point $x \in U$ and so $\mathscr{P} = 0$. But this contradicts the existence of the surjection $\mathscr{P} \to \mathcal{O}_X$.

b

Exercise 6.3. *Let X be a noetherian scheme, and let $\mathscr{F}, \mathscr{G} \in \mathfrak{Mod}(X)$.*

- a *If $\mathscr{F}, \mathscr{G}$ are both coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent, for all $i \geq 0$.*
- b *If $\mathscr{F}$ is coherent and $\mathscr{G}$ is quasi-coherent, then $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is quasi-coherent, for all $i \geq 0$.*

Solution. a We immediately reduce to the affine case since by definition $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})$ is coherent if and only if for every open affine subset $U = \operatorname{Spec} A$ of X, the sheaf $\mathscr{E}xt^i_X(\mathscr{F}, \mathscr{G})|_U = \mathscr{E}xt^i_U(\mathscr{F}|_U, \mathscr{G}|_U)$ (Proposition III.6.2) is coherent and similarly, for $\mathscr{F}$ and $\mathscr{G}$. So since $X = \operatorname{Spec} A$ is affine, the sheaves $\mathscr{F}$ and $\mathscr{G}$ correspond to finitely generated A-modules M and N. From Exercise III.6.7 we then have $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) = \operatorname{Ext}^i_A(M, N)^\sim$ so we know that $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N})$ is at least quasi-coherent, so we have proven part (b). Now since M is finitely generated and A is noetherian, we can construct inductively a resolution of M by finite rank free A-modules $\cdots \to A^{n_1} \to A^{n_0} \to M \to 0$. We then have $\operatorname{Ext}^i_A(M, N) = h^i(\hom_A(A^{n_\bullet}, N)) = h^i(N^{n_\bullet})$. Since N is finitely generated, so are the N^{n_i} and consequently, so are the $h^i(N^{n_\bullet}) = \operatorname{Ext}^i_A(M, N)$. Hence, $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) = \operatorname{Ext}^i_A(M, N)^\sim$ is quasi-coherent.

b Was proven in part (a).

Exercise 6.4. *Let X be a noetherian scheme, and suppose that every coherent sheaf on X is a quotient of a locally free sheaf. Then for any $\mathscr{G} \in \mathfrak{Mod}(X)$ show that the δ-functor $\mathscr{E}xt^i(\cdot, \mathscr{G})$ from $\mathfrak{Coh}(X)$ to $\mathfrak{Mod}(X)$ is a contravariant universal δ-functor.*

Remark. We assume the hypothesis "every coherent sheaf on X is a quotient of a locally free sheaf of *finite rank*" was intended.

Solution. By Theorem III.1.3A we just need to show that $\mathscr{E}xt^i(\cdot, \mathscr{G})$ is coeffaceable. Since every coherent sheaf $\mathscr{F}$ is the quotient of a locally free sheaf *of finite rank*, $\mathscr{L} \to \mathscr{F} \to 0$, it is enough to show that $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ for $\mathscr{L}$ locally free *of finite rank*. From Proposition III.6.2, to see that $\mathscr{E}xt^i(\mathscr{L}, \mathscr{G}) = 0$ it is enough to show that $\mathscr{E}xt^i(\mathscr{L}|_U, \mathscr{G}|_U) = 0$ for every U in an open cover of X. Choose a cover such that for each U we have $U = \operatorname{Spec} A$ for some A and $\mathscr{L}|_U \cong \oplus_{i=1}^n \mathcal{O}_U$. Then we must show that $\mathscr{E}xt^i_{\mathcal{O}_U}(\oplus_{i=1}^n \mathcal{O}_U, \mathscr{G}|_U) = 0$ for all $i > 0$. Take an injective resolution $0 \to \mathscr{G}|_U \to \mathscr{I}^\bullet$ of $\mathscr{G}|_U$. Then we have $\mathscr{E}xt^i(\oplus_{i=1}^n \mathcal{O}_U, \mathscr{G}|_U) = h^i(\mathscr{H}om(\oplus_{i=1}^n \mathcal{O}_U, \mathscr{I}^\bullet)) = h^i(\oplus_{i=1}^n \mathscr{H}om(\mathcal{O}_U, \mathscr{I}^\bullet)) = \oplus_{i=1}^n h^i(\mathscr{H}om(\mathcal{O}_U, \mathscr{I}^\bullet)) = \oplus_{i=1}^n h^i(\mathscr{I}^\bullet) = 0$ for $i > 0$.

Exercise 6.5. *Let X be a noetherian scheme, and assume that $\mathfrak{Coh}(X)$ has enough locally frees. Show*

- a *$\mathscr{F}$ is locally free if and only if $\mathscr{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$ for all $\mathscr{G} \in \mathfrak{Mod}(X)$;*
- b *$\operatorname{hd}(\mathscr{F}) \leq n$ if and only if $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$;*

c $\mathrm{hd}(\mathscr{F}) = \sup_x \mathrm{pd}_{\mathcal{O}_X} \mathscr{F}_x$.

Remark. Again, we assume the hypothesis "every coherent sheaf on X is a quotient of a locally free sheaf of *finite rank*" was intended.

Solution. a If $\mathscr{F}$ is locally free *of finite rank* then by Proposition III.6.5 we have $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > 0$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$. Conversely, suppose that $\mathscr{E}xt^1(\mathscr{F}, \mathscr{G}) = 0$ for all $i > 0$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$. Taking stalks, we have by Proposition III.6.8 that $0 = \mathscr{E}xt^1(\mathscr{F}, \mathscr{G})_x \cong \mathrm{Ext}^1_{\mathcal{O}_x}(\mathscr{F}_x, \mathscr{G}_x)$ for $i > 0$. This is a criterion for $\mathscr{F}_x$ to be projective, and finitely generated modules over local rings are projective if and only if they are free (Proposition 6 at the end of this section). So $\mathscr{F}_x$ is free for each x. Hence, $\mathscr{F}$ is locally free (Exercise II.5.7(b)).

b First suppose that $\mathrm{hd}(\mathscr{F}) \leq n$. Then there exists a locally free resolution $\dots \to 0 \to \mathscr{E}_n \to \mathscr{E}_{n-1} \to \dots \to \mathscr{E}_0 \to \mathscr{F} \to 0$ of $\mathscr{F}$ of length n. We can use this to calculate $\mathscr{E}xt$ by Proposition III.6.5 and so we find that $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$. We prove the converse by induction on n. The case $n = 0$ has the same proof as part (a) of this question. Consider $n > 0$ and suppose that $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$ and all $\mathscr{G} \in \mathfrak{Mod}(X)$. Express $\mathscr{F}$ as the quotient of a locally free sheaf $\mathscr{E}$ and consider the resulting short exact sequence $0 \to \mathscr{F}' \to \mathscr{E} \to \mathscr{F} \to 0$. This gives rise to a long exact sequence

$$\dots \to \mathscr{E}xt^n(\mathscr{E}, \mathscr{G}) \to \mathscr{E}xt^n(\mathscr{F}', \mathscr{G}) \to \mathscr{E}xt^{n+1}(\mathscr{F}, \mathscr{G}) \to \mathscr{E}xt^{n+1}(\mathscr{E}, \mathscr{G}) \to \dots$$

Since $\mathscr{E}$ is locally free and $n > 0$ part (a) of this exercise tells use that the two outer groups vanish and so we have an isomorphism $\mathscr{E}xt^n(\mathscr{F}', \mathscr{G}) \cong \mathscr{E}xt^{n+1}(\mathscr{F}, \mathscr{G})$. By hypothesis $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i > n$ and so we find that $\mathscr{E}xt^i(\mathscr{F}', \mathscr{G}) = 0$ for all $i > n - 1$ which by the inductive hypothesis implies that $\mathrm{hd}\,\mathscr{F}' \leq n - 1$. So there is a locally free resolution $\dots \to 0 \to \mathscr{E}_{n-1} \to \dots \to \mathscr{E}_0 \to \mathscr{F}' \to 0$ of $\mathscr{F}'$ of length $n - 1$. The exact sequence $\dots \to 0 \to \mathscr{E}_{n-1} \to \dots \to \mathscr{E}_0 \to \mathscr{E} \to \mathscr{F} \to 0$ where $\mathscr{E}_0 \to \mathscr{E}$ is the composition $\mathscr{E}_0 \to \mathscr{F}' \to \mathscr{E}$ then gives us a resolution of length n and so $\mathrm{hd}\,\mathscr{F} \leq n$.

c Given a locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ of $\mathscr{F}$ of length n, taking stalks gives a free (and hence projective) resolution of length $\leq n$ of $\mathscr{F}_x$ for each point x, hence $\mathrm{hd}(\mathscr{F}) \geq \sup_x \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$. Suppose equality doesn't hold. Then for every point x we have $\mathrm{hd}(\mathscr{F}) > \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$. By Proposition III.6.10A this means that $\mathrm{Ext}^i(\mathscr{F}_x, N) = 0$ for all points x, all $i \geq \mathrm{hd}\,\mathscr{F}$ and all $\mathcal{O}_x$-modules N. Using Proposition III.6.8 this says that $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G})_x = 0$ for all points x, all $i \geq \mathrm{hd}\,\mathscr{F}$ and all $\mathcal{O}_X$-modules $\mathscr{G}$, and so $\mathscr{E}xt^i(\mathscr{F}, \mathscr{G}) = 0$ for all $i \geq \mathrm{hd}\,\mathscr{F}$ and all $\mathcal{O}_X$-modules $\mathscr{G}$. By part (b) of this exercise, this implies that $\mathrm{hd}(\mathscr{F}) < \mathrm{hd}(\mathscr{F})$ which is clearly a contradiction. Hence, the inequality $\mathrm{hd}(\mathscr{F}) \geq \sup_x \mathrm{pd}_{\mathcal{O}_x} \mathscr{F}_x$ is actually an equality.

Exercise 6.6. *Let A be a regular local ring, and let M be a finitely generated A-module. In this case, strengthen the result (6.10A) as follows.*

a *M is projective if and only if* $\operatorname{Ext}^i(M, A) = 0$ *for all* $i > 0$.

b *Use (a) to show that for any n,* $\operatorname{pd} M \leq n$ *if and only if* $\operatorname{Ext}^i(M, A) = 0$ *for all* $i > n$.

Solution. a If M is projective then $\operatorname{Ext}^i(M, A) = 0$ for $i > 0$ since $\operatorname{Ext}^i(M, A)$ can be defined as the ith left derived functor of $\hom(-, A)$. Conversely, suppose that $\operatorname{Ext}^i(M, A) = 0$ for all $i > 0$. Let N be a finitely generated A-module. Then we have an exact sequence $0 \to K \to A^n \to N \to 0$ and a corresponding long exact sequence

$$\cdots \to \underbrace{\operatorname{Ext}^{i-1}(M, A^n)}_{=0} \to \operatorname{Ext}^{i-1}(M, N) \to \operatorname{Ext}^i(M, K) \to \underbrace{\operatorname{Ext}^i(M, A^n)}_{=0} \to \dots$$

(one of the many ways to see $\operatorname{Ext}^i(M, A^n) = 0$ is by considering the long exact sequence associated to $0 \to A^{n-1} \to A^n \to A \to 0$ and using induction). So if the statement:

(S_i) $\operatorname{Ext}^i(M, N) = 0$ for all finitely generated A-modules N

is true with $i > 2$ then (S_{i-1}) is also true. The statement (S_i) for all $i > \dim A$ is true as a consequence of Proposition III.6.11A and so by induction we have the verity of (S_i) for all $i \geq 1$. In particular, consider the exact sequence $0 \to K \to A^n \to M \to 0$ and the corresponding exact sequence $\cdots \to \operatorname{Ext}^0(M, A^n) \to \operatorname{Ext}^0(M, M) \to \operatorname{Ext}^1(M, K) \to \dots$. Since $\operatorname{Ext}^1(M, K)$ is zero, the morphism $\operatorname{Ext}^0(M, A^n) \to \operatorname{Ext}^0(M, M)$ is surjective and so the identity $M \to M$ is a composition $M \to A^n \to M$. In otherwords, M is a direct summand of A^n. This is one criteria for M to be projective.

b If $\operatorname{pd} M \leq n$ then we can calculate $\operatorname{Ext}^i(M, A)$ using a projective resolution of M of length $\leq n$ which implies that $\operatorname{Ext}^i(M, A) = 0$ for $i > n$. Conversely, suppose that $\operatorname{Ext}^i(M, A) = 0$ for $i > n$, and suppose that $\operatorname{Ext}^i(M', A) = 0$ for $i > n - 1$ implies that $\operatorname{pd} M' \leq n - 1$. If $n = 0$ then we have $\operatorname{pd} M \leq 0$ by part (a). If not then take a finite set of generators of M and the associated short exact sequence $0 \to N \to A^k \to M \to 0$. This gives a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}(A^k, A) \to \operatorname{Ext}^{i-1}(N, A) \to \operatorname{Ext}^i(M, A) \to \operatorname{Ext}^i(A^k, A) \to \dots$$

Since A^k is already free, we have $\operatorname{Ext}^i(A^k, A) = 0$ for all $i > 0$ and so $\operatorname{Ext}^{i-1}(N, A) \cong \operatorname{Ext}^i(M, A)$ for $i > 1$. This means that $\operatorname{Ext}^i(N, A)$ for $i > n - 1$ and so by the inductive hypothesis $\operatorname{pd} N \leq n - 1$. So there exists a projective resolution $0 \to P_{n-1} \to \cdots \to P_0 \to N \to 0$ of length $n - 1$ and from this we obtain a projective resolution $0 \to P_{n-1} \to \cdots \to P_0 \to A^k \to M \to 0$ of length $n - 1$ where $P_0 \to A^k$ is the composition $P_0 \to N \to A^k$. Hence, $\operatorname{pd} M \leq n$.

Exercise 6.7. *Let $X = \operatorname{Spec} A$ be an affine noetherian scheme. Let M, N be A-modules, with M finitely generated. Then*

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)$$

and

$$\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N}) \cong \operatorname{Ext}^i_A(M, N)^\sim$$

Solution. Since M is finitely generated and A noetherian, we can find a resolution of M by finite rank free modules $\cdots \to A^{n_1} \to A^{n_0} \to M \to 0$. The A-modules $\operatorname{Ext}^i_A(M, N)$ can then be calculated as $h^i(\hom_A(A^{n_\bullet}, N))$. Now compare the two functors

$$\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{\cdot}) \qquad\qquad h^i(\hom_A(A^{n_\bullet}, \cdot))$$

that map A-mod to A-mod. Since $(\cdot)^\sim$ is an exact equivalence between A-mod and $\mathfrak{Qco}(X)$ the functor $\operatorname{Ext}^i_X(\widetilde{M}, \widetilde{\cdot})$ is a derived functor and therefore automatically a universaly δ-functor (Corollary III.1.4). Since $\hom_A(A^n, \cdot) \cong (\cdot)^n$ is exact for finite n, the functors $h^i(\hom_A(A^{n_\bullet}, \cdot))$ are also a δ-functor (use the Snake Lemma). Since A-mod has enough injectives, and $\hom_A(\cdot, I)$ is exact for any injective I, the functors $h^i(\hom_A(A^{n_\bullet}, \cdot))$ are effaceable for $i > 0$ and therefore form a universal δ-functor. Now $\operatorname{Ext}^0_X(\widetilde{M}, \widetilde{N}) \cong \hom_A(M, N)$ and $h^0(\hom_A(A^{n_\bullet}, N) = \hom_A(M, N)$. So the two sequences of functors are the isomorphic.

Now consider $\mathscr{E}xt^i_X(\widetilde{M}, \widetilde{N})$ and $\operatorname{Ext}^i_A(M, N)^\sim$. We use the same resolution $\cdots \to A^{n_1} \to A^{n_0} \to M \to 0$ and get a finite rank free resolution of $\widetilde{M}$ which can be used to calculate $\mathscr{E}xt$ by Proposition III.6.5 as $\mathscr{E}xt^i(\widetilde{M}, \widetilde{M}) \cong h^i(\mathscr{H}om(\widetilde{A^{n_\bullet}}, \widetilde{N}))$. Now since M is finitely generated and A noetherian, we have $\hom_A(M, N)^\sim \cong \mathscr{H}om(\widetilde{M}, \widetilde{N})$. [1] Hence we have

$$\begin{aligned}\mathscr{E}xt^i(\widetilde{M}, \widetilde{M}) &\cong h^i(\mathscr{H}om(\widetilde{A^{n_\bullet}}, \widetilde{N})) \cong h^i(\hom_A(A^{n_\bullet}, N)^\sim) \\ &\cong h^i(\hom_A(A^{n_\bullet}, N))^\sim = \operatorname{Ext}^i_A(M, N)^\sim\end{aligned}$$

Exercise 6.8. *Prove the following theorem of Kleiman: if X is a noetherian, integral, separated, locally factorial scheme, then every coherent sheaf on X is a quotient of a locally free sheaf (of finite rank).*

a First show that open sets of the form X_s for various $s \in \Gamma(X, \mathscr{L})$, and various invertible sheaves $\mathscr{L}$ on X, form a base for the topology of X.

b Now use (II, 5.14) to show that any coherent sheaf is a quotient of a direct sum $\bigoplus \mathscr{L}_i^{n_i}$ for various invertible sheaves $\mathscr{L}_i$ and various integers n_i.

[1] This doesn't hold in general. Consider the values of these two sheaves on a basic open $D(f)$. On the left we have $(\hom_A(M, N))_f$ and on the rights $\hom_{A_f}(M_f, N_f)$. There is a clear morphism $(\hom_A(M, N))_f \to \hom_{A_f}(M_f, N_f)$ but in general this morphism is neither injective nor surjective (consider the ring $\oplus_{i=1}^\infty k[x, y]/(x^i)$ with $M = N = \oplus_{i=1}^\infty k[x]/(x^i)$ localized at $f = x$). If M is finitely generated though, the morphism is an isomorphism. It is also natural with respect to inclusions of basic open affines, and so the sheaves are isomorphic.

Solution. a We show that given a closed point $x \in X$ and an open neighbourhood U of x, there is an $\mathscr{L}$ and s such that $x \in X_s \subseteq U$. Let $Z = X - U$ and $Z = \cup_{i=1}^n Z_i$ be the decomposition of Z into its irreducible components. If the statement is true for each $U_i = X - Z_i$ then we can take the global section $s = s_1 \otimes \cdots \otimes s_n$ of $\mathscr{L}_1 \otimes \cdots \otimes \mathscr{L}_n$. Then we have $(s_1 \otimes \cdots \otimes s_n)_y \notin \mathfrak{m}_y \mathscr{L}_y$ if and only if $(s_i)y \notin \mathfrak{m}_y \mathscr{L}_y$ for all i. So $X_s = \cap X_{s_i}$ and so $x \in X_s \subseteq \cap U_i = U$. So we can assume that Z is irreducible. Therefore it is a prime Weil divisor and by Proposition II.6.11 corresponds to a Cartier Divisor D. That is, a global section of $\mathscr{K}^*/\mathcal{O}^*$. This is represented by a (finite) cover $\{U_i\}$ and for each U_i an element $f_i \in K$ such that $f_i/f_j \in \mathcal{O}_X^*(U)$. By construction these f_i also satisfy: for any codimension one irreducible subscheme Z' we have $f_i \in \mathfrak{m}_{Z'}\mathcal{O}_{X,Z'}$ if and only if $Z' = Z$. We then have Proposition II.6.13 which gives us an invertible sheaf $\mathscr{L}(D)$, constructed as the sub-$\mathcal{O}_X$-module of $\mathscr{K}^*$ generated locally by f_i^{-1}. The local sections $f_i f_i^{-1} \in \Gamma(U_i, \mathscr{L}(D))$ then glue together to give a global section $s \in \Gamma(X, \mathscr{L}(D))$ such that under the isomorphisms $\Gamma(U_i, \mathscr{L}(D)) \cong \Gamma(U_i, \mathcal{O}_X)$ defined by $ff_i^{-1} \leftrightarrow f$ we have $s|_{U_i} \leftrightarrow f_i$ and so $X_s = U$. Hence we have found $\mathscr{L}, s$ such that $x \in X_s \subseteq U$.

b Let $\mathscr{F}$ be a coherent sheaf on X. Then there is a cover by open affines $U_i = \operatorname{Spec} A_i$, such that on U_i, we have $\mathscr{F}|_{U_i} \cong \widetilde{M_i}$ for some finitely generated A_i-module M_i. This means that $\mathscr{F}_{U_i}$ is generated by finitely many sections $m_{ij} \in M_i = \Gamma(U_i, \mathscr{F}|_{U_i})$. Now take a refinement of this cover consisting of open set of the form $X_{s_{ik}} \subseteq U_i$ for some $s_{ik} \in \Gamma(X, \mathscr{L}_{ik})$ and some $\mathscr{L}_{ik}$. Then each m_{ij} is a section of $\Gamma(X_{s_{ik}}, \mathscr{F})$ and so by Lemma II.5.14 there is some n_{ij} such that $s_{ik}^{n_{ij}} m_{ij}$ extends to a global section of $\mathscr{L}^{n_{ik}} \otimes \mathscr{F}$. This global section defines a morphism $\mathcal{O}_X \to \mathscr{L}^{n_{ik}} \otimes \mathscr{F}$ and tensoring with $\mathscr{L}^{-n_{ik}}$ we obtain a morphism $\mathscr{L}_{ik}^{-n_{ik}} \to \mathscr{F}$. Take the direct sum of these morphisms $\bigoplus \mathscr{L}_{ik}^{-n_{ik}} \to \mathscr{F}$. On the open set $X_{s_{ik}}$ the section m_{ij} is in the image of the morphism $\mathscr{L}^{-n_{ik}} \to \mathscr{F}$ and so since the m_{ij} generate $\mathscr{F}$ locally, the morphism $\bigoplus \mathscr{L}_{ik}^{-n_{ik}} \to \mathscr{F}$ is surjective.

Exercise 6.9. *Let X be a noetherian, integral, separated, regular scheme. Show that the natural group homomorphism $\varepsilon : K_{vec}(X) \to K_{coh}(X)$ from the Grothendieck group of the category of locally free finite rank sheaves, to the Grothendieck group of the category of coherent sheaves is an isomorphism as follows.*

a *Given a coherent sheaf $\mathscr{F}$, use (Ex. 6.8) to show that it has a locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$. Then use (6.11A) and (Ex. 6.5) to show that it has a finite locally free resolution*

$$0 \to \mathscr{E}_n \to \cdots \to \mathscr{E}_1 \to \mathscr{E}_0 \to \mathscr{F} \to 0$$

b *For each $\mathscr{F}$, choose a finite locally free resolution $\mathscr{E}_\bullet \to \mathscr{F} \to 0$, and let $\delta(\mathscr{F}) = \sum(-1)^i \gamma(\mathscr{E}_i)$ in $K_{vec}(X)$. Show that $\delta(\mathscr{F})$ is independent of the resolution chosen, that it defines a homomorphism of $K_{coh}(X)$ to $K_{vec}(X)$, and finally, that it is an inverse to ε.*

Solution. a By Exercise III.6.8 every coherent sheaf is a quotient of a locally free sheaf of finite rank (regular implies locally factorial; this is a hard theorem [Matsumura Theorem 48, page 142]), and so $\mathfrak{Coh}(X)$ has enough locally frees. Hence, we can define the homological dimension of $\mathscr{F}$ and by Exercise III.6.5(c) we have $\operatorname{hd}\mathscr{F} = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x$. Since X is regular, each $\mathcal{O}_x$ is regular and so by Proposition III.6.11A we have $\operatorname{pd}\mathscr{F}_x \leq \dim \mathcal{O}_{X,x} \leq \dim X$. Hence $\operatorname{hd}\mathscr{F} = \sup_x \operatorname{pd}_{\mathcal{O}_x} \mathscr{F}_x \leq \dim X$ and so there exists a finite locally free finite rank resolution of $\mathscr{F}$.

b

Remark. This proof mimicks that found in [Borel, Serre - Théorème de Riemann-Roch].

Lemma 1. *Suppose we have a diagram* $0 \leftarrow \mathscr{F} \leftarrow \mathscr{F}'' \to \mathscr{F}' \to 0$ *in the category of coherent sheaves. Then there is a commutative square with* $\mathscr{E}$ *locally free and all morphisms surjective*

$$\begin{tikzcd} \mathscr{E} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[d] \\ \mathscr{F}' \arrow[r] & \mathscr{F} \end{tikzcd}$$

Proof. Let $\mathscr{G}$ be the kernel of the canonical morphism $\mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}$. Since $\mathscr{F}' \to \mathscr{F}$ and $\mathscr{F}'' \to \mathscr{F}$ are both surjective, the same is true of the compositions with projections $\mathscr{G} \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}'$ and $\mathscr{G} \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}''$. So the two morphisms $\mathscr{G} \to \mathscr{F}'$ and $\mathscr{G} \to \mathscr{F}''$ are surjective. Then we express $\mathscr{G}$ as the quotient of a locally free sheaf $\mathscr{E} \to \mathscr{G}$ and take the compositions $\mathscr{E} \to \mathscr{G} \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}'$ and $\mathscr{E} \to \mathscr{G} \to \mathscr{F}' \oplus \mathscr{F}'' \to \mathscr{F}''$. □

Lemma 2. *Suppose we have the commutative exact diagram of solid arrows in the category of coherent sheaves with* $\mathscr{E}, \mathscr{E}'$ *locally free. Then we can find* $\mathscr{G}''$ *and* $\mathscr{E}''$ *with* $\mathscr{E}''$ *locally free and extend the diagram to a commutative exact diagram with the dashed arrows.*

$$\begin{tikzcd} & 0 & 0 & 0 & \\ 0 \arrow[r] & \mathscr{G}' \arrow[r] \arrow[u, dashed] & \mathscr{E}' \arrow[r] \arrow[u, dashed] & \mathscr{F}' \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r, dashed] & \mathscr{G}'' \arrow[r, dashed] \arrow[u, dashed] \arrow[d, dashed] & \mathscr{E}'' \arrow[r, dashed] \arrow[u, dashed] \arrow[d, dashed] & \mathscr{F}'' \arrow[r, dashed] \arrow[u] \arrow[d] & 0 \\ 0 \arrow[r] & \mathscr{G} \arrow[r] \arrow[d, dashed] & \mathscr{E} \arrow[r] \arrow[d, dashed] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

Proof. First we use Lemma 1 above to obtain the comutative square on the left with $\mathscr{E}_1$ locally free, and then again to obtain the commutative square in the center with $\mathscr{E}_2$ locally free. Note that this gives us the diagram on the right where all morphism are epimorphisms.

$$
\begin{tikzcd}
\mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[d] \\
\mathscr{E} \arrow[r] & \mathscr{F}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{E}_2 \arrow[rr] \arrow[d] & & \mathscr{E}' \arrow[d] \\
\mathscr{E}_1 \arrow[r] & \mathscr{F}'' \arrow[r] & \mathscr{F}'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathscr{E}' \arrow[r] & \mathscr{F}' \\
\mathscr{E}_2 \arrow[u, "a"] \arrow[r, "c"] \arrow[d, "b"'] & \mathscr{F}'' \arrow[u] \arrow[d] \\
\mathscr{E} \arrow[r] & \mathscr{F}
\end{tikzcd}
$$

Then we take expressions for $\mathscr{G}$ and $\mathscr{G}'$ as quotients of locally free sheaves $\mathscr{E}_3 \xrightarrow{d} \mathscr{G}$ and $\mathscr{E}_4 \xrightarrow{e} \mathscr{G}'$. Now we have a diagram

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \mathscr{G}' \arrow[u] \arrow[r, "h"] & \mathscr{E}' \arrow[u] \arrow[r] & \mathscr{F}' \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \ker(c) \oplus \mathscr{E}_3 \oplus \mathscr{E}_4 \arrow[u, "a|_{\ker c}+0+e"'] \arrow[d, "b|_{\ker c}+d+0"] \arrow[r] & \mathscr{E}_2 \oplus \mathscr{E}_3 \oplus \mathscr{E}_4 \arrow[u, "a+0+he"'] \arrow[d, "b+gd+0"] \arrow[r, "c+0+0"] & \mathscr{F}'' \arrow[u] \arrow[d] \arrow[r] & 0 \\
0 \arrow[r] & \mathscr{G} \arrow[d] \arrow[r, "g"] & \mathscr{E} \arrow[d] \arrow[r] & \mathscr{F} \arrow[d] \arrow[r] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

which satisfies the requirements. $\square$

Corollary 3. *For any two locally free resolutions $\mathscr{E}_\bullet \xrightarrow{\varepsilon} \mathscr{F} \to 0$ and $\mathscr{E}'_\bullet \xrightarrow{\varepsilon'} \mathscr{F} \to 0$ of a coherent sheaf $\mathscr{F}$ there is a third locally free resolution $\mathscr{E}''_\bullet \xrightarrow{\varepsilon''} \mathscr{F} \to 0$ together with a commutative diagram where the vertical morphisms are all surjective.*

$$
\begin{tikzcd}
\mathscr{E}'_\bullet \arrow[r, "\varepsilon'"] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
\mathscr{E}''_\bullet \arrow[u] \arrow[d] \arrow[r, "\varepsilon''"] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
\mathscr{E}_\bullet \arrow[r, "\varepsilon"] & \mathscr{F} \arrow[r] & 0
\end{tikzcd}
$$

Proof. We construct $\mathscr{E}''_\bullet$ inductively. From Lemma 2 we get a diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker\varepsilon' \arrow[r] & \mathscr{E}'_0 \arrow[r, "\varepsilon'"] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\varepsilon'' \arrow[u] \arrow[d] \arrow[r] & \mathscr{E}''_0 \arrow[u] \arrow[d] \arrow[r, "\varepsilon''"] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \ker\varepsilon \arrow[r] & \mathscr{E}_0 \arrow[r, "\varepsilon"] & \mathscr{F} \arrow[r] & 0
\end{tikzcd}
$$

with surjective vertical morphisms. For the inductive step we use Lemma 2 to get a the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker d_i' \arrow[r] & \mathscr{E}_i' \arrow[r, "d_i'"] & \ker d_{i-1}' \arrow[r] & 0 \\
0 \arrow[r] & \ker d_i'' \arrow[r] \arrow[u] \arrow[d] & \mathscr{E}_i'' \arrow[r, "d_i''"] \arrow[u] \arrow[d] & \ker d_{i-1} \arrow[r] \arrow[u] \arrow[d] & 0 \\
0 \arrow[r] & \ker d_i' \arrow[r] & \mathscr{E}_i \arrow[r, "d_i"] & \ker d_{i-1}'' \arrow[r] & 0
\end{tikzcd}$$

with surjective vertical morphisms. □

Proof of independence of the chosen resolution. Now we have the results we need to show that the class $\sum(-1)^i[\mathscr{E}_i]$ in $K_{vec}(X)$ is independent of the resolution chosen. Suppose that we have a second resolution $\mathscr{E}_\bullet' \to \mathscr{F} \to 0$ as in Corollary 3. Then we get a third resolution $\mathscr{E}_\bullet'' \to \mathscr{F} \to 0$ which "dominates" the other two and so we have an exact commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
\cdots \arrow[r] & \mathscr{G}_1 \arrow[r] \arrow[d] & \mathscr{G}_0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & \mathscr{E}_1'' \arrow[r] \arrow[d] & \mathscr{E}_0'' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \mathscr{E}_1 \arrow[r] \arrow[d] & \mathscr{E}_0 \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and and analogous one for $\mathscr{E}_\bullet'$ (denote the kernels in this analgogous diagram by $\mathscr{G}_i'$ instead of $\mathscr{G}_i$). If the $\mathscr{G}_i$ are locally free then we get

$$\begin{aligned}
\sum(-1)^i\mathscr{E}_i &= \sum(-1)^i(\mathscr{E}_i'' - \mathscr{G}_i) \\
&= \sum(-1)^i\mathscr{E}_i'' - \sum(-1)^i\mathscr{G}_i \\
&= \sum(-1)^i\mathscr{E}_i'' \\
&= \sum(-1)^i\mathscr{E}_i'' - \sum(-1)^i\mathscr{G}_i' \\
&= \sum(-1)^i\mathscr{E}_i'
\end{aligned}$$

in $K_{vec}(X)$ and so we just need to prove:

□

δ defines a morphism $K_{coh}(X) \to K_{vec}(X)$. We must show that formal sums of coherent sheaves that are zero in $K_{coh}(X)$ get sent to zero in

$K_{vec}(X)$. For this it is enough to show that for any short exact sequence $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ of coherent sheaves, we have $\delta(\mathscr{F}) = \delta(\mathscr{F}') + \delta(\mathscr{F}'')$ in $K_{vec}(X)$. To show this we will see that there exist resolutions for $\mathscr{F}, \mathscr{F}', \mathscr{F}''$ that themselves form an exact sequence, so we have an exact commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}'_\bullet \arrow[r] \arrow[d] & \mathscr{E}_\bullet \arrow[r] \arrow[d] & \mathscr{E}''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}$$

As in the proof of Corollary 3 we build the sequences step by step. Each step uses the following lemma.

Lemma 4. Suppose that $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ *is an exact sequence of coherent sheaves. Then there is an exact sequence of locally free sheaves* $0 \to \mathscr{E}' \to \mathscr{E} \to \mathscr{E}'' \to 0$ *together with a surjective morphism to the original sequence.*

Proof. Expressing $\mathscr{F}''$ as a quotient of a locally free sheaf $\mathscr{E}'' \to \mathscr{F}''$ we obtain $\mathscr{E}''$. Now use Lemma 1 to obtain a commutative diagram of surjective morphisms

$$\begin{tikzcd}
\mathscr{G} \arrow[r, "a"] \arrow[d, "b"] & \mathscr{E}'' \arrow[d] \\
\mathscr{F} \arrow[r] & \mathscr{F}''
\end{tikzcd}$$

Express $\mathscr{F}'$ as a quotient of a locally free sheaf $\mathscr{E}' \xrightarrow{c} \mathscr{F}'$ and we end up with a diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker a \oplus \mathscr{E}' \arrow[r] \arrow[d, "b|_{\ker a}+c"] & \mathscr{G} \oplus \mathscr{E}' \arrow[r, "a+0"] \arrow[d, "b+dc"] & \mathscr{E}'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r, "d"] & \mathscr{F} \arrow[r] & \mathscr{F}'' \arrow[r] & 0
\end{tikzcd}$$

with the desired properties. $\square$

Now using this lemma and given the ith step of the resolutions, we can construct the $(i+1)$th step by forming the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}'_{i+1} \arrow[r] \arrow[d] & \mathscr{E}_{i+1} \arrow[r] \arrow[d] & \mathscr{E}''_{i+1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker d'_i \arrow[r] & \ker d_i \arrow[r] & \ker d''_i \arrow[r] & 0
\end{tikzcd}$$

where $d_i : \mathscr{E}_i \to \mathscr{E}_{i-1}$ and similarly for d_i' and d_i''. Hence we get a commutative exact diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathscr{E}'_\bullet \arrow[r] \arrow[d] & \mathscr{E}_\bullet \arrow[r] \arrow[d] & \mathscr{E}''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

and so we have $\delta(\mathscr{F}) = \sum(-1)^i[\mathscr{E}_i] = \sum(-1)^i([\mathscr{E}_i']+[\mathscr{E}_i'']) = \sum(-1)^i[\mathscr{E}_i']+ \sum(-1)^i[\mathscr{E}_i''] = \delta(\mathscr{F}') + \delta(\mathscr{F}'')$.

δ provides an inverse to ε. Clearly, if $\mathscr{E}$ is a locally free sheaf then we can take the resolution $\cdots \to 0 \to \mathscr{E} \to \mathscr{E} \to 0$ and so $\delta(\varepsilon(\mathscr{E})) = [\mathscr{E}]$. Conversely, for any counded exact sequence $0 \to \mathscr{F}_n \to \cdots \to \mathscr{F}_0 \to 0$ in $\mathfrak{Coh}(X)$ we have the relation $\sum(-1)^i[\mathscr{F}_i]$ in $K_{coh}(X)$ and so if $\mathscr{E}_\bullet \to \mathscr{F} \to 0$ is a bounded resolution by locally free sheaves then $\varepsilon(\delta(\mathscr{F})) = \varepsilon(\sum(-1)^i[\mathscr{E}_i]) = \sum(-1)^i[\mathscr{E}_i] = [\mathscr{F}]$.

Exercise 6.10. Duality for a Finite Flat Morphism.

a *Let $f : X \to Y$ be a finite morphism of noetherian schemes. For any quasi-coherent $\mathcal{O}_Y$-module $\mathscr{G}$, $\mathscr{H}om_Y(f_*\mathcal{O}_X, \mathscr{G})$ is a quasi-coherent $f_*\mathcal{O}_Y$-module, hence corresponds to a quasi-coherent $\mathcal{O}_X$-module, which we call $f^!\mathscr{F}$.*

b *Show that for any coherent $\mathscr{F}$ on X and any quasi-coherent $\mathscr{G}$ on Y, there is a natural isomorphism*

$$f_*\mathscr{H}om_X(\mathscr{F}, f^!\mathscr{G}) \xrightarrow{\sim} \mathscr{H}om_Y(f_*\mathscr{F}, \mathscr{G})$$

c *For each $i \geq 0$, there is a natural map*

$$\phi_i : \operatorname{Ext}^i_X(\mathscr{F}, f^!\mathscr{G}) \to \operatorname{Ext}^i_Y(f_*\mathscr{F}, \mathscr{G})$$

d *Now assume that X and Y are separated, $\mathfrak{Coh}(X)$ has enough locally frees, and assume that $f_*\mathcal{O}_X$ is locally free on Y. Show that ϕ_i is an isomorphism for all i, all $\mathscr{F}$ coherent on X, and all $\mathscr{G}$ quasi-coherent on Y.*

7 The Serre Duality Theorem

Exercise 7.1. *Let X be an integral projective scheme of dimension ≥ 1 over a field k, and let $\mathscr{L}$ be an ample invertible sheaf on X. Then $H^0(X, \mathscr{L}^{-1}) = 0$.*

Exercise 7.2. *Let $f : X \to Y$ be a finite morphism of projective schemes of the same dimension over a field k, and let ω_Y° be a dualizing sheaf for Y.*

a *Show that $f^!\omega_Y^\circ$ is a dualizing sheaf for X.*

b If X and Y are both nonsingular, and k algebraically closed, conclude that there is a natural trace map $t : f_*\omega_X \to \omega_Y$.

Exercise 7.3. *Let $X = \mathbb{P}^n_k$. Show that $H^q(X, \Omega^p_X) = 0$ for $p \neq q$, k for $p = q$, $0 \leq p, q, \leq n$.*

Solution. Consider the exact sequence of Theorem 8.13. From Exercise II.5.16(d) we have a filtration for each r

$$\wedge^r(\mathcal{O}(-1)^{n+1}) = F^0 \supseteq F^1 \supseteq \cdots \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients $F^p/F^{p+1} \cong \Omega^p \otimes \wedge^{r-p}\mathcal{O}$. Since $\wedge^{r-p}\mathcal{O} \cong 0$ for $r - p \neq 0, 1$ and $\wedge^{r-p}\mathcal{O} \cong \mathcal{O}$ for $r - p = 0, 1$ we see that $F^p = F^{p+1}$ for $p \neq r, r-1$ so our filtration is $\wedge^r(\mathcal{O}(-1)^{n+1}) \supseteq F^r \supseteq F^{r+1} = 0$. The quotient $F^r/F^{r+1} = F^r$ is $\Omega^r \otimes \wedge^{r-r}\mathcal{O} \cong \Omega^r$ and the quotient $F^{r-1}/F^r = \wedge^r(\mathcal{O}(-1)^{n+1})/\Omega^r$ is $\Omega^{r-1} \otimes \wedge^{r-(r-1)}\mathcal{O} \cong \Omega^{r-1}$ so the filtration is actually an exact sequence:

$$0 \to \Omega^r \to \wedge^r(\mathcal{O}(-1)^{n+1}) \to \Omega^{r-1} \to 0$$

Now for any line bundle $\mathscr{L}$ on any ringed space we have $\wedge^r(\mathscr{L}^{\oplus m}) \cong (\mathscr{L}^{\otimes r})^{\oplus \binom{r}{m}}$ (one way of showing this is to take a trivializing cover, choose a local basis, and then look at the transition morphisms) and so our exact sequence is

$$0 \to \Omega^r \to \mathcal{O}(-r)^{\oplus N} \to \Omega^{r-1} \to 0$$

for suitable N that we don't care about. This gives rise to a long exact sequence on cohomology. Since $H^i(X, \mathcal{O}(-r)) = 0$ for $i < n$ or $r < n+1$ (Theorem III.5.1) we have isomorphisms $H^i(X, \Omega^r) \cong H^{i-1}(X, \Omega^{r-1})$ for $1 \leq i$ if $r < n+1$. If $r \geq n+1$ then we still have isomorphisms but only for $1 \leq i < n$.

Now we know that $H^0(X, \Omega^0) \cong H^0(X, \mathcal{O}_X) \cong k$ (Theorem III.5.1) and so using these isomorphisms we see that $H^i(X, \Omega^i) \cong k$ for $0 \leq i \leq n$. Again, using Theorem III.5.1 we know the cohomology of $\Omega^n \cong \mathcal{O}(-n-1)$, and in particular, that $H^i(X, \Omega^n) \cong 0$ for $i < n$. Using our isomorphisms above, this tells us that $H^i(X, \Omega^r) = 0$ in the region $i < r, 0 \leq r \leq n$. All that remains to show is the region $i > r, 0 \leq i \leq n$ and this follows from Corollary III.7.13.

Exercise 7.4.

8 Higher Direct Images of Sheaves

Exercise 8.1. *Let $f : X \to Y$ be a continuous map of topological spaces. Let $\mathscr{F}$ be a sheaf of abelian groups on X, and assume that $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$. Show that there are natural isomorphisms, for each $i \geq 0$,*

$$H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$$

Solution. Take an injective resolution $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ of $\mathscr{F}$ on X. Then $0 \to f_*\mathscr{F} \to f_*\mathscr{I}^\bullet$ is an injective resolution of $f_*\mathscr{F}$ on Y. A priori, this complex is not necassarily exact but the hypothesis $R^i f_*(\mathscr{F}) = 0$ for all $i > 0$ says that it is infact exact. By definition the cohomology of $\mathscr{F}$ is the cohomology of the complex $\Gamma(X, \mathscr{I}^\bullet)$ which is actually the same complex as $\Gamma(Y, f_*\mathscr{I}^\bullet)$. Hence, $H^i(X, \mathscr{F}) = H^i(Y, f_*\mathscr{F})$.

Exercise 8.2. *Let $f : X \to Y$ be an affine morphism of schemes, with X noetherian and let $\mathscr{F}$ be a quasi-coherent sheaf on X. Show that the hypothesis of Exercise III.8.1 are satisfied, and hence that $H^i(X, \mathscr{F}) \cong H^i(Y, f_*\mathscr{F})$ for each $i \geq 0$.*

Solution. By Proposition III.8.1 we know that $R^i f_* \mathscr{F}$ is the sheaf associated to $V \mapsto H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)})$. Since f is affine, $f^{-1}(V)$ is affine for every open subscheme V of Y (Exercise II.5.17). Theorem III.3.7 then tells us that $H^i(f^{-1}(V), \mathscr{F}|_{f^{-1}(V)}) = 0$ for $i > 0$. Hence, $R^i f_* \mathscr{F} = 0$ for $i > 0$.

Exercise 8.3. *Let $f : X \to Y$ be a morphism of ringed spaecs, let $\mathscr{F}$ be an $\mathcal{O}_X$-module, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Prove the projection formula*

$$R^i f_*(\mathscr{F} \otimes f^*\mathscr{E}) \cong R^i f_*(\mathscr{F}) \otimes \mathscr{E}$$

Solution. Let $0 \to \mathscr{F} \to \mathscr{I}^\bullet$ be an injective resolution of $\mathscr{F}$. Using the natural isomorphisms from Exercise II.5.1(d) we get an isomorphism of chain complexes

$$f_*(\mathscr{I}^\bullet \otimes f^*\mathscr{E}) \cong f_*(\mathscr{I}^\bullet) \otimes \mathscr{E}$$

Consider the cohomology sheaves of these chain complexes. The pullback $f^*\mathscr{E}$ is locally free and so by Proposition III.6.7 $0 \to \mathscr{F} \otimes f^*\mathscr{E} \to \mathscr{I}^\bullet \otimes f^*\mathscr{E}$ is an injective resolution of $\mathscr{F} \otimes f^*\mathscr{E}$ and so can be used to calculate the right derived functors of f_* (tensoring with locally free sheaves is exact: check stalks). So the cohomology sheaves of $f_*(\mathscr{I}^\bullet \otimes f^*\mathscr{E})$ are $R^i f_*(\mathscr{F} \otimes f^*\mathscr{E})$.

Now $R^i f_*(\mathscr{F})$ are the cohomology sheaves of $f_*\mathscr{I}^\bullet$. More explicitely, $R^i f_*(\mathscr{F}) = \operatorname{coker}(f_*\mathscr{I}^{i-1} \to \ker(f_*\mathscr{I}^i \to f_*\mathscr{I}^{i+1}))$. As tensoring with a locally free sheaf is exact, it follows that $R^i f_*(\mathscr{F}) \otimes \mathscr{E}$ are isomorphic to the cohomology sheaves of $f_*(\mathscr{I}^\bullet) \otimes \mathscr{E}$.

Hence, the isomorphisms of cohomology sheaves induced by our isomorphism of complexes above are the desired isomorphisms.

Exercise 8.4. *Let Y be a noetherian scheme, and let $\mathscr{E}$ be a locally free $\mathcal{O}_Y$-module of rank $n+1$, $n \geq 1$. Let $X = \mathbb{P}(\mathscr{E})$, with the invertible sheaf $\mathcal{O}_X(1)$ and the projection morphism $\pi : X \to Y$.*

- a *Then $\pi_*(\mathcal{O}(l)) \cong S^i(\mathscr{E})$ for $l \geq 0$, $\pi_*(\mathcal{O}(l)) = 0$ for $l < 0$; $R^i\pi_*(\mathcal{O}(l)) = 0$ for $0 < i < n$ and $l \in \mathbb{Z}$; and $R^n\pi_*(\mathcal{O}(l)) = 0$ for $l > -n-1$.*
- b *Show there is a natural exact sequence*

$$0 \to \Omega_{X/Y} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0$$

and conclude that the relative canonical sheaf $\omega_{X/Y} = \wedge^n \Omega_{X/Y}$ is isomorphic to $(\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$. Show furthermore that there is a natural isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$

c *Now show, for any $i \in \mathbb{Z}$, that*

$$R^n\pi_*(\mathcal{O}(l)) \cong \pi_*(\mathcal{O}(-l-n-1))^\vee \otimes (\wedge^{n+1}\mathscr{E})^\vee$$

d *Show that $p_a(X) = (-1)^n p_a(Y)$ and $p_g(X) = 0$,*

e *In particular, if Y is a nonsingular projective curve of genus g, and $\mathscr{E}$ a locally free sheaf of rank 2, then X is a projective surface with $p_a = -g, p_g = 0$, and irregularity g.*

Solution. a Let $\{U_i\}$ be a trivializing cover on X for $\mathscr{E}$ such that each U_i is affine, and consequently the spectrum of a noetherian ring A_i. So we have $\mathscr{E}(U_i) \cong \mathcal{O}_X^{n+1}$ for each U_i and hence $\pi^{-1}(U_i) \cong \mathbb{P}^n_{A_i}$. This means in particular that $H^j(\pi^{-1}U_i, \mathcal{O}(l)|_{\pi^{-1}U_i}) = H^j(\mathbb{P}^n_{A_i}, \mathcal{O}(l)|_{\pi^{-1}U_i})$ which is zero for $0 < j < n$ after Theorem III.5.1. As a consequence of this, $R^j\pi_*\mathcal{O}(l) = 0$ for $0 < j < n$ after Proposition III.8.1. By the same reasoning, $R^n\pi_*\mathcal{O}(l) = 0$ for $l > -n-1$ since $H^n(\mathbb{P}^n_{A_i}, \mathcal{O}(l)) = 0$ for $l > -n-1$.

b Part (b) of Theorem II.7.11 gives us a natural surjection $\pi^*\mathscr{E} \to \mathcal{O}(1)$. Consider exact sequence arising from the twist of this by $\mathcal{O}(-1)$

$$0 \to \mathscr{F} \to (\pi^*\mathscr{E})(-1) \to \mathcal{O} \to 0$$

Let $U = \operatorname{Spec} A$ be any open affine subscheme of Y on which $\mathscr{E}$ is isomorphic to $\mathcal{O}_Y^{n+1}$. Then $\pi^{-1}U \cong \mathbb{P}^n_A$ and the restriction of this exact sequence looks like

$$0 \to \mathscr{F}|_{\mathbb{P}^n_A} \to \mathcal{O}(-1)_{\mathbb{P}^n_A} \to \mathcal{O}|_{\mathbb{P}^n_A} \to 0$$

which is easily recognisable as the exact sequence from Theorem II.8.13. So we have isomorphisms $\mathscr{F}|_{\mathbb{P}^n_A} \cong \Omega_{\mathbb{P}^n_A/U}$. These isomorphisms are compatible with restrictions to smaller affine subsets and so we obtain a global isomorphism $\mathscr{F} \cong \Omega_{X/Y}$.

The isomorphism $\wedge^n\Omega_{X/Y} \cong (\pi^* \wedge^{n+1} \mathscr{E})(-n-1)$ is a consequence of Exercise II.5.16. If we then cover X with open subets of the form $U_i = \mathbb{P}^n_{A_i}$ where $\operatorname{Spec} A_i$ are opens of Y on which $\mathscr{E} \cong \mathcal{O}_Y^{n+1}$ (and so $\pi^{-1}U \cong \mathbb{P}^n_A$), then restricting to these we get isomorphisms $\omega_{X/Y}|_{\pi^{-1}U} \cong \mathcal{O}_{\pi^{-1}U}(-n-1)$ via the isomorphisms just mentioned. So we have $R^n\pi_*(\omega_{X/Y})|_{\operatorname{Spec} A} \cong R^n\pi_*(\omega_{X/Y}|_{\mathbb{P}^n_A}) \cong H^n(\mathbb{P}^n_A, \omega_{\mathbb{P}^n_A/A})^\sim \cong A^\sim = \mathcal{O}_{\operatorname{Spec} A}$ (Corollary III.8.2, Proposition III.8.5, and Theorem III.5.1). Since these isomorphisms are all natural, we obtain the desired isomorphism $R^n\pi_*(\omega_{X/Y}) \cong \mathcal{O}_Y$.

c

d

e There is nothing to show.

9 Flat Morphisms

Exercise 9.1. *A flat morphism $f : X \to Y$ of finite type of noetherian schemes is open.*

Solution. We need to show that for any open subscheme $U \subset X$ the image $f(U)$ is open in Y. Since the induced morphism $U \to Y$ is also of finite type we can restrict to the case when $U = X$. By Exercise II.3.18 we know that $f(X)$ is constructible, and so if it is closed under generization, then it will be open. That is, we need to show that given a generization $y' \in Y$ of a point $y \in f(X)$ there is some point $x' \in X$ whose image is y'. let $\operatorname{Spec} B$ be an open affine neighbourhood of y. The scheme $\operatorname{Spec} B$ also contains y', and the induced morphism $f^{-1} \operatorname{Spec} B \to \operatorname{Spec} B$ is still a flat morphism of finite type of noetherian schemes. Let x be a point whose image is y, and let $\operatorname{Spec} A$ be an open affine neighbourhood of y. By Proposition III.9.1A(d) A is a flat B-module.

So now we have a homomorphism $\phi : B \to A$ of noetherian rings where A is a finitely generated B-algebra and flat as a B-module. We have two primes $\mathfrak{p}' \subset \mathfrak{p}$ of B, a prime $\mathfrak{q}$ of A such that $\phi^{-1}\mathfrak{q} = \mathfrak{p}$ and we are looking for a prime $\mathfrak{q}' \subset \mathfrak{q}$ such that $\phi^{-1}\mathfrak{a}' = \mathfrak{p}'$. This is a commutative algebra result that can be found in Matsumara.

Exercise 9.2. *Do the calculation of (9.8.4) for the curve of (I, Ex. 3.14). Show that you get an embedded point at the cusp of the plane cubic curve.*

Solution. The curve has parametric coordinates $(x, y, z, w) = (t^3, t^2u, tu^2, u^3)$ and projection is from the point $(0, 0, 1, 0)$. That is, we are considering the family of curves (t^3, t^2u, atu^2, u^3) projecting to the projective plane $z = 0$. We are interested in what happens at the cusp $(0, 0, 0, 1)$ of the projected curve so we only need to consider the affine space $w \neq 0$.

X_a has the parametric equations

$$\begin{cases} x = t^3 \\ y = t^2 \\ z = at \end{cases}$$

To get the ideal $I \subseteq k[a, x, y, z]$ of the total family $\overline{X}$ extended over all of $\mathbb{A}^1$ we eliminate t from the parametric equations, and make sure a is not a zero divisor in $k[a, x, y, z]/I$, so that $\overline{X}$ will be flat. We find

$$I = (y^3 - x^2, z^2 - a^2y, z^3 - a^3x, zy - ax, zx - ay^2)$$

Setting $a = 0$ we obtain the ideal $I_0 \subseteq k[x, y, z]$ of X_0 which is

$$I_0 = (y^3 - x^2, z^2, zx, zy)$$

So X_0 has support equal to the curve $x^2 = y^3$ in $\operatorname{Spec} k[x, y]$. Now at points where $\mathfrak{p}$ with $x \notin \mathfrak{p}$ we have $z \in \mathfrak{p}$ since $xz = 0 \in \mathfrak{p}$ and so these local rings are reduced. At the prime $\mathfrak{p} = (x, y)$ however, z is not zero and so $A_{\mathfrak{p}}$ has a nonzero nilpotent element.

Exercise 9.3. *Some examples of flatness and nonflatness.*

a *If $f : X \to Y$ is a finite surjective morphism of nonsingular varieties over an algebraically closed field k, then f is flat.*

b *Let X be a union of two planes meeting at a point, each of which maps isomorphically to a plane Y. Show that f is not flat. For example, let $Y = \operatorname{Spec} k[x, y]$ and $X = \operatorname{Spec} k[x, y, z, w]/(z, w) \cap (x + z, y + w)$.*

c *Again let $Y = \operatorname{Spec} k[x, y]$, but take $X = \operatorname{Spec} k[x, y, z, w]/(z^2, zw, w^2, xz - yw)$. Show that $X_{red} \cong Y$, X has no embedded points, but that f is not flat.*

Solution. a

b Suppose x is the intersection point. The morphism is finite and so for it to be flat, $\mathcal{O}_{x,X}$ must be a finite rank free $\mathcal{O}_{f(x),Y}$-module (Proposition III.9.1A(f)). We have $\mathcal{O}_{x,X}/\mathfrak{m}_{f(x),Y}\mathcal{O}_{x,X} \cong k$ and so if $\mathcal{O}_{x,X}$ is a finite rank free $\mathcal{O}_{f(x),Y}$-module then it has rank one and therefore we would have an isomorphism $\mathcal{O}_{f(x),Y} \xrightarrow{\sim} \mathcal{O}_{x,X}$ as $\mathcal{O}_{f(x),Y}$-modules. Let $f \in \mathcal{O}_{x,X}$ be the image of 1 under this isomorphism. Then $z = gf$ for some $g \in \mathcal{O}_{f(x),Y}$. But z can't be expressed in this way in $\mathcal{O}_{x,X}$. Hence, the isomorphism doesn't exist and the morphism is not flat.

Exercise 9.4.

Exercise 9.5.

Exercise 9.6.

Exercise 9.7. *let $Y \subseteq X$ be a closed subscheme, where X is a scheme of finite type over a field k. Let $D = k[t]/(t^2)$ be the ring of dual numbers, and define an infinitesimal deformation of Y as a closed subscheme of X, to be a closed subscheme $Y' \subseteq X \times_k D$, which is flat over D, and whose closed fibre is Y. Show that these Y' are classified by $H^0(Y, \mathscr{N}_{Y/X})$, where*

$$\mathscr{N}_{Y/X} = \mathscr{H}om_{\mathcal{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y)$$

Solution. First a lemma to make the affine case easier to deal with.

Lemma 5. *Consider ideals $I \subset A$ and $I' \subset A[t]$. Then $\operatorname{Spec} A[t]/I'$ is an infinitesimal deformation of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$ if and only if*

a *$t^2 \in I'$;*

b *under the map $A[t] \to A$ sending t to zero, the image of I' is I; and*

c *the kernel of the composite morphism $A \to A[t]/I' \xrightarrow{t} A[t]/I'$ is contained in I'.*

Proof. Condition (a) just says that $A[t]/I'$ is a D-algebra. Condition (b) is equivalent to saying that the composition $(\operatorname{Spec} A[t]/I') \otimes_D k \to A/I$ is an isomorphism. Condition (c) is equivalent to saying that $\operatorname{Spec} A[t]/I'$ is flat over D. To see this consider the criteria of Proposition III.9.1A(a). Since D has a unique nonzero ideal, we only need to test (t). Furthermore, by writing every element of $A[t]/I' \otimes_D (t)$ as $a \otimes t$ we reduce to showing that for $a \in A$, it holds that $at = 0$ implies $a \otimes t = 0$. Hence, the condition. □

Now given a ring A, an ideal I, and a homomorphism $\phi \in \hom_{A/I}(I/I^2, A/I)$, define an ideal $I' \subset A[t]$ to be the set of polynomials $a_0 + a_1 t + \cdots + a_n t^n \in A[t]$ such that $a_0 \in I$ and $\phi(a_0) = a_1$ or 0 in A/I. It is fairly straightforward to check that the conditions of the lemma are fulfilled and so we have an infinitesimal deformation of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$. Conversely, given an infinitesimal deformation of $\operatorname{Spec} A/I$ in $\operatorname{Spec} A$, we can define a morphism $\phi \in \hom_{A/I}(I/I^2, A/I)$ as follows. Given an element $a \in I$, consider elements of the form $a + bt \in I'$. There must be at least one, for otherwise condition (b) of the lemma does not hold. Define $\phi(a) = b$. Note that if $a + b't \in I'$ is a different choice, then $(b' - b)t \in I'$, so $(b' - b) \in I'$ by condition (c), so $(b - b') \in I$ by condition (b) and so we end up with the same morphism $I/I^2 \to A/I$. We still need to show that ϕ is A/I-linear. That is, we must show that $\phi(ax + by) = a\phi(x) + b\phi(y)$ for $a, b \in A/I$ and $x, y \in I/I^2$. Given our definition of ϕ, this amounts to showing that for any elements $(ax + by) + zt$, $x + x't$ and $y + y't$ in I', we have $z - ax' - by' \in I$. We know that $ax + ax't$ and $by + by't$ are in I' and so $(ax + by) + zt - (ax + ax't) - (by + by't) = (z - ax' - by')t \in I'$ and this implies that $z - ax' - by' \in I$ using conditions (b) and (c) of the lemma. So we have given an isomorphism

$$\hom_{A/I}(I/I^2, A/I) \to \mathfrak{Inf}(\operatorname{Spec}(A/I)/\operatorname{Spec} A)$$

and its inverse where $\mathfrak{Inf}(Y/X)$ is the set of infinitesimal deformations of Y as a subscheme of X.

Now that the affine case is done, we prove the general case by glueing in the usual way by glueing. The first thing to notice is that if we have ideals $I \subset A$, $J \subset B$, and ring homomorphism $\psi : A \to B$ such that $\psi^{-1} J \subset I$ then we get a commutative square

$$\begin{tikzcd}
\hom_{A/I}(I/I^2, A/I) \arrow[r, "\sim"] \arrow[d] & \mathfrak{Inf}(\operatorname{Spec}(A/I)/\operatorname{Spec} A) \arrow[d] \\
\hom_{B/J}(J/J^2, B/J) \arrow[r, "\sim"] & \mathfrak{Inf}(\operatorname{Spec}(B/J)/\operatorname{Spec} B)
\end{tikzcd}$$

So in the general case, since both sides are sheaves, and we have natural isomorphisms for affine opens, we can glue to get a global isomorphism

$$\hom_{\mathcal{O}_Y}(\mathscr{I}_Y/\mathscr{I}_Y^2, \mathcal{O}_Y/\mathscr{I}_Y) \cong \mathfrak{Inf}(Y/X)$$

Exercise 9.8.

Exercise 9.9. *Let $A = k[x,y,z,w]/(x,y) \cap (z,w)$, and show that A is rigid.*

Solution. Let $P = k[x,y,z,w]$ and $J = (x,y) \cap (z,w)$ as in the previous exercise. By the previous exercise, we must show that the morphism

$$\hom_A(\Omega_{P/k} \otimes A, A) \to \hom_A(J/J^2, A)$$

is surjective. We do this explicitely.

We have $\Omega_{P/k} \cong P^4$ with basis dx, dy, dz, dw and so $\Omega_{P/k} \otimes A \cong A^4$ with the same basis and $\hom_A(\Omega_{P/k} \otimes A, A) \cong A^4$ with the dual basis dx^*, dy^*, dz^*, dw^*. The ideal J is generated by xz, xw, yz, yw as a P-module and since A is a quotient of A, these elements represent generators of the A-module J/J^2. So any morphism $\phi \in \hom_A(J/J^2, A)$ is determined by its value on xz, xw, yz, yw and in this way we get $\hom_A(J/J^2, A) \subset A^4$, by identifying a morphism with its value on xz, xw, yz, yw.

The morphism $J/J^2 \to \Omega_{P/k} \otimes A$ sends f to $df \otimes 1$ and so using

$$\hom_A(\Omega_{P/k} \otimes A, A) \cong A^4 \qquad \hom_A(J/J^2, A) \subset A^4$$

we can represent the morphism $\hom_A(\Omega_{P/k} \otimes A, A) \to \hom_A(J/J^2, A)$ as a matrix. The morphism in $\hom_A(\Omega_{P/k} \otimes A, A)$ that sends dx to 1 and all other generators to zero gets sent to $(z, w, 0, 0)$ in $\hom_A(J/J^2, A)$ since $d(xz) = zdx + xdz$, $d(xw) = \dots$. Continuing like this we find the matrix to be

$$\begin{pmatrix} z & w & 0 & 0 \\ 0 & 0 & z & w \\ x & 0 & y & 0 \\ 0 & x & 0 & y \end{pmatrix}$$

We want to show that the morphism induced by this matrix is surjective.

Consider an element (b_1, b_2, b_3, b_4) of $\hom_A(J/J^2, A) \subset A^4$ where, recall that b_1 (resp. b_2, b_3, b_4) is the image of xz (resp. xw, yz, yw). We have $yb_1 = xb_3$. Since xz, xw, yz, yw are all zero in A, multiplying by x or y kills all the terms with z or w in them, but "preserves" any terms without, x sending $x^i y^j$ to $x^{i+1} y^j$ and y sending it to $x^i y^{j+1}$. So $b_1 = \frac{x}{y} b_3 + b_1'$ where $b_1' \in (z,w)k[z,w]$. Similarly, from the relation $wb_1 = zb_2$ we see that $b_1 = \frac{z}{w} b_2 + b_1''$ where $b_1'' \in (x,y)k[x,y]$. Putting these two together we see that $b_1 = \frac{z}{w} b_2 + \frac{x}{y} b_3$. We use a similar argument for b_2, b_3, b_4 to find that

$$\begin{aligned} b_1 &= \frac{z}{w} b_2 + \frac{x}{y} b_3 \\ b_2 &= \frac{x}{y} b_4 + \frac{w}{z} b_1 \\ b_3 &= \frac{y}{x} b_1 + \frac{z}{w} b_4 \\ b_4 &= \frac{y}{x} b_2 + \frac{w}{z} b_3 \end{aligned}$$

abd consequently, (b_1, b_2, b_3, b_4) is in the image of $\hom_A(\Omega_{P/k} \otimes A, A) \to \hom_A(J/J^2, A)$. Hence, it is surjective, and so $T^1(A) = 0$ and therefore, the k-algebra A is rigid.

Exercise 9.10. *a Show that $\mathbb{P}^1_k$ is rigid.*

b

c

Solution. a By (9.13.2) the infinitesimal deformations are classified by $H^1(X, \mathscr{T}_X)$. When $X = \mathbb{P}^1_k$ we know that $\Omega_{X,k} \cong \mathcal{O}(-2)$ and so $\mathscr{T}_X = \mathcal{O}(2)$ and we have already calculated the cohomology of this sheaf. We find that $H^1(X, \mathscr{T}_X) = H^1(X, \mathcal{O}(2)) = 0$. Hence, there are no infinitesimal deformations.

Proposition 6. *Let M be a finitely generated module over a local ring $(A, \mathfrak{m})$. Then M is projective if and only if M is free.*

Proof. For any module over any ring, free implies projective so we need only prove the converse. Since M is finitely generated $M/\mathfrak{m}M$ is a finite dimensional $(A/\mathfrak{m})$-vector space. Take a set of elements $m_1, \dots, m_n$ in M whose image in $M/\mathfrak{m}M$ is a basis. Then by Nakayam's Lemma, the m_i generated M and so we get an exact sequence $0 \to N \to A^n \to M \to 0$. Since M is projective, this sequence splits and we see that $A^n \cong M \oplus N$. Now we have $A^n/\mathfrak{m}A^n \cong M/\mathfrak{m}M \oplus N/\mathfrak{m}N$. But both $A^n/\mathfrak{m}A^n$ and $M/\mathfrak{m}M$ are finite dimensional vector spaces of the same dimension. Hence $N/\mathfrak{m}N = 0$ which implies $\mathfrak{m}N = N$ and Nakayama's Lemma says that this implies $N = 0$. So $A^n \cong M$. □

Corollary 7. *If $\mathscr{E}_1 \to \mathscr{E}_0$ is a surjective morphism of locally free coherent sheaves then the kernel is also locally free.*

Proof. Let $\mathscr{G}$ be the kernel. At each point x we get an exact sequence of $\mathcal{O}_{X,x}$-modules, and since the $\mathscr{E}_i$ are locally free, this has the form $0 \to \mathscr{G}_x \to \mathcal{O}^n_{X,x} \to \mathcal{O}^m_{X,x} \to 0$. Since finite rank free modules are projective, the sequence splits and so $\mathscr{G}_x$ is a direct summand of the free module $\mathcal{O}^n_{X,x}$, and hence projective. But $\mathscr{G}_x$ is a finitely generated module over a local ring and so being projective is equivalent to being free (Proposition 6 above). □