Problems

1. (1996 January #3) Show that $f(z) = \cot z - \frac{1}{z}$ has a removable singularity at 0
2. (Gamelin) Suppose that $f(z)$ is an entire function such that $f(z)/z^n$ is bounded for $|z| > R$. Show that $f(z)$ is a polynomial of degree at most n. What can be said if $f(z)/z^n$ is bounded on the entire complex plane?
3. (2016 Jan. #4) Let f be meromorphic on $\mathbb{C}$ but not entire. Let $g(z) = e^{f(z)}$. Is g meromorphic? Is g entire?
4. (1996 Sept. #5) If $f(z)$ is an entire function which assumes the values 0 and 1, show that for any complex number a and any real number $\epsilon > 0$ there is a point z_0 such that $|f(z_0) - a| < \epsilon$.
5. (2014 September #2) Show that if f has an isolated singularity at z_0, then $e^{f(z)}$ cannot have a pole at z_0.
6. (2017 September #4)(a) Classify the singularities of $f, \exp(f), \exp(1/f)$
7. (Gamelin) Show that if u is a harmonic function on $\mathbb{R}$ that is bounded above, then u is constant.
8. (2021 January #4)(a) For $a, b \in \mathbb{C}$ linearly independent over $\mathbb{C}$, define the lattice $\Lambda = \{na + mb \colon (n, m) \in \mathbb{Z}^2\}$. A meromorphic function on $\mathbb{C}$ is elliptic (for the lattice Λ) if it satisfies $f(z) = f(z + \omega)$ for all $z \in \mathbb{C}, \omega \in \Lambda$. Show that an elliptic function which does not have poles is constant.
9. (Gamelin) Show that
$$\int_{-\infty}^{\infty} e^{-zt^2+2wt} dt = \sqrt{\frac{\pi}{z}} e^{\frac{w^2}{z}}, \quad z, w \in \mathbb{C}\,\mathrm{Re}(z) > 0$$
where we take the principle branch of the square root. (Hint: you can use the fact that this integral equals 1 for $w = 0, z = 1$.)
10. (January 2021 #1)Find all holomorphic functions on $\mathbb{C}$ such that
$$f(1 + \frac{1}{n}) = \frac{1}{n}, \quad n \in \mathbb{N}$$
11. (1991 September # 6) Find all functions $f(z)$ satisfying
    - $f(z)$ is analytic on $\mathrm{Im}(z) > 0$
    - $f(z)$ is continous on $\{\mathrm{Im}\, z \geq 0\}$
    - $f(z)$ is real on the real axis
    - $|f(z)| > |\sin z|$ on $\mathrm{Im}\, z > 0\}$
12. (Fall 2020 #1) For every positive integer p, classify the singularities of the function $f_p(z) = \frac{1}{z^p} - \frac{1}{(\sin z)^p}$
13. (Fall 2020 #2) Find all entire functions f such that $|f(z)| \leq e^{xy}$ for all $z = x + iy \in \mathbb{C}$.
14. (1995 Septempter: P5) Find a function $f(z)$ that satisfies

- $f(z)$ is analytic in the upper half plane, $\mathrm{Im}(z) > 0$,and continuous up to the real axis except at the origin
- $f(z)$ is real when x is real and $x \neq 0$
- $|f(z)| \leq \frac{C}{|x|^3}$ when $\mathrm{Im}(z) > 0$
- $f(i) = 4i$

Is this function unique? Why?