NUMERICAL CERTIFICATE FOR PRIME GAPS AT MOST 186

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Abstract. We give the numerical certificate used in the accompanying proof of prime gaps at most 186. The calculation uses an explicit 77-coefficient rational trial in 40 variables on a grid with 98,304 cells per coordinate. We specify the source supports and all 97 positive loss components, and prove the finite kernel and directed-arithmetic enclosures. The recorded bounds give a normalized margin greater than 1/50000. The accompanying Python script reproduces the certificate from the exact inputs.

1. The numerical certificate

The accompanying main paper [8] proves the distribution estimates, arithmetic realization and sieve criterion. We give its finite certificate: the rational trial, support comparisons, coordinate enclosures, directed arithmetic and all 97 component bounds. The data archive [2] contains the exact inputs, source code and production records. We first fix the trial and source masks, then derive the positive loss cover. §2 constructs the coordinate enclosures and finite contractions, records the component bounds, and concludes with the numerical margin.

Write γ for Euler's constant, ρ_{D} for the Dickman function, and $\mathcal{L}$ for the probability-law operator. For $c > 0$, let Π_c be the Poisson process on $(0, c]$ with intensity $\mathrm{d}v/v$, and put

$$\nu_c := e^{\gamma} c\, \mathcal{L}(\Pi_c), \qquad \bar{\nu}_c(\mathrm{d}t) := \rho_{\mathrm{D}}(t/c)\, \mathrm{d}t.$$

The second measure is the total-size pushforward of the first [1, (2.4)–(2.6)]. For $0 < b \le c$, the probability of no points in $(b, c]$ is b/c, so $\nu_c|_{\{\max X \le b\}} = \nu_b$. We write $\nu = \nu_\zeta$, where the ambient cap ζ is specified in (1.13); smaller profile caps are restrictions of this same measure. A fragment configuration is a multiset X, with total size $|X| := \sum_{p \in X} p$. For a vector $\mathbf{Y} = (X_j)_{j \ne i}$, let $\mathbf{Y} \oplus_i X$ denote insertion of X in coordinate i. The erased-coordinate operator is

$$E_i F(\mathbf{Y}) := \int F(\mathbf{Y} \oplus_i X)\, \mathrm{d}\nu(X).$$

For an activation ξ, put $M_\xi(X) := \max(\{p \in X : p > \xi\} \cup \{0\})$. For a nonnegative function ϕ, the inclusive obstruction is

$$H_{\phi,\xi}(X) := \max\left(\left\{ \sum_{q \in X,\, q \ge p} q + \phi(p) : p \in X,\ p > \xi \right\} \cup \{0\} \right).$$

Thus equal fragment sizes remain in the inclusive prefix; an empty maximum is zero throughout. All form and support notation below agrees with the accompanying main paper.

1.1. The trial function.

For the normalized fragment measure, a coordinate total under a largest-fragment cap c has density $\rho_{\mathrm{D}}(t/c)\, \mathrm{d}t$. Totals exceeding c are allowed. We take

$$\begin{aligned} k &:= 40, & S &:= \frac{2742997}{2624989}, & \rho_* &:= \frac{2624989}{10000000}, \\ T_0 &:= \frac{2499106033}{2624989000}, & T_1 &:= \frac{2510000}{2624989}, & \rho &:= \frac{262499}{1000000}. \end{aligned} \tag{1.1}$$

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Thus, $0 < T_0 < T_1 < S$, with physical radii

$$\rho_* S = \frac{2742997}{10^7}, \qquad \rho_* T_0 = \frac{2499106033}{10^{10}}, \qquad \rho_* T_1 = \frac{251}{1000}. \tag{1.2}$$

The cap cells below fix the trial domains. Their source conditions are verified in §1.3; the common order-two source for inner squares justifies the physical radius $\rho_* T_1 > 1/4$. Use the grid

$$\begin{gathered} N := 98304, \qquad h := S/N, \qquad n := N - 40 = 98264, \\ C_j := [jh, (j+1)h), \qquad t_j^\circ := (j + \tfrac{1}{2})h. \end{gathered} \tag{1.3}$$

A d-coordinate cell with index sum r is assigned to a radial shell $(u, v]$ precisely when

$$u < (r + d)h \le v, \qquad 0 \le r < n. \tag{1.4}$$

Round the fragment cap down to a multiple of h. Let $\mathscr{H}_O, \mathscr{H}_0, \mathscr{H}_1, \mathscr{H}_{\mathrm{f}}$ be the outer, base, enlarged and full domains specified by the masks below, where j is the sum of the coordinate indices.

domain	j-range	z/h
outer ($d = 40$)	$0 : 89196$	68225
	$89197 : 95598$	49152
	$95599 : 98263$	46580
base ($d = 39$)	$0 : 84930$	68225
	$84931 : 87194$	44781
	$87195 : 89524$	35265
enlarged ($d = 39$)	$0 : 85161$	68225
	$85162 : 87249$	44976
	$87250 : 89914$	35419
full ($d = 39$)	$0 : 98263$	68225

(1.5)

Here $a : b$ denotes the inclusive integer interval. The masks in (1.5) have largest retained outer and full-face totals $(N-1)h$ and $(N-2)h$, respectively; the actual least common multiple bounds use these distinct endpoints.

The cap domains satisfy $\mathscr{H}_0 \subseteq \mathscr{H}_1 \subseteq \mathscr{H}_{\mathrm{f}}$. Indeed, their radial endpoints in (1.5) satisfy $89524 \le 89914 \le 98263$. On every shell the base cap is at most the enlarged cap, which is at most the full cap $68225h$; the outer cap indices $68225, 49152, 46580$ also do not exceed 68225. For $\mathbf{t} = (t_i)_{i=1}^{40}$, with $t_i = |X_i|$, put $s := \sum_i t_i$, $P_j(\mathbf{t}) := \sum_i t_i^j$ and $P_\sigma := \prod_{j \in \sigma} P_j$, with $P_\varnothing := 1$. The ordered signatures are

$$\mathscr{S} := (\varnothing, (2), (3), (4), (5), (6), (2,2), (2,3), (2,4), (3,3), (2,2,2)). \tag{1.6}$$

For radial degrees $0 \le d \le 6$, use the positive profile

$$g(t) := \frac{21/200}{1 + t/100} + \frac{179/200}{1 + (907/5)t}. \tag{1.7}$$

The 77 rational coefficients $c_{\sigma,d}$, in exactly this order, are printed in §1.2. Define

$$\widetilde{F}(\mathbf{X}) := \mathbf{1}_{\mathscr{H}_O}(\mathbf{X}) \prod_{i=1}^{40} g(t_i) \sum_{\sigma \in \mathscr{S}} \sum_{d=0}^{6} c_{\sigma,d} (s - 9/10)^d P_\sigma(\mathbf{t}). \tag{1.8}$$

The trial used below is the step function F obtained from $\widetilde{F}$ on the grid (1.3). On each cell, evaluate the profile and polynomial at the rational midpoint, using $(r + 20)h - 9/10$ as the radial argument. All face forms use the same outer trial and unrestricted face.

Put $Z := \sum_{j<n} g(t_j^\circ)^2 > 0$. Throughout this paper, every quadratic form, including displayed squared norms, pairings and loss integrals, is divided by $(hZ)^{40}$; we retain the same notation for these normalized forms. This common factor cancels from all quotients and multiplies both sides of every restoration inequality. With $V_i := E_i F$, define

$$I_H \coloneqq \|F\|_2^2, \qquad J_0 \coloneqq \sum_i \int_{\mathscr{H}_0} |V_i|^2 \, \mathrm{d}\nu^{39}, \qquad J_+ \coloneqq \sum_i \int_{\mathscr{H}_1 \setminus \mathscr{H}_0} |V_i|^2 \, \mathrm{d}\nu^{39}, \tag{1.9}$$

$$J_{\mathrm{t}} \coloneqq \sum_i \int_{\mathscr{H}_{\mathrm{f}} \setminus \mathscr{H}_1} |V_i|^2 \, \mathrm{d}\nu^{39}. \tag{1.10}$$

In the square of V_i, the two integrated coordinates are independent copies conditional on the same retained fragment configuration. This convention is used in every face calculation.

1.2. **Trial coefficients.** Use the signatures in (1.6), radial degrees $0 \le d \le 6$, and the exact coefficients in Table 1.1:

$$c_{\sigma,d} \coloneqq 10^{-10} C_{\sigma,d}. \tag{1.11}$$

They specify the fixed trial (1.8); no coefficient is selected during evaluation. Every quadratic form and loss in this paper uses the common normalization $(hZ)^{-40}$ of §1.1.

TABLE 1.1. Integer coefficients of the trial, with $c_{\sigma,d} = 10^{-10} C_{\sigma,d}$.

σ	$C_{\sigma,0}$	$C_{\sigma,1}$	$C_{\sigma,2}$
$\varnothing$	10000000000	-264598476112	834262268474
(2)	71070047507	7222861788586	-48747932657986
(3)	6457252424873	-61201446212885	28811649792090
(4)	-16779263512274	128033707910825	169290603857215
(5)	28114161526671	-276375633435566	-690482702304933
(6)	-21150553032771	379875924448266	-929062247569514
$(2,2)$	-19004216224617	136674974139652	-183344485344333
$(2,3)$	31171802814567	-97428388152051	526635309233054
$(2,4)$	-53602626608739	329397902441121	1613505134333316
$(3,3)$	27653330903418	-290549334488305	-1847330641348475
$(2,2,2)$	12374547113901	-244168600145684	1603694896120437

σ	$C_{\sigma,3}$	$C_{\sigma,4}$
$\varnothing$	-3540575351215	5377491111325
(2)	290976672545723	-1136422724027134
(3)	-2058803084231281	9970156747759406
(4)	7359669931312727	-35090795379920588
(5)	-15120456749385986	47650276584638031
(6)	-13967219843969236	168416050564605519
$(2,2)$	1114879023072977	1119583327002910
$(2,3)$	1222047580792220	15846796161434448
$(2,4)$	13893066541430270	-126262434123562668
$(3,3)$	28351866831729468	-61505472221886320
$(2,2,2)$	-21603130476787649	-30285492734943698

σ	$C_{\sigma,5}$	$C_{\sigma,6}$
$\varnothing$	116705356572254	121730820431102
(2)	-2058910631434711	1375878942948547
(3)	38278849934023144	34806920812932737
(4)	-120997133235510923	-141527585901304670
(5)	372137534144492224	2191913505882230103
(6)	-248058724714138769	-4161797957833172083
$(2,2)$	-47568062965320963	2537631525616777
$(2,3)$	-165623953461271580	-2358019996221938729
$(2,4)$	239934302929501943	-1506861482386002243
$(3,3)$	-424266003419714347	2585236507449911535
$(2,2,2)$	381987976419637874	3605516061450295448

1.3. **Source supports.** For activated fragments use the inclusive obstruction

$$H_{m,\xi}(X) \coloneqq \max_{p\in X,\ p>\xi} \left(\sum_{q\in X,\ q\geq p} q + (m-1)p \right), \qquad m \in \{2, 5/2\}. \tag{1.12}$$

The inclusive convention is retained when physical prime cells coincide. The source masks are determined by rational data. Put

$$\begin{gathered} g_0 \coloneqq 10^{-7}, \qquad \epsilon_{\mathrm{o}} \coloneqq 10^{-6}, \qquad \epsilon_{\mathrm{n}} \coloneqq 10^{-7}, \qquad \tau \coloneqq 10^{-10}, \\ \sigma_0 \coloneqq \frac{100001}{10^6}, \qquad \sigma_{m,\tau} \coloneqq \frac{1}{2} - \frac{40481}{100000} + \tau, \\ \Omega_{\mathrm{o}} \coloneqq \frac{12499}{10^6}, \qquad \Omega_{\mathrm{n}} \coloneqq \frac{253}{20000}, \qquad \rho \coloneqq \frac{1}{4} + \Omega_{\mathrm{o}}, \qquad \rho_* \coloneqq \rho - g_0, \\ S \coloneqq \frac{2742997}{10^7 \rho_*}, \qquad T_1 \coloneqq \frac{251}{1000\rho_*}, \\ T_0 \coloneqq 2 - \frac{3}{1000} - S, \qquad e \coloneqq \frac{g_0}{\rho}, \qquad \zeta \coloneqq \frac{19037}{100000\rho}. \end{gathered} \tag{1.13}$$

The subscripts o, n denote the old full-prime and new minorant ladders, with $T_{\mathrm{o}} \coloneqq T_0$ and $T_{\mathrm{n}} \coloneqq T_1$. Define

r	$(c_{\mathrm{o}r}, s_{\mathrm{o}r})$	$(c_{\mathrm{n}r}, s_{\mathrm{n}r})$
1	$((1-5\sigma_0)/15, 18/5)$	$((1-5\sigma_{m,\tau})/15, 18/5)$
2	$((1-4\sigma_0)/16, 7/2)$	$((1-4\sigma_{m,\tau})/16, 7/2)$
3	$(3/80, 3)$	$((1-2\sigma_{m,\tau})/20, 16/5)$

(1.14)

Let r_t be $1, 2, 3$ on the blocks $0 \leq t < 12$, $12 \leq t < 24$, and $t \geq 24$, respectively. Starting at $\omega_{\nu,-1} \coloneqq 0$, set $E_\nu \coloneqq \rho(S + T_\nu) - 1/2$ and iterate

$$\begin{aligned} \omega_{\nu,t} &\coloneqq \min\left\{ \Omega_\nu, \frac{c_{\nu r_t} - \epsilon_\nu - E_\nu + 2\omega_{\nu,t-1} - g_0}{s_{\nu r_t}} \right\}, \\ \delta_{\nu,t} &\coloneqq c_{\nu r_t} - s_{\nu r_t}\omega_{\nu,t} - \epsilon_\nu. \end{aligned} \tag{1.15}$$

Stop at the first occurrence of Ω_ν. The old and new ladders have 29 and 43 rows, with order counts $(12, 12, 5)$ and $(12, 12, 19)$. Indeed, within an order block, if $W_{\nu r} \coloneqq (c_{\nu r} - \epsilon_\nu - E_\nu - g_0)/(s_{\nu r} - 2)$, the untruncated value after j steps from v is $W_{\nu r} + (2/s_{\nu r})^j (v - W_{\nu r})$.

For each row define

$$\begin{gathered} B_{\nu,t} \coloneqq \frac{1/2 + 2\omega_{\nu,t-1}}{\rho}, \qquad B^+_{\nu,t} \coloneqq \frac{1/2 + 2\omega_{\nu,t}}{\rho}, \qquad \xi_{\nu,t} \coloneqq \delta_{\nu,t}/\rho, \\ a_{\nu,t} \coloneqq B_{\nu,t} - T_\nu, \qquad b_{\nu,t} \coloneqq B_{\nu,t} - S, \qquad \eta_{\nu,t} \coloneqq \begin{cases} \xi_{\nu,t}, & r_t \leq 2, \\ (\xi_{\nu,t} + S + T_\nu - B_{\nu,t})/2, & r_t = 3, \end{cases} \\ A_{\nu,t} \coloneqq a_{\nu,t} + \eta_{\nu,t}, \qquad C_{\nu,t} \coloneqq b_{\nu,t} + \eta_{\nu,t}. \end{gathered} \tag{1.16}$$

In order three, put $L_{\nu,t} \coloneqq 23C_{\nu,t}/40$, $\phi_D(u) \coloneqq \min(3u/2, L_{\nu,t})$, and $\phi_E(u) \coloneqq 3u - \phi_D(u)$. For every nonterminal row, $\delta_{\nu,t} - E_\nu + 2\omega_{\nu,t-1} = g_0$; hence the retained thresholds are

$$(A_{\nu,t}, C_{\nu,t}) = \begin{cases} (S + e, T_\nu + e), & r_t \leq 2, \\ (S + e/2, T_\nu + e/2), & r_t = 3. \end{cases} \tag{1.17}$$

The continuous fragment caps are as follows; all radial intervals are lower-open and upper-closed except those starting at zero. The caps apply to each fragment, not to coordinate totals.

domain	radial interval	$z(s)$
outer	$[0,a_{\mathrm{n},0}]$	ζ
	$(a_{\mathrm{n},0},a_{\mathrm{n},24}]$	$(S+e)/2$
	$(a_{\mathrm{n},24},S]$	$S+e/2-23(T_1+e/2)/40$
base	$[0,b_{\mathrm{o},12}]$	ζ
	$(b_{\mathrm{o},12},b_{\mathrm{o},24}]$	$(T_0+e)/2$
	$(b_{\mathrm{o},24},T_0]$	$63(T_0+e/2)/160$
enlarged	$[0,b_{\mathrm{n},12}]$	ζ
	$(b_{\mathrm{n},12},b_{\mathrm{n},24}]$	$(T_1+e)/2$
	$(b_{\mathrm{n},24},T_1]$	$63(T_1+e/2)/160$
full	$[0,S]$	ζ

(1.18)

Use the grid (1.3). For d coordinates whose indices sum to j, a shell $(l,u]$ retains exactly

$$\max\{0,\lfloor l/h\rfloor-d+1\}\le j\le\min\{98263,\lfloor u/h\rfloor-d\},\tag{1.19}$$

with fragment cap $h\lfloor z/h\rfloor$. This uses the full upper endpoint $(j+d)h$ and gives the complete masks in (1.5). The largest retained totals are $S_\square:=98303h$, $T_{0,\square}:=89563h$, and $T_{1,\square}:=89953h$. The actual-LCM rule retains precisely

$$\begin{aligned}&\mathcal{T}_\nu:=\{(\nu,t):B_{\nu,t}<S_\square+T_{\nu,\square}\},\\&\mathcal{T}_{\mathrm{o}}=\{(\mathrm{o},t):0\le t\le 27\},\qquad\mathcal{T}_{\mathrm{n}}=\{(\mathrm{n},t):0\le t\le 38\}.\end{aligned}\tag{1.20}$$

For a retained row, abbreviate (ν,t) by t. Its predicates on a root of total s are

$$\begin{aligned}\mathscr{O}_t&:=\begin{cases}\{s\le a_t\}\cup\{H_{2,\xi_t}\le A_t\}, & r_t\le 2,\\ \{s\le a_t\}\cup\{H_{\phi_D,\xi_t}\le A_t,\phi_E(M_{\xi_t})\le C_t\}, & r_t=3,\end{cases}\\ \mathscr{I}_t&:=\begin{cases}\text{the whole space}, & r_t=1,\\ \{s\le b_t\}\cup\{H_{2,\xi_t}\le C_t\}, & r_t=2,\\ \{s\le b_t\}\cup\{H_{\phi_E,\xi_t}\le C_t,\phi_D(M_{\xi_t})\le A_t\}, & r_t=3.\end{cases}\end{aligned}\tag{1.21}$$

Set $P_\nu^D:=\prod_{t\in\mathcal{T}_\nu}\mathbf{1}_{\mathscr{O}_t}$ and $P_\nu^E:=\prod_{t\in\mathcal{T}_\nu}\mathbf{1}_{\mathscr{I}_t}$. On the outer cap, $P_O=P_{\mathrm{o}}^DP_{\mathrm{n}}^D$. The raw inner predicates $\mathscr{L}_{\mathrm{old}}$ and $\mathscr{L}_{\mathrm{new}}$ have the following characteristic functions on face i, of total s_i:

$$L_{\mathrm{old},i}:=\mathbf{1}_{\{s_i\le T_0,\ M_0\le\zeta\}}P^E_{\mathrm{o},i},\qquad L_{1,i}:=\mathbf{1}_{\{s_i\le T_1,\ M_0\le\zeta\}}P^E_{\mathrm{n},i}.$$

The actual inner domains are $\mathscr{L}_1=\mathscr{H}_1\cap\mathscr{L}_{\mathrm{new}}$ and $\mathscr{L}_0=\mathscr{H}_0\cap\mathscr{L}_{\mathrm{old}}\cap\mathscr{L}_{\mathrm{new}}$; in particular, the base domain must satisfy the new conditions too. On face i, let $H_{a,i}$ and $R_{a,i}$ be multiplication by $\mathbf{1}_{\mathscr{H}_a}$ and $\mathbf{1}_{\mathscr{L}_a}$, respectively, for $a=0,1$. Thus $R_{0,i}=H_{0,i}L_{\mathrm{old},i}L_{1,i}$ and $R_{1,i}=H_{1,i}L_{1,i}$. The old-only predicate is distinct from the complete base predicate, and no unretained row is imposed.

The six error groups are $O_2,O_{5/2},I_2^{\mathrm{old}},I_{5/2}^{\mathrm{old}},I_2^{\mathrm{new}},I_{5/2}^{\mathrm{new}}$. In the same order, their dimensions are $40,40,39,39,39,39$. Their source rows are

$$\begin{aligned}&\{(\mathrm{o},t),(\mathrm{n},t):0\le t\le 23\},\qquad\{(\mathrm{o},t):24\le t\le 27\}\cup\{(\mathrm{n},t):24\le t\le 38\},\\&\{(\mathrm{o},t):12\le t\le 23\},\qquad\{(\mathrm{o},t):24\le t\le 27\},\\&\{(\mathrm{n},t):12\le t\le 23\},\qquad\{(\mathrm{n},t):24\le t\le 38\}.\end{aligned}\tag{1.22}$$

For a group G, the subscript is m_G, the activation is ξ_G, the threshold is U_G, and its radial interval is $(b_G, t_G]$. The aligned cap z_G and splitting point p_G are fixed by

$$\begin{array}{c|c|c|c} G & \xi_G & U_G & b_G \\ \hline O_2 & \xi_{\mathrm{o},23} & S+e & a_{\mathrm{n},0} \\ O_{5/2} & \xi_{\mathrm{n},38} & S+e/2 & a_{\mathrm{n},24} \\ I_2^{\mathrm{old}} & \xi_{\mathrm{o},23} & T_0+e & b_{\mathrm{o},12} \\ I_{5/2}^{\mathrm{old}} & \xi_{\mathrm{o},27} & T_0+e/2 & b_{\mathrm{o},24} \\ I_2^{\mathrm{new}} & \xi_{\mathrm{n},23} & T_1+e & b_{\mathrm{n},12} \\ I_{5/2}^{\mathrm{new}} & \xi_{\mathrm{n},38} & T_1+e/2 & b_{\mathrm{n},24} \end{array} \tag{1.23}$$

$$\begin{array}{c|c|c|c} G & t_G & z_G/h & p_G/h \\ \hline O_2 & a_{\mathrm{n},24} & 49152 & 24576 \\ O_{5/2} & 98303h & 46580 & 19660 \\ I_2^{\mathrm{old}} & b_{\mathrm{o},24} & 44781 & 22390 \\ I_{5/2}^{\mathrm{old}} & 89563h & 35265 & 17912 \\ I_2^{\mathrm{new}} & b_{\mathrm{n},24} & 44976 & 22488 \\ I_{5/2}^{\mathrm{new}} & 89953h & 35419 & 17990 \end{array} \tag{1.24}$$

In particular, $p_G = h\lfloor U_G/(2m_G h)\rfloor$. Intersect the appropriate rows of (1.5) with

$$\lfloor b_G/h\rfloor - d_G + 1 \le j \le \lfloor t_G/h\rfloor, \qquad d_G \in \{39, 40\}, \tag{1.25}$$

to retain every box meeting the radial event. For a low bin ending at u, include every originating row, also those of the preceding order-two group when $m_G = 5/2$. If c_t, V_t are its radial core and threshold, and $\mu_t = 2$ for source orders one and two, $\mu_t = 5/2$ for order three, the safe lower boundary is

$$c_G(u) := \max\left\{b_G, \min_{t:\, \xi_t < u} \max\{c_t, V_t - (\mu_t - 1)u\}\right\}. \tag{1.26}$$

An empty eligible set gives an empty event; otherwise retain $j \ge \lfloor c_G(u)/h\rfloor - d_G + 1$. In particular, an absorbed row keeps its own order in this formula.

Lemma 1.1 (Source-data comparisons)**.** *Every retained row satisfies its strict distribution inequalities, actual-LCM transfer guards, and the level and density retreat to ρ_*. The retained bands cover all mixed moduli on the physical cells, and all bounds for the largest fragment and opposite root hold. In order three,* $\max(A_t, C_t) \le 7L_t/3$*. The six groups satisfy $\xi_G > 2h$, $\xi_G < p_G \le z_G$, and $z_G + m_G p_G \le U_G$. The new order-two row* 12 *also implies the common inner self-square source.*

Proof. The block formula after (1.15) gives the row counts and these rational enclosures:

	$r = 1$	$r = 2$	$r = 3$
$10^6\omega_{\mathrm{o},t}$	$[2533, 5696]$	$[7051, 8855]$	$[10332, 12499]$
$10^6\omega_{\mathrm{n},t}$	$[2676, 6018]$	$[7267, 8932]$	$[10326, 12650]$

At the endpoints, the old rows satisfy the corresponding bilinear conditions of [8, Prop. 2.16] in orders one and two, and [8, (2.14)] in order three, together with the Type III condition [8, (2.12)]. Use $\sigma = \sigma_0$ in orders one and two, and $\sigma = 1/10 + (3 - 240\omega - 80\delta)/40$ in order three. For the new rows, verify [8, Prop. 2.22] with $\sigma = \sigma_{m,\tau}$ in the bilinear bounds [8, (2.11)] and [8, (2.14)], as appropriate to the order, and with $68\omega + 14\delta < 1$ in orders one and two. Reduce all three upper bounds in [8, (2.24)] by τ. In particular, the Type III parameter can be chosen below $19/200 - \tau$. These comparisons are affine in ω. The least positive density is $89999/10^{11}$, and every required strict affine margin exceeds 10^{-8}.

By (1.17), the lower-order transfer guards reduce to $S - T_\nu + e \ge 0$ and $\min(2T_\nu - S, 2S - T_\nu) + e \ge 0$. In retained order three, $\eta_t = \xi_t - e/2 > 0$, $A_t + C_t = B_t + \xi_t$, and $A_t \ge C_t > L_t$, with $A_t \le 11C_t/10 < 161C_t/120 = 7L_t/3$. The bounds for the largest fragment and opposite root are therefore $A_t - L_t$ and $63C_t/160$; the lower-order cap is half the threshold. Each exceeds its activation. Taking their running minimum after each core, together with ζ, gives (1.18). Inward cap rounding and full upper cell endpoints give (1.5), including cells crossing a core; the terminal rows lower no cap.

The four boundary comparisons are

(ν, t)	$(\mathrm{o}, 27)$	$(\mathrm{o}, 28)$	$(\mathrm{n}, 38)$	$(\mathrm{n}, 39)$
$\lfloor B_{\nu,t}/h \rfloor$	187773	188087	188254	188256

The pair maxima are $187866h, 188256h$, and $B^+_{\nu,t} = B_{\nu,t+1}$. These comparisons prove the retained-row rule and complete band coverage. Since $\rho_* < \rho$, every level and density endpoint, including the initial Bombieri–Vinogradov band, is strictly retreated. Also,

$$\min_{t \in \mathcal{T}_\mathrm{o} \cup \mathcal{T}_\mathrm{n}} \xi_t > 33 \cdot 10^{-6} > 22 \cdot 10^{-6} > 2h.$$

The minimum thresholds and activations give the six-group tables; their rational entries verify the splitting guards. In each role the later order-5/2 activation and threshold are no larger than the earlier order-two ones. For the common inner-square source, use the fixed parameters of [8, (4.15)]:

$$\omega_s := \frac{31}{10000}, \qquad \delta_s := \frac{21319}{800000},$$
$$\xi_s := \delta_s/\rho, \qquad c_s := \frac{1}{2\rho} - T_1.$$

Substitution in the prime and minorant distribution inequalities gives positive margins for both weights, and

$$\frac{1}{2} + 2\omega_s - 2\rho T_1 = \frac{55122259}{13124945000} > 0, \qquad c_s + \xi_s > 0.$$

The rational source recurrence gives $c_s - b_{\mathrm{n},12} > 4/100$ and $\xi_s - \xi_{\mathrm{n},12} > 5/100$. Since $C_{\mathrm{n},12} = b_{\mathrm{n},12} + \xi_{\mathrm{n},12}$, their sum also gives $C_{\mathrm{n},12} < c_s + \xi_s$. Thus new row 12 implies the common source. $\square$

Remark 1.2 (Old and new inner supports). Take $s := (b_{\mathrm{o},25} + b_{\mathrm{n},25})/2$, 42 fragments of size 11/500, and one fragment of size $s - 231/250 > 0$. This root lies in the base cap, satisfies every old predicate and fails new row 25, hence belongs to $I^{\mathrm{new}}_{5/2}$. Small distinct perturbations preserve these strict inequalities, so the new-inner condition on the base face cannot be omitted.

1.4. **The hybrid form.** Use the deficit bound and hybrid parameters from [8]:

$$\begin{gathered}\kappa_{\mathrm{def}} := \frac{1}{50000}, \qquad m_0 := 1-\kappa_{\mathrm{def}} = \frac{49999}{50000},\\ K_{\mathrm{ex}} := \frac{17}{50}, \qquad \lambda := \frac{1}{125}.\end{gathered} \tag{1.27}$$

The hybrid coefficients are

$$\begin{aligned} a_h &:= m_0^2 - m_0\lambda = \frac{2479900401}{2500000000},\\ b_h &:= (1-m_0/\lambda)\kappa_{\mathrm{def}}K_{\mathrm{ex}} = -\frac{843183}{1000000000},\\ d_0 &:= 1-a_h-b_h = \frac{44415113}{5000000000}.\end{aligned} \tag{1.28}$$

The signed cap numerator is formed from disjoint positive layers:

$$J_{\lambda,H} = J_0 + (a_h+b_h)J_+ + b_h J_{\mathrm{t}}. \tag{1.29}$$

Its quotient is $\rho_* J_{\lambda,H}/I_H$.

Charge the failures of the actual inner domains separately:

$$\beta_{\mathrm{old}} := \sum_i \int_{\mathscr{H}_0\setminus\mathscr{L}_{\mathrm{old}}} |V_i|^2\,\mathrm{d}\nu^{39}, \qquad \beta_{\mathrm{new}} := \sum_i \int_{\mathscr{H}_1\setminus\mathscr{L}_1} |V_i|^2\,\mathrm{d}\nu^{39}. \tag{1.30}$$

Their coefficients are d_0 and $d_0 + a_h = 1 - b_h = 1000843183/10^9$, respectively. On a retained face put

$$h_i(\mathbf{Y}) := \begin{cases} 1, & \mathbf{Y}\in\mathscr{H}_0,\\ \max\{|a_h+b_h|,|b_h|\}, & \mathbf{Y}\in\mathscr{H}_1\setminus\mathscr{H}_0,\\ |b_h|, & \mathbf{Y}\notin\mathscr{H}_1.\end{cases} \tag{1.31}$$

If $\{\mathscr{B}_j\}$ is a positive cover of the bad outer set and $c_j>0$, Young's inequality gives the nonnegative loss

$$\mathcal{E}_O := \sum_{i,j}\int_{\mathscr{B}_j} h_i(\mathbf{Y})\left(c_j|F(\mathbf{Y}\oplus_i X_i)|^2 + \frac{|V_i(\mathbf{Y})|^2}{c_j}\right)\mathrm{d}\nu^{40}. \tag{1.32}$$

Positive majorants of event indicators may be used in this formula.

Work in $L^2(\mathscr{H}_O,\nu^{40})$, extending functions by zero outside the outer cap. With E_i^* the Hilbert-space adjoint of E_i, the true face operator is

$$\mathcal{B} := \sum_i E_i^*(d_0R_{0,i} + a_hR_{1,i} + b_h)E_i.$$

The negative term acts on the unrestricted face.

Proposition 1.3 (Signed quadratic-form restoration)**.** *For the operator $\mathcal{B}$ defined above,*

$$\rho_*\langle P_OF,\mathcal{B}P_OF\rangle - \|P_OF\|_2^2 \ge \rho_*\left(J_{\lambda,H} - d_0\beta_{\mathrm{old}} - (1-b_h)\beta_{\mathrm{new}} - \mathcal{E}_O\right) - I_H. \tag{1.33}$$

Proof. The difference between the cap and true face operators is $\sum_i E_i^*(d_0(H_{0,i}-R_{0,i}) + a_h(H_{1,i}-R_{1,i}))E_i$, which is positive. Its value at F is bounded by the two separately charged losses in (1.30). Apply [8, Prop. 4.6] and [8, (4.33)]. The discarded outer-norm term has coefficient

$$1-4\rho_*|b_h| = \frac{2497786653900013}{2500000000000000} > 0.$$

The negative unrestricted-face coefficient is unchanged throughout; no commutation of a face operator with P_O is used. $\square$

1.5. **A positive cover of the support failures.** For the next lemma, suppress the row index in A_t, C_t, L_t for a retained order-three row. The allocation functions are

$$\phi_D(p) := \min(3p/2, L), \qquad \phi_E(p) := 3p - \phi_D(p), \qquad L := 23C/40. \tag{1.34}$$

Lemma 1.4. *After the bounds for the largest fragment and opposite root have been imposed, every retained order-three failure is a nonlargest $H_{5/2}$ failure at the corresponding threshold.*

Proof. Lemma 1.1 gives $\max(A, C) \leq 7L/3$. Below $2L/3$ both allocation functions equal $3p/2$. Above $2L/3$, a nonlargest witness has inclusive prefix at least $2p$; the two nonlinear obstructions and the balanced obstruction are at least $2p + L$, $5p - L$, and $7p/2$, all greater than $7L/3$. Thus, the predicates fail together. The imposed caps exclude a largest witness. □

The rank-one comparisons of Lemma 1.1 and Lemma 1.4 reduce all failures to

$$O_2, \qquad O_{5/2}, \qquad I_2^{\text{old}}, \qquad I_{5/2}^{\text{old}}, \qquad I_2^{\text{new}}, \qquad I_{5/2}^{\text{new}}. \tag{1.35}$$

The activation, threshold, cap, radial interval, splitting point, and every originating source row are given in (1.22)–(1.24). We now fix the witness intervals and verify their cover by rank.

For a group G write its parameters as (d, m, ξ, U, z, p_G).

1.5.1. *Witness intervals.* Write $(G_1, \ldots, G_6)$ for the six groups in their stated order, and in each list below use $\xi = \xi_G$, $p = p_G$. Put $a_j := (6/5)^j/20$ for $0 \leq j \leq 9$, and let $\|$ concatenate lists. The complete low-bin boundaries are

$$\begin{aligned}
\mathcal{U}_1 &:= (\xi, 3\xi/2, a_0) \| (a_j)_{j=4}^{8} \| ((a_8 + a_9)/2, a_9, p), \\
\mathcal{U}_2 &:= (2^j \xi)_{j=0}^{8} \| (1/100, 3/200, 9/400, 27/800) \\
&\qquad \| (a_j)_{j=0}^{7} \| ((a_7 + p)/2, p), \\
\mathcal{U}_3 &:= (\xi, a_0, a_4, a_6, a_8, p), \\
\mathcal{U}_4 &:= (\xi, 2\xi, 1/100, 27/800, a_3, a_5, a_6, p), \\
\mathcal{U}_5 &:= (\xi, a_1, a_4, a_5, a_7, a_8, p), \\
\mathcal{U}_6 &:= (2^j \xi)_{j \in \{0,1,4,6,8\}} \| (9/400, a_1, a_3, a_5, a_6, a_7, p).
\end{aligned}$$

Consecutive endpoints u_{j-1}, u_j define L_j, with slope $\lceil 7/u_j \rceil$, except for slope 120 at (G_1, L_3) and slope $\lceil 9/u_j \rceil$ throughout G_6. For rank-two bins put $q_0 := U_G/(m_G + 1)$. Consecutive t_{j-1}, t_j below define the interval $(q_0 + t_{j-1}(z_G - q_0), q_0 + t_j(z_G - q_0)]$:

$$\begin{aligned}
\mathcal{T}_1 &:= (j/6)_{j=0}^{6}, & \mathcal{T}_2 &:= (j/16)_{j=0}^{8} \| (5/8, 3/4, 7/8, 1), \\
\mathcal{T}_3 &:= (0, 1), & \mathcal{T}_4 &:= (0, 1/2, 1), \\
\mathcal{T}_5 &:= (0, 1/6, 1/2, 2/3, 1), & \mathcal{T}_6 &:= (0, 1/8, 3/8, 1/2, 3/4, 1).
\end{aligned}$$

Each group also has its single third-factorial component above p_G.

1.5.2. *Low witnesses.* A low witness in row t implies both $s > c_t$ and $s > V_t - (\mu_t - 1)u$, proving (1.26).

If a witness belongs to a low bin $(\ell, u]$, then

$$L_{\leq \ell} < s + (m - 1)u - U. \tag{1.36}$$

Consequently, for the positive rational slope θ, its event is covered by the positive Palm sum

$$\sum_{p \in X \cap (\ell, u]} \exp(\theta \{s + (m - 1)u - U - L_{\leq \ell}\}). \tag{1.37}$$

Lemma 2.3 supplies every small-fragment completion and every possible number of remaining large fragments.

Lemma 1.5 (Full-radius factorization)**.** *If $u^+ = h\lceil u/h\rceil$, the residual radial factor is*

$$W_r = \exp\Big(\theta\{rh - U + (m-1)u^+ + dh\}\Big)\,. \tag{1.38}$$

For an erased coordinate of index v, use W_{r+v} before summing over that coordinate.

Proof. The identity $\prod_i e^{\theta(j_i h - U/d)} = e^{\theta(rh-U)}$ together with the full-cell correction proves the formula. On an erased-coordinate term the full index is $r+v$, giving exactly (2.20). ☐

Lemma 1.6 (Containment of the low-witness intervals)**.** *The radial restrictions in* (1.26)*, applied to the displayed* 61 *low intervals, retain every source failure assigned to those intervals.*

Proof. For an eligible originating row of effective order μ, a witness below u implies both $s > c_t$ and $s > V_t - (\mu-1)u$. Taking the minimum over all eligible rows, including absorbed order-two rows, gives exactly (1.26). The outward full-cell test then retains every cell meeting this event. The finite schedule also has the following useful direct check. Of the 40 order-5/2 low intervals, the first $11, 2, 5$ in the outer, old-inner, and new-inner groups, respectively, end below every absorbed order-two activation. On the remaining $11, 5, 6$ intervals the order-three minimum is the group's original lower radial boundary. These assertions follow by substituting the displayed endpoint lists in (1.26); thus they impose no additional restriction on an eligible absorbed order-two event. Each of the 61 intervals is nonempty on this grid. ☐

1.5.3. *Rank-two witnesses.* If the second-largest witness exceeds p_G, let q be the largest fragment. A necessary rank-two condition is

$$q > \frac{U}{m+1}, \qquad p > d(q) := \max\{p_G, \xi, (U-q)/m\}, \qquad q \le z. \tag{1.39}$$

The six exact groups have $\max(p_G, \xi) \le U/(m+1)$, so $d(q) < q$ on this range. Omitting p_G, ξ from the maximum enlarges the event and gives the positive kernels in Lemma 2.4. At fixed coordinate totals, write $A_i := D_q(t_i)$ and $B_i := D_{d(q)}(t_i)$, shifting t_j to $t_j - q$ when coordinate j contains the largest fragment, where $D_c(t) := \rho_{\mathrm{D}}(t/c)$ for $t \ge 0$. Terms with $t_j < q$ vanish. The one-largest Palm density is

$$\sum_j \int \left(A_j \prod_{i\ne j} A_i - B_j \prod_{i \ne j} B_i \right) \frac{\mathrm{d}q}{q}. \tag{1.40}$$

All its differences are nonnegative. A convenient positive upper bound for the integrand belonging to j is

$$(A_j - B_j) \prod_{i \ne j} A_i + B_j \sum_{i \ne j} (A_i - B_i) \prod_{v \ne i,j} A_v. \tag{1.41}$$

Let P be the ordinary measure, A the measure for the coordinate containing the largest fragment, B that for a secondary fragment in a different coordinate, and C that for both fragments in the same coordinate. Put

$$Q := P + xA + yB + xyC \qquad \text{in } \mathbb{R}_{\ge 0}[x,y]/(x^2, y^2). \tag{1.42}$$

The two marks occupy one coordinate or two ordered distinct coordinates, so

$$[xy]Q^d = dP^{d-1}C + d(d-1)P^{d-2}AB. \tag{1.43}$$

On separating an erased coordinate, the four placements are

$$C_e P_r + P_e (xy)_r + A_e y_r + B_e x_r. \tag{1.44}$$

Lemma 1.7 (Feasible logarithmic mass)**.** *In a largest-fragment interval* $(q_-, q_+]$*, put* $L \coloneqq \log(q_+/q_-)$ *and*

$$\mathscr{L}(v) \coloneqq \begin{cases} 0, & v \le q_-, \\ \log(v/q_-), & q_- < v < q_+, \\ L, & v \ge q_+. \end{cases} \tag{1.45}$$

For a physical cell with upper endpoint t^+*, its largest-fragment coordinate and same-coordinate measures may be multiplied, respectively, by outward upper bounds for*

$$\frac{\mathscr{L}(t^+)}{L}, \qquad \frac{\mathscr{L}((mt^+ - U)/(m-1))}{L}. \tag{1.46}$$

The second fraction may also be capped by the first. Charge the common factor L *once.*

Proof. A largest fragment requires $q < t^+$. If the second fragment is on the same coordinate, then $t^+ > q + (U-q)/m$, whence $q < (mt^+ - U)/(m-1)$. Integrating $\mathrm{d}q/q$ over these feasible parts of the original interval gives the two fractions. The other factors have been bounded over the whole original interval, and each monomial in (1.43) contains exactly one factor carrying the largest fragment. Thus, its fraction is applied exactly once. A positive lower bound for L is used in the denominator of its outward evaluation. □

1.5.4. *Higher ranks.* Every remaining witness above p_G has at least three fragments above p_G, and hence $\mathbf{1}_{N_{>p_G} \ge 3} \le \binom{N_{>p_G}}{3}$. The identity

$$\binom{\sum_i N_i}{3} = \sum_i \binom{N_i}{3} + \sum_{i \ne j} \binom{N_i}{2} N_j + \sum_{i<j<\ell} N_i N_j N_\ell \tag{1.47}$$

is realized by the factorial kernels in (2.8), with the factorial divisor included once. The low and rank-two partitions and one factorial term per group therefore cover every offending rank. All lists increase strictly between the required endpoints. Together with (1.26) and Lemma 1.6, they therefore omit no witness interval, including events absorbed from lower-order rows.

2. The finite calculation

2.1. **Coordinate kernels.** Use $C_j = [jh, (j+1)h)$, $g_j \coloneqq g((j+\frac{1}{2})h)$, $Z_0 \coloneqq 1$, $Z_2 \coloneqq Z \coloneqq \sum_{j<n} g_j^2$, and $w_j^{(e)} \coloneqq g_j^e / Z_e$ for $e \in \{0, 2\}$. All polynomials here are reduced modulo X^N; only $j < n$ enter the final arrays. Arrows denote outward dyadic extraction at 160-bit real-ball precision. For a source upper bound, its positive coordinate measures are fixed before a signed square is expanded; the cap forms retain two-sided enclosures. Indeed, $0 \le \mu \le \mu^+$ implies $\int |P|^2 \,\mathrm{d}\mu \le \int |P|^2 \,\mathrm{d}\mu^+$, even when P has signed coefficients. This comparison is made for one fixed positive measure before contracting its signed moments.

Lemma 2.1 (Elementary enclosures)**.** *The logarithms and exponentials used below admit finite rational, and hence outward dyadic, enclosures of the prescribed accuracy.*

Proof. After binary scaling, logarithms reduce to $1 \le y \le 2$, with $v \coloneqq (y-1)/(y+1) \le 1/3$. The standard series bounds are

$$0 \le \log y - 2 \sum_{r=0}^{M-1} \frac{v^{2r+1}}{2r+1} \le \frac{2v^{2M+1}}{(2M+1)(1-v^2)},$$

$$0 \le e^v - \sum_{r=0}^{M} \frac{v^r}{r!} \le \frac{v^{M+1}}{(M+1)!\{1 - v/(M+2)\}}.$$

For exponentials take $v := |x|/2^b \in [0, 1/2]$, restore $e^{|x|}$ by squaring, and invert if $x < 0$. Treat $\log 1$ and e^0 exactly; otherwise refine to relative width at most 2^{-164}, excluding zero first for a logarithm. Use monotonicity for interval arguments, then round outwards. $\square$

Put $D_c(t) := \rho_{\mathrm{D}}(t/c)$ for $t \geq 0$, and zero for $t < 0$. Its cell mass is $A_j(c) := \int_{C_j} D_c(t)\,\mathrm{d}t$; the normalized fragment law requires no further conditioning on the cap.

Lemma 2.2 (Dickman cell enclosures)**.** *For $c = Mh$, $M \in \mathbb{Z}_{\geq 1}$, initialize $\underline{d}_j = \overline{d}_j = 1$ for $0 \leq j \leq M$, and set*

$$\underline{d}_{j+1} := \max\left\{0, \downarrow\left(\underline{d}_j - \frac{\overline{d}_{j-M}}{2j} - \frac{\overline{d}_{j+1-M}}{2(j+1)}\right)\right\},$$
$$\overline{d}_{j+1} := \min\left\{1, \uparrow\left(\overline{d}_j - \frac{\underline{d}_{j-M}}{2j} - \frac{\underline{d}_{j+1-M}}{2(j+1)} + \frac{1}{2M^3}\right)\right\} \qquad (j \geq M). \tag{2.1}$$

Then $\underline{d}_j \leq D_c(jh) \leq \overline{d}_j$. Accept only ordered intervals in $[0,1]$ with both endpoint sequences nonincreasing. For $j < M$, $A_j(c) = h$; for $j \geq M$,

$$\max\left\{0, \downarrow\left(\frac{h}{2}(\underline{d}_j + \underline{d}_{j+1}) - \frac{h^3}{12c^2}\right)\right\} \leq A_j(c) \leq \uparrow\left(\frac{h}{2}(\overline{d}_j + \overline{d}_{j+1})\right). \tag{2.2}$$

The last enclosure may be intersected with $[0, h]$.

Proof. The Dickman delay equation [1, (2.4)–(2.6)], $D_c'(t) = -D_c(t-c)/t$, gives $0 \leq -D_c' \leq c^{-1}$ and $0 \leq D_c'' \leq c^{-2}$ on smooth intervals above c. For $K(t) := D_c(t-c)/t$, $0 \leq K'' \leq 6c^{-3}$ on each such grid cell; its derivative changes at $2c, 3c$ lie at endpoints. Thus, the trapezoid excess for K lies in $[0, h^3/(2c^3)]$. Integrating the delay equation proves (2.1) inductively, since all delayed endpoints are available. The trapezoid excess for D_c is at most $h^3/(12c^2)$, giving (2.2). Clipping uses $0 \leq D_c \leq 1$. On the first delay interval, $D_c(t) = 1 - \log(t/c)$ also provides an independent consistency check. $\square$

For an unaligned cap use $c^- := h\lfloor c/h \rfloor$ below and $c^+ := h\lceil c/h \rceil$ above, since D_c increases with c. Every lower cap used here contains a positive cell. The cap arrays enclose $w_j^{(e)} A_j(c)/h$, and the erased marginal uses $g_j A_j(c)/h$: these are exact cell integrals for the step trial, not midpoint quadrature.

For small caps use the positive renewal majorant

$$R_j := 1 \qquad (j \leq M), \qquad R_j := \frac{1}{j}\sum_{a=j-M}^{j-1} R_a \qquad (j > M), \qquad M \geq 2. \tag{2.3}$$

The identity $u\rho_{\mathrm{D}}(u) = \int_{\max(0,u-1)}^{u} \rho_{\mathrm{D}}(v)\,\mathrm{d}v$ and a left Riemann sum give $R_j \geq \rho_{\mathrm{D}}(j/M)$. The exact rational R_j is carried in a two-sided interval: initialize the sliding sum to $[M, M]$, divide outwards by j, then add the new interval and subtract the departing one. Reject a nonpositive lower endpoint. Only once its upper bound is at most a lower bound for 10^{-40} may the majorant be set to the constant 10^{-40} thereafter; monotonicity justifies this tail. For low kernels choose the delay recurrence when $M \geq 64$ and its acceptance tests pass, and otherwise the renewal recurrence. Write $d_j^\star(Mh)$ for the chosen exact majorant and $I_j(Mh)$ for its enclosure. In the delay branch these are $\overline{d}_j$ and its singleton interval; in the renewal branch the full two-sided intervals must remain through the products below.

For a low bin $(\ell_0, u_0]$ and hard cap z_0, put $L := \lfloor \ell_0/h \rfloor$, $H := \lceil u_0/h \rceil$, $K := \lceil z_0/h \rceil$, and $(\ell, u, z) := (Lh, Hh, Kh)$; the data satisfy $2 \leq L < H \leq K \leq N$. For its rational slope $\theta > 0$, define

$$Q(X) := \sum_{a=L}^{K-1} \frac{X^a}{a}, \qquad M(X) := \sum_{a=L}^{H-1} \frac{X^a}{a}, \qquad D_\theta(X) := h\sum_{j<N} d_j^\star(\ell) e^{-\theta jh} X^j. \tag{2.4}$$

The enclosure of D_θ uses the complete intervals $I_j(\ell)$, not their upper endpoints. Let $E_r(X) \coloneqq r!^{-1}\sum_{a=0}^{r-1}\operatorname{Eu}(r,a)X^a$ and $R_r(X) \coloneqq 1 + X + \cdots + X^{r-1}$, where

$$\operatorname{Eu}(1,0) = 1, \qquad \operatorname{Eu}(r,a) = (a+1)\operatorname{Eu}(r-1,a) + (r-a)\operatorname{Eu}(r-1,a-1), \tag{2.5}$$

and out-of-range indices are zero. Put $q_r \coloneqq Q^r/r!$ and $J \coloneqq \min(32, \lfloor (N-1)/L \rfloor)$. For $e = 0,1$,

$$T_e \coloneqq \sum_{r\ge 0} q_r R_{r+e+1} = \frac{\exp Q - X^{e+1}\exp(XQ)}{1-X} \pmod{X^N},$$
$$\widehat{T}_e \coloneqq \sum_{r=0}^{J} q_r E_{r+e+1} + \uparrow\left(T_e - \sum_{r=0}^{J} q_r R_{r+e+1}\right).$$

The positive omitted sum is enclosed by subtracting the enclosed prefix from an upper enclosure of T_e; a negative upper endpoint is rejected. It vanishes if $J = \lfloor (N-1)/L \rfloor$. Define the physical-mass polynomials

$$\mathsf{B}_\theta \coloneqq D_\theta \widehat{T}_0, \tag{2.6}$$
$$\mathsf{M}_\theta \coloneqq D_\theta M \widehat{T}_1. \tag{2.7}$$

Lemma 2.3 (Eulerian cell kernels). *The coefficients of $\mathsf{B}_\theta, \mathsf{M}_\theta$ majorize, respectively, the physical-cell masses of*

$$(e^{-\theta t} D_\ell(t)\,\mathrm{d}t) * \exp_*\left(\mathbf{1}_{(\ell,z]}(v)\frac{\mathrm{d}v}{v}\right),$$
$$\mathbf{1}_{(\ell,u]}(v)\frac{\mathrm{d}v}{v} * (e^{-\theta t} D_\ell(t)\,\mathrm{d}t) * \exp_*\left(\mathbf{1}_{(\ell,z]}(v)\frac{\mathrm{d}v}{v}\right).$$

No small-fragment mass or high-fragment count is omitted.

Proof. Independence of disjoint Poisson restrictions [7, Thm. 5.2] gives, on total sizes, $\bar\nu_z = \bar\nu_\ell * \exp_*(\mathbf{1}_{(\ell,z]}\mathrm{d}v/v)$: the void factor ℓ/z cancels the change in normalization. Reduced Mecke marking [7, Thm. 4.5] adds the designated factor. On C_a, $\mathrm{d}v/v$ is bounded by constant density $1/(ah)$, of mass $1/a$; on C_j, the tilted low density is bounded by $d_j^\star(\ell)e^{-\theta jh}$. For r uniform fractional cell coordinates, unit-cube inclusion-exclusion gives carry mass

$$\frac{1}{r!}\sum_{b=0}^{a+1}(-1)^b\binom{r+1}{b}(a+1-b)^r = [X^a]E_r.$$

Thus, the carry is E_{r+1}, or E_{r+2} with a designated fragment. Since $0 \le E_r \le R_r$ coefficientwise, the replicated remainder bounds every omitted term. Terms with $rL \ge N$ vanish, so the construction is finite and every comparison is positive. □

Fix the upward endpoints of $w_j^{(e)}[X^j]\mathsf{B}_\theta/h$ and $w_j^{(e)}[X^j]\mathsf{M}_\theta/h$ as the ordinary and designated arrays. Apply (1.38) at the full index before erasure.

For the third-factorial bound above p_G, take $L \coloneqq \lfloor p_G/h \rfloor$, $K \coloneqq \lceil z_0/h \rceil$, form Q as above and set $D_0 \coloneqq h\sum_{j<N}\overline{d}_j(Lh)X^j$. Define, for $a = 0,1,2,3$,

$$\mathcal{F}_a \coloneqq D_0\frac{Q^a}{a!}\sum_{b=0}^{\lfloor (N-1-aL)/L\rfloor}\frac{Q^b}{b!}E_{a+b+1} \pmod{X^N}, \tag{2.8}$$

with empty sums zero. The fixed array is $\uparrow(w_j^{(e)}[X^j]\mathcal{F}_a/h)$. Reduced factorial Palm marking and the same carry calculation show that it bounds the cell measure weighted by $\binom{N_{>p_G}}{a}$. The factor $1/a!$ is charged once. Extracting total degree three therefore includes every higher rank and all placements in physical coordinates.

For a largest-fragment bin $(a,b]$, assume $U/(m+1) \le a < b \le z_0 < U$, $m \in \{2, 5/2\}$. Drop the splitting and activation cutoffs to enlarge the rank-two event, and put

$$r(q) := (U-q)/m \le q, \qquad r_- := (U-b)/m, \qquad r_+ := (U-a)/m,$$
$$s_j^- := \max(0, j - \lceil b/h \rceil), \qquad s_j^+ := \max(0, j+1-\lfloor a/h \rfloor).$$

Let $t_j^+ := (j+1)h$, $\widehat{w}_j^{(e)} :=\uparrow w_j^{(e)}$, and let $\beta_j = 0$ for $jh < b$, $\beta_j = \underline{d}_{s_j^+}(r_-^-)$ otherwise. With upward caps in upper grids and downward caps in lower grids, set

$$\begin{aligned}
\mathsf{P}_j^{(e)} &:=\uparrow (\widehat{w}_j^{(e)} \overline{d}_j(b^+)),\\
\mathsf{A}_j^{(e)} &:= \mathbf{1}_{t_j^+ > a} \uparrow (\widehat{w}_j^{(e)} \overline{d}_{s_j^-}(r_+^+)),\\
\mathsf{B}_j^{(e)} &:=\uparrow (\widehat{w}_j^{(e)} [\overline{d}_j(b^+) - \underline{d}_{j+1}(r_-^-)]_+),\\
\mathsf{C}_j^{(e)} &:= \mathbf{1}_{t_j^+ > a + r_+} \uparrow (\widehat{w}_j^{(e)} [\overline{d}_{s_j^-}(b^+) - \beta_j]_+).
\end{aligned} \tag{2.9}$$

Here $[v]_+ := \max(v, 0)$, and differences are enclosed upward before clipping.

Lemma 2.4 (Whole-bin Palm envelopes)**.** *These arrays bound the ordinary, largest-fragment coordinate, distinct-secondary, and same-coordinate coordinates, respectively. Put $\Lambda := \log(b/a)$, and let $\lambda(v) := \min\{1, \max\{0, \log(v/a)/\Lambda\}\}$ for $v > 0$, with $\lambda(v) := 0$ for $v \le 0$. Multiply the arrays for the largest fragment and for coincident marks by outward upper bounds on the respective fractions*

$$\lambda(t_j^+), \qquad \lambda\Big(\min\{t_j^+, (mt_j^+ - U)/(m-1)\}\Big),$$

respectively. The external factor Λ is charged once.

Proof. Off the coordinate containing the largest fragment use $A = D_q(t)$, $B = D_{r(q)}(t)$; on that coordinate use $A = D_q(t-q)$, $B = D_{r(q)}(t-q)$, with $t \ge q$. All differences are positive since $r(q) \le q$. The product difference splits as in (1.41), giving the four coordinate roles. Monotonicity in cap and nonnegative total, together with $r_- \le r(q) \le r_+$ and $t - q \ge \max(0, jh - b)$, gives the upper endpoints. For $jh \ge b$, also $t - q \le t_j^+ - a$, giving the lower endpoint in the same-coordinate difference. A cell containing the largest fragment is empty if $t_j^+ \le a$, and a same-coordinate cell if $t_j^+ \le a + r_+$, since $q + r(q)$ increases. No negative residual total is evaluated. Finally, a largest fragment requires $q < t_j^+$, and two marks in the same coordinate also require $q < (mt_j^+ - U)/(m-1)$. Integrating $\mathrm{d}q/q$ over these feasible parts gives the stated fractions, each applied once to the sole factor carrying its largest fragment. Use a positive lower bound for Λ in the denominator, clip at one, and bound the second fraction by the first if helpful. Lemma 2.1 supplies the enclosures. $\square$

2.2. **Finite contractions.** Write $t_j := (j + \frac{1}{2})h$, $x_r := (r+20)h - 9/10$, $p_\sigma(r) := \sum_{d=0}^{6} c_{\sigma,d} x_r^d$, and $d_c(j) := A_j(c)/h$. All radial polynomials in this subsection are reduced modulo X^n; $\sigma \sqcup \tau$ denotes sorted concatenation of signatures. For a nonnegative coordinate array K, also allowing coefficients in a positive marking ring, set

$$\begin{aligned}
P_e[K](X) &:= \sum_{j<n} K_j t_j^e X^j,\\
M_{d,\sigma}[K](X) &:= \sum_{\pi \in \mathrm{Part}(\sigma)} (d)_{|\pi|} P_0[K]^{d-|\pi|} \prod_{B \in \pi} P_{\sum_{v \in B} v}[K].
\end{aligned} \tag{2.10}$$

Here $(d)_r$ is the falling factorial. Equal entries in σ remain distinct labels in its set partitions; the empty signature contributes $P_0[K]^d$. Assigning each power-sum occurrence to a coordinate and collecting equal coordinates proves the identity, with nonnegative integer multiplicities. For cap moments put $K_j(c) \coloneqq g_j^2 d_c(j)/Z$ and $M^c_{d,\sigma} \coloneqq M_{d,\sigma}[K(c)]$, and define

$$Q_\eta(r) \coloneqq \sum_{\substack{\sigma,\tau\in\mathscr{S}\\ \sigma\sqcup\tau=\eta}} p_\sigma(r)p_\tau(r). \tag{2.11}$$

The pairs are ordered, so off-diagonal products occur twice. If c_s, $s = 1, 2, 3$, are the decreasing outer caps and χ_s their radial masks in (1.5), then

$$I_H = \sum_{s=1}^{3}\sum_{r<n}\chi_s(r)\sum_{\eta} Q_\eta(r)[X^r]M^{c_s}_{40,\eta}. \tag{2.12}$$

Each signed square uses one coherent cap measure.

For a labelled subset A of a signature σ, put $e(A) \coloneqq \sum_{j\in A}\sigma_j$, and write $\sigma \setminus A$ for the remaining signature. The affine rows of the erased marginal are

$$v_{s,\tau}(r) \coloneqq \sum_{\substack{\sigma\in\mathscr{S},\ 0\le d\le 6\\ A\subseteq\{1,\dots,|\sigma|\},\ \sigma\setminus A=\tau}} c_{\sigma,d}\sum_{a=0}^{n-1-r}\chi_s(r+a)x^d_{r+a}g_a d_{c_s}(a)t_a^{e(A)}. \tag{2.13}$$

This is $P_j(\mathbf{t}, t_a) = P_j(\mathbf{t}) + t_a^j$, with repeated entries counted by their labelled subsets. Put $v^{(l)}_\tau \coloneqq \sum_{s\le l} v_{s,\tau}$ and $c_4 \coloneqq 0$. When the largest shared fragment lies in $(c_{l+1}, c_l]$, the actual marginal is $v^{(l)}$; form this signed sum before squaring. For affine rows v, moment rows M, and radial mask A, set

$$\mathcal{S}(v, M; A) \coloneqq \sum_{r<n} A(r)\sum_{\tau,\upsilon} v_\tau(r)v_\upsilon(r)M_{\tau\sqcup\upsilon}(r). \tag{2.14}$$

List all distinct positive cap values increasingly as $b_1 < \cdots < b_t$, put $M^{b_0} \coloneqq 0$, and set $\Delta M^a \coloneqq M^{b_a}_{39,\cdot} - M^{b_{a-1}}_{39,\cdot}$. These are positive layer measures. Let $l(a) \coloneqq \#\{s : c_s \ge b_a\}$, and let $\chi_{0,a}, \chi_{1,a}, \chi_{\mathrm{f},a}$ be the three face masks in layer a. Then

$$\begin{aligned}
J_0 &= \frac{40h}{Z}\sum_{a:l(a)>0}\mathcal{S}(v^{(l(a))}, \Delta M^a; \chi_{0,a}),\\
J_+ &= \frac{40h}{Z}\sum_{a:l(a)>0}\mathcal{S}(v^{(l(a))}, \Delta M^a; \chi_{1,a} - \chi_{0,a}),\\
J_{\mathrm{t}} &= \frac{40h}{Z}\sum_{a:l(a)>0}\mathcal{S}(v^{(l(a))}, \Delta M^a; \chi_{\mathrm{f},a} - \chi_{1,a}).
\end{aligned} \tag{2.15}$$

Nestedness makes these masks nonnegative. Enclose each layer difference from both sides and intersect with the interval from zero to its cumulative upper bound.

For a source component, take the ordinary and marked arrays from §2.1. Use profile power two on retained coordinates and zero on an additional erased coordinate. The marking rings and extracted coefficients are

component	coordinate polynomial	coefficient
L_j	$P + yD \pmod{y^2}$	$[y]$
P_j	$P + xA + yB + xyC \pmod{x^2, y^2}$	$[xy]$
H	$F_0 + yF_1 + y^2F_2 + y^3F_3 \pmod{y^4}$	$[y^3]$

(2.16)

The entries denote coordinate arrays. Write $[*]$ for the selected coefficient, $[X^r*] \coloneqq [X^r][*]$, $K^{(2)}, K^{(0)}$ for the profile choices, and $M_{d,\eta} \coloneqq M_{d,\eta}[K^{(2)}]$. All coordinate placements, including coincident marks, remain. The external factor is $\Lambda \coloneqq \log(q_+/q_-)$ for a rank-two bin and $\Lambda \coloneqq 1$ otherwise.

Let χ_G indicate the full-cell intervals in (1.25), clipped by (1.26) for a low bin. Its radial weight is $W(r) := \exp(\theta\{rh - U_G + (m_G - 1)u^+ + d_G h\})$, where $u^+ := h\lceil u/h\rceil$; otherwise put $W := 1$. Only indices with $\chi_G = 1$ are evaluated, and their exponent is less than 700. For an outer low component, select the dyadic upper array once and use it in both source forms. For an inner low component, keep the same two-sided enclosure of the exact weight in every signed term. The root-square bound is

$$R_G := 40\Lambda \sum_{r<n} \chi_G(r)W(r) \sum_{\eta} Q_\eta(r)[X^r*]M_{40,\eta}. \tag{2.17}$$

The factor 40 bounds the sum of the absolute face multipliers, each at most one. Using the largest group cap and discarding further outer cap restrictions only increases this integral of a square.

For $(u, v] \subseteq (0, c_1]$, put

$$\mathcal{P}(u, v) := \{l \in \{1, 2, 3\} : \max(u, c_{l+1}) < \min(v, c_l)\}. \tag{2.18}$$

The actual signed marginal is one of these prefixes. Use $\mathcal{P} := \mathcal{P}(q_-, q_+)$ for an inner rank-two component and $\mathcal{P} := \mathcal{P}(0, z_G)$ otherwise. The sum of their complete squares therefore gives

$$\beta_G := \frac{40h\Lambda}{Z} \sum_{l\in\mathcal{P}} \mathcal{S}(v^{(l)}, ([X^r*]M_{39,\eta})_{\eta,r}; \chi_G W) \tag{2.19}$$

as a positive inner bound. For an outer component, form in the marking ring

$$C(r) := \sum_{a=0}^{n-1-r} \chi_G(r+a)W(r+a)K_a^{(0)}, \qquad H_\eta(r) := [*](C(r)[X^r]M_{39,\eta}). \tag{2.20}$$

The event and radial weight remain at the full index $r + a$; the erased coordinate carries no extra profile factor. This product contains all four rank-two placements and all factorial degrees of the erased coordinate. With the radial upper bound

$$w(r) := \begin{cases} 1, & (r + 39)h \le T_0, \\ \max\{|a_h + b_h|, |b_h|\}, & T_0 < (r + 39)h \le T_1, \\ |b_h|, & (r + 39)h > T_1, \end{cases}$$

the second outer form is

$$V_G := \frac{40h^2\Lambda}{Z} \sum_{l\in\mathcal{P}(0,z_G)} \mathcal{S}(v^{(l)}, H; w). \tag{2.21}$$

The factors $40h/Z$ and $40h^2/Z$ result from dividing the face square, and its additional erased-coordinate integral, by the same $(hZ)^{40}$. There is no conditional-cap normalization.

2.3. **Interval evaluation.** §2.1 gives finite recurrences and error bounds for every one-coordinate measure, including the Dickman cell integrals, tilted Eulerian measures, largest-fragment terms, and factorial terms. §2.2 specifies the exact polynomial expressions for the cap and source forms. We record the arithmetic rules needed for their signed contractions.

For a positive polynomial and a binary scale $Q = 2^b$, store integer polynomials A^-, A^+ with $0 \le A^-/Q \le A \le A^+/Q$ coefficientwise. Truncated products are enclosed by

$$(AB)^- = \left\lfloor \frac{A^-B^-}{Q} \right\rfloor, \qquad (AB)^+ = \left\lceil \frac{A^+B^+}{Q} \right\rceil. \tag{2.22}$$

Every product on the right is an exact integer polynomial product; all divisions are coefficientwise.

Coefficientwise upper division of P, for $\deg P < r$, is

$$\left\lceil \frac{P}{Q} \right\rceil = \left\lfloor \frac{P + (Q-1)(1 + X + \cdots + X^{r-1})}{Q} \right\rfloor. \tag{2.23}$$

Exact zero prefixes may be removed before multiplication: if $A = X^{v_A}\widetilde{A}$, $B = X^{v_B}\widetilde{B}$ and $v_A + v_B < n$, apply (2.22) to $\widetilde{A}\widetilde{B}$ modulo $X^{n-v_A-v_B}$, then restore $X^{v_A+v_B}$; otherwise the product modulo X^n is zero. Nonnegativity makes the omitted lower coefficients zero too, so this commutes with truncation and coefficientwise division in every marking ring used here.

Reversed positive correlations use the identity

$$\sum_{r=j}^{n-1} I_r K_{r-j} = [X^{n-1-j}] \left(\sum_{r=0}^{n-1} I_{n-1-r} X^r \right) K(X). \tag{2.24}$$

The cap moments use $b = 224$, and the source moments use $b = 192$. Their positive coordinate endpoints are fixed before any signed polynomial is expanded. This makes the set-partition identity (2.10) a two-sided enclosure of one coherent positive measure.

2.3.1. *Signed shell sums.* In a cap layer, the signed sum of the allowed outer-shell marginals is squared before integration. For a positive source bound, the actual marginal is one of the finitely many sums in (2.18); the sum of their complete squares is a valid upper bound. Neither operation replaces signed shell coefficients by their absolute values.

2.3.2. *Final reductions.* The final reductions use binary radix, 53-bit significands, round-to-nearest with ties to even, gradual underflow to 2^{-1074}, and no overflow. These conditions are checked in the producing process. Dyadic endpoints are converted outwards by one predecessor or successor after binary scaling. An integer too large to convert before scaling is first replaced by

$$v_- := \lfloor v/2^r \rfloor, \qquad v_+ := \lceil v/2^r \rceil, \tag{2.25}$$

using a bounded integer prefix, and then scaled by 2^{r-b}. The final outward step includes gradual underflow. Enclose all four endpoint products and take their interval hull.

Put $u := 2^{-53}$ and $\gamma_r := ru/(1-ru)$. If $s = \mathrm{fl}(\sum_{j=1}^{r} x_j)$ and $t = \mathrm{fl}(\sum_{j=1}^{r} |x_j|)$, a lower endpoint is the binary64 predecessor of $\mathrm{fl}(s - \mathrm{fl}(\gamma_r t))$, and an upper endpoint is the corresponding successor with a plus sign. The factors γ_r are evaluated by the same round-to-nearest operations. The required strict rounding slack is

$$(1-u)^2(1-\gamma_{r-1})\gamma_r - \gamma_{r-1} > 0 \qquad (1 \le r \le 98264). \tag{2.26}$$

After removing the positive factor $u/((1-ru)(1-(r-1)u))$, its numerator is

$$1 - (r^2 + r)u + (4r^2 - 3r)u^2 - (2r^2 - 2r)u^3.$$

For $M = 98264$ this is at least $1 - (M^2 + M)u - 3Mu^2 - 2M^2u^3 > 0$, an exact rational comparison. Here ru and $1 - ru$ are exactly representable. Any summation tree of length r has error at most $\gamma_{r-1} \sum |x_j|$. The two factors $1 - u$ account for rounding the division defining γ_r and rounding its product with the absolute sum; $1 - \gamma_{r-1}$ accounts for that sum. Subnormal addition is exact whenever its exact result is subnormal. An underflowing correction has absolute error at most 2^{-1075}, absorbed, together with final-addition rounding, by the adjacent endpoint. Thus, these endpoint reductions enclose the signed scalar products.

All elementary transcendental operations are validated as in Lemma 2.1. The real-ball calculations use precision 160 and the published Arb method [4]. The recorded production used python-flint 0.9.0 and FLINT 3.6.0 with the published signed-FFT conversion correction [3, 6, 5]; the inputs and directed outputs are included in [2].

Construct the coordinate enclosures at 160 bits, then use (2.22) with 224 fractional bits for cap moments and 192 for source moments. Fix source coordinate upper endpoints only after the kernel constructions; evaluate the signed marginal rows by outward real-ball arithmetic, sum their shells before squaring, and use the signed reductions of §2.3. Reject nonfinite or reversed endpoints, negative positive-upper bounds, and any failure of the machine hypotheses. Apply the exact Young parameters and round raw bounds upwards to 10^{-18} and component bounds to 10^{-12}. Acceptance requires (2.33) and all the rational inequalities described below. The index is bounded by N, the factorial degree by three, and every formal exponential is truncated; hence every operation is finite.

2.4. **The component bounds.** For an outer component, let R_j, V_j be the positive upper forms in (2.17) and (2.21). Its loss is

$$E_j \coloneqq c_j R_j + V_j/c_j, \qquad c_j > 0. \tag{2.27}$$

For an old or new inner component use the positive form (2.19), multiplied respectively by

$$d_0 = \frac{44415113}{5000000000}, \qquad 1 - b_h = \frac{1000843183}{1000000000}. \tag{2.28}$$

Let L^+ be the sum of all these 97 component bounds. The positive covers and their common normalizations give

$$L^+ \geq d_0 \beta_{\mathrm{old}} + (1 - b_h)\beta_{\mathrm{new}} + \mathcal{E}_O. \tag{2.29}$$

Set $I_H^- \coloneqq 23685317816 \cdot 10^{-24}$. Tables 2.1–2.6 record all 97 component enclosures and all 52 rational Young parameters. For an outer component their integers q, A, B, E mean

$$\begin{gathered} c \coloneqq q/10^6, \qquad R/I_H^- \leq A/10^{18}, \qquad V/I_H^- \leq B/10^{18}, \\ cA/10^{18} + B/(10^{18}c) \leq E/10^{12}. \end{gathered} \tag{2.30}$$

For an inner component the recorded A, E mean

$$\beta_G/I_H^- \leq A/10^{18}, \qquad \kappa_G A/10^{18} \leq E/10^{12}, \qquad \kappa_G \coloneqq \begin{cases} d_0, & G = I_m^{\mathrm{old}}, \\ 1 - b_h, & G = I_m^{\mathrm{new}}. \end{cases} \tag{2.31}$$

All table entries are upward-rounded integers. Each last-column comparison can be checked by integer multiplication and division.

Table 2.1: Component bounds for $G = O_2$.

component	q	A	B	E
L_1	961904	11	10	1
L_2	502424	2285	577	1
L_3	483341	11432060	2670744	12
L_4	547373	3056104728	915663654	3346
L_5	563915	37877639997	12045112668	42720
L_6	583181	300901046806	102336788484	350961
L_7	604629	2682803914309	980771899210	3244207
L_8	620671	3338737765461	1286194297547	4144522
L_9	629321	7260461043003	2875471614189	9138326
L_{10}	635211	1211995036896	489032601185	1539747
P_1	616326	8286469691008	3147682021553	10214338
P_2	593862	4616001082128	1627937050440	5482540
P_3	573977	2353287968619	775291485464	2701470
P_4	553178	1146587714775	350863740368	1268537
P_5	531463	529511465762	149562603056	562833
P_6	508862	229315416929	59379253693	233381
H	459016	631278927	133008010	580
total				38927522

Table 2.2: Component bounds for $G = O_{5/2}$.

component	q	A	B	E
L_1	7266522	27	1426	1
L_2	1241454497	1	821	1
L_3	1208324400	1	1392	1
L_4	1152630107	1	1765	1
L_5	1126190783	1	3334	1
L_6	1096246679	1	5753	1
L_7	1058983690	1	10303	1
L_8	967816560	1	18815	1
L_9	867471653	1	2089	1
L_{10}	603785822	1	11427	1
L_{11}	32188902	16	16011	1
L_{12}	1308239	14362	24579	1
L_{13}	386321	7065761	1054524	6
L_{14}	373849	17914115	2503735	14
L_{15}	377891	260216687	37159538	197
L_{16}	385136	3305952377	490372043	2547
L_{17}	395013	38054077523	5937779759	30064
L_{18}	405835	352112119115	57993623042	285799
L_{19}	419505	3006707964277	529135146833	2522662
L_{20}	432001	19352692647427	3611707956032	16720799
L_{21}	445139	14498518468563	2872865686933	12907719
L_{22}	457321	28197429960534	5897287451435	25790569
P_1	525975	57148020076132	15810035599715	60116961
P_2	518733	69886316496332	18805329967080	72504766
P_3	515168	69366993102523	18409874222209	71471327
P_4	512357	62684551010344	16455348517458	64233828
P_5	509770	53862830801099	13997130877252	54915393
P_6	507320	44981355032435	11577030541299	45639918
P_7	504951	36911787941323	9411611503675	37277308
P_8	502604	29975466544992	7572139370727	30131606
P_9	503256	50573740961589	12808689167903	50903176
P_{10}	498048	32438336646873	8046407139897	32311736
P_{11}	492222	20308616081603	4920425453752	19992702
P_{12}	485810	12358345921158	2916712121993	12007621
H	433769	15056954296612	2833062492447	13062511
total				622829241

Consequently,

$$\frac{L^+}{I_H} \leq \frac{L^+}{I_H^-} \leq \frac{696075110}{10^{12}} < \frac{697}{10^6}. \tag{2.32}$$

Table 2.3. Bounds for $G = I_2^{\text{old}}$.

component	A	E
L_1	25777	1
L_2	1511410893	14
L_3	18120016651	161
L_4	903601038105	8027
L_5	425243194887	3778
P_1	4871216699917	43272
H	23946432	1
total		55254

Table 2.4. Bounds for $G = I_{5/2}^{\text{old}}$.

component	A	E
L_1	1	1
L_2	3229104	1
L_3	29825526	1
L_4	77797373079	692
L_5	131978724894	1173
L_6	292684783730	2600
L_7	5548294545493	49286
P_1	30283518217418	269010
P_2	12009121688668	106678
H	686922192553	6102
total		435544

Table 2.5. Bounds for $G = I_2^{\text{new}}$.

component	A	E
L_1	467789	1
L_2	381747797	383
L_3	386210860	387
L_4	99885644276	99970
L_5	247732013063	247941
L_6	381057139991	381379
P_1	266162792752	266388
P_2	337097314828	337382
P_3	34427294106	34457
P_4	36820947233	36852
H	18106118	19
total		1405159

Table 2.6. Bounds for $G = I_{5/2}^{\text{new}}$.

component	A	E
L_1	2	1
L_2	107126908277	107218
L_3	1	1
L_4	61	1
L_5	137	1
L_6	177471603	178
L_7	327802576	329
L_8	50667881720	50711
L_9	143104919759	143226
L_{10}	1323952422879	1325069
L_{11}	697854132745	698443
P_1	4234127556194	4237698
P_2	11632061739670	11641870
P_3	3641610451935	3644681
P_4	6136054632765	6141229
P_5	3690866567521	3693979
H	737132501820	737755
total		32422390

Table 2.7. Group totals, including the Young or inner restoration coefficients.

G	low	rank two	high	$10^{12} L_G^+/I_H^- \leq$
O_2	10	6	1	38927522
$O_{5/2}$	22	12	1	622829241
I_2^{old}	5	1	1	55254
$I_{5/2}^{\text{old}}$	7	2	1	435544
I_2^{new}	6	4	1	1405159
$I_{5/2}^{\text{new}}$	11	5	1	32422390
total	61	30	6	696075110

2.5. **The numerical inequality.**

Lemma 2.5 (Finite evaluation)**.** *For the exact trial and physical cells above,*

$$\begin{aligned} 23685317816 \cdot 10^{-24} \le I_H \le 23685317890 \cdot 10^{-24}, \\ J_{\lambda,H} \ge 90248755123 \cdot 10^{-24}. \end{aligned} \tag{2.33}$$

The complete positive source loss satisfies

$$\frac{L^+}{I_H} \le \frac{696075110}{10^{12}} < \frac{697}{10^6}, \qquad \frac{\rho_* J_{\lambda,H}}{I_H} > \frac{500103}{500000}. \tag{2.34}$$

Proof. Evaluate (2.12) and (2.15) with the directed rules above, using the 77 printed coefficients and the exact cap cells. The resulting outward inequalities are (2.33). Since $a_h + b_h > 0$ and $b_h < 0$, the lower numerator uses lower endpoints for J_0, J_+ and the upper endpoint for J_{t}. Substitution of the three integer bounds gives

$$\frac{\rho_*\, 90248755123}{23685317890} - \frac{500103}{500000} = \frac{18806715247}{2368531789000000000} > 0.$$

For each component use the coordinate kernels of §2.1 and the finite contractions in §2.2.

Tables 2.1–2.6 give the directed raw bounds, rational Young parameters and weighted bounds for all 97 components; §2.4 records their six group totals. These sum to $L^+/I_H^- \le 696075110/10^{12}$, where $I_H^- := 23685317816 \cdot 10^{-24} \le I_H$. All bounds use the common normalization and the complete positive cover proved above. □

Corollary 2.6 (Certified margin)**.** *The stated trial and support masks satisfy the source conditions in Lemma 1.1. Their cap forms and complete positive loss cover obey*

$$\begin{aligned} \frac{\rho_*(J_{\lambda,H}^- - L^+) - I_H^+}{I_H^+} &> \frac{500103}{500000} - 1 - \frac{2624989}{10^7}\frac{697}{10^6} = \frac{230382667}{10^{13}} \\ &> \frac{1}{50000}. \end{aligned} \tag{2.35}$$

Here $I_H^-, I_H^+, J_{\lambda,H}^-$ are the endpoints in (2.33)*, and L^+ is the sum of the* 97 *weighted component bounds.*

Proof. Lemma 2.5 gives the cap quotient and $L^+/I_H^- \le 696075110/10^{12} < 697/10^6$. Since $I_H^+ \ge I_H^- > 0$, replacing the denominator of the loss by I_H^+ preserves this upper bound. The displayed rational subtraction proves the margin. The source and containment lemmas verify the supports and retain every assigned failure. □

This is the numerical input to the restoration proposition in [8, Prop. 4.6]. The arithmetic conclusion is proved there.

The data archive [2] fixes the exact trial, masks, complete schedule, source code, rational endpoint records, and corrected runtime provenance. Its recorded production used 160-bit Arb arithmetic through python-flint, 224-bit cap convolutions, and 192-bit source convolutions, with the FLINT signed-FFT conversion correction [4, 6, 3, 5]. The independent exact replay checks identities, coverage, and rational arithmetic from those records; it does not recompute their integrals. A full run of the accompanying script reproduces the complete certificate from the exact inputs.

Remark 2.7 (Scope of the accompanying Lean development). The Lean development uses the same trial coefficients, cells, source schedule and common normalization. Its explicit numerical input consists of the 149 upper inequalities for the physical source integrals indexed by Tables 2.1–2.6: two for each of the 52 outer rows and one for each of the 45 inner rows, together with the three cap bounds in (2.33). The physical source integrals are the quantities majorized by the computational forms (2.17), (2.19) and (2.21); these are not identified by definition.

From that input, Lean proves the rational Young and inner-weight rounding inequalities, the group totals, support restoration and the resulting positive margin. It also proves general Dickman, elementary-enclosure and Eulerian identities and bounds for the first outer low-bin coordinate kernels. It does not yet verify all the finite contractions or the recorded Arb/FLINT production evaluation. This describes the current scope of the formal development.

2.6. A rational bound for the minorant mass.

Lemma 2.8 (Mass of the minorant)**.** *For the exceptional sums $\mathfrak{b}_1, \mathfrak{b}_2$ in* [8, (2.20)–(2.22)]*, the limiting deficit exists and satisfies*

$$0 \le \kappa_{\text{true}} \le \frac{146365385252734375}{15379362287924882625792} < \frac{1}{50000}. \tag{2.36}$$

Proof. Put $t := 481/100000$ and $\xi := 9519/50000$. By [8, Prop. 2.21], both exceptional sums count five primes at least x^{ξ}. The prime number theorem and partial summation give limiting density $1/(\alpha_1 \cdots \alpha_5)$ on $\sum \alpha_i = 1$, in the four independent exponent coordinates. Refine a fixed partition of the exponent region: all prime scales are bounded below by a fixed power of x, and the integrand is bounded. Repeated-prime tuples contribute $O(x^{1-\xi}(\log x)^C)$ and do not change the limit.

Put $z_i := (\alpha_i - 1/5)/t$ and eliminate the last variable using $\sum z_i = 0$. The two four-dimensional polytopes are

$$\begin{aligned}
\mathcal{P}_1 := \{&z_i \ge -2, \qquad z_1 \ge z_2 \ge z_3 \ge z_4, \qquad z_5 \ge z_4,\\
&z_1 + z_2 \le 1, \qquad z_2 + z_3 + z_4 \ge -1, \qquad \textstyle\sum_{i=1}^{5} z_i = 0\},\\
\mathcal{P}_2 := \{&z_i \ge -2, \qquad z_2, z_4 \ge z_3, \qquad z_6 \ge z_5,\\
&z_2 + z_4 \le 1, \qquad z_2 + z_3 + z_5 \ge -1, \qquad \textstyle\sum_{i=2}^{6} z_i = 0\}.
\end{aligned}$$

The remaining upper bounds are redundant, and strict faces have measure zero. In coordinates (z_1, z_2, z_3, z_4) for $\mathcal{P}_1$ and (z_2, z_3, z_4, z_5) for $\mathcal{P}_2$, their vertices, in the indicated order, are

i	$\mathbf{v}_i \in \mathcal{P}_1$	$\mathbf{w}_i \in \mathcal{P}_2$
0	$(-1/3, -1/3, -1/3, -1/3)$	$(-2, -2, -2, 3)$
1	$(0, 0, 0, 0)$	$(-1/3, -1/3, 4/3, -1/3)$
2	$(1/2, 1/2, -3/4, -3/4)$	$(1/2, -2, 1/2, 1/2)$
3	$(1/2, 1/2, -1/3, -1/3)$	$(1/2, 1/2, 1/2, -2)$
4	$(1/2, 1/2, 1/2, -2)$	$(1/2, 1/2, 1/2, -3/4)$
5	$(1/2, 1/2, 1/2, -3/4)$	$(3, -2, -2, -2)$
6	$(4/3, -1/3, -1/3, -1/3)$	$(3, -2, -2, 1/2)$

Coning the facets from the zeroth vertex gives the following triangulations; each five-digit string lists a four-simplex by its vertex indices:

	simplices	their volumes
$\mathcal{P}_1$	01356, 02356, 02456	$125/5184, 625/20736, 625/6912$
$\mathcal{P}_2$	01256, 01346, 01356	$625/288, 625/576, 625/288$

The vertex and facet claims follow by substitution in the displayed linear inequalities; each volume is the absolute determinant of its four edge vectors divided by 4!. Hence,

$$\operatorname{vol}(\mathcal{P}_1) = \frac{125}{864}, \qquad \operatorname{vol}(\mathcal{P}_2) = \frac{3125}{576}.$$

For five numbers $\alpha_i \ge \xi$ of sum 1, their product is at least $\xi^4(1 - 4\xi)$: moving mass between two entries above ξ until one reaches ξ cannot increase their product. The two integral losses are thus at most

$$\frac{t^4}{\xi^4(1-4\xi)} \frac{125}{864}, \qquad \frac{t^4}{\xi^4(1-4\xi)} \frac{3125}{576}.$$

Their sum is the rational bound in (2.36). □

Appendix A. Code availability

The accompanying files `prime_gap_186_certificate.py` and `PrimeGaps186.lean` contain the Python certificate and Lean development, respectively. The certificate uses Python 3.12, NumPy and Python-FLINT linked to FLINT containing the correction in [5]; run `python3 prime_gap_186_certificate.py -workers 4`.

References

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