The matrix multiplication exponent is nine fourths

OpenAI

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Abstract

We prove that the exponent of square matrix multiplication is 9/4 over every field. The upper bound was known. For each $n \geq 2$, a formal arithmetic circuit multiplying two $n \times n$ matrices needs at least $n^{9/4}/2$ multiplication or division gates. The proof shows that a smaller exponent would compress tensor networks enough to violate a lower bound for testing whether the entries in each of several independent lists are distinct.

1 The exponent of matrix multiplication

The product of two square matrices of side n has n^2 entries. Computing the dot product for each entry separately takes order n^3 arithmetic operations. Strassen's seven-product identity showed that sharing work among the entries can save a power of n. How far can those savings go? The conjecture $\omega = 2$ asks whether, for every $\varepsilon > 0$, the product can be computed in $O(n^{2+\varepsilon})$ operations. We show that the arithmetic exponent exceeds the exponent of the output size by exactly one quarter.

Theorem 1.1. *Over every field F, the exponent of square matrix multiplication is $\omega_F = 9/4$.*

Here ω_F is the infimum of the real numbers s for which two $n \times n$ matrices can be multiplied in $O_{F,s}(n^s)$ scalar operations. We use formal arithmetic circuits. Their inputs are independent variables, their constants lie in F, and their gates can add, subtract, multiply or divide by a nonzero rational function. Each output must agree with the corresponding entry of the product as an identity of rational functions. This convention applies also over finite fields.

The 9/4 upper bound was established over $\mathbb{C}$ by OpenAI [41] and over every field by Amanov, van den Berg and Zuiddam [5], following Lü's extension to arbitrary characteristic [39]. We prove the matching lower bound. In fact, for every $n \geq 2$ a circuit for two $n \times n$ matrices requires at least $\lceil n^{9/4}/2 \rceil$ multiplication or division gates, even if linear operations are free. The upper bound gives circuits of size $O_{F,\varepsilon}(n^{9/4+\varepsilon})$ for each $\varepsilon > 0$.

1.1 Rank at each matrix size

A bilinear algorithm forms products of linear combinations of entries from the first matrix with linear combinations from the second. It then combines these products linearly to give all the output entries. The least number of products is the rank of the matrix multiplication tensor

$$T_n = \sum_{i,j,k=1}^{n} e_{ij} \otimes e_{jk} \otimes e_{ki}.$$

The third factor pairs the output at position (i, k) with e_{ki}. Grouping corresponding factors gives $T_n^{\otimes j} \simeq T_{n^j}$. Consequently an improvement at one fixed matrix size can be reused at arbitrarily large sizes.

Geometric border rank allows the tensor to belong to the Zariski closure, over $\overline{F}$, of tensors computed with a given number of products. Asymptotic rank over F is the limit of the j-th roots of the ranks over F of the j-th tensor powers.

Corollary 1.2. *For every $n \geq 2$ and every field F, the geometric border rank of T_n is at least $n^{9/4}$, and the asymptotic rank of T_n over F equals $n^{9/4}$. Every formal arithmetic circuit over F for this product has at least $\lceil n^{9/4}/2 \rceil$ nonscalar multiplication or division gates.*

In particular, an exact bilinear algorithm at side n uses at least $\lceil n^{9/4} \rceil$ products. The factor of two for general circuits comes from extracting the bilinear part of a gate. A multiplication or division introduces at most two new products of linear forms (Lemma 2.4).

The square bound also restricts how wide a rectangular product can be while its cost still has exponent two. Write $T_{a,b,c}$ for the multiplication tensor of an $a \times b$ matrix and a $b \times c$ matrix. Let α_F be the supremum of the $\beta \geq 0$ for which multiplying an $n \times \lfloor n^\beta \rfloor$ matrix by a $\lfloor n^\beta \rfloor \times n$ matrix has exponent two in n.

Corollary 1.3. *For all positive integers a, b, c, the geometric border rank of $T_{a,b,c}$ and its asymptotic rank over F are at least*

$$\max\{ab, bc, ca, (abc)^{3/4}\}.$$

Over every field, $1/2 \leq \alpha_F \leq 2/3$.

Three copies of the rectangular tensor, with the roles of the factors cycled, multiply to the square tensor T_{abc}. This explains the power $3/4$ in Corollary 1.3. The lower endpoint for α_F comes from the known algorithm for inner dimension $\lfloor \sqrt{n} \rfloor$ [5, 55]. Section 2.4 proves the two corollaries and explains the consequence for Strassen's conjecture on tight tensors.

1.2 Why a faster matrix product would test lists too cheaply

Suppose one could save a fixed power of n below $9/4$. A single bilinear algorithm witnessing a strict saving could then be specialized to a finite field. Over that field we compare two estimates for a different computation. Given a list of 2^w words of length $2w$ bits, return one if all the words are distinct and zero if any word repeats. Sorting the words and checking adjacent pairs already gives a small network for this test. The question is how much of its cost can be shared among several lists whose inputs vary independently.

We measure cost by counting the vertices and wires in a tensor network with at most three bits incident at each vertex. The network sums over its internal bits and, for each

setting of its exterior bits, must give the exact zero-one answer. With t lists it must give the product of their t answers. Its internal wiring may connect different lists. Even with that freedom, its volume per input bit exceeds w^σ for some fixed $\sigma > 0$, uniformly for $1 \le t \le 2^w$ and all large w (Theorem 2.2). The proof reduces the problem to routing tagged records. The companion results on tensor geometry and network complexity control this routing or turn cheap routing into a violation of a counting lower bound [42, 43].

The hypothetical matrix multiplication algorithm gives an estimate in the opposite direction. Consider a trivalent network on e exterior bits with size at most $e^{1.05}$, where e is large. We can represent a tensor power of its table in one new network. Per copy, the new network saves a power of the logarithm on the old size and charges an arbitrarily small power of the logarithm per exterior bit (Theorem 2.1). The replicated bits remain individually accessible, so we can apply the construction again. Starting with the sorting test and compressing a fixed number of times gives volume $O(w^\eta)$ per input bit for any chosen $0 < \eta < \sigma$. The batch still has at most 2^w lists. Section 2 gives the short contradiction and the passage back to arbitrary fields. Most of the proof is devoted to deriving this compression from a strict saving in matrix multiplication.

1.3 Three wires and a shared label

To see how the exponent governs network identities, consider a number assigned to each tensor which is multiplicative under tensor products, additive under direct sums and monotone under linear maps on the factors. With the usual normalization this is a spectral point in Strassen's sense [52]. Its logarithm is a *price*. A wire shares an index between two factors. Its price per bit defines a distance between their sites. The companion paper [42] extends these distances to spaces of nonpositive curvature. Section 3 recalls the properties used here.

Figure 1 gives two descriptions of T_n. Each of i, j, k occurs at exactly two factors. The price of the tensor is therefore the sum of three wire lengths times $\log_2 n$. A rank decomposition with r terms supplies the same tensor from a single shared label, the choice of term, with r possible values. If the price per bit of this three-way label is D, monotonicity gives

$$(\text{sum of the three wire lengths}) \log_2 n \;\le\; D \log_2 r.$$

Thus the ratio of a triangle's perimeter to its shared-label rate is a lower bound on the exponent. Prices coming from Euclidean flats give ratio two [42]. A different geometry is needed to reach $9/4$.



Three indices of T_n One label for a rank term

Figure 1. The three tensor factors correspond to the two inputs and the output paired against a third matrix. On the left, A, B, C see $(i, j), (j, k), (k, i)$. On the right, all three see the same label $\ell \in [r]$ and apply their own linear maps.

The critical ratio occurs at three tips of a tripod with unit arms. The distances between tips are all two. If the shared-label rate is $8/3$, the ratio is $6/(8/3) = 9/4$. Under the strict-saving hypothesis such a tripod cannot occur. Part I exploits this exclusion. If varying the parameters of a tensor were to move its preferred site in a certain family of prices, the motion would produce a homogeneous tripod. A rectangular multiplication decoder first determines the limiting rate when one arm is long. The rigidity of polynomial multiplication then carries this equality to the tripod with three equal arms (Proposition 8.1). Ruling out that motion lets us price an entire family of tensors in a single extension of the sites. This synchronization is the geometric input to compression. Once the lower bound is proved, we can also identify the three wire lengths at every price maximizing the value of T_n (Corollary 11.1).

1.4 From prices to a smaller network

The family we use consists of sphere tensors. Their entries are determined by marked points. When these points collide, a tensor expands into contractions through legs of controlled dimension. Under the strict-saving hypothesis we can also determine the largest prices these tensors attain. For prescribed leg widths (logarithms of dimensions), they attain asymptotically the cost of preparing a tensor at the cheapest common site and transporting each leg to its destination (Section 14). The marked points that specify the entries and the sites that determine prices play different roles. Figure 2 shows how this calculation enters the two parts of the network construction.

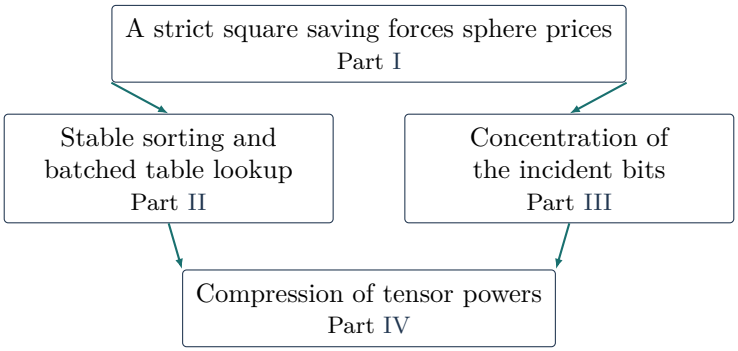

Figure 2. The implication needed for the lower bound. Sorting and concentration give complementary estimates on the same network. Repeated compression would violate the independent-list lower bound of Theorem 2.2.

The compression must pay for each table of a network and for the records that say how tables are connected. Direct lookup in a table of u incident bits can cost 2^u. In a large batch, sorting queries together with table entries amortizes that cost. Part II uses incidence inequalities from sphere tensors to route records. For 2^u records of width $O(u)$ the resulting stable sort costs $O(u^p)$ per record for some $p < 2$. Wider records admit a separate estimate that keeps the cost close to linear in their width.

The strict inequality $p < 2$ matters because the quantity controlled by collisions is proportional to the *square* of a leg width. Part III realizes the wiring with sphere tensors, moves insertions across their seams and merges them. The use of fermionic contractions to reduce wiring is related to the crossing construction of [32, 42] and to compression in subexponential satisfiability algorithms [17, 40]. Merging concentrates the incident bits at

fewer stations while controlling the sum of squared widths. Part IV applies these identities in rounds. The expansion has many summands with different wirings. A single network must describe the possible wirings, route the records, look up the entries and sum the contributions. Counting all four operations gives the logarithmic saving in Theorem 2.1 and completes the contradiction.

1.5 Earlier bounds for matrix multiplication

Strassen applied his seven-product identity recursively to blocks, giving exponent $\log_2 7$ [50]. Subsequent algorithms found savings by working with several matrix products at once. Pan aggregated trilinear forms [46], Bini showed how interpolation converts approximate bilinear identities into exact algorithms [10], and Schönhage's asymptotic sum inequality turned a direct sum of matrix products into an upper bound on the exponent of a single product [47]. Coppersmith and Winograd studied recursive extraction of disjoint products [23]. Strassen's laser method extracts them from tensor powers [51], and Coppersmith and Winograd designed an auxiliary tensor for which progression-free sets accomplish the extraction [24].

The successive improvements depended on how many products could be kept independent. Analyses of higher tensor powers [25, 37, 49, 56], refined extraction and asymmetric hashing [2, 3, 27, 57], and numerical optimization [28] sharpened the bounds obtained from the Coppersmith–Winograd tensor. Staggered extraction gave $\omega_F < 2.371055$ over every field [44]. A separate line of work embedded matrix products in group algebras [18, 20] and in the algebras of coherent configurations [19].

The upper bound $9/4$ comes from polynomial multiplication. In Strassen's asymptotic spectrum [52], asymptotic rank is the maximum of normalized functions that are additive on direct sums, multiplicative on tensor products and monotone under restriction. Over $\mathbb{C}$, the upper-bound proof tracks how these functions value polynomial multiplication as the two input degrees grow. Degenerations give a lower bound on the growth of this profile. Interpolation gives an upper bound that depends on the value at matrix multiplication. Comparing the two bounds yields $9/4$ [41, Theorem 1.1]. The central degeneration separates blocks that have independent variables on two legs and share the third. Partial labels and semidirect sums give related formulations of this step [38, 45]. Lü proved the separation step in arbitrary characteristic [39]. Amanov, van den Berg and Zuiddam obtained the bound over every field [5, Theorem 1.1]. See also the tensor-value proof [26] and Shpilka's explicit algorithms [48]. The rectangular version of the argument [5, 55] connects these estimates to the dual exponent studied in [22, 57].

On the lower-bound side, Bläser proved that the ordinary rank of T_n is at least $\frac{5}{2}n^2 - 3n$ over every field [11]. Over $\mathbb{C}$, Landsberg gave an ordinary-rank bound of $3n^2 - o(n^2)$ [35], and Landsberg and Michałek gave a border-rank bound of $2n^2 - \lceil \log_2 n \rceil - 1$ [36]. A constant factor above n^2 disappears on taking roots along the sequence $T_n^{\otimes j} \simeq T_{n^j}$. Barriers for the laser method and bounds from tensor irreversibility address algorithms that pass through specified intermediate tensors [6, 14]. For polynomial multiplication, Amanov, van den Berg and Zuiddam similarly proved that $9/4$ is optimal for a specified set of degenerations and combination rules [5, Theorem 5.4]. Theorem 1.1 gives the matching lower bound for unrestricted arithmetic circuits. Related work on the spectrum constructs quantum functionals [15] and explicit points over arbitrary fields [4], and describes the sublevel sets of asymptotic rank by polynomial equations [16]. Wigderson and Zuiddam survey this framework [59].

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2 The obstruction from list distinctness

Fix a finite field k. Consider the test that accepts a list precisely when its records are pairwise distinct. Sorting gives one way to perform the test. Our lower bound applies to every tensor network computing it, even when a single network tests many lists at once. The contradiction will come from estimating the cost per input bit. A matrix multiplication exponent below $9/4$ would drive that cost below the lower bound after a fixed number of compressions.

2.1 Two estimates for networks on bits

A *bit tensor* of arity u is a function $\{0,1\}^u \to k$. In a tensor network each vertex carries such a table. A wire pairs two argument positions, and each unpaired position is a named exterior bit. For a fixed setting of the exterior bits, multiply the vertex entries and sum over all settings of the internal wires. This gives the tensor represented by the network. A network is *trivalent* if each vertex has arity at most three. Its *size* counts the vertices and exterior bits. Its *volume* counts the vertices and all wires, including the exterior wires. These counts agree up to an absolute factor by trivalency. For networks of unrestricted arity we count vertices plus their total arity.

The first estimate starts with an arbitrary trivalent network. Its conclusion gives a network for a tensor power, with separate exterior bits for each copy. On these bits it must compute the product of the original answers.

Theorem 2.1 (Compression of tensor powers)**.** *Suppose $\omega_k < 9/4$. For every $\eta > 0$ there is a $\gamma > 0$ such that the following holds for all sufficiently large n. Write $h = \lceil \log_2 n \rceil$. If a trivalent k-network of size $m \le n^{1.05}$ with $e \ge n$ exterior bits represents V, then for some integer $d \ge 1$ its power $V^{\otimes d}$ has a trivalent k-network with de distinct exterior bits and size at most*

$$d(mh^{-\gamma} + eh^{\eta}). \tag{2.1}$$

Here $d \le n^{o(1)}$ uniformly in the input network for fixed k, η.

Section 27 proves Theorem 2.1. Per copy, the first term saves a power of h on the original size. The second charges h^{η} for each exterior bit. The choice of η lets us compare this charge with the cost of list distinctness.

For $w \ge 1$, let D_w be the zero-one tensor for a list of 2^w words of $2w$ bits each. Its value is one exactly when the words are pairwise distinct. It has $n_w = 2w\,2^w$ exterior bits. The tensor $D_w^{\otimes t}$ tests distinctness separately within each of t independently specified lists. The network may connect vertices across lists. Let vol_k denote minimum trivalent volume over k.

Theorem 2.2 (Volume of list distinctness)**.** *There is a constant $\sigma > 0$, depending only on k, such that for all sufficiently large w and every integer $1 \le t \le 2^w$,*

$$\mathrm{vol}_k(D_w^{\otimes t}) > t n_w w^{\sigma}. \tag{2.2}$$

Section 4 proves Theorem 2.2. Its uniformity in t is essential because the replication factor in Theorem 2.1 is chosen by the compression.

2.2 Comparison at a fixed input length

Proposition 2.3. *Over every finite field k, $\omega_k \geq 9/4$.*

Proof. A sorting network on 2^w records uses $O(2^w w^2)$ comparisons [8]. Comparing and conditionally exchanging two $2w$-bit words takes $O(w)$ Boolean gates. After sorting, the conjunction of the inequalities between adjacent words computes D_w. To realize this circuit as a tensor network, use the zero-one table of each gate, copy bits at fan-outs with equality tables, and pin the output to one. For each accepted input there is exactly one consistent assignment of the internal wires. This gives a trivalent network over k of size $m_0 = O(n_w w^2)$.

Suppose $\omega_k < 9/4$. Choose $0 < \eta < \sigma$, take γ from Theorem 2.1, and fix an integer $J > 2/\gamma$. Set $n = n_w$ and $h = \lceil \log_2 n \rceil = w + O(\log w)$. These parameters stay fixed through all J applications of the theorem. At step j write t_j for the number of lists, m_j for the network size and $x_j = m_j/(t_j n)$ for the size per exterior bit. Initially $t_0 = 1$ and $x_0 = O(w^2)$. With replication factor d_j, (2.1) gives

$$t_{j+1} = d_j t_j, \qquad x_{j+1} \leq h^{-\gamma} x_j + h^{\eta}.$$

Inductively, at each of the first J steps the network has at least n exterior bits and $t_j = n^{o(1)}$, since the bound on d_j is uniform and J is fixed. The recurrence gives $x_j = O(w^2 + h^{\eta})$, so $m_j = t_j n x_j \leq n^{1.05}$ for large w. This verifies the size hypothesis at each step. Summing the recurrence gives

$$x_J \leq h^{-J\gamma} x_0 + h^{\eta} \sum_{r=0}^{J-1} h^{-r\gamma} = O(w^{\eta}).$$

Since $\log_2 t_J = o(w)$, Theorem 2.2 applies to these t_J lists. Their network has volume $O(t_J n w^{\eta})$ by trivalency, which violates (2.2) for sufficiently large w. □

2.3 Passing from finite fields to arbitrary fields

The step that permits specialization is to extract a single bilinear identity with a strict rank saving. Its coefficients generate a finitely generated ring, and its equations survive reduction to a finite field.

Lemma 2.4 (Bilinearization of a formal circuit)**.** *If a formal circuit for multiplying two s-by-s matrices over F uses q nonscalar multiplication or division gates, then*

$$\operatorname{rank}_{\overline{F}}(T_s) \leq 2q. \tag{2.3}$$

Here q counts a multiplication when both inputs are nonconstant and a division when its divisor is nonconstant.

Proof. Over the infinite field $\overline{F}$, choose matrices A, B at which all intermediate rational functions are regular and every divisor is nonzero. Evaluate the circuit on $A + uX, B + vY$ with $u^2 = v^2 = 0$. Each gate has a unique expansion

$$f = f_{00} + u f_{10} + v f_{01} + uv f_{11}.$$

Here f_{00} is a scalar, f_{10} is linear in X, and f_{01} is linear in Y. At a product gate $f = ab$, the mixed coefficient satisfies

$$f_{11} = a_{00} b_{11} + a_{11} b_{00} + a_{10} b_{01} + a_{01} b_{10}.$$

Only the last two terms require new products of linear forms. At a quotient gate $f = a/b$, use $bf = a$ and $b_{00} \neq 0$ to obtain

$$b_{00} f_{11} = a_{11} - b_{11} f_{00} - b_{10} f_{01} - b_{01} f_{10}.$$

This again requires at most two new products. Linear operations, including multiplication or division by a field scalar, contribute none. The mixed coefficients of the outputs are the entries of XY. They all belong to the span of at most $2q$ products of an X-linear form and a Y-linear form, giving (2.3). □

Lemma 2.5 (Finite specialization)**.** *If $\omega_F < 9/4$ over a field F, then $\omega_k < 9/4$ over some finite field k.*

Proof. The circuit bound has the form $O(s^{9/4-\epsilon})$ for some fixed $\epsilon > 0$. By Lemma 2.4, at a sufficiently large side $s \geq 2$ we have a rank decomposition of T_s over $\overline{F}$ with $r < s^{9/4}$ terms. Let $R \subseteq \overline{F}$ be the unital ring generated over $\mathbb{Z}$ by all of its coefficients. The decomposition identities are polynomial equations over R with target coefficients zero and one.

Choose a maximal ideal of R. Its residue field k is finitely generated as a $\mathbb{Z}$-algebra. In positive characteristic Zariski's lemma makes k a finite extension of its prime field. In characteristic zero the same lemma over $\mathbb{Q}$ would make k a number field. Multiplying its algebra generators by a common positive integer M would then make them algebraic integers. Consequently the generated ring would lie in the ring of integers of k with M inverted. This ring omits $1/\ell$ for every prime $\ell \nmid M$, whereas the field k contains all of $\mathbb{Q}$. Thus k is finite. Reducing the identities gives $\operatorname{rank}_k(T_s) \leq r$.

Finally, take the j-th power of this fixed bilinear algorithm. Each of the three linear maps can be applied one tensor coordinate at a time. Since $s^2 \leq r$ by flattening rank, every intermediate array has at most r^j entries. The linear work and the r^j scalar products together cost $O_{r,s}(jr^j)$ operations at side s^j. Padding to the next power of s gives $\omega_k \leq \log_s r < 9/4$. □

Proof of Theorem 1.1. Proposition 2.3 and Lemma 2.5 give the lower bound over every field. The upper bound over every field is due to Amanov, van den Berg and Zuiddam [5]. □

2.4 Rank bounds from the exponent

We use the passage from an algorithm at one fixed size to an algorithm on tensor powers, together with the closure argument for border rank [10, 13, 52].

Proof of Corollary 1.2. Fix $n \geq 2$. A rank-r decomposition of T_{n^j} over any field K gives $\omega_K \leq \log_{n^j} r$ by the recursion and padding in Lemma 2.5. Theorem 1.1 therefore gives $\operatorname{rank}_K(T_{n^j}) \geq n^{9j/4}$ at each j. Taking roots gives the asymptotic-rank lower bound. At $K = \overline{F}$ and $j = 1$, Lemma 2.4 now gives the claimed lower bound of $\lceil n^{9/4}/2 \rceil$ on the number of nonlinear gates.

For the geometric border bound, work in the fixed tensor space V over $\overline{F}$ containing T_n. Suppose T_n is in the Zariski closure of the rank-at-most-r tensors. For each j, their j-th powers span a space containing $T_n^{\otimes j}$, since membership of a tensor power in this span is a closed condition. The number of coefficients of a homogeneous polynomial of degree j in $\dim V$ variables bounds the dimension of the span. We can thus express $T_n^{\otimes j}$ using

$O_n(j^{\dim V-1})$ such powers, each of rank at most r^j. Comparison with the lower bound at n^j and passage to j-th roots give $r \geq n^{9/4}$.

For the asymptotic-rank upper bound fix $\varepsilon > 0$. At side n^j the known circuit upper bound and Lemma 2.4 give a decomposition of rank $r_j = O_{F,\varepsilon}(n^{j(9/4+\varepsilon)})$ over $\overline{F}$. Its coefficients belong to a finite extension E_j/F, of degree d_j. For each t, power this decomposition t times. Expand each of the three vectors in each simple tensor in an F-basis of E_j containing 1, then project the products of basis coefficients onto the coordinate of 1. The tensor being represented is defined over F, so this gives

$$\operatorname{rank}_F(T_n^{\otimes jt}) \leq d_j^3 r_j^t.$$

The limit defining asymptotic rank exists by submultiplicativity. First take jt-th roots and let $t \to \infty$ with j fixed. Then let $j \to \infty$ and $\varepsilon \downarrow 0$. The resulting upper bound is $n^{9/4}$ over the original field. □

At finite sizes, Corollary 1.2 also conflicts with the dyadic border-rank estimate proposed in the first version of Sun's preprint [54]. At $n = 2^8$ the proposed upper bound is 2^{17} and our lower bound is 2^{18}.

The square tensor also decides a prediction about tight three-tensors. A support is tight when each local basis admits distinct integer weights whose sum is zero on every support triple. In T_n, take $B > n$ and assign weights $i - Bj$, $Bj - B^2k$ and $B^2k - i$ to e_{ij}, e_{jk}, e_{ki}, respectively. Each assignment is injective on its factor and the three weights sum to zero. Each one-factor flattening has rank n^2. Strassen's conjecture for tight tensors predicts asymptotic rank n^2 in this case [53, Conjecture 5.3]. Corollary 1.2 gives $n^{9/4}$.

Proof of Corollary 1.3. The three one-factor flattenings have ranks ab, bc, ca. These are lower bounds for geometric border rank by the closedness of matrix-rank conditions, and for asymptotic rank by multiplicativity of flattening ranks. Grouping corresponding factors in

$$T_{a,b,c} \otimes T_{b,c,a} \otimes T_{c,a,b} \simeq T_{abc}$$

turns each of the three indices into a tuple with abc possibilities. Both ranks are invariant under cyclic relabeling of the factors and submultiplicative under tensor product. Hence each of their values on $T_{a,b,c}$, cubed, is at least $(abc)^{9/4}$ by Corollary 1.2. The case $abc = 1$ is immediate.

For $a = c = n$ and $b = \lfloor n^\beta \rfloor$, the geometric border bound is $\Omega(n^{3(2+\beta)/4})$. The bilinearization in Lemma 2.4 applies entrywise to rectangular products as well. A quadratic exponent therefore requires $3(2 + \beta)/4 \leq 2$. This gives $\alpha_F \leq 2/3$, and the upper bound for multiplication at $\beta = 1/2$ gives $\alpha_F \geq 1/2$ [5]. □

3 Tensor prices and wire distances

A bilinear algorithm of rank r constructs its tensor from r independent scalar products by local linear maps. A *spectral point* assigns a value to tensors that can only decrease under such maps. Tensoring multiplies the values, so their logarithms add. We call these logarithms *prices*. Applied to a wire, the price measures the cost of sharing an index between its two sites. This is how a spectral point gives a geometry on the sites of a tensor network.

3.1 From tensors to distances

Fix a field and a finite set S of at least two sites. Each tensor has one finite-dimensional vector space at each site. A tensor T is a *restriction* of U if one linear map at each site takes U to T. For a tensor product, group the factors at each site. For a direct sum, take the direct sum of the spaces at every site, so the two tensors occupy separate blocks. The unit $\mathbf{1}$ is the tensor 1 in one-dimensional spaces.

A spectral point φ is a restriction-monotone map to $\mathbb{R}_{\geq 0}$ that preserves zero and the unit and satisfies

$$\varphi(T \oplus U) = \varphi(T) + \varphi(U), \qquad \varphi(T \otimes U) = \varphi(T)\varphi(U).$$

Every nonzero tensor restricts to $\mathbf{1}$, so its value is at least one. The price at scale $D > 0$ is $H(T) = D \log \varphi(T)$, with $H(0) = -\infty$. In this section and Part I logarithms are natural. Thus the prices of [42] are multiplied by $\log 2$ at the same scale.

For a nonempty $A \subseteq S$, the tensor

$$\Delta_A(q) = \sum_{j=1}^{q} \bigotimes_{i \in A} e_j$$

shares a q-valued index among the sites of A. Its other legs are one-dimensional. Tensoring diagonals multiplies their dimensions, and a local projection can discard any set of labels. Comparing powers of dimensions q and 2 therefore gives

$$H(\Delta_A(q)) = \delta(A) \log q, \qquad q \geq 2.$$

We call $\delta(A)$ the common-label rate on A. In particular $\Delta_S(q)$ is a direct sum of q units, so $\delta(S) = D$. For $a \neq b$, $\Delta_{\{a,b\}}(q)$ is a wire of dimension q. Write $d(a, b) = \delta(\{a, b\})$ and set $d(a, a) = 0$. For example, matrix rank across a fixed cut of S is a spectral point. At scale one its wire distance is one for sites on opposite sides of the cut and zero for sites on the same side.

Joining two wires by contracting their common endpoint gives a wire between their other endpoints. Monotonicity and additivity of price give the triangle inequality for d. A q_i-dimensional wire can also move a leg of dimension q_i from x_i to y_i. Moving each leg of a nonzero tensor T, then applying the same argument in reverse, gives the transport bound

$$|H(T; x) - H(T; y)| \leq \sum_i d(x_i, y_i) \log q_i. \tag{3.1}$$

Here $x = (x_i)$ and $y = (y_i)$ are the placements of the legs. Group the legs assigned to one site. When all legs are at a single site, any nonzero tensor there is equivalent to the unit and has price zero.

3.2 Geometric extension and spectral comparison

A *spectral space* is a metric space with compatible prices for all finite placements, whose wire rates are the given distances. Compatibility allows trivial legs to be added and legs at coincident sites to be grouped. On a finite set of positive diameter the scale is its common-label rate, which may vary with the set. At a single point the scale is zero. Identifying sites at zero wire distance turns a spectral price on finitely many sites into a spectral space.

Every spectral space embeds, with all its prices, in a complete CAT(0) spectral space. In the extension every pair a, b has a unique midpoint m, and every point o satisfies

$$d(o,m)^2 \le \tfrac{1}{2}d(o,a)^2 + \tfrac{1}{2}d(o,b)^2 - \tfrac{1}{4}d(a,b)^2.$$

Prices are jointly convex as their legs move along geodesics. Distances and prices may be rescaled together. In a pointed metric ultralimit of these spaces, the price of a fixed tensor on a bounded sequence of placements is the ultralimit of its prices [42, Sections 2 and 2.3, Theorem 4.3 and Proposition 3.3]. We also use convexity of distance and nonexpansiveness of nearest-point projection onto a nonempty closed convex set. A finitely supported probability measure has a unique barycenter b minimizing its mean squared distance. Evaluating the same mean at z increases it by at least $d(z,b)^2$ [12, 42].

Over an algebraically closed field, price sublevel sets in a fixed tensor format are Zariski closed. Under a field extension, the possible joint values of spectral points on tensors defined over the original field are preserved [42, Section 2]. Finally, if two nonzero tensors on the same sites satisfy $\varphi(T) < \varphi(U)$ for every spectral point, there is a fixed integer C such that, for every $j \ge 1$, $T^{\otimes j}$ is a restriction of the direct sum of C copies of $U^{\otimes j}$. We use this strict form of spectral comparison to turn price inequalities into exact tensor restrictions [42, Lemma 2.2].

4 The cost of testing many lists

Two distinctness tests suffice to check that records have reached their specified destinations. This gives a common starting point for the two cases of Theorem 2.2. In a field without ports, the possible destinations force many wires to have large total length in a suitable spectral space. In a finite field with ports, cheap distinctness tests would make it possible to compress a family of tensors with a known lower bound.

4.1 Checking a tagged permutation

Write $h = w - 1$, $N = 2^h$, and index records by the h-bit words. The tagged-permutation tensor $\mathrm{Tag}(h,h)$ of [42, Definition 8.1] takes N records (j_i, x_i) and an ordered list of N payloads y_j, with h bits per tag or payload. Its value is one when $i \mapsto j_i$ is a permutation and $x_i = y_{j_i}$ for every i. Its other entries are zero.

The first D_w test checks the tags. Encode each j_i as $(0, j_i, 0^{h+1})$ and fill the remaining N positions with the distinct words $(1, j, 0^{h+1})$. All $2N$ words have length $2w$, and the test passes precisely when the tags form a permutation π.

The second D_w test checks the payloads. Its first N words are $(j_i, 0, x_i, 0)$. At the position for j in the second half, sum over two choices, the reserve word $(j, 1, 0^h, 0)$ with weight 1 and the comparison word $(j, 0, y_j, 0)$ with weight -1. For tags forming π, the only possible collision at address j is between the comparison word and the word from $i = \pi^{-1}(j)$. Thus the signed sum factors as

$$\prod_j (1 - \mathbf{1}_{\{x_{\pi^{-1}(j)} \ne y_j\}}) = \prod_j \mathbf{1}_{\{x_{\pi^{-1}(j)} = y_j\}}.$$

This identity holds in every characteristic. A summed selector bit for each address controls its word and its weight. Copying the tags, selecting the words, and pinning the constants

cost $O(Nh)$ trivalent volume. Consequently, if $D_w^{\otimes t}$ has a realization of volume V, two uses of that realization give a network for $\mathrm{Tag}(h,h)^{\otimes t}$ of volume at most $2V + O(tNh)$, including when the realization shares work among lists.

4.2 Fields without ports

A class of Hadamard spaces has *ports* if it admits regular graphs of unbounded order and common bounded degree with the following property. For every placement of their vertices into any space of the class, the mean squared distance of two independent uniform vertices is at most a fixed constant times the mean squared edge length. A field has ports when its Hadamard spectral spaces have this property [42, Definition 11.1].

Lemma 4.1 (Uniform destinations)**.** *Let the field have no ports. Consider a fixed trivalent topology with r input bits and r output bits. Suppose that for every permutation in the support of a distribution, some choice of vertex tables realizes the r independent wires joining each input to its assigned output. If each input has a uniform destination on a set of $q \geq 2$ outputs, the topology has size $\Omega(r\sqrt{\log q})$. The destination sets may depend on the input.*

Proof. Give each exterior bit its own equality leaf. Connect the resulting components in a path by adding edges to the graph used to measure distances. This increases the maximum degree by at most two and gives m edges, with m at most a constant times the network size. Let α be the law of a uniform oriented edge, and let μ be the joint law of a uniform input and its random destination. For any placement in a spectral space with distance d,

$$r\,\mathbb{E}_\mu d \;\leq\; m\,\mathbb{E}_\alpha d.$$

Indeed, for each permutation the network tensors locally contract the product of its internal bit wires to the prescribed exterior wires. Monotonicity and additivity of prices bound the sum of the destination distances by the sum of the internal edge lengths. Average over the permutations and include the extra edges on the right.

We need a placement where many destinations are far away at once. The class of Hadamard spectral spaces is closed under isometry, rescaling, closed convex subspaces, and pointed ultralimits (Section 3). The product theorem [43, Theorem 6.1] gives a subclass with the same properties and no ports, closed also under finite ℓ_2 products. Let $\mathcal{K}$ be the squared distance tables $D = d^2$ realized in this subclass with $\mathbb{E}_\alpha D \leq 1$. Products of rescaled placements give convex combinations of tables. Connectivity bounds every entry, and pointed ultralimits realize coordinatewise limits. Hence $\mathcal{K}$ is convex and compact.

For a probability law $\nu \leq 2\mu$ on ordered pairs, let $\beta_\nu = \max_{D\in\mathcal{K}} \mathbb{E}_\nu D$. By rescaling, $\mathbb{E}_\nu d^2 \leq \beta_\nu \mathbb{E}_\alpha d^2$ for every placement in the subclass. Write X, Y for the endpoints under ν. The graph entropy bound [43, Corollary 5.2] and the bounded degree of our graph give $H_\nu(Y \mid X) \leq C(1 + \beta_\nu)$, with entropy in bits. Every pair has ν-mass at most $2/(rq)$, so

$$H_\nu(Y \mid X) = H_\nu(X,Y) - H_\nu(X) \geq \log_2(rq/2) - \log_2 r = \log_2(q/2).$$

It follows that $\beta_\nu \geq c \log q$ for all sufficiently large q, uniformly in ν.

Bilinear minimax on $\mathcal{K}$ and the compact convex set of laws $\nu \leq 2\mu$ gives a single $D = d^2 \in \mathcal{K}$ for which $\mathbb{E}_\nu D \geq c \log q$ for every such law. If the pairs with $D < c \log q$ had μ-mass at least $1/2$, conditioning μ on those pairs would violate this bound. Hence $\mathbb{E}_\mu d \geq \frac{1}{2}\sqrt{c \log q}$, while $\mathbb{E}_\alpha d \leq 1$ by Cauchy–Schwarz. The price inequality gives the result. For bounded q, counting exterior bits suffices. □

In a network for $\mathrm{Tag}(h,h)^{\otimes t}$, pin each list's tags to a uniform permutation. The topology realizes a permutation of $r = tNh$ payload bits. Each input bit has $q = N$ possible destinations, at its own bit position in its own list, all equally likely. By Lemma 4.1, $2V + O(tNh) \geq c\,tNh\sqrt{h}$. Since $n_w = 4N(h+1)$, this proves (2.2) for any fixed $0 < \sigma < 1/2$ and all sufficiently large w, uniformly in t.

4.3 Finite fields with ports

Fix a finite field with ports. We use the signed counting tensor U_ℓ^* on ℓ exterior bits from the circuit lower-bound paper. Its entries count the signed computation paths of a polynomially clocked universal machine, reduced in the field. The polynomial-size verifier gives an exact trivalent network of size $\ell^{O(1)}$ for one copy. Sum over its guess strings, requiring the unused bits on each computation path to be zero, and weight the outcome by its sign. The exact batch lower bound [43, Theorem 12.1] gives $\delta, \xi > 0$ such that, for all sufficiently large ℓ,

$$\mathrm{vol}_k\big((U_\ell^*)^{\otimes j}\big) > j\ell^{1+\delta} \qquad (1 \leq j \leq 2^{\ell^\xi}).$$

The descent with tagged permutations [42, Proposition 14.1] supplies a constant $a > 0$. For all sufficiently large h, given a trivalent network of size s in the range $N \leq s \leq N^4$, with e exterior bits, it constructs a network for d copies of its tensor using trivalent gates and $\mathrm{Tag}(h,h)$ tests. Its size S, charging every incident bit including those at tests, satisfies

$$S \leq Cd\big(sh^{-a} + e(\log h)^C\big), \qquad \log_2 d = O\big((\log h)^2\big).$$

These bounds are uniform in the network. The normalization required for descent attaches equality leaves and subdivides self-loops. It costs only a constant factor, and the cited bound holds on every fixed polynomial size range.

Choose $b > \max\{2, 2/\xi\}$ and $0 < \sigma < \min\{a/4, b\delta/4, 1/4\}$. Suppose there are arbitrarily large $w = h + 1$ for which some $1 \leq t \leq 2N$ admits a realization of $D_w^{\otimes t}$ of volume at most $tn_w w^\sigma$. The two-list construction gives a network for t tagged-permutation tests of size $O(tNh^{1+\sigma})$.

Group the tests in a descent network into batches of t and substitute this realization for each batch. In the last batch pin unused tests to the identity permutation with zero payloads, where each has value one. Since the substituted network realizes the full product tensor, the replacement stays exact under every contraction of the surrounding wires, including wires joining tests in one batch. Each full batch contributes $3tNh$ incident bits to S. Paying separately for the last batch therefore gives trivalent size

$$O(h^\sigma S + tNh^{1+\sigma}).$$

Set $\ell = \lceil h^b \rceil$ and begin with $j = N^2$ copies of the polynomial-size network for U_ℓ^*. At each descent let $x = s/(j\ell)$ be the size per exterior bit. The new batch has dj copies. Its size per bit, for all sufficiently large h, satisfies

$$\begin{aligned} x' &\leq C'\left(h^{-a+\sigma}x + h^\sigma(\log h)^C + \frac{tNh^{1+\sigma}}{dj\ell}\right) \\ &\leq h^{-a/2}x + h^{2\sigma}. \end{aligned}$$

Here $j \ge N^2$, $d \ge 1$, and $t \le 2N$, so the fraction is $O(h^{1+\sigma-b})$. This is the reason for starting with N^2 copies.

Initially $x \le h^A$ for some fixed A. Choose a fixed integer $L > 2A/a$. At every one of these L steps, the product of the replication factors is $N^{o(1)}$. Inductively, $j = N^{2+o(1)}$ and x stays polynomial in h, so the current network and its normalization have size in the descent range. Iterating the last inequality gives $x = O(h^{2\sigma})$. Now $j \le 2^{\ell^\xi}$ because $\log_2 j = (2+o(1))h$, while $h^{2\sigma} = o(\ell^\delta)$ by the choice of σ. The final trivalent volume is therefore $o(j\ell^{1+\delta})$, contradicting the exact batch lower bound. This proves Theorem 2.2 in the remaining case.

Part I
Sphere tensors and their prices

5 Price extensions and simultaneous minimization

The compression argument introduces new sites at which to place tensor legs. A price on the old sites already puts a lower bound on the price of any proposed placement. Contract the legs at the new sites and evaluate the tensor that remains. One can do this on any tensor power, with each new site contracting all its copies at once. Theorem 5.1 shows that these lower bounds are sharp and gives a criterion for satisfying several price bounds at the same placement. In Section 12 the tensors to be placed will depend on a parameter. To control specialization of that parameter, we prove the criterion over valued fields. For a product of ordinary wires incident to one new site, the minimum is computed in [42, Theorem 3.1].

5.1 Integral restrictions

Let K be a field with a real additive nonarchimedean valuation ν, normalized by $\nu(0) = +\infty$. Its valuation ring is $\mathcal{O} = \{c \in K : \nu(c) \ge 0\}$. We use the given coordinate bases at all sites. A restriction is *integral* if each local matrix has entries in $\mathcal{O}$. A *valued price* p of scale $D > 0$ on S (here $|S| \ge 2$, as before) is real on nonzero tensors, has $p(0) = -\infty$, $p(\mathbf{1}) = 0$, and is nonincreasing under integral restriction. Its scalar, product, and direct-sum rules are

$$p(cT) = p(T) - \nu(c) \quad (c \neq 0), \qquad p(T \otimes U) = p(T) + p(U),$$
$$p\Big(\bigoplus_j T_j\Big) = D \log \sum_j e^{p(T_j)/D}. \tag{5.1}$$

Here log is the natural logarithm. Thus multiplying by a scalar of positive valuation lowers the price. The trivial valuation recovers ordinary prices.

We shall construct prices using a convenient equivalent form of the direct-sum rule. For tensors in common spaces, the right side of (5.1) bounds $p(\sum_j T_j)$ by local addition of blocks. Conversely, suppose this ordinary sum bound holds and $p(\Delta_S(q)) = D \log q$ for every q. The block inclusions express $\bigoplus_j T_j$ as an ordinary sum and give the desired upper bound. For the lower bound, take the m-th power of the direct sum and project at every site onto the words in which the j-th summand occurs mr_j times. Local permutations identify the result with $\bigotimes_j T_j^{\otimes mr_j}$ times a diagonal of size $m!/\prod_j (mr_j)!$. For rational probability vectors r on the nonzero summands, take m through multiples of a common denominator. The price per copy is at least $\sum_j r_j p(T_j) - D \sum_j r_j \log r_j$. Maximization gives the asserted identity.

Write q_i for the local dimensions of T and $\nu_{\min}(T)$ for the minimum valuation of an entry. For $T \neq 0$,

$$-\nu_{\min}(T) \leq p(T) \leq -\nu_{\min}(T) + D \sum_i \log q_i. \tag{5.2}$$

For the lower bound project to an entry of minimum valuation. For the upper bound expand T in at most $\prod_i q_i$ coordinate tensors and use the ordinary sum inequality. Thus prices of fixed scale form a compact space in the topology of pointwise convergence, since their axioms are closed in the product of the intervals (5.2). The wire and transport rules above require only integral operations and hold also for p. At one site the price of a nonzero vector is $-\nu_{\min}$, since after division by a minimum-valuation entry it is integrally equivalent to the unit. When K contains a trivially valued field F, restriction to tensors over F gives the ordinary spectral geometry of the wire distances.

5.2 A test for extending a price

Fix p on S, and adjoin a finite set L of sites. Denote by $\widehat{p}$ an extension to $S \sqcup L$ of the same scale which retains p on tensors with trivial legs at L. Such extensions exist by placing the added sites at old ones. Restriction to any intermediate list of sites containing S gives a valued price of scale D. Indeed the diagonal on that list restricts to the base diagonal and is a restriction of the full diagonal, so the diagonal test applies.

A *closure* applies one integral linear functional at each site of L and leaves all base legs free. The *assisted cost* of T is

$$\beta_p(T) = \sup_{m \geq 1} \frac{1}{m} \sup_{\ell} p\big(\ell(T^{\otimes m})\big). \tag{5.3}$$

At a site of dimension q the functional in this formula acts on K^{q^m}, so it may couple the m copies arbitrarily. Every extension satisfies $\widehat{p}(T) \geq \beta_p(T)$ by monotonicity under closure and additivity on powers.

For example, in the ordinary case take the product of two size-q wires from a to y and from y to b, where a, b are old sites and y is new. Already on one copy, closing the two legs at y by the pairing $e_i \otimes e_j \mapsto \delta_{ij}$ recovers the wire from a to b. This gives the lower bound $d(a, b) \log q$ for the product. Placing y at a attains it. More generally, placing y anywhere on the geodesic from a to b attains the bound by additivity of distance along that segment.

Theorem 5.1 (Price extension)**.** *For every tensor T on $S \sqcup L$, $\beta_p(T) = \min_{\widehat{p}} \widehat{p}(T)$. For an arbitrary family of nonzero tensors T_i and real numbers h_i, a single extension satisfies $\widehat{p}(T_i) \leq h_i$ for all i if and only if*

$$\beta_p(T_{i_1} \otimes \cdots \otimes T_{i_j}) \leq h_{i_1} + \cdots + h_{i_j} \tag{5.4}$$

for every finite list of indices, with repetitions allowed.

Proof. Minimizing one tensor. The zero tensor is immediate. For a prescribed nonzero T we first assign prices to a finite list of distinct nonzero tensors $A_1, \ldots, A_s$ containing T. The exponents of their mixed products will serve as variables. For $x \in \mathbb{Q}^s_{\geq 0}$ set

$$b(x) = \frac{1}{n} \beta_p\Big(\bigotimes_i A_i^{\otimes nx_i}\Big), \qquad nx \in \mathbb{Z}^s.$$

In (5.3) the supremum can be restricted to multiples of any given integer by repeating each closure independently. Thus assisted cost is homogeneous under powers and b is well defined. Independent closures, taken at common multiples, make b superadditive, hence concave by homogeneity.

The increment per unit of coordinate i lies in the price interval (5.2) for A_i. For the lower estimate isolate a minimum-valuation coefficient of every added factor, by coordinate projections at the base sites as well as the added sites. For the upper estimate expand each added factor in coordinates. Even for a closure coupling all copies, fixing the added indices of each term induces an integral closure on the old factors. The coordinate factors remaining at base sites have price zero, and the ordinary sum bound pays at most $D\sum_j \log q_j$ per added factor of format (q_j). This proves the increment bounds after taking powers. In particular b extends continuously to the real orthant.

At a strictly positive rational x choose a supporting linear form u for this homogeneous concave function, so that $b(y) \le u \cdot y$ and $b(x) = u \cdot x$. The two-sided increment estimates bound u_i in the same price interval. We claim that the numbers u_i respect the price relations on the list. For example, if $A_i = A_j \otimes A_k$, then $b(x + t(e_j + e_k - e_i)) = b(x)$ for both signs of sufficiently small rational t. The supporting inequality in these two directions forces $u_i = u_j + u_k$. A scalar change shifts b by minus its valuation per copy, giving the scalar relation by the same argument. If A_i restricts integrally to A_j, replacing copies of A_j by copies of A_i can only increase b. The supporting inequality in the direction $e_i - e_j$ therefore gives $u_i \ge u_j$. A base tensor increases b by its exact price per unit, since its factors pass unchanged through every closure. For $\Delta_{S\sqcup L}(q)$, closing its added legs by all-ones functionals yields $\Delta_S(q)$, and expanding its q diagonal entries gives the reverse bound. Thus its increment is $D \log q$.

To check the ordinary sum bound, let $A_i = A_j + A_k$ in the list. Expand $n\sigma$ copies of A_i in a product with multiplicities nx, for a rational $0 < \sigma \le x_i$. A term choosing a copies of A_j has price after closure at most $nu \cdot x - n\sigma u_i + au_j + (n\sigma - a)u_k$. The ordinary sum bound on the base and the binomial formula bound the price after closure of the original tensor by

$$nu \cdot x - n\sigma u_i + n\sigma D \log(e^{u_j/D} + e^{u_k/D}).$$

Taking the supremum per n through admissible multiples yields $b(x) = u \cdot x$ on the left. This proves the required bound on u_i. Relations involving zero summands hold automatically.

Let x tend to the unit vector corresponding to T. The support vectors have a convergent subsequence and continuity gives $u_T = \beta_p(T)$ in the limit. Every finite collection of price relations and the equality at T is therefore feasible by applying the construction to the list of all tensors occurring in those constraints. Compactness of the product of intervals gives a simultaneous solution for all such relations. The diagonal test following (5.1) makes this solution a valued price. This proves attainment for one tensor.

Satisfying several bounds. Any extension obeying the proposed bounds satisfies (5.4) by additivity on products. Conversely, take a finite list of k bounds. Form U as a direct sum of N_i copies of $T_i^{\otimes n}$ for each i. In a power of U each word of summands is a mixed product. Restricting a closure to the blocks of that word gives an integral closure, whose price is bounded by (5.4). Sum over the words on the base and take the supremum per copy to obtain

$$\beta_p(U) \le D \log \sum_i N_i e^{nh_i/D}.$$

Choose $C > \max_i h_i$ and $N_i = \lceil e^{n(C-h_i)/D} \rceil$. These multiplicities bring every $N_i e^{nh_i/D}$ into the interval $[e^{nC/D}, 2e^{nC/D}]$. At an extension minimizing U, the direct-sum identity then gives

$$D \log N_i + n\widehat{p}(T_i) \le \beta_p(U) \le nC + D\log(2k), \qquad \widehat{p}(T_i) \le h_i + \frac{D\log(2k)}{n}.$$

Let n tend to infinity and apply compactness, first to the finite list and then, by the finite intersection property, to the whole family. □

Two consequences will be used below. Given several extensions of p, there is a single extension whose evaluations are bounded above by any fixed convex combination of theirs. Indeed, each extension bounds the assisted cost of every mixed product, so the averaged bounds pass (5.4). We may also enlarge the base before applying the test. Suppose an enlarged base carries a price agreeing with p on the old base. A tensor supported on the old base and the unknown sites has exactly the same closures, with trivial legs on the extra base sites. Thus its assisted cost is unchanged. This observation also applies when the enlarged base has larger scale. Theorem 5.1 then gives an extension at that larger scale.

5.3 Geodesic placement

Assume henceforth in this section that K contains a trivially valued subfield F. Given pairs of sites a_j, b_j, place new sites y_j first at the a_j and then at the b_j. The interpolation consequence of Theorem 5.1 gives an extension bounding the price of every tensor by the affine interpolation of these two evaluations, with weights $1-\theta, \theta$. In particular

$$d(a_j, y_j) \le \theta\, d(a_j, b_j), \qquad d(y_j, b_j) \le (1-\theta) d(a_j, b_j).$$

The triangle inequality forces equality in both bounds. Thus y_j is the point a fraction θ of the way along the geodesic from a_j to b_j in a complete CAT(0) spectral realization over F. The same extension provides the interpolation bound for all tensors over K.

Lemma 5.2 (Geodesic determination and submean inequality)**.** *Ordinary price data on any subset of a complete* CAT(0) *spectral space determine all prices on its closed geodesic convex hull up to a price-preserving isometry fixing the given data. Moreover, suppose the prices at a central placement are at most the weighted average of prices at finitely many end placements, for every tensor on the original slots. In each placement adjoin points on corresponding geodesic segments. If the two distances of each new point to its endpoints at the center are the weighted averages of their values at the ends, the price inequality holds on the enlarged placements.*

Proof. For each new site, prescribe the prices of its size-two wires to its two endpoints. These prices are the two geodesic distances multiplied by $\log 2$. We claim that the price of any $T \neq 0$ on the enlarged list is the smallest number h for which the bound h and all the wire bounds pass the product test. The actual price passes by closure. Conversely, enlarge the base to include the actual geodesic points, and apply Theorem 5.1 to duplicate sites with those bounds. In a common metric realization, the triangle inequality forces both endpoint distances of each duplicate to be the prescribed distances. Uniqueness of geodesics identifies the duplicate with the actual point. Transport (3.1) therefore gives the assertion. For a collapsed base all its prescribed segments are constant and the assertion follows directly.

Iterating this characterization determines prices, including distances, on all finite geodesic constructions. Completing their union gives the closed convex hull by continuity of geodesic interpolation, and (3.1) extends the price identifications to the completion. An infinite original subset causes no change since each finite construction starts from finitely many points.

For the second assertion, a collapsed central placement is immediate. Otherwise fix a product in the test, a power and a closure on its new sites. The original submean inequality bounds the central price of the closed tensor by the average of its closed prices at the ends. At each end this is at most the price of the unclosed power by restriction, including a collapsed end, where every nonzero tensor has price zero. Divide by the power and take the supremum at the center. The averaged actual upper bounds on the enlarged placements thus pass the product test at the center, and the averaged wire bounds agree with the prescribed ones there. Apply the characterization just proved. □

5.4 Unique evaluations at minimizing sites

A price minimum can admit several placements, as the two-wire example shows. To compare minimizers as a parameter varies, we add a strictly convex penalty at each unknown site. Fix o on the base. For a new site y, form the direct sum of $\mathbf{1}$ and the size-two wire from o to y. Its direct-sum label is visible at every site in the configuration, so its price is

$$f(r) = D\log(1+\exp(r\log(2)/D)), \qquad r = d(o,y).$$

For a nonzero tensor U on the base and a list of new sites, fix an integer $M \geq 1$. Its *pin objective* V is the product of $U^{\otimes M}$ with one such penalty tensor for each new site. For several objectives, each is formed on the base and its own list, then padded by trivial legs at the other lists.

Lemma 5.3 (Pinned evaluations)**.** *For a finite family of objectives on pairwise disjoint lists of new sites, there is a same-scale extension minimizing each objective. All such extensions have the same evaluations on every tensor on the joint list. These evaluations agree on overlaps between subfamilies. For any such family, including an infinite one, the resulting evaluations on tensors over F are realized in a complete* $\mathrm{CAT}(0)$ *spectral space.*

Let W be the product of a finite family, $G = \sum_V \beta_p(V)$, and h_T the pinned evaluation of a nonzero test tensor T. For some $c_D > 0$ depending only on D, and every $n \geq 1$,

$$0 \leq h_T - \big(\beta_p(W^{\otimes n} \otimes T) - nG\big) \leq \frac{\sum_j (\log q_j)^2}{8c_D n}, \tag{5.5}$$

where q_j are the dimensions of the test legs grouped at the added sites, including dimension one for an unused site.

Proof. Successively enlarge the base and minimize the next objective using Theorem 5.1. Each minimization attains its original minimum by the base-enlargement property.

To compare two assignments within a common extension, interpolate all corresponding sites at their midpoints, with valued price on U bounded by the mean of its two prices. For any two sites a, b and their midpoint m,

$$f(d(o,m)) \leq \tfrac{1}{2}\big(f(d(o,a)) + f(d(o,b))\big) - c_D d(a,b)^2. \tag{5.6}$$

Indeed wire distances are at most D, by restriction of the full diagonal. Choose $c_D > 0$ small enough that $f(r) - 4c_D r^2$ is convex and nondecreasing on $[0, D]$, using the positive lower bounds for f' and f'' there. Apply midpoint convexity to this function and the squared-distance midpoint inequality to $4c_D r^2$. The sites of each objective contain the original base, so its direct-sum penalty has scale D even in the larger configuration. If two assignments minimize that objective, the midpoint is bounded above by the same minimum minus c_D times the sum of the squared distances between corresponding new sites. These distances must all vanish.

For two minimizing extensions P, P', enlarge P by a second copy of the entire list, imposing upper bounds on its evaluations from P'. These bounds pass the product test on the old base and therefore on the enlarged one. Each copied objective attains its minimum, so its sites are at distance zero from their counterparts. Transport shows $P(T) \le P'(T)$ for every test tensor. Interchanging P, P' gives equality. Restricting to any subfamily still minimizes its objectives. The resulting compatible ordinary evaluations on all finite lists give a spectral space after quotient by zero distances, and hence the claimed complete extension.

Finally, adjoin an optimizing copy to an arbitrary competing assignment as above, and let d_j be the distances between corresponding sites. The midpoint objective is at least G, and (5.6) gives $p_{\mathrm{comp}}(W) - G \ge 2c_D \sum_j d_j^2$. Transport gives $p_{\mathrm{comp}}(T) \ge h_T - \sum_j d_j \log q_j$. Multiply the first inequality by n, add the second, and complete the squares in the d_j. Taking the minimum by Theorem 5.1 proves the upper bound in (5.5). The optimizing assignment itself gives the lower bound. □

6 A valued lift and a submean inequality

In the synchronization argument, one parameter can approach several different values. We need to compare the rate at which the tensor price falls near each of them. The scalar model is a polynomial with several roots. The valuation records how closely the parameter approaches a root, and the multiplicity gives the rate of decrease of the scalar price. For tensors, Lemma 6.2 bounds the sum of these rates by the growth of the price on an enclosing disk. The scalar comparison is familiar from the potential theory of the nonarchimedean projective line [7]. Our tensor inequality follows by interpolation once we construct a price over a field with a valuation.

6.1 The Gauss lift

Let H be an ordinary price of scale $D > 0$ over a field E, and put $\phi(A) = \exp(H(A)/D)$, with $\phi(0) = 0$. The ring $E[t^{\mathbb{Q}}]$ consists of finite sums of rational powers of t. On its fraction field K take the additive valuation ν given by the least exponent on nonzero elements of the ring. Thus $\nu(t^r) = r$, and a nonzero constant in E has valuation zero. Write $[t^j]T$ for the coefficient tensor of t^j. A monomial tensor $t^j A$ should have price $H(A) - j$. For a general tensor we form powers using the same t in every factor and take the largest coefficient price with this valuation adjustment.

Theorem 6.1 (Gauss lift)**.** *There is a valued price p of scale D on K extending H. On tensors with entries in $E[t^{\mathbb{Q}}]$ it is given by*

$$p(T) = \lim_{n\to\infty} \frac{1}{n} \max_{j\in\mathbb{Q}} \bigl(H([t^j]T^{\otimes n}) - j\bigr). \tag{6.1}$$

Proof. Choose a common denominator m for the exponents and write $x = t^{1/m}$. Temporarily assign x an arbitrary weight $b > 0$. For a nonzero Laurent tensor $A(x)$ consider the weighted coefficient sum and its regularization

$$N_b(A) = \sum_i e^{-bi/D}\phi([x^i]A), \qquad p_b(A) = \lim_{n\to\infty} \frac{D}{n} \log N_b(A^{\otimes n}).$$

The sum bound for ϕ makes N_b subadditive. Applying the same bound to coefficient convolution and using multiplicativity of ϕ shows that N_b is submultiplicative. Hence the limit exists by subadditivity of the logarithms. The lowest coefficient of $A^{\otimes n}$ is the n-th power of the lowest coefficient of A, which gives a finite lower bound for the normalized logarithms.

There are $O_A(n+1)$ exponents in $A^{\otimes n}$. Replacing the sum by its largest term therefore costs $O_A(\log(n+1)/n)$ in normalized logarithmic price. It follows that $p_b(A)$ is a limit of maxima of affine functions of b, whose slopes lie in a fixed bounded interval. In particular, $p_b(A)$ is convex and Lipschitz. Setting $b = 1/m$ gives Equation (6.1), independently of the common denominator.

Separating the maximal coefficients. Submultiplicativity gives $p_b(T \otimes U) \le p_b(T) + p_b(U)$. For the reverse inequality, a maximal coefficient from each power may contribute to a convolution coefficient in which other terms cancel it. We isolate that pair by varying x in the first factor alone. Multiplying by x^a shifts p_b by $-ba$, so it suffices to consider nonzero polynomials T, U.

First fix b at which $b \mapsto p_b(T)$ is differentiable. For each n, choose indices $i = i_n$ and $j = j_n$ maximizing the weighted coefficient sizes of $T^{\otimes n}$ and $U^{\otimes n}$. An index a fixed fraction of n away from i incurs a uniform deficit. Indeed, for each $\xi > 0$ there is a $\delta > 0$ such that for all sufficiently large n,

$$\frac{H([x^h]T^{\otimes n}) - bh}{n} \le p_b(T) - \delta \qquad (|h - i| \ge \xi n). \tag{6.2}$$

Along any asymptotically maximizing sequence with $h/n \to v$, comparison at b' gives $p_{b'}(T) \ge p_b(T) - (b' - b)v$. Differentiability forces $v = -\frac{d}{db}p_b(T)$. Compactness of the interval of possible h/n gives both convergence of i/n to this number and the deficit in Equation (6.2).

At $k = i + j$, introduce a variable w and write

$$\begin{aligned} C_s &= [x^k w^s](T((1+w)x)^{\otimes n} \otimes U(x)^{\otimes n}) \\ &= \sum_h \binom{h}{s} [x^h]T^{\otimes n} \otimes [x^{k-h}]U^{\otimes n}. \end{aligned}$$

Changing s changes the integer multipliers of the same coefficient products. Consecutive values of s suffice to isolate any one index in a short interval in every characteristic.

Fix $0 < \eta < 1/4$. Let I_n consist of the nonnegative indices within $\eta n/3$ of i, and put $m_n = |I_n|$. For large n, $m_n - 1 \le \eta n$. The matrix $(\binom{h}{s})_{0 \le s < m_n,\ h \in I_n}$ has determinant one over the integers. Indeed, its columns are the coefficients of consecutive powers of $1 + w$ modulo w^{m_n}. Multiplication by the first power is unitriangular, and the powers with exponents $0, \dots, m_n - 1$ form a unitriangular basis. Inverting this matrix over E gives a linear combination C of $C_0, \dots, C_{m_n - 1}$ with multiplier one at i and zero on $I_n \setminus \{i\}$. The isolated product has weighted size $\exp(n(p_b(T) + p_b(U) + o(1))/D)$. By Equation (6.2) with $\xi = \eta/3$, each of the $O(n)$ remaining products has weighted size smaller by an exponential

factor. Their scalar multipliers in E have no cost. Applying subadditivity of ϕ to the isolated product, written as C minus these remaining terms, gives

$$e^{-bk/D}\phi(C) \;\geq\; \exp\Big(\frac{n}{D}\big(p_b(T)+p_b(U)-o(1)\big)\Big). \tag{6.3}$$

To bound the same combination from above, put $B_\ell(x)=[w^\ell](T((1+w)x)\otimes U(x))$ and $d=\deg T$. Then $B_0=T\otimes U$, and the term contributing to C_s with shift degrees $\ell_1,\dots,\ell_n$ has at most s nonzero ℓ_a. If there are l of them, its weighted coefficient size is at most

$$N_b(B_0^{\otimes(n-l)})\,\max(1,N_b(B_1),\dots,N_b(B_d))^l,$$

where the maximum is 1 if $d=0$. Here we have permuted copies simultaneously at every site to group the unchanged factors. The number of choices for $l\leq\eta n$ is at most $\sum_{l\leq\eta n}\binom{n}{l}(1+d)^l$. Its logarithm is $O_d(n\eta\log(1/\eta)+\log(n+1))$. Moreover $D\log N_b(B_0^{\otimes(n-l)})=(n-l)p_b(B_0)+o(n)$ uniformly in this range. The $O(n)$ coefficients used to form C have logarithmic cost $O(\log(n+1))$. Taking logarithms in Equation (6.3), dividing by n/D and letting $n\to\infty$ gives

$$p_b(T)+p_b(U)\leq p_b(T\otimes U)+O_{b,T,U}\big(\eta\log(1/\eta)\big).$$

Let $\eta\downarrow 0$. Differentiability points are dense, so continuity establishes the identity for every $b>0$.

Scalars and integral maps. For a nonzero scalar Laurent polynomial the lowest coefficient of each power has the largest weighted size, since all nonzero scalar coefficients have ordinary price zero. Thus $p(f)=-\nu(f)$. Multiplicativity now gives a well-defined extension to fractions by $p(A/f)=p(A)+\nu(f)$.

Consider a local map with polynomial entries of nonnegative degree. After taking its n-th power, the matrix at each site has degree $O(n)$. Each output coefficient of degree j is a sum over the degrees chosen at the finitely many sites. There are polynomially many terms in n, and each is an ordinary restriction of an input coefficient of degree at most j. The input coefficient controls its term by monotonicity of H, and the degree weight only decreases. The resulting logarithmic loss is $O(\log(n+1))$, which vanishes after normalization. For a matrix of integral rational entries, choose a common denominator and remove its lowest power from both numerator and denominator. The numerator matrix then has only nonnegative powers and the denominator has valuation zero. Doing this at each site proves monotonicity for every integral restriction.

For the ordinary sum bound, the binomial expansion, with local permutations of copies, gives

$$N_b((T+U)^{\otimes n})\leq\sum_{j=0}^{n}\binom{n}{j}N_b(T^{\otimes j})N_b(U^{\otimes(n-j)}).$$

For a fixed nonzero T and all $0\leq j\leq n$ we have $D\log N_b(T^{\otimes j})=jp_b(T)+o(n)$ uniformly. The binomial formula yields $\exp(p(T+U)/D)\leq\exp(p(T)/D)+\exp(p(U)/D)$. Common denominators give this inequality on K. The lift agrees with H on constants, including every diagonal tensor, so the diagonal test of Section 5 gives the direct-sum identity. Set $p(0)=-\infty$. □

6.2 Prices on disks

We now need a generic value of the price on each disk of parameters. Take an algebraically closed field F and an ordinary price on a fixed finite set of sites over F. Enlarge the coefficient field to $E = F(\mathcal{U})$ for an uncountable set $\mathcal{U}$ of independent indeterminates. For a rational tensor depending on finitely many of these variables, define its ordinary price over E as the supremum over its F-valued specializations on the domain of definition.

The fixed-format bound in Equation (5.2) makes this supremum finite for every nonzero rational tensor. Its fixed-format price sublevel sets over F are Zariski closed (Section 3). Hence the specializations within any positive tolerance of the supremum contain a nonempty open set. Finitely many such sets meet on the common domain of definition. Applying the price axioms at these simultaneous specializations and letting the tolerance tend to zero gives the axioms over E. The extension is consistent on subfields generated by subsets of $\mathcal{U}$ and invariant under permutations of $\mathcal{U}$.

Apply Theorem 6.1 over E. Write $\mathcal{O}$ for the valuation ring of $K = \mathrm{Frac}(E[t^{\mathbb{Q}}])$. A *fresh* $u \in \mathcal{U}$ occurs in none of the data being compared. For $r \in \mathbb{Q}$ the disk with center $a \in K$ and valuation bound r is $a + t^r\mathcal{O}$. Its generic price is the price at $a + t^r u$. Increasing r shrinks the disk.

Lemma 6.2 (Specialization and the star inequality)**.** *For a polynomial tensor $T(c)$ over K, let $P(c) = p(T(c))$ on the base sites, or let $P(c) = \beta_p(T(c))$ when T has additional sites. With a fresh u and $P_* = P(u)$, one has*

$$P(c) \le P_* \quad (c \in \mathcal{O}), \qquad P_* = \sup_{a \in F} P(a).$$

The generic price stays the same after an affine change of variable $c \mapsto a + bc$ with $a \in \mathcal{O}$ and $b \in \mathcal{O}^\times$. Here u is fresh also for a and b.

Suppose T is nonzero of degree d. Choose finitely many centers $c_j \in \mathcal{O}$ with distinct residues and positive rational numbers r_j, r_∞. Let

$$P_j = P(c_j + t^{r_j} u), \qquad P_\infty = P(t^{-r_\infty} u).$$

Then

$$\sum_j \frac{P_* - P_j}{r_j} \;\le\; \frac{P_\infty - P_*}{r_\infty} \;\le\; d. \tag{6.4}$$

Equivalently, the first inequality bounds the center price P_* by the weighted average of the prices at the ends of the star. The weights are proportional to $1/r_j$ and $1/r_\infty$. We call the values on the star harmonic *if the center equals this average. The weighted-average inequality also holds for the zero family.*

For instance, a scalar polynomial $T(c) = \prod_j (c - c_j)^{m_j}$ with distinct $c_j \in F$ has generic price zero on the unit disk. On the inner disk at c_j its price is $-m_j r_j$, giving a drop of m_j per unit of valuation length. On the outer disk its price is $r_\infty \sum_j m_j$. Both inequalities in Equation (6.4) are equalities.

Proof. The zero family is immediate. First let $P(c) = p(T(c))$ with $T \neq 0$, and write $Q(c) = T(c)^{\otimes n}$. Its degree is at most nd. Interpolation on $nd + 1$ fresh indeterminates uses

only integral Lagrange weights at any $c \in \mathcal{O}$, since the nodes have unit differences. All the samples have price nP_* by permutation symmetry, so the ordinary sum bound gives

$$p(Q(c)) \le nP_* + D\log(nd+1).$$

Dividing by n and letting $n \to \infty$ proves specialization. Conversely, interpolate at $nd+1$ distinct nodes of F and evaluate at u. The same argument gives $P_* \le \sup_{a\in F} P(a)$. Applying specialization to an integral affine change of variable and its integral inverse proves the asserted invariance. Specialization on nested disks also gives $P_j \le P_* \le P_\infty$.

The inverse Vandermonde matrix at these nodes has integral entries, so interpolation controls coefficients as well as evaluations. On the unit disk it gives $p([c^\ell]Q(c)) \le nP_* + D\log(nd+1)$. Evaluating at $t^{-r_\infty}u$ can increase any coefficient price by at most ndr_∞. The price of their sum incurs another $D\log(nd+1)$, proving $P_\infty - P_* \le dr_\infty$ in the limit.

To prove the other inequality, we use interpolation on each inner disk to bound the low Taylor coefficients at its center and on the outer disk to bound the high coefficients at infinity. More precisely, interpolating $Q(c_j + t^{r_j}v)$ and $Q(t^{-r_\infty}v)$ in v and removing the scalar power of t gives

$$\begin{aligned} p([v^\ell]Q(c_j+v)) &\le nP_j + r_j\ell + D\log(nd+1), \\ p([c^\ell]Q(c)) &\le nP_\infty - r_\infty\ell + D\log(nd+1). \end{aligned} \tag{6.5}$$

Suppose the sum of the inner drops per unit length exceeds the outer growth per unit length. Choose a_j strictly between zero and $(P_* - P_j)/r_j$ on the arms with positive drop and zero on the other arms, so that

$$\sum_j a_j > b > (P_\infty - P_*)/r_\infty$$

for some b. Put $m = \lceil nb \rceil$ and $m_j = \lfloor na_j \rfloor$. The strict inequalities and Equation (6.5) give $\epsilon > 0$ such that, for all large n, the coefficients

$$[c^\ell]Q(c) \quad (m \le \ell \le nd), \qquad [v^\ell]Q(c_j+v) \quad (0 \le \ell < m_j)$$

all have price at most $n(P_* - \epsilon)$. Together they determine $Q(u)$ by an integral linear combination. Indeed, subtract the high-degree part of Q. The remaining polynomial has degree $< m$ and its Taylor coefficients through order $m_j - 1$ give its residue modulo $(c - c_j)^{m_j}$ at each center. For large n, the total multiplicity $\sum_j m_j$ is at least m. The moduli are pairwise comaximal over $\mathcal{O}$ because the center differences are units. The Chinese remainder theorem thus recovers the low-degree polynomial by an $\mathcal{O}$-linear map in monomial coordinates. Computing the Taylor coefficients of the subtracted part and evaluating at u are integral operations as well. These identities hold coefficientwise for tensors over K.

The reconstruction expresses $Q(u)$ using only $O(n+1)$ coefficients from the displayed list. The sum bound would give $nP_* = p(Q(u)) \le n(P_* - \epsilon) + O(\log(n+1))$, a contradiction.

For the assisted cost, fix a power m in Equation (5.3) and write $A(c, \mathbf{z})$ for the closure of $T(c)^{\otimes m}$ with functional entries $\mathbf{z} = (z_1, \dots, z_s)$. These entries occur polynomially. For any fixed c, the supremum of $p(A(c,\mathbf{z}))$ over $\mathcal{O}^s$ equals the supremum over F^s. Indeed, interpolate $A(c,\mathbf{z})^{\otimes n}$ on a rectangular grid in F^s. For fixed m, there are $O_m(n+1)$ nodes per coordinate and polynomially many nodes in total. Evaluation at integral arguments has integral weights, so the logarithmic cost per tensor copy tends to zero.

Consequently $\beta_p(T(c))$ is the supremum of the functions $p(A(c,\mathbf{z}))/m$ over m and constant closing data. Omit identically zero tests when $T \neq 0$. The other tests are nonzero at all the generic points. Their specialization inequalities pass to the supremum, and taking suprema over $a \in F$ and over tests in either order gives the generic-value identity. Their weighted-average inequalities pass to the supremum because each weight is positive. Finally, each test satisfies the outer growth bound dr_∞ after division by m, since $\deg_c A \leq md$. This common bound also holds for the supremum. □

7 A moving family forces a lower bound of 9/4

A tensor depending on a scalar parameter can have a different price-minimizing placement at each parameter value. Section 12 will construct these placements on a disk. We show that a nonconstant family satisfying the disk comparisons forces square matrix multiplication to have exponent at least 9/4. The first step is geometric. At the depth where a translation first moves a point, a rescaled neighborhood contains a tree in which two branches meet at the valuation of the difference of their parameters. The second step uses prices on four equal branches of that tree. With the fourth tip at a generic parameter, a label shared by all four tips has price at most 3/2 times the price of a wire spanning the diameter, to first order. Section 8 uses collisions between the tips to pass from this estimate to a three-site comparison for matrix multiplication rank.

Let F be algebraically closed. Take an uncountable set $\mathcal{U}$ of indeterminates, and put

$$K_0 = \mathrm{Frac}(F(\mathcal{U})[t^{\mathbb{Q}}]), \qquad \mathcal{O} = \{z : \nu(z) \geq 0\}, \qquad J = \{z : \nu(z) > 0\},$$

with the lowest-exponent valuation and $\nu(0) = +\infty$. An indeterminate is *fresh* for finite data in K_0 if it does not occur in those data. Consider a family of sites x_z, $z \in J$, in a spectral space over F, subject to the following conditions.

1. Adding the same $w \in J$ to all indices, and permuting $\mathcal{U}$, preserve the prices of all tensors over F.

2. For a nonzero tensor on any finite list of slots, let $P(c)$ be its price at the sites $x_{a_i+b_ic}$, where $a_i, b_i \in J$. With u fresh, $P(c) \leq P(u)$ for $c \in \mathcal{O}$. For $c_j \in \mathcal{O}$ of distinct residues and positive rational lengths s_j, s_∞, the same test satisfies

$$\sum_j \frac{P(c_j + t^{s_j}u) - P(u)}{s_j} + \frac{P(t^{-s_\infty}u) - P(u)}{s_\infty} \;\geq\; 0 \tag{7.1}$$

   whenever the indices at the outer end lie in J. Here u is fresh also for the star data. Thus the price at the generic point of the disk is at most the weighted mean of its end prices, with weights proportional to the reciprocals of the corresponding lengths. The two comparisons are the ordinary-price versions of specialization and the submean inequality in Lemma 6.2.

3. Some two sites have positive distance, and $d(x_0, x_z) \to 0$ as $\nu(z) \to +\infty$.

The inequalities apply in particular to distances, by testing a two-site wire and dividing its price by the logarithm of its size.

Theorem 7.1 (Rigidity of analytic motion)**.** *If such a family exists, square matrix multiplication over F has exact rank at least $n^{9/4}$ for each side n.*

7.1 The threshold for displacement

To find a scale at which motion persists after rescaling, fix a point under all translations of sufficiently high valuation. Its orbit starts to move when the valuation falls below the threshold for that point. We shall choose the point so that its displacement has a strictly negative left derivative at the threshold.

By translation invariance $x_w \neq x_0$ for some $w \in J$. Choose rational $0 < b < \nu(w)$ and replace x_z by $x_{t^b z}$. This still gives a nonconstant family satisfying all three conditions. Specialization of the wire $d(x_0, x_{t^b c})$ bounds the radius of the new family by its value at a fresh u. Thus we can work in the bounded closed convex hull X of the new sites, taken in a complete CAT(0) spectral extension. By the geodesic determination of prices in Section 5, the symmetries act on X by price-preserving isometries. Write g_w for the action of translation.

We first extend the affine comparisons in Condition 2 to sites $g_{a_i+b_i c}A_i$ with $A_i \in X$. If A_i is obtained by a finite sequence of geodesic interpolations between original sites, every segment in its construction undergoes the same translation as A_i. Its length and the two distances specifying the interpolation point are constant across the comparison. Lemma 5.2 therefore extends the star inequality through each interpolation. It extends specialization in the same way, with the specialized placement as center and the generic placement as its sole end. Now approximate arbitrary A_i by such points and apply the transport inequality (3.1). The approximating sequences for these finitely many sites involve only countably many indeterminates, so one choice of u serves throughout. If every A_i is invariant under permutations, permuting indeterminates while fixing the finite test data makes the comparison available at any fresh u.

For $r > 0$ rational and $A \in X$, take u outside a countable support for A, such as the variables in a sequence of finite geodesic constructions converging to A. Specialization gives

$$R(A, r) := \sup_{\nu(w) \geq r} d(A, g_w A) = d(A, g_{t^r u} A). \tag{7.2}$$

For any given w with $\nu(w) \geq r$, specialization bounds its displacement by the generic displacement at a variable fresh for both A and w. Permuting that variable to u while fixing a countable support for A gives the same bound for every w. Conversely, $t^r u$ itself belongs to the subgroup. Apply (7.1) to the distance $d(A, g_{t^r c}A)$ with one inner end at zero. For positive rational $r - s, r, r + b$ with $s, b > 0$, this gives

$$\frac{R(A, r) - R(A, r - s)}{s} \;\leq\; \frac{R(A, r + b) - R(A, r)}{b}.$$

Thus $R(A, \cdot)$ is convex. It lies between zero and $\operatorname{diam} X$, so it extends continuously to $r > 0$ with a Lipschitz bound on each compact subinterval independent of A. Squeezing the supremum in (7.2) between its values at rational depths identifies it with this extension at real depths.

The common fixed set $X_r = \{A : R(A, r) = 0\}$ of the subgroup $\nu(w) \geq r$ is closed and convex. The circumcenter of any orbit of this subgroup belongs to X_r. Since the orbit of A contains A and fits in the ball of radius $R(A, r)$ about A, this gives $d(A, X_r) \leq R(A, r)$. In particular, $f(r) = d(x_0, X_r)$ is positive whenever the subgroup moves x_0, decreases with r, and tends to zero by Condition 3. For $0 < h < r$, the projection A_r of x_0 to X_r satisfies

$$0 \leq f(r - h) - f(r) \leq R(A_r, r - h) - R(A_r, r).$$

The common Lipschitz bound makes f locally Lipschitz. Its strict decrease somewhere gives a depth r_* with $f'(r_*) < 0$. Fix $A = A_{r_*}$. Every permutation of the indeterminates fixes x_0 and preserves X_{r_*}, so it fixes this unique projection. The last display bounds the left derivative of the convex function $R(A, \cdot)$ at r_* by $f'(r_*)$. To the right of r_* it vanishes identically. Writing $-k < 0$ for the left derivative gives

$$R(A, r_* + h) = k(-h)_+ + o(|h|). \tag{7.3}$$

7.2 Rescaling at the threshold

Take rational $\varepsilon \downarrow 0$ and choose rational $r_0 = r_* - \varepsilon + o(\varepsilon)$. In the pointed spectral ultralimit at A, rescale distances and prices by $2/(k\varepsilon)$. For each $z \in K_0$ take the point L_z represented by

$$g_{t^{r_0} z(t^\varepsilon)} A,$$

where the substitution fixes $F(\mathcal{U})$. For fixed z the translation index is in J eventually and the rescaled displacement is bounded by (7.3). Finite initial terms can be chosen arbitrarily. Additivity of the index substitution gives translation invariance for all $z \in K_0$. Permutation invariance follows since every permutation fixes A and commutes with the substitution. To transfer the star comparison for arbitrary affine expressions in c, observe that their coefficients become $t^{r_0} a_i(t^\varepsilon), t^{r_0} b_i(t^\varepsilon)$ before passing to the limit. The centers $c_j(t^\varepsilon)$ still have distinct residues, each star length is multiplied by ε, and all the tested indices lie in J eventually. Thus the common factors in (7.1) cancel and the inequality passes to the ultralimit. The same argument applies to specialization.

At a generic parameter the limiting distance follows directly from (7.2) and (7.3),

$$d(L_0, L_{t^r u}) = 2(1 - r)_+ \qquad (r \in \mathbb{Q}).$$

The same formula holds at every parameter of that valuation. More precisely,

$$d(L_z, L_{z'}) = 2(1 - \nu(z - z'))_+. \tag{7.4}$$

For distinct indices write $z' - z = t^r v$ with $\nu(v) = 0$. Translation and specialization give the upper bound, which is already equality for $r \geq 1$. If $r < 1$, test $d(L_0, L_{t^r c})$ on a star with two inner ends centered at zero and at v. The slope on the zero arm of any length $0 < a < 1 - r$ is -2. On the outer arm it is $+2$, by the generic distance formula. These slopes cancel in (7.1), leaving a nonnegative slope on the v-arm. Specialization makes that slope nonpositive, so its end distance equals the generic value $2(1 - r)$. Choose the length $b \geq 1 - r$ on this arm. The upper bound already proved identifies $L_{t^r(v + t^b u)} = L_{t^r v}$, which gives equality in (7.4).

7.3 Geodesics of the disk tree

The distance formula places the leaves. To locate the branch points, we prove that the midpoint of two leaves lies on each geodesic joining one of them to a third leaf at least as far away as the other.

Let $B = L_b$, $C = L_{b+v}$, $d(B, C) = 2h > 0$, and let m be their midpoint. Take a star with inner ends $0, 1$, each of length h, and outer length s. On $[B, C]$, prescribe the point m at the center and the outer end, and B, C at the two inner ends. Its distances to each endpoint are harmonic for the weights proportional to the inverse end lengths. Lemma 5.2 therefore

permits comparing its distance to L_{b+cv}. At the inner ends this distance is zero by (7.4). At the outer end it is at most $3h + 2s$ by going through B. Consequently the generic center distance $J_* = d(m, L_{b+uv})$ satisfies

$$0 \le -\frac{2J_*}{h} + \frac{3h + 2s - J_*}{s}.$$

Sending s to infinity yields $J_* \le h$, and specialization gives the same bound throughout the unit disk. More generally,

$$d(m, L_{b+cv}) \le h + 2s \qquad (\nu(c) \ge -s,\ s \ge 0 \text{ rational}). \tag{7.5}$$

Indeed the generic value for $c = t^{-s}u$ is convex in s by one-inner-end comparisons. It starts at most h and is bounded above by $3h + 2s$ throughout. Its secant slopes from zero are thus at most 2. Specialization proves the assertion.

If $d(B, L_z) = 2h + 2s$ with $s \ge 0$, apply (7.5) with $c = (z - b)/v$. The triangle inequality forces equality and $m \in [B, L_z]$. In particular every leaf at distance $2h$ from B gives the same midpoint. For each rational $\ell \le 1$, denote by $P(z, \ell)$ the point of $[L_z, L_{z'}]$ at distance $1 - \ell$ from L_z for any sufficiently distant $L_{z'}$. For $\ell < 1$ choose first z' at distance $2(1 - \ell)$ and use the midpoint just proved. For $\ell = 1$ take L_z. Two distinct leaves $L_z, L_{z'}$ at distance $2h$ meet at depth $1 - h = \nu(z - z')$. Choose a third leaf sufficiently far from both. The midpoint test applied from each original leaf places their common midpoint on both geodesics to the third. Above the meeting depth the two paths lie in opposite halves of $[L_z, L_{z'}]$. Below it they lie on the same segment from the midpoint to the third leaf, by uniqueness of geodesics. Coincident leaves have the same path at every depth. This gives

$$d(P(z, \ell), P(z', \ell')) = \ell + \ell' - 2\min\{\ell, \ell', \nu(z - z')\}. \tag{7.6}$$

The point $P(z, \ell)$ thus depends only on the disk $\{z' : \nu(z' - z) \ge \ell\}$. Its ancestors correspond to the larger disks containing it.

In a finite star comparison we can now prescribe points $P(a + bc, \ell)$ with depths harmonic across that star. Choose a far leaf $L_{t^{-N}}$ with N sufficiently large. Its distance to every L_{a+bc} in the finite test is the same, and the corresponding segment contains the prescribed site. The prescribed distances to these two endpoints are $1 - \ell$ and its complement in their common distance. Lemma 5.2 applies. It also gives specialization for constant depths. In particular the price of a fixed nonzero tensor along a ray $c = \xi + t^{\rho}u$ is convex in ρ if each depth is affine in ρ, since any three rational radii are compared by a star with one inner end at zero. At $\rho = 0$, the generic values obtained from u and $\xi + u$ coincide for integral ξ, by the two specializations in opposite directions.

7.4 The price of a common label at four sites

Place the common label $\Delta_{[4]}(q) = \sum_{j=1}^{q} e_j^{\otimes 4}$, $q \ge 2$, at the tips of four equal branches of the disk tree. Choose the first three centers $z_1 = 0, z_2 = 1, z_3 \in F \setminus \{0, 1\}$ and let the fourth center $z_4 = c$ vary. At a fresh $c = u$, the four tips $P(z_i, \ell)$ at rational depth $0 < \ell < 1$ have common origin $P(0, 0)$ and pairwise distance 2ℓ. Denote the price of their label divided by $\log q$ by D_ℓ. It vanishes at $\ell = 0$. Joint geodesic convexity and the upper bound 4ℓ from four size-q wires to the common origin give a finite one-sided slope

$$\underline{D} = \lim_{\ell \downarrow 0} \frac{D_\ell}{2\ell}.$$

The denominator is the rate of a single wire between two of the tips. We estimate the ratio by moving the fourth center toward each of the three fixed centers in turn.

On the arm toward z_i, let $c = z_i + t^r u$ for $0 \le r \le 2\ell$. Increase the depths of $i, 4$ by $r/2$ and decrease those of the other pair by $r/2$. Take $2\ell \le 1$, so all four depths remain in the tree. By (7.6) the two within-pair distances are $h = 2\ell - r$, and all four distances between pairs are 2ℓ. The path joining the pair midpoints grows by precisely the amount the within-pair distances shrink. Figure 3 depicts the collision with z_1. Relabeling gives the collision with z_3 used in Section 8. When $r = 2\ell$, both pairs have merged. The grouped vectors $e_j \otimes e_j$ are linearly independent at each end, so the label is equivalent under local maps to one size-q wire. Its rate at the collision is exactly 2ℓ.

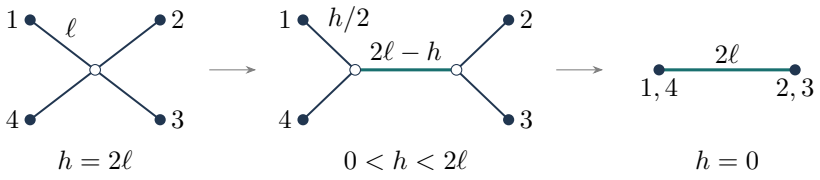
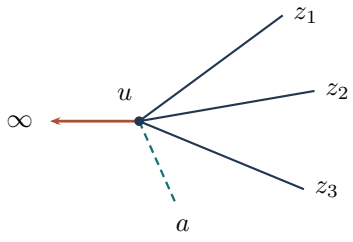


Figure 3. Collision of the fourth tip with the first. On the middle tree the distance inside each pair is h, and the segment between pair midpoints has length $2\ell - h$. At the right endpoint the common label is a single wire of length 2ℓ.

Across these three arms the depth slopes sum to $-1/2$ at each of the first three sites and to $3/2$ at site 4. To make all depths harmonic, add two arms of length ℓ. On an unused finite arm $a \in F \setminus \{z_1, z_2, z_3\}$ increase every depth at slope $1/2$. By specialization its end rate is at most $D_{3\ell/2}$. On the outer arm use $c = t^{-r}u$. Only the fourth depth changes, at slope -2. At $r = \ell$ this site is $P(0, -\ell)$. Write D_ℓ^- for the resulting label rate. The ray price functions are continuous at their origins. Indeed for small signed r the same affine depth prescriptions and t^r or t^{-r} give a convex extension across zero.

Figure 4 records the five depth changes in a single comparison. The sum down each column is zero, which is the harmonicity condition for the corresponding site.



	1	2	3	4	end estimate
z_1	1	−1	−1	1	2ℓ
z_2	−1	1	−1	1	2ℓ
z_3	−1	−1	1	1	2ℓ
a	1	1	1	1	$D_{3\ell/2}$
∞	0	0	0	−4	D_ℓ^-

Figure 4. The five arms based at the generic point u of the parameter disk. The three collision arms have length 2ℓ, the unused finite arm a and the outer arm have length ℓ. The table gives twice the four depth slopes and an upper bound for the label rate at each end.

We compare D_ℓ^- to the generic rate. Start the outer ray at a fixed small depth $\lambda > 0$ and stop at $0 < R < \lambda$, with all lengths rational. Write $D(\lambda, R)$ for the rate. The first three sites have radius λ about $P(0, 0)$. By (7.6) the fourth site also has radius λ, with its first R units of geodesic going toward negative depths. Contract all four radii from λ to $\ell \le R$. This gives

the placement defining D_ℓ^-, so convexity gives

$$\frac{D_\ell^-}{2\ell} \le \frac{D(\lambda, R)}{2\lambda}.$$

Taking the upper limit in ℓ, then $R \downarrow 0$ by ray continuity, then $\lambda \downarrow 0$, yields $\limsup D_\ell^-/(2\ell) \le \underline{D}$. Substituting the upper bound for the fourth finite end into (7.1) gives

$$0 \le 3\frac{2\ell - D_\ell}{2\ell} + \frac{D_\ell^- - D_\ell}{\ell} + \frac{D_{3\ell/2} - D_\ell}{\ell}. \tag{7.7}$$

The three collision arms contribute $3(1 - \underline{D})$ in the limit. The outer arm has upper limit at most zero, and the last term tends to $\underline{D}$ since $D_{3\ell/2}/(3\ell) \to \underline{D}$. Hence $0 \le 3(1 - \underline{D}) + \underline{D}$, giving

$$\underline{D} \le 3/2. \tag{7.8}$$

The rectangular comparison of Section 8 will use this bound near the endpoint $h = 0$ in Figure 3. There the three fixed tips form a tripod with a pair of short arms and one long arm. Propagating the resulting limiting price equality to the balanced tripod gives a common-label rate of $8/3$. The three matrix multiplication wires on that tripod have total length 6, so the rank comparison has exponent $6/(8/3) = 9/4$.

8 A lower bound from the price of a shared label

Consider a square matrix product of side n, with its two inputs and its output at the tips of a tripod with three unit arms. Each of the three matrix indices is shared by two tips at distance 2, so the product tensor has price $6 \log n$. A rank-r decomposition supplies that tensor from a label with r values shared by all three tips. If the rate of this shared label is $8/3$, its price is $(8/3) \log r$, and price monotonicity gives $r \ge n^{9/4}$. We will find such a tripod in the family constructed in Section 7. We first test the shared-label rate when one arm is much longer than the other two. The following principle carries the resulting equality to three equal arms.

Proposition 8.1 (Tripod principle)**.** *Over an algebraically closed field, place three sites at pairwise distances $a_i + a_j$ for $i \ne j$, where $a_1, a_2, a_3 \ge 0$. Their common-label rate $D(a)$ satisfies*

$$D(a) \;\ge\; D_*(a) := a_1 + a_2 + a_3 - \big(a_1^{-1} + a_2^{-1} + a_3^{-1}\big)^{-1}.$$

The reciprocal term is zero when any $a_i = 0$.

Now fix three infinite rays in a spectral space that form a metric tripod with origin O, and suppose that their prices are homogeneous under radial dilation. In $D(a)$, place site i on ray i at distance a_i from O. If $D(a) - D_*(a)$ vanishes at one positive triple, it vanishes at every positive triple. The same conclusion holds if the defect tends to zero as one arm tends to infinity with the other two held at fixed positive lengths.

The first assertion applies to any placement with the specified distances. The second compares placements on a single tripod. Sections 9–11 prove both assertions by studying how the price of polynomial multiplication changes with its degree and with the positions of its coefficients.

8.1 A decoder for rectangular multiplication

Multiply an $s \times q$ matrix by a $q \times s$ matrix over the algebraically closed field F of Section 7, taking $s = N^{1/2}$ and $q = N^{1/4}$ for a fourth power N. The product has N entries. Place its output at A and its inputs at C, D', with respective arm lengths c, a, b. The outer indices give wires of size s on AC, AD', and the inner index gives a wire of size q on CD'. The price of the product, divided by $\log N$, therefore satisfies

$$c + \tfrac{3}{4}(a+b) \;\le\; c + a + b - \frac{ab}{a+b} \;\le\; D_*(c,a,b) \;\le\; D(c,a,b).$$

Here $ab/(a+b) = 0$ if $a = b = 0$. This gives the rectangular bound associated with half-width multiplication [5, 55].

In particular, for every $\eta > 0$ there are arbitrarily large such N for which the exact tensor rank is at most $N^{1+\eta}$. Indeed, start with a sufficiently large fourth power N_0 and a full diagonal of size $\lceil N_0^{1+\eta/2} \rceil$. The displayed inequality at every spectral point gives strict spectral comparison with this diagonal. Each tensor power of the product is a restriction of a bounded number of copies of the corresponding power of the diagonal. At sufficiently large powers the exponent $1 + \eta$ absorbs both this multiplicity and the rounding.

Lemma 8.2 (A decoder with short rows)**.** *For every $\epsilon > 0$ there is a $\delta > 0$ such that for unbounded fourth powers N the N output bilinear forms B_e of this product admit decompositions*

$$B_e = S_e + R_e, \qquad \operatorname{rank} S_e \le N^{\epsilon}, \qquad \dim\langle R_e : 1 \le e \le N \rangle \le N^{1-\delta}.$$

The two bounds have different uses. A short sum of simple products computes each S_e once the output index e is shared. The corrections R_e may be expensive individually, but their entire span can be sent across a single wire. To obtain the bounds uniformly in the base format, we implement a tensor power of a decoder using one work register. For related nonrigidity estimates for Kronecker products, see [1, 33].

Proof. It suffices to take $0 < \epsilon < 1$. Fix an integer $K > 10/\epsilon$. For a base format with d output entries and rank $d \le r \le d^{1+\eta}$, let M be the $d \times r$ matrix that expresses its outputs in terms of r simple bilinear products. We will choose η below. The Km-th tensor power has $N = d^{Km}$ entries. Pad each decoder block $M^{\otimes m}$ with zero rows and columns to side 2^t, where $t = \lceil m \log_2 r \rceil$. The full padded decoder has side $T = 2^{Kt}$. An index consists of K registers of t bits each.

Apply the first block matrix to the first register. For each subsequent block, swap its register with the first, apply the block matrix there, and swap back. An exchange of registers performs t simultaneous exchanges of pairs of bits. A bit exchange transposes two of four letters, 01 and 10. Its matrix factors into a fixed number of elementary shears and diagonal signs, over any field. Thus a register exchange consists of a fixed number of stages, each applying the same local shear or diagonal sign at every bit position.

For a shear, let the high letter be the target of its off-diagonal entry. Immediately before and after the stage, project onto the coordinates whose register pair contains between $t/4 - \rho t$ and $t/4 + \rho t$ high letters, for a fixed small $\rho > 0$. The discarded fraction of coordinates is at most $e^{-c_\rho t}$ for large t, by the binomial tail. Telescoping over the $O(K)$ projections bounds the rank of the discarded part of the decoder by $O(K) T e^{-c_\rho t} \le T^{1-\gamma}$, with $\gamma = \gamma(K, \rho) > 0$. On

a path retained at a shear stage, every change creates a high letter, so at most $2\rho t$ positions change.

Only the first register is acted on by a block matrix. Every other register participates in just two exchanges, so its input lies within Hamming distance $C\rho t$ of its output on a retained path, for an absolute constant C. Each output row of the retained decoder therefore has at most

$$2^t \left(\sum_{j \le C\rho t} \binom{t}{j} \right)^{K-1} \le T^{1/K + o_{\rho \downarrow 0}(1)}$$

nonzero entries. Choose ρ so that this exponent is below $\epsilon/2$, then choose η sufficiently small. Since $T \le 2^K N^{1+\eta}$, for all large m the two bounds become N^ϵ entries per retained row and rank $N^{1-\delta}$ for the discarded matrix, with $\delta = \gamma/2$. Remove the zero padding and apply both matrices to the list of simple bilinear products. The retained rows give the forms S_e, and the image of the discarded row space spans the R_e. □

8.2 Two copies of the output near a collision

Choose the collision in Section 7 where site 4 approaches site 3, and denote sites $1, 2, 3, 4$ by A, B, C, D'. At parameter r write $L = 2\ell$ and $h = L - r$, with $0 < h < L$. The distances within AB and CD' are both h, and all four cross distances are L. Write $D_\ell(h)$ for the four-site common-label rate. At the collision $D_\ell(0) = L$, and at the generic four-site placement $D_\ell(L) = D_\ell$. Convexity gives

$$D_\ell(h) \le L + \frac{h}{L}(D_\ell - L).$$

Duplicate the output index $e = (i, i')$ of the rectangular product at A and B, leaving its two inputs at C, D'. The resulting tensor is

$$U_N = \sum_e e_e \otimes e_e \otimes B_e = \sum_{\substack{1 \le i, i' \le s \\ 1 \le j \le q}} e_{ii'} \otimes e_{ii'} \otimes e_{ij} \otimes e_{ji'}.$$

The index i is a shared label at A, B, C, the index i' is a shared label at A, B, D', and j travels on the short wire CD'. Writing $D_{ABC}, D_{ABD'}$ for the two triple rates, we have

$$\frac{H(U_N)}{\log N} = \frac{D_{ABC} + D_{ABD'}}{2} + \frac{h}{4}.$$

Both triples have arms $(h/2, h/2, L - h/2)$. Their reciprocal correction in Proposition 8.1 is at most $h/4$, so both rates are at least $L + h/4$.

Use the decomposition $B_e = S_e + R_e$ from Lemma 8.2. In units of $\log N$, the tensor formed from the S_e costs at most $D_\ell(h) + \epsilon h$. Share the label e at all four sites, then use a wire of size at most N^ϵ on CD' to supply the simple products for the specified e. The tensor formed from the R_e has matrix rank at most $N^{1-\delta}$ across the cut $AB \mid CD'$. A wire between A and C supplies this rank, and wires within the two pairs deliver the legs at B and D', each of dimension at most N. This costs at most $(1 - \delta)L + 2h$.

Take $h/L < \delta/2$. The correction then costs less than L, and $D_\ell(h) \ge L$ by restriction to a cross pair. For a fixed placement the ordinary-sum price bound costs an additive constant

independent of N. Sending N to infinity along the formats of Lemma 8.2 and then using Equation (7.8) at fixed h/L gives

$$\frac{D_{ABC} + D_{ABD'}}{2} + \frac{h}{4} \;\le\; D_\ell(h) + \epsilon h \;\le\; L + \big(\tfrac{1}{2} + \epsilon + o_{\ell\downarrow 0}(1)\big)h.$$

Subtract the lower bound $D_{ABD'} \ge L + h/4$ to obtain

$$D_{ABC} \le L + h/4 + \big(2\epsilon + o_{\ell\downarrow 0}(1)\big)h.$$

The first three sites lie on the fixed rays z_1, z_2, z_3 out of the depth-zero point. Take the tangent tripod there by placing sites at depths ta_i, dividing prices by t, and letting $t \downarrow 0$ in a spectral ultralimit. On these fixed rays the rescaled price of every nonzero tensor has a limit, since its original price is convex in t and vanishes at zero. The limiting prices are homogeneous under radial dilation.

Set $t = h/2$ and hold $b = 2L/h - 1$ fixed. The three depths become $t(1, 1, b)$. Dividing the last inequality by t gives $D(1, 1, b) \le b + 3/2 + 4\epsilon$, whereas

$$D_*(1, 1, b) = b + \tfrac{3}{2} + \frac{1}{4b + 2}.$$

Thus $0 \le D(1, 1, b) - D_*(1, 1, b) \le 4\epsilon$ for every sufficiently large rational b, and for real b by continuity. Sending b to infinity and then ϵ to zero gives contact at infinity on this fixed tripod. Proposition 8.1 now gives $D(1, 1, 1) = 3 - 1/3 = 8/3$. The three-wire price comparison at the start of this section yields $r \ge n^{9/4}$, proving Theorem 7.1.

9 Polynomial multiplication near contact

The tripod principle compares the price D of a common label at the three terminals with the lower bound $D_*(a) = \sum_i a_i - (\sum_i a_i^{-1})^{-1}$ for their arm lengths $a_i > 0$. We call equality *contact* and write $\Delta = D - D_*(a)$ for the defect. To propagate contact to other triples of arm lengths, we need to know how a price changes when just one input width varies. Polynomial multiplication supplies a tensor for which we can measure this change. Its price as one input grows has a limiting slope and intercept after exponentiation. Proposition 9.2 determines the intercepts to within $O(\Delta)$. Section 10 then compares different placements, and Section 11 uses the comparison to propagate equality.

Throughout these three sections the field is algebraically closed and prices use natural logarithms with the distances held fixed. A label with q values shared at the terminals costs $D \log q$. On an embedded tripod, including its origin gives the same price. A duplicate at either of two terminals is free, and convexity bounds its cost at every point on the geodesic between them.

9.1 The continuum profile

Place a finite nonempty set of inputs and one output in a spectral space. Write H for price and $D > 0$ for the common-label rate of the placement. For integer lengths $L_i \ge 1$, let $M(L)$ multiply univariate polynomials of degrees less than L_i, retaining all output coefficients. The following normalization also fixes the constant in the energy formula below.

Lemma 9.1. *There is a continuous coordinatewise nondecreasing function P on the positive orthant such that, if $L_i/N \to t_i > 0$, then*

$$H(M(L)) = D\log N + P(t) + o(1), \qquad P(ct) = P(t) + D\log c.$$

For polynomials in m variables with total input degrees at most $\lfloor Nt_i \rfloor$, the corresponding price is $mD\log N + mP(t) - D\log(m!) + o(1)$ for each fixed m.

Proof. Evaluation and interpolation bound $H(M(L))$ by $D\log(\sum_i L_i)$. In the other direction, $M(kL)$ degenerates to k direct copies of $M(L)$. Indeed, divide $[0,1]$ into k equal intervals and choose a continuous convex piecewise affine function with a strict change of slope at each division. An input monomial of degree d_i has normalized position $(d_i + 1/2)/(kL_i)$. Give it weight kL_i times the function there. Give the output the opposite weight at the weighted mean of those positions, with coefficient $k\sum_i L_i$. The output weight depends only on the sum of degrees. Jensen's inequality retains precisely the terms whose inputs lie in one interval. Different intervals use disjoint input and output coordinates, and each surviving block is a translated copy of $M(L)$. The weight equalities define a rational linear subspace. Rational approximation within it preserves the finitely many strict inequalities, and scaling gives integral weights. Rescaling the coordinates and using closedness of price sublevels therefore gives the asserted degeneration.

For $L_i = \lfloor Nt_i \rfloor$ the normalized cost at N therefore dominates that at any fixed sufficiently large M, up to $D\log(M\lfloor N/M \rfloor/N)$. Taking first the lower limit in N and then the upper limit in M proves convergence. Changing the unit of N gives log homogeneity, first with floors by squeezing the scales and then for real N. Coordinate monotonicity and squeezing between $(1 \pm \epsilon)t$ give continuity and allow $o(N)$ errors in the lengths.

For the assertion about total degree, divide the exponent vector at input i by Nt_i and at the output by $N\sum_i t_i$. All inputs now range over lattice points in the unit simplex. To cut this domain by a hyperplane, first discard input points on the plane. Apply the same convex weighting as above, using the positive part of the defining affine form and coefficients Nt_i. Jensen's inequality retains terms whose inputs are on the same side. Their outputs are on that side as well, so the two blocks form a direct sum. One may repeat the cut within each block.

On a translated axis-parallel cube of side s, multiplication factors over the m coordinates. The univariate limit shows that its exponentiated price, divided by $N^m \exp(mP(t)/D)$, tends to its volume s^m. Translation just relabels exponents and rounding a cube side changes widths by $O(1)$. Packing the simplex with interior grid cubes thus bounds its normalized price from below by $1/m!$, since $\exp(H/D)$ adds on direct sums. For the upper bound, start with an enclosing cube and cut off a slightly enlarged simplex, so all input points of the desired tensor survive in one cell. Pack the complementary cells with interior grid cubes. The enclosing tensor dominates the direct sum of the desired tensor and all packed cubes. Refining the grid and shrinking the enlargement accounts for the whole complementary volume and gives the matching upper bound $1/m!$. Taking logarithms proves the formula. □

9.2 A label visible at two sites

We also need a lower bound for a sum whose block label is visible at only two of three sites. Suppose $T = \sum_b T_b$ with $T_b \neq 0$ and $T_b \in X \otimes Y_b \otimes Z_b$, where the Y_b are direct subspaces at

Y, and the Z_b are direct subspaces at Z. For every probability law q on the block labels, writing $\mathrm{Ent}(q) = -\sum_b q_b \log q_b$, we have

$$H(T) \geq \sum_b q_b H(T_b) + d(Y,Z)\,\mathrm{Ent}(q). \tag{9.1}$$

To see the entropy term, take a rational q and a power n for which each nq_b is integral. At Y and Z, retain the $M = n!/\prod_b (nq_b)!$ words of type q. Each word has price $n\sum_b q_b H(T_b)$. The shared-leg separation of [41, Proposition 3.1], in the interpolation form of [39, Proposition 2.1], degenerates $2M-1$ direct copies of the retained tensor to the full direct sum of the M word blocks, each tensored with an M-dimensional wire on Y, Z. Interpolation uses $2M-1$ distinct field elements, available in every algebraically closed field. Comparing the two prices gives

$$nH(T) + D\log(2M-1) \;\geq\; n\sum_b q_b H(T_b) + (D + d(Y,Z))\log M,$$

where D is the common-label rate of the three sites. Now $\log((2M-1)/M) \leq \log 2$ and $n^{-1}\log M \to \mathrm{Ent}(q)$. Division by n and continuity in q give (9.1).

9.3 Slopes and intercepts on a triangle

Use i, j, k for the three sites in any order and suppose their arms a_i are positive. Put $d_i = d(j,k) = a_j + a_k$, $D = \delta(\{i,j,k\})$, and write $P_k(x,y)$ for the profile with output k and input widths x at i and y at j. We identify each argument by its site. For a fixed first input define

$$F_{ik}(t) = \exp\bigl(P_k(t_i = 1, t_j = t)/d_i\bigr).$$

The determinant filtration and the shifted three-block degeneration for polynomial multiplication [41, proofs of Lemmas 4.1 and 4.2] give

$$F_{ik} \text{ is concave}, \qquad F_{ik}(3t+1) \geq 2F_{ik}(t) + F_{ij}(t).$$

In the first degeneration a two-dimensional wire on j, k supplies two blocks, with the length at j increased and decreased by one. The blocks are direct on that pair. Optimizing the pair-label inequality over the block probabilities adds their exponentiated prices at scale d_i. This gives midpoint concavity on the lattice of lengths, hence concavity in the limit. In the second degeneration the longer length is $3L_j + L_i - 1$, and one of the three blocks exchanges j and k. The same pair-label inequality and Lemma 9.1 give the second displayed assertion. Both degenerations work in every characteristic. For the first, injectivity of multiplication by the determinant and a dimension count give the degree-one Clebsch–Gordan determinant sequence (see [34]). Lifting monomials to its filtration suffices. The second uses a monomial grading, and reversal of the order on a leg identifies the middle block.

Positive increasing concave functions have finite limiting slopes at infinity. Dividing the three-block inequalities by t shows that the slopes of F_{ik} and F_{ij} are equal. For $S = F_{ik} + F_{ij}$ we have $S(3t+1) \geq 3S(t)$. Iterating along $t_n = 3^n(t + 1/2) - 1/2$ gives $S(t)/(t+1/2) \leq \lim_{n\to\infty} S(t_n)/(t_n + 1/2)$, twice the common slope. In particular the common slope is positive. Write it as $\exp(c_i/d_i)$. For either function the excess over the line through zero of that slope is nonnegative and increasing. The bound on S therefore gives numbers $\beta_{ik}, \beta_{ij} \geq 0$, with $\beta_{ik} + \beta_{ij} \leq 1$, such that

$$t \leq e^{-c_i/d_i} F_{ik}(t) \leq t + \beta_{ik}.$$

Here β_{ik} is the limiting excess after division by the slope. In particular P_k is d_i-Lipschitz as a function of $\log y$ with x fixed, since every one-sided derivative of F_{ik} lies between zero and $F_{ik}(t)/t$.

Set $h = (\sum_i 1/a_i)^{-1}$, $b_i = a_i - h$, $u_i = b_i/a_i$, $\beta^*_{ik} = a_k/d_i$, and $\Delta = D - D_*(a)$, where $D_*(a) = \sum_i a_i - h$. Since $P_k(x,y) = D\log x + P_k(1, y/x)$ and $D - d_i = b_i + \Delta$, the intercept bound gives an upper envelope valid at every pair of widths,

$$P_k(x,y) \le U_{ik}(x,y) := c_i + (b_i + \Delta)\log x + d_i \log(y + \beta_{ik}x). \tag{9.2}$$

Proposition 9.2 (Profile near contact)**.** *We have $\Delta \ge 0$. If $\min_i u_i \ge \eta > 0$, there is a number C common to all three sites such that*

$$d_i(\beta_{ik} - \beta^*_{ik}) = O_\eta(\Delta), \qquad c_i + b_i \log u_i = C + O_\eta(\Delta). \tag{9.3}$$

In the sector $y/x \ge u_j/u_i$,

$$0 \le U_{ik}(x,y) - P_k(x,y) \le O_\eta(\Delta). \tag{9.4}$$

Proof. Let $E_{ik} = U_{ik} - P_k$ and $z_{ik} = d_i(\beta_{ik} - \beta^*_{ik})$. Differentiating with respect to y and bounding the derivative of F_{ik} below by its limiting slope yields

$$\partial_y P_k(x,y) \;\ge \frac{d_i}{y + \beta_{ik}x} \exp(E_{ik}(x,y)/d_i).$$

There is a second bound from interchanging i, j. At differentiability points, $x\partial_x P_k + y\partial_y P_k = D$. The profiles are locally Lipschitz, so continuity gives the sum of the two derivative bounds at $(x,y) = (u_i, u_j)$ even if a derivative fails to exist there. At this point $u_j + u_i\beta^*_{ik} = 1$ and $u_j d_i + u_i d_j = D_*$. Use $(1+r)^{-1} = 1 - r + r^2/(1+r)$ with $r = u_i z_{ik}/d_i$, and $e^s \ge 1 + s$ for $s \ge 0$. Since $\eta \le u_j + u_i\beta_{ik} \le 2$, this gives, for some $c_\eta > 0$,

$$\frac{\Delta}{u_i u_j} \ge -(z_{ik} + z_{jk}) + c_\eta \left(E_{ik} + E_{jk} + \frac{z_{ik}^2}{d_i} + \frac{z_{jk}^2}{d_j} \right),$$

where all the gaps are evaluated at the specified widths. The row sums $z_{ik} + z_{ij}$ are nonpositive since $\beta^*_{ik} + \beta^*_{ij} = 1$. Summing over output sites k gives $\Delta \ge 0$ and bounds each nonnegative remainder by $O_\eta(\Delta)$. Each row sum thus lies between $-O_\eta(\Delta)$ and zero. Each column sum is at least $-O_\eta(\Delta)$ by the displayed inequality. Since the columns have the same total as the rows, each column sum is also at most $O_\eta(\Delta)$.

Comparing the two envelopes for $P_k(u_i, u_j)$ now gives

$$c_i + b_i \log u_i + u_i z_{ik} = c_j + b_j \log u_j + u_j z_{jk} + O_\eta(\Delta).$$

Here the logarithmic Taylor remainders are controlled by z_{ik}^2/d_i and z_{jk}^2/d_j. The row and column estimates show that, up to $O_\eta(\Delta)$, the entries $(z_{12}, z_{13}, z_{23}, z_{21}, z_{31}, z_{32})$ alternate between z_{12} and $-z_{12}$. Sum the three comparisons of envelopes for $(i,j) = (1,2), (2,3), (3,1)$. The terms $c_i + b_i \log u_i$ cancel, leaving $2(u_1 + u_2 + u_3)z_{12} = O_\eta(\Delta)$. This proves (9.3). The gap E_{ik} depends only on y/x. The difference between $t + \beta_{ik}$ and $e^{-c_i/d_i}F_{ik}(t)$ decreases with t, and the latter function increases. Their log ratio hence decreases, which proves (9.4). □

For example, at contact on the unit tripod the predicted intercept is $\beta^*_{ik} = 1/2$ and $u_i = u_j = 2/3$. Thus $e^{-c_i/d_i}F_{ik}(t) = t + 1/2$ for $t \ge 1$. More generally, (9.3) remains uniform as a single arm tends to infinity while the other two stay in a compact subset of $(0, \infty)$. If an arm has length zero, $D \ge D_*(a)$ follows from the diameter bound.

10 Energy and radial contraction

Fix an output and a nonempty finite set A of inputs with common-label rate $D > 0$ on all the terminals. For costs $\ell_i > 0$ per unit of input width, define the energy

$$Q_A(\ell) = D + \sup_{t_i>0}\bigl(P(t) - \ell \cdot t\bigr). \tag{10.1}$$

It is finite by the evaluation bound, convex in ℓ, and satisfies $Q_A(c\ell) = Q_A(\ell) - D\log c$ by log homogeneity of P. A common surcharge s penalizes a candidate width vector by $s\sum_i t_i$. Hence the slope of the optimized energy along a common cost increase measures total width. We will use the finite difference $Q_A(\ell) - Q_A(\ell + s\mathbf{1})$. Lemma 10.3 shows that this drop can only decrease when inputs move towards the output and their respective costs decrease in proportion. We prove the comparison first for homogeneous degrees, where we can move an input between two polynomial multiplications, and then recover Q_A by rescaling those degrees.

To compare different placements, first charge a cost $f_i > 0$ per degree of a homogeneous polynomial. Write $M_m^{\hom}(r)$ for multiplication of homogeneous polynomials in m variables, with input degrees $r = (r_i)_{i\in A}$. Define

$$B_A(f) = \sup_{m\ge 1,\ r_i\in\mathbb{Z}_{\ge 0}} \frac{H(M_m^{\hom}(r)) - \sum_i f_i r_i}{m}.$$

This is finite. Evaluation and interpolation at a basis of evaluation functionals on the output give rank at most its dimension. With $R = \sum_i r_i$ and $q = \exp(-\min_i f_i/D)$, the exponential of the cost-subtracted price at scale D is bounded by $\binom{m-1+R}{R}q^R \le (1-q)^{-m}$. The supremum is unchanged if m is required to be a multiple of a prescribed integer. Indeed, using disjoint groups of variables recovers any power of the original tensor as a restriction of multiplication with the combined degrees.

Lemma 10.1 (Adding an input)**.** *If $\varnothing \ne A' \subseteq A$ and $s > 0$, then*

$$B_A(f) - B_A(f + s\mathbf{1}) \ge B_{A'}(f) - B_{A'}(f + s\mathbf{1}), \tag{10.2}$$

with costs restricted to the indicated input set. If all sites in a restricted placement coincide, its energy is zero.

Proof. It suffices to add a single input e. Rewrite the claim with $B_{A'}(f) + B_A(f + s\mathbf{1})$ on the right, and choose two multiplications approaching those suprema. They can use the same number m of variables by taking powers in separate groups. Name their variables respectively $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_m)$. At each common input the tensor product of the two multiplications accepts a bihomogeneous polynomial in (u, v) of prescribed bidegree. At e it accepts a polynomial $h(v)$ of degree q. Write (U, V) for the total bidegree, including q.

We realize this tensor product by a sum of multiplications in which e belongs instead to the u group. Rename $h(v)$ to $h(u)$ and substitute $u - v$ for u at the common inputs. Multiply, substitute $u + v$ for u on the output, and project onto bidegree (U, V). The substitutions on the common inputs cancel, and the only term of $h(u + v)$ that survives the projection is $h(v)$. This identity uses no division. Expand the intermediate inputs into bidegrees before multiplying. If a total of k degrees from the common inputs move from u to v, the

intermediate v-degree is $V - q + k$. The final substitution can only increase this degree, so a surviving sector has $k \le q$. The total degree at each input is preserved. Thus the base cost of every sector is unchanged and its surcharge on v is at most sV. Each sector factors through a multiplication on u with all inputs and one on v with inputs A', with local pre- and postprocessing.

Apply this construction to powers in nm variables per group with all degrees multiplied by n. There are polynomially many bidegree sectors as functions of n, since A is fixed. The price bound for an ordinary sum costs only $O(\log n)$ beyond the largest sector price. After division by nm and passage to the limit, the right side is bounded by $B_A(f) + B_{A'}(f + s\mathbf{1})$, as claimed. □

10.1 From degrees to widths

Charging ℓ_i/N per degree makes degrees of order N correspond to widths of order one. The factor $1/m!$ for total-degree polynomials in Lemma 9.1 accounts for the additive D in (10.1).

Lemma 10.2 (Continuum limit of the degree energy)**.** *At a placement with common-label rate $D > 0$, one has*

$$B_A(\ell/N) = D \log N + Q_A(\ell) + o(1) \qquad (N \longrightarrow \infty). \tag{10.3}$$

Proof. Write $S_m(r)$ for polynomial multiplication with total degree *at most* r_i on each input. Its price minus degree costs, divided by m, is a lower bound on B_A. Indeed, n copies on disjoint variable groups restrict from homogeneous multiplication with $nm + 1$ variables and degrees nr_i. Use the last variable to homogenize and set it to one on the output. Hence $B_A(f) \ge n(H(S_m(r)) - f \cdot r)/(nm + 1)$, and we let $n \to \infty$. For $r_i = \lfloor Nmt_i \rfloor$, Lemma 9.1 now gives

$$\liminf_{N\to\infty} (B_A(\ell/N) - D \log N) \ge P(t) - \ell \cdot t + D\big(\log m - m^{-1} \log(m!)\big).$$

The last summand tends to D. Taking the supremum over t gives the required lower bound.

For the upper bound take integer N. In the supremum defining B_A we may use Km variables grouped into K sets of size m. At each input, record the total degree in each group by its bin $[(b_i - 1)N, b_i N)$, $b_i \ge 1$. Within a fixed bin pattern the multiplication restricts from the product of $S_m(Nb)$ over the groups, and the cost in each group is at least $\sum_i \ell_i(b_i - 1)$. The price of the sum over patterns is bounded by the logarithm of the sum of their exponentiated prices at scale D. Summing over all bins independently in each group enlarges this last sum to a Kth power. Dividing its logarithm by Km and taking the supremum gives

$$B_A(\ell/N) \le \frac{D}{m} \log \sum_{b \in \mathbb{Z}^A_{>0}} \exp\left(\frac{H(S_m(Nb)) - \sum_i \ell_i(b_i - 1)}{D} \right).$$

For fixed m, after removing N^m from the sum, the terms are dominated by a constant times $(m + \sum b_i)^m \exp(-\ell \cdot b/D)$, by the output-dimension bound. Lemma 9.1 and dominated convergence bound the upper limit of $B_A(\ell/N) - D \log N$ by

$$D\left(\log m - \frac{\log(m!)}{m}\right) + \frac{\sum_i \ell_i}{m} + \frac{D}{m} \log \sum_b \exp\left(\frac{mP(b/m) - \ell \cdot b}{D} \right).$$

For $0 < \epsilon < 1$, use the supremum of $P(t) - (1-\epsilon)\ell \cdot t$ to bound the last term, with a remainder at most $(D/m) \log \sum_b \exp(-\epsilon \ell \cdot b/D)$. Let $m \to \infty$, then $\epsilon \downarrow 0$, using the scaling relation for Q_A. Monotonicity in N gives the same limit at real scales. □

Lemma 10.3 (Radial contraction)**.** *Move each input i towards the output along its geodesic, to a fraction $0 < \lambda_i \leq 1$ of its original distance. Write A' for the resulting placement and $\ell'_i = \lambda_i \ell_i$. For $s > 0$ one has*

$$Q_A(\ell) - Q_A(\ell + s\mathbf{1}) \geq Q_{A'}(\ell') - Q_{A'}(\ell' + s\mathbf{1}) \tag{10.4}$$

when both placements have positive common-label rate.

Proof. Duplicating an input at its own site and cost leaves B unchanged, by local multiplication. An additional input at the output at zero cost also leaves B unchanged, by local multiplication on the output. The reverse inequalities in both statements use degree zero at the added input. For each fixed multidegree the price is convex as the added input moves on a geodesic, and the subtracted cost is affine. Thus an added input at radial fraction λ_i and cost at least $\lambda_i f_i$ leaves B unchanged, first for one addition and then for all of them. This applies both at $f = \ell/N$ and after increasing every cost by s/N, since $\lambda_i \ell_i + s \geq \lambda_i(\ell_i + s)$. Insert the moved inputs at their specified costs and use Lemma 10.1 to delete the original inputs. The energy drop for the resulting placement is bounded by the drop for the original one. Let $N \to \infty$ in (10.3). The leading $D \log N$ cancels separately at each placement, which proves (10.4). □

11 Propagation of equality on a tripod

Fix three infinite rays forming a metric tripod with origin O. Prices on these rays are homogeneous under simultaneous radial dilation. Proposition 9.2 gives $D(a) \geq D_*(a)$ at the three tips of lengths a_1, a_2, a_3. We now prove the rigidity assertion of Proposition 8.1. Equality at a single triple of lengths forces equality at every triple. The same conclusion follows from equality approached along one arm as its length tends to infinity with the other two lengths fixed and positive.

Use the reciprocal lengths

$$v_i = 1/a_i, \qquad h(a) = \frac{1}{v_1 + v_2 + v_3}, \qquad \Delta(v) = D(a) - D_*(a).$$

An infinitely long arm now corresponds to a coordinate face. The proof first compares integrals of the defect along the three coordinate rays out of v. At a zero of Δ these integrals agree. Radial homogeneity then forces their common value to vanish.

11.1 Comparing the three directional integrals

The profile lower bound and the spanning tree through O give $D_*(a) \leq D(a) \leq \sum_i a_i$. Equivalently,

$$0 \leq \Delta(v) \leq \frac{1}{v_1 + v_2 + v_3}, \qquad \Delta(rv) = r^{-1}\Delta(v) \quad (r > 0). \tag{11.1}$$

On the positive octant Δ is locally Lipschitz and locally semiconvex. The latter means that adding $C|v|^2$ makes it convex on a given small ball for a suitable C. Indeed, D is convex in the arm lengths by joint geodesic convexity. Transport of a single tip through distance t changes D by at most t. The change of variables $a_i = 1/v_i$ has bounded first and second derivatives on interior compact sets, and D_* is smooth there.

As $v_1 \downarrow 0$, the function $D(a) - \sum_i a_i$ is nonincreasing by the same transport bound, and is bounded below by $-h(a)$. Since $h(a)$ also has a limit there, Δ extends to the open face $v_1 = 0$. The Lipschitz and semiconvexity constants in the other two variables are uniform when those variables range in compact subsets of $(0, \infty)$, including as $v_1 \downarrow 0$. Thus the extension is continuous and has the same tangential regularity. The other faces are treated by relabeling.

For each interior v and coordinate vector e_i, set

$$J_i(v) = \int_0^\infty \left[\Delta(v + te_i) - \mathbf{1}_{\{t\le 1\}}\Delta(v)\right] \frac{dt}{t}.$$

The subtraction makes the integral convergent at zero by local Lipschitz continuity. Equation (11.1) gives convergence at infinity. We will prove the estimate

$$|J_i(v) - J_j(v)| \le C\Delta(v). \tag{11.2}$$

Here C is locally uniform up to each open face. On a face the same estimate holds for its two tangential directions.

An energy derivative. Keep v fixed and use $a_i = 1/v_i$ as the locations of three input tips, with the output at O. Let Q be their energy from Equation (10.1). For a variable $s \in (0, \infty)^3$, put

$$G_v(s) = Q((a_i s_i)_i) + \sum_i a_i \log s_i - h(a) \log\Big(\sum_i s_i\Big).$$

Moving s in direction v raises every cost $a_i s_i$ at the same unit rate. On the same tripod place tip i at distance r/s_i with cost r on every input. The ratio of cost to distance on arm i is still s_i. For small r the comparison placement lies inside the fixed one, and for large r it lies outside. By Lemma 10.3, the energy drops at these inner and outer placements bound the drop at the fixed placement from below and above, respectively.

At the comparison placement the common-label rate is $rD((1/s_i)_i)$. Adding O to the three label sites costs nothing. Increasing all costs from r to $r + t$ therefore lowers the energy by $rD((1/s_i)_i)\log(1 + t/r)$, by the scaling law for Q. After division by t, both bounds tend to $D((1/s_i)_i)$ as $t \downarrow 0$. Differentiating the two explicit logarithmic terms of G_v adds $\sum_i 1/s_i - 1/\sum_i s_i = D_*((1/s_i)_i)$. Hence

$$\left.\frac{d}{dt}\right|_{t=0+} G_v(s + tv) = -\Delta(s). \tag{11.3}$$

The finite convex function Q is locally Lipschitz, so this identity may be integrated along any ray in direction v.

Starting the rays near their axes. Let $s_i = 1$, $s_j = s_k = \varepsilon$. Recall $b_i = a_i - h(a)$ and the intercept c_i of Section 9. Write $B_i = b_i + \Delta(v)$ and $d_i = a_j + a_k$. The limit of G_v is

$$\lim_{\varepsilon \downarrow 0} G_v(s) = c_i + B_i \log \frac{B_i}{a_i}. \tag{11.4}$$

To prove the upper bound, let the three input widths be x, t_j, t_k and denote their profile with output at O by $P(x, t_j, t_k)$. The point O divides the segment from j to k in the ratio $a_j : a_k$. Convexity in the output position bounds P by the weighted average of the profiles with output at k and at j, with respective weights a_j/d_i and a_k/d_i. At either endpoint, the input at the output site can be multiplied in locally. Equation (9.2) and $\beta_{ik}, \beta_{ij} \le 1$ give

$$P(x, t_j, t_k) \le c_i + B_i \log x + a_j \log(t_j + x) + a_k \log(t_k + x).$$

In the energy formula use costs $(a_i, \varepsilon a_j, \varepsilon a_k)$. Changing variables to $x, t_j + x, t_k + x$ makes the effective cost of x equal to $a_i - \varepsilon d_i > 0$ for small ε. Maximizing the three terms separately and using $B_i + d_i = D(a)$ gives

$$Q \le c_i + B_i \log \frac{B_i}{a_i - \varepsilon d_i} + d_i \log(1/\varepsilon).$$

For a matching lower bound take $x = B_i/a_i$, $t_j = 1/\varepsilon$, $t_k = x + t_j$. At O extract a single coefficient of the product. Choosing complementary degrees and reversing the basis at k makes this coefficient the pairing of the third input with the product of the first two. An endpoint rounding by one leaves the continuum profile unchanged. Hence $P(x, t_j, t_k) \ge P_k(x, t_j)$, and the asymptotic slope from Section 9 gives

$$P_k(x, t_j) = c_i + B_i \log x + d_i \log(1/\varepsilon) + o(1).$$

The total width cost of this choice tends to $B_i + d_i = D(a)$, which cancels the additive constant in the energy formula. Subtracting $d_i \log(1/\varepsilon)$ through the logarithmic terms of G_v gives Equation (11.4).

Comparison by integration. Start two rays near e_i and e_j, at points of the form used in Equation (11.4). Integrate Equation (11.3) in direction v on both rays and subtract. Their terminal G_v values have difference tending to zero at infinity, since $G_v(rs) = G_v(s) - \Delta(v) \log r$ and G_v is continuous at v. Now send the starting points to e_i and e_j. The two integrands are uniformly bounded near zero by Equation (11.1). At infinity their difference is $O(t^{-2})$ by homogeneity and the Lipschitz bound near v. The difference of the two axis limits in Equation (11.4) therefore equals

$$\int_0^\infty \left[\Delta(e_i + tv) - \Delta(e_j + tv)\right] dt = J_i(v) - J_j(v).$$

For the last identity use $\Delta(e_i + tv) = t^{-1}\Delta(v + t^{-1}e_i)$ and substitute $u = 1/t$.

By Equation (9.3), the quantities $c_i + b_i \log(b_i/a_i)$ agree up to $O(\Delta(v))$. Replacing b_i by B_i changes each quantity by $O(\Delta(v))$ as well. Indeed, b_i/a_i is bounded away from zero when the other two reciprocal lengths stay in compact subsets of $(0, \infty)$ and v_i approaches zero; more generally, all three ratios stay bounded away from zero in a neighborhood of any open face point. Also $B_i/a_i \le 1$ by Equation (11.1). The derivative of $b \mapsto b \log(b/a_i)$ is uniformly bounded throughout the interval in question. This proves Equation (11.2) with the asserted uniformity. For the two tangential directions at a face, dominated convergence uses the tangential Lipschitz bound at zero and Equation (11.1) at infinity.

11.2 The strong minimum principle

Completion of the proof of Proposition 8.1. Suppose $\Delta(v) = 0$. Work either in the interior or on an open face, using just its positive coordinates, indexed $1, \dots, k$ after relabeling. All $J_i(v)$ equal a common $J \geq 0$ by Equation (11.2). A simultaneous dilation gives the exact change $J_i((1+\varepsilon)v) = J/(1+\varepsilon)$. To compare this decrease with the variation on the individual rays, perform the dilation one coordinate at a time. Its intermediate points are

$$v^i = v + \varepsilon \sum_{j \leq i} v_j e_j \qquad (0 \leq i \leq k).$$

At each intermediate point the cost of switching from J_i to J_{i+1} is bounded by $C\Delta(v^i)$. Thus

$$S_\varepsilon = \sum_{i=1}^{k} \left(J_i(v^i) - J_i(v^{i-1}) \right) = \frac{J}{1+\varepsilon} - J + O\Big(\max_{0<i<k} \Delta(v^i) \Big).$$

Write $K = \limsup_{\varepsilon \downarrow 0} \max_{0<i<k} \Delta(v^i)/\varepsilon$, finite by local Lipschitz continuity, and set $g_i(t) = \Delta(v + te_i)$. We claim that

$$-J + CK \;\geq\; \liminf_{\varepsilon \downarrow 0} \frac{S_\varepsilon}{\varepsilon} \;\geq\; \sum_{i=1}^{k} v_i \int_0^\infty g_i'(t)\, \frac{dt}{t}. \tag{11.5}$$

To justify the second inequality, write all k integrals with the same variable t. For almost every $t > 0$, the increment of Δ on $[v^{i-1} + te_i, v^i + te_i]$, divided by ε, tends to $v_i g_i'(t)$. Indeed, after adding a local quadratic, the slope on this segment lies between the slopes of adjoining segments in direction e_i of any fixed small length. Send ε to zero, then shrink the adjoining segments at each point where g_i is differentiable. The unshifted increments that are subtracted for $t \leq 1$ sum to $\Delta((1+\varepsilon)v) - \Delta(v) = 0$.

These subtractions also control the singularity at zero. Pair each shifted increment with its own unshifted increment. Adding a local quadratic makes the shifted increment at least the unshifted one, because both segments have the same length and direction. Removing the quadratic gives a lower bound $-C\varepsilon t$ for the difference. For large t, moving the starting point by $\varepsilon v_i e_i$ changes the corresponding arm length by $O(\varepsilon t^{-2})$. Transport bounds the change in D, and the same bound follows from the formula for D_*. These bounds control the negative part uniformly after division by εt. On a face the same estimates apply in the tangential variables. Fatou's lemma now proves Equation (11.5). Each limiting integral has finite negative part since g_i is semiconvex near zero and takes its minimum there.

The right derivatives $g_i'(0+)$ exist and are nonnegative. A positive derivative would, by semiconvexity, give a positive lower bound for g_i' near zero and hence an infinite integral in Equation (11.5). Thus every $g_i'(0+)$ vanishes. The directional derivative of Δ at v is sublinear by semiconvexity and nonnegative by minimality. Since it vanishes on the positive coordinate vectors, it vanishes on their positive span. In particular $K = 0$. Integrating by parts gives

$$-J \;\geq\; \sum_{i=1}^{k} v_i \int_0^\infty \frac{g_i(t)}{t^2}\, dt \;\geq\; 0.$$

The boundary terms vanish because $g_i(t) = o(t)$ at zero and Equation (11.1) holds at infinity. As $J \geq 0$, continuity gives $g_i(t) = 0$ for every i and $t \geq 0$. Applying the same

argument successively along the coordinate rays reaches every point above v coordinatewise. Any positive triple, or any point of the same open face, can be dilated into this region. Homogeneity gives zero defect throughout the interior or the face in which we started.

It remains to move a face zero into the interior. Suppose $v_1 = 0$ and $\Delta(v) = 0$. At $v^{(r)} = v + re_1$, tangential dominated convergence gives $J_2(v^{(r)}) \to J_2(v) = 0$. By the uniform Equation (11.2) and continuity of Δ, $J_1(v^{(r)}) \to 0$ as well. Although we have only tangential Lipschitz bounds at the face, transport shows that $\Delta - h = D - \sum_i a_i$ is nondecreasing in v_1. Since the normal derivative of h stays bounded near this point,

$$\Delta(v^{(r)} + te_1) - \Delta(v^{(r)}) \geq -Ct \qquad (0 < t \leq 1).$$

Fatou's lemma in the normal integral now yields $\int_0^\infty \Delta(v + te_1)\, dt/t \leq 0$. The continuous nonnegative integrand vanishes at every $t > 0$, giving the interior zero required above. In particular a homogeneous tripod with contact in either regime has $D(1,1,1) = 3 - 1/3 = 8/3$, the equal-arm value used in Section 8. □

11.3 The three wire lengths at the maximal price

The unconditional inequality of Proposition 8.1 gives a geometric proof of the known square upper bound and identifies the wire prices when equality holds. Work at scale one on three sites, so that a label shared by all three has rate one. Write the three wire lengths as $a_i + a_j$ for $i \neq j$. The triangle inequalities give $a_i \geq 0$. Over an algebraically closed field the Proposition and the arithmetic–harmonic mean inequality give

$$1 \;\geq\; \sum_i a_i - \Big(\sum_i a_i^{-1}\Big)^{-1} \;\geq\; \frac{8}{9}\sum_i a_i. \tag{11.6}$$

The reciprocal term is zero if an arm vanishes. The joint values of the three wires are preserved on passing to an algebraic closure (Section 3), so the same bound holds over every field. Since T_n consists of one n-dimensional wire on each side, every spectral point assigns it value $n^{2\sum_i a_i} \leq n^{9/4}$. Strassen's maximum formula for asymptotic rank [52] gives the upper bound. Fixed-size recursion as in Lemma 2.5 then gives $\omega_F \leq 9/4$.

Corollary 11.1. *Fix $n \geq 2$ and a field F. At every scale-one spectral point maximizing the value of T_n, all three wire lengths are $3/4$. There exists such a point, and it assigns $T_{a,b,c}$ the value $(abc)^{3/4}$ for every choice of positive integers a, b, c.*

Proof. The spectral maximum is attained. Corollary 1.2 identifies its value as $n^{9/4}$, so $\sum_i a_i = 9/8$. In (11.6) both inequalities must be equalities. All three arms are positive, since a zero arm would give $1 \geq 9/8$. Equality of the arithmetic and harmonic means gives $a_i = 3/8$ for each i. Each wire therefore has rate $3/4$. The rectangular tensor has one wire of each dimension a, b, c, and their values multiply. □

12 Synchronization of tensor parameters

Theorem 5.1 tests whether one price extension works for an entire family. Its tests allow one to choose any tensors in the family, multiply them and close their extra legs jointly. For a family depending on a marked point s, two factors in such a test can therefore have different

marked points. The sphere calculation in Section 14 will give estimates when all factors use the same marked point. We show that under a strict saving for square multiplication this diagonal test already controls every choice of the points.

Fix an algebraically closed field F, an ordinary price H of scale $D > 0$ on base sites S, and a finite set L of additional sites, called openings. Recall from Equation (5.3) that β_H measures the largest price per copy obtained by taking a tensor power and applying one joint functional at each opening. For a polynomial tensor family $T(\mathbf{s})$, $\mathbf{s} = (s_1, \dots, s_d)$, write

$$\beta_H^{\mathrm{gen}}(T(\mathbf{s})) = \sup_{\mathbf{c} \in F^d} \beta_H(T(\mathbf{c})).$$

This is the assisted cost at independent indeterminates for the generic extension of H in Section 6. Indeed, fix a power and a closure with rational function coefficients. One can specialize its entries together with the tensor on a common open set where the resulting price is arbitrarily close to its generic value. Conversely, a fixed closure over F witnessing a strict lower bound at one point witnesses the same bound on a nonempty open set, by closedness of price sublevels. In particular the supremum is unchanged on restricting to any dense open parameter domain.

Theorem 12.1 (Synchronization)**.** *Suppose $\omega_F < 9/4$. For any polynomial tensor families $U_1, \dots, U_a$ on F^d with common base and openings,*

$$\beta_H^{\mathrm{gen}}\left(\bigotimes_{i=1}^{a} U_i(\mathbf{s}_i)\right) = \beta_H^{\mathrm{gen}}\left(\bigotimes_{i=1}^{a} U_i(\mathbf{s})\right).$$

On the left the a parameter vectors vary independently. On the right they are equal. Both products group their legs at each opening, so a closure on either side may couple all the factors.

Proof. Choosing equal parameters gives the inequality from right to left. For the reverse inequality we first take copies of one univariate family $U(s)$. An identically zero factor makes the assertion immediate, so we may assume that the families under consideration are nonzero. Write $B = \beta_H^{\mathrm{gen}}(U(s))$. The diagonal cost of a copies is aB by homogeneity under tensor powers. We first control different parameters lying in a disk above the same residue s, then recover independent parameters by letting the valuation depth of that disk tend to zero.

Pinning above one residue. Take an uncountable set $\mathcal{U}$ of independent indeterminates, and let s be a further indeterminate. Put

$$E = F(\mathcal{U}, s), \qquad K_0 = \mathrm{Frac}(F(\mathcal{U})[t^{\mathbb{Q}}]), \qquad J = \{z \in K_0 : \nu(z) > 0\}.$$

The Gauss lift of the generic extension of H to E is a valued price p on $K_0(s)$ (Theorem 6.1). For $z \in J$, translation by z preserves the unit disk. Apply Lemma 6.2 to this translation and its inverse, taking s as the fresh residue variable. It gives

$$\beta_p(U(s+z)) = \beta_p(U(s)) = B. \tag{12.1}$$

For the second equality the Gauss lift agrees with the ordinary generic price on coefficients independent of t. As in the proof of Lemma 6.2, it suffices to use closing functionals with constant entries in F.

Fix $M \geq 1$ and an anchor in S. For each $z \in J$, let V_z be the product of $U(s+z)^{\otimes M}$ with the pin penalties of Lemma 5.3, one for each opening. By translation invariance its minimum $b = \beta_p(V_z)$ is the same for every z. Give each objective its own copy of L. Lemma 5.3 realizes its minimizing sites $x_{z,\ell}$, $\ell \in L$, in one complete CAT(0) spectral space over F. For any finite list of objectives, a valued extension simultaneously realizes their minima.

The minimizing sites cannot move. For each ℓ we check that a nonconstant map $z \mapsto x_{z,\ell}$ would satisfy the hypotheses of Theorem 7.1. The bridge from these ordinary sites to the valued disk is the pinned evaluation formula. For every nonzero test tensor T over F placed at sites from a finite list $z_1, \ldots, z_m$, its price is

$$\lim_{n\to\infty} \left[\beta_p\big((V_{z_1} \otimes \cdots \otimes V_{z_m})^{\otimes n} \otimes T\big) - nmb\right]. \tag{12.2}$$

The m copies of L here are distinct. If two parameters coincide, the corresponding minimizing sites are at distance zero, by uniqueness of pinning.

Permutation of the indeterminates in $\mathcal{U}$ preserves the prices in Equation (12.2) after relabeling the z_j. Translating every z_j by $w \in J$ replaces s by $s + w$ in a polynomial family over K_0, so the disk comparison preserves each finite-n expression as well. For an affine test $z_j = a_j + b_j c$ in Theorem 7.1, the tensor inside β_p is polynomial in c. Lemma 6.2 therefore gives specialization and the weighted submean inequality at the disk and its inner and outer ends. Subtracting the same constant nmb at every end preserves these inequalities, and taking the limit transfers them to the pinned prices.

It remains to check continuity at zero. At any one placement of the openings, $U(s+z)$ and $U(s)$ both have an entry of valuation zero, because their common reduction $U(s)$ is nonzero. Each tensor has nonnegative price there. The entries of their difference have valuation at least $\nu(z)$. Applying the entry bound (5.2) to the difference and the sum inequality in both directions shows that the prices of the two tensors differ by at most

$$\eta(z) = D\log(1 + \exp((C - \nu(z))/D))$$

for a constant C depending only on the format and scale. Evaluate V_0 at the sites minimizing V_z. Its price is at most $b + M\eta(z)$ since the pin penalties are identical at that placement. In a common minimizing extension for the two lists, the midpoint estimate (5.6) applied to V_0 gives

$$2c_D \sum_{\ell \in L} d(x_{z,\ell}, x_{0,\ell})^2 \leq M\eta(z).$$

The right side tends to zero as $\nu(z) \to \infty$ at fixed M. Distances are bounded by the scale D. A nonconstant family of pinned sites would now force exact matrix multiplication rank at least $n^{9/4}$ for every n, by Theorem 7.1. Lemma 2.4 would then give $\omega_F \geq 9/4$. Thus for every ℓ, all $x_{z,\ell}$ coincide.

Removing the penalty and recovering independent parameters. The sum of the penalties is between zero and a constant C_0 depending only on D and $|L|$. Using a placement minimizing $U(s+z)$ as a competitor gives $b \leq MB + C_0$ by Equation (12.1). Hence at its pinned sites $U(s+z)$ has price at most $B + C_0/M$. For any $z_1, \ldots, z_a \in J$, take a common valued extension minimizing their pin objectives. The corresponding openings have distance zero. Transport therefore identifies their prices also on tensors with coefficients in $K_0(s)$. Product additivity at these common openings gives

$$\beta_p\left(\bigotimes_{i=1}^{a} U(s+z_i)\right) \leq a(B + C_0/M).$$

Let $M \to \infty$ to obtain aB as an upper bound throughout the disk.

Take independent $u_1, \dots, u_a \in \mathcal{U}$. Fix a power h and a closure over F, and apply this closure to $(\bigotimes_i U(s + vu_i))^{\otimes h}$, obtaining a polynomial tensor $P(v)$ over E on the base sites. For rational $r > 0$ the parameters $s + t^r u_i$ lie above s, so $p(P(t^r)) \le haB$. Write H_E for the ordinary generic extension of H to E. If $P(1) \neq 0$, the Gauss formula gives

$$H_E(P(1)) \le p(P(t^r)) + r \deg P.$$

Indeed, the coefficients of $P(v)^{\otimes n}$ sum to $P(1)^{\otimes n}$. There are at most $n \deg P + 1$ coefficients, and the Gauss penalty for any of them is at most $rn \deg P$. The loss from adding them is at most $D \log(n \deg P + 1)$. Divide by n and pass to the limit using Equation (6.1). Letting $r \downarrow 0$ gives $H_E(P(1)) \le haB$. The substitution $u_i \mapsto s + u_i$ is an invertible change of variables over $F(s)$, so $P(1)$ tests the independent generic parameters. Taking all powers and constant closures proves the identical-family case.

Different families and several coordinates. For arbitrary univariate families U_i, apply the result just proved to $U(s) = \bigotimes_i U_i(s)$. With independent $s_1, \dots, s_a$, regroup $\bigotimes_j U(s_j)$ as the product of a tensors $\bigotimes_i U_i(s_{i+j})$, reading indices cyclically. Each of these has the same assisted cost at the generic price on $F(s_1, \dots, s_a)$ because that price is invariant under permutations of the indeterminates. Superadditivity of assisted cost on products follows by taking separate closures of the factors on a common multiple of their witnessing powers. Thus a times the independent cost for the U_i is at most the independent cost for a copies of U, which we have bounded by $a\beta_H^{\mathrm{gen}}(U(s))$.

For several coordinates choose any $\mathbf{c}_1, \dots, \mathbf{c}_a \in F^d$. Coordinatewise interpolation gives a single polynomial curve $\gamma : \mathbb{A}^1 \to \mathbb{A}^d$ through these points at distinct scalar arguments. The univariate result applied to $U_i \circ \gamma$ bounds the assisted cost of this tuple by the common-parameter cost along γ, hence by the common-parameter cost on F^d. Taking the supremum over tuples completes the proof. □

Corollary 12.2 (Simultaneous pricing)**.** *Under the hypotheses of Theorem 12.1, let $\{U_i(\mathbf{s})\}_{i \in I}$ be a collection of polynomial families with common openings. For every finite list $i_1, \dots, i_m$, allowing repetitions, suppose every closure of $\bigotimes_j U_{i_j}(\mathbf{s})$ at a shared generic parameter has price at most $\sum_j h_{i_j}$. Then one ordinary extension $\widehat{H}$ of H, of the same scale, satisfies $\widehat{H}(U_i(\mathbf{c})) \le h_i$ for every i and every $\mathbf{c} \in F^d$. The conclusion also holds for rational families on a common nonempty open domain of regularity. In particular, for one such family U on its domain Ω,*

$$\min_{\widehat{H} \supset H} \sup_{\mathbf{c} \in \Omega} \widehat{H}(U(\mathbf{c})) = \sup_{\mathbf{c} \in \Omega} \beta_H(U(\mathbf{c})).$$

Proof. Repeating each finite list gives the assumed bound on closed powers and hence on its synchronized assisted cost. By Theorem 12.1 this bound also holds at independently chosen parameters. Theorem 5.1 now yields the single price extension.

For a finite list of rational families, clear scalar denominators on a dense open set where they are nonzero. An ordinary price is unchanged by nonzero scalar multiples, so the polynomial result gives the same product tests there. If such a test failed at a regular tuple, a fixed power and closure would witness a strict violation. Closedness of price sublevels makes this violation persist on an open neighborhood of the tuple, which meets the dense open set. The tests now hold throughout the domain, as required by Theorem 5.1.

For the minimax identity use the supremum of the pointwise assisted costs as the bound for the single family. If the family is nonzero this is finite by the fixed-format bound. Products at a shared parameter satisfy the tests by homogeneity of β_H on powers, giving an extension with the stated upper bound. Every extension majorizes β_H pointwise, giving the reverse inequality. The zero family has both sides equal to $-\infty$. □

For the sphere tensor, s specifies the additional marked point and the family index specifies its energies. Corollary 12.2 places its leg once for all regular positions and all energies. Each collision with an original marked point can then measure a distance from this same site.

13 Fermionic sphere tensors

We need a tensor whose legs can be combined when their positions coincide. Start with marked points $z_1, \dots, z_k$ in an affine coordinate on the projective line. At each point the input space has a half-integral *energy* h_i. The energies fix the dimensions of the legs, while the marked points determine the entries of the tensor. The metric sites at which we will price these legs can be chosen independently of the z_i.

Two features of this construction enter the proof. When a cluster of marked points collides, the Laurent coefficient of a given order passes through a single leg, whose energy the order determines when the other sites stay fixed (Proposition 13.1). Conversely, at a two-point collision, varying the first input across a bounded range of energies realizes every linear map from the second input to the fused leg. Lemma 13.3 gives that energy bound. Together these facts let Section 14 compare the price of a sphere tensor with the price of moving an independent wire.

The half-integral grading comes from the free fermion [31, Section 4]. Khoruzhii used its modes to decompose a crossing of two wires [32]. The crossing argument of [42, Theorem 4.1] uses the same calculus in arbitrary characteristic.

13.1 The kernel and its coefficients

Let F be a field and take the exterior algebra on generators ψ_m, $m = 0, 1, \dots$. Give ψ_m energy $m + \frac{1}{2}$, and denote by V_h the subspace of energy $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$. Its basis consists of the products $\psi_A = \bigwedge_{m \in A} \psi_m$, in increasing order, with $\sum_{m \in A}(m + \frac{1}{2}) = h$. In particular $V_0 = F$. Every such product has exterior degree congruent to $2h$ modulo two.

For distinct $z_1, \dots, z_k$ in F we define a multilinear form $C(\mathbf{h}; \mathbf{z})$ on $\prod_i V_{h_i}$. For basis inputs, list all generators in site order. The value is a signed sum over perfect matchings of this list (Wick's rule [58]). Its factors are as follows. Two generators at one site have contraction zero. At different sites, the contraction of ψ_m at z_i with ψ_n at z_j, in that order, is

$$[x^m y^n] \frac{1}{z_j + y - z_i - x} = \frac{(-1)^n \binom{m+n}{m}}{(z_j - z_i)^{m+n+1}}. \tag{13.1}$$

Pair the entries of each oriented matching consecutively, multiply their contractions, and insert the sign of the permutation from the original list to this paired list. Reversing a pair reverses both the contraction and the permutation sign. Changing the order of pairs preserves the sign. This sum is the Pfaffian of the alternating matrix of contractions. The empty list has value one and an odd list has value zero.

For example, put one generator ψ_0 at each of four sites and write $K_{ij} = (z_j - z_i)^{-1}$. The value on these four inputs is

$$K_{12}K_{34} - K_{13}K_{24} + K_{14}K_{23}.$$

At a collision of z_1 with z_2 the first matching has an internal pair and a pole, while the other two matchings connect the cluster to the outside. This distinction governs the general fusion expansion.

These formulas use integer coefficients and commute with change of field. Two repeated generators at a site give canceling pairs of matchings, by exchanging their partners. Thus the forms descend to exterior powers in characteristic two as well. Interchanging sites i, j in adjacent positions contributes the sign $(-1)^{(2h_i)(2h_j)}$. Each coefficient, viewed as a rational function, is homogeneous of degree $-\sum_i h_i$ in the positions, since a contraction has degree minus the sum of its two energies. Its only possible poles are site collisions. As a function of z_i it is bounded at infinity, and vanishes there when $h_i > 0$.

13.2 Fusing a cluster at a marked point

Partition the sites into consecutive clusters $J_1, \dots, J_s$, reordering the sites first if necessary. In cluster J_j write $z_i = p_j + ua_i$, where the centers p_j are distinct and the offsets within each cluster are distinct. A singleton with offset zero stays fixed. Its fusion map is the identity at its original energy and zero at every other energy.

Proposition 13.1 (Fusion)**.** *For each cluster J there are maps*

$$I_e^J : \bigotimes_{i \in J} V_{h_i} \longrightarrow V_e \qquad (e \in \tfrac{1}{2}\mathbb{Z}_{\geq 0})$$

depending only on its offsets and input energies. Put $H_J = \sum_{i \in J} h_i$. The maps vanish unless $e - H_J \in \mathbb{Z}$. The Laurent expansion at $u = 0$ is

$$C(\mathbf{h}; \mathbf{z}) = \sum_{\mathbf{e}} u^{\sum_j e_j - \sum_i h_i} C(\mathbf{e}; \mathbf{p}) \circ (I_{e_1}^{J_1}, \dots, I_{e_s}^{J_s}). \tag{13.2}$$

Only finitely many terms contribute to each coefficient.

Proof. Group the perfect matchings by the pairs they use inside each cluster. Fix one cluster J and one such partial matching. Contract its pairs using (13.1) at positions a_i. Move the matched pairs to the front, with their permutation sign, keeping all unmatched generators in order. Replace each unmatched generator at offset a_i by

$$\psi_m \longmapsto \sum_{\ell \geq m} \binom{\ell}{m} a_i^{\ell - m} \psi_\ell. \tag{13.3}$$

Wedge the resulting vectors in their original order and keep only energy e. Each translation may be truncated to the modes of energy at most e, so this is an operation on finite-dimensional spaces. Sum over all internal matchings with their signed factors to obtain I_e^J. If a generator is repeated in one input and either copy is contracted, interchanging the copies pairs terms of opposite signs. If both are unmatched, their translated wedge vanishes. This

cancellation works over every field. Removing pairs and raising mode indices preserve energy modulo $\mathbb{Z}$.

Decompose each full matching into its internal edges and its edges between clusters. An internal contraction at the original positions contributes the stated contraction at the offsets times u raised to minus the energy of the pair. On an edge between clusters the kernel is regular at $u = 0$. Taylor expansion in the two offsets uses $[x^m](ua_i + x)^\ell = \binom{\ell}{m}(ua_i)^{\ell-m}$, which gives (13.3) and one factor of u per index increase. For a given output energy e, the total power from the cluster is therefore $e - H_J$. Moving the internal pairs of all clusters to the front contributes only the signs already included in I_e^J, since an ordered pair can cross any earlier cluster without changing sign. What remains is the sign of the matching between the output wedges. This proves (13.2). At fixed total power the sum of the nonnegative e_j is fixed, so the coefficient is finite. □

For example, fuse two generators ψ_0, ψ_0 at offsets $0, 1$. Their total energy is 1. The internal contraction produces the vacuum at energy 0 with coefficient u^{-1}. If the two generators remain unmatched, their first nonzero translated wedge is $\psi_0 \wedge \psi_1$, of energy 2 and order u^1. This second term can pair with generators at other centers.

13.3 Energy and dimension

A leg of dimension $d_h = \dim V_h$ carries width $\log d_h$ when $d_h > 0$. A basis wedge chooses a subset of distinct modes, and its energy is the sum of their half-integer weights. Counting these subsets gives width asymptotic to $c\sqrt{h}$, where $c = \pi/\sqrt{3}$ and the logarithm is natural.

Lemma 13.2 (Dimension of a level)**.** *For every $h \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ one has $d_h \leq e^{c\sqrt{h}}$. As $h \to \infty$ through all half-integral levels,*

$$d_h \sim \frac{\exp(c\sqrt{h})}{2^{5/2}3^{1/4}h^{3/4}}.$$

Proof. For $t > 0$ the generating function is

$$Z(t) = \sum_h d_h e^{-th} = \prod_{m\geq 0}(1 + e^{-t(m+1/2)}).$$

The function $f(x) = \log(1 + e^{-x})$ is convex on $[0, \infty)$. The midpoint estimate on each interval $[mt, (m + 1)t]$ gives $\log Z(t) \leq b/t$, with $b = \int_0^\infty f(x)\,dx = \sum_{j\geq 1}(-1)^{j+1}/j^2 = \pi^2/12$. Thus $d_h \leq \exp(th + b/t)$. Minimization in t gives the bound for $h > 0$, and $d_0 = 1$.

Doubling the generator energies identifies basis wedges at energy h with partitions of $2h$ into distinct odd parts. These are in bijection with the self-conjugate Young diagrams of size $2h$. Each diagonal hook has length $2m + 1$, where m is its arm length, and the strictly decreasing sequence of arm lengths reconstructs the diagram. The asymptotic formula for the number of self-conjugate partitions of n is $\exp(\pi\sqrt{n/6})/(2^{7/4}3^{1/4}n^{3/4})$ to leading order [60, Section 5.1]. Set $n = 2h$. □

13.4 The width needed to transmit a leg

Consider a two-site cluster at offsets $0, 1$ with incoming energies h, g. Write I_e for its fusion map to energy e. Fixing the first input turns this map into an operator on the second input. To recover a wire in Section 14, we need every matrix unit from V_g to V_e. The next lemma constructs them with first-input width at most $c(\sqrt{e} + \sqrt{g}) + O(1)$ per level.

Lemma 13.3 (Fusion width)**.** *For $e, g \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ put*

$$H(e,g) = (\sqrt{e} + \sqrt{g})^2 + (\sqrt{e} + \sqrt{g})/\sqrt{2}.$$

The map

$$\bigoplus_{h \leq H(e,g)} V_h \longrightarrow \mathrm{Hom}(V_g, V_e), \qquad (v_h)_h \longmapsto \sum_h I_e(v_h, -)$$

is surjective. In the wedge bases, each matrix unit has an integral preimage.

Proof. Fix mode sets A, B of energies e, g, respectively. We construct an input at the first point that sends ψ_B to ψ_A and annihilates every other basis vector of V_g. If $e = g = 0$, the first input is the vacuum. Otherwise write $w = \sqrt{e}$ and $v = \sqrt{g}$. An input at offset zero will be a wedge of test vectors. Represent a test $\sum_m t_m \psi_m$ by its Laurent polynomial $f(X) = \sum_m t_m X^{-m-1}$. In a matrix coefficient of I_e, the test has two possibilities. It can contract with input index b at offset one, with coefficient $[y^b]f(1+y)$ by (13.1). Or it can occupy output index ℓ, with coefficient t_ℓ, because translation at offset zero is the identity. This follows by expanding the output wedge. Each test must occupy one slot of one of these two kinds, and different tests use different slots.

We choose tests that can use their designated slots or slots of strictly larger weight. The target slots will account for the entire available weight, forcing every test into place. Give output index ℓ the weight $\tau = (\ell + \frac{1}{2})/w$ and input index b the weight $\tau = (b + \frac{1}{2})/v$. A side of energy zero contains no indices and requires no division. The test for each target slot is given in the table.

target	$f(X)$	r
$\ell \in A$	$X^{-\ell-1}(1 - X^{-1})^r$	$\lfloor v\tau - \frac{1}{2} \rfloor + 1$
$b \in B$	$X^{-r-1}(1 - X^{-1})^b$	$\lfloor w\tau - \frac{1}{2} \rfloor + 1.$

For an output target the coefficient t_ℓ is one, and the zero of order r at $X = 1$ forces every input slot it can occupy to have larger weight. Its other possible output indices exceed ℓ. For an input target, $f(1+y) = y^b(1+y)^{-r-b-1}$, so its leading input coefficient is one. All other possible input slots have larger indices, and the choice of r forces all possible output slots to have larger weight. These conclusions hold over the integers.

Form the wedge of all the tests in a fixed order. In every matrix coefficient of the indicated energies the sum of the weights of all available slots is $w + v$, which is also the sum of the target weights. The construction gives each test a slot of at least its target weight. Since the slots have positive weights, every nonzero term must use every slot and attain equality for each test. The table admits only the designated slot at that weight, even if an opposite-side slot has equal weight. In particular every generator of the second input has contracted with a test. The input and output sets are therefore B and A, and the pairing and wedge assignment are unique. Thus the desired coefficient is 1 up to sign, which can be absorbed into the chosen input wedge.

The largest generator energy in each test of weight τ is at most $(w + v)\tau + \frac{1}{2}$, by the displayed choice of r. A wedge of s generators has energy at least $\sum_{m=0}^{s-1}(m + \frac{1}{2}) = s^2/2$, so there are at most $\sqrt{2}(w + v)$ tests. Adding their individual bounds shows that every homogeneous component of the input wedge has energy at most $H(e, g)$. Taking these components gives the claimed preimage. □

The sum of square roots in this cutoff also follows from a dimension count. If the map in Lemma 13.3 is surjective with cutoff H, then $d_e d_g \le \sum_{h \le H} d_h \le (2H+1)e^{c\sqrt{H}}$. If $H \le (\sqrt{e}+\sqrt{g})^2$, taking logarithms and using Lemma 13.2 gives $\sqrt{H} \ge \sqrt{e}+\sqrt{g}-O(\log(e+g+2))$ along nonzero levels with $e+g \to \infty$. The same lower bound holds immediately for larger H. Lemma 13.3 attains this leading width in every characteristic.

14 The price of a sphere tensor

How expensive can a tensor be when the dimensions of its legs are prescribed? Prepare the tensor at a site o, then send each leg to its destination x_i along a wire. In the logarithmic price H, a wire costs its length times the logarithm of its dimension. In particular, for every nonzero sphere tensor of Section 13,

$$H(C(\mathbf{h};\mathbf{z});x_1,\dots,x_k) \le \sum_{i=1}^{k} d(o,x_i)\log\dim V_{h_i}. \tag{14.1}$$

The marked points z_i determine the tensor's entries. The sites x_i determine its price. For legs of log dimension at most Na_i, the cheapest such center gives an upper bound of $N\inf_o \sum_i a_i d(o,x_i)$. Under the standing hypothesis $\omega_F < 9/4$, sphere tensors attain this bound to leading order (Corollary 14.3).

We also need to know which energies make a sphere tensor expensive at a given placement. We put a quadratic penalty on each leg width. Squared width is proportional to the energy recorded by the order of a collision. Theorem 14.1 identifies the optimizing widths with distances from a weighted barycenter, whose weights are set by the penalties.

Fix an algebraically closed field F, distinct $z_1,\dots,z_k \in F$, and sites $\mathbf{x}=(x_1,\dots,x_k)$ in a complete CAT(0) spectral space X over F, with $k \ge 2$. Write $H_{\mathbf{x}}$ for the price at this placement and D for its scale. Let $c=\pi/\sqrt{3}$ be the constant from Lemma 13.2. For $N \ge 1$ an integer, define the normalized width of energy h by $w_N(h)=c\sqrt{h}/N$. For positive penalties $\lambda_1,\dots,\lambda_k$, set

$$B_N = \max_{\mathbf{h}} \left\{ N^{-1}H_{\mathbf{x}}(C(\mathbf{h};\mathbf{z})) - \sum_i \lambda_i w_N(h_i)^2 \right\}, \tag{14.2}$$

$$E = \min_{o\in X} \sum_i \frac{d(o,x_i)^2}{4\lambda_i}. \tag{14.3}$$

Zero tensors may be discarded in the maximum. The unique minimizer o in (14.3) is the barycenter with weights proportional to $1/\lambda_i$. It lies in the closed convex hull of the x_i. Write $r_i=d(o,x_i)$ and $\rho_N=\sqrt{\log(2N)/N}$.

Theorem 14.1 (Sphere prices)**.** *Suppose $\omega_F < 9/4$. Then*

$$0 \le E - B_N \le O(\rho_N). \tag{14.4}$$

For every choice of maximizing energies $\mathbf{e}$ at each N,

$$w_N(e_i) = \frac{r_i}{2\lambda_i} + O(\sqrt{\rho_N}), \qquad \frac{H_{\mathbf{x}}(C(\mathbf{e};\mathbf{z}))}{N} = 2E + O(\sqrt{\rho_N}). \tag{14.5}$$

The constants are uniform for fixed k, bounded scales D, and all λ_i in a fixed compact subinterval of $(0,\infty)$.

For two sites at distance L, the barycenter divides the joining segment in the ratio $\lambda_1 : \lambda_2$. Both limiting widths are therefore $L/[2(\lambda_1 + \lambda_2)]$. More generally, to weight site i by a mass $a_i > 0$, choose $\lambda_i = 1/(2a_i)$. Its width then approaches $a_i r_i$, and the normalized price approaches $\sum_i a_i r_i^2$. Section 15 uses these individual widths to measure the effect of several simultaneous collisions.

By Lemma 13.2, $\log \dim V_h \le c\sqrt{h}$. Completing squares in (14.1) gives, for any nonzero $C = C(\mathbf{h}; \mathbf{z})$,

$$\frac{H_{\mathbf{x}}(C)}{N} - \sum_i \lambda_i w_N(h_i)^2 \le E - \sum_i \lambda_i \left(w_N(h_i) - \frac{r_i}{2\lambda_i} \right)^2 . \tag{14.6}$$

The vacuum tensor, with all energies zero, has price zero. Thus the maximum exists, $0 \le B_N \le E$, and the optimizing widths are bounded uniformly in the range of Theorem 14.1, since the original sites have diameter at most D and $r_i \le D$. If all sites coincide, zero energies attain the maximum and (14.6) proves the theorem. For the remaining case, the proof finds one extra site y_N satisfying

$$d(x_i, y_N) \le 2\lambda_i w_N(e_i) + O(\rho_N) \qquad (1 \le i \le k)$$

for every fixed maximizing tuple $\mathbf{e}$. Squaring and summing these distances gives a lower bound on the optimizer's penalty. The change in its penalty from scale N to $N + 1$ then gives the lower bound on B_N.

14.1 Pricing one additional leg

The extension must work for all collisions at once. Synchronization reduces that requirement to a price bound for products of sphere tensors in which all additional legs have the same marked position s. A principal part at $s = z_i$ then allows all the additional legs to be closed at the single site x_i.

Lemma 14.2. *Suppose $\omega_F < 9/4$, fix $R > 0$ and put $\mu = \max_i \lambda_i$. For every $N \ge 2$ the base price has an extension of scale D to one more site y_N such that*

$$H_{\mathbf{x},y_N}(C((\mathbf{h}, g); (\mathbf{z}, s))) \le N \left(B_N + \sum_j \lambda_j w_N(h_j)^2 + \mu w_N(g)^2 \right) + O_{k,R}(D \log(2N)) \tag{14.7}$$

simultaneously for all $h_j, g \le RN^2$ and $s \in F \setminus \{z_j\}$.

Proof. Let the uth factor in a product of m sphere tensors have energies $\mathbf{h}_u, g_u$ and additional position s. A closure inserts one joint state in the tensor product of the m additional argument spaces. By Corollary 12.2 for rational families, it suffices to bound every such closure at the shared generic parameter. A strict violation would persist on an open set of values of s and of the closure coordinates, by closedness of price sublevels. We may therefore fix any joint state over F, keep it constant as s varies, and bound the closed product at every regular value.

If all $g_u = 0$, the closed product is a scalar multiple of the base product and (14.2) applies. Otherwise a factor with $g_u > 0$ must contract at least one generator at s with a generator at a fixed point in every matching. Those contractions vanish at infinity. Thus the closed

product is a rational tensor vanishing at infinity, with poles only at the z_i, and equals the sum of its principal parts.

At the pole z_i, fuse the additional point in each factor with its ith point, using offsets 1 and 0, respectively. A term with intermediate energies e_u has Laurent order $\sum_u (e_u - h_{ui} - g_u)$, by Proposition 13.1. After inserting the joint state, this term is a restriction of a product of base sphere tensors in which the ith energies have been replaced by e_u. Indeed all m fusion maps and the joint state now act at the single site x_i. The price of a term of negative order is therefore at most

$$\sum_u N\left(B_N + \sum_j \lambda_j w_N(h_{uj})^2 + \lambda_i(w_N(e_u)^2 - w_N(h_{ui})^2)\right) \\ \le \sum_u N\left(B_N + \sum_j \lambda_j w_N(h_{uj})^2 + \mu w_N(g_u)^2\right).$$

Indeed a principal-part term has $\sum_u e_u < \sum_u (h_{ui} + g_u)$. The identity $w_N(h)^2 = c^2 h/N^2$ converts this single inequality on total energy to the displayed bound, including factors whose individual Laurent orders are positive. The integers $2e_u$ have sum at most $4RmN^2$, giving at most $\binom{m+\lfloor 4RmN^2 \rfloor}{m}$ possibilities for each i. Thus the ordinary-sum price inequality costs $O_{k,R}(mD\log(2N))$ for all principal parts. This proves the required product test with the error in (14.7) per factor. □

14.2 Recovering a wire by a collision

Proof of Theorem 14.1. Fix maximizing energies $\mathbf{e}$, abbreviate $u_i = w_N(e_i)$, and write $Q_N = \sum_i \lambda_i u_i^2$. Choose a half-integer g nearest to $N\log(2N)$. Lemma 13.2 gives

$$v = w_N(g) \asymp \sqrt{\frac{\log(2N)}{N}}, \qquad \log\dim V_g = Nv - O(\log(2N)) \tag{14.8}$$

for sufficiently large N. We use this small but growing space to probe the distance from y_N to each x_i. The choice $v \asymp \rho_N$ balances the quadratic cost v^2 of the added width with the normalized interpolation cost $O(\log(2N)/N)$.

Hold $h_j = e_j$ for $j \neq i$. Lemma 13.3 says that the fusion maps at z_i give a surjection

$$\bigoplus_h V_h \longrightarrow \mathrm{Hom}(V_g, V_{e_i}), \qquad (a_h)_h \longmapsto \sum_h I_{e_i}(a_h, -), \tag{14.9}$$

using only h with $\sqrt{h} \le \sqrt{e_i} + \sqrt{g} + O(1)$. Choose R in Lemma 14.2 large enough for these energies and all other e_j. The bound on optimizing widths makes one R sufficient throughout the uniform range of the theorem. Choose the site y_N from that lemma. It works simultaneously for every choice of i, h and every regular s. In the single sphere tensor with $h_i = h$, extract the coefficient of $(s - z_i)^{e_i - h - g}$ whenever the parities agree. Fusion identifies this coefficient, up to a scalar sign, with $C(\mathbf{e}; \mathbf{z})$ precomposed at i by $I_{e_i} : V_h \otimes V_g \to V_{e_i}$.

The required order $e_i - h - g$ can be positive. Each coordinate function is bounded at infinity, and fusion bounds its pole orders at the z_j by $2RN^2$. A common denominator of degree $O_{k,R}(N^2)$ consequently turns every entry into a polynomial of at most that degree.

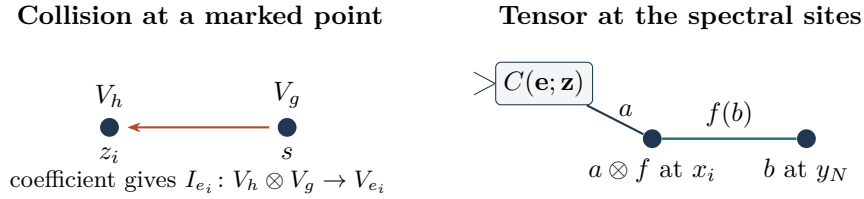


Figure 5. The collision at $s = z_i$ supplies a fusion map for each admissible h. Their span contains $b \mapsto f(b)a$. Inserting this map gives the original tensor with input a and a wire joining the two spectral sites. The other legs keep their placements.

Interpolating at distinct regular points of F recovers any Laurent coefficient as the same linear combination of the corresponding tensor evaluations.

Identify $a \otimes f \in V_{e_i} \otimes V_g^*$ with the map $b \mapsto f(b)a$. A right inverse in (14.9) writes this map as a sum of fusion maps, linearly in $a \otimes f$. Precompose the recovered coefficients with the components of this inverse at x_i and add. With $b \in V_g$ at y_N, the resulting form is

$$(a \otimes f, b) \longmapsto f(b)\, C(\mathbf{e};\mathbf{z})(\dots, a, \dots).$$

Figure 5 illustrates the resulting factorization. The tensor is the optimizer times a wire of dimension $\dim V_g$ between x_i and y_N. Its price is the sum of their logarithmic prices. All h and interpolation values together contribute only polynomially many summands, so the ordinary-sum inequality costs $O(\log(2N))$. By (14.7), producing the wire can increase the old ith width from u_i to at most $u_i + v + O(N^{-1})$ and adds the new width v. Subtract the optimizer's price $N(B_N + Q_N)$ to get

$$\begin{aligned} d(x_i, y_N)\frac{\log \dim V_g}{N} &\le \lambda_i\big((u_i + v + O(N^{-1}))^2 - u_i^2\big) + \mu v^2 + O\!\left(\frac{\log(2N)}{N}\right) \\ &\le 2\lambda_i u_i v + O\!\left(v^2 + \frac{\log(2N)}{N}\right). \end{aligned}$$

The extended scale bounds $d(x_i, y_N) \le D$. Hence (14.8) yields $d(x_i, y_N) \le 2\lambda_i u_i + O(v)$. All constants here are uniform as stated.

The extension can be realized in a complete CAT(0) spectral space. The closed convex hull of the old sites is canonically isometric to their original hull by Lemma 5.2. Projection onto it weakly decreases distances to the old sites. Consequently

$$E \le \sum_i \frac{d(x_i, y_N)^2}{4\lambda_i} \le Q_N + O\!\left(\sqrt{\frac{\log(2N)}{N}}\right). \tag{14.10}$$

This is a lower bound on the penalty paid by every optimizer. To turn it into a lower bound on the optimized price, reuse the same energies in the objective defining B_{N+1}. Their price is unchanged, while their penalty in $(N+1)B_{N+1}$ is $N^2 Q_N/(N+1)$, compared with NQ_N in NB_N. Thus

$$(N+1)B_{N+1} - NB_N \ge \frac{N}{N+1} Q_N.$$

Sum this inequality and (14.10) up to scale N. The errors sum to $O(\sqrt{N \log(2N)})$ and E is uniformly bounded. Together with $B_N \le E$ this proves (14.4). Equation (14.6) now gives the width estimates in (14.5). The widths and their targets are bounded, so $Q_N = E + O(\sqrt{\rho_N})$. Adding B_N gives the asserted price. □

14.3 Prescribed widths and uniform comparison

Corollary 14.3 (Prescribed widths)**.** *Under the hypotheses of Theorem 14.1, fix positive width caps a_i. Then*

$$\lim_{N\to\infty}\frac{1}{N}\max_{\mathbf{h}:\, w_N(h_i)\le a_i} H_{\mathbf{x}}(C(\mathbf{h};\mathbf{z})) = \inf_{o\in X}\sum_i a_i d(o,x_i). \tag{14.11}$$

Convergence is uniform over all placements of bounded scale.

Proof. Transport gives the upper bound. Fix $\tau>0$ and minimize $\sum_i a_i\sqrt{d(o,x_i)^2+\tau^2}$ on the closed convex hull of the sites. For r between zero and the hull diameter the function $r\mapsto\sqrt{r^2+\tau^2}-\epsilon r^2$ is convex and increasing for sufficiently small $\epsilon>0$. The CAT(0) midpoint inequality then makes every minimizing sequence Cauchy, giving a minimizer by completeness. Metric projection shows that this is also a minimizer on X. Write $r_i=d(o,x_i)$ at this minimizer and set $2\lambda_i=\sqrt{r_i^2+\tau^2}/a_i$. The tangent upper bound for the concave function $s\mapsto\sqrt{s+\tau^2}$ shows that o minimizes $\sum_i d(o,x_i)^2/(4\lambda_i)$ as well. Indeed the tangent majorant for the smoothed objective agrees with it at o and differs from this quadratic objective by a constant.

At scale bounded by D_0, we have $r_i\le D_0$. For fixed τ these penalties stay in a compact positive interval. The limiting widths in (14.5) are $a_i r_i/\sqrt{r_i^2+\tau^2}$, bounded strictly below a_i uniformly. Thus every maximizing tuple at large N meets the caps. Its normalized price is

$$2E+o(1)=\sum_i a_i\frac{r_i^2}{\sqrt{r_i^2+\tau^2}}+o(1)\ \ge\ \inf_{z\in X}\sum_i a_i d(z,x_i)-\tau\sum_i a_i+o(1).$$

Let $\tau\downarrow 0$ after taking $N\to\infty$. □

A maximizing energy tuple in Corollary 14.3 can depend on the placement. The local construction of Section 20.2 compares a single tensor against all spectral prices simultaneously. Taking a direct sum over the admissible energies gives that tensor at a logarithmic cost in each leg dimension.

Corollary 14.4 (A common comparison tensor)**.** *Suppose $\omega_F<9/4$. Fix the k marked positions and positive caps $a_1,\dots,a_k$. Let A_N be the direct sum of all nonzero $C(\mathbf{h};\mathbf{z})$ with $w_N(h_i)\le a_i$ at every leg. Then*

$$\frac{H_{\mathbf{x}}(A_N)}{N}=\inf_{o\in X}\sum_i a_i d(o,x_i)+o(1)$$

uniformly over placements of bounded scale in complete CAT(0) spectral spaces over F. The ith leg of A_N has log dimension at most $Na_i+O_{k,\mathbf{a}}(\log(2N))$.

Proof. Each leg admits $O_{\mathbf{a}}(N^2)$ half-integer energies, so there are $O_{k,\mathbf{a}}(N^{2k})$ summands. Each summand is a restriction of A_N, and the direct-sum inequality bounds the price of A_N by the largest summand price plus $O_{k,\mathbf{a}}(D\log(2N))$. Corollary 14.3 now gives the stated limit. Each ith summand leg has dimension at most e^{Na_i} by Lemma 13.2, so summing those dimensions gives the claimed bound. □

15 Incidence expansion

Let S be a finite set of distinct points in an affine plane, and let a *secant* be a line containing at least two of the points. Write $\mathcal{L}$ for the secants and t_i for the number through point i. For centered real labels v_i, meaning $\sum_i v_i = 0$, we have

$$\sum_{L\in\mathcal{L}} \Big| \sum_{i\in S\cap L} v_i \Big|^2 = \sum_{i\in S} (t_i - 1)|v_i|^2.$$

On expanding the squares, each diagonal term occurs t_i times and each ordered pair of distinct points occurs once. Centering gives the identity. We need a version for points placed in a spectral metric space, where one can measure how far an entire secant pulls a common center in a given direction. This version will bound the mean squared length of an incidence edge in the graphs used for permutation routing (Section 17).

15.1 An inequality for secant lines

Take $S \subset F^2$ with $|S| \geq 2$. For each $i \in S$ choose a site x_i in a complete CAT(0) spectral space X over F and a mass $a_i > 0$. Let o be the minimizer of $\sum_i a_i d(o, x_i)^2$ and set $r_i = d(o, x_i)$. Two points of S may be placed at the same site.

Suppose the legs for a subset $C \subset S$ have widths $a_i r_i$. Sending them from a common site m to their destinations costs $\sum_{i\in C} a_i r_i d(m, x_i)$. By first variation at $m = o$, the supremum of the initial savings per unit distance, allowing the common site to stay at o, is

$$K_C = \sup_{y\neq o} \left(\sum_{i\in C} a_i r_i \cos \angle_o(x_i, y) \right)_+ .$$

Here $y \in X$, $\angle_o$ is the Alexandrov angle and the summand is zero if $r_i = 0$. For a one-point space set $K_C = 0$. Write $K_L = K_{S\cap L}$. On a real line this is the absolute value of the secant sum above, with $v_i = a_i(x_i - o)$.

Theorem 15.1 (Incidence inequality)**.** *If F is algebraically closed and $\omega_F < 9/4$, then for every such finite configuration of affine points, weighted sites and secants,*

$$\sum_{L\in\mathcal{L}} K_L^2 \leq \sum_{i\in S} (t_i - 1) a_i^2 r_i^2. \tag{15.1}$$

Proof. Write $i = (u_i, v_i)$ and project the points to the sphere positions $z_i(s) = u_i s + v_i$. As s varies, a pair collides exactly once on $\mathbb{P}^1(F)$, with the projection $i \mapsto u_i$ used at infinity. Let D consist of the finite collision parameters together with ∞. For each $\alpha \in D$ let $\mathcal{P}_\alpha$ be the partition into fibers of its projection. The nonsingleton fibers are precisely the intersections with secants, each occurring at the parameter for its direction. Thus i is a singleton at exactly $|D| - t_i$ parameters, and $K_{\{i\}} = a_i r_i$. Adding these singleton contributions to both sides of (15.1) gives the equivalent estimate

$$\sum_{\alpha\in D} \sum_{C\in\mathcal{P}_\alpha} K_C^2 \leq (|D| - 1) \sum_i a_i^2 r_i^2. \tag{15.2}$$

Choosing the external widths. Fix $s_0 \notin D$. The positions $z_i(s_0)$ are distinct, so Theorem 14.1 with $\lambda_i = 1/(2a_i)$ gives energies $h_i = h_i(N)$ with

$$w_N(h_i) \longrightarrow a_i r_i, \qquad \frac{1}{N} H_{\mathbf{x}}(C(\mathbf{h}; \mathbf{z}(s_0))) \longrightarrow P := \sum_i a_i r_i^2. \tag{15.3}$$

Here $w_N(h) = c\sqrt{h}/N$, $c = \pi/\sqrt{3}$. The cost of supplying the external legs from o, in these limiting widths, is exactly P. If $P = 0$ all K_C vanish. Otherwise take N large enough that the chosen tensor is nonzero. Its total energy $H_0 = \sum_i h_i$ is $O(N^2)$ and is an integer, since every generator in any nonzero coefficient is paired.

A joint bound on the collision energies. Write $T(s) = C(\mathbf{h}; \mathbf{z}(s))$. In the chart at a finite α use $s - \alpha$ as coordinate. At infinity use $u = 1/s$ and the positions $u_i + uv_i$. Homogeneity in the positions gives the tensor $u^{-H_0} T(1/u)$ in this chart. In either chart the points of each fiber C have a common center and distinct linear offsets. Proposition 13.1 expands the coefficient of order l as a sum over intermediate energies $e_C \geq 0$ satisfying $\sum_C e_C = H_0 + l$.

Choose temporarily a linear functional ℓ with $f(s) = \ell(T(s))$ nonzero. Let l_α be the order of its first nonzero coefficient in the corresponding tensor chart. In particular, $l_\infty = \operatorname{ord}_\infty f - H_0$. The ordinary orders of f on $\mathbb{P}^1$ sum to zero. Since there are no poles off D, the energy totals at these orders satisfy

$$\sum_{\alpha \in D} (H_0 + l_\alpha) \leq (|D| - 1) H_0. \tag{15.4}$$

Each summand is nonnegative. Using the *same* functional for all directions is what gives one bound on their total intermediate energy.

We now require the selected coefficients to retain the price in (15.3). In every chart collect the fusion terms with total intermediate energy at most $(|D| - 1)H_0$. There are only polynomially many in N. Each term is specified by one nonnegative half-integer per fiber, and the number of fibers and directions is fixed. Let W be the span of all collected terms whose price is less than $H_{\mathbf{x}}(T(s_0)) - \sqrt{N}$. The ordinary-sum inequality bounds the price of every nonzero tensor in W by this threshold plus $O(\log N)$. For large N, $T(s_0) \notin W$. Take ℓ vanishing on W and nonzero on $T(s_0)$. At each α the coefficient at its first nonzero order l_α has a fusion term surviving ℓ. By (15.4) such a term was collected, so its price is at least $H_{\mathbf{x}}(T(s_0)) - \sqrt{N}$. Select one surviving term per direction.

Pass to a common subsequence along which the selected widths $w_N(e_C)$ converge to limits $b_{\alpha,C}$. There are finitely many widths, all bounded by (15.4). Multiplication of that inequality by c^2/N^2 gives

$$\sum_{\alpha \in D} \sum_{C \in \mathcal{P}_\alpha} b_{\alpha,C}^2 \leq (|D| - 1) \sum_i a_i^2 r_i^2.$$

The width needed by one fiber. Fix α, a fiber C, and $y \neq o$. Supply the selected term by preparing the fused tensor at o. Carry the intermediate leg for C a distance t along $[o, y]$ to a site m, apply its fusion map there and send the external legs from m to their sites. Split all other legs at o. By the transport bound (3.1) and $\log \dim V_h \leq c\sqrt{h}$, the price of the selected term is at most the cost of this supply. After division by N, the $\sqrt{N}$ price loss vanishes. Passing to the limit using (15.3) gives

$$P \leq \sum_{i \notin C} a_i r_i^2 + b_{\alpha,C} t + \sum_{i \in C} a_i r_i d(m, x_i).$$

At $t = 0$ equality holds. The derivative from the right is therefore nonnegative. First variation in a CAT(0) space gives $b_{\alpha,C} \geq \sum_{i \in C} a_i r_i \cos \angle_o(x_i, y)$ [12, Chapter II.3]. Taking the supremum of the positive part yields $b_{\alpha,C} \geq K_C$, proving (15.2). □

15.2 The sharp local link inequality

Suppose F contains $\mathbb{F}_q$. Take the q^2 points of the affine plane $\mathbb{F}_q^2$ and retain the lines in s parallel classes, $1 \leq s \leq q+1$. A point belongs to s retained lines and each line contains q points. Place the vertices of the resulting bipartite incidence graph in X. Let e be the mean squared distance across a uniform edge. Give each side half the total mass, uniformly distributed over its vertices, and let V be the minimum mean squared distance of this distribution to a common center.

Corollary 15.2 (Affine incidence links)**.** *Under the hypotheses of Theorem 15.1,*

$$e \geq 2(1 - s^{-1/2})V.$$

The coefficient is optimal in every target containing a nontrivial geodesic segment. Omitting one parallel class gives $s = q$ and a q-regular graph on $2q^2$ vertices.

Proof. Take o to be the barycenter of the point sites alone. Write r_i, r_L for the distances from o of a point site x_i and a line site y_L. The mean squared distances on the two sides are

$$A = q^{-2} \sum_i r_i^2, \qquad B = (sq)^{-1} \sum_{L \text{ retained}} r_L^2.$$

For each incidence, triangle comparison bounds its squared distance from below by $r_i^2 + r_L^2 - 2 r_i r_L \cos \angle_o(x_i, y_L)$. A zero radius gives a zero cross term. By (15.1) with all masses one, $\sum_L K_L^2 \leq q \sum_i r_i^2$ even for the full set of lines. Thus the mean cross term over the sq^2 retained incidences is at most

$$\frac{2}{sq^2} \sum_{L \text{ retained}} r_L K_L \leq \frac{2}{sq^2} \sqrt{q \sum_i r_i^2 \sum_{L \text{ retained}} r_L^2} = 2\sqrt{AB/s}.$$

Consequently

$$e \geq A + B - 2\sqrt{AB/s} \geq (1 - s^{-1/2})(A + B).$$

Evaluation at o gives $V \leq (A+B)/2$, which proves the corollary.

To attain it, choose one retained parallel class and assign its q lines real labels of mean zero and positive variance. Give each point the label of its line. Give each line site of that class $\sqrt{s}$ times its label and every other line site zero. Both sides now have mean zero and the same mean square A. A fraction $1/s$ of the edges belongs to the chosen class, so the mean product of the two endpoint labels is $A/\sqrt{s}$. Hence $V = A$ and $e = 2(1 - s^{-1/2})A$. Translate and rescale the labels into any nontrivial segment of X. Nearest-point projection onto the segment ensures that the ambient minimum has the same value. □

The bound in terms of $A + B$ keeps the point barycenter fixed. Section 17 uses that form for affine links, and uses the secant inequality directly for projective links.

Part II
Sorting and table lookup

16 Stable sorting under a strict square saving

A tensor network may have to move a table to the place where it will be used, or bring together all the queries addressed to one table. Moving a table involves long records, while gathering its queries requires sorting many short ones. We show that a strict improvement on the exponent $9/4$ would make both operations cheap enough for the final assembly in Sections 26 and 27.

Fix the finite field k from Section 2 and let $n = 2^h$. The tagged permutation tensor from Section 4 allows the destination of every record to be part of the input. Write j_i for the h-bit destination of the w-bit source record x_i, and y_j for the proposed record at position j. Then $\mathrm{Tag}(h, w)$ is the indicator of the event that the j_i are distinct and $y_{j_i} = x_i$ for every i. Similarly, the stable-sort tensor is the indicator that a proposed output sorts the input records by a designated key and preserves the input order within each key. Both indicators include the payload bits in the equality tests.

Theorem 16.1 (Conditional stable sorting)**.** *Suppose $\omega_k < 9/4$, and let $n = 2^h$.*

1. *For some $1 < p < 2$, $\mathrm{Tag}(h, h)$ has an exact trivalent network over k of size $O(nh^p)$. The same bound holds for stable sorting of records of at most ch bits, for every fixed c.*

2. *For every $\zeta > 0$ there is $a > 1$ such that, for every $w \geq h^a$, both $\mathrm{Tag}(h, w)$ and stable sorting of w-bit records have exact trivalent networks over k of size $O_\zeta(nwh^\zeta)$.*

16.1 Evaluating a table by sorting its queries

At a station of arity t, the incident bits give the address of an entry in a table of length 2^t. When many copies of the station are evaluated together, only the multiplicity of each address matters to the product of the selected entries. Sorting the queries recovers these multiplicities and spreads the cost of the table over the batch.

Lemma 16.2 (Batched table evaluation)**.** *Assume the ordinary-width stable-sort bound of Theorem 16.1 over k, with exponent $p > 1$. Let a network of binary wires and K-valued tables represent a tensor T, where K/k is any field extension. If $t_v \leq u$ is the arity of station v and $s = \sum_v (1 + t_v)$, then $T^{\otimes 2^u}$ has an exact trivalent K-network of size*

$$O\big(2^u s (1 + u)^{p-1}\big).$$

The constant is independent of K and of the entries in the tables.

Proof. First consider one table $f\colon \{0,1\}^t \to K$, with $t \geq 1$, and expose the 2^t query addresses $x_1, \ldots, x_{2^t}$ as tensor legs. If c_z is the number of occurrences of address z, the required product is $\prod_z f(z)^{c_z}$, with an empty product at an unused address. Create one dictionary row for each z and guess its count in $t + 1$ bits. On count bit j, for $0 \leq j \leq t$, place a unary factor with entries

$$\big(1,\ f(z)^{2^j}\big).$$

The position of this factor already determines z. Thus its two entries are fixed coefficients of the network.

Give each query its own row, with zeros in the count field. Sort the 2^{t+1} rows by address and then by row type, so that the dictionary row follows all queries to that address. A counter, initially zero, increments on every query. On a dictionary row it must agree with the carried count and then resets to zero. This checks the count even at addresses with no queries. For fixed exposed addresses there is exactly one valid choice of counts, sorted rows and counter states. The contraction therefore equals $\prod_i f(x_i)$ in every characteristic, also when a table entry vanishes.

The counter uses $O(2^t(1+t))$ gates by ripple-carry addition. Sorting the $O(t+1)$-bit rows costs $O(2^t(1+t)^p)$. The sorting network realizes a zero–one tensor over k and hence realizes the same tensor over K by scalar extension.

Group the 2^u copies of each station of arity t into 2^{u-t} batches of 2^t. A station of arity zero supplies a scalar directly. Connect the exposed wires according to the original network in each copy. The result contracts to $T^{\otimes 2^u}$ and its size is bounded by a constant times

$$2^u \sum_v (1+t_v)^p \;\le\; 2^u (1+u)^{p-1} \sum_v (1+t_v).$$

□

16.2 The two scales of the sorting proof

Assume $\omega_k < 9/4$ throughout Part II. We first route wide records. A hop that permutes records within a smaller address set can be handled recursively. The objective is to shorten the address by an arbitrarily large constant factor while keeping the total width transported in each summand within a constant multiple of the input width, independently of that factor. Repeating such a reduction gives the wide-record estimate in Theorem 16.1. Once that estimate is available, two levels of radix sorting give the ordinary-width estimate. A short key selects a digit within a block, and whole packs of records with the same digit then travel together.

Section 17 uses the incidence inequalities of Part I to compare the price of a target permutation with the prices of hops on smaller sets of addresses. The comparison first holds for an action on a CAT(0) space. Local price data sufficient to realize such an action give the same bound. If the data fail a local consistency test, spectral comparison instead supplies a tensor identity that reduces the cost. Section 18 combines these two conclusions to route payloads and to bound the number of recursive calls in a fixed network.

Throughout Parts II–IV we use base-two logarithms for ordinary prices. In these units the distance d is the price of carrying one bit. On a fixed finite list of sites, price data with bounded scale, including scale zero, form a compact set by (5.2) and Section 3. A bound on diameter suffices here. Equality along a spanning tree of binary wires realizes the full diagonal and bounds its price.

Distances between sites obtained by finitely many geodesic interpolations at fixed fractions depend continuously on the original prices. The enlarged configurations still have bounded scale. Along any convergent subnet the interpolation equalities pass to the limit, where Lemma 5.2 determines the distances from the original data.

17 From incidence to permutation actions

A permutation can be factored into hops that change only a short part of the address. Adding their prices gives a bound proportional to the number of factors. Sorting with wide records requires a bound independent of that number. We first obtain such a bound on displacement in a metric space. We then show that finite tests on the addresses suffice to bound the mean price of the payload wires.

17.1 Two incidence inequalities

Fix a power $q \geq 100$ of the characteristic of k. In a placement of a bipartite incidence graph, a vertex on either side is assigned a site. The incidence energy e is the mean squared distance between sites at the ends of a uniform incidence.

Lemma 17.1. *Suppose $\omega_k < 9/4$. In any complete* CAT(0) *spectral space over k, consider a placement of either of the following graphs.*

1. *The points and lines of the projective space* $\mathrm{PG}(\ell - 1, q)$*, where $\ell \geq 3$.*
2. *The points of $\mathbb{F}_Q^2$ and the affine lines in the $q+1$ directions defined over $\mathbb{F}_q$, where $Q = q^R$.*

Let o be the barycenter of the point sites. Let A and B be the mean squared distances to o on the point and line sides, respectively, with uniform weights on each side. Then

$$e \geq (1 - q^{-1/2})(A + B).$$

Proof. The field $\bar{k}$ contains every $\mathbb{F}_Q$ in the statement and satisfies $\omega_{\bar{k}} \leq \omega_k$. To apply Theorem 15.1 there, recall that scalar extension preserves the possible joint spectral evaluations on any finite placement of k-tensors (Section 3). In particular, each finite subset of our metric space has an isometric realization in a complete spectral space over $\bar{k}$. Take a pointed ultralimit of these realizations over the finite subsets containing a fixed basepoint, using an ultrafilter containing every tail under inclusion. Each original site is eventually present with all its prescribed distances. Completeness and uniqueness of geodesics make the resulting isometric image closed and convex. For a bounded finite distribution in the ultralimit, the barycentric variance inequality identifies its barycenter with the limit of the coordinate barycenters. For a distribution in the image, nearest-point projection places its barycenter in the image as well. It therefore suffices to prove the inequalities on placements over $\bar{k}$.

For the affine plane, apply the unit-mass estimate in the proof of Corollary 15.2 to the configuration $\mathbb{F}_Q^2 \subset \bar{k}^2$, retaining $q+1$ directions. It bounds the cross term in squared-distance comparison by $2\sqrt{AB/(q+1)}$, giving the assertion.

For the projective space, map $\bar{k}^\ell$ linearly into $\bar{k}^3$, preserving the ranks of sets of up to three projective representatives. Such maps form a nonempty open set. Indeed, preserving each rank is a nonempty open condition and there are only finitely many conditions. In the resulting projective plane choose an affine chart containing all the images. Preservation of pairs and triples makes the images distinct and preserves exactly which triples are collinear. The secants of the affine configuration thus represent the lines of the original projective space.

There are j lines through each point and $s = q + 1$ points on each line. Write $r_i = d(o, x_i)$ for point sites and $r_L = d(o, y_L)$ for line sites. Set

$$K_L = \sup_{z \neq o} \Big(\sum_{i \in L} r_i \cos \angle_o(x_i, z) \Big)_+ ,$$

with zero summands at $r_i = 0$. The singleton space satisfies the claim directly. In every other case Theorem 15.1 gives $\sum_L K_L^2 \le (j-1) \sum_i r_i^2$. If M is the number of incidences, triangle comparison and Cauchy–Schwarz bound the mean cross term by

$$\frac{2}{M} \sum_L r_L K_L \le 2 \sqrt{\frac{j-1}{js} AB} \le q^{-1/2}(A + B).$$

Here M is both j times the number of points and s times the number of lines. Subtracting the cross term from $A + B$ proves the claim. □

17.2 Permutations on short address sets

Choose an integer $\ell \ge 3$ and reserve m auxiliary coordinates, where m will depend only on q. For $n = 2^h$ put

$$r = \left\lfloor \frac{\log_q n}{\ell + m} \right\rfloor , \qquad U = \mathbb{F}_q^r, \quad N = q^r, \quad c = \left\lceil n / N^{\ell+m} \right\rceil , \qquad \mathcal{I} = U^\ell \times [c] \times U^m .$$

Here $[c] = \{1, \dots, c\}$. Identify the original n addresses with the first n labels in a fixed ordering of $\mathcal{I}$ and extend the input permutation π by fixed points. The padded address set satisfies $n \le |\mathcal{I}| < 2n$.

A *permutation function* takes a height h and a valid input permutation as parameters and outputs a permutation of $\mathcal{I}$. We require a single deterministic program using $nh^{O(1)}$ operations on words of $O(h)$ bits, with the usual word operations of polynomial bit complexity. These functions form a countable group G under pointwise composition. An explicitly listed permutation has a linear-time inverse, and π itself is an element of G.

Identify U^ℓ with $\mathbb{F}_q^\ell \otimes U$. For a fixed subspace $V \le \mathbb{F}_q^\ell$ of dimension at most two, let K_V be the subgroup preserving every fiber of the projection $\mathcal{I} \to (\mathbb{F}_q^\ell / V) \otimes U$. All auxiliary labels belong to the fiber. Its size is $cN^{m + \dim V}$, so every hop in K_V permutes sets of size at most $2^{th + O(1)}$, where

$$t = \frac{m + 2}{m + \ell}. \tag{17.1}$$

The additive constant follows from $c \le q^{m+\ell}$. Increasing ℓ shortens the address within a fiber.

The subgroups K_V generate G. We give the coordinate factorization of a rearrangeable network [9] in algorithmic form because the routing protocol will use its factors. Treat $[c] \times U^m$ as one coordinate. To eliminate a coordinate of size a, form a bipartite multigraph on the remaining input and output names, with one edge per requested transfer. Each vertex has degree a. Give the edges a colors, using each color once at each vertex. In the first stage, permute the chosen coordinate to its edge color. In the last stage, permute that color to the destination coordinate. Between these stages the remaining coordinates are permuted separately for each color. Iterating gives a bounded number of stages for fixed ℓ, m. Each ordinary coordinate stage is in its coordinate line subgroup, and each auxiliary stage is in K_0.

For completeness, the coloring takes deterministic time linear in the edge count up to logarithmic factors (see also [21]). An even-regular bipartite multigraph splits into two regular halves by alternating colors around Euler tours. At an odd degree a we first remove one perfect matching. To find it, let v be the number of vertices on each side and take the least H with $2^H > av$. Multiply each edge multiplicity by $\lfloor 2^H/a \rfloor$ and complete the graph to degree 2^H with copies of an arbitrary dummy matching. The total dummy multiplicity is less than $av < 2^H$. Split into regular halves repeatedly, keeping the half with fewer dummy copies and breaking ties deterministically. Each split divides even multiplicities equally and alternates the remaining odd copies around Euler tours, distinguishing dummy from original edges. After H splits the retained degree-one graph has no dummy copy. Only the original edges and the dummy matching need be stored, with multiplicities fitting in a word. After the matching is removed the residual even degree splits in half, so the full coloring recursion has logarithmically many levels.

17.3 Uniform displacement

A bound obtained by adding displacements along the coordinate factorization would depend on ℓ. The fixed point in the next lemma bounds the displacement of every product of hops by a single absolute constant.

Lemma 17.2. *Suppose G acts isometrically on a complete* CAT(0) *space Y in which the inequalities of Lemma 17.1 hold for all placements. There is an absolute constant C_0 such that, if $d(x, gx) \le 1$ for all $g \in \bigcup_V K_V$, then G fixes a point within $C_0/2$ of x. In particular $d(x, \pi x) \le C_0$.*

Proof. Use a triangle for each flag $0 \subset L \subset P$, where L is a vector line and P a vector plane in $\mathbb{F}_q^\ell$. We use the link-energy argument of Garland and its metric fixed-point version [29, 30]. Choose the flag uniformly and, within it, a vertex or edge uniformly. A placement assigns each V a point y_V fixed by K_V. For each V we seek another such point z_V so that, in a uniform flag through V,

$$\mathbb{E}\, d(z_V, y_W)^2 \le \tfrac{3}{4}\, \mathbb{E}\, d(y_{W_1}, y_{W_2})^2. \tag{17.2}$$

The vertices W_1, W_2 lie opposite V, and W is uniform among them. At $V = 0$, the opposite edges are the incidences of projective points L and projective lines P. The barycenter of the y_L is fixed by K_0. Lemma 17.1 bounds its mean squared distance to a neighbor by $1/[2(1 - q^{-1/2})]$ times the opposite-edge energy. At $V = L$, the barycenter of the y_P with $P \supset L$ is fixed by K_L. Its variance inequality with test point y_0 gives the coefficient $1/2$.

The plane vertex. When $V = P$, the opposite edge joins 0 to a line $L \subset P$. Here y_0 and the y_L need not be fixed by K_P. We use translates of these sites to build an affine incidence graph of the same edge energy. The point barycenter of this graph will be nearly fixed by K_P.

Choose coordinates $(x_1, x_2) \in U^2$ along P in each fiber modulo P, using a fixed basis and linear complement. Each coordinate pair names a class of cN^m labels. Consider any other partition of each fiber into classes with these same names and cardinalities, computed uniformly at all heights and inputs in the time defining G. It is obtained from the coordinate partition by some $g \in K_P$, by sorting and matching equal names within fibers. Explicitly, if p is the coordinate name, the transported name is $p \circ g^{-1}$. Assign the new partition the site gy_0. Any other such g differs on the right by an element of K_0, so this site is well defined.

Projecting the names modulo $L \otimes U$ gives a coarser partition to which we assign gy_L. The stabilizer of the reference projected partition is K_L. Both assignments commute with the action of K_P. Thus a partition and its projection have site distance exactly $d(y_0, y_L)$.

Let $Q = q^R$, where $R \le m$ will be chosen below. Use R auxiliary U-coordinates as the values of an $\mathbb{F}_q$-linear map $u : \mathbb{F}_Q \to U$ on a fixed basis of $\mathbb{F}_Q$. For a uniform label in a fiber, u is uniform independently of (x_1, x_2). For each $(a, b) \in \mathbb{F}_Q^2$ form the partition with name $(x_1 + u(a), x_2 + u(b))$. The name is uniform on U^2. Along an affine line of direction L defined over $\mathbb{F}_q$, the name modulo $L \otimes U$ is constant as a function of (a, b), by linearity of u. We therefore obtain sites for all points and all retained lines. The incidence energy is $e_P = \mathbb{E}_{L \subset P} d(y_0, y_L)^2$. At the point barycenter b_0, Lemma 17.1 now gives $(1 - q^{-1/2})(A + B) \le e_P$.

A fixed center. For d_* independent samples from the point-site distribution, form a successive geodesic mean by moving a fraction $1/i$ toward sample i at step i. Its final value M_{d_*} satisfies

$$\mathbb{E}\, d(M_{d_*}, b_0)^2 \le A/d_*. \tag{17.3}$$

Indeed, condition on the previous mean. Squared-distance convexity and the variance inequality at b_0 bound the new expected squared error by $(1 - 1/i)^2$ times the old error plus A/i^2.

Call a tuple of partitions independent if its joint names are uniform in every fiber at every height and input. This is a property of a realized tuple. Choose a large absolute integer d_* and set $m = R = 4d_*$. For $4d_*$ independent uniform samples from $\mathbb{F}_Q$, the probability of a linear dependence over $\mathbb{F}_q$ is at most $\sum_{j=0}^{R-1} q^{j-R} < 1/(q-1)$. With probability at least $1/2$, the two coordinates of each of $2d_*$ sampled affine points give $4d_*$ linearly independent field elements. Evaluation of u on them is then jointly uniform even conditional on (x_1, x_2), so the $2d_*$ partitions are independent.

The group K_P is transitive on the efficiently specified independent tuples of this length, by matching joint names within each fiber. Consequently the distance between the means of the two halves is the same for all such tuples. Its unconditional expected square is at most $4A/d_*$ by (17.3) and the triangle inequality. For each independent tuple the square is therefore at most $8A/d_*$. The same conditioning gives an independent d_*-tuple of affine partitions whose mean M satisfies $d(M, b_0)^2 \le 2A/d_*$.

Fix $g \in K_P$. The tuple giving M and its g-translate may be dependent. Construct a third tuple jointly independent with each of them separately. In each fiber join the old name at a label to its translated name by one edge for that label. Both sides of the resulting bipartite graph have N^{2d_*} vertices, each of degree cN^{m+2-2d_*}. The degree is divisible by N^{2d_*}. Color its edges properly as above and group the colors into N^{2d_*} sets of equal size. Regard the color set of an edge as its third name in U^{2d_*}. At every vertex, each third name occurs equally often, which proves the two required independence statements. The coloring also supplies the third tuple in the time defining G.

The mean of this tuple lies within $\sqrt{8A/d_*}$ of both M and gM. Hence

$$d(b_0, gb_0) \le 2\big(\sqrt{2A/d_*} + \sqrt{8A/d_*}\big) \le 0.04\sqrt{A}$$

on choosing d_* sufficiently large. The circumcenter z_P of the K_P-orbit of b_0 is fixed by K_P and lies within $0.04\sqrt{A}$ of b_0. With half the mass on each side of the affine incidence graph, the root mean squared distance to z_P is at most $\sqrt{(A+B)/2} + 0.04\sqrt{A}$. In this average an

affine point site is a translate of y_0, and an affine line site of direction L is a translate of y_L. Since z_P is fixed, these distances are precisely the distances from z_P to y_0 and y_L. The inequality $(1/\sqrt{2}+0.04)^2/(1-q^{-1/2}) < 3/4$ proves (17.2) at P.

Iteration on the flags. Initialize y_V at the circumcenter of $K_V x$, within distance one of x. Replace each y_V by its midpoint with z_V, which is still fixed by K_V. Write e for the mean squared edge length of a flag. By convexity of distance between geodesics, the squared distance between the new endpoints at V, W is at most $\big(d(y_V, z_W)^2 + d(z_V, y_W)^2\big)/2$. Averaging (17.2) over flags thus gives $e_{\mathrm{new}} \le 3e/4$. By the triangle inequality through a random old neighbor, the root mean squared movement per vertex is at most $(\sqrt{e} + \sqrt{3e/4})/2 \le \sqrt{e}$. Initially $e \le 4$. The movements are summable in the finite weighted metric product, and each edge in its limit has length zero. The flag graph is connected, so the limiting placement has a single value y_*. This value is fixed by all the K_V and hence by G. Summing the movements gives $d(x, y_*) \le 1 + 2/(1-\sqrt{3/4})$. We may take $C_0 = 100$. □

17.4 Finite tests on permutation orbits

We now return to the finite set of addresses. In the routing protocol a proposed price is assigned separately to each payload wire. We will test these prices in balls of bounded radius. To compare price realizations on overlapping balls we must also test the other tensors on their common sites.

Take a finite symmetric set of elements of $\bigcup_V K_V$ containing the factors of a fixed coordinate factorization of π and their inverses. Call these the *core generators*. For each input they define an undirected graph on $\mathcal{I}$, with a *core link* for the hop from a label to its image under a generator. Its degree is bounded by the size of the set. Each i has a path to $\pi(i)$ along the factors, of length bounded by a fixed rational a_{p} independently of h and the input.

The four test parameters are the core generators, a ball radius, a maximum number of sites, and a maximum local dimension. Give each payload wire its own commodity index. Also index every nonzero k-tensor of permitted format on an ordered list of distinct sites in a common test ball, using fixed coordinate bases. Assign this index on the whole graph, so two balls containing the same tensor on the same ordered sites see one price. There are $O(|\mathcal{I}|)$ indices because each ball has bounded size and the field is finite.

Assign each commodity a nonnegative price. Cap a payload price at a_{p}. For each other format choose a uniform rational cap sufficient to create the tensor at one site and transport its legs in binary coordinates along core links in the ball. On a test ball compare the assigned prices of all its commodities with an ordinary spectral price on its sites. The error is the largest absolute discrepancy on a commodity or positive excess above unit cost on a core link with both endpoints in the ball. Duplicate commodities are compared separately with the spectral price of the same tensor. The minimum error over ordinary prices, allowing the collapsed price, is the *discrepancy* of the test. This minimum is attained. Indeed, the core links connect the ball, so a spanning tree bounds the scale of any price with bounded error, and prices at bounded scale form a compact set. On a singleton use the collapsed price. A center *passes* to tolerance ρ if this minimum is at most ρ.

Lemma 17.3 (Finite detection)**.** *For each ℓ there are finite test parameters and $\rho > 0$ such that the following holds for all sufficiently large h and every valid permutation input. If all*

but a fraction ρ of the centers pass to tolerance ρ, the assigned payload prices satisfy

$$|\mathcal{I}|^{-1} \sum_{i \in \mathcal{I}} x_{i,\pi(i)} \leq C_0 + 1.$$

Here C_0 is the absolute constant of Lemma 17.2.

Proof. If the assertion fails, choose a sequence of counterexamples with $h \to \infty$. Enlarge the finite symmetric sets of core generators to exhaust the countable union of the K_V, and let the radii and format bounds tend to infinity. Take the tolerance and permitted exceptional fraction tending to zero, keeping the original factorization of π and the payload cap fixed. We use these counterexamples to construct an action in which each hop has displacement at most one and the target permutation has displacement at least $C_0 + 1$.

The marked sites. At each passing center i take a complete CAT(0) spectral realization of a price attaining the test. At every other center use the one-point space. For $g \in G$ mark the site $g^{-1}i$ in the chart at i. A fixed word in subgroup elements represents g^{-1}, so eventually these hops are tested and the ball contains the whole path from i to $g^{-1}i$. Thus the distance from the center to this mark is uniformly bounded for each fixed g. Use the center itself when the mark is unavailable.

At each index take the metric product over all centers with the root mean square distance, pointed at the vector of centers. Take a pointed ultralimit of the products and let Y be the closed convex hull of the resulting marks. The product and ultralimit are complete CAT(0) spaces, as one sees from the midpoint inequality. Both incidence bounds of Lemma 17.1 hold in Y. A product barycenter is coordinatewise, so the squared inequalities average over the coordinates. In the ultralimit the variance inequality identifies the limit of barycenters with the barycenter of a bounded finite placement. Finally, projection onto a closed convex subset preserves the barycenter of a distribution in that subset.

Translation of the marks. We claim that left multiplication $g \mapsto ug$ extends to an isometry of Y. A point of a dense subset of Y is given by a finite expression of geodesic interpolations of marks. Fix two such expressions. At center i the translated expressions have the same underlying address labels as the original expressions at center $u^{-1}i$, since $(ug)^{-1}i = g^{-1}u^{-1}i$.

For sufficiently large indices, both balls contain these labels. Both centers pass outside a fraction tending to zero, since $i \mapsto u^{-1}i$ preserves uniform measure. For every fixed tensor on the distinct labels the two realizations then differ by at most twice the tolerance once the format bounds include that tensor. Their evaluations use its common commodity index. It follows that the distances between the two expressions in the two charts differ by a quantity tending uniformly to zero on such pairs. Indeed, a violating sequence has a subsequence with a fixed pattern of coincident labels. The finitely many factor words bound the diameter of the marked list in both charts, so the tree estimate bounds their price scales. By compactness take limiting prices on the list. They agree on every tensor by the format exhaustion. Continuity of distances between geodesic expressions, from Lemma 5.2, gives equal limiting distances, a contradiction.

All expression distances are bounded even on exceptional pairs. In a passing chart the fixed word lengths bound all the marks, and a failed chart has collapsed to a point. The exceptional pairs therefore make a vanishing contribution to the squared mean. Reindexing by $i \mapsto u^{-1}i$ proves the claimed equality of distances in the limit. The assignment on expressions thus extends to Y. Its inverse comes from u^{-1}, and the group law holds first on expressions and then by continuity on Y.

Let x be the identity mark. For each individual $g \in K_V$ its hop and inverse eventually belong to the tested set, so $d(x, gx) \leq 1$. The mark $\pi^{-1}x$ at center i corresponds to the address $\pi(i)$. In a passing chart its distance from the center agrees up to the tolerance with the assigned price of the payload wire $i \to \pi(i)$. In a failed chart the distance is zero while the price is at most a_{p}. Thus $d(x, \pi^{-1}x)$ is the ultralimit of the root mean square payload price. This is at least the limiting lower bound $C_0 + 1$ on the arithmetic means of the counterexamples. Since $d(x, \pi^{-1}x) = d(x, \pi x)$, this contradicts Lemma 17.2. □

18 Bounded-width permutation routing

18.1 From a failed price test to a routing identity

Fix the core generators and the test parameters of Lemma 17.3. Write $M = |\mathcal{I}|$, so that $n \leq M < 2n$. Sending a bit along the designated path from i to $\pi(i)$ pays for every hop on that path. We seek an exact sum of networks in which the average transport cost per payload bit has an absolute bound, even as the parameter ℓ increases.

Proposition 18.1. *Suppose $\omega_k < 9/4$ and fix $H \geq 1$. For all sufficiently large h, every valid set of tags and every $h \leq w \leq h^H$, the associated permutation tensor for w-bit payloads is an exact sum over k of networks with the following properties.*

1. *Each label has a bounded number of fixed slots per core generator. A used slot carries $O(w)$ bits to a neighbor along the generator or its inverse. Both ends belong to the same prescribed group of at most $2^{th+O(1)}$ labels. The sum of the carried widths in any summand is at most Cnw, where C is independent of ℓ and t.*

2. *At each label a fixed trivalent network of size $O(w(1 + \log w)^{O(1)})$ implements the local maps. Its configuration consists of a bounded list of menu indices, widths and offsets.*

3. *A summand is specified by $O(n \log(2w))$ bits. Given these bits and the tags, a deterministic controller computes the local configurations and matchings in $nh^{O(1)}$ word operations. The exponents and slot capacities are independent of H. The running time may have a multiplicative constant depending on H.*

Proof. We use the local replacement argument of [42, Section 9]. A *request* for a commodity leaves that tensor to be supplied at specified legs. Its cap a_i bounds the cost of supplying it directly, by preparing the tensor at one site and sending its legs along designated core paths. We encode each leg in binary and pad its unused coordinates by zero. Multilinearity permits us to substitute a tensor identity for several requests at any stage of a contraction.

The identity on a ball. Let $x \in \prod_i [0, a_i]$ be a price array with positive discrepancy on a ball. Request one copy of each commodity with $x_i > 0$ on the ball. Their tensor product D has assigned total price $L = \sum_{i : x_i > 0} x_i > 0$, since the zero array passes the test. The available supplies T_j are all the commodities on the ball, at their assigned prices, and its binary core links, at price one. On a singleton we can prepare every demand locally. On a larger ball there are prices l_j strictly above the supply prices and a number $0 < L_- < L$ such that every normalized spectral log price P satisfies

$$P(D) < L_- \max_j \frac{P(T_j)}{l_j}.$$

Indeed a spanning tree of binary core links supplies the full binary diagonal, of normalized log price one. Thus the maximum ratio is at least the reciprocal of the sum of the l_j over the tree. If the strict inequality with L failed for perturbations decreasing to the original supply prices, rescale each violating P by dividing by its maximum ratio. The tree bound keeps the scales bounded. A limit price then satisfies every supply bound and gives D price at least L. Additivity on products forces equality at each positive-priced commodity, and the remaining commodities have price zero. This limit would pass the test on the ball. For a fixed successful perturbation, compactness permits the decrease from L to L_- with a uniform strict margin.

Choose an integer b_0 and form the direct sum

$$U = \bigoplus_j T_j^{\otimes \lfloor b_0 L_- / l_j \rfloor}.$$

Rounding costs a bounded amount in log price. For sufficiently large b_0, every normalized spectral price gives U a higher value than $D^{\otimes b_0}$. Strict spectral comparison in Section 3 now gives an exact restriction of $D^{\otimes b_0 r}$ from at most $K(b_0)$ direct copies of $U^{\otimes r}$ for every $r \geq 1$. Expand the direct sums and apply the restricting maps. The resulting ordinary sum supplies $b = b_0 r$ copies of every demand. Each term uses supplies of total price at most bL_-. Its index specifies a direct copy and an ordered word of r supply choices. With s supply types this takes at most $\lceil \log_2(K(b_0) s^r) \rceil$ bits. The ratio of this length to b can be made arbitrarily small by first increasing b_0 and then r.

Repeat the identity J times. Use only the *histogram* of the J term indices as the outer selector. This costs $O(\log(2J))$ bits because the alphabet is fixed. Within the corresponding network a root sums over ordered words, sends each word along the core tree and enforces its prescribed histogram. The histogram fixes the number of copies of every supply. The word assigns these copies to the individual applications of the restricting maps. Each ordered word contributes exactly once with its original coefficient over k. Choose the rate of the term indices low enough that the tree transmission costs at most $Jb(L - L_-)/2$. The replacement thus saves a positive amount per block of demands, after paying for its own control bits.

These demands remain positively priced, and the replacement retains a strict saving, throughout a neighborhood of the original array. There are finitely many ball types since the field, radius, degrees and formats are fixed. Compactness of the price arrays of discrepancy at least $\rho/2$ gives a finite menu of replacements. On this set, the saving per unit of demand and the selected demand prices both have uniform positive lower bounds.

Bounding the total width. Follow one summand in the expansion. Let wz_i be its integer stock of outstanding requests for commodity i, and let S be its installed core width divided by Mw. Give the summand the account $S + F(z)$, where

$$F(z) = \frac{1}{M} \max_{0 \leq x \leq a} \sum_i (z_i x_i - \tfrac{1}{2} \mu x_i^2).$$

There are $O(M)$ coordinates with uniformly bounded caps. Choose a positive rational μ such that $\mu \sum_i a_i^2/(2M) \leq 1$. The maximizing price $x_i = \min(a_i, z_i/\mu)$ satisfies $z_i \geq \mu x_i$. Its Lipschitz bound gives, for $z + v \geq 0$,

$$F(z + v) - F(z) \leq \frac{1}{M} \sum_i x_i v_i + \frac{1}{2\mu M} \sum_i v_i^2.$$

Fix an integer $B > \max_i \mu a_i$. Whenever a stock exceeds wB, supply the excess directly. On the interval $z_i \geq B$ the marginal price equals a_i, so this operation does not increase the account.

If the mean maximizing price of the M payloads exceeds $C_0 + 2$, Lemma 17.3 gives more than ρM balls of discrepancy greater than ρ. A greedy packing selects $\Omega(M)$ disjoint such balls, allowing a fixed slack in the test. Indeed the discrepancy is 1-Lipschitz in the array of prices. A sufficiently fine rational grid can assign a valid menu entry to each cell meeting the set of discrepancy at least ρ, with the entry valid throughout the cell. These finite tables suffice to select and pack eligible balls.

On each packed ball apply its replacement with $J = \lfloor \alpha w / b \rfloor$, for a small fixed rational $\alpha > 0$. The bound $z_i \geq \mu x_i$ and the positive lower bound on selected demand prices ensure sufficient stock. Charge each new core link to S, and enter other supplies as new requests. In the linear bound on the account, each ball saves a fixed positive amount per retired demand block, including transmission of the word. For large w the packed balls therefore save at least $c\alpha$ in the normalized account for some $c > 0$. The finite menu and the disjoint balls give $\max_i |v_i| = O(\alpha)$, so the quadratic error is $O(\alpha^2)$. Fix α small enough that each parallel round lowers the account by a fixed positive amount in every resulting summand. Supply any excess over the stock cap after the round.

Starting from zero, introduce all w copies of each payload in equal increments on a fixed fine mesh, rounding to integers with exact total w. Before each increment use parallel rounds until the mean payload price is at most $C_0 + 2$. The account is nonnegative. The increments have total linear cost at most $C_0 + 2$, and their summed quadratic cost is at most one for a sufficiently fine mesh and sufficiently large w. Supply excess stock after each increment. Every summand therefore undergoes only boundedly many rounds, and its final account is at most $C_0 + 3$. Supply all the remaining requests directly. Taking $x = a$ in the definition of F gives

$$\frac{1}{M} \sum_i a_i z_i \leq F(z) + 1.$$

The total width is at most $(C_0 + 4)Mw \leq 2(C_0 + 4)nw$. This last coefficient is absolute.

Realizing the summands. A label lies in at most one packed ball per round. Every direct supply that uses a given edge originates within bounded distance of it. Bounded degree, boundedly many commodity types per neighborhood, and the stock cap thus give boundedly many slots of width $O(w)$ per generator. The generator prescribes the group containing both ends of the slot.

At the sites of a commodity, keep its requests in common order, retiring them from the front and appending new requests at the back. For each replacement the histogram determines the lengths of the lists. Inside its network, prefix counts of the ordered word assign supply positions to repeated applications. Comparison sorting on addresses of $O(1 + \log w)$ bits realizes these permutations and partial matchings with $O(w(1 + \log w)^{O(1)})$ trivalent gates at a site. Complete partial matchings deterministically and set idle coordinates to zero. Each individual block map has fixed size over k and admits a constant-size trivalent implementation. Only boundedly many local fields select the maps, lengths and offsets.

The controller enumerates the bounded neighborhoods, computes prices from the stocks, uses the finite tables to pack eligible balls, and assigns the slots. There are $O(n)$ histograms over the boundedly many rounds, with a unique encoding of $O(\log(2w))$ bits each. Enforce valid encodings and zeros in unused selector positions. Only local counts and indices enter

the controller, while the ordered words are summed inside the networks. The enumeration and stock updates take $nh^{O(1)}$ word operations on tags of h bits and counts of $O(\log(2w))$ bits. This proves the three bounds. □

18.2 From local transports to stable sorting

Write $S(h, w)$ for the least trivalent size of $\mathrm{Tag}(h, w)$ over k, and put $n = 2^h$. Proposition 18.1 bounds the transport used in each summand. To derive a recurrence for S, we must fit the calls from all summands into a single network. Different summands can use the same call if their data can reach it at low cost.

Sharing a fixed pool of calls. Fix the dimension ℓ, the address reduction factor $t = (m + 2)/(m + \ell)$ and the protocol of Proposition 18.1. In each term the protocol has at most Ln links of total width at most Cnw. Of these two constants, C is independent of ℓ. Choose $0 < Y \le 1$ with $L \le C/Y$ and split each link into chunks of $w' = \lceil Yw \rceil$ bits. Pin the padding of a final chunk to zero at both ends. The number of nonempty chunks in any term is at most

$$Cnw/w' + Ln \;\le\; m_*n, \qquad m_* = 2C/Y.$$

For this fixed protocol there are $O(n)$ candidate slots, grouped according to the short-address sets within which they can communicate. Duplicating a group for the boundedly many link ends or chunks it handles changes its size only by a constant factor.

We use the assignment graphs of [42, Lemma 10.2]. Such a graph connects at most cK candidates to K slots. Every active subset of at most $K/2$ candidates can be assigned injectively to adjacent slots. The degrees are bounded in terms of c, and the assignment takes $O_c(K)$ deterministic word operations with the graph as fixed advice.

Choose a dyadic capacity $U = 2^{th+O(1)}$ at least twice the size of the largest candidate group. Pack whole groups greedily into lists of capacity U, closing a list when the next group would exceed that capacity. Each closed list is at least half full, so there are $O(n/U + 1)$ lists. Fix a small dyadic fraction K/U. In a list, divide its active chunks in order into batches of size $K/2$, allowing a shorter last batch. With K slots reserved for each nonempty batch, the total reservation is at most

$$2m_*n + O((n/U + 1)K) \;\le\; 3m_*n.$$

Here $U = o(n)$ and we choose K/U after fixing the packing constants.

For each list install enough candidate batches to handle every chunk in it. Use one assignment graph to place the source ends of a batch in K slots and another for the target ends. A second assignment step handles whole batches. Its targets are a shared pool of $\lceil 6m_*n/K \rceil$ physical modules, each with K input and output positions. At most half the pool is active, and the number of candidate batches is a bounded multiple of the pool size. Thus the two lists of each active batch can be moved together to its assigned module. Including idle modules, at most $7m_*n$ positions are installed for all large h.

Every module uses $\mathrm{Tag}(h', w')$ with $h' = \log_2 K = th + O(1)$. Fill unmatched positions in order, pin unused bits to zero and give an idle module the identity tags and zero payloads. The assignments and padding are deterministic once the term and its tags are given. Because the assignment graphs have bounded degree, the switches cost $O(nw)$ in total, including those that move two lists of Kw' bits at a time.

The remaining control uses only tags, selectors and slot descriptions. It can be checked by guessing a time-stamped execution, verifying each instruction, and comparison-sorting all memory transactions by address, time and access order within each instruction to check every read against the last write. Fixed initial memory and padding make the accepting execution unique for every legal selector. A conventional comparison network also checks the original tags for distinctness and rejects invalid permutations. For each fixed H and $h \le w \le h^H$, the size outside the shorter calls is

$$D_{\mathrm{control}}(h,w) = O_H\big(nw(\log(2h))^{O(1)} + nh^J\big), \qquad J > 2. \tag{18.1}$$

The first term pays for the local maps with their polynomial overhead in $\log(2w)$. The second pays for $nh^{O(1)}$ deterministic word operations, their bit implementation and the memory check. The exponents can be chosen independently of H. We have obtained the recurrence

$$S(h,w) \;\le\; D_{\mathrm{control}}(h,w) + \frac{7m_*n}{2^{h'}}\, S(h',w').$$

Wide payloads. It suffices to prove the estimate for $0 < \zeta < 1$. Choose ℓ large enough that $14Ct^{\zeta} < 1/4$, and then fix the resulting Y and J. Shortening an address from h to h' reduces the allowed overhead h^{ζ} by a factor approaching t^{ζ}. To account for calls close to the width at which conventional sorting takes over, we let the overhead exponent decrease gradually with the width.

Choose a smooth nonincreasing $u : [1,\infty) \to [\zeta, 2]$, equal to 2 on $[1, J]$ and eventually equal to ζ. Take such a long transition interval that $b = -u'$ satisfies $0 \le b < 1$ and $Y^{-b} < 3/2$ everywhere. Fix H in the interior of the constant tail. For $h \ge 2$ and $h \le w \le h^H$ we claim

$$S(h,w) \;\le\; Dnwh^{u(a)}, \qquad a = \log_h w. \tag{18.2}$$

If $a \le J$, Batcher's bitonic sorting network [8] uses $O(nh^2)$ comparators of bit cost $O(h+w)$ to put the tagged sources in destination order. Comparing their tags with the prescribed positions and their payloads with the proposed outputs gives (18.2).

For $a > J$ induct on h. For large heights $h' < h$ and $w' \ge h'$. Writing $a' = \log_{h'} w'$, we have uniformly on $J \le a \le H$,

$$a' - a = \frac{\log Y - a\log t + o(1)}{\log h}, \qquad \frac{w'(h')^{u(a')}}{wh^{u(a)}} = (1+o(1))\, t^{u(a)+ab(a)} Y^{1-b(a)}.$$

The second formula is Taylor expansion on a compact interval. If $w' > (h')^H$, split the payload evenly into $\lceil w'/\lfloor (h')^H \rfloor \rceil$ slices, using the same tags for each. The number of slices is bounded, their widths sum to w', and their width exponents are $H + o(1)$, where $u = \zeta$. Copying the tags costs $O(h')$ per slot and slice, within (18.1). The same per-slot estimate therefore applies.

After multiplying by the ratio $7m_* = 14C/Y$ of installed to original positions, the recursive calls use at most the fraction $(1+o(1))14Ct^{\zeta}Y^{-b(a)} < 1/2$ of the claimed bound. The extra cost (18.1) is $o(nwh^{u(a)})$ uniformly, since $a \ge J$ and $u \ge \zeta$. Enlarging D to cover bounded heights closes the induction. For $w > h^H$, split into slices as evenly as possible with maximum width $\lfloor h^H \rfloor$. Each width exponent is $H + o(1)$ for large h, and tag copying costs at most a constant times the total data volume. At bounded heights the bitonic network costs $O_h(w)$. This proves, for some $A > 1$,

$$S(h,w) = O_{\zeta}(2^h wh^{\zeta}) \qquad (w \ge h^A). \tag{18.3}$$

From tagged permutations to stable sorting. For 2^s records of v bits, a stable sort uses one $\mathrm{Tag}(s, v)$ and $O(2^s(s + v))$ additional gates. Give each proposed output a guessed destination in the original input list. Use the tagged permutation to check that this sends every output back to an equal input. Along the output list check strictly increasing pairs (key, guessed input position) in lexicographic order. All records with a common key must then be exactly the corresponding input subsequence. This forces the order and every guessed index, even for identical records. The stable-sort relation thus inherits the wide-record bound.

Packing ordinary-width records. We use the wide-record bound twice in each radix pass. First take $\zeta = 1/2$ in (18.3), with threshold $a > 1$. Then take $\zeta = 1/(2a)$ there, with threshold $B > 1$. Start with $O(n)$ records whose keys and payloads have $O(h)$ bits. Choose a radix digit of $s = \lfloor h^{1/a} \rfloor$ bits. Run $O(h/s)$ stable passes starting with the least significant digit, zero-padding the last digit. Address lengths for the two sorts will be about $2s$ and h respectively.

For one pass, pad the list to blocks of 2^{2s} slots. Sort each block by a dummy flag and the digit, with dummies last in their old order. Choose a power of two $P \in [h^{B-1}, 2h^{B-1})$. Extend each run of a single real digit to a multiple of P, inserting dummies at its end. Each pack of P consecutive slots now contains records of at most one real digit. When the packs are sorted stably by digit, equal-digit records retain their order both within a block and across blocks. Empty packs sort last. Figure 6 shows one pass with two blocks.

Input block in order	Packs of digit 0	Packs of digit 1
$a_1\, b_0\, c_1\, d_1$	$[b, \cdot]$	$[a, c]$ $[d, \cdot]$
$e_0\, f_1\, g_0\, h_0$	$[e, g]$ $[h, \cdot]$	$[f, \cdot]$

Figure 6. One radix pass with two records per pack. A subscript is the digit being sorted and a dot is a dummy slot. Stable sorting of whole packs collects the middle column, then the right column, each from top to bottom. Reading the real records gives b, e, g, h, a, c, d, f.

Within a block a fixed allowance of $O((2^s + 1)P)$ extra slots covers the padding for all real runs, retains the old dummies and rounds the total length to a multiple of P. Scans find the length of each run, round it up, and assign destinations to all old records, including the old dummies. A second local tagged permutation checks these destinations. Append new dummies and require their guessed destinations to be strictly increasing. Only the unused positions are available to them, so there is exactly one valid filling. Additional temporary padding to a power of two is removed after the call. Scan each pack to obtain its digit before the global stable sort. Dummies, including any inside a pack, are ignored when reading the real order.

The list remains of size $O(n)$ across all passes. In one pass its multiplicative growth is $1 + O(2^{-s}P)$, with an additive cost $O(2^{2s} + 2^s P)$ for the last block. Over $O(h/s)$ passes the multiplicative factors have bounded product because P is polynomial in h and $s = \lfloor h^{1/a} \rfloor$. The accumulated additive cost is $o(2^h)$ since $s = o(h)$.

Each local call has address length $2s + O(1)$. Padding its records to a sufficiently large constant times h meets the first width threshold, so it costs $O(hs^{1/2})$ per slot. The global sort has $O(n/P)$ packs of $O(hP)$ bits and address length $h + O(\log h)$. Since $hP \geq h^B$, constant-factor padding meets the second threshold and the global cost is $O(nh^{1+1/(2a)})$. The

scans, copies and order checks fit the same bound. All passes together cost

$$O\big((h/s)\,nh^{1+1/(2a)}\big) = O\big(nh^{\,2-1/(2a)}\big).$$

To realize $\mathrm{Tag}(h,h)$, take the union of the n source records and the n proposed target records. Give each source its claimed destination and each target its prescribed position, and radix-sort the union by destination, breaking ties by source before target. Scan the real records, skipping dummies. For each destination in order the scan requires one source followed by one target with equal payloads. An $O(h)$-bit register suffices. This scan rejects repeated or missing source tags and requires no rearrangement of the final dummy slots.

Thus $\mathrm{Tag}(h,h)$ has exponent $p = 2 - 1/(2a)$, with $1 < p < 2$. For width ch, split the payload across boundedly many tagged calls with the same tags, padding when necessary. The reduction from stable sorting to tagged permutations now yields the ordinary-width assertion and completes the proof of Theorem 16.1.

Part III
Concentrating arity on spheres

19 Concentrating the arity of a tensor network

Our aim is to gather the large tables of a tensor network at a few stations while preserving its contraction. The relevant budget is the sum of the squared arities. For example, W/T stations of arity T have budget WT. Gathering D at a time into stations of arity DT would multiply that budget by D. Lemma 19.1 keeps the budget of the large stations at $O(WT)$ while reducing their number to $O(W/(DT))$. Their total arity is then $O(W/\sqrt{D})$. The transformation also introduces small tables. Part IV shortens the wires of this small-table network between successive concentrations, then uses batched table lookup to evaluate the result.

A *station* is a tensor given by an arbitrary table on an ordered list of bits. Its arity counts the external bits as well as its ends of internal wires. Group the stations into cells of a host graph of maximum degree g. Wires run within a cell, between adjacent cells, or to the exterior. The same host may carry several networks. A cell used in two networks counts twice. A station designated *immobile* retains its external bits and its cell. All external bits must belong to such stations.

Lemma 19.1 (Local concentration)**.** *Fix a finite field k with $\omega_k < 9/4$, a degree bound g, and an integer $\ell \geq 1$. For each sufficiently large fixed integer D, the following holds for all sufficiently large integers T. Use an extension K/k containing a fixed finite extension, with $|K|$ at least a sufficiently large polynomial in T and $[K : k] = O_D(\log T)$. The input tables may already be over K.*

Suppose there are at most $W/(DT)$ occupied cells and at most $W/(DT)$ immobiles in total. In any one network each cell has at most $100D$ stations of total arity at most $100DT$, and over all stations of all networks $\sum_i t_i^2 \leq WT$. The tensor of each network is a sum over K of networks with the same exposed bits. Every choice of one term per network, taken together, satisfies the following bounds.

1. *There are $O(W/(DT))$* frozen stations*, including the holders of the original external bits in their original cells. Their maximum arity is $O(DT)$, their squared arities sum to $O(WT)$, and their arities sum to $O(W/\sqrt{D})$.*

2. *All other stations, called the* core*, have arity at most $D^{-\ell}T$ and total arity $O(W(1+\log D)^2)$.*

3. *A common host replaces each original cell by $D^{O(1)}$ positions, with possible contacts only within and between neighboring cells. The induced host on the core positions has absolute bounded degree. No two frozen stations touch by a wire. For some $c = c(g,\ell,k)$ all active internal bundles can have width at least $\lceil T/D^c \rceil$, and the bounds in the first two conclusions continue to hold when every station is charged at least that arity.*

4. *The outer summation uses $O_D(\log(T+1))$ control bits per occupied cell. For its computational bound, represent each input table by a label and its ports by ordered bundles, one per station pair, specified by lengths and endpoint slots. Bits agree in order along a bundle. Any local reordering is part of the supplied table label. Given a fixed coloring with $O_D(1)$ colors separating cells within a bounded distance depending on D, g, the validity of the indices, output layout, and tracking of exposed sublists can be determined in $O_D(1)$ rounds of communication on the old host. With input labels of at most A bits, the work and communication per occupied cell are $O_D(A + (1+\log T)^{O(1)})$ bit operations and bits. Each output table has a label of length $O_D(A + \log(T+1))$ in a dictionary fixed by the input dictionary and the parameters.*

The implicit constants in the first three conclusions can depend on k, g, ℓ, independently of D, T, W. A fixed change of the numerical input bound 100 *changes only these constants.*

The proof uses two sphere forms in each occupied cell. The first joins neighboring cells along pairs of legs called *seams*. It represents a station by an insertion at a marked point. The energy of this insertion bounds the square of the width needed to encode its state. Moving insertions across seams and merging them at common marks will concentrate their energy at the frozen stations. The seams have a separate energy budget. A crossing spends some of this budget, and at the end each seam is divided among many small tables.

To begin the motion we must realize the actual wires of the input network. The second sphere, which stays in its cell, serves this purpose. Corollary 14.4 allows its insertions to encode the prescribed wiring on the first sphere. Both sphere legs participate in mergers, but only the first can cross a seam. The two spheres use different numbers of species and different energy scales. These choices make the private leg left behind at a crossing thin.

19.1 Sphere forms and a transposition identity

We use the algebraic free-fermion construction of Section 13, now with many species. Write $V_e^{(r)}$ for the energy-e subspace of the exterior algebra on generators $\psi_{s,m}$ with species $1 \le s \le r$, mode $m \ge 0$, and energy $m + \frac{1}{2}$. Use the wedge basis in lexicographic order on (s,m). Every basis vector has length congruent to $2e \pmod 2$.

Put $c_0 = \pi/\sqrt{3}$. For $q > 0$ the single-species generating function is $Z(q) = \prod_{m\ge 0}(1 + \exp(-q(m+1/2)))$. Since $t \mapsto \ln(1+\exp(-t))$ is convex, integration on the successive

intervals of length q gives $\ln Z(q) \le q^{-1}\int_0^\infty \ln(1+\exp(-t))\,dt = \pi^2/(12q)$. For r species, $\dim V_e^{(r)} \le \exp(qe)Z(q)^r$. Minimizing over q for $e>0$, and using the vacuum for $e=0$, gives

$$\dim V_e^{(r)} \le \exp(c_0\sqrt{re}). \tag{19.1}$$

We encode a leg by the binary index of its basis state, padding unused codes with zero weights.

At distinct points $z_i \in K$, the sphere form C_r sends a tuple of wedges to a Pfaffian. List their generators in site order. Pairings at a common site or between different species vanish. For a single species the pairing of modes m,n at distinct sites i,j, in that order, is

$$[x^m y^n]\,(z_j + y - z_i - x)^{-1}.$$

The Pfaffian of the empty matrix is one and an odd list gives zero. Exchanging adjacent homogeneous sites of energies e,d costs the sign $(-1)^{(2e)(2d)}$. In particular all such signs are scalar when the energies are fixed. These definitions include characteristic two by reduction of the integral alternating pairing formula.

If two adjacent sites in that order at $z, z+u$ have energies e,m, respectively, their coefficient at $u^{e'-e-m}$ is obtained by replacing the pair by one site of energy e' at z, precomposed with the map

$$I_{e'} : V_e^{(r)} \otimes V_m^{(r)} \longrightarrow V_{e'}^{(r)}.$$

Only $e' \ge 0$ with $e'-e-m \in \mathbb{Z}$ occur. To compute $I_{e'}$, sum over partial internal pairings between the two sites at offsets $0,1$, translate the unmatched generators by expanding around zero, and wedge them in order at energy e'. More precisely, the translation at offset a replaces $\psi_{s,d}$ by $\sum_{f\ge d}\binom{f}{d}a^{f-d}\psi_{s,f}$. This is the fusion formula of Proposition 13.1 extended to many species. Its proof decomposes a full matching into its internal and external pairs and applies species by species. As a function of any one position, the form has poles only at the other positions and is bounded at infinity.

To join two spheres, contract the legs at two specified sites using equality on the wedge basis at an energy e. We call the pair a *seam*. To move an insertion through the seam, we need to transpose a fusion coefficient against this ordinary equality pairing.

Lemma 19.2 (Transposition at a seam)**.** *Let a,b be basis states at energies e,e', and $x \in V_m^{(r)}$. There are parity-preserving maps $S_j : V_m^{(r)} \to V_j^{(r)}$ for $j \le m$, independent of a,b, and a nonzero scalar $\alpha = \alpha(e,e',m)$ such that*

$$[I_{e'}(a,x)]_b = \alpha \sum_{j\le m} [I_e(b, S_j x)]_a. \tag{19.2}$$

The coefficient is taken in the indicated wedge basis.

Proof. Include a square root i of -1 in the fixed extension of k. Use y as the local coordinate at infinity, so the position is $1/y$. Before extracting a mode as a coefficient in y, multiply the kernel once for each generator there by i/y. The pairing between finite coordinate U and that chart is then $i/(1-Uy)$. Consequently the two-point form at zero and infinity pairs equal basis states of length d with coefficient

$$(-1)^{d(d-1)/2} i^d = \begin{cases} 1 & d \text{ even}, \\ i & d \text{ odd}, \end{cases}$$

and all other coefficients vanish. In the three-point form at $0, 1, \infty$ on a, x, b, first take the internal matchings at zero and one. Pairing the remaining generators with infinity performs exactly the translation and wedge defining $I_{e'}(a, x)$. Thus the three-point value is its b-coefficient times the displayed parity factor.

Inverting the finite coordinate by $V = 1/U$ preserves the two-point kernel when each argument receives the factor $-i/U$. Indeed $(-i/U_1)(-i/U_2)/(1/U_2 - 1/U_1) = 1/(U_2 - U_1)$. The old infinity chart becomes an ordinary finite chart with factor one, and the old zero becomes an infinity chart with one minus sign per generator. At the middle site, take the exterior power of the single-generator substitution

$$\psi_{s,d} \longmapsto \sum_{f=0}^{d} \left([X^d] \, \frac{-i}{1+X} \left(\frac{1}{1+X} - 1 \right)^f \right) \psi_{s,f}.$$

Projection to energy j gives S_j. Each term retains the length and has energy at most m. Reverse the site order and evaluate the three-point form with b at zero and a at infinity. The order reversal, the signs from the old zero, and the two diagonal parity factors depend only on the three original parities. Taking their ratio proves (19.2). □

For one species take $a = 1$ and $b = x = \psi_0$. At energy $1/2$ the near-side fusion passes x unchanged. After inversion the middle insertion is $-i\psi_0$, and its far-side contraction with b at offsets $0, 1$ gives $-i$ at energy zero. The factor $\alpha = i$ recovers the original coefficient one. The same exchange of input and output levels will carry a general insertion across a seam.

20 Planting the wiring at marked points

A station in the input may have wires to many other stations. To move it as one insertion, we encode its ends of those wires in a single leg. The sphere form must then test whether the bits at opposite ends agree. We implement that test separately in each occupied cell of each network.

There are two sphere forms in a cell. The form A carries the station legs and the seams to neighboring cells. Extra marks on A supply a joint state that tests the wiring. The private form B realizes this joint state through a separate insertion table at each extra mark. Thus, after planting, each insertion has just one leg on each sphere, along with any exposed or control bits. At original stations and seams the B leg is the vacuum. Figure 7 shows the two kinds of insertions.

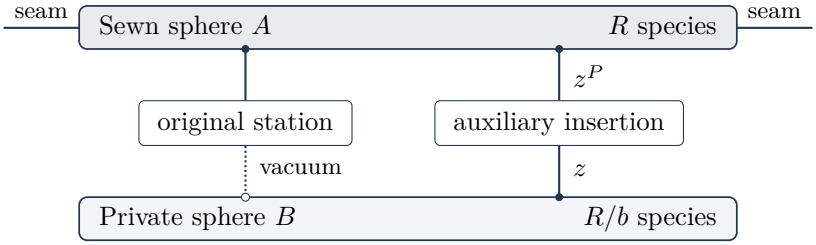


Figure 7. The station table encodes its incident internal wire bits on A. The auxiliary tables couple A to B and together impose equality along the wires. Only A is sewn to neighboring cells.

A mark at z on B corresponds to the mark at z^P on A, where P is a power of the characteristic. Moving z by t moves z^P by t^P. In units of the Laurent order on A, the energy of the pair is therefore $e_A + P^{-1}e_B$. A given energy budget allows P times as much energy on B. We keep the corresponding leg narrow by using fewer species on B. This will bound the width left behind when a station crosses a seam.

Fix g, ℓ as in Lemma 19.1. Let F be the first characteristic power at least $D^{1000(\ell+1)}$ and let $a = \lceil F^3 \rceil$. For a chunk count M and a sufficiently large characteristic power $b > F$, take R to be the largest multiple of bM at most T/a. Set

$$R' = R/b, \qquad P = b/F, \qquad \kappa = P^{-1}, \qquad H = Ra^2. \tag{20.1}$$

Here R and R' count the species on A and B, respectively, and $T/(2a) \le R \le T/a$ for sufficiently large T. If $e_A + \kappa e_B \le h$, Equation (19.1) bounds the widths by constant multiples of $\sqrt{Rh}$ and $\sqrt{R'Ph} = \sqrt{Rh/F}$. In bits, up to rounding,

$$w_A = O(T\sqrt{h/H}), \qquad w_B = O(T\sqrt{h/(FH)}). \tag{20.2}$$

20.1 The wiring form for one species

Combine the wires between each pair of distinct stations into one ordered bundle, absorbing endpoint permutations into the station tables. Contract self-loops in those tables first. Split each internal bundle into packets of R bits, padding the last packet with zeros. Species s will carry bit s of every packet. A station meets $O_g(D)$ bundles, and a cell together with its incident edges meets $O_g(D^2)$. At a station of arity t_i, one species therefore carries at most $t_i/R + O_g(D)$ bits.

Choose distinct wedge basis states at an integer energy h_i to encode the bit strings at station i. The logarithm of the single-species dimension is asymptotic to $(\pi/\sqrt{3})\sqrt{h_i}$ through integer levels by Lemma 13.2. Since $R \ge T/(2a)$ and $a \ge D$, we may take

$$h_i = O_g(a^2(1 + t_i/T)^2).$$

At seam j the packets across the interface contribute $O_g(DT/R + D^2) = O_g(Da)$ bits to one species. An integer energy $q_j = O_g(D^2a^2)$ suffices. The two cells use the same bit order and the same basis codes there. For fixed D the packet counts per species, energies and codes range over finite sets.

Define the multilinear form L on these station and seam levels by its coefficients. Give a basis tuple coefficient one if all codes are valid and both ends of every bit agree. For a wire to another cell, compare its station end with its seam end. Every other coefficient is zero. We will sew a multispecies seam at a fixed *total* energy. To recover the correct energy in each individual species, put $Q = \sum_j q_j$ and extend L by zero to all other tuples of nonnegative half-integer seam energies of sum at most Q. The station energies stay fixed.

Lemma 20.1 (Auxiliary marks of bounded weight)**.** *Inserting a joint state at polynomially many extra marks on a single-species sphere realizes the extended form L. At each extra mark the state uses only the vacuum and ψ_0. In every word in its support the number of occupied marks is even and at most $O_g(Da)$.*

Proof. A wedge of length d has energy at least $\frac{1}{2} + \frac{3}{2} + \cdots + (d - \frac{1}{2}) = d^2/2$. At the stations the total length is $O_g(a\sum_i(1 + t_i/T)) = O_g(Da)$. At the at most g seams the same bound

follows from Cauchy–Schwarz and their total energy bound $Q = O_g(D^2a^2)$. List all pairs (v, m) of original marks and modes occurring in the resulting input spaces. The list has polynomial size in D, a.

An insertion of ψ_0 at a new point z pairs with those modes by the covector with coordinates $(z - z_v)^{-m-1}$. Distinct principal parts give linearly independent rational functions in every characteristic. Choose distinct regular points whose covectors span the dual mode space. Indeed, a nonzero linear combination annihilated by the points already chosen is nonzero at some unused point of the algebraic closure. At most one point per listed mode is needed, all in a finite extension. Add vacua if necessary to obtain at least three auxiliary marks.

For a subset of d new marks, insert ψ_0 there and the vacuum at the others. Consider first the part of the resulting Pfaffian in which every auxiliary generator pairs with an original generator. This part is itself a linear combination of insertion tests on at most d marks. Indeed, order all auxiliary generators last and group the matchings by their exact set of internal auxiliary pairs. Each nonempty set contributes its contracted scalar, with a sign depending only on those paired positions, times the part on the remaining marks with no internal auxiliary pairs. Subtracting these contributions inductively isolates the desired part.

Now take a nonzero vector in the even part of the direct sum of input spaces, and let d be its greatest nonzero total wedge length. A test on d marks with no internal auxiliary pairs vanishes on every shorter wedge. On the length-d component it evaluates the exterior product of the chosen d covectors, up to a common sign. The original basis tuples are distinct basis wedges in the exterior algebra on the list of pairs (v, m). Some choice of d covectors therefore detects the component. The resulting even tests span the full dual of the even part. Each actual insertion used has even weight at most $d = O_g(Da)$ and vanishes on odd vectors. The prescribed integer energies give even lengths, so the zero extension of L also vanishes on the odd part. A linear combination of insertion tests now gives the required joint state. □

Use one copy of this state in each of the R species. At station i the insertion table enforces level h_i in every species, decodes the packets, checks that padded bits are zero and supplies the original table entry, with its exposed bits. At seam j we sew by basis equality at total energy Rq_j. To see why this recovers the original wiring, group the Pfaffian by species. Order the original and auxiliary marks in two consecutive lists and reorder each list by species. The auxiliary reordering sign can be absorbed in the joint state. Interleaving the lists costs no further sign because every species has even auxiliary weight. The original reordering sign depends only on the input wedges. In any nonzero product after the auxiliary state has been contracted, the seam tuple of each species is either the prescribed tuple or has sum greater than Q, by the zero extension. The sums over all species add to RQ, so every species has the prescribed tuple. All the original wedges then have even length and the remaining reordering sign is one. The product tests exactly the equality of the bits along each wire. We also place one vacuum collector in each cell for the motion step.

20.2 Realizing the joint auxiliary state

Lemma 20.1 prescribes one tensor shared by all auxiliary marks. For motion we need a separate insertion table at each mark. The private sphere supplies the shared tensor through local maps. One B species will serve a block of b species on A.

Partition the R sewn species into R' consecutive blocks. A block has *type* $\tau = (d_i)_{i=1}^{m}$ if d_i of its modes ψ_0 are occupied at auxiliary mark i. Projecting the block tensor onto these local

subspaces gives a tensor J_τ, including the sign for reordering within the block. Its ith leg has dimension $\binom{b}{d_i}$. For a fixed ordered word of block types, the joint tensor is the product of the J_τ up to a known sign. Exchanging two site lists from different blocks contributes only the product of their lengths modulo two to the sign exponent. Retain the types with $J_\tau \neq 0$.

Write H_2 for binary entropy in bits, with $H_2(0) = H_2(1) = 0$. Round each $H_2(d_i/b)$ up to a strictly larger point u_i of a fixed positive grid of mesh $1/m$. Then

$$\log_2 \binom{b}{d_i} \leq b u_i, \qquad \max_i u_i \leq 2, \qquad \sum_i u_i = O_{g,\ell}(Da \log D). \tag{20.3}$$

Indeed, Lemma 20.1 gives $s = \sum_i d_i/b = O_g(Da)$. Concavity bounds the sum of the entropies by $s \log_2(em/s)$, with value zero at $s = 0$. The m marks are polynomially many in D for fixed ℓ and characteristic, and rounding adds at most one to the sum.

At the corresponding B marks, take the direct sum S_τ of all nonzero single-species sphere forms with energy at most $C(bu_i)^2$ at site i, where C is an absolute constant chosen below. For sufficiently large b there is a constant c_τ such that for every $d \geq 1$ a product of local linear maps gives

$$\bigoplus_{s=1}^{c_\tau} S_\tau^{\otimes d} \longrightarrow J_\tau^{\otimes d}. \tag{20.4}$$

The independence of c_τ from d will let one selector index serve every occurrence of the type.

For the comparison, take natural logarithms and work over the algebraic closure, where the exponent is still strictly below $9/4$. For a normalized spectral price at $x_1, \dots, x_m$ put $M_u = \inf_o \sum_i u_i\, d(o, x_i)$. Preparing J_τ at o and transporting its legs bounds its price by $b(\log 2) \sum_i u_i d(o, x_i)$ by Equation (20.3). Taking the infimum gives $b(\log 2) M_u$. Applying the same transport bound to the size-two diagonal, whose price is $\log 2$, gives $\inf_o \sum_i d(o, x_i) \geq 1$. Hence $M_u \geq \min_i u_i > 0$. Apply Corollary 14.4 at scale $N = b$ with width caps $c_0\sqrt{C}\, u_i$. Its comparison tensor is precisely S_τ, whose price is $b(c_0\sqrt{C}\, M_u + o(1))$ uniformly in the normalized price. Choose $c_0\sqrt{C} > 2 \log 2$. For fixed D there are finitely many grid vectors and choices of A positions, all defined over a fixed finite extension. Taking Pth roots permutes that field, so the possible B positions also range over a finite set as b varies. It follows that S_τ strictly dominates J_τ in every price for all sufficiently large characteristic powers b. The field-extension and strict comparison results of Section 3 give Equation (20.4) over the fixed field of definition. The maps extend to the field of the input tables.

Fix the type word. For each type of multiplicity $d > 0$, expand the restriction in Equation (20.4) by choosing one copy index s and one B-energy tuple for each of the d factors. Assign the factors to private species in the original block order. Their product is the multispecies sphere form up to the sign for permuting the homogeneous lists. Each site map now supplies a local table coupling the A and B legs at that mark. It may act jointly on all blocks of a given type *at that mark*. Summing the choices and type words produces the required auxiliary tensor. The B legs at the original stations and seams stay in the vacuum.

Control and storage. A *private selector* records the ordered type word, the per-block B energies and one index s per occurring type. A single table checks validity, including zero padding of unused slots, and supplies the permutation sign. The types and twice the energies cost $O(m \log(2b))$ bits per block, since $d_i \leq b$ and the doubled energies are $O(b^2)$. The copy indices cost $O_{D,b}(1)$ bits in total. With R/b blocks, choosing first b and then T large makes the selector shorter than ϵT for any fixed $\epsilon > 0$.

On A every occupied auxiliary mode costs energy $1/2$, giving total auxiliary energy $O_g(RDa)$. On one B species the sum of the caps is $O(b^2 \sum_i u_i^2) = O_{g,\ell}(b^2 Da \log D)$ by Equation (20.3). Multiplying by R' and converting to A units gives the cell bound

$$\sum_i (e_{A,i} + \kappa e_{B,i}) = O_{g,\ell}(RDa + FRDa \log D) \le DH$$

for sufficiently large D, since $H = Ra^2$ and $a \ge F^3$. For storage, use the actual spaces of the selected block. At mark i either leg needs only $O(bu_i + 1)$ bits, by Equations (20.3) and (19.1). Over all blocks, each auxiliary insertion needs $O(R) = O(T/a)$ bits for these two legs. Summed over the marks, this costs $O_{g,\ell}(DT \log D)$, taking $b \ge m$ to absorb rounding.

The selector is summed inside the network, but the outer layout must know the energy and width at each mark. Split the blocks into M equal consecutive chunks. In the outer sum, specify for each chunk the histogram of its pairs (type, B-energy tuple). A histogram records the multiplicities of the pairs, while the private selector records their order. The alphabet is fixed independently of T, so the histograms cost $O_{D,b,M}(\log(T+1))$ bits per cell. They fix the energy and the concatenated code length of each leg in each chunk. The table checks the histograms against the selector and uses its block order to decode. Each valid selector occurs in exactly one outer term. Thus M can be chosen for the chunk layout, then ϵ for its selector routing, and finally b for the comparison and selector width, all before T grows.

21 Moving insertions through a sewn surface

The planting construction expresses the original tables on pairs of spheres in each cell. We now combine these tables at a small number of marks. Two outcomes of a Laurent expansion suffice. At an insertion mark, a coefficient fuses the moving table with the table already there. At a seam, a sufficiently negative coefficient transfers the moving table to the neighboring cell and lowers the energy of the seam. This decrease bounds both the number of crossings and the amount of energy that any cell can receive.

In each cell, designate the marks of the original immobile stations and one additional vacuum mark as *collectors*. They stay in their cell. Process the original movable stations in turn, skipping those already absorbed. The moving table keeps its ordinary bit ports. If its current legs on the sewn sphere A and private sphere B have energies m_A, m_B, put

$$m = m_A + m_B/P, \qquad \mu = m + H, \qquad L = \lceil 2P\mu \rceil.$$

Here P is the exchange rate of Equation (20.1). The reserve H stays with this move from its start until it ends, while m is updated after every fusion or crossing.

21.1 Reconstructing an evaluation from collisions

Fix the branch of the outer sum and, for the following identity, the private selectors. The branch determines the energies, so the same identity works for all allowed selector values. Hold all other positions and insertion tables fixed. Each entry of the contraction is a rational function of the mover's B-coordinate z. Its A-coordinate is z^P. The sphere kernels have poles only at other marks on the current sphere and are bounded at infinity. Near a mark with B-coordinate z_0, write $s = z - z_0$. Since P is a power of the characteristic, the corresponding coordinate difference on A is s^P. Hence an A-fusion of order i and a B-fusion of order v

contribute to order $j = Pi + v$ of the contraction. At an insertion of weighted energy m_0 the merged energy is

$$m_{\mathrm{new}} = m_0 + m + j/P. \tag{21.1}$$

Indeed each fusion adds its order to the two incident energies, by Proposition 13.1. Nonnegativity of the two output energies bounds i and v below. Thus the pole orders are bounded, and only finitely many pairs (i, v) contribute at each j.

To reconstruct the value at the current position, use the following coefficients at the other marks of the cell.

- at every insertion, its entire principal part;
- at the designated vacuum collector, also the orders $0, \dots, gL$, even after this collector has absorbed a table;
- at each seam, only the orders $j \le -L$ within its pole bound.

A function on which all these coefficients vanish has poles only at the at most g seams, of order at most $L - 1$ each. It has a zero of order at least $gL + 1$ at the designated collector and no pole at infinity. The degree of the divisor of a nonzero rational function on the projective line is zero, so this function vanishes identically. On the space with the prescribed pole bounds, evaluation at the mover's position is therefore a linear combination of the listed coefficients. This combination acts entrywise, independently of the tables.

A coefficient at an insertion is a sum over the compatible pairs (i, v). For each pair, apply the two fusion maps and contract them with the moving and stationary tables, including the sign of their chosen order. A collision with a collector ends the move. At any other insertion only $j < 0$ is used, so the combined table continues with at most the sum of the two original weighted energies. This step can absorb an auxiliary insertion, an original station awaiting its turn, or a private insertion left by an earlier crossing.

21.2 Crossing a seam

Suppose the selected coefficient has order $j \le -L$ at a seam of energy e. Only sphere A is sewn there. The insertion on B at the seam is the vacuum, so the B-expansion has Taylor orders $v \ge 0$. For each such v with $i = (j - v)/P$ integral and in the fusion range, the new A-energy at the seam satisfies

$$e' = e + m_A + i \le e + m_A - 2\mu \le e - \mu. \tag{21.2}$$

Terms with $e' < 0$ vanish. With i fixed, interpolate the B Taylor coefficient using evaluations at fresh regular positions. Each resulting term has one private insertion of energy m_B left behind, with vacuum on A. Its index is carried to the original table by an ordinary equality wire. We call this private insertion the *residue* of the crossing.

To identify the new moving table, fix its old A-state x by linearity. Let a, b range over wedge basis states of energies e, e', respectively, and denote the two remaining sphere forms by $C_{\mathrm{near}}, C_{\mathrm{far}}$, suppressing their spectator arguments. Lemma 19.2 gives the identity

$$\begin{aligned} &\sum_b C_{\mathrm{near}}(b) \sum_a [I_{e'}(a, x)]_b \, C_{\mathrm{far}}(a) \\ &\quad = \alpha \sum_{t \le m_A} \sum_b C_{\mathrm{near}}(b) \sum_a [I_e(b, S_t x)]_a \, C_{\mathrm{far}}(a). \end{aligned}$$

On the right, the inner sum is the Laurent coefficient of order $e - e' - t$ for an insertion $S_t x$ approaching b on the far-side A-sphere. This order is integral, since $e' - e - m_A = i$ and $m_A - t$ are integers for nonzero terms. Interpolate the coefficient by far-side regular evaluations, with vacuum in the corresponding B-position. The sum over b sews the two spheres at energy e'. The new moving table, now on the far side with energies $(t, 0)$, retains its ordinary ports and the wire to the residue (Figure 8). All reordering signs depend only on the specified energies, and the identity remains valid when x is coupled to the private index or to a spectator.

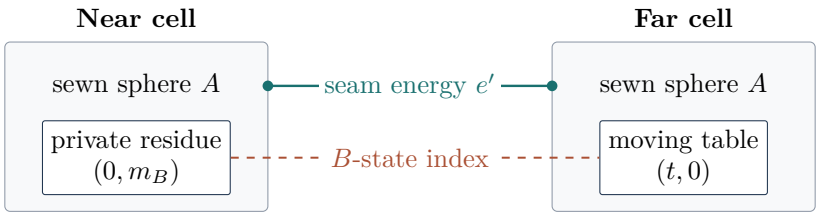


Figure 8. A moving insertion of energies (m_A, m_B) crosses from the near cell to the far cell. The solid wire identifies seam states of energy $e' \le e - \mu$. The dashed wire carries the old private state to the new moving table, whose sewn energy is $t \le m_A$.

Both interpolations can be made over the same finite field K as the original contraction. Multiplying by a common denominator embeds the space of rational functions bounded at infinity with prescribed poles into the polynomials of degree at most the total pole bound. Evaluation at more than that many regular field points is injective, so these evaluations span the dual and recover each required Laurent coefficient. Exclude all occupied positions. The map $z \mapsto z^P$ permutes K, so regular evaluations can use corresponding positions on both spheres. The accounting below bounds the degrees by $O_D(T)$ and the number of occupied positions per cell by a polynomial in D. One can first obtain those bounds over the algebraic closure and use them to choose K before any interpolation. Fix an ordering of the field elements and, for each allowed set of regular points in this order, store the entries of one reconstruction matrix in the table dictionary. The local control need only name the selected point and check its availability.

Allow at most $U = D^5$ crossings in a move. When this limit is reached, its current mark becomes a collector. Nonterminal fusions between successive crossings remove one stationary insertion each. A completed move never releases a new mover, so this procedure terminates in every summand.

21.3 Accounting for motion

We first bound the energy stored in the final insertions. An original station of arity t_i has initial weighted energy $O((1+t_i/T)^2 H)$. The auxiliaries contribute at most DH per occupied cell. Also allocate one reserve H to each original movable station. The initial insertion energy and reserves are $O(D^2 H)$ per cell. Over all cells, including all networks in the batch, the bound is

$$O\left(H\left(D\,\#\{\text{occupied cells}\} + \#\{i\} + T^{-2}\sum_i t_i^2\right)\right) = O(WH/T).$$

A continuing fusion cannot increase the sum of participating energies, by Equation (21.1). Keep only the reserve of the current move. In a crossing the energy m pays for $t \le m_A$ on the far side and m_B/P at the near-side residue. At a final fusion the collector may gain up to $m + gL/P \le (1+3g)\mu$, since $H \ge 1$ and $P \ge 1$. Count energy at collectors with weight $1/(1+3g)$ and energy elsewhere, including reserves, with weight one. The sum cannot increase at any of the three operations or when a move expires. The total insertion energy in every resulting summand is consequently $O_g(WH/T)$.

There is also a bound in each cell. A crossing brings in at most $t + H \le \mu$ of charge. By Equation (21.2) the seam loses at least this much energy. A cell has at most g seams, each initially of energy $O_g(D^2H)$. Adding the initial charge within the cell bounds its current insertion energy by $O_g(D^2H)$ at every step. Moreover, each crossing decreases the seam energy by at least H, so throughout the construction a cell has $O_g(D^2)$ crossing incidences. Only these incidences can introduce new stationary residues. With $H = O_D(T)$ and P fixed, these energy and mark bounds give the asserted $O_D(T)$ total pole bound. The number of expired moves over all cells is at most the total number of crossings divided by U. Including the initial immobiles and designated collectors, the total number of collectors is $O(W/(DT))$.

It remains to route the ordinary wires created in the expansion. By Equation (20.2), the private state left at a crossing uses $O_g(DT/\sqrt{F}+1)$ bits. One endpoint follows the mover. The other stays at the residue until another insertion absorbs it, then travels with that insertion. A residue that can still be absorbed carries only the endpoint created at its own crossing. Once a moving endpoint finishes a move, its collector stays fixed. A move visits at most $U+1$ cells counting repetitions, collecting at most $O_g(D^2)$ residue endpoints per visit and creating at most U of its own. Hence it carries $O_g((U+1)D^2)$ endpoints along any one hop. Using the bound on incident crossings and including terminations gives the looser bound $O_g((U+1)D^6)$ local wire segments per cell. Successive motions within a single cell require only one local connection. With F from Section 20, these segments have total width $O(D^{-2\ell-1}T) + O_D(1)$ there.

The private-selector wires start at marks that read them and at the stations checking the selectors and evaluating the sphere forms. Their endpoints follow the same paths. Each cell starts with polynomially many such ports in D and the number of chunks, so the path and crossing bounds give a polynomial bound in the same quantities on incident selector segments and copies. This count is independent of b and of the selector rate. After fixing the number of chunks, choose the rate small enough that the total selector width, including equality copies, is at most $D^{-2\ell}T$ per cell. The next section converts the remaining energy levels into table arities.

22 Accounting for the new arities

After the motion, each collector has two homogeneous sphere legs and may also hold external bits or thin wires. A seam can still have large energy. We divide the species into groups so that each group meets a small table, even when one group receives the entire energy of a seam.

Choose M, a power of D specified below. Both species counts R, R' can be taken divisible by M in Equation (20.1). Divide the species of each sphere into M consecutive equal chunks, preserving whole blocks in the private construction. Expand each remaining homogeneous leg as a direct sum over its tuple of chunk energies, using the prescribed tuple at each untouched

auxiliary insertion and the same tuple on both sides of a seam. Every basis state occurs once in this expansion. For fixed chunk energies, the sphere form factors into the chunk forms. Reordering the generators contributes a single sign determined by the parity of each chunk at each site.

Consider a generic leg of energy e on r species, and write e_j for the energy in chunk j, so $\sum_j e_j = e$. By Equation (19.1) and Cauchy–Schwarz, the total coded width satisfies

$$\sum_{j=1}^{M} \left\lceil \frac{c_0}{\ln 2} \sqrt{(r/M)e_j} \right\rceil \le \frac{c_0}{\ln 2} \sqrt{re} + M. \tag{22.1}$$

A layout can use the integer width $\lceil 4\sqrt{(r/M)e_j} \rceil$ instead. This amounts to a rounded integer square root since $2e_j$ is an integer, and the same bound holds with coefficient four.

An untouched auxiliary leg uses the actual product subspace specified by its selector. Within a chunk, the total number of bits can be read from the counts of block types and block energies, since a block of type $(d_i)_i$ with rounded entropy bounds u_i needs at most $O(bu_i)$ bits on leg i. Under a fixed selector these codes inject into the corresponding tensor product of homogeneous spaces. The table for the chunk form decodes them and assigns weight zero to unused words.

One arbitrary table implements the form at each cell and chunk on the first and, separately, the private sphere. The number of legs in a cell is polynomial in D. Equation (22.1), applied to a single chunk, bounds a generic port there by $O(DT/\sqrt{M}+1)$, and the bounds on actual auxiliary subspaces give $O(R/M+1)$ per auxiliary leg and chunk. Thus the total arity at one of these tables is at most $D^{c_1}T/\sqrt{M} + O_D(1)$, before the small selector wires are supplied. The exponent c_1 can be chosen independently of M, b and the selector rate. Take M to be a sufficiently large power of D to put this quantity below $D^{-\ell}T/10$. Use selector rate small enough for the routing bounds of Section 21 with this M, then choose the block size b in the planting. This order permits all three choices to be made before T grows.

Frozen stations. The collectors become the frozen stations. At one collector of weighted energy e, the total width of its two sphere legs is $O(T\sqrt{e/H}) + O(M)$ by Equations (20.2) and (22.1). The motion bounds give $\sum e = O(WH/T)$ over all collectors and $e = O(D^2H)$ at any one of them. Thus the energy term in the width has squared sum $O(WT)$ and maximum $O(DT)$. The sum over collectors of the squares of their total external widths is controlled by the input bound, since every immobile retains its own external ports. All additional wires from motion and selectors together have total width $O(D^{-2\ell}T)$ per cell. Their squared widths added at collectors cost at most $O(D^{-4\ell}T^2)$ per cell, even if all end at the same collector. Additive rounding costs from constantly many chunks, with D fixed, obey the same bounds for sufficiently large T. Combining the three charges using $(x+y+z)^2 \le 3(x^2+y^2+z^2)$ proves the squared-arity bound, including the original external widths. The collector count and Cauchy–Schwarz give

$$\sum_{\text{frozen}} t_i \le \left(O\Big(\frac{W}{DT}\Big)\, O(WT) \right)^{1/2} = O(W/\sqrt{D}).$$

Core stations. To bound the total arity at the chunk tables, first sum their port widths over all chunks. Untouched auxiliary insertions have total coded width $O(DT(1+\log D))$

per cell by Section 20.2. The seams have at most g ports per cell of total width $O_g(DT)$ after summing over chunks in Equation (22.1). A cell also has $O_g(D^2)$ residual private legs. Each costs $O_g(DT/\sqrt{F}) + O(M)$ across all chunks by Equations (20.2) and (22.1), so all residual legs together cost $O(T)$ per cell for large T. Legs paired with collectors contribute at most the global frozen arity. The ordinary wires from motion, the selector wires, and any scalar tables contribute $O(T)$ per cell. Thus core arity before controlling the host degree is $O(W(1 + \log D))$. Untouched auxiliaries have individual arity $O(T/a)$, the residues have individual arity $O(DT/\sqrt{F})$, and the chunk tables and routing wires have each been made smaller than $D^{-\ell}T$ by a fixed margin. These and the selector checks exhaust the core tables.

To build a common host, reserve slots in each cell for each kind of table and bundle that can occur simultaneously in a single summand. This requires $D^{O(1)}$ slots per cell. The block size changes table entries and selector lengths, while the same slots serve every block type. Combine scalar factors in their cell. Initially allow all connections between slots in the same cell or in neighboring cells. Split any direct connection between collectors with a core identity table on the thin wire carried along their paths. For each core station, attach a fixed balanced binary tree with one leaf per possible bundle. Its root receives the table's ordered ports, and each internal vertex separates its list of bits into the sublists destined for its two children. The corresponding equality tables pass each bit along a unique path of length $O(1 + \log D)$ and no internal arity exceeds twice the original station arity. This proves the asserted degree and total core bounds. Finally take c larger than the exponents in these counts by a margin depending on ℓ. Padding a nonempty bundle to $\lceil T/D^c \rceil$, or making this minimum charge for any one station, then uses at most $O(D^{-2\ell}T)$ extra bits per cell. Unique zero padding preserves the tensor. It also absorbs any bounded number per cell of weight-only tables in these counts. This proves the first three conclusions of Lemma 19.1.

Indices of the outer sum. Encode each bundle by its endpoint slots, its length, and its position in the ordered ports at its ends. The input label already includes any local permutation of bits. No individual exterior address is needed when a bundle is carried as an ordered list. The branching data at one occupied cell comprise the initial chunk histograms, the motion and fusion choices (positions, energies, interpolation points and orders), and the final chunk splittings. Their number of entries is bounded for fixed D, including the size of the alphabet of the histograms. Each integer ranges over $T^{O_D(1)}$. A field point uses $O_D(\log T)$ bits. All private selector bits are wires inside the summand and are excluded from this count. Zero unused entries give a unique padding of a term index.

Use the supplied coloring of the $(4D^5 + 20)$-th power of the host. Greedy coloring gives such a coloring with $O_D(1)$ colors for fixed g. Start motions successively by color and station slot. Concurrent motions affect disjoint neighborhoods, since each has at most D^5 crossings. A complete padded schedule therefore has length bounded in terms of D, g only. In each step the cells may send their local descriptions to their neighbors. A merger concatenates the descriptions of its two input tables and the operation parameters. A crossing adds the named interpolation, the fusion, and the new endpoints of its thin wire. The history of any resulting table draws on a bounded number (depending on D) of these records. Thus a label of length $O_D(A + \log(T + 1))$ specifies each entry as a function of its ports through the fixed dictionary. The dictionary may use arbitrary interpolation weights, restriction maps and internal contractions. Generating its labels requires no evaluation of its entries.

It remains to check that the short computations never require inspecting the selector bits. Counts in a chunk histogram fix the energies and coded widths. Range and total checks admit every realizable histogram; the selector check gives zero whenever an admitted histogram is unrealizable. Interpolation may use a fixed ordered collection of all regular field points and weights indexed by the local pole bounds. The scheduler only checks avoidance of marks, the ranges of the indices, and the rules of fusion and motion. Zero coefficients may be retained. The checks and length estimates use arithmetic on $O_D(\log(T+1))$-bit integers or field coordinates, including division with remainder and rounded square roots. The scheduler copies old labels as strings in fixed padded slots. At each cell the number of records and operations on them is bounded independently of T, giving $O_D(A+(1+\log T)^{O(1)})$ bit operations and bits communicated. This proves the final conclusion of Lemma 19.1.

Part IV
From arity to network volume

23 Reducing the total arity of a tensor power

The concentration estimate of Lemma 19.1 divides a network into a few large stations and a core of small stations. To lower the cost of evaluating the network we must also reduce the *total* arity. Indeed, if u_v is the arity at station v, the batched evaluation in Lemma 16.2 costs, per tensor copy, at most a constant times the number of stations plus

$$\sum_{v:\,u_v>0} u_v^{p} \le \bigl(\max_v u_v\bigr)^{p-1} \sum_v u_v.$$

Here $1<p<2$. We can therefore afford some growth in the largest arity if enough incident bits disappear. The construction below makes this trade repeatedly, keeping a bound on the sum of squared arities so that each new network again meets the concentration hypothesis.

Throughout this part k is a finite field with $\omega_k < 9/4$, and logarithms are binary. A *table network* over a field K has bit wires and a table $\{0,1\}^s \to K$ at each station of arity s. Its size is the number of stations plus the sum of their arities, including external legs. Parallel bits between two stations form an ordered *bundle*. A label names the station's table in a dictionary fixed for the network.

Lemma 23.1 (Compression of a tensor power)**.** *Let $h=\lceil \log_2 n\rceil$, $0<\delta\le 1/10$, and $0<\beta\le b\le 1/2$. For every sufficiently small fixed $a>0$ depending only on k,δ,β, and all sufficiently large n, the following holds. Work over an extension K/k of degree $O(\log h)$ large enough for Lemma 19.1 at the scales used below. Suppose that an input table network has maximum arity h^b or less, $(\log h)^{O(1)}$-bit table labels, size at most m with $n\le m\le n^{1.1}$, and $e\le mh^{-2a}$ external bits. Some d with $1\le d=n^{o(1)}$ admits a table network over K for its dth tensor power, with all de external bits separately accessible, such that*

$$\text{size} \le md\,h^{-(1-\delta)a}, \quad \text{maximum arity} \le h^{b+(1+\delta)a}.$$

The output labels again have $(\log h)^{O(1)}$ bits. The cutoff for n may depend on the fixed exponent in the input label-length bound.

Call one application of Lemma 23.1 a *macro*. Before the macro the lookup bound is $O(mh^{b(p-1)})$ per copy. After it, the bound acquires the factor

$$h^{-a\{(1-\delta)-(1+\delta)(p-1)\}}.$$

Since $p < 2$, choosing δ small makes the exponent negative. Section 27 accumulates this saving before evaluating the tables.

The macro expands a power of the original tensor as a sum of products. Each product has many factors, called *blocks*. We combine blocks in groups of r to pack stations or rearrange their wires. Each round first concentrates the squared arities, then uses a fixed number of wire reductions to shorten the core, and finally restores bounded degree so the next round can start. A wire reduction, called a *beat*, consists of several contractions followed by a reset of the core degree. The number of rounds grows with h. Once D is fixed, all repetition counts *within* a round are constant. Table 1 collects the budgets and the quantities they measure.

Different summands can use different wirings. The bounds in Sections 23–25 apply to each summand. Section 26 represents their sum on one fixed network and completes the proof of Lemma 23.1.

23.1 The common graph and its initial loads

Each summand is a product of *blocks*, with disjoint lists of external bits. In a single summand its N_b blocks use the same simple graph of possible adjacencies, the *host*. An occurrence is a pair consisting of a host position and a block. It can be empty. Each actual internal wire joins stations in the same block, and the local port order tells which bits at its ends correspond. Thus a host edge can carry different bundles in different blocks. We permit the host to depend on the summand.

At a grouping step, relabel the blocks and group them r at a time, with r a power of two. In each group the old blocks are multiplied and transformed together. New positions may be indexed by an old position, a dyadic subinterval of the r indices, and a bounded number of other choices. There are fewer than $2r$ intervals. If H denotes the number of host positions, the padding allowed in any one step will have

$$H' \le crH, \qquad N'_b = N_b/r, \tag{23.1}$$

Quantity	Meaning within one summand
$H,\ N_b$	positions of the common host and blocks using it
r	number of blocks combined at each grouping step
$S,\ T_0$	initial blocks and original copies per block, with $d = ST_0$
$L,\ T$	total arity budget and working arity scale
D	concentration parameter, fixed for all rounds
$X,\ Z$	width between core stations and total frozen charge
$\Delta,\ \nabla$	degree bounds for the core host and the full host
$B,\ u$	maximum packing load and unit of internal width

Table 1. Parameters for the compression of a tensor power. The budgets L, X and Z each cover all blocks of the summand.

where c is fixed throughout all repetitions of that step. Constants here may depend on the round design. Transformations at a single position using a fixed number of slots can be included in the adjacent grouping step.

At the beginning of each round, every live occurrence holds one station. The host has degree bounded by an absolute constant, and external bits are fixed at specified ports. With T a dyadic width and L a total arity budget, the charged arity of each live station of actual arity s_v is $z_v = \max(s_v, \theta T)$, where $\theta > 0$ is a small fixed dyadic constant chosen below. Our invariants are

$$\sum_v z_v \le L, \qquad \sum_v z_v^2 \le LT, \qquad \max_v z_v \le DT/2. \tag{23.2}$$

All three bounds cover all blocks of the summand. The first pays for surviving bits. The second is the input to the next local concentration, and the third lets us pack several stations without overflowing one cell. The floor θT also bounds how many table descriptions can enter a pack of a given charge. Remove every scalar station and retain its entry as a factor of the summand.

First replace the input by a graph of bounded degree. On the ports of each station of positive arity s use a balanced binary tree. The root receives its s bits as a bundle, and an identity at each split divides one ordered bundle in two, of lengths differing by at most one. Place an identity at each leaf and attach the original wires and external bits at the leaves. Connecting two of the resulting leaves also handles an input wire that was a loop. The arities on either end of a tree edge have bounded ratio. The leaf-to-leaf connections have matching unit width. There are $O(\log(2+h))$ levels and $O(m\log(2+h))$ total arity. All the new tables have short labels.

Choose a dyadic $T_0 \in [4h^b, 8h^b]$. Stack T_0 copies of the network just constructed. At a station of arity s let v be the largest dyadic integer at most T_0/s. Divide the copies into consecutive intervals of length v. Give each interval its own position carrying the product of the corresponding tables. Its arity sv belongs to $(T_0/2, T_0]$. Comparable arities on the two ends of each old edge imply that each interval meets only boundedly many intervals at the other end. Each edge inherits the order of the old bundle and then of the copies, so its two endpoints use the same bit order. The stacked graph has bounded degree and $H_0 = O(m\log(2+h))$ positions, since a station of arity s gives at most $2s$ intervals. The table at an interval is specified by the old table label, the interval length, and a bounded number of ordered port lengths.

Take S blocks of this stack and pad H_0 to order $m\log(2+h)$. Include the dth power of the product of the removed input scalars as a factor of the final tensor. The starting parameters are

$$d = T_0 S, \qquad T = T_0, \qquad L = L_0 = CmT_0 S\log(2+h). \tag{23.3}$$

The total charged arity is at most L_0 for a sufficiently large absolute C. Each station has arity in $(T_0/2, T_0]$, so its square is at most T_0 times its arity. This gives all three bounds in (23.2). The ed exterior bits retain their individual names throughout the construction.

24 Reducing the width between unfrozen stations

After concentration, some stations remain frozen for the rest of this round. The others form its *core*. We shall reduce the width between core stations before restoring bounded degree

and passing all the stations to the next round. Each core occurrence carries one tensor. Each frozen occurrence holds a list of tensors and the charge assigned to each when it was frozen. All exterior legs are on frozen tensors. We discard host adjacencies between frozen stations, since they carry no wires.

Let X be the sum, over all blocks, of the widths of the core–core wires, counting each wire once. Let Z be the total frozen charge. The core–frozen wires, called *tethers*, have total width at most Z. For packing, give each core occurrence load equal to its arity and each frozen occurrence load equal to the sum of its charges times a fixed dyadic factor in $(0, 1]$. Write B for an upper bound on one load. The sum of all loads satisfies $\mathcal{M} \le 2X + 2Z$. Each internal width is a multiple of an integer $u \le B$.

The host records possible contacts across all blocks. Write $\Delta \ge 3$ for a bound on its core degree and $\nabla \ge \Delta$ for a bound on its full degree. A core station can meet different frozen stations in different blocks, so even a core of degree three can have many potential tethers. The width estimate below is uniform in their number.

Lemma 24.1 (Reduction of core width)**.** *Fix a bound on the incoming core degree and assume the hypothesis of Theorem 16.1. There is a fixed sequence of grouping and contraction steps, called a* beat*, with the following bounds in each summand, for slack $\sigma > 0$.*

$$X_{\mathrm{out}} \le \tfrac{1}{2}X + C(Z + \sigma), \qquad \Delta_{\mathrm{out}} \le 3, \qquad Z_{\mathrm{out}} = Z.$$

The identity retains all exterior legs separately. The maximum load, the reciprocal bundle unit and $\log(2\nabla)$ *increase by at most fixed factors. These factors and C are independent of the incoming full degree.*

At a step with H host positions and N_b blocks, a power of two, choose dyadic $r \mid N_b$. Grouping leaves N_b/r blocks and uses at most crH positions. It suffices to take $r \ge c_*$, $c_* H N_b B/r \le \sigma$ at each step and a starting unit divisible by the product of the subdivision factors. The position and slack constants c, c_* may depend on the degree bounds at the step in question.

A beat consists of several wire-saving subrounds and one reset. Each subround trades some core width for more possible core contacts. The reset routes the surviving wires through a core of degree three. We shall choose the number of subrounds so that their combined saving pays for this routing.

24.1 A wiring identity

Place six parties in the triangular array of Figure 9. A pair is *permitted* if it lies in one row or one column. The desired equality wires AB and CD lie in neither. We shall express their product as a sum of networks on permitted pairs and then contract an entire row or column partition. This leaves a choice of two layouts for the same summand.

An s-bit wire is the equality tensor on the two s-bit indices. For a scale-one ordinary price over k, let m be the greatest price-distance on a permitted pair. A spanning tree of the permitted graph supplies one bit shared by all six parties, by equating its incident bits at each vertex. Thus $1 \le 5m$. Both demanded distances are at most $2m$. If $d(C, D) = 2m$, the paths C, M, D and C, N', D make M and N' the same midpoint in a CAT(0) extension. If also $d(A, B) = 2m$, then C and $M = N'$ are both midpoints of A, B. This would give $C = M$

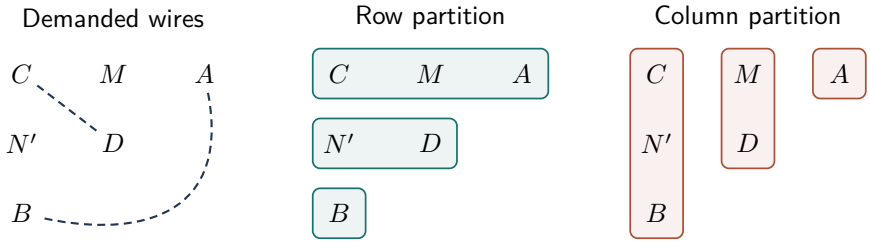


Figure 9. An equality wire between C and M, for example, disappears when the rows are contracted and survives when the columns are contracted. Every permitted wire disappears in one layout. Across all copies of the identity in a summand, the cheaper of the two common layouts retains at most half the replacement width.

although $d(C, M) = m > 0$. Compactness of scale-one prices therefore gives a uniform $\lambda_* < 2$ with

$$d(A, B) + d(C, D) \leq 2\lambda_* m.$$

Choose $\lambda_* < \lambda_0 < 2$. For fixed $\eta' > 0$ and arbitrarily large b, we claim that the two width-b demands admit an ordinary sum of at most $2^{\eta' b}$ networks, each with total permitted width at most $2\lambda_0 b$. Indeed, choose h_0 large enough that $\lfloor 2\lambda_0 h_0 \rfloor > 2\lambda_* h_0$. As supply, take the direct sum over permitted pairs of a single wire on that pair of width $\lfloor 2\lambda_0 h_0 \rfloor$. At each spectral point the price of this supply exceeds that of the two width-h_0 demands. Strict spectral comparison makes the width-b demands, for $b = h_0 r$, a restriction of boundedly many direct copies of the rth supply power. Expanding the direct sums and applying the local restriction maps gives the claimed networks. With E_{perm} the set of permitted pairs, a summand index takes $r \log_2 |E_{\mathrm{perm}}| + O_{h_0}(1)$ bits. Choose h_0 and then r large to fit $\eta' b$. The identity over k extends to every field K/k. For the subround we need a fixed number of choices even when each wire carries many copies of a bit.

Lemma 24.2 (A fixed choice of widths)**.** *There are constants $b \in \mathbb{N}$ and $\lambda < 2$ such that for every $z \geq 1$ the two width-bz demands are the sum of a bounded number of networks on the same six parties. Each uses at most $2\lambda b$ bundles of z bits on permitted edges. Its local tensors are specified by z, the party and the index of the summand. The index set and choices of widths (in units of z) can be fixed independently of z.*

Proof. Raise the finite identity above to its zth tensor power. A term is initially indexed by a sequence of z selectors, with at most $\lceil \eta' b \rceil$ bits per selector. For each permitted edge round up the total number of bits on it to a multiple of z. Allowing empty groups, group the sequences by the vector of rounded numbers in units of z. This vector belongs to a fixed finite set, and rounding costs at most one unit per permitted edge. Within each group choose a spanning tree of the connected permitted graph and send the full sequence on each tree edge, using at most $\lceil \eta' b \rceil$ extra units per edge. A local table checks identical valid sequences on all incident tree edges, membership in the group, and zeros in padding positions. It uses the sequence to split positions on each original edge into the prescribed slices and evaluates the product of the z local tables. Connectivity makes the encodings of a sequence unique once the sequence is specified. Summing the tree variables therefore gives precisely the sum for that group. The total in units of z is at most $2\lambda_0 b + |E_{\mathrm{perm}}| + 5\lceil \eta' b \rceil$. Choose η' sufficiently small and then b sufficiently large to bound this by $2\lambda b$ with $\lambda < 2$. □

24.2 Subrounds

In a subround we group occurrences at each position into packs. Two outgoing units from different packs can play the roles AB and CD. We first arrange enough such pairs, with a bound on how many packs a single contraction can merge.

Balanced partitions. Within each group of r block indices use the dyadic intervals in their common order. Give every host edge e a propagation distance

$$s_e = 64\lceil \log_2(2x_e) \rceil, \qquad x_e = \begin{cases} \Delta & \text{on core edges,} \\ \nabla & \text{on tethers.} \end{cases}$$

At each position mark every interval of load greater than AB, with $2 \le A \le r$ dyadic. Marks require splitting. Propagate each mark to its parent at the same position and to its ancestor s_e levels higher at the other end of each incident edge e, whenever the ancestor exists. The leaves, called colonies, have load at most AB and length at least A. Intersecting colonies at opposite ends of e differ in length by a factor at most 2^{s_e}, by propagation from the parent of the smaller colony.

A common permutation of the indices inside each group gives a total colony count

$$Q \le C_1 \left(H N_b / r + \mathcal{M}/(AB) \right) \tag{24.1}$$

with C_1 absolute. To see this, identify the indices with $\mathbb{F}_r$ in a binary basis and multiply by a uniformly random nonzero element. The probability that two distinct indices fall in one depth-j interval is at most 2^{-j}. At a source position in one group let $x_i \le 1/A$ be the loads divided by AB, and put $x = \sum_i x_i$. An initially marked interval has at least $1/2$ in ordered cross products of distinct normalized loads. Thus the expected number of initial marks at depth j is at most $2x^2 2^{-j}$. The sum of $2^{-s_e/16}$ over incident edges is at most $1/2$ at every vertex. Summing the resulting path weights shows that at most $2 \cdot 2^{s/16}$ positions are reachable with total propagation distance at most s. Fix the total upward displacement s, allowing steps to parents at the same position. At each reachable position, depth-j source marks then reach at most 2^{j-s} distinct ancestors. The expected count there, summed over $j \ge s$, is at most

$$\sum_{j \ge s} \min\{2^{j-s}, 2x^2 2^{-j}\} = O(x 2^{-s/2}).$$

Summing over distances, sources and groups bounds the number of marks in expectation by $O(\mathcal{M}/(AB))$. Each binary partition has one more leaf than marked internal nodes. Choose a multiplier minimizing the actual count.

A wire-saving subround. Orient all core edges. Color each block thick or thin, using one affine functional over $\mathbb{F}_2$ on its within-group index. Every position in a block receives the same color. To pack tensors at one position, take their tensor product and retain their wires. Any loops created by later mergers are contracted into the local table. At each core colony gather all thick tensors in one pack. In order, close a thin pack once it reaches load B. There are $O(A)$ thin packs, each of load at most $2B$. At a frozen colony concatenate the lists. Let $P = 2^{64\lceil \log_2(2\Delta) \rceil}$ and $K_1 = \Delta P$. Pairwise independence of the colors bounds the expected absolute imbalance of thick and thin outgoing widths in a colony by $(B \sum_i a_i)^{1/2}$,

where a_i is the outgoing core amount of its ith occurrence. Summing gives a coloring of total imbalance at most $\sqrt{BQX}$.

For a source colony, group thick outgoing units by their destination pack. There are at most K_1 bins by the length ratio along core edges. Treat the outgoing units of each thin pack as one item and match the entire item into any bin with enough remaining capacity. At the first failure every remaining capacity is below $2B$, so the unmatched part of the smaller total is at most $2BK_1$. The one-sided matched width therefore satisfies

$$I \geq \tfrac{1}{2}X - \tfrac{1}{2}\sqrt{BQX} - 2BK_1Q.$$

The colony bound (24.1) and the slack condition with $c_* \geq A$ give $BQ = O((X + Z + \sigma)/A)$. Take A a sufficiently large dyadic multiple, up to rounding, of $1+K_1$. Then $I \geq X/4-O(Z+\sigma)$ with absolute implicit constant.

Within each bin pair its matched thin units with distinct thick units. Use Lemma 24.2 with $b_* = b$ and $z = u/b_*$, an integer by the subdivision hypothesis. In each pair the thick wire plays AB, with A its source, and the thin wire plays CD, with C its source. The sources share a colony. The destinations B, D belong to thick and thin packs, respectively, because a wire stays in one block and the color is common to that block. Introduce fresh auxiliary parties M, N' for each pair. Across all pairs, the two demanded widths have total $2I$, and the replacement wires have total width at most $2\lambda I$ in each product summand.

In one such summand, add the widths that survive the row contractions over all pairs. Do the same for the column contractions. Every replacement wire is internal to one of the two layouts, so the two totals sum to at most $2\lambda I$. Perform the cheaper contraction in every pair. Further identifications can only decrease the surviving width. The unmatched wires contribute at most $X - 2I$, hence

$$X' \leq X - (2 - \lambda)I \leq \rho X + C(Z + \sigma), \qquad \rho = 1 - (2 - \lambda)/4 < 1.$$

Matching each thin item in its entirety controls the growth in both load and degree. The merged old packs form stars. A thin pack in its source role joins its own thick pack in the row layout or its one chosen thick destination in the column layout. In its destination role it joins only an auxiliary party. Thus one thick center absorbs at most $O((1 + K_1)A)$ thin packs. Every auxiliary is absorbed into a demand endpoint, to which its new incidences can be charged at a constant multiple of the matched width. A dyadic load multiplier g, polynomial in Δ, bounds both the resulting core loads and the scaled loads of the frozen lists. The new unit is u/b_*.

Reserve pack slots at every potential dyadic interval. Possible core contacts run between packs of one colony or of intersecting neighboring intervals with ratio at most P. There are $O(P + \log(2P))$ possible intervals per neighbor for each given interval. The stars and the four matched endpoints have bounded diameter in this graph. Taking a fixed graph power gives a common core support of degree at most $C_2\Delta^{C_2}$. A tether needs just one old tether adjacency with its own length ratio, followed by a bounded number of core adjacencies. Hence the full degree is at most $C_2\nabla^{C_2}$, with absolute C_2 enlarged if necessary. There are fewer than $2r$ potential intervals per position. This gives the claimed bound on positions and keeps tethers attached to their respective frozen tensors.

24.3 Resetting the core degree

Use H, N_b, Δ, ∇ for the parameters immediately before the reset, and put $P = 2^{64\lceil \log_2(2\Delta)\rceil}$, $K_1 = \Delta P$. The reset replaces the core by a degree-three host. Its principal cost is a permutation network at each core base. Fix a trivalent network of size $c_0 Y w_0$ that realizes all permutations of Y rows of width w_0 by fixing its tags, where $4K_1 \le Y$ and $c_0 \ge 1$. Call this network the *rotor*.

Figure 10 shows the two routes. First partition the blocks at each position into intervals called *bases*, using their mean loads, and reserve capacity between each pair of adjacent core bases. Mixing the blocks makes most bases fit these reservations. A wire between two such bases crosses a direct link between dedicated rotor outputs. Every core wire incident to an exceptional base takes a longer route through binary trees. We must bound the total load on those bases before charging the extra hops.

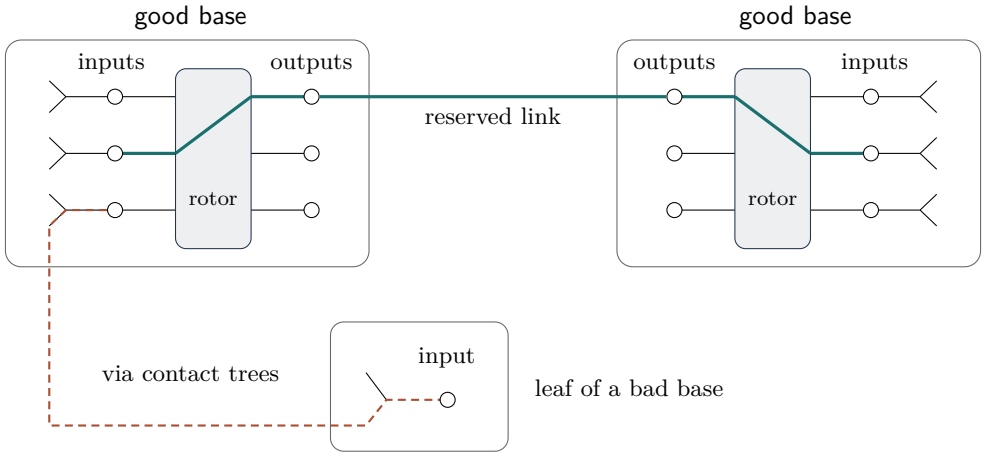


Figure 10. Routing during a reset. Between good bases, each end of a wire reaches a dedicated output through a rotor (solid). A contact with a bad base joins input terminals through binary contact trees (dashed). Old tensors and their tethers stay at the inputs. Only selected terminals are shown.

Bases and capacities. Choose dyadic $F \ge Y^2 w_0$; require $c_* \ge F$ in this step. For the mean load μ_v at each position over the N_b blocks, put

$$\nu_v = \max\left\{ FB/r,\ \max_{\pi : y \to v} \mu_y \, 2^{-\sum_{e \in \pi} s_e/2} \right\},$$

allowing a path of length zero. The sum of the attenuated weights of paths from each position is bounded by an absolute constant, as in the colony count. Hence $\sum_v \nu_v \le HFB/r + C \sum_v \mu_v$. At position v partition into bases of the largest dyadic length $s_v \le FB/\nu_v$, between F and r. All partitions of block indices respect their grouping into consecutive r's. Neighboring base depths differ by at most $s_e/2$. Their total number is $O((\mathcal{M} + \sigma)/(FB))$. For each pair of intersecting bases at neighboring core positions v, w, reserve capacity

$$\overline{y}_{vw} \min(s_v, s_w) + FB/K_1,$$

where $\overline{y}_{vw}$ is the mean core amount of that contact per block. The first terms sum to at most $s_v \mu_v \le FB$ at any base, and there are at most K_1 reservations. Hence total reserved

capacity at a base is at most $2FB$. Base lengths and capacities are fixed from the means before permuting the blocks. In each group the output terminals for a given pair of base positions will therefore occupy the same host slots.

Mixing. Permute all block indices by one invertible binary matrix. Within the bases, mark dyadic subintervals (including possibly the bases themselves) of load above $2FB$, propagating with the same edge distances. Leaves have load at most $2FB$, length at least F, and the previous length-ratio bounds (base splits above the marked region already satisfy propagation by comparison of the base depths; propagate as in colony packing). Call a base bad if it splits or an incident core reservation overflows. For any $\delta_* > 0$, choosing F sufficiently large in terms of Δ, Y, δ_* gives a permutation with

$$\sum_{\text{bad bases}} \text{load} \le \delta_*(\mathcal{M} + \sigma). \tag{24.2}$$

To prove this uniformly in the full degree, count how far one overloaded interval can propagate. For a given position having $n_v = N_b/s_v$ bases, normalize old loads by FB, obtaining $x_i \le 1/F$. At depth j below the bases let a_J be the normalized interval loads and $\overline{a}$ their mean, at most 2^{-j}. Both $V_j = \sum_J (a_J - \overline{a})^2$ and $U_j = \sum_J a_J^2 - \sum_i x_i^2$ bound the number of overloaded intervals and half their (normalized) load. The prefix collision probability for distinct indices under a random invertible binary matrix is at most the reciprocal of the number of intervals. Consequently

$$\mathbb{E}V_j \le \sum_i x_i^2 \le n_v/F, \qquad \mathbb{E}U_j \le n_v 2^{-j}.$$

A mark at depth j can force a base split only along paths with $\sum s_e \le 2j$, since neighboring base depths change by at most half the edge distance. For this source mark, dyadic nesting gives at most one affected base at each reachable position. The path count used in (24.1) therefore bounds the number affected by $2 \cdot 2^{j/8}$. At depth j, use V_j if $1/F \le 2^{-j}$ and U_j otherwise. Weight this estimator by $4 \cdot 2^{\lceil j/8 \rceil}$. Split the sum at $j = \log_2 F$. The expected charge is at most a constant times $n_v \sum_{j \ge 0} 2^{j/8} \min\{F^{-1}, 2^{-j}\} = O(n_v F^{-7/8})$. Bases with load above $2FB$ are paid for directly by the depth-zero estimate. For core capacity overflows use the variance estimator for the contact width divided by FB at the depth of the base intersections. The FB/K_1 buffer above the mean ensures that each overflow contributes at least $1/K_1^2$. Summing the unweighted variances in expectation over core edges costs at most $X/(F^2 B)$. Each overflow affects at most two bases. Those whose load exceeds $2FB$ have already been paid for at depth zero, and every other affected base costs at most $2FB$. Let Φ be the sum of the weighted interval estimators and K_1^2 times the contact variances. We have bounded total bad load deterministically by $O(FB\Phi)$ and have shown

$$FB\,\mathbb{E}\Phi = O(F^{-7/8}(\mathcal{M} + \sigma) + K_1^2 X/F).$$

A matrix with estimator at most the mean proves (24.2). Section 26 finds such a permutation by computing collision scores.

The routes. After mixing, group consecutive runs of r blocks. Give every core leaf Y input terminals. Assign its old tensors whole to a least-loaded terminal successively, so each terminal receives at most $2FB/Y + B \le 3FB/Y$. Keep all tethers at their assigned terminals, with their other ends in the concatenated frozen list. For every good core base the

rotor connects its input terminals to Y output terminals. A reservation of capacity c receives $\lceil cY/(8FB) \rceil$ pairs of dedicated output terminals, joined by direct links. This needs at most $Y/4 + K_1 \le Y$ terminals per base. Divide the units of a good–good contact evenly among its pairs. Their output loads are at most $8FB/Y + u \le 9FB/Y$.

At a good base write m_{ij} for the number of units routed from input i to output j. The row and column sums of $(\lceil m_{ij}/w_0 \rceil)$ are at most

$$9FB/(Yuw_0) + Y \le 10FB/(Yuw_0).$$

Bring every row and column sum to the largest sum by adding edges. Hall's Theorem decomposes the resulting regular bipartite multigraph into perfect matchings. Each matching configures a layer of the rotor with bit edges powered to width u. Pin unused inputs and corresponding outputs to zero. Take the tensor product of the layer tables at each rotor position. This uses only the prescribed equality tensor obtained by pinning the rotor tags. Balanced binary trees at each rotor row collect its w_0 ports, at $O(\log(2w_0))$ hops per unit. Including direct links this costs $O((c_0 + \log(2w_0))FB)$ per good base.

For a core contact with a bad endpoint use balanced binary contact trees attached to its input terminals. The possible neighboring leaf intervals have length ratios at most P, with Y terminals per leaf, giving depth $O(\log(2\Delta Y))$. Route ordered lists through the trees by equality tensors. The assigned lists at both ends give a unique internal assignment to the transmitted bits. Choose δ_* in (24.2) small enough to pay this routing depth. Thus for an absolute A_0,

$$X' \le A_0(1 + c_0 + \log(2w_0))(X + Z + \sigma).$$

This counts every core link in the rotors, collecting trees, direct connections and contact trees. The unit remains u. At every input terminal, output terminal, tree vertex, and rotor vertex the load is $O(FB)$, as follows from the terminal and layer bounds. For a frozen leaf its scaled list load is at most $2FB$. The core has degree at most three. Rotor vertices have this degree; input terminals have at most rotor-row (via one edge to a row tree) and contact-tree connections within the core; output terminals have at most a rotor-row connection and a direct link. Each binary tree has degree at most three. For tethers the common host allows every input terminal at every possible intersecting core leaf of the prescribed length ratio. The resulting full degree is bounded by a fixed power of 2∇, with constants depending on this reset. Reserving this hardware at all possible interval slots costs at most crH positions.

24.4 Paying for the reset

Write

$$\kappa = A_0(1 + c_0 + \log(2w_0))$$

for the width multiplier of the reset. Take t to be the least positive integer such that $\kappa\rho^t \le 1/2$. Repeated use of the subround estimate gives core width at most $\rho^t X + C(Z + \sigma)/(1 - \rho)$ just before the reset. After it, the coefficient of the incoming X is at most one half. The remaining cost is a fixed multiple of $Z + \sigma$, provided we can choose a rotor with $Y \ge 4K_1$ for the degree reached after t subrounds.

Starting at bounded core degree, the recurrence $\Delta' \le C_2\Delta^{C_2}$ gives

$$\log_2(4K_1) \le C^{t+1} \le C''(2 + c_0 + \log(2w_0))^{C'}.$$

Here C, C', C'' depend only on the incoming core degree and the subround constants. For $Y = 2^y$, Theorem 16.1 gives $w_0 = \lceil y^{a_\zeta} \rceil$ and $c_0 = O_\zeta(y^\zeta)$ for each $\zeta > 0$. Fix $\zeta < 1/(2C')$, then take y large. The displayed polynomial bound is at most y, so this rotor covers the required core degree.

The number of subrounds t, the rotor, F and δ_* depend only on the incoming core-degree bound and the conditional sorting construction. These parameters bound the growth of the maximum load. The full degree governs tether propagation and the number of host positions to reserve. All subrounds together divide u by b_*^t, and their polynomial degree bounds, followed by the reset, multiply $\log(2\nabla)$ by at most a fixed factor.

This completes the proof of Lemma 24.1.

25 The arity budgets after one round

The frozen stations produced by concentration have total arity $O(L/\sqrt{D})$. The core has small maximum arity but may still use more than L bits. Repeated beats reduce its internal width until the total charge fits a smaller budget L/G. To resume concentration we will also need a bound on the squared charges and an absolute degree bound on the full host.

25.1 From concentration to bounded degree

Apply Lemma 19.1 with arity exponent $\ell = 10$. Its absolute degree bound for the core fixes the parameters of a beat in Lemma 24.1. Choose $\epsilon \in (0, 1/100]$ so small that in

$$J = \lceil 2\epsilon \log D \rceil$$

beats the products of the per-beat increases of the maximum load, of the reciprocal bundle unit, and of the logarithm of the overall degree are each at most $D^{0.03}$ when D is sufficiently large. These three per-beat factors are independent of D, which permits this choice. Increase D below as needed in terms of k, δ, ϵ.

We require $ed \le L/D$. For each grouping in this round we also require

$$HN_bT/r \le c_D L \tag{25.1}$$

with $c_D > 0$ sufficiently small, using the current host counts and the round's starting L, T. The choices of S, r meeting these conditions in every round are made below.

Concentrating the squared arities. Group r blocks and apply the dyadic partition estimate (24.1) to the charged loads at absolute host degree, with maximum load $DT/2$ and capacity DT. By (25.1) there are $O(L/(DT))$ occupied intervals over all positions and groups. Regard each interval as one vertex of the graph to which we apply Lemma 19.1. Include all candidate intervals as positions, and allow an edge to an interval at an old neighbor only when the two intersect and satisfy the absolute length-ratio bound of the dyadic partition. All actual contacts are present. Dyadic nesting bounds the simple degree of this new host by an absolute constant.

Inside each occupied interval retain each station of charged arity greater than T. Pack the other stations in their order, closing a pack when its total charged arity reaches T. The last pack can be smaller, and each pack has arity at most $2T$. Its table is the product of its

members, with any internal indices contracted. An interval contains $O(D)$ resulting stations. Move its external bits, if any, to one additional identity station, keeping their order and joining the duplicate ports to the original incidences. Designate these exterior identities as the immobile stations. Their number is at most the number of occupied intervals. The old stations and packs had total arity at most DT per interval, and the identity adds at most $2DT$. The old large stations contribute at most LT to the square sum. Each new pack has arity at most $2T$, so the packs contribute at most $2TL$. If f_i is the external width of interval i, the new identities contribute at most

$$4\sum_i f_i^2 \le 4DT\,ed \le 4LT.$$

With $W = C_1 L$ and C_1 absolute and sufficiently large, these are precisely the occupation, immobility, and square bounds of Lemma 19.1. In particular there are at most D retained large stations, $D+1$ packs and one exterior identity in an occupied interval, within its bound of $100D$.

Expand using Lemma 19.1. Write Z for the total charged arity of the frozen stations, and X for the total width of the internal wires between core stations, counted once per wire. In every summand the frozen charges have total $Z = O(L/\sqrt{D})$, square sum $O(LT)$ and maximum $O(DT)$. The core stations have individual arities at most $D^{-10}T$ and total arity $O(L(\log D)^{O(1)})$. The constants in these estimates are uniform in D. All positive internal bundles have width at least $T/D^{O(1)}$.

To apply the packing operations, take an integer unit u below the minimum guaranteed internal width, still $u \ge T/D^{O(1)}$, divisible by every subdivision prescribed in the J beats. For fixed D this is possible once T is large, by rounding down to a multiple of their product. Pad each positive internal bundle to the next multiple of u with bits pinned to zero. The arity and its square grow by at most factors two and four. Raise each frozen charge to the larger of its old charge and its padded arity. Its charge grows by at most a factor of two, so the total Z and the square budget still obey the bounds above. Choose a dyadic load bound B of order $D^{-10}T$ covering every core station. For packing only, scale the frozen charges by a fixed dyadic multiplier of order $B/(DT)$, small enough that each is at most B. This changes neither the actual widths nor the frozen square bound. We take u small enough also to ensure $u \le B$.

Removing the core wires. Apply Lemma 24.1 J times, with slack $\sigma = L/\sqrt{D}$. The initial full host degree is polynomial in D. The load bounds at intermediate steps are fixed multiples, depending only on D, of T, so (25.1) supplies the required slack each time. In the recurrence of Lemma 24.1 the fixed error $C(Z+\sigma)$ is multiplied by a geometric series of sum less than two. Thus

$$X_{\mathrm{end}} \le 2^{-J} X_{\mathrm{start}} + O(Z+\sigma) = O\big(LD^{-2\epsilon}(\log D)^{O(1)}\big).$$

Frozen stations retain their charges, although one occurrence can now hold a list of them. Each list has $D^{O(1)}$ members. Indeed, its total scaled load is bounded by the final load bound, while the positive charge floor gives a lower bound on the scaled load of each member. Each core station has arity at most $D^{-9}T$. The final bundle unit is still at least $T/D^{O(1)}$, the core degree is at most three, and the overall degree ∇ satisfies $\log(2\nabla) \le D^{0.04}$ for D sufficiently large. The latter estimate uses the initial $O(\log D)$ bound on the logarithm of the degree.

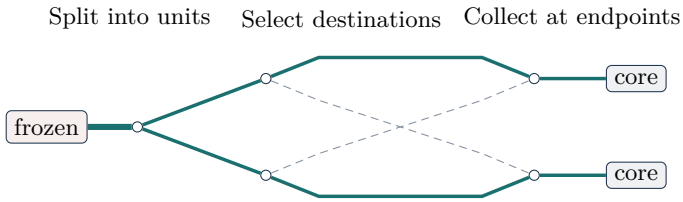


Figure 11. Two units leave one frozen station. The solid routes are active in this block. The dashed routes allow either outlet to reach the other endpoint in a different block. A splitting tree on the left may carry the full frozen arity for $O(\log D)$ levels. In the middle each active branch carries a single unit. At the right its width is bounded by the tether width at the receiving core station.

Restoring bounded degree. An individual station can meet few *actual* neighbors and still have many *possible* neighbors in the common host. At a frozen position reserve a slot for each member of its list. The only frozen contacts are tethers to the core. Route them in three stages (Figure 11). First split its active tether units into separate outlets, in order. There are $D^{O(1)}$ outlets, by the maximum width and minimum unit. Next, from each outlet use a binary tree addressing any potential core endpoint. At each core endpoint use a binary tree for the potential incoming outlets. Link corresponding leaves. An internal branching table is the identity on the concatenated strings of its active children. Inactive branches need no station. Each block uses the trees with precisely its own tether assignments, and the trees and slots themselves are fixed on the common host. The first trees have depth $O(\log D)$, and the others have depth $O(\log(2\nabla) + \log D) = O(D^{0.04})$. Their host size increases by a factor depending only on D. A core endpoint retains at most three core contacts and receives one tree contact, giving overall degree at most four.

For the linear budget, the total extra arity is $O(ZD^{0.04})$, since each tether bit travels along one route of that length. For the square budget, the first splitting tree is bounded in terms of the frozen arity. On a fixed level of that tree the total width is at most the arity of its frozen station. The sum of squared arities at that level is at most four times the square of this arity. Summing first over the frozen stations and then over levels thus gives $O(LT \log D)$. Every other new vertex carries at most the width of one unit or all the tether bits of one core station, so has arity at most $2D^{-9}T$. Its contribution to the square sum is at most $2D^{-9}T$ times its arity. The old core stations have total arity at most $2X_{\mathrm{end}} + Z$ and obey the same estimate.

Together these bounds give charged arities z_v at the end of a round satisfying

$$\begin{aligned} \sum_v z_v &= O\big(LD^{-2\epsilon}(\log D)^{O(1)}\big), \\ \sum_v z_v^2 &= O\big(LT(\log D)^{O(1)}\big), \qquad \max_v z_v = O(DT). \end{aligned} \tag{25.2}$$

For frozen stations we kept the positive floor from Lemma 19.1. Every other remaining station meets a positive internal bundle of width at least $T/D^{O(1)}$. Thus all charges in (25.2) can respect a floor of this order. The constants and the powers of $\log D$ in (25.2) are independent of D. In particular the term $ZD^{0.04} = O(LD^{-0.46})$ fits the first estimate.

Set $G = 2^{\lfloor \epsilon \log D \rfloor}$ and choose a dyadic q with $G^{1+\delta/4} \le q \le G^{1+\delta/3}$. This interval contains a dyadic integer for D sufficiently large. Restart with

$$L' = L/G, \qquad T' = qT. \tag{25.3}$$

Indeed $D^{-2\epsilon}(\log D)^{O(1)} = o(1/G)$, and $L'T' \ge LTG^{\delta/4}$ absorbs the logarithmic loss in the squares. The maximum in (25.2) is at most $DT'/2$ for large D. Finally fix a dyadic $\theta > 0$ so small that θqT is bounded by the positive floor just obtained. None of the estimates used in choosing G, q, D requires a lower bound on θ, since the grouping used the total load cap and repacked the small stations before applying the local lemma. This proves (23.2) for the next round.

25.2 Iteration and the description of the final tables

Fix the round just constructed, including D, G, q and its s grouping steps. To save a factor h^a in the total charge, take

$$j = \left\lceil \frac{a \log h}{\log G} \right\rceil, \qquad L_i = L_0 G^{-i}, \qquad T_i = T_0 q^i \qquad (0 \le i \le j).$$

Here L_i, T_i are the budgets after i completed rounds. It remains to choose r and the initial number S of blocks. A large r makes partially occupied packs inexpensive. A large S supplies enough groups for all sj steps and leaves enough blocks to amortize the common host in Section 26.

Packing slack and exterior width. In one grouping step N_b decreases by r while H increases by at most cr. Their product, which counts allocated occurrences, increases by at most c. The fixed expansions of the host can be included in c. Thus at every stage of the construction,

$$H N_b \le C^{j+1} m S \log h, \tag{25.4}$$

where C depends only on the design of one round. Since $L_0 = C_0 m T_0 S \log(2+h)$, the ratio that controls the slack during the round starting from L_i, T_i is bounded by

$$\frac{H N_b T_i}{r L_i} \le \frac{C_1 C^{j+1} (qG)^i}{r}.$$

Choose r to be the least power of two at least $C_2 (CqG)^{j+1}$. The round design fixes C_2 so that this ratio is at most c_D in (25.1) and r meets the lower bound in every step. In particular $r = h^{O_D(a)+o(1)}$, independently of S. The external width has a separate bound. The ed exposed bits are present in every round, and $G^j < G h^a$. Hence for $i \le j$,

$$\frac{ed}{L_i} = \frac{e G^i}{C_0 m \log(2+h)} = O_D\left(\frac{h^{-a}}{\log h} \right).$$

For sufficiently large h this gives $ed \le L_i / D$, the width allowance for the immobile exterior identities.

Copies and terminal budgets. Let R be the least power of two at least h^M, for any fixed $M > 0$. Taking $S = R r^{sj}$ leaves exactly R blocks after j rounds. Each initial block contains T_0 copies, so $d = T_0 S$ and $\log d = O((\log h)^2)$. In particular $d = n^{o(1)}$. Now $h^a \le G^j < G h^a$, $q \le G^{1+\delta/3}$ and $T_0 = \Theta(h^b)$. It follows from (23.3) and (25.4) that

$$\begin{aligned} L_j &= O(md \log h \, h^{-a}), \quad T_j = O\big(h^{b+(1+\delta/3)a}\big), \\ H N_b &\le m S h^{O_D(a)+o(1)}. \end{aligned} \tag{25.5}$$

The occurrence bound holds throughout, with a constant in $O_D(a)$ independent of M, a, b, β. These scales also justify using a single working field. They range from $T_0 \geq 4h^{\beta}$ to a fixed power of h. Choose the extension degree divisible by the degree of the fixed extension in Lemma 19.1, and large enough to meet the cardinality requirement at T_j. Its degree is $O(\log h)$ and hence $O(\log T_i)$ for every i.

Descriptions of tables. An occurrence carries a label for its table, the widths of its bundles, and their positions in its port order. The cost of naming its incident host edges is bounded in terms of D because each edge can be named by its place in the local neighbor list. Constants indexed by D may depend on the round design. These local data have total length per occurrence at most

$$C_3^{j+1}(\log h)^{O(1)} = h^{O_D(a)+o(1)}. \tag{25.6}$$

At initialization the stack at a position has a fixed table label, specified by its original table, its stack length and the locations of its ordered sublists. Its description has length $(\log h)^{O(1)}$. Within one round the following facts bound the number of old labels that can enter any new label.

- A colony of capacity DT_i contains at most D/θ live entering stations, since each is charged at least θT_i. The local expansion in Lemma 19.1 uses a bounded number of old labels, discrete choices and field elements in each output description. Its additional description length is $O_D(\log h)$.

- In a subround or reset, both the bound on load per leaf and the bound on the number of leaves contributing to one resulting pack are fixed multiples, depending on D, of T_i and 1, respectively. Each live core contributes at least one internal unit of width $u \geq T_i/\operatorname{poly}(D)$. The charged load of an individual frozen station has the same kind of lower bound even after the fixed rescaling. Scalar stations are withdrawn as soon as they form. It follows that a contributing pack has a bounded number of old labels and a bounded number of units. Lists of unit pairings, rotor layers and ordered sublists at a port therefore have bounded numbers of entries. Each entry needs at most $O_D(\log h + \log r)$ bits. This also describes the routing on the final trees at the frozen stations.

Thus if A bounds the current label length, one step gives a bound $C_4A + C_4(\log h)^{C_5}$, including the descriptors of locally withdrawn scalar factors. The multiplicative constant counts old labels, independently of the field degree and the length of the input dictionary labels. Iteration through the sj steps proves Equation (25.6). Global host addresses enter the control of a whole position only. They never have to be appended to an individual table description.

The tensor represented by the sum. Each term now consists of R surviving blocks and the scalar factors withdrawn along its history. Regrouping permutes the exterior lists along with their stations and preserves each product entry by entry. A local tensor identity replaces a product by a sum with the same entries on those ed named ports. Colorings, packings, and routings use fixed tie-breaking rules. Only choices from local tensor identities

index the expansion. The final internal wire assignments are summed when the R blocks are contracted.

For this evaluation one dictionary suffices. A key records the type, stage, description and arity of a table, followed by its incident bits. Length fields or fixed padding give a unique encoding. The input dictionary and the rules of the construction assign a field value to each valid key. The values at unused keys may be fixed arbitrarily. In particular, a key for a withdrawn scalar has no incident bits. A longest key has length

$$O_D(T_j) + h^{O_D(a)+o(1)}$$

by the maximum-arity invariant and Equation (25.6). A derived table description can exceed $(\log h)^{O(1)}$ bits. Section 26 places that description in the *argument* of a dictionary call. The label of the installed table need only specify the key format and length. This accounts separately for the arity of a lookup and the short label required as input to the next macro.

26 Realizing the expansion on a fixed network

Different summands in Section 25 can finish with different wires and different tables. To represent their sum on one network, we encode a choice of summand, together with an assignment to its final wires, as a transcript. A local check determines which transcripts contribute, and a dictionary supplies their tensor entries. The size bound depends on how the host is shared. All blocks at one host position use the same address. The bits carried on a wire vary independently across blocks.

26.1 From a transcript to a tensor

Let K be the working extension of k, and put $h = \lceil \log_2 n \rceil$. Fix $P \le n^{O(1)}$ numbered processors. Each has binary advice of at most h^c bits and a fixed ordered list of exterior ports. Its local record includes guesses and at most h^c incoming and outgoing messages. A message consists of an $O(h)$-bit key identifying source, destination and port, together with a bit string. Communication is valid if every used key occurs exactly once at each end with the same string.

One fixed finite bit-RAM program checks the local record against the advice and exterior bits. It may emit strings of length at most U as arguments to a common dictionary W with values in K. Its decisions depend only on the strings. For fixed exterior bits, the transcript tensor is the sum over accepted transcripts of the product of all values returned by W. The record format, including length fields and padding, is fixed. All size-dependent program data enter through the advice.

The charge at a processor includes its deterministic running time, advice, exterior bits, guesses, messages with headers, and emitted strings. Count at least one unit per record or item. This charge is determined by the record. A local address or time uses $(\log h)^{O(1)}$ bits. Bit arithmetic on words of that length adds at most a factor $(\log h)^{O(1)}$ to the bounds below.

Lemma 26.1 (Realizing a transcript)**.** *Suppose the charge of each local record is at most $V_{\max} = 2^{(\log h)^{O(1)}}$. Let $V \le n^{O(1)}$ bound the total charge of every accepted transcript and the volume of the fixed advice and exterior interface, including one unit per processor. If $U \le h$*

and the wide-record bound of Theorem 16.1 holds, then for every fixed $\tau > 0$ the transcript tensor has a fixed network of size at most

$$h^{\tau}V + Ph^{C_0}.$$

The gates are trivalent or are dictionary tables of arity $O(1 + U + \log h)$. Their labels use $(\log h)^{O(1)}$ bits. The constant C_0 depends only on τ, c, the polynomial bounds in n and the wide-record estimate. The exponent in $V_{\max}$ occurs only in factors $(\log h)^{O(1)}$ before absorption.

Proof. Allocate space by record length. For each dyadic capacity t, from 1 to the least power of two at least $\min(V, V_{\max})$, reserve $N_t = \min(P, \lfloor 2V/t \rfloor)$ slots. A processor with charge s uses the least $t \geq s$. Every record of this class has charge greater than $t/2$, so N_t slots suffice. Guess their processor identifiers and records. Check the identifier range and the charge, require occupied slots to form an increasing prefix in identifier order, and pin all unused entries and padding. The two relevant capacity bounds are

$$\sum_t tN_t = O\big(V(1 + \log V_{\max})\big), \qquad \sum_t N_t = O\big(P(1 + \log V_{\max})\big). \tag{26.1}$$

Check the computation and its dictionary arguments. In a slot of capacity t, guess the deterministic execution for at most t steps. Time-ordered checks verify its instructions, registers, counter, addresses and halt flag, with fixed padding after halt. To check the memory trace, sort the reads, writes and initial contents by address and execution order. Scan each address to verify that a read gives the most recent value, starting at zero for unused cells. This is the trace construction of OpenAI [42, Section 16.2]. A comparison network on $O(t)$ accesses uses $O(t \log^2(2t))$ comparisons of $(\log h)^{O(1)}$-bit words. Long inputs, advice, messages and emitted strings occupy addressed arrays and are processed in pieces. The local cost is $t(\log h)^{O(1)}$ plus $h^{O(1)}$ per slot for long identifiers and framing.

Within the slot, group the emitted strings by least dyadic padded length u, counting an empty string as length one. Reserve $\lfloor 2t/u \rfloor$ calls in that class and take them in issuance order. The checker writes each emitted bit at the address given by its length class, ordinal and bit offset. The same memory-trace test checks these arrays against the inputs of the installed calls, including their active flags, actual lengths and zero padding. Each address has $O(1 + \log t + \log h)$ bits. Accounting for all the padded seats adds $t(\log h)^{O(1)}$ to the checking cost. A seat returns the value of W on its indicated active string and returns 1 when inactive. Its table is named by the padded length. The arity is $O(1 + U + \log h)$ and the total table size per slot is $O(t \log(2 + U) \log(2 + h))$.

Match the messages. Fix $w = \lceil h^{C_1} \rceil$, where $C_1 > 1$ will be chosen for the wide-record sort. Split each message into w-bit chunks, padding the last one and reserving one for an empty message. Its chunk record contains an active flag, key, direction, total length, chunk number and payload. Add a fixed sender for each processor. Its key belongs to a reserved namespace and supplies that processor's advice and exterior bits with their component lengths. An occupied slot must receive the whole supply for its claimed identifier even when the supply is empty.

Check each outgoing source and incoming destination against the slot identifier, and check the splitting, piece order, lengths and padding. Sort the records by active flag, key, chunk number and direction. In the active part, require every key and chunk to occur as one

send and one receive of equal length and payload. A duplicate message on one side collides at chunk zero. In particular each private supply is matched to exactly one slot, so every processor occurs once. For a given input the stable-sort relation has just one output, which can be summed out after these checks. Its zero-one tensor over k extends to the same relation over K.

A slot reserves $O(t/w + h^{c} + 1)$ chunk records. The $O(h)$-bit header is copied once per chunk, so constructing them costs $(t + h^{O(1)})(\log h)^{O(1)}$. Charge a fixed sender for its actual supply and one partial chunk. By Equation (26.1) the reserved bit volume, including padding at message boundaries, is

$$V(\log h)^{O(1)} + Ph^{O(1)}.$$

Pad the list to a power of two. Its length is polynomial in n, so its logarithm is $O(h)$. Theorem 16.1 with tolerance $\tau/3$ permits a fixed C_1 large enough that sorting costs at most $O(h^{\tau/3})$ times this volume. The subsequent scan has linear size. Equation (26.1) gives the same volume bound for the local computations and the dictionary tables. Absorb the powers of $\log h$ into h^{τ} and the per-processor costs into Ph^{C_0}.

The checking and sorting gates are trivalent over k, and each dictionary gate is named by its length class. These names have the stated length. For each accepted transcript the slot capacities, identifiers, order, traces, calls and chunks are uniquely specified. Conversely the local and communication checks recover an accepted transcript. Contracting the installed relations therefore gives its weighted sum exactly over K. □

26.2 The transcript of the rounds

We now apply Lemma 26.1 to the expansion in Section 25. Initially there are S blocks of T_0 copies, so $d = T_0 S$. At every stage of the j rounds there are $N_b \geq R$ blocks on a host with H allocated positions. Give each position one processor, including at the terminal stage, and give each step $O(H)$ more processors for balanced aggregation trees. An occurrence consists of one position in one block.

One address for all blocks. A position holds the ordered list of its host neighbors and an array of occurrence records in block order. Its advice specifies the parent and child naming conventions, integer parameters, field presentation, and the initial convention for ordering exterior ports. At an initial position it also gives the input table label. Exterior bits enter through fixed ports in stack order.

In one step every host degree is bounded in terms of the fixed round design. A ball of radius $O_D(1 + \log r)$ thus contains $r^{O_D(1)}$ positions. Exchange neighbor lists to that radius, remove repetitions by identifier and order the resulting names. This locates each possible child and contact by an index in a local list, at a cost of $h^{O(1)}$ common operations per position. A message to a contact can now carry an entire array of occurrence records under one $O(h)$-bit header. Within that array only local indices are needed. There are $h^{O(1)}$ phases and declarations per position. By Equation (25.6), handling the arrays costs $N_b r^{O_D(1)} (\log h)^{O(1)} h^{O_D(a)} = N_b h^{O_D(a)+o(1)}$ per position. The coefficients in $O_D(a)$ are independent of a, b, β.

For decisions shared by the host, each position computes its local contributions to the candidate scores. Sum the integer arrays along a balanced tree and broadcast the first minimizing candidate. At height i the width of a sum of initially z-bit entries is at most

$z + i$, while the number of positions falls by a factor of order 2^i. Since $\sum_{i \geq 0} 2^{-i}(z+i)^{O(1)} = O((1+z)^{O(1)})$, aggregation preserves the preceding bit-cost bound. At the root allow an additional $2^{(\log h)^{O(1)}}$ operations to choose the minimizer. A greedy coloring of a given host can be checked locally as well. In fixed identifier order, require each color to be the least one absent from its lower neighbors. Induction on identifiers makes the checked assignment unique.

Checking the changes of host. Each of the operations in a round supplies its contacts from these neighbor lists and its common parameters from additive scores.

1. For colony packing, try the multipliers of the r slots, propagate the overloaded marks to ancestors and neighbors, and sum the candidate colony counts. A propagation moves upward by at least one dyadic level, so the radius is $O_D(1 + \log r)$. Relative interval endpoints, loads and dyadically scaled charges all have short encodings. Summing the local scores also chooses the affine coloring for whole-item pairing.

2. At a local concentration step, the colony interval and its old contacts give the possible neighbors of each sphere. In the checked color order, the movement and interpolation choices of Lemma 19.1 specify the new tables, withdrawn scalars and ordered bundles. Its description bounds pay for the choices and labels. The port orders identify the exposed sublists.

3. In a wire-saving subround the source colony and bin determine the seats of its matched units. Record one summand choice for each actual gadget. Sum the surviving replacement widths under each of the two contractions and take the same minimizer for all blocks. The widths before identifying coincident endpoints suffice for this decision. Bounded neighborhoods of the colonies then determine the new contacts, port orders and table labels.

4. For a reset, compute each position's mean load over the blocks. The smoothed mean has floor FB/r and a propagated contribution decays by at least a factor of two per edge. The starting load is at most B, so a neighborhood of radius $O(1 + \log r)$ determines the bases and their capacities.

   For the common mixer, let x_i be the normalized load at a position or the normalized width at an edge in block $i \in \mathbb{F}_2^{\log_2 N_b}$. At prefix length s the part of the collision estimator of Section 24.3 that depends on a candidate matrix A is a multiple of

   $$\sum_{z \neq 0} \mathbf{1}_{(Az)_{1:s}=0} \sum_i x_i x_{i+z}.$$

   Two indices have the same output prefix exactly when their difference z satisfies the displayed condition. The diagonal and centering terms are constant in A, so they can be omitted in choosing a minimizer. The denominators of the remaining x_i are cleared by FB times the fixed dyadic load denominator. The level and overflow weights are integers after fixed scaling. In particular the choice uses integer arithmetic in every characteristic.

   Compute the autocorrelations $\sum_i x_i x_{i+z}$ simultaneously by squaring the Walsh coefficients and inverting the transform. This takes $O(N_b \log N_b)$ operations on $(\log h)^{O(1)}$-bit

local integers. Aggregate the coefficients by difference and prefix length. At the root one can enumerate all invertible matrices in dimension $\log_2 N_b = O((\log h)^2)$ within $2^{(\log h)^{O(1)}}$ bit operations. The first minimizer satisfies the bound proved by averaging. Positions permute their block arrays accordingly. The bounded length ratios of intersecting leaves and the bounded number of units per leaf then allow the terminal assignments, rotor settings and new labels to be determined from neighboring records.

5. The final splitting trees for tethers have bounded size, depending on D, per possible station and contact. Adjacent positions determine the active sublists. An edge between new children is supported by a bounded path between their parents. In the next phase the children use their own host adjacency, so every neighborhood collected is local to its current step.

Declare messages in a fixed order of phases, ports and blocks. Enforce each successive array and common host from the preceding data and the local summand choices. Each term of a local identity, including a term of weight zero, has one prescribed encoding. The intermediate arrays therefore introduce no further multiplicities.

Independent bits and tensor entries. Transfer the exterior bits in the order specified by the splits and concatenations. The source and destination share a processor header for all pieces sent along one contact. Their array positions and bit offsets have length $(\log h)^{O(1)}$. When a sphere collects bits at its exterior station, the station is frozen during the ensuing steps. When the blocks are reordered, permute its lists with them. Neighborhood collection transports descriptions. Each of the ed exterior bits travels only along its specified route, at most once per transfer. The $O(\log h)$ steps therefore use $ed(\log h)^{O(1)}$ payload work.

At the terminal stage, guess one bit per internal wire at its designated endpoint and send it to the other endpoint in contact and bundle order. A table call consists of the station description and its ordered port bits. The terminal payload costs $L_j(\log h)^{O(1)}$ by Equation (25.5). A scalar withdrawn earlier issues its call at withdrawal, with just the short description. The input scalars are fixed. If their product is c, one nullary table of value c^d supplies their contribution to the entire expansion. Naming this element of K costs $O(\log h)$ bits. For fixed summand choices, the free payload guesses now range exactly over the terminal wire assignments. The product of call values is their contraction weight. Summing over these assignments and then over summands gives the input tensor power on its ordered ed exterior bits.

Size of the fixed network. At each stage Equation (25.5) gives $HN_b \le mSh^{O_D(a)+o(1)}$. There are $O(\log h)$ steps, and $N_b \ge R$. The bounds for processors and for their total charge are therefore

$$P \le mSh^{O_D(a)+o(1)}/R,$$
$$V \le Ph^{C_2} + mSh^{O_D(a)+o(1)} + (L_j + ed)h^{o(1)} + 2^{(\log h)^{O(1)}}. \tag{26.2}$$

The first two terms pay for the common work per position and the short records per occurrence. The third term pays for independent wire bits, and the last for the root computations. Every local record has charge at most $2^{(\log h)^{O(1)}}$, since $N_b \le S$, $\log S = O((\log h)^2)$ and an occurrence has only polynomially many bits in h. The exponent C_2 and the exponent bounding the

declaration count can be fixed independently of M in $R = h^{M+o(1)}$. Common work uses $O(h)$-bit addresses and $r^{O_D(1)}$ candidate contacts. The dependence on M is confined to factors polynomial in $\log S$ and the additive root cost.

The longest dictionary call uses $O_D(T_j) + h^{O_D(a)+o(1)}$ bits. Choose a small enough that $Ca + 2a < \beta$ for each coefficient C in the short-work and label bounds, and $b + (1+\delta)a < 1$ for all allowed b. Equation (25.5) gives $T_j = O(h^{b+(1+\delta/3)a})$, so the calls have length below h. Use Lemma 26.1 with $\tau = a\delta/3$, then take M sufficiently large to absorb both Ph^{C_0} and $Ph^{C_2+\tau}$. For the three other charges in Equation (26.2), the costs per md before the factor h^τ are

$$\frac{mSh^{Ca+o(1)}}{md} = h^{-b+Ca+o(1)}, \qquad \frac{ed}{md} \le h^{-2a}, \qquad \frac{L_j}{md} = O(h^{-a}\log h).$$

Here $d = T_0 S$ and $T_0 = \Theta(h^b)$. The additive root cost is $n^{o(1)}$ while $m \ge n$. Consequently the fixed network has size at most $md\,h^{-a+\tau+o(1)} \le md\,h^{-(1-\delta)a}$ for large n. Its maximum table arity, including the activity and length fields, is at most $h^{b+(1+\delta)a}$. A dictionary table retains its short length-class label, ready for another application of Lemma 23.1. This completes the proof of that lemma.

27 Trivalent compression and the exponent

Each application of Lemma 23.1 reduces the size per copy and raises the maximum arity. We stop while the saving in size still exceeds the cost of evaluating the larger tables. Two operations then recover a trivalent network over the original field. Batched evaluation uses Lemma 16.2, and a single coordinate projection removes the field extension used in the local identities.

Lemma 27.1 (Descent of a trivalent network)**.** *Let K/k have finite degree D. If a trivalent K-network of size s represents a tensor with entries in k, the same tensor has a trivalent k-network of size $O(sD^2)$ with the same exterior positions.*

Proof. Choose a k-basis of K whose first element is 1. For $\lambda \in K$, let M_λ represent multiplication by λ in that basis. Order the stations along a chain and give each of them two extra ports carrying the incoming and outgoing basis indices. If an assignment to its original ports has weight λ, the enlarged table returns the indicated entry of M_λ. Pin both ends of the chain to the first index. For any fixed assignment to the original wires, contracting along the chain now gives the first coordinate of the product of the weights. This coordinate map is k-linear. Summing over the original internal bits therefore takes the first coordinate of the old tensor entry, which belongs to k.

Encode each basis index in $\lceil \log_2 D \rceil$ bits, giving weight zero to unused codes. The enlarged table has at most $r = 3 + 2\lceil \log_2 D \rceil$ bits. Every table on r bits has a trivalent realization of size $O(2^r)$. In the binary prefix tree compute, at each node, the indicator that the input has that prefix. At depth j, copying the next input bit to the 2^j nodes and testing the two children of each node costs $O(2^j)$ gates. Give each leaf a unary weight equal to its table entry when its indicator is one and to 1 otherwise. Exactly one leaf supplies the selected entry. Summing the level costs gives $O(2^r) = O(D^2)$ per original station. □

Proof of Theorem 2.1. Fix the finite field k with $\omega_k < 9/4$ and the exterior-bit allowance $\eta > 0$. Write $h = \lceil \log_2 n \rceil$. The ordinary-width sort in Theorem 16.1 supplies an exponent

$1 < p < 2$. Put $\rho = p-1$. A total arity increase from h^β to h^{b_*} will cost $O(h^{\rho b_*})$ per surviving incidence in the final lookup. Choose $0 < \delta \le 1/10$ so that $1 - \delta > \rho(1+\delta)$. Take $A > 0$ sufficiently small, and then $\beta > 0$ sufficiently small, that $b_* = \beta + (1+\delta)A$ satisfies

$$b_* < \tfrac{1}{2}, \qquad \rho b_* < \tfrac{1}{2}\eta, \qquad \Delta = (1-\delta)A - \rho b_* > 0. \tag{27.1}$$

The positive gap Δ is the net saving on the original network size. Choose a fixed integer J large enough that $a = A/J$ satisfies Lemma 23.1 uniformly for $\beta \le b \le 1/2$ and $2a < \eta/2$.

One extension K/k suffices for these J applications. Each width scale T lies between a constant times h^β and a fixed power of h. The local identities require a fixed constant extension and a field of cardinality at least a fixed power of T. Choose $[K : k]$ large enough for the largest such power and divisible by the degrees of the required constant extensions. Since k is finite, we can do this with $[K : k] = O(\log h) = O(\log T)$ at every scale.

Let s_j count stations and their total arity after j applications, let d_j be the cumulative number of copies, and put $e_j = d_j e$. Initially $d_0 = 1$, $s_0 = O(m)$, the arity is at most $3 \le h^\beta$, and the trivalent k-tables have a fixed finite dictionary. At step $j < J$ use the size allowance

$$M_j = \max\{s_j, \lceil e_j h^{2a} \rceil\}.$$

This allows the construction to continue even when the exterior bits already account for most of the current size. We have $M_j \ge n$, $e_j \le M_j h^{-2a}$ and maximum arity at most $h^{\beta+(1+\delta)aj}$. If this application makes r_j copies, Lemma 23.1 preserves the short table labels, gives $d_{j+1} = r_j d_j$ and yields

$$s_{j+1} \le r_j M_j q, \qquad q = h^{-(1-\delta)a}.$$

Its construction in Section 25 has $\log_2 r_j = O((\log h)^2)$ uniformly in the input network. Inductively, the original bound $s_0, e_0 = O(n^{1.05})$, multiplied by a fixed number of these copy factors and fixed powers of h, gives $M_j \le n^{1.05+o(1)} \le n^{1.1}$ at every invocation.

In terms of size per exterior bit, $x_j = s_j/e_j$, the replication factor cancels. For large h,

$$x_{j+1} \le q \max\{x_j, 2h^{2a}\}, \qquad x_J \le \max\{q^J x_0, 2qh^{2a}\}.$$

Indeed $q < 1$, so every contribution arising from an exterior-bit allowance is bounded by the most recent such allowance.

Let u be the final maximum arity. Applying Lemma 16.2 makes 2^u copies and multiplies the size per copy by $O((1+u)^\rho)$. The resulting tensor is the dth power of the original k-tensor, with

$$d = 2^u d_J, \qquad \log_2 d \le h^{b_*} + O(J(\log h)^2) = o(h).$$

Thus $d = n^{o(1)}$ uniformly at fixed k, η. Each of its de exterior positions is retained through lookup and through Lemma 27.1. Descent costs $O([K : k]^2) = h^{o(1)}$. Combining these bounds, the trivalent size over k divided by de is at most

$$O(m/e)\, h^{-\Delta+o(1)} + h^{2a+\rho b_*+o(1)}.$$

Choose $\gamma = \Delta/2$. The strict inequalities $\Delta > \gamma$ and $2a + \rho b_* < \eta$ absorb the constants and lower-order terms for large n. Multiplication by de gives Equation (2.1). □

For the distinctness tensor of Theorem 2.2, the independent exterior positions are the entries of separate lists. Choosing $\eta < \sigma$, Proposition 2.3 iterates the compression until the volume per exterior bit is $O(w^\eta)$, below the required w^σ. This rules out $\omega_k < 9/4$ over every finite field. Lemma 2.5 transfers the lower bound to any field, where the matching upper bound gives Theorem 1.1.

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