EE4323 Industrial Control Systems
Homework Assignment 1
Dynamics model of a DC motor with gear train

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The objective is to model the dynamics of a DC servo motor with gear train, Fig. 1, and to deduce two equilibrium points.

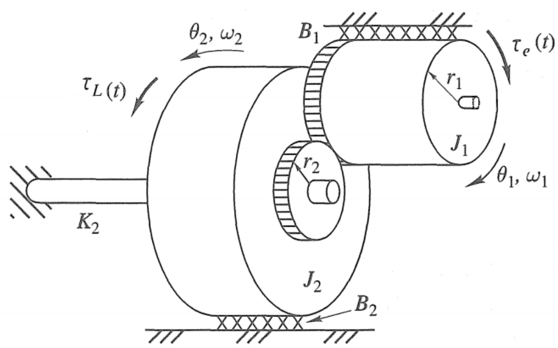

Figure 1: DC servo motor with gear train.

1 Free-body diagram analysis

The system can be decomposed in two sections: a rotational mechanical, and an electro-nmechanical. The rotational mechanical can be derived as follows,

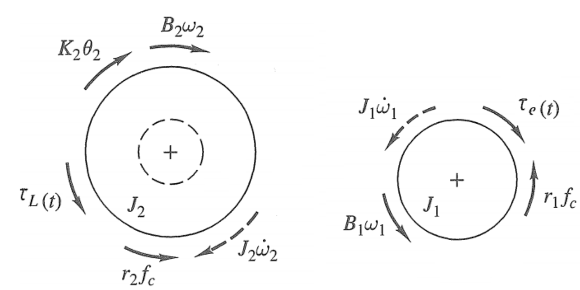

Figure 2: Rotational mechanical free-body diagram.

where θ is the angular displacement, ω is the angular speed, B is the rotational viscous-damping coefficient, K is the stiffness coefficient, J is the moment of inertia, f_c is the contact force between two gears, and r is the gear radius.

The electro-mechanical section (DC motor) is

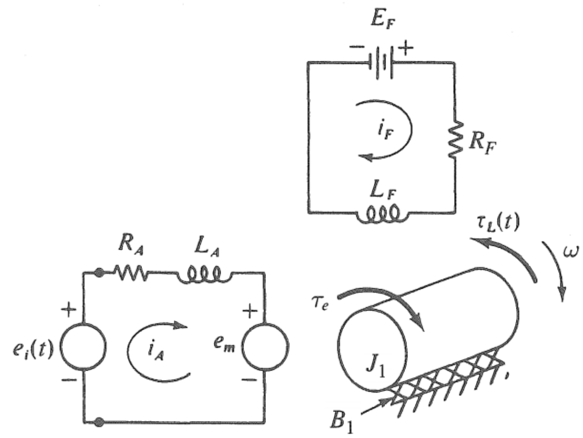

Figure 3: Electromechanical free-body diagram.

where R_F is the field resistance, L_F is the field inductance, E_F is the applied constant field voltage, and i_F is the input field current. R_A is the stationary resistance, L_A is the stationary inductance, and e_m is the induced voltage, i_A is the input stationary current, and $e_i(t)$ is the applied armature voltage, and τ_e is the electro-mechanical driving torque exerted on the rotor.

If the flux density $\mathcal{B}$ is

$$\mathcal{B} = \frac{\phi(i_F)}{A} \tag{1}$$

the torque on the rotor is

$$\begin{aligned} \tau_e &= \mathcal{B} la \; i_A \\ \tau_e &= \frac{la}{A}\phi(i_F) i_A \end{aligned} \tag{2}$$

where $\phi(i_F)$ is the flux induced by i_F, A is the cross-sectional area of the flux path in the air gap between the rotor and stator, l is the total length of the armature conductors within the magnetic field, and a is the radius of the armature.

Also, the voltage induced in the armature e_m can be written as

$$e_m = \frac{la}{A}\phi(i_F)\omega \tag{3}$$

where both, τ_e and e_m, depend on the geometry of the DC motor.

2 Dynamic system

We begin applying D'Alembert's law (restatement of Newton's law) to the rotational mechanical section.

$$\sum \tau_{all} = 0$$

$$J_1\dot{\omega}_1 + B_1\omega_1 + r_1 f_c = \tau_e(t) \tag{4}$$

$$J_2\dot{\omega}_2 + B_2\omega_2 + K_2\theta - r_2 f_c = \tau_L(t) \tag{5}$$

where τ_{all} are the torques acting on a body, $K\theta$ is the stiffness torque, $B\omega$ is the viscous-frictional torque, $J\dot{\omega}$ is the inertial torque, $\tau_e(t)$ is the driving torque, $\tau_L(t)$ is the load torque, and rf_c is the contact torque.

Due to the relation between gears,

$$\theta_1 = N\theta_2$$
$$\omega_1 = N\omega_2$$
$$\dot{\omega}_1 = N\dot{\omega}_2$$
$$N = \frac{r_2}{r_1}$$

where N is the gear radius relation. We solve (4) and (5) in terms of ω_2 and θ_2,

$$(J_2 + N^2 J_1)\dot{\omega}_2 + (B_2 + N^2 B_1)\omega_1 + K_2\theta_2 - N\tau_e(t) - \tau_L(t) = 0$$

defining the relations

$$J_{eq} = J_2 + N^2 J_1$$
$$B_{eq} = B_2 + B^2 B_1$$

it becomes in

$$J_{eq}\dot{\omega}_2 + B_{eq}\omega_2 + K_2\theta_2 - N\tau_e(t) - \tau_L(t) = 0 \tag{6}$$

Now, let us derive the equations of the electro-mechanical section using Kirchoff's law.

$$\sum V_{all} = 0$$
$$e_m + V_{L_A} + V_{R_A} = e_i(t) \tag{7}$$

where V_{all} are the induced voltages on the rotor and stator, V_{L_A} is the stationary resistance voltage, V_{R_A} is the stationary inductance voltage.

If i_F is defined as constant, then (2) is

$$\tau_e(t) = \left(\frac{la}{A}\phi(i_F)\right) i_A(t)$$
$$\tau_e(t) = \alpha i_A(t) \tag{8}$$

where α is the internal parameters of the DC motor.

Then, simplifying and using (6) and (7) the dynamic system is,

$$J_{eq}\dot{\omega}_2 + B_{eq}\omega_2 + K_2\theta_2 - N\tau_e - \tau_L = 0 \tag{9}$$
$$L_A\dot{i}_A + R_A i_A + \alpha\omega_1 - e_i = 0 \tag{10}$$

3 State-space equations

Let us define the state-space equations for $x = \begin{bmatrix}\theta_2 & \dot{\theta}_2 & i_A\end{bmatrix}^\intercal$. From the dynamic system,

$$J_{eq}\ddot{\theta}_2 + B_{eq}\dot{\theta}_2 + K_2\theta_2 - N\alpha i_A - \tau_L = 0$$
$$L_A\dot{i}_A + R_A i_A + \alpha\omega_1 - e_i = 0$$

reordering,

$$\ddot{\theta}_2 = -\frac{B_{eq}}{J_{eq}}\dot{\theta}_2 - \frac{K_2}{J_{eq}}\theta_2 + \frac{N\alpha}{J_{eq}}i_A - \frac{1}{J_{eq}}\tau_L$$
$$\dot{i}_A = -\frac{R_A}{L_A}i_A - \frac{N\alpha}{L_A}\dot{\theta}_2 + \frac{1}{L_A}e_i$$

defining the states as

$$\begin{cases} x_1 &= \theta_2, \quad \dot{x}_1 = \dot{\theta}_2 = x_2 \\ x_2 &= \dot{\theta}_2, \quad \dot{x}_2 = \ddot{\theta}_2 = -\frac{B_{eq}}{J_{eq}} x_2 - \frac{K_2}{J_{eq}} x_1 + \frac{N\alpha}{J_{eq}} x_3 - \frac{1}{J_{eq}} \tau_L \\ x_3 &= i_A, \quad \dot{x}_3 = \dot{i}_A = -\frac{R_A}{L_A} x_3 - \frac{N\alpha}{L_A} x_2 + \frac{1}{L_A} e_i \end{cases}$$

then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_2}{J_{eq}} & -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ 0 & -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{J_e q} & 0 \\ 0 & 0 & \frac{1}{L_A} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 0 \\ \tau_L \\ e_i \end{bmatrix}}_{\mathbf{u}} \tag{11}$$

$$\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} \tag{12}$$

The output $y = \dot{\omega}_2$ can be defined as

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}}_{D} e_i \tag{13}$$

$$y = C\dot{\mathbf{x}} \tag{14}$$

4 Equilibrium point $\mathbf{x_0}$

Using $\dot{\mathbf{x}} = 0$ in (12), the equilibrium point $\mathbf{x_0}$ can be calculated as

$$0 = A\mathbf{x_0} + B\mathbf{u} \tag{15}$$

$$\mathbf{x_0} = -A^{-1}B\mathbf{u} \tag{16}$$

$$\begin{bmatrix} x_{1_0} \\ x_{2_0} \\ x_{3_0} \end{bmatrix} = - \begin{bmatrix} 0 & 1 & 0 \\ -\frac{K_2}{J_{eq}} & -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ 0 & -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{J_e q} & 0 \\ 0 & 0 & \frac{1}{L_A} \end{bmatrix} \begin{bmatrix} 0 \\ \tau_L \\ e_i \end{bmatrix} \tag{17}$$

Solving for no external torque $\tau_L = 0$, constant applied armatrue voltage $e_i = E_0$, and $K_2 \neq 0$,

$$\begin{aligned} 0 &= x_{2_0} \\ 0 &= -\frac{K_2}{J_{eq}} x_{1_0} - \frac{B_{eq}}{J_{eq}} x_{2_0} + \frac{N\alpha}{J_{eq}} x_{3_0} \\ 0 &= -\frac{N\alpha}{L_A} x_{2_0} - \frac{R_A}{L_A} x_{3_0} + \frac{1}{L_A} E_0 \end{aligned}$$

due to $x_{2_0} = 0$, we have

$$\begin{aligned} 0 &= -\frac{K_2}{J_{eq}} x_{1_0} + \frac{N\alpha}{J_{eq}} x_{3_0} \\ 0 &= -\frac{R_A}{L_A} x_{3_0} + \frac{1}{L_A} E_0 \end{aligned}$$

then

$$\begin{aligned} x_{1_0} &= \frac{N\alpha}{K_2 R_A} E_0 \\ x_{3_0} &= \frac{1}{R_A} E_0 \end{aligned}$$

therefore the equilibrium point is

$$\mathbf{x_0} = \begin{bmatrix} x_{1_0} \\ x_{2_0} \\ x_{3_0} \end{bmatrix} = \begin{bmatrix} \frac{N\alpha}{K_2 R_A} \\ 0 \\ \frac{1}{R_A} \end{bmatrix} E_0 \tag{18}$$

This equilibrium point indicates that a **constant angular displacement (twist)** produced by $x_{1_0} = \theta_{2_0}$ is sufficient to balance the constant applied armature voltage $e_i = E_0$.

On the other hand, if we solve for no external torque $\tau_L = 0$, constant applied armature voltage $e_i = E_0$, and no stiffness $K_2 = 0$. The problem is,

$$\begin{bmatrix} x_{1_0} \\ x_{2_0} \\ x_{3_0} \end{bmatrix} = - \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ 0 & -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{J_e q} & 0 \\ 0 & 0 & \frac{1}{L_A} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E_0 \end{bmatrix}$$

if we eliminate x_{1_0} because the first column of A^{-1} has zeros, the problem reduces to

$$\begin{bmatrix} x_{2_0} \\ x_{3_0} \end{bmatrix} = - \begin{bmatrix} -\frac{B_{eq}}{J_{eq}} & \frac{N\alpha}{J_{eq}} \\ -\frac{N\alpha}{L_A} & -\frac{R_A}{L_A} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{1}{J_e q} & 0 \\ 0 & \frac{1}{L_A} \end{bmatrix} \begin{bmatrix} 0 \\ E_0 \end{bmatrix} \tag{19}$$

solving, we have

$$\begin{bmatrix} x_{2_0} \\ x_{3_0} \end{bmatrix} = \begin{bmatrix} \frac{N\alpha}{B_{eq} R_A + (N\alpha)^2} \\ \frac{-B_{eq}}{B_{eq} R_A + (N\alpha)^2} \end{bmatrix} E_0 \tag{20}$$

which indicates that a **constant angular speed** produced by $x_{2_0} = \dot{\theta}_{2_0}$ is needed to balance the constant applied armature voltage $e_i = E_0$.

References

[1] Close, Charles M. and Frederick, Dean K. and Newell, Jonathan C., *Modeling and Analysis of Dynamic Systems*, 2001, ISBN 0471394424.