

# Constrained Model Predictive Control using Feedforward Steady-State Target Optimization tracking approach for an Isolated Power System

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**Abstract**— A constrained Linear Quadratic Model Predictive Control (LQ-MPC) with offset-free tracking and disturbance rejection using the Feedforward Steady-State Target Optimization (SSTO) approach is presented in this paper. The MPC control law uses Finite-Receding-Horizon method in dual-mode. Moreover, the stability and recursive feasibility are guaranteed by off-line dual-mode and on-line optimization. The results are presented by simulation.

## I. STATE-SPACE MODEL OF THE SYSTEM

The isolated power system [1] to be studied is a linear Single-Input Single-Output (SISO) model with the addition of the disturbance in the output, Fig. 1.

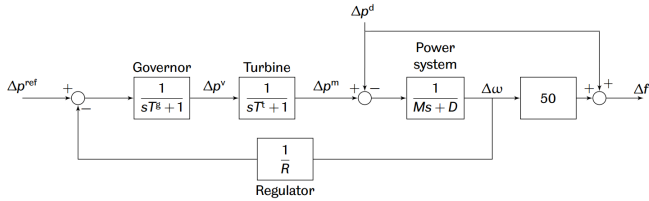


Fig. 1: Isolated power system

The linear continuous-time state-space model is as follows,

$$\begin{bmatrix} \Delta\dot{\omega} \\ \Delta\dot{p}^m \\ \Delta\dot{p}^v \end{bmatrix} = \begin{bmatrix} -D/M & 1/M & 0 \\ 0 & -1/T^t & -1/T^t \\ -1/(RT^g) & 0 & -1/T^g \end{bmatrix} \begin{bmatrix} \Delta\omega \\ \Delta p \\ \Delta p^v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/T^g \end{bmatrix} \Delta p^{ref} + \begin{bmatrix} -1/M \\ 0 \\ 0 \end{bmatrix} \Delta p^d \quad (1)$$

$$\Delta f = \begin{bmatrix} 50 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta\omega \\ \Delta p \\ \Delta p^v \end{bmatrix} + \Delta p^d$$

where  $\Delta f$  is the output values,  $\Delta p^{ref}$  is the input, and  $\Delta p^d$  is the disturbance, the rest of the parameters can be consulted at [1].

Thus, the Discrete Linear Time-Invariant (LTI) state-space system is

$$\begin{aligned} x(k+1) &= A x(k) + B u(k) + B_d d(k) \\ y(k) &= C x(k) + D_d d(k), \quad k = 0, 1, 2, \dots \end{aligned} \quad (2)$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^m$ ,  $y \in \mathbb{R}^p$ , and  $d \in \mathbb{R}^q$  are the states, input, output, and disturbance vectors respectively.  $A, B, C$ , are the state, input, and output matrices, and  $B_d, D_d$  are the disturbance matrices in the input, and in the output.

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## II. FORMULATION OF THE LQ-MPC TRACKING PROBLEM

### A. Feedforward tracking with SSTO

The Feedforward tracking approach by SSTO requires a target optimizer that calculates the steady-state values for the states  $x_{ss}$  and the input  $u_{ss}$  given a reference  $r$  in the presence of a disturbance  $d$ , Fig. 2.

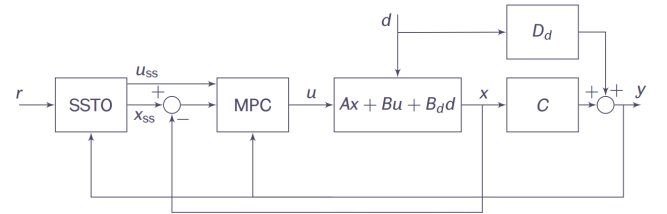


Fig. 2: MPC and SSTO model

### B. Cost function

Because it is desired that  $x \rightarrow x_{ss}$  and  $u \rightarrow u_{ss}$ , new variables can be defined as  $\xi \equiv x - x_{ss}$  and  $v \equiv u - u_{ss}$ , where  $\xi \rightarrow 0$  and  $v \rightarrow 0$ .

Therefore, the constrained LQ-MPC problem is to minimize the cost function

$$\mathbb{P}_N(v(k)) : \min_{v(k)} \sum_{j=0}^{N-1} (\xi^T(k+j|k) Q \xi(k+j|k) + v^T(k+j|k) R v(k+j|k)) + \xi^T(k+N|k) P \xi(k+N|k) \quad (3)$$

subject to, for  $j = 0, 1, 2, \dots, N-1$

$$\xi(k+1+j|k) = A \xi(k+j|k) + B v(k+j|k)$$

$$\xi(k|k) = \xi(k)$$

$$P_x \xi(k+j|k) \leq q_x - P_x x_{ss} \quad (4)$$

$$P_u v(k+j|k) \leq q_u - P_u u_{ss} \quad (5)$$

$$P_{x_N} \xi(k+N|k) \leq \tilde{q}_{x_N} \quad (6)$$

where  $Q$  and  $R$  are weight matrices,  $P$  is the terminal cost matrix, (4), (5), (6) are the linear inequality (polyhedra) constraints, and  $N$  is the finite-horizon value. Note that  $B_d, D_d$  and  $d$  are not present because the SSTO will manage the disturbance rejection.

### C. Defining the inequality constraints

Given  $|\Delta p^{ref}| = |u| \leq u_{min,max}$ , and  $|\Delta f| = |y| \leq y_{min,max}$ , for  $y = Cx$ ,

$$\begin{cases} \begin{bmatrix} C \\ -C \end{bmatrix} x(k+j|k) \leq \begin{bmatrix} y_{max} \\ -y_{min} \end{bmatrix} \\ \begin{bmatrix} I \\ -I \end{bmatrix} u(k+j|k) \leq \begin{bmatrix} u_{max} \\ -u_{min} \end{bmatrix} \end{cases}$$

can be written in terms of  $\xi$  and  $v$  as follows,

$$\begin{cases} \underbrace{\begin{bmatrix} C \\ -C \end{bmatrix}}_{P_x} \xi(k+j|k) \leq \underbrace{\begin{bmatrix} y_{max} \\ -y_{min} \end{bmatrix}}_{q_x} - \underbrace{\begin{bmatrix} C \\ -C \end{bmatrix}}_{P_x} x_{ss} \\ \underbrace{\begin{bmatrix} I \\ -I \end{bmatrix}}_{P_u} v(k+j|k) \leq \underbrace{\begin{bmatrix} u_{max} \\ -u_{min} \end{bmatrix}}_{q_u} - \underbrace{\begin{bmatrix} I \\ -I \end{bmatrix}}_{P_u} u_{ss} \end{cases} \quad (7)$$

and using a proper deadbeat mode-2 terminal inequality constraint to ensure a fast convergence to zero,

$$\underbrace{\mathbf{I}_{n \times n} \otimes \begin{bmatrix} P_x \\ P_u \ K_\infty \end{bmatrix}}_{P_{x_N}} \begin{bmatrix} (A+BK_\infty)^0 \\ \vdots \\ (A+BK_\infty)^{N-1} \end{bmatrix} \xi(k+N|k) \leq \mathbf{1}_n \otimes \underbrace{\begin{bmatrix} q_x - P_x x_{ss} \\ q_u - P_u u_{ss} \end{bmatrix}}_{\tilde{q}_{x_N}} \quad (8)$$

where  $K_\infty$  is the deadbeat mode-2 gain calculated using Linear Quadratic Regulator (LQR) method.

### D. Target equilibrium optimization under constraints (SSTO)

The offset-free equilibrium points  $(x_{ss}, u_{ss})$  that satisfies the constraints exists if only if  $x_{ss} = Ax_{ss} + Bu_{ss} + B_d d$  and  $y_{ss} = Cx_{ss} + D_d d = r$ .

Thus, the SSTO problem is

$$\min_{\{x_{ss}, u_{ss}\}} \|C x_{ss} + D_d d - r\| + \rho \|u_{ss}\| \quad (9)$$

subject to,

$$\begin{cases} \underbrace{\begin{bmatrix} I-A & -B \\ C & 0 \end{bmatrix}}_{A_{eq}=T} \underbrace{\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix}}_{x_{eq}} = \underbrace{\begin{bmatrix} B_d \\ r - D_d \end{bmatrix}}_{b_{eq}} d \\ \underbrace{\begin{bmatrix} P_x & 0 \\ 0 & P_u \end{bmatrix}}_{A_{neq}} \underbrace{\begin{bmatrix} x_{ss} \\ u_{ss} \end{bmatrix}}_{x_{neq}} \leq \underbrace{\begin{bmatrix} q_x \\ q_u \end{bmatrix}}_{b_{neq}} \end{cases} \quad (10)$$

## III. SOLVING THE LQ-MPC OPTIMIZATION PROBLEM

The solution of the LQ-MPC problem (3) in the compress form can written as,

$$\min_{v(k)} \frac{1}{2} \mathbf{v}^\top(k) H \mathbf{v}(k) + \xi^\top(k) L^\top \mathbf{v}(k) + \xi^\top(k) M \xi(k) \quad (11)$$

subject to

$$P_c \mathbf{v}(k) \leq q_c + S_c \xi(k) \quad (12)$$

where  $H, L, M$  are the cost matrices given  $Q$  and  $R$ .  $P_c, q_c, S_c$  are the stacked inequality constraints for  $P_x, P_u, P_{x_N}, q_u, q_x, \tilde{q}_{x_N}$  given the predictions matrices  $\mathbf{x} = Fx(k) + G\mathbf{u}(k)$ , [1]. Note that the prediction matrices are constructed using the states and input without the tracking model.

The solution of (11) is the optimal control input sequence,

$$\mathbf{v}^* = \{v^*(k|k), v^*(k+1|k), \dots, v^*(k+N-1|k)\}$$

which can be solved numerically.

Because MPC uses the receding-horizon principle, only the first optimized control input is needed

$$v(k) = v^*(k|k) = K_N(\xi(k))$$

but the real control input applied to the system is

$$u(k) = u_{ss} + v^*(k|k) \quad (13)$$

### A. Stability

In order to ensure stability it is necessary to compute a terminal cost matrix  $P$  that satisfies the Lyapunov equation

$$(A+BK)^\top P(A+BK) - P + (Q+K^\top R K) = 0 \quad (14)$$

which solution can be computed numerically. A good approach is to use a deadbeat mode-2 control gain  $K$  that converges the states to the origin  $x = 0$  as long as  $(A+BK)$  is stable, which means that the eigenvalues  $\lambda$  are inside the unit circle,  $|\lambda| < 1$ .

## IV. ALGORITHM FOR THE IMPLEMENTATION

Now that the solution of the LQ-MPC is established and the stability can be guaranteed, there are basic conditions to meet such as,

- $(A, B)$  is stabilizable which can be proved by the controllability, in Matlab `rank(ctrb(A, B))`.
- $Q \succeq 0$ , can be  $C^\top C$
- $R \succ 0$ , changed later for tuning.

The following steps reflect the Matlab implementation algorithm

*Off-line mode:*

1. Discretization of the model (1) using zero holder (zoh), in Matlab will be `sys=c2d(idss(Ac, Bc, C, D, Bd, x0, 0), Ts)`, where  $T_s$  is the sampling time.
2. Construct the target equilibrium constraints (10)
3. Define the optimization equation as `z=[xss; uss]` and `fun=@(z) C(1)*z(1)+Dd*d-r+rho*z(4)`
4. Solve the optimization problem for SSTO using `f=fmincon(fun, z0, Aneq, bneq, Aeq, beq)` in Matlab, where `z0=[r, 0, 0, 0]`, `fun` is (9), and `f=[xss, uss]` is the optimized vector solution.
5. Construct the LQ-MPC inequality constraints (11), and compute the deadbeat mode-2 terminal constraint using `K=-dlqr(A, B, Q, R)`.

6. Construct the prediction matrices  $F, G$  using  $[F, G]=\text{predict\_mats}(A, B, N)$  and the constraint matrices (12) using  $[Pc, qc, Sc]=\dots \text{constraint\_mats}(F, G, Pu, qu, Px, qx, PxN, qxN)$ ;
7. Compute the deadbeat mode-2 control  $Kl$  using  $Kl=-\text{acker}(A, B, [0, 0, 0])$
8. Calculate the Lyapunov solution of (14) with  $P=\text{dlyap}((A+B*K) ', Q+K' *R*K)$ , which gives a terminal cost matrix  $P$  that guaranties stability.
9. Finally before the on-line mode is initiated, set the initial conditions as  $x=[0; 0; 0; 0]$

*On-line mode: iteration for  $k$  steps*

1. Recall steps 3-4 of the off-line mode for the SSTO optimization but at each iteration  $z0=[x; us(1, k)]$ , where  $us(1, k)$  is the first optimized control input.
2. Calculate  $\xi(k)$  and  $v(k)$  using  $xi(:, k)=x-x_{ss}$   $v(:, k)=us(:, k)-u_{ss}$ .
3. Compute the constraints (10) with the new  $x_{ss}$  and  $u_{ss}$
4. Compute (12) like it was done in the off-line mode step 6.
5. Store the states  $xs(:, k)=x$  for the iteration process.
6. Solve the LQ-MPC optimization problem (11) without using the matrix  $M$  as follows,  $[v, fval, flag]=\text{quadprog}(H, L*xi(:, k), \dots Pc, qc+Sc*xi(:, k))$  where  $v$  is the optimal control input  $v(k)$
7. Check feasibility using  $flag$ , if it is  $< 1$  the solution is infeasible.
8. Use (13) as the real control input applied to the system,  $us(:, k)=v(1)+u_{ss}$  using only the first optimized value (receding principle).
9. Close the loop (2),  $x=A*x+B*us(1, k)+Bd*d$   $y=C*x+Dd*d$
10. Calculate the cost value of (11) using the matrix  $M$ .
11. Return to step 1.

## V. DESIGN REQUIREMENTS

For the simulation the following specifications are demanded,

- Reference  $r = 0.3$
- Frequency settles to  $|\Delta f| = |y| \leq 0.01$  within 2 [s] of the disturbance initiaion.
- $|\Delta p^{ref}| = |u| \leq 0.5$ , and  $|\Delta f| = |y| \leq 0.5$ .
- Zero steady-state error in the output.

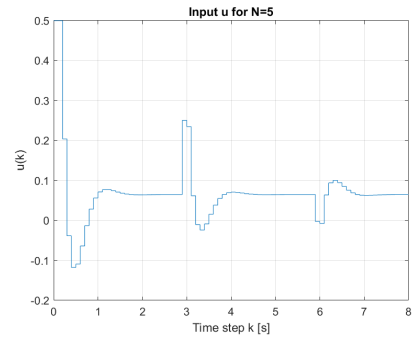
## VI. SIMULATION AND RESULTS

The tuning and design parameters, as well as the simulation value results are shown in Table I

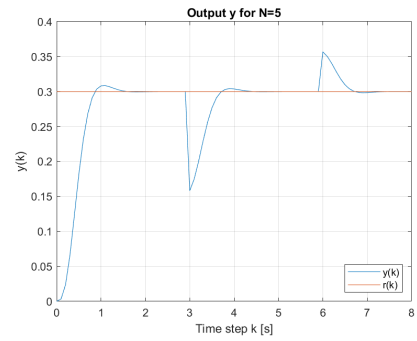
Fig. 3a shows that the input constraint is satisfied, and in Fig. 3b it can be seen that the zero offset-free tracking is also satisfied rejecting disturbances at different times. The low value of  $R$  is to minimize the overshoot as much as possible which is 0.6%, the best  $Q$  is  $C^T C$  where the settling time  $t_s$  is around 0.7. Also, the selection of  $N = 3$  was lowest possible to increase performance, Fig. 3c.

Parameter	Value
$Q$	$C^T C$
$R$	0.08
$N$	3
$d_1$	0.5 at $t = 30[s]$
$d_2$	-0.2 at $t = 60[s]$
$r$	0.3
$x_0$	$[0; 0; 0]^T$
$t_s$	0.7[s]
overshoot	0.6%

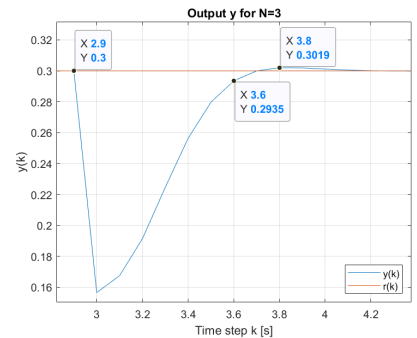
TABLE I: Tuning and design parameters



(a)  $u(k)$



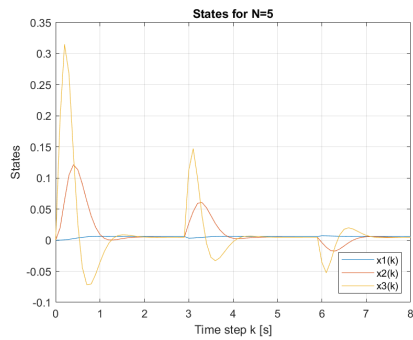
(b)  $y(k)$  vs  $r(k)$



(c)  $y(k)$  vs  $r(k)$

Fig. 3: Input and output

Fig. 4 shows that the constraints are satisfied for the three states. And the cost value shown in Fig. 5 is relatively low due the low value of  $R$ .



(a) States vs step (k)

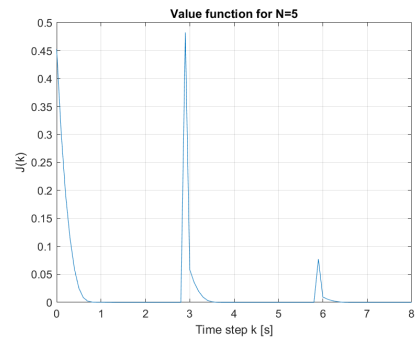
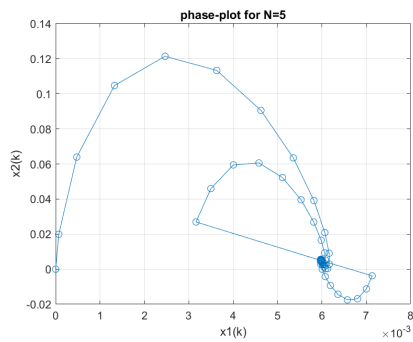
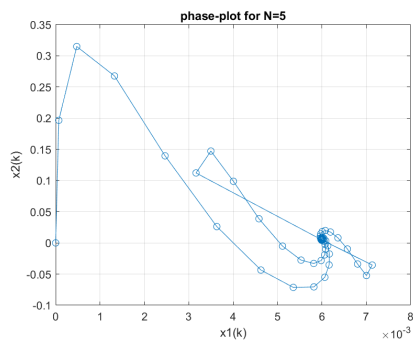


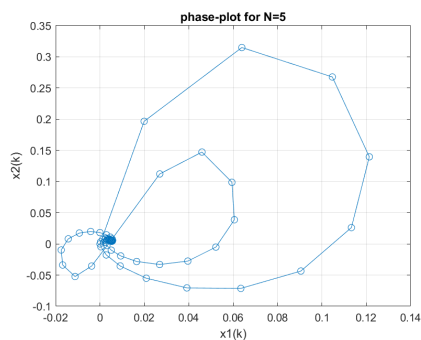
Fig. 5: Cost value  $J(k)$  function



(b)  $x_1(k)$  vs  $x_2(k)$



(c)  $x_1(k)$  vs  $x_3(k)$



(d)  $x_2(k)$  vs  $x_3(k)$

Fig. 4: States and phase plots

One of the drawbacks of the SSTO approach is that it does not have robustness because it needs a perfect model of the plant. Also, when  $d > 0.5$ , the solution is infeasible, in order to handle bigger disturbances it is necessary to change the cost function, [2] shows an alternative solution.

#### REFERENCES

- [1] P. Trodden, Lecture Notes ACS616 2018/19
- [2] S. Dughman, J.A. Rossiter, A survey of guaranteeing feasibility and stability in MPC during target changes 9th International Symposium on Advanced Control of Chemical Processes, June 2015.