Introduction to Lean

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There are many proof assistants. In this workshop, we will use *Lean*.

Quick demo

Why formalize

Formalizing a mathematical result is hard work.

Terence Tao: “each line in the paper such as ‘this can be rearranged as’ is currently taking me 20 minutes to formalize properly.”

Why bother?

Verifying correctness

A theorem accepted by a proof assistant is correct with a very high level of confidence.

Sometimes having such a verification is desirable. I will give some examples.

The Liquid Tensor Experiment

In 2020, Peter Scholze challenged the formalization community to formalize one of his recent results about condensed mathematics. He wrote:

“I think the theorem is of utmost foundational importance, so being 99.9% sure is not enough. As it will be used as a black box, a mistake in this proof could remain uncaught.”

“I spent much of 2019 obsessed with the proof of this theorem, almost getting crazy over it. In the end, we were able to get an argument pinned down on paper, but I think nobody else has dared to look at the details of this, and so I still have some small lingering doubts.”

“I think this may be my most important theorem to date. (It does not really have any applications so far, but I'm sure this will change.) Better be sure it's correct. . . ”

The Liquid Tensor Experiment

A team led by Johan Commelin set out to formalize Scholze's results.

Half a year later, Scholze announced "the Experiment has verified the entire part of the argument that I was unsure about".

The proof was found to be correct, though "One day I was sweating a little bit. The proof walks a fine line, so if some argument needs constants that are quite a bit different from what I claimed, it might have collapsed."

A year later, the entire challenge was completed.

The consistency of New Foundations

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In 2024, the project was completed by Sky Wilshaw, and the consistency of New Foundations is no longer an open question.

The Erdős-Graham problem and the PFR conjecture

In late 2021, Thomas Bloom announced a proof of an open problem by Erdős and Graham from 1980.

By mid-2022, Bhavik Mehta and Thomas Bloom had formalized the proof (including prerequisites from analytic number theory).

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On November 9, 2023, Timothy Gowers, Ben Green, Freddie Manners and Terence Tao announced a proof of the PFR conjecture from 1999.

Tao asked for help with formalizing the result. With the help of 30 people, the project was completed on December 5.

Correcting mistakes

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There is an error in the proof of the Artin-Tate lemma in Atiyah-Macdonald that is very easy to miss and that (as far as we know) was first discovered when formalizing the proof in Lean.

Improving proofs

Formalizing a mathematical result often leads to new mathematical insights.

The Liquid Tensor Experiment

Scholze's proof used an object called the Breen-Deligne resolution that requires a lot of machinery to construct.

During the formalization, Johan Commelin explicitly worked out the properties of the Breen-Deligne resolution that are used in Scholze's proof.

He then managed to write down a much simpler object (the "Commelin complex") that also satisfied these axioms, greatly reducing the prerequisites of the proof.

Scholze writes: "This makes the rest of the proof of the Liquid Tensor Experiment considerably more explicit and more elementary, removing any use of stable homotopy theory. I expect that Commelin's complex may become a standard tool in the coming years."

The Liquid Tensor Experiment

Scholze writes:

"[I also learned] what actually makes the proof work! When I wrote the blog post half a year ago, I did not understand why the argument worked [...]. But during the formalization, a significant amount of convex geometry had to be formalized, and this made me realize that actually the key thing happening is a reduction from a non-convex problem over the reals to a convex problem over the integers."

Refactoring libraries

Mathematics is a complex web of results that depend on each other.

If I generalize a fifty-year-old result by remvoving an assumption, which theorems in the literature since then are automatically also generalized?

No one would go through the literature themselves, going through every invocation of a result and checking if the assumption is needed elsewhere.

With Lean, it only takes a second.

Data and visualization

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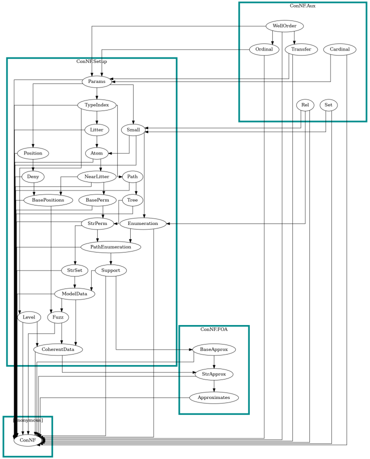

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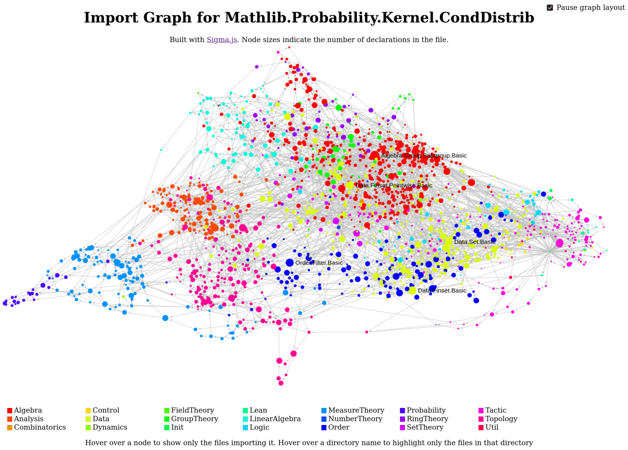

Search

A digital library of mathematics can be searched efficiently. Here are some questions that we will be able to answer:

- Which separation axioms is a certain topological space known to satisfy?
- What are all known results that relate two graph parameters (or any other two mathematical objects)?
- Does a certain claim follow straightforwardly from a known result?
- Does one result logically depend on another result (based on the proofs present in the system)?

Teaching and learning

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Formalization may also be able to improve teaching materials, as demonstrated in the informalization project by Patrick Massot and Kyle Miller.

Computation

Computer-aided proofs are traditionally difficult to check because they rely on the correctness of the software.

In 2024, Heule and Scheucher used a SAT solver to show that every set of 30 points in the plane contains an empty convex hexagon.

The proof relies on a SAT solver, a very complex piece of software.

Subercaseux, Nawrocki, Gallicchio, Codel, Carneiro and Heule verified the reduction to a SAT problem and ran a verified checker on the SAT solver's proof.

Style and aesthetics

Mathematicians have an intuitive sense of aesthetics for mathematical proofs.

The best written proof is not always the most suited for formalization.

Formalization invites us to think about proofs in different ways.

Towards AI in math

Proof assistants are a potential way to integrate AI into mathematical reasoning.

Many smart people are working on this.

Recently, the DeepMind's AlphaProof system showed silver medal level performance on a formal-to-formal version of the 2024 International Mathematical Olympiad.

Exciting things to come!

Collaborating

Mathematics is growing more collaborative.

Formalization projects see teams of 30+ mathematicians working together seamlessly today. Pietro will show how they do it on Friday.

Some of the things learned about digiatal collaboration during formalization projects will be useful for collaborative mathematical discovery.

One day, proof assistants might play a direct role in collaborative mathematical discovery.

Summary

Formalizing mathematics is useful for

- ensuring correctness on large and small scales,
- gaining additional insight into mathematical proofs,
- advancing the way we work and teach, and
- taking advantage of future developments in artificial intelligence.

Further reading: Jeremy Avigad: *Mathematics and the formal turn*. Bulletin of the AMS.