Lemma 1. *For $s > 1$, the Dirichlet eta function*

$$\eta(s) := (1 - 2^{1-s})\zeta(s)$$

is nondecreasing.

Proof. For $s > 1$,

$$\eta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}}{e^x + 1}\, dx.$$

Let X_s have the gamma distribution with density

$$\frac{x^{s-1}e^{-x}}{\Gamma(s)} \qquad (x > 0),$$

and put

$$h(x) := \frac{1}{1 + e^{-x}}.$$

Then

$$\eta(s) = \mathbb{E}[h(X_s)].$$

Differentiating under the integral sign gives

$$\eta'(s) = \operatorname{Cov}\big(h(X_s), \log X_s\big).$$

Since both $h(x)$ and $\log x$ are increasing functions of x,

$$\operatorname{Cov}\big(h(X_s), \log X_s\big) = \frac{1}{2}\mathbb{E}\Big[\big(h(X_s) - h(Y_s)\big)\big(\log X_s - \log Y_s\big)\Big] \geq 0,$$

where Y_s is an independent copy of X_s. Hence $\eta'(s) \geq 0$. □

Lemma 2. *For every $r > 0$,*

$$-\frac{\zeta'}{\zeta}(1 + r) \leq \frac{\log 2}{2^r - 1}.$$

Proof. For $s > 1$,

$$\frac{\eta'}{\eta}(s) = \frac{\zeta'}{\zeta}(s) + \frac{\log 2}{2^{s-1} - 1}.$$

By the previous lemma, $\eta'(s) \geq 0$, and since $\eta(s) > 0$, we have $\eta'(s)/\eta(s) \geq 0$. Therefore

$$-\frac{\zeta'}{\zeta}(s) \leq \frac{\log 2}{2^{s-1} - 1}.$$

Taking $s = 1 + r$ gives the claim. □

Theorem 1. *Let Λ denote the von Mangoldt function. For every $n \geq 1$,*

$$\sum_{q \geq 1} \frac{\Lambda(q)}{q \log(nq) \log(2nq)} \leq \frac{1}{\log(2n)}.$$

Equivalently, since $\Lambda(1) = 0$, the sum may be read as a sum over $q \geq 2$.

Proof. For $x > 1$,

$$\frac{1}{\log x} = \int_0^\infty x^{-u}\, du.$$

Using Tonelli's theorem and the identity

$$-\frac{\zeta'}{\zeta}(s) = \sum_{q \geq 1} \frac{\Lambda(q)}{q^s} \qquad (s > 1),$$

we obtain

$$\begin{aligned}\sum_{q\ge1}\frac{\Lambda(q)}{q\log(nq)\log(2nq)} &= \int_0^\infty\int_0^\infty n^{-u-v}2^{-v}\sum_{q\ge1}\frac{\Lambda(q)}{q^{1+u+v}}\,du\,dv\\ &= \int_0^\infty\int_0^\infty n^{-u-v}2^{-v}\left(-\frac{\zeta'}{\zeta}(1+u+v)\right)du\,dv.\end{aligned}$$

Put $r=u+v$. Then $0\le v\le r$, and

$$\int_0^r 2^{-v}\,dv=\frac{1-2^{-r}}{\log 2}.$$

Hence

$$\sum_{q\ge1}\frac{\Lambda(q)}{q\log(nq)\log(2nq)}=\frac{1}{\log 2}\int_0^\infty n^{-r}(1-2^{-r})\left(-\frac{\zeta'}{\zeta}(1+r)\right)dr.$$

By the previous lemma,

$$-\frac{\zeta'}{\zeta}(1+r)\le\frac{\log 2}{2^r-1}.$$

Therefore

$$\begin{aligned}\sum_{q\ge1}\frac{\Lambda(q)}{q\log(nq)\log(2nq)} &\le \frac{1}{\log 2}\int_0^\infty n^{-r}(1-2^{-r})\frac{\log 2}{2^r-1}\,dr\\ &= \int_0^\infty n^{-r}\frac{1-2^{-r}}{2^r-1}\,dr\\ &= \int_0^\infty n^{-r}2^{-r}\,dr\\ &= \int_0^\infty (2n)^{-r}\,dr\\ &= \frac{1}{\log(2n)}.\end{aligned}$$

This proves the theorem. ☐

Proposition 1. *Let $A\subseteq 2\mathbb{N}$ be primitive, meaning that no two distinct elements of A divide one another. Then*

$$\sum_{a\in A}\frac{1}{a\log a}\le\frac{1}{2\log 2}.$$

Consequently, 2 is Erdős-strong.

Proof. Let

$$V:=\{1\}\cup 2\mathbb{N}.$$

Define a downward weighted directed graph on V as follows. First put one special edge

$$2\longrightarrow 1$$

of weight

$$w(2\to 1):=\frac{1}{2\log 2}.$$

For every $n\ge 1$ and every $q\ge 2$, put an edge

$$2nq\longrightarrow 2n$$

of weight

$$w(2nq\to 2n):=\frac{\Lambda(q)}{2nq\,\log(nq)\,\log(2nq)}.$$

For $x \in V$, let $O(x)$ be the total weight of the edges leaving x, and let $I(x)$ be the total weight of the edges entering x.

We first compute $O(2n)$. If $n = 1$, the only outgoing edge from 2 is $2 \to 1$, so

$$O(2) = \frac{1}{2 \log 2}.$$

If $n > 1$, then the outgoing edges from $2n$ are obtained by writing $2n = 2(n/d)d$, where $d \mid n$ and $d \geq 2$. Therefore, using

$$\sum_{d \mid n} \Lambda(d) = \log n$$

and $\Lambda(1) = 0$, we get

$$\begin{aligned} O(2n) &= \sum_{\substack{d \mid n \\ d \geq 2}} \frac{\Lambda(d)}{2n \, \log n \, \log(2n)} \\ &= \frac{1}{2n \, \log n \, \log(2n)} \sum_{d \mid n} \Lambda(d) \\ &= \frac{1}{2n \log(2n)}. \end{aligned}$$

Thus, for every $n \geq 1$,

$$O(2n) = \frac{1}{2n \log(2n)}.$$

Next compute $I(2n)$. The incoming edges are

$$2nq \longrightarrow 2n \qquad (q \geq 2),$$

so

$$\begin{aligned} I(2n) &= \sum_{q \geq 2} \frac{\Lambda(q)}{2nq \, \log(nq) \, \log(2nq)} \\ &= \frac{1}{2n} \sum_{q \geq 2} \frac{\Lambda(q)}{q \log(nq) \log(2nq)}. \end{aligned}$$

By the theorem,

$$I(2n) \leq \frac{1}{2n \log(2n)} = O(2n).$$

Now reverse the edges and define a sub-Markov chain. Put

$$C := O(2) = \frac{1}{2 \log 2}.$$

The chain starts at 1, then moves deterministically from 1 to 2. From a vertex $2n$, it moves to $2nq$, where $q \geq 2$, with probability

$$\mathbb{P}(2n \to 2nq) := \frac{w(2nq \to 2n)}{O(2n)}.$$

The total probability of leaving $2n$ is

$$\sum_{q \geq 2} \mathbb{P}(2n \to 2nq) = \frac{I(2n)}{O(2n)} \leq 1.$$

The remaining probability is interpreted as killing the chain.

Let $P(2n)$ be the probability that the chain ever visits $2n$. We claim that

$$P(2n) = \frac{O(2n)}{C} \qquad (n \geq 1).$$

For $n = 1$, this is immediate, because the chain moves from 1 to 2 with probability 1, and $O(2) = C$.

Assume $n > 1$, and suppose the identity has been proved for all proper divisors of n. The immediate predecessors of $2n$ in the reversed graph are the vertices $2n/d$, where $d \mid n$ and $d \geq 2$. Hence

$$\begin{aligned} P(2n) &= \sum_{\substack{d \mid n \\ d \geq 2}} P(2n/d)\, \mathbb{P}(2n/d \to 2n) \\ &= \sum_{\substack{d \mid n \\ d \geq 2}} \frac{O(2n/d)}{C} \, \frac{w(2n \to 2n/d)}{O(2n/d)} \\ &= \frac{1}{C} \sum_{\substack{d \mid n \\ d \geq 2}} w(2n \to 2n/d) \\ &= \frac{O(2n)}{C}. \end{aligned}$$

This proves the claim by induction.

Let $B \subseteq A$ be finite. Along every non-killed path of the reversed chain, the current integer strictly increases by divisibility:

$$2n \longmapsto 2nq \qquad (q \geq 2).$$

Since A is primitive, such a path can visit at most one element of A. Therefore the events

$$\{\text{the chain visits } b\}, \qquad b \in B,$$

are pairwise disjoint. Using the formula for $P(2n)$, we get

$$\sum_{b \in B} \frac{O(b)}{C} = \sum_{b \in B} P(b) \leq 1.$$

Thus

$$\sum_{b \in B} O(b) \leq C.$$

Since $b \in 2\mathbb{N}$, the outflow formula gives

$$O(b) = \frac{1}{b \log b}.$$

Hence

$$\sum_{b \in B} \frac{1}{b \log b} \leq \frac{1}{2 \log 2}.$$

Taking the supremum over all finite subsets $B \subseteq A$, equivalently applying monotone convergence, yields

$$\sum_{a \in A} \frac{1}{a \log a} \leq \frac{1}{2 \log 2}.$$

Equality is attained by the primitive set $A = \{2\}$. Therefore 2 is Erdős-strong. □