All logarithms are natural.

Define the von Mangoldt function Λ by

$$\Lambda(q) = \begin{cases} \log p, & q = p^k \text{ for some prime } p \text{ and integer } k \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

For $N, M \in \mathbb{Z}_{>0}$, define $v(N, M)$ as follows. If $N > 1$, q is a prime-power divisor of N, and

$$M = \frac{N}{q},$$

then

$$v(N, M) = \frac{\Lambda(q)}{N \log^2 N}.$$

In all other cases, set

$$v(N, M) = 0.$$

Define $w(N, M)$ by modifying $v(N, M)$ only in the following cases. For every prime p and every integer $k \geq 2$, set

$$w(p^k, 1) = 0$$

and

$$w(p^k, p^{k-1}) = v(p^k, p^{k-1}) + v(p^k, 1).$$

For all other pairs (N, M), set

$$w(N, M) = v(N, M).$$

For $N \geq 1$, define

$$\operatorname{Out}(N) = \sum_{M \geq 1} w(N, M)$$

and

$$\operatorname{In}(N) = \sum_{K \geq 1} w(K, N).$$

Claim 1. *For every $N > 1$,*

$$\operatorname{Out}(N) = \frac{1}{N \log N}.$$

Proof. Fix $N > 1$. The passage from v to w only moves mass within a fixed row N. If N is not of the form p^k with p prime and $k \geq 2$, then

$$w(N, M) = v(N, M)$$

for every M. If $N = p^k$ with $k \geq 2$, then the only changed entries in row N are $M = 1$ and $M = p^{k-1}$, and

$$w(p^k, 1) + w(p^k, p^{k-1}) = 0 + \big(v(p^k, p^{k-1}) + v(p^k, 1)\big).$$

Thus in every case,

$$\sum_{M \geq 1} w(N, M) = \sum_{M \geq 1} v(N, M).$$

Now compute the row sum of v. By definition,

$$\sum_{M \geq 1} v(N, M) = \sum_{\substack{q \mid N \\ q \text{ prime power}}} \frac{\Lambda(q)}{N \log^2 N}.$$

Since $\Lambda(q) = 0$ unless q is a prime power,

$$\sum_{M\geq 1} v(N, M) = \frac{1}{N\log^2 N}\sum_{q|N}\Lambda(q).$$

If

$$N = \prod_{i=1}^{r} p_i^{a_i},$$

then the prime-power divisors contributing to $\sum_{q|N}\Lambda(q)$ are

$$p_i, p_i^2, \ldots, p_i^{a_i},$$

and each contributes $\log p_i$. Therefore

$$\sum_{q|N}\Lambda(q) = \sum_{i=1}^{r}\sum_{j=1}^{a_i}\log p_i = \sum_{i=1}^{r} a_i \log p_i = \log N.$$

Hence

$$\mathrm{Out}(N) = \sum_{M\geq 1} w(N, M) = \sum_{M\geq 1} v(N, M) = \frac{\log N}{N\log^2 N} = \frac{1}{N\log N}.$$

□

Lemma 1. *For $s > 1$, define*

$$A(s) = \sum_{q\geq 2}\frac{\Lambda(q)}{q^s}.$$

Then, for every prime p,

$$A(s) + \frac{\log p}{p^s} \leq \frac{1}{s-1}.$$

Proof. For $s > 1$,

$$A(s) = -\frac{\zeta'(s)}{\zeta(s)},$$

where

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Use the standard estimates

$$-\zeta'(s) = \sum_{n=2}^{\infty}\frac{\log n}{n^s} \leq \int_1^{\infty}\frac{\log x}{x^s}\,dx = \frac{1}{(s-1)^2}$$

and

$$\zeta(s) \geq \frac{1}{s-1} + \frac{1}{2}.$$

The second estimate follows from the convexity of x^{-s}: for each integer $n \geq 1$,

$$\int_n^{n+1} x^{-s}\,dx \leq \frac{n^{-s} + (n+1)^{-s}}{2},$$

and summing over $n \geq 1$ gives

$$\frac{1}{s-1} = \int_1^{\infty} x^{-s}\,dx \leq \zeta(s) - \frac{1}{2}.$$

Therefore

$$A(s) = -\frac{\zeta'(s)}{\zeta(s)} \leq \frac{\frac{1}{(s-1)^2}}{\frac{1}{s-1} + \frac{1}{2}} = \frac{2}{s^2-1}.$$

Also, for every prime p,

$$\frac{\log p}{p^s} \le \max_{x>1} \frac{\log x}{x^s} = \frac{1}{es} \le \frac{1}{s+1}.$$

Hence

$$A(s) + \frac{\log p}{p^s} \le \frac{2}{s^2-1} + \frac{1}{s+1} = \frac{1}{s-1}.$$

□

Claim 2. *For every $N > 1$,*

$$\mathrm{Out}(N) \ge \mathrm{In}(N).$$

Proof. Fix $N > 1$, and write

$$L = \log N.$$

Then $L > 0$.

A nonzero v-flow into N comes from a source Nq, where $q \ge 2$. For every $q \ge 2$,

$$v(Nq, N) = \frac{\Lambda(q)}{Nq \log^2(Nq)}.$$

Thus the v-contribution to the inflow into N is

$$\sum_{q \ge 2} \frac{\Lambda(q)}{Nq \log^2(Nq)}.$$

The modification from v to w can add extra inflow into $N > 1$ only when N is a prime power. If N is not a prime power, then

$$N\,\mathrm{In}(N) = \sum_{q \ge 2} \frac{\Lambda(q)}{q(L + \log q)^2}.$$

If

$$N = p^a$$

with p prime and $a \ge 1$, then the extra inflow into N comes from the modified flow

$$p^{a+1} \longrightarrow p^a.$$

The extra amount is

$$v(p^{a+1}, 1) = \frac{\log p}{p^{a+1} \log^2(p^{a+1})}.$$

Multiplying by $N = p^a$, this extra contribution becomes

$$Nv(p^{a+1}, 1) = \frac{\log p}{p(L + \log p)^2}.$$

Therefore, when $N = p^a$,

$$N\,\mathrm{In}(N) = \sum_{q \ge 2} \frac{\Lambda(q)}{q(L + \log q)^2} + \frac{\log p}{p(L + \log p)^2}.$$

Use the identity

$$\frac{1}{Y^2} = \int_0^\infty te^{-tY}\,dt \qquad (Y > 0).$$

If $N = p^a$, then

$$N\,\mathrm{In}(N) = \int_0^\infty te^{-Lt} \left(\sum_{q \ge 2} \frac{\Lambda(q)}{q^{1+t}} + \frac{\log p}{p^{1+t}} \right) dt.$$

By Lemma 1, with $s = 1 + t$,

$$\sum_{q \geq 2} \frac{\Lambda(q)}{q^{1+t}} + \frac{\log p}{p^{1+t}} \leq \frac{1}{t}.$$

Thus

$$N \operatorname{In}(N) \leq \int_0^\infty te^{-Lt} \cdot \frac{1}{t}\, dt = \int_0^\infty e^{-Lt}\, dt = \frac{1}{L}.$$

Hence

$$\operatorname{In}(N) \leq \frac{1}{N \log N}.$$

If N is not a prime power, the same argument applies without the extra term, using

$$\sum_{q \geq 2} \frac{\Lambda(q)}{q^{1+t}} \leq \frac{1}{t}.$$

Therefore again

$$\operatorname{In}(N) \leq \frac{1}{N \log N}.$$

By Claim 1,

$$\operatorname{Out}(N) = \frac{1}{N \log N}.$$

Therefore

$$\operatorname{Out}(N) \geq \operatorname{In}(N).$$

□

Claim 3. *For every $N > 1$,*

$$w(N, 1) > 0 \iff N \text{ is prime.}$$

Proof. A nonzero v-flow from N to 1 can occur only if

$$1 = \frac{N}{q},$$

so

$$q = N.$$

Thus, for $N > 1$,

$$v(N, 1) = \frac{\Lambda(N)}{N \log^2 N},$$

and this is nonzero exactly when N is a prime power.

If N is not a prime power, then

$$\Lambda(N) = 0,$$

so

$$v(N, 1) = 0.$$

No exceptional modification applies, and therefore

$$w(N, 1) = 0.$$

If $N = p^k$ with p prime and $k \geq 2$, then $v(N, 1) > 0$, but this is exactly one of the exceptional cases in the definition of w. Hence

$$w(p^k, 1) = 0.$$

Finally, if $N = p$ is prime, then no exceptional modification applies, so

$$w(p,1) = v(p,1) = \frac{\Lambda(p)}{p\log^2 p} = \frac{\log p}{p\log^2 p} = \frac{1}{p\log p} > 0.$$

Therefore

$$w(N,1) > 0 \iff N \text{ is prime.}$$

□

Let

$$\mathbb{N}_{\geq 2} := \{2, 3, 4, \dots\},$$

and let $\mathcal{P}$ denote the set of primes. A set

$$A \subseteq \mathbb{N}_{\geq 2}$$

is called *primitive* if no two distinct elements of A divide one another.

A *divisibility flow* is a nonnegative function $W(m,n)$ on pairs of positive integers such that

$$W(m,n) = 0$$

unless

$$n \mid m \qquad \text{and} \qquad n < m.$$

Conjecture 1. *There exists a divisibility flow w such that, for every $r \geq 2$,*

$$\mathrm{Out}(r) \geq \frac{1}{r\log r},$$

with equality whenever r is prime;

$$\mathrm{Out}(r) \geq \mathrm{In}(r);$$

and the only nonzero flows into 1 are those coming from primes:

$$w(m,1) = 0 \qquad \textit{whenever } m \geq 2 \textit{ is composite.}$$

Proof. The function w defined above is nonnegative and is zero unless $M \mid N$ and $M < N$. Thus it is a divisibility flow.

Claim 1 gives

$$\mathrm{Out}(r) = \frac{1}{r\log r}$$

for every $r \geq 2$. In particular,

$$\mathrm{Out}(r) \geq \frac{1}{r\log r},$$

with equality whenever r is prime.

Claim 2 gives

$$\mathrm{Out}(r) \geq \mathrm{In}(r)$$

for every $r \geq 2$.

Claim 3 gives

$$w(m,1) = 0$$

for every composite $m \geq 2$. Hence all three asserted properties hold. □

Theorem 1 (Main result)**.** *For every primitive set $A \subseteq \mathbb{N}_{\geq 2}$,*

$$\sum_{a\in A} \frac{1}{a\log a} \leq \sum_{p\in\mathcal{P}} \frac{1}{p\log p}.$$

Consequently,

$$\sum_{n\in A} \frac{1}{n\log n}$$

is maximized over primitive sets $A \subseteq \mathbb{N}_{\geq 2}$ by taking $A = \mathcal{P}$.

Proof. First suppose that A is finite. Define

$$\Omega := \{d \geq 2 : \ d \mid a \text{ for some } a \in A\}.$$

Then Ω is finite, $A \subseteq \Omega$, and Ω is downward closed above 1: if $d \in \Omega$, $e \mid d$, and $e \geq 2$, then $e \in \Omega$.

For this finite set Ω, define

$$\mathrm{Out}(\Omega) := \sum_{\substack{m\in\Omega,\ n\notin\Omega \\ n\mid m,\ n<m}} w(m,n)$$

and

$$\mathrm{In}(\Omega) := \sum_{\substack{m\notin\Omega,\ n\in\Omega \\ n\mid m,\ n<m}} w(m,n).$$

The finite divergence identity is

$$\sum_{r\in\Omega} \big(\mathrm{Out}(r) - \mathrm{In}(r)\big) = \mathrm{Out}(\Omega) - \mathrm{In}(\Omega).$$

Indeed, after expanding the left-hand side, every term $w(m,n)$ with both $m, n \in \Omega$ occurs once with a positive sign and once with a negative sign. Terms leaving Ω occur once with a positive sign, and terms entering Ω occur once with a negative sign.

We first compute the possible nonzero terms in $\mathrm{Out}(\Omega)$. Let $m \in \Omega$, and suppose $n \notin \Omega$, $n \mid m$, and $n < m$. If $n \geq 2$, then the downward closure of Ω gives $n \in \Omega$, a contradiction. Hence $n = 1$. By Conjecture 1, the only nonzero flows into 1 come from primes. Therefore

$$\mathrm{Out}(\Omega) = \sum_{\substack{p\in\Omega \\ p \text{ prime}}} w(p,1).$$

If p is prime, then its only proper divisor is 1, so

$$w(p,1) = \mathrm{Out}(p).$$

By Conjecture 1,

$$\mathrm{Out}(p) = \frac{1}{p\log p}.$$

Thus

$$\mathrm{Out}(\Omega) = \sum_{\substack{p\in\Omega \\ p \text{ prime}}} \frac{1}{p\log p} \leq \sum_{p\in\mathcal{P}} \frac{1}{p\log p}.$$

Next, the finite divergence identity gives

$$\mathrm{Out}(\Omega) = \mathrm{In}(\Omega) + \sum_{r\in\Omega} \big(\mathrm{Out}(r) - \mathrm{In}(r)\big).$$

By Conjecture 1,

$$\mathrm{Out}(r) - \mathrm{In}(r) \geq 0$$

for every $r \geq 2$. Since $A \subseteq \Omega$,

$$\mathrm{Out}(\Omega) \geq \mathrm{In}(\Omega) + \sum_{a \in A} \big(\mathrm{Out}(a) - \mathrm{In}(a)\big).$$

We claim that every nonzero flow entering a point $a \in A$ enters Ω from outside Ω. Such a flow has the form

$$m \longrightarrow a$$

with

$$a \mid m \qquad \text{and} \qquad m > a.$$

If $m \in \Omega$, then $m \mid b$ for some $b \in A$. Hence

$$a \mid m \mid b.$$

Since A is primitive and $a, b \in A$, this forces $a = b$. Then $m \mid a$, contradicting $m > a$. Therefore $m \notin \Omega$.

It follows that the full inflow into every $a \in A$ is counted in $\mathrm{In}(\Omega)$. Hence

$$\mathrm{In}(\Omega) \geq \sum_{a \in A} \mathrm{In}(a).$$

Combining this with the preceding inequality gives

$$\mathrm{Out}(\Omega) \geq \sum_{a \in A} \mathrm{In}(a) + \sum_{a \in A} \big(\mathrm{Out}(a) - \mathrm{In}(a)\big) = \sum_{a \in A} \mathrm{Out}(a).$$

By Conjecture 1,

$$\mathrm{Out}(a) \geq \frac{1}{a \log a}$$

for every $a \in A$. Therefore

$$\sum_{a \in A} \frac{1}{a \log a} \leq \sum_{a \in A} \mathrm{Out}(a) \leq \mathrm{Out}(\Omega) \leq \sum_{p \in \mathcal{P}} \frac{1}{p \log p}.$$

This proves the desired inequality when A is finite.

Now let $A \subseteq \mathbb{N}_{\geq 2}$ be an arbitrary primitive set. For every finite subset $A_0 \subseteq A$, the finite case gives

$$\sum_{a \in A_0} \frac{1}{a \log a} \leq \sum_{p \in \mathcal{P}} \frac{1}{p \log p}.$$

Taking the supremum over all finite subsets $A_0 \subseteq A$ gives

$$\sum_{a \in A} \frac{1}{a \log a} \leq \sum_{p \in \mathcal{P}} \frac{1}{p \log p}.$$

The set of primes $\mathcal{P}$ is primitive, and for $A = \mathcal{P}$ the left-hand side is equal to the right-hand side. Hence the maximum over primitive sets is attained by the set of primes. $\square$