When is the ring-to-ring growth ratio $I(d+1)/I(d)$ greater than 1?

1 Setup

For an integer $d \geq 2$, define

$$I(d) \;=\; \int_{2^{d-1}}^{2^d} \frac{dt}{(\ln t)^2}.$$

This is the integral that appears in the Hardy–Littlewood-style density model for the number of twin-prime primes in the d-bit ring $[2^{d-1}, 2^d)$; the empirical count blue(d) tracks $4C_2 \cdot I(d)$ to within about 0.2% for d in the range checked against real data.

The ring-to-ring growth ratio is

$$\text{growth}(d) \;=\; \frac{I(d+1)}{I(d)}.$$

Question. Is growth(d) > 1 for every d?

Answer. No. It fails at $d = 2$: numerically $I(3)/I(2) \approx 0.6889 < 1$. But there is a clean proof that it *does* hold for all $d \geq 4$.

2 The proof (valid for $d \geq 4$)

Theorem 1. *For every integer $d \geq 4$, $I(d+1) > I(d)$.*

Proof. Start from

$$I(d+1) \;=\; \int_{2^d}^{2^{d+1}} \frac{dt}{(\ln t)^2}.$$

Substitute $t = 2s$, so $dt = 2\,ds$; the bounds $t = 2^d$ and $t = 2^{d+1}$ become $s = 2^{d-1}$ and $s = 2^d$. Then

$$I(d+1) \;=\; \int_{2^{d-1}}^{2^d} \frac{2\,ds}{(\ln 2 + \ln s)^2}.$$

This is now an integral over exactly the same range as $I(d) = \int_{2^{d-1}}^{2^d} \frac{ds}{(\ln s)^2}$, so the two can be compared pointwise:

$$I(d+1) - I(d) \;=\; \int_{2^{d-1}}^{2^d} \left[\frac{2}{(\ln 2 + \ln s)^2} - \frac{1}{(\ln s)^2} \right] ds.$$

Sign of the integrand. Let $x = \ln s$. The bracketed term has the same sign as

$$2x^2 - (x + \ln 2)^2 \;=\; x^2 - 2\ln 2 \cdot x - \ln^2 2,$$

a quadratic in x with roots

$$x \;=\; \ln 2 \left(1 \pm \sqrt{2}\right).$$

Since $s \geq 2$ on the domain of interest, $x = \ln s > 0$, so only the positive root is relevant:

$$x^* \;=\; \ln 2\,(1 + \sqrt{2}), \qquad s^* \;=\; e^{x^*} \;=\; 2^{\,1+\sqrt{2}} \;\approx\; 5.330.$$

Because the quadratic opens upward, the integrand is **negative** for $s < s^*$ and **positive** for $s > s^*$.

When the whole range clears the threshold. The integration range is $[2^{d-1}, 2^d]$. If its left endpoint already exceeds s^*, i.e.

$$2^{d-1} \;\geq\; 2^{\,1+\sqrt{2}} \quad\Longleftrightarrow\quad d \;\geq\; 2 + \sqrt{2} \;\approx\; 3.414,$$

then *every* point of the range has a positive integrand, and the integral $I(d+1) - I(d)$ is a strictly positive quantity. Since d is an integer, this holds for all

$$d \;\geq\; 4.$$

Hence $I(d+1) > I(d)$ for all $d \geq 4$. $\square$

3 Why it fails at $d = 2$

d	$I(d)$	$I(d+1)$	ratio
2	1.9224	1.3243	0.6889
3	1.3243	1.3424	1.0136
4	1.3424	1.6229	1.2090

Table 1: $I(d)$, $I(d+1)$, and the growth ratio near the boundary of the proof.

At $d = 2$ the range $[2, 4]$ lies entirely *below* the threshold $s^* \approx 5.33$, so the integrand is negative at every point of the range – exactly what the proof's own case analysis predicts, and exactly what the numbers confirm: $I(3)/I(2) \approx 0.6889$.

At $d = 3$ the range $[4, 8]$ straddles s^*: part of the integrand is negative, part positive. The proof does not determine the sign of the total in this case; direct computation shows the positive part wins, but only barely (1.0136).