

Assume two trackers  $A$  and  $B$  in different spaces (i.e. different reference frames) have a set of paired observations with a constant relative transform between them: positions  $\{\vec{p}_{A0}, \vec{p}_{A1}, \dots, \vec{p}_{AN}\}$  and  $\{\vec{p}_{B0}, \vec{p}_{B1}, \dots, \vec{p}_{BN}\}$  with orientations  $\{Q_{A0}, Q_{A1}, \dots, Q_{AN}\}$  and  $\{Q_{B0}, Q_{B1}, \dots, Q_{BN}\}$  (where  $Q$ 's are rotation matrices).

Let  $F$  be the constant transform from  $B$ 's reference frame to  $A$ 's and  $P$  be the constant transform from  $B$ 's position and orientation to  $A$ 's. For shorthand let  $F_r$  and  $P_r$  be the 3x3 rotation submatrices of  $F$  and  $P$ .

Then:

$$\forall i : \vec{p}_{Ai} = PF\vec{p}_{Bi} \quad (1)$$

$$\forall i : Q_{Ai} = P_r F_r Q_{Bi} \quad (2)$$

Trying to solve (1) on its own is not possible since there is ambiguity between  $P$  and  $F$ . Solving (1) and (2) simultaneously might be possible, but the expansion gets very tedious and I don't know enough linear algebra to know of an easy linear least squares type of solution.

We can keep going though and come up with some other properties.

$$\forall i, j : Q_{Ai}^T (\vec{p}_{Ai} - F\vec{p}_{Bi}) = Q_{Aj}^T (\vec{p}_{Aj} - F\vec{p}_{Bj}) \quad (3)$$

$$\forall i, j : |\vec{p}_{Ai} - F\vec{p}_{Bi}| = |\vec{p}_{Aj} - F\vec{p}_{Bj}| \quad (4)$$

(3) states the difference in position for a pair of samples is constant after compensating for rotation. (4) states the distance between the trackers is constant. With these we remove the need to solve for  $F$  and  $P$  simultaneously. It's tempting to try solving (4) but this ends up in quadratic land if done directly. Maybe you know a good way. If we try to solve (3) we get:

$$\forall i, j : Q_{Ai}^T \vec{p}_{Ai} - Q_{Aj}^T \vec{p}_{Aj} = Q_{Ai}^T F\vec{p}_{Bi} - Q_{Aj}^T F\vec{p}_{Bj} \quad (5)$$

In (5) the left side is known but I still don't know how to solve for  $F$  since the relative positions are dependent on the rotation. Next, let's try to look for some rotation properties since it doesn't depend on position.

$$\forall i, j : Q_{A_i} Q_{A_j}^T = F_r Q_{B_i} Q_{B_j}^T \quad (6)$$

Now this looks solvable! The idea is that the change in rotation between pairs of samples should be the same for both trackers. Some sort of SVD-based least squares solution should work. If you take the axis of rotation of  $Q_{A_i} Q_{A_j}^T$  and  $Q_{B_i} Q_{B_j}^T$  you can produce pairs of vectors as inputs to the Kabsch algorithm to produce the rotation matrix  $F_r$ .

Once we have  $F_r$  and apply it to the calibration points, (4) becomes much easier to solve. With  $\vec{F}_p$  representing the translation of  $F$ :

$$\forall i, j : Q_{A_i}^T (\vec{p}_{A_i} - \vec{p}_{B_i} - \vec{F}_p) = Q_{A_j}^T (\vec{p}_{A_j} - \vec{p}_{B_j} - \vec{F}_p) \quad (7)$$

$$\forall i, j : (Q_{A_j}^T - Q_{A_i}^T) \vec{F}_p = Q_{A_j}^T (\vec{p}_{A_j} - \vec{p}_{B_j}) - Q_{A_i}^T (\vec{p}_{A_i} - \vec{p}_{B_i}) \quad (8)$$

(8) is a linear equation, solvable with standard methods.