Degree bounds for closure properties of D-algebraic functions

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Can we predict the "size" (degree of the polynomial) of those equations?

D-algebraic functions

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A morphism of differential rings is a ring homomorphism that commutes with the derivation of each ring.

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- $R\{y\} := R[y, y', y'', y^{(3)}, \dots]$ with the unique derivation extending that of R and $\partial(y^{(i)}) = y^{(i+1)}$.

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Universal evaluation morphism

Let $\varphi : R \to R'$. For any $f \in R'$ there exists a unique differential morphism $\tilde{\varphi}_f : R\{y\} \to R'$ extending φ and mapping y to f.

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For any $P \in R\{y\}$ we write $P(f) := \tilde{\varphi}_f(P) = P(f, f', f'', \dots)$.

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- On the nature of the generating series of walks in the quarter plane. [Dreyfus, Hardouin, Roques, Singer, *Inventiones mathematicae* 2018]

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What about degree bounds ?

Degree bounds (for D-regular sequences)

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Theorem

If $P_1, \ldots, P_n$ is D-regular at order $r - r_l$ for $r \geqslant r_{\min}$ then $\langle P_1, \ldots, P_n \rangle^{(r-r_l)} \cap K[y_l, y_l', \ldots, y_l^{(r)}]$ has a non zero element of degree k if

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Previous works

- An elimination theory for differential algebra. [A. Seidenberg, *Univ. California Publ. Math III* 1956]
- Standard Bases in Differential Algebra. [François Ollivier, *Springer*, 1991]
- A survey on differential Gröbner bases. [G. Carra-Ferro, *Gröbner bases in Symbolic Analysis*, 2007]
- Differential elimination by differential specialization of Sylvester style matrices. [Sonia Rueda, *Advances in Applied Mathematics 72*, 2016]

Previous works

- D-algebraic functions. [Manssour, Sattelberger, Teguia Tabuguia, *Journal of Symbolic Computation*, 2025]

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- Projecting dynamical systems via a support bound. [Mukhina, Pogudin, *Projecting dynamical systems via a support bound*, 2025]

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$$I^{(k)}$$

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$$I^{(\infty)} := \bigcup_{k\in\mathbb{N}} I^{(k)}$$

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Couterexample: (Mansour, Sattelberger, Teguia Tabuguia)

$$P_1(y_1) = y_1 y_1'' - (y_1')^2,\ P_2(y_2) = y_2^2 + y_2'^4,\ \alpha(y_1, y_2) = y_1 + y_2.$$

No nontrivial intersection in $I^{(3)}$.

Hilbert function, polynomial, dimension

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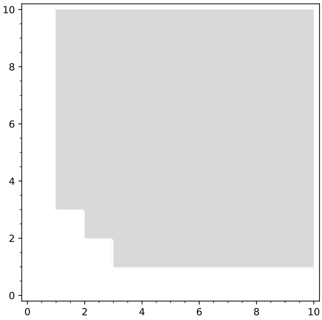

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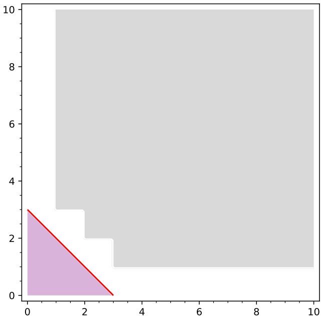

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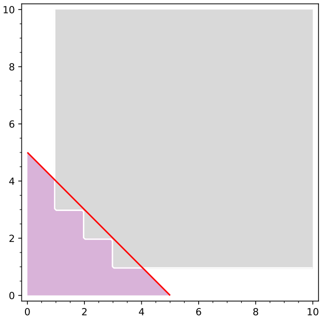

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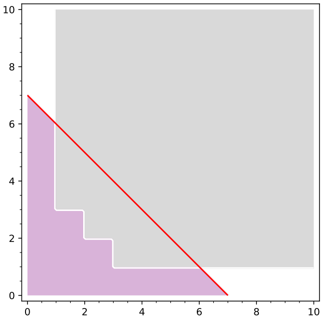

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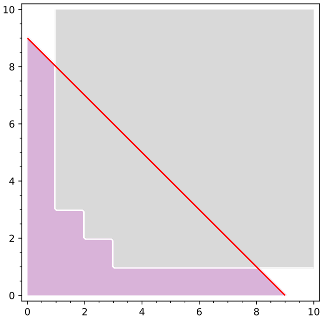

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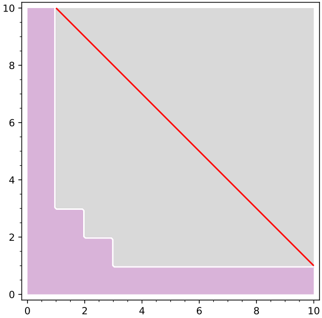

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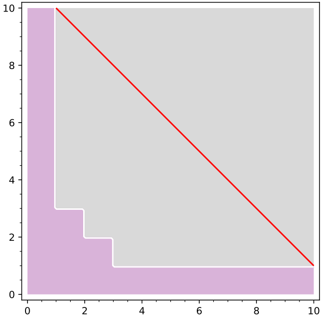

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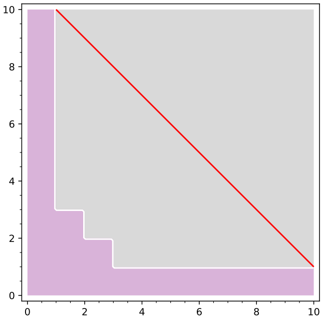

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- $\mathrm{HS}_I(t) = \sum_{k\in\mathbb{N}} \mathrm{HF}_I(k) t^k$.

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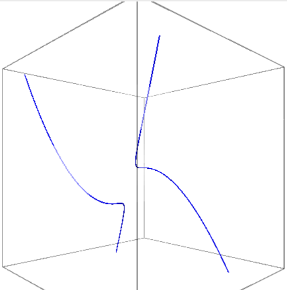

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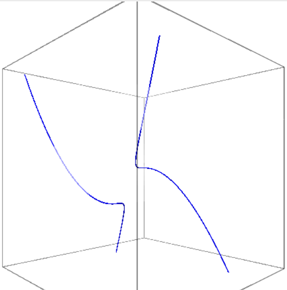

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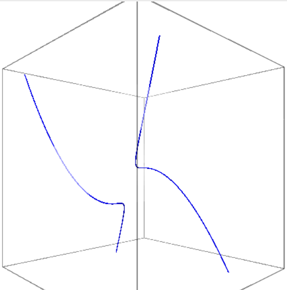

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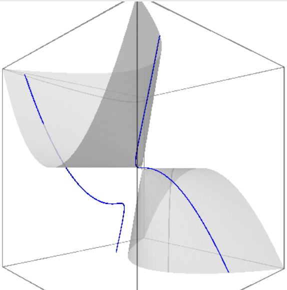

$\mathbb{V}(P_1, P_2)$
$\to P_3$?

Regular sequences

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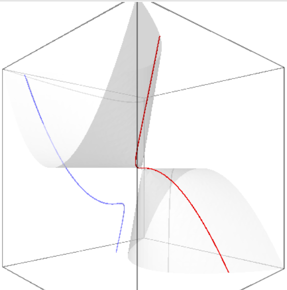

$\mathbb{V}(P_1, P_2, P_3)$ contains a curve of codimension 2. **WRONG**

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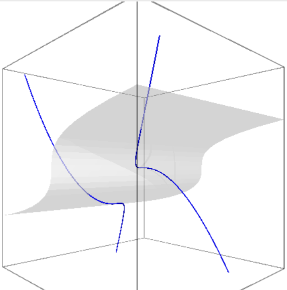

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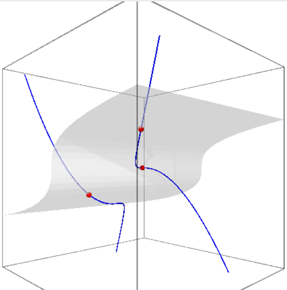

$\mathbb{V}(P_1, P_2, P_3)$ is a collection of points of codim 3.

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Lemma (Folklore)

If $P_1, \dots, P_l \in R$ is a regular sequence of homogeneous polynomials of degree $d_1, \dots, d_l$ respectively then

$$\mathrm{HS}_{\langle P_1, \dots, P_n\rangle}(t) = \frac{(1-t^{d_1})\dots(1-t^{d_l})}{(1-t)^n}$$

General idea

Given: $I = \langle P_1, \ldots, P_n \rangle \subset K[y_1^{(k\leqslant r_1)}, \ldots, y_n^{(k\leqslant r_n)}]$.

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If everything is regular, for some l, more variables in $K[y_i^{(k \leqslant r_i+l)}]$ than $\dim I^{(l)}$.

Degree bounds (for D-regular sequences)

Let $P_1, \ldots, P_n \in K[y_1^{(k \leqslant r_1)}, \ldots, y_n^{(k \leqslant r_n)}]$ of degree $d_1, \ldots, d_n$ respectively

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We say that $P_1, \ldots P_n \in K\{y_1, \ldots, y_n\}$ is D-regular at order ρ iff the tuple $(P_i^{(j)})_{\substack{1 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant \rho}}$ is a regular sequence.

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Theorem

If $P_1, \ldots, P_n$ is D-regular at order $r - r_l$ for $r \geqslant r_{\min}$ then $\langle P_1, \ldots, P_n \rangle^{(r - r_l)} \cap K[y_l, y_l', \ldots, y_l^{(r)}]$ has a non zero element of degree k if

$$k > (r+1)(d^{1 + (r_{\min} - r_l)/(r + 1 - r_{\min})} - 1).$$

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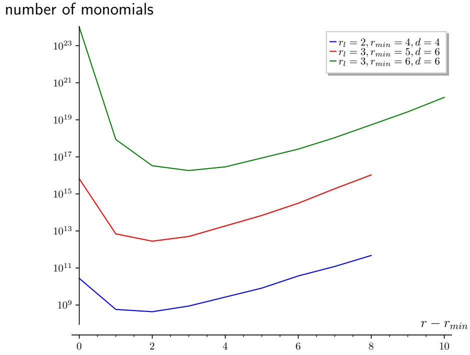

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Corollary

Let $f_1, \ldots, f_n$ be D-algebraic functions, given with equations $P_i \in K[y_i^{(k\leqslant r_i)}]$ and $Q \in K[y_1^{(k\leqslant r_1)}, \ldots, y_n^{(k\leqslant r_n)}]$.

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$Q(f_1, \cdots, f_n)$ is a solution of a D-algebraic equation of order r and degree k if

$$k > (r+1)((\deg(Q)d)^{1+r_{\min}/(r-r_{\min}+1)} - 1).$$

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Suppose that P_1 is D-regular at order r_2 and P_2 is D-regular at order r_1.

Then $f_1 \circ f_2$ is solution of an equation of order $r_1 + r_2$ and degree k if

$$k > (r_1 + r_2 + 1)((r_1 + r_2 + 1)! d_1^{r_2} d_2^{r_1} - 1).$$

Open questions

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- Getting rid of the D-regular hypothesis.

Thank you for your attention