Factoring linear differential operators in positive characteristic

JNCF 2023

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Algebra of differential operators

Object of study

Differential operators in $\mathbb{F}_p(x)\langle\partial\rangle = \{a_n(x)\partial^n + \cdots + a_1(x)\partial + a_0(x)\}$.

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Running example:

$(2x^6 + 2)\partial^6 + (x^6 + 2)\partial^3 + 2x^6 + 2x^3 + 2 \in \mathbb{F}_3(x)\langle\partial\rangle$

State of the art

- M. van der Put. Modular methods for factoring differential operators. Unpublished manuscript, 1997.
- M. Giesbrecht, Y. Zhang, Factoring and decomposing Ore polynomials over $\mathbb{F}_p(t)$, ISSAC 2003.
- T. Cluzeau, factorisation of differential systems in characteristic p, ISSAC 2003.
- X. Caruso, J. Le Borgne. A new faster algorithm for factoring skew polynomials over finite fields. JSC 2017.
- J. Gomez-Torrecillas, F. J. Lobillo, G. Navarro, Computing the bound of an Ore polynomial. Applications to factorisation, JSC 2019.

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Notation

$C := \mathbb{F}_p(x^p)$ is the field of constants of $\mathbb{F}_p(x)$.
$C[\partial^p]$ is the center of $\mathbb{F}_p(x)\langle\partial\rangle$.

Main tools

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Contribution

An algorithm able to fully factor central differential operators which will extend to operators with coefficients in finite separable extensions of $\mathbb{F}_p(x)$.

A guideline: studying the submodules of $\mathbb{F}_p(x)\langle\partial\rangle/\mathbb{F}_p(x)\langle\partial\rangle L$

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- $\mathcal{D}_L := \mathbb{F}_p(x)\langle\partial\rangle/\mathbb{F}_p(x)\langle\partial\rangle L$.
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Direct consequence of $\mathbb{F}_p(x)\langle\partial\rangle$ being a left principal ideal domain.

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Facts

$\mathcal{D}_{N(\partial^p)}$ is a C_N-algebra ($Y \mapsto \partial^p$)

Structure results

Proposition (van der Put)

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Corollary

$\mathcal{D}_{N(\partial^p)}$ is either a division algebra or isomorphic to $M_p(C_N)$.

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Fact

If L is a divisor of $N(\partial^p)$ then L is irreducible if and only if $\text{ord}(L) = \deg(N)$.

Extending the field of constants

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Recall: $\mathcal{D}_{N(\partial^p)} = {}^{\mathbb{F}_p(x)\langle\partial\rangle}/_{N(\partial^p)}$

$$\begin{tikzcd}
\mathbb{F}_p(x)\langle\partial\rangle \arrow[r, hook] \arrow[d] & K_N\langle\partial\rangle \arrow[d, two heads] \\
{}^{\mathbb{F}_p(x)\langle\partial\rangle}/_{N(\partial^p)} \arrow[r, dashed, "\varphi_N"] & {}^{K_N\langle\partial\rangle}/_{\partial^p - y_N}
\end{tikzcd}$$

"p-Riccati" equation

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Lemma (P., 2022)

Let $f \in K_N$ be such that $f^{(p-1)} + f^p = y_N$. Then

$$\partial^p - y_N = \mathrm{lclm}_{i=1}^{p}\left(\partial - f - \frac{i}{x}\right)$$

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The case when $L \neq N(\partial^p)$ (after van der Put)

$$L_N := \mathrm{gcrd}(\varphi_N(L), \partial^p - y_N)$$

is a non trivial divisor of $\partial^p - y_N$.

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Fact

If $L_N = \partial^m + a_{m-1}\partial^{m-1} + \cdots + a_0$ then $-\frac{a_{m-1}}{m}$ is a solution of the p-Riccati equation.

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The space of solutions of p-Riccati is an affine space

Factoring $N(\partial^p)$: when $K_N = \mathbb{F}_p(x)$

Suppose that $K_N = \mathbb{F}_p(x)$, $y_N = g^p \in \mathbb{F}_p(x^p)$
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Step 3: Solve an $\mathbb{F}_p$-linear system.

Factoring $N(\partial^p)$: general case

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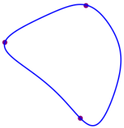

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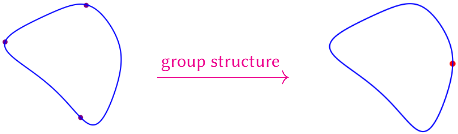

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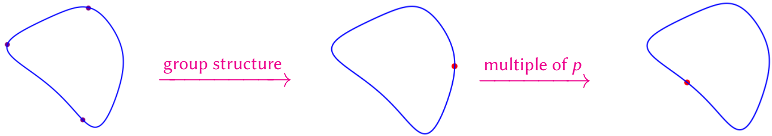

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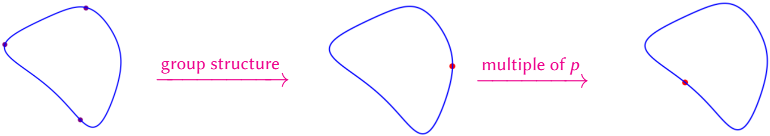

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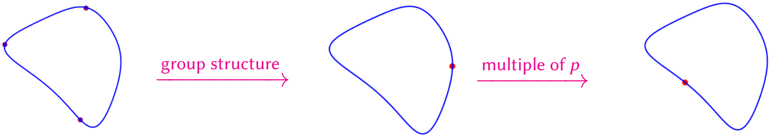

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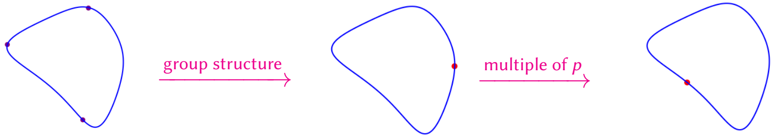

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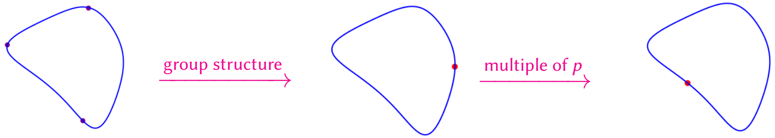

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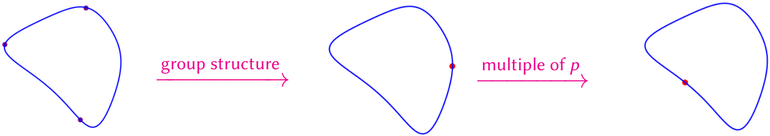

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Theorem (P., 2022)

If the p-Riccati has a solution in K_N then one of its solution has its poles located in the poles of y_N, in ramified places of K_N, a chosen place of degree 1 and in a set of places generating the cokernel of the multiplication by p on the Jacobian.

Algorithm

- Compute $f \in K_N$ such that $f^{(p-1)} + f^p = y_N$.

- Compute $L = \operatorname{gcrd}(\varphi_N^{-1}(\partial - f), N(\partial^p))$.

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  - Construct a divisor A_N such that we know that a solution lives in $\mathcal{L}(A_N)$.
  - Computes the Riemann-Roch space $\mathcal{L}(A_N)$.
  - Solve a $\mathbb{F}_p$-linear system over $\mathcal{L}(A_N)$
- Compute $L = \mathrm{gcrd}(\varphi_N^{-1}(\partial - f), N(\partial^p))$.

Running example

$$L_1 = \partial^2 + \left(\frac{2x^5 + x^4 + x^3 + 2x^2 + x + 1}{x^5 + x^4 + x^2 + 2x}\right)\partial + \frac{x^3 + x^2 + 2}{x^3 + x^2 + 2x}$$

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$$= \partial^2 + \left(\frac{2x^5 + x^4 + x^3 + 2x^2 + x + 1}{x^5 + x^4 + x^2 + 2x}\right)\partial + \frac{x^3 + x^2 + 2}{x^3 + x^2 + 2x}$$

Running example

$$(2x^6+2)\partial^6 + (x^6+2)\partial^3 + 2x^6 + 2x^3 + 2$$

$$= \partial^2 + \left(\frac{2x^5+x^4+x^3+2x^2+x+1}{x^5+x^4+x^2+2x}\right)\partial + \frac{x^3+x^2+2}{x^3+x^2+2x}$$

$$\times \partial^2 + \left(\frac{2x^3+x^2+1}{x^3+x}\right)\partial + \frac{x^{10}+x^9+x^8+x^5+x^4+2x^2+2}{x^{10}+2x^9+x^8+2x^7+x^5+x^4+x^3+x^2}$$

Running example

$$
\begin{aligned}
&(2x^6+2)\partial^6+(x^6+2)\partial^3+2x^6+2x^3+2\\
=&\partial^2+\left(\frac{2x^5+x^4+x^3+2x^2+x+1}{x^5+x^4+x^2+2x}\right)\partial+\frac{x^3+x^2+2}{x^3+x^2+2x}\\
\times&\partial^2+\left(\frac{2x^3+x^2+1}{x^3+x}\right)\partial+\frac{x^{10}+x^9+x^8+x^5+x^4+2x^2+2}{x^{10}+2x^9+x^8+2x^7+x^5+x^4+x^3+x^2}\\
\times&\left(2x^6+2\right)\partial^2+\left(\frac{x^8+x^7+2x^6+x^5+2x^3+x^2+2x+2}{x^2+x+2}\right)\partial\\
&+\frac{2x^{12}+2x^{10}+2x^9+2x^7+2x^6+x^5+2x+2}{x^6+2x^5+2x^4+x^3+x^2}
\end{aligned}
$$

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Lemma

Let $E_y = \left(\frac{\mathrm{d}}{\mathrm{d}y}N\right)(y_N)^{1/p}$ and N_k be the quotient of the euclidian division of N by y^k. Then for any $f = \sum_{i=0}^{d-1} f_i y_N^{i/p} \in K_N$ we have

$$f_k = \mathrm{Tr}\left(\frac{N_{k+1}(y_N)^{1/p}}{E_y} \cdot f\right)$$

Size of the p-Riccati solution

Let $\deg_x(N) = r$ and $\deg_y(N) = d$.

Heuristic

With the same notations as the previous slides, supposing $\mathcal{R}_N = 0$ (usual case in experiments), we observe

$$\mathcal{L}(A_N) \subset \frac{\mathbb{F}_p[x, y_N^{1/p}]_{\leqslant r, <d}}{E_y}$$

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  inverting $d \times d$ **polynomial matrix with coefficients of degree** $O(pr)$.
- Computing gcrd of $N(\partial^p)$ and $\partial - L(\partial^p)$ with $N \in C[Y]$ and $L \in \mathbb{F}_p(x)[Y]$.
  naive approach manipulates objects of size $O(p^2 rd)$

Future works

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Thank you for your attention