

Testing the irreducibility of central differential operators with a local-to-global principle

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Algebra of differential operators

Object of study

Differential operators in $k(t)\langle\partial\rangle = \{a_n(t)\partial^n + \cdots + a_1(t)\partial + a_0(t)\}$.

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If k is a field of characteristic p , $k(t^p)[\partial^p]$ is the center.

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Fact:

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Assumption:

$L \in \mathbb{F}_q(t)\langle\partial\rangle$ is a divisor of $N(t^p, \partial^p)$ where $N \in \mathbb{F}_q[X, Y]$ is irreducible.

State of the art on factorisation

- M. van der Put. Modular methods for factoring differential operators. Unpublished manuscript, 1997.
- M. Giesbrecht, Y. Zhang, Factoring and decomposing Ore polynomials over $\mathbb{F}_p(t)$, ISSAC 2003.
- T. Cluzeau, factorisation of differential systems in characteristic p , ISSAC 2003.
- J. Gomez-Torrecillas, F. J. Lobillo, G. Navarro, Computing the bound of an Ore polynomial. Applications to factorisation, JSC 2019.

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Theorem (van der Put)

If $N(t^p, \partial^p)$ is reducible, the irreducible divisors of $N(t^p, \partial^p)$ are parametrized by the solutions of

$$f^{(p-1)} + f^p = y_N$$

in K_N . If $N(t^p, \partial^p)$ is irreducible then this equation has no solution in K_N .

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Theorem (van der Put)

Let L be a non trivial divisor of $N(t^p, \partial^p)$ and

$\text{GCRD}(L, \partial^p - y_N) = \partial^m + l_{m-1}\partial^{m-1} + \cdots + l_0 \in K_N\langle\partial\rangle$. Then

$$l_{m-1}^{(p-1)} + l_{m-1}^p = -my_N.$$

Theorem (Pagès 2024)

*Let S be a set of places of K_N generating $\text{Pic}(K_N)/p\text{Pic}(K_N)$.
 If $N(t^p, \partial^p)$ is reducible then a solution of $f^{(p-1)} + f^p = y_N$ lives in the
 Riemann-Roch space*

$$\mathcal{L}\left(\max\left(\sum_{\mathfrak{P} \in S} \mathfrak{P} + \text{Diff}(K_N) - 2(t)_{\infty}, p^{-1} \cdot (y_N)_{\infty}\right)\right)$$

Problem: Computing $\text{Pic}(K_N)/p\text{Pic}(K_N)$ is hard !

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- Let y_N be a root of $N(t^p, Y)$ in a separable closure of $\mathbb{F}_q(t)$.
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$\mathcal{D}_N$ is a central simple C_N -algebra of dimension p^2 .

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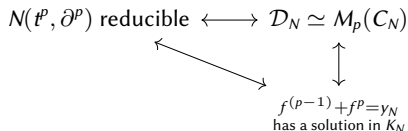
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Remember that $\mathbb{F}_{q'}((T))$ is the completion of K_N in a place $\mathfrak{P}$. If $\mathfrak{P}$ is tamely ramified, e is the ramification index of $\mathfrak{P}$.

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- $(T' \frac{d}{dT})^{(p-1)} f_1 + f_1^p = y_N + O(T^{p(pn+(p-1)e)})$.

Reduction to the poles of t and y_N

Corollary

If $(T' \frac{d}{dT})^{(p-1)} f + f^p = y_N + O(T^{p\nu(T)})$ has a solution in $\mathbb{F}_{q'}((T))$ then $(T' \frac{d}{dT})^{(p-1)} f + f^p = y_N$ has a solution in $\mathbb{F}_{q'}((T))$.

Reduction to the poles of t and y_N

Corollary

If $(T' \frac{d}{dT})^{(p-1)} f + f^p = y_N + O(T^{p\nu(T')})$ has a solution in $\mathbb{F}_{q'}((T))$ then $(T' \frac{d}{dT})^{(p-1)} f + f^p = y_N$ has a solution in $\mathbb{F}_{q'}((T))$.

In particular if $p^{-1} \cdot \nu(y_N) \geq \nu(T')$ then a solution exists.

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The equation $f^{(p-1)} + f^p = y_N$ has a solution in $K_{N,\mathfrak{P}}$ for all places $\mathfrak{P}$ which are not poles of y_N or infinity places.

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The equation $f^{(p-1)} + f^p = y_N$ has a solution in $K_{N,\mathfrak{P}}$ for all places $\mathfrak{P}$ which are not poles of y_N or infinity places.

For the remaining places checking the existence of a solution is a matter of solving a non invertible linear system of $(\nu(T') - p^{-1} \cdot \nu(y_N)) \log_p(q')$ equations in just as many variables.

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Thank you for your attention.