Electromagnetic Wave Propagation in Plasmas and applications in Laboratory Plasma Diagnostics

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In this article, we derive the general electromagnetic wave dispersion relation in a plasma, subject to a background magnetic field, using Maxwell's equations and the plasma fluid conservation laws. The plasma fluid conservation laws are derived from the Boltzmann equation. Different modes of electromagnetic wave propagation are derived from the non-trivial solution of the dispersion relation. Electromagnetic wave propagation is important for laser-based diagnostics of laboratory plasmas, in order to obtain plasma properties such as plasma density and magnetic field. A theoretical framework for the use of interferometry to determine the line-integrated plasma density, and the use of Faraday polarimetry to determine the magnetic field, is presented. Finally, the methodology of determination of the line-averaged plasma density from interferometry is also presented.

I. INTRODUCTION AND BACKGROUND

A plasma can be defined as ionized gas; a plasma has at least two components - delocalized negatively charged electrons, and positively charged ions[1]. Plasmas are the most ubiquitous state of matter in the universe - stellar media, interstellar matter, solar wind and solar corona, and cosmic rays, are all composed of plasma[4]. When the energy of the electron is high, electrons are able to overcome the Coulombic potential well of the atom, and become delocalized. On earth, plasmas are produced naturally by lightning, and are found in the earth's ionosphere[4]. Artificial plasmas are of particular interest in fusion reactors[1], as well as in plasma-synthesized nanotechnology[6].

Laboratory plasmas are also crucial to understanding the physics of astrophysical plasmas[52]. In the laboratory, plasmas can be generated by focusing a high-intensity laser beam onto a surface, or by introducing a large current through a cylindrical wire (a configuration known as a Z-pinch). A common approach is to use cylindrical wire arrays in pulsed-power systems, which drive fast rising ($\sim 100ns$) and high-amplitude ($\sim 1MA$) current pulses through inductive loads. When a large current is applied to an a solid wire, it begins to boil, vaporize and ionize, and eventually ablates plasma. The strong current also generates strong azimuthal magnetic fields around each wire; the magnetic field $B_\theta = \mu_0 I/(2\pi r)$ varies linearly with the current I and inversely with the radial distance r from a cylindrical load. The Lorentz force ($\mathbf{j}\times\mathbf{B}$) accelerates the ablated plasma radially outwards or inwards (depending on the geometry of the wire array). The magnetic fields are advected by the accelerated plasma, creating high velocity high energy density magnetized plasma flows. These high-energy density plasmas are critical to the investigation of plasma turbulence, magnetic reconnection and plasma shocks. Laser-based refractive index diagnostic enable us to visualize and determine fluctuations in plasma density, velocity, and magnetic field.

A plasma is said to exhibit collective behavior and quasi-neutrality[1]. Collective behavior means that the particles in a plasma interact over large distances via long-range electromagnetic forces. Quasi-neutrality means that although the plasma contains charged particles, the effective charge density in a plasma is zero. Any electric potential introduced into the plasma is shielded out exponentially over a characteristic length scale called the Debye length[1]. For quasi-neutrality to apply, the length scale of the plasma must be much larger than the Debye length, and there must be enough particles within a Debye sphere to shield out introduced potentials[1]. The Debye length λ_d is can be shown to be:

$$\lambda_d = \left(\frac{\varepsilon_0 k_B T_e}{ne^2}\right)^{1/2} \tag{1}$$

Here, ε_0 is the vacuum permittivity, k_b is the Boltzmann constant, T_e is the electron temperature, n is the plasma density, and e is the charge of a proton. In a plasma with singly-charged ions, quasi-neutrality implies that:

$$n_e \approx n_i \approx n \tag{2}$$

This is known as the plasma approximation.

In a plasma, quantum effects can be largely ignored if the electron wavelength is small compared to the average inter-particle separation. We assume the electron distribution function in an ionized plasma to be Maxwellian.

$$f(v) = n\left(\frac{m_e}{2\pi k_B T_e}\right)^{1/2} \exp\left(-\frac{v^2}{v_{th}^2}\right) \tag{3}$$

Here, $v_{th} = \sqrt{2k_B T_e/m_e}$ is the thermal velocity, and m_e is the electron mass. The thermal de Broglie electron wavelength λ_{th} is then:

$$\lambda_{th} = \frac{h}{mv_{th}} = h(2k_b T m_e)^{-1/2} \tag{4}$$

Here, h is the Plank constant. Thus, quantum size effects can be neglected when:

$$h(2k_B T m_e)^{-1/2} n^{1/3} \ll 1 \tag{5}$$

The refractive index of a plasma is different than that of vacuum. Thus, an electromagnetic wave propagating through a plasma accumulates a phase relative to a wave that propagates through vacuum. The phase difference can be used to determine the plasma density. In addition, the polarization of a linearly polarized wave is also changed by the action of a magnetised plasma. This is called the Faraday effect, and can be used to determine the magnetic field in a plasma. The goal of this article is to provide a theoretical description of electromagnetic wave propagation in plasmas, and the use of refractive-index based diagnostic methods to determine properties of the plasma through which the wave propagates.

This article is structured as follows. The plasma fluid mass and momentum conservation laws are derived from the zero-th and first moments of Boltzmann equation respectively. These conservation equations are then used together with Maxwell's equations and Ohm's law to derive expressions from the conductivity and dielectric tensors for a plasma. Then, the general dispersion relation in a plasma, for an electromagnetic wave propagating at an angle to the background magnetic field, is derived. The dispersion relation is solved to determine the principal modes of propagation and the refractive indices parallel and perpendicular to the unperturbed magnetic field. The refractive indices are then used to develop a theoretical framework for the use of laser interferometry to determine the line-integrated plasma density, and the use of Faraday polarimetry to determine the magnetic field. Finally, a methodology of determination of the line-averaged plasma density from the analysis of interferograms is presented.

II. CONSERVATION LAWS IN PLASMAS

In this section, we derive the plasma continuity and momentum equations from the Boltzmann equation. The Boltzmann equation relates the total rate of change of the distribution function $f(\mathbf{r},\mathbf{v},t)$ to a collisional source term $(\partial f/\partial t)_c$.

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{\mathbf{F}}{m}\cdot\nabla_{\mathbf{v}} f = \left(\frac{\partial f}{\partial t}\right)_c \tag{6}$$

Here, $\mathbf{r}$ and $\mathbf{v}$ are the position and velocity, $\mathbf{F}$ is the external force, and m is the mass.

In a plasma, the external force is given by the Lorentz force $\mathbf{F} = q(\mathbf{E}+\mathbf{v}\times\mathbf{B})$. Thus, the Boltzmann equation for a plasma becomes:

$$\begin{aligned}\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f \\ = \left(\frac{\partial f}{\partial t}\right)_c\end{aligned} \tag{7}$$

We can now take moments of Equation (7) to determine the plasma continuity and momentum conservation equations.

A. Continuity Equation

The density is defined as:

$$n(\mathbf{r},t) = \int_{-\infty}^{\infty} f d\mathbf{v} \tag{8}$$

Therefore, we take the zero-th moment of Equation (7):

$$\int \frac{\partial f}{\partial t} d\mathbf{v} + \int \mathbf{v}\cdot\nabla f d\mathbf{v} + \int \frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f d\mathbf{v} = \int \left(\frac{\partial f}{\partial t}\right)_c d\mathbf{v} \tag{9}$$

We evaluate the integrals term-by-term. The first term is just the time rate of change of the density n.

$$\int \frac{\partial f}{\partial t} d\mathbf{v} = \frac{\partial}{\partial t}\int f d\mathbf{v} = \frac{\partial n}{\partial t} \tag{10}$$

The second term can be written as:

$$\int \mathbf{v}\cdot\nabla f d\mathbf{v} = \int \nabla\cdot\mathbf{vf} - f\nabla\cdot\mathbf{v} \tag{11}$$

The second term on the right hand side vanishes because in phase space $\mathbf{v}$ is not a function of $\mathbf{r}$. Therefore, we get: The second term can be written as:

$$\int \mathbf{v}\cdot\nabla f d\mathbf{v} = \nabla\cdot\int \mathbf{v} f = \nabla\cdot(n\mathbf{u}) \tag{12}$$

Here, $\mathbf{u}$ is the mean velocity $\bar{\mathbf{v}}$.

Next, we look at the contribution of the electric field.

$$\int \mathbf{E}\cdot\nabla_{\mathbf{v}} f d\mathbf{v} = \int \nabla_{\mathbf{v}}\cdot\mathbf{E} f d\mathbf{v} = \int_S \mathbf{E} f\cdot d\mathbf{S} \tag{13}$$

Where, Gauss' theorem is used to convert the volume integral to a surface integral. Since $f\to 0$ falls to zero as $v\to\infty$ for a distribution function with finite energy, the term goes to 0.

The contribution of the magnetic field is:

$$\begin{aligned}\int (\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f d\mathbf{v} &= \int \nabla_{\mathbf{v}}\cdot f(\mathbf{v}\times\mathbf{B}) d\mathbf{v} \\ &\quad - \int f\nabla_{\mathbf{v}}\cdot(\mathbf{v}\times\mathbf{B}) d\mathbf{v}\end{aligned} \tag{14}$$

The second term vanishes since $\nabla_{\mathbf{v}}\cdot(\mathbf{v}\times\mathbf{B}) = \mathbf{0}$, and the first term can again be converted to a velocity surface integral evaluated at $v\to\infty$.

$$\int (\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f d\mathbf{v} = \int_S f(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S} = 0 \tag{15}$$

Finally, the zero-th moment of the collisonal term is 0 because particle collisions conserve particle number.

$$\int \left(\frac{\partial f}{\partial t}\right)_c d\mathbf{v} = 0 \tag{16}$$

Therefore, the continuity equation is:

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{u}) = 0 \tag{17}$$

B. Momentum Conservation Equation

The momentum is defined as:

$$\mathbf{p} = m\mathbf{u} = m\int_{-\infty}^{\infty} \mathbf{v} f d\mathbf{v} \tag{18}$$

Therefore, we take the first moment of Equation (7).

$$\begin{aligned} &m\int \mathbf{v}\frac{\partial f}{\partial t}d\mathbf{v} + m\int \mathbf{v}\mathbf{v}\cdot\nabla f d\mathbf{v} \\ &+ m\int \frac{q}{m}\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f d\mathbf{v} = m\int \mathbf{v}\left(\frac{\partial f}{\partial t}\right)_c d\mathbf{v} \end{aligned} \tag{19}$$

Again, we evaluate the equation term-by-term. The first term gives us the time rate of change of momentum density.

$$m\int \mathbf{v}\frac{\partial f}{\partial t}d\mathbf{v} = m\frac{\partial}{\partial t}\int \mathbf{v} f d\mathbf{v} = mn\frac{\partial \mathbf{u}}{\partial t} \tag{20}$$

The second term can we written as:

$$m\int \nabla\cdot(\mathbf{v}\mathbf{v}f)d\mathbf{v} = \nabla\cdot mn < \mathbf{v}\mathbf{v} > \tag{21}$$

Expressing the total velocity $\mathbf{v}$ as the superposition of a mean velocity $\mathbf{u}$ and the thermal velocity $\mathbf{w}$, we get:

$$\begin{aligned} mn < (\mathbf{u}+\mathbf{w})(\mathbf{u}+\mathbf{v}) > &= mn < \mathbf{u}\mathbf{u}+\mathbf{w}\mathbf{w}+\mathbf{u}\mathbf{w} > \\ &= mn\mathbf{u}\mathbf{u} + \mathbf{P} \end{aligned} \tag{22}$$

Here, we use $< \mathbf{w} >= 0$ and $\mathbf{P} = mn < \mathbf{w}\mathbf{w} >$ is the stress tensor.

The force term becomes:

$$\begin{aligned} &\int q\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f d\mathbf{v} \\ &= \int \nabla_{\mathbf{v}}\cdot q\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f d\mathbf{v} \\ &- \int f q(\mathbf{E}+\mathbf{v}\times\mathbf{B})d\mathbf{v} \end{aligned} \tag{23}$$

The first term becomes zero, as $f \to 0$ when $\mathbf{v} \to \infty$. The second term becomes the total Lorentz force.

$$\int_{S\infty} \nabla_{\mathbf{v}}\cdot q\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f d\mathbf{v} = \int_S q\mathbf{v}(\mathbf{E}+\mathbf{v}\times\mathbf{B}) f \cdot d\mathbf{S} = 0 \tag{24}$$

$$\int f q(\mathbf{E}+\mathbf{v}\times\mathbf{B})d\mathbf{v} = qn(\mathbf{E}+\mathbf{v}\times\mathbf{B}) \tag{25}$$

Finally the collision term represents the momentum gained by the fluid due to collisions with other fluids in the system.

$$m\int \mathbf{v}\left(\frac{\partial f}{\partial t}\right)_c d\mathbf{v} = \mathbf{P_{ij}} \tag{26}$$

Collecting terms, and using the plasma continuity equation, we get the plasma momentum equation:

$$mn\left(\frac{\partial \mathbf{u}}{\partial t} + \partial\mathbf{u}\cdot\nabla\partial\mathbf{u}\right) = -\nabla\cdot\mathbf{P} + qn(\mathbf{E}+\mathbf{v}\times\mathbf{B}) + \mathbf{P_{ij}} \tag{27}$$

III. ELECTROMAGNETIC DISPERSION RELATION

An electromagnetic wave obeys Maxwell's equations:

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \tag{28}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{29}$$

Here, μ_0 is the magnetic constant, $\boldsymbol{j}$ is the current density, $\boldsymbol{B}$ is the magnetic field, and $\boldsymbol{E}$ is the electric field.

Taking the curl of Equation (29) and substituting Equation (28), we get:

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}\nabla\times\mathbf{B} \tag{30}$$

$$\nabla\times\nabla\times\mathbf{E} = -\frac{\partial}{\partial t}(\mu_0\mathbf{j} + \mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t}) \tag{31}$$

If we express the electric and magnetic fields as a superposition of uniform homogeneous zero-th order components, and time and space varying first order components (of the form $\boldsymbol{E}_1, \boldsymbol{j}_1 \sim e^{-i\omega t}e^{i\mathbf{k}\cdot\mathbf{x}}$), and keep only the first-order terms, we can express the wave equation (31) as:

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = -\frac{i\omega}{c^2}\left(\frac{\mathbf{j}}{\varepsilon_0} - i\omega\mathbf{E}\right) \tag{32}$$

Here, we have made the substitution $c^2 = 1/\mu_0\varepsilon_0$. $\boldsymbol{k}$ and ω are the wavevector and frequency of the perturbed electric field and charge density. The subscript 1 has been dropped from first-order terms for simplicity. Using the identity $\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = \mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - \mathbf{k^2}\mathbf{E}$, we get:

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} + \frac{\omega^2}{c^2}\mathbf{E} + \frac{i\omega}{c^2\varepsilon_0}\mathbf{j} = 0 \tag{33}$$

Equation (33) represents the linearized electromagnetic wave equation. In a vacuum, $\mathbf{j} = 0$ and $\nabla\cdot\mathbf{E} = 0$, so the dispersion relation for an electromagnetic wave propagating in vacuum becomes:

$$\omega^2 = c^2k^2 \tag{34}$$

In a medium, using Ohm's law, the current density $\mathbf{j} = \boldsymbol{\sigma}\cdot\mathbf{E}$ can be written as the dot product of the conductivity tensor and the electric field. Thus, we get:

$$\left(\mathbf{k}\mathbf{k} + (\frac{\omega^2}{c^2} - k^2)\mathbf{I} + \frac{i\omega}{c^2\varepsilon_0}\boldsymbol{\sigma}\right)\cdot\mathbf{E} = 0 \tag{35}$$

Introducing the dielectric tensor $\boldsymbol{\varepsilon} = \mathbf{I} + i\boldsymbol{\sigma}/(\omega\varepsilon_0)$, we can write:

$$\left(\mathbf{kk} - k^2 \boldsymbol{I} + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}\right) \cdot \mathbf{E} = 0 \tag{36}$$

We can determine an expression for the conductivity tensor $\boldsymbol{\sigma}$, and hence for the dielectric tensor $\boldsymbol{\varepsilon}$ from:

$$\boldsymbol{j} = \sum_s \boldsymbol{j}_s = \sum_s n_s q_s \boldsymbol{v}_s \equiv \sum_s \boldsymbol{\sigma}_s \cdot \mathbf{E} \tag{37}$$

Where n_s, q_s and $\boldsymbol{v_s}$ are the number density, charge and the velocity of species s (electron and ion) in the plasma. From the plasma fluid momentum equation, we can determine the velocity as a function of the electric field. Here, we assume a cold plasma ($\nabla p \approx 0$) and neglect collisions.

$$m_s \frac{D\boldsymbol{v}_s}{Dt} = q_s\left(\mathbf{E} + \boldsymbol{v}_s \times \mathbf{B}\right) \tag{38}$$

Assuming the zero-th order magnetic field is $\boldsymbol{B}_0 = B_0\hat{\boldsymbol{z}}$, and that the zero-th order velocity and electric fields are zero, we linearize the momentum equation to get:

$$m_s \frac{\partial \boldsymbol{v}_s}{\partial t} = q_s\left(\mathbf{E} + \boldsymbol{v}_s \times \mathbf{B}_0\right) \tag{39}$$

Here, the subscript 1 has been dropped from first-order terms for simplicity.

$$-im_s\omega \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}_s = q_s \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} + q_s \begin{pmatrix} v_y B_0 \\ -v_x B_0 \\ 0 \end{pmatrix}_s \tag{40}$$

Or:

$$v_{x,s} = \frac{q_s}{m_s}\left(\frac{i\omega E_x - \Omega_s E_y}{\omega^2 - \Omega_s^2}\right) \tag{41}$$

$$v_{y,s} = \frac{q_s}{m_s}\left(\frac{i\omega E_y + \Omega_s E_x}{\omega^2 - \Omega_s^2}\right) \tag{42}$$

$$v_{z,s} = \frac{q_s}{m_s}\left(\frac{iE_z}{\omega}\right) \tag{43}$$

Here, $\Omega = qB_0/m$ is the cyclotron frequency, and represents the frequency with which a charged particle gyrates in the plane perpendicular to the background magnetic field.

The current density $\boldsymbol{j}$ is:

$$j_{x,s} = \varepsilon_0 \frac{q_s^2 n_s}{m_s \varepsilon_0}\left(\frac{i\omega E_x - \Omega_s E_y}{\omega^2 - \Omega_s^2}\right) = \varepsilon_0 \omega_{p_s}^2 \left(\frac{i\omega E_x - \Omega_s E_y}{\omega^2 - \Omega_s^2}\right) \tag{44}$$

$$j_{y,s} = \varepsilon_0 \frac{q_s^2 n_s}{m_s \varepsilon_0}\left(\frac{i\omega E_y + \Omega_s E_x}{\omega^2 - \Omega_s^2}\right) = \varepsilon_0 \omega_{p_s}^2 \left(\frac{i\omega E_y + \Omega_s E_x}{\omega^2 - \Omega_s^2}\right) \tag{45}$$

$$j_{z,s} = \varepsilon_0 \frac{q_s^2 n_s}{m_s \varepsilon_0}\left(\frac{iE_z}{\omega}\right) = \varepsilon_0 \omega_{p,s}^2 \left(\frac{iE_z}{\omega}\right) \tag{46}$$

Here, $\omega_p = \sqrt{q^2 n/(\varepsilon_0 m)}$ is the plasma frequency, and represents a high-frequency electrostatic plasma oscillation. Thus, the current density $\boldsymbol{j}$ can be represented as:

$$\boldsymbol{j}_s = \boldsymbol{\sigma}_s \cdot \mathbf{E} = \varepsilon_0 \omega_{p,s}^2 \begin{bmatrix} \frac{i\omega}{\omega^2 - \Omega_s^2} & \frac{-\Omega_s}{\omega^2 - \Omega_s^2} & 0 \\ \frac{\Omega_s}{\omega^2 - \Omega_s^2} & \frac{i\omega}{\omega^2 - \Omega_s^2} & 0 \\ 0 & 0 & i/\omega \end{bmatrix} \cdot \boldsymbol{E} \tag{47}$$

And the dielectric tensor $\boldsymbol{\varepsilon}$ is:

$$\boldsymbol{\varepsilon} = \mathbf{I} + i\frac{\sum_s \boldsymbol{\sigma}_s}{\omega \varepsilon_0} = \begin{bmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{bmatrix} \tag{48}$$

Where:

$$S = 1 - \sum_s \frac{\omega_{p,s}^2}{\omega^2 - \Omega_s^2} \tag{49}$$

$$D = \sum_s \frac{\Omega_s}{\omega}\frac{\omega_{p,s}^2}{\omega^2 - \Omega_s^2} \tag{50}$$

$$P = 1 - \sum_s \frac{\omega_{p,s}^2}{\omega^2} \tag{51}$$

In the case, where the ion contribution is small, i.e. $\omega\ \Omega_i >> 1$, we can approximate S, P and D as functions of the electron contribution only.

Assuming the wavevector k is in the yz plane and forms an angle θ with the unperturbed magnetic field $B_0\hat{z}$, the dyadic product kk becomes:

$$kk = k^2 \begin{bmatrix} \sin^2\theta & 0 & \sin\theta\cos\theta \\ 0 & 0 & 0 \\ \sin\theta\cos\theta & 0 & \cos^2\theta \end{bmatrix} \tag{52}$$

We can now construct the dispersion matrix as:

$$\mathbf{D} = \left(\mathbf{kk} - k^2\mathbf{I} + \frac{\omega^2}{c^2}\boldsymbol{\varepsilon}\right) \tag{53}$$

$$\mathbf{D} = \begin{bmatrix} k^2\sin^2\theta - k^2 + \frac{\omega^2}{c^2}S & -iD\frac{\omega^2}{c^2} & k^2\sin\theta\cos\theta \\ iD\frac{\omega^2}{c^2} & -k^2 + \frac{\omega^2}{c^2}S & 0 \\ k^2\sin\theta\cos\theta & 0 & k^2\cos^2\theta - k^2 + \frac{\omega^2}{c^2}P \end{bmatrix} \tag{54}$$

$$\mathbf{D} = \begin{bmatrix} -k^2\cos^2\theta + \frac{\omega^2}{c^2}S & -iD\frac{\omega^2}{c^2} & k^2\sin\theta\cos\theta \\ iD\frac{\omega^2}{c^2} & -k^2 + \frac{\omega^2}{c^2}S & 0 \\ k^2\sin\theta\cos\theta & 0 & -k^2\sin^2\theta + \frac{\omega^2}{c^2}P \end{bmatrix} \tag{55}$$

Multiplying with c^2/ω^2 throughout, and using the relation $\mathbf{N} = \mathbf{k}c/\omega$ for the refractive index $\mathbf{N}$:

$$\mathbf{D_N}\cdot\mathbf{E} = \begin{bmatrix} -N^2\cos^2\theta + S & -iD & N^2\sin\theta\cos\theta \\ iD & -N^2 + S & 0 \\ N^2\sin\theta\cos\theta & 0 & -N^2\sin^2\theta + P \end{bmatrix}\cdot\mathbf{E} = 0 \tag{56}$$

For non-trivial solutions the determinant of the dispersion matrix must be zero, i.e $\det(\mathbf{D_N}) = 0$. Thus, the dispersion relation for an electromagnetic wave propagating an a plasma at an angle θ compared to the background magnetic field is:

$$\det\begin{bmatrix} -N^2\cos^2\theta + S & -iD & N^2\sin\theta\cos\theta \\ iD & -N^2 + S & 0 \\ N^2\sin\theta\cos\theta & 0 & -N^2\sin^2\theta + P \end{bmatrix} = 0 \tag{57}$$

For the special case where the unperturbed magnetic field is small, , i.e. $\Omega/\omega \approx 0$, and ignoring the ion contribution, we get:

$$S = 1 - \frac{\omega_{p,e}^2}{\omega^2 - \Omega_e^2} \approx 1 - \frac{\omega_{p,e}^2}{\omega^2} = P \tag{58}$$

$$D = \frac{\Omega_e}{\omega}\frac{\omega_{p,e}^2}{\omega^2 - \Omega_e^2} \approx 0 \tag{59}$$

$$P = 1 - \frac{\omega_{p,e}^2}{\omega^2} \tag{60}$$

The dispersion relation is:

$$\det\begin{bmatrix} -N^2\cos^2\theta + P & 0 & N^2\sin\theta\cos\theta \\ 0 & -N^2 + P & 0 \\ N^2\sin\theta\cos\theta & 0 & -N^2\sin^2\theta + P \end{bmatrix} = 0 \tag{61}$$

The only non-trivial solution is:

$$N^2 = P \tag{62}$$

Or

$$\omega^2 = c^2k^2 + \omega_p^2 \tag{63}$$

This wave propagating through the plasma is termed the ordinary wave.

For the case where the wave vector is perpendicular to the unperturbed magnetic field, i.e. $\theta = \pi/2$, the determinant of the dispersion tensor is:

$$\det\begin{bmatrix} S & -iD & 0 \\ iD & -N^2 + S & 0 \\ 0 & 0 & -N^2 + P \end{bmatrix} = 0 \tag{64}$$

$$(P - n^2)\left(S(S - n^2) - D^2\right) = 0 \tag{65}$$

The solutions are:

$$N^2 = P \tag{66}$$

$$N^2 = \frac{S^2 - D^2}{S} \tag{67}$$

The first solution is the same as that for the ordinary wave. The ordinary wave is a transverse ($\mathbf{k} \perp \mathbf{E}_1$) wave propagating perpendicular to the background magnetic field ($\mathbf{k} \perp \mathbf{E}_1$), with the perturbed electric field parallel to the background magnetic field ($\mathbf{E}_1 \parallel \mathbf{B}_0$). The second solution represents the extraordinary (X) wave, in which both the wavevector and the perturbed electric field are perpendicular to the background magnetic field ($\mathbf{k} \perp \mathbf{B}_0$, $\mathbf{E}_1 \perp \mathbf{B}_0$), and the perturbed electric field has components both perpendicular (transverse) and parallel (longitudinal) to the direction of propagation. For the extraordinary wave, the dispersion relation is:

$$\frac{c^2k^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}\left(\frac{\omega^2 - \omega_p^2}{\omega^2 - \omega_h^2}\right) \tag{68}$$

Here, $\omega_h^2 = \omega_p^2 + \omega_c^2$ is terms the upper hybrid frequency.

Similarly, when the wave vector is along the unperturbed magnetic field $\theta = 0$, we get:

$$\det\begin{bmatrix} -N^2 + S & -iD & 0 \\ iD & -N^2 + S & 0 \\ 0 & 0 & P \end{bmatrix} = 0 \tag{69}$$

$$P((S - N^2)^2 - D^2) = 0 \tag{70}$$

The solutions are:

$$P = 0, \quad N^2 = S + D, \quad N^2 = S - D \tag{71}$$

The solutions represent the electron plasma waves, and the right and left circularly polarized waves respectively. We can verify that the last two expressions result in circularly polarized waves by substituting $N^2 = S \pm D$ into the dispersion relation.

$$\frac{E_x}{E_y} = \frac{iD}{S - (S \pm D)} = \pm i \tag{72}$$

The dispersion relation for the right and left circularly polarized waves can be written as:

$$R = S + D = 1 - \frac{\omega_{pe}^2}{\omega(\omega + \Omega_e)} \tag{73}$$

$$L = S - D = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \Omega_e)} \tag{74}$$

Here, we have assumed that the ion contribution is small compared to the electron contribution.

These solutions apply for the case of a constant and uniform background medium. The Fourier transformation $\sim e^{-i\omega t}e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ no longer applies in media with spatial gradients. However, for the case where the medium is slowly varying, the WKB (Wentzel, Kramers Brillouin) approximation can be used:

$$E \sim \exp\left(\int \mathbf{k}\cdot\mathbf{dl}\right)\exp(-i\omega t) \tag{75}$$

The WKB approximation is valid when:

$$\frac{|\nabla k|}{k^2} << 1 \tag{76}$$

IV. REFRACTIVE INDEX BASED PLASMA DIAGNOSTICS

A. Laser Interferometry

Laser interferometry is a technique used to measure the line integrated plasma density, and is based on the principle of interference of two electromagnetic waves with different phases. In a typical interferometer, a monochromatic beam is split into a reference and a probe beam using a beam splitter. The probe beam is then allowed to propagate through the plasma, while the reference beam propagates via vacuum. Since the refractive index of the plasma is different than that of vacuum, the probing beam accumulates a different phase compared to the reference beam as it propagates through the plasma. The reference and probe beams, which now have a relative phase difference between them, are allowed to recombine and form an interference pattern at the target. The phase difference between the two beams can be determined form the interferogram, and the phase difference can then be related to the plasma density.

Let us consider two monochromatic electromagnetic waves:

$$\mathbf{E}_r = \mathbf{E}_{r,0}e^{-i\omega t}e^{i\mathbf{k_r}\cdot\mathbf{x}} \tag{77}$$

$$\mathbf{E}_p = \mathbf{E}_{p,0}e^{-i\omega t}e^{i\mathbf{k_p}\cdot\mathbf{x}} \tag{78}$$

The reference and probing waves are allowed to interfere at the target. The total field becomes:

$$\mathbf{E}_t = \mathbf{E}_r + \mathbf{E}_p = \left(\mathbf{E}_{r,0}e^{i\mathbf{k_r}\cdot\mathbf{x}} + \mathbf{E}_{p,0}e^{i\mathbf{k_p}\cdot\mathbf{x}}\right)e^{-i\omega t} \tag{79}$$

The total intensity is:

$$I_t = \mathbf{E}_t \cdot \mathbf{E}_t^* \tag{80}$$

$$\begin{aligned} I_t = &|\mathbf{E}_{r,0}|^2 + |\mathbf{E}_{p,0}|^2 \\ &+ \mathbf{E}_{r,0}\cdot\mathbf{E}_{p,0}\left(e^{i(\mathbf{k_1}-\mathbf{k_2})\cdot\mathbf{x}} + e^{-i(\mathbf{k_1}-\mathbf{k_2})\cdot\mathbf{x}}\right) \end{aligned} \tag{81}$$

Using the identity $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, we get:

$$I_t = |\mathbf{E}_{r,0}|^2 + |\mathbf{E}_{p,0}|^2 + 2\mathbf{E}_{r,0}\cdot\mathbf{E}_{p,0}\cos((\mathbf{k_1}-\mathbf{k_2})\cdot\mathbf{x}) \tag{82}$$

Or:

$$I_t = (|\mathbf{E}_{r,0}|^2 + |\mathbf{E}_{p,0}|^2)\left(1 + \frac{2\mathbf{E}_{r,0}\cdot\mathbf{E}_{p,0}}{|\mathbf{E}_{r,0}|^2 + |\mathbf{E}_{p,0}|^2}\cos\Delta\phi\right) \tag{83}$$

Where $\Delta\phi = (\mathbf{k_1}-\mathbf{k_2})\cdot\mathbf{x}$ is the phase difference between the reference and probe beams. Thus, we expect to observe a periodic fluctuation of the total intensity of the interferogram based on the value of the phase difference $\Delta\phi$ between the reference and probe beams.

Let us consider the situation where the plasma has a weak magnetic field. In this case, the ordinary wave is the predominant mode of propagation. From Equation (63), the dispersion relation is:

$$N^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega^2} \tag{84}$$

When $\omega = \omega_p$, the wave is cut-off. When $\omega < \omega_p$, the wave vector k becomes imaginary and the wave cannot propagate. The wave is said to be evanescent and is damped exponentially. The cutoff plasma density n_c is:

$$n_c = \frac{\omega^2 m\varepsilon_0}{e^2} \tag{85}$$

And the refractive index can be written as:

$$N^2 = 1 - n_e/n_c \tag{86}$$

The phase difference $\Delta\phi$ between the reference beam and the probing beam, using the WKB approximation, is:

$$\Delta\phi = \int(\mathbf{k_1}-\mathbf{k_2})\cdot\mathbf{dl} = \frac{\omega}{c}\int 1 - \sqrt{1-\omega_p^2/\omega^2}dl \tag{87}$$

In the limit where $\omega_p^2/\omega^2 << 1$, we can write:

$$\Delta\phi = \frac{1}{2c\omega}\int\omega_p^2 dl = \frac{\omega}{2cn_c}\int n_e dl \tag{88}$$

Thus, the phase difference between the reference and probe beams can be used to determine the line integrated plasma density.

B. Faraday Polarimetry

Faraday polarimetry is used to determine the magnetic field in a plasma. This technique is based on the Faraday effect, which causes a linearly polarized wave's electric field vector to rotate as it propagates through a medium where the principal modes are circularly polarized in the presence of a magnetic field.

Assuming that the direction of propagation of the wave is parallel to the background magnetic field, i.e. $\mathbf{B} = B\hat{\mathbf{z}}, \mathbf{k} \parallel \mathbf{B}$, then the perturbed electric field lies in the x-y plane. We assume that the wave is initially linearly polarized with $\mathbf{E} = E\hat{\mathbf{x}}$. The transverse components of the circularly polarized modes satisfy:

$$\frac{E_x}{E_y} = \pm i \tag{89}$$

The total electric field is the superposition of the right and left polarized waves:

$$\mathbf{E} = \frac{E}{2}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{ik_+ z} + \frac{E}{2}\begin{pmatrix} 1 \\ i \end{pmatrix} e^{ik_- z} \tag{90}$$

Here, k_+ and k_- are the wave vectors of the right and left polarized waves respectively, and $E = E_0 e^{-i\omega t}$ is the time varying amplitude of the wave.

The x-component of $\mathbf{E}$ can be written as:

$$E_x = \frac{E}{2}(e^{ik_+ z} + e^{ik_- z}) \tag{91}$$

The sum of the exponential terms can be written as:

$$\begin{aligned} e^{im} + e^{in} &= \cos m + \cos n + i(\sin m + \sin n) \\ &= 2\cos(\frac{m+n}{2})\cos(\frac{m-n}{2}) \\ &+ 2i\left(\sin(\frac{m+n}{2})\cos(\frac{m-n}{2})\right) \\ &= 2\exp\left(i\frac{m+n}{2}\right)\cos(\frac{m-n}{2}) \end{aligned} \tag{92}$$

Thus,

$$E_x = \frac{E}{2}(e^{ik_+ z} + e^{ik_- z}) = E\exp\left(iz\frac{k_+ + k_-}{2}\right)\cos\left(\frac{k_+ - k_-}{2}z\right) \tag{93}$$

Similarly, the y-component of the electric field can be shown to be:

$$E_y = i\frac{E}{2}(-e^{ik_+ z} + e^{ik_- z}) = E\exp\left(i\frac{k_+ + k_-}{2}z\right)\sin\left(\frac{k_+ - k_-}{2}z\right) \tag{94}$$

In terms of the refractive indices $N_\pm$ the propagating electric field is:

$$\mathbf{E} = E\exp\left(i\frac{N_+ + N_-}{2}\frac{\omega}{c}z\right)\begin{pmatrix} \cos(\Delta\phi/2) \\ \sin(\Delta\phi/2) \end{pmatrix} \tag{95}$$

Thus, the polarization of the electric field vector of a linearly polarized wave is rotated by an angle $\Delta\phi(z)$ with respect to the initial polarization of the wave. The Faraday rotation angle is:

$$\alpha(z) = \frac{\Delta\phi}{2} = \frac{N_+ - N_-}{2}\frac{\omega}{c}z \tag{96}$$

In a plasma with a magnetic field $B\hat{\mathbf{z}}$ and the wave vector $\mathbf{k} \parallel \mathbf{B}$, the right and left circularly polarized waves, as described by Equations (73) and (74), are obtained. The Faraday rotation angle can then be calculated from Equation (96).

$$\begin{aligned} \alpha(z) &= \left(\frac{R^{1/2} - L^{1/2}}{2}\right)\frac{\omega}{c}z = \frac{\Omega_e \omega_{pe}^2}{2c(\omega^2 - \Omega_e^2)}z \\ &\approx \frac{\Omega_e \omega_{pe}^2}{2c\omega^2}z \approx \frac{e^3}{2\varepsilon_0 m^2 \omega^2 c}(\mathbf{B}\cdot\hat{\boldsymbol{k}})n_e z \end{aligned} \tag{97}$$

In a slowly-varying plasma, and with an angle θ between the wave vector and the magnetic field, the rotation angle can be written as[3]:

$$\begin{aligned} \alpha &= \frac{\Delta\phi}{2} = \frac{1}{2}\int \frac{\omega_{pe}^2 \Omega_e \cos\theta}{\omega^2\left(1 - \omega_{pe}^2/\omega^2\right)^{1/2}}\frac{dl}{c} \\ &\approx \frac{e^3}{2c\varepsilon_0 m^2 \omega^2}\int n_e \mathbf{B}\cdot\mathbf{dl} \qquad \text{for } \frac{\omega_{pe}}{\omega} \ll 1 \end{aligned} \tag{98}$$

Thus, we can determine the line integrated product of the plasma density and magnetic field from Faraday polarimetry.

V. DETERMINATION OF PLASMA DENSITY USING INTERFEROMETRY

In this section, we provide an example to show how interferometry can be used to determine the plasma density in laboratory plasmas. In a simple interferometry setup, a beamsplitter splits a coherent laser beam into a reference beam and a probing beam. The probing beam is allowed to travel through the plasma, while the reference beam travels through vacuum. The reference and probe beams are then allowed to recombine at the target, which is typically a CCD. If the probing and reference beams are misaligned, we get an interference pattern. The interference pattern in the absence of the plasma is compared with that in the presence of the plasma to determine the phase shift introduced by the plasma.

In Figure 1, synthetic background and spot interferograms are shown. These synthetic interferograms are produced by calculating the total intensity of two plane electromagnetic waves whose wavefronts are misaligned by a small angle α, that interfere at the target. In order to determine the phase shift, we first trace the background and spot interferograms. Each bright of dark fringe represents a constant phase difference. The fringes are then indexed, and the region between the fringes is interpolated, as shown in Figure 2. The phase shift is determined by subtracting the spot and background phase maps, which is then converted into a line-averaged density map using Equation (88), as shown in Figure 3.

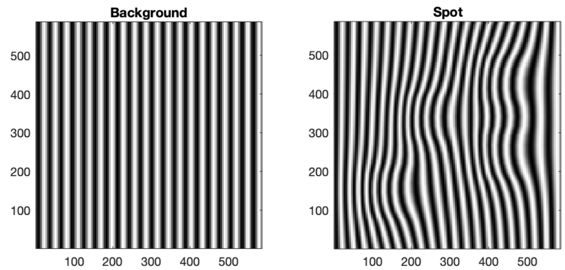

FIG. 1. Interference patterns from interference of reference and probing beams through vacuum (left) and plasma (right)

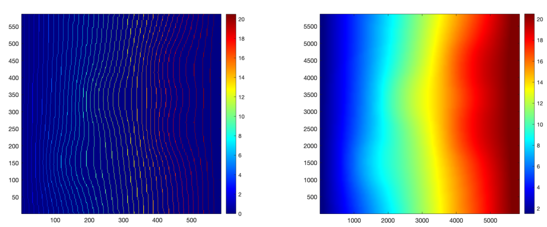

FIG. 2. The bright and dark fringes are traced and the area between them is interpolated

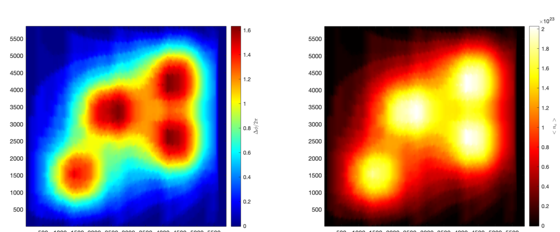

FIG. 3. Subtracting the spot phase from the background phase determines the phase map (left) which is then converted to an electron density map(right)

VI. CONCLUSIONS

In this article, we derive the plasma continuity and momentum equations from the Boltzmann equation. These conservation laws are then used to determine the conductivity tensor. The conductivity tensor is used in the wave equation to determine the dispersion relation for electromagnetic waves. The non-trivial solutions of the dispersion tensor give the principal modes of electromagnetic wave propagation through the plasma. The refractive index of a plasma is different compared to that of vacuum, so an electromagnetic wave propagating through the plasma accumulates a phase relative to a wave which propagates through vacuum. We show that this phase difference can be used to determine the line averaged plasma density using interferometry. When the principal modes of propagation are circularly polarized, the electric field vector of the wave is rotated as is propagates through the plasma. We show that this effect can be used to measure the line averaged magnetic field in a plasma.

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