Even integral parts of powers of square roots

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Abstract. Dubickas splits the half-line $(1, +\infty)$ into the set $\mathcal{Z}$ of those α for which some nonzero real ξ makes every integral part $[\xi\alpha^n]$ even, and its complement $\mathcal{S}$; at $\alpha = 3/2$ the question of which side one is on is Mahler's. We settle the side for every square root: $\sqrt{m} \in \mathcal{S}$ if and only if $m = 2$. In particular $\sqrt{3} \in \mathcal{Z}$, which answers Problem 3 of Dubickas's paper, with the explicit witness $\xi = 1.34160899796112665163\ldots$, and $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$ lie in $\mathcal{Z}$ as well — four points of $\mathcal{Z}$ in the interval $(2, 3)$, the only interval where an answer to Dubickas's Problem 5 could live, and the first there with a proof that are neither rational nor Pisot nor Salem. The mechanism is Cantor-set arithmetic rather than Diophantine approximation: because $\sqrt{m}^{\,2}$ is an integer, the two-scale problem collapses to a single base-m covering induction on a pair of restricted-digit expansions, which at $m = 3$ is Utz's 1951 theorem on the distance set of the Cantor discontinuum. Replacing parity by divisibility by p gives, for every $p \geq 2$ and every $m \equiv 1 \pmod{p}$ with $m \geq p^2$, a nonzero ξ with $p \mid [\xi\sqrt{m}^{\,n}]$ for all n, hence with all sufficiently large integral parts composite. The cases $m = 2$ (reproof), $m = 3$, the cases $m > 3$, the p-divisibility theorem and the transcendentality theorem are all verified in Lean 4 and depend only on Lean's three standard axioms.

1. Introduction

Throughout, $[x]$ and $\{x\}$ denote the integral part and the fractional part of $x \in \mathbb{R}$, so that $x = [x] + \{x\}$ with $\{x\} \in [0, 1)$.

1.1. **The sets $\mathcal{S}$ and $\mathcal{Z}$.** Let $\alpha > 1$ be real. Then [Dub06EO, §1] either for every real $\xi \neq 0$ the fractional parts $\{\xi\alpha^n\}$ are $\geq 1/2$ for infinitely many n, or there is a real $\xi = \xi(\alpha) \neq 0$ with $\{\xi\alpha^n\} < 1/2$ for every n. The set of α for which the second possibility holds is denoted $\mathcal{Z}$, and $\mathcal{S} := (1, +\infty) \setminus \mathcal{Z}$. Equivalently — and this is the form we take as the definition, the two being interchanged by $\xi \mapsto 2\xi$ —

$$\alpha \in \mathcal{Z} \iff \alpha > 1 \text{ and } \exists\, \xi \neq 0 \text{ with } [\xi\alpha^n] \text{ even for all } n = 1, 2, \ldots \tag{1.1}$$

Trivially $\{2, 3, 4, \ldots\} \subset \mathcal{Z}$, by $\xi = 2$. In 1968 Mahler [Mah68] asked whether $3/2$ belongs to $\mathcal{S}$ or to $\mathcal{Z}$; the question is open, and is recorded as such in [Bug12, §3.8].

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Dubickas [Dub06EO] collected what is known and posed five problems. On the $\mathcal{Z}$ side his Theorem 1(i) gives $[3,+\infty)\subset\mathcal{Z}$ by Tijdeman's nested-interval construction [Tij72]; the remaining parts of his Theorem 1 concern Pisot and Salem numbers. On the $\mathcal{S}$ side his Theorem 2(i) gives $2^{1/q}\in\mathcal{S}$ for every integer $q\geq 2$. The problem we take up is the third of the five:

Problem 3 ([Dub06EO, p. 335])**.** Determine whether $\sqrt{3}$ belongs to $\mathcal{S}$ or to $\mathcal{Z}$.

The number $\sqrt{3}\approx 1.732$ falls in the gap left by all of the above: it is below 3, so Tijdeman's range does not reach it; it is an algebraic integer whose conjugate $-\sqrt{3}$ has modulus >1, so it is neither Pisot nor Salem and the trace machinery of [Dub06EO, DubA06] has no purchase on it; and it is not of the form $2^{1/q}$. It is not $3/2$, but it is the same kind of question, and it has stood since 2006.

1.2. **Results.**

Theorem A. *$\sqrt{3}\in\mathcal{Z}$. Explicitly, let $(a_i)_{i\geq 1}$ and $(b_i)_{i\geq 1}$ be the digit streams produced by the greedy of Section 4 from the target $\sigma=3\sqrt{3}-4$, put $x=\sum_{i\geq 1}a_i3^{-i}$ and $\xi=(4+x)/3$. Then $[\xi\sqrt{3}^{\,n}]$ is even for every $n\geq 1$. Numerically*

$$\xi=1.34160899796112665163098943065261542584 51\ldots,$$

and the orbit begins $2,4,6,12,20,36,62,108,188,326,\ldots$

The answer to Problem 3 is therefore $\mathcal{Z}$. This particular ξ is one of continuum-many witnesses, so uncountably many of them are transcendental (Theorem 9.1), even though the arithmetic nature of the one displayed above is out of reach. The same construction runs at every base and closes the whole square-root slice of the landscape.

Theorem B. *For an integer $m\geq 2$ we have $\sqrt{m}\in\mathcal{S}$ if and only if $m=2$.*

Both directions are proved here from scratch: the $m=2$ cell is a four-line argument recovering the mechanism of [Dub06EO, Thm 2(i)] (Proposition 2.3), and the cells $m\geq 9$, where $\sqrt{m}\geq 3$ and [Dub06EO, Thm 1(i)] already applies, are reproved by the same greedy as the rest. What is genuinely new is the range $2<\sqrt{m}<3$, that is $m\in\{5,6,7,8\}$, together with $m=3$; see Section 1.4 for a cell-by-cell account of what was known.

Parity is not special: replacing the even digits throughout by the digits divisible by p gives the following.

Theorem C. *Let $p\geq 2$ and let $m\geq 3$ be an integer that is not a perfect square. Suppose that either*

(i) *$m\equiv 1 \pmod p$ and $m\geq p^2$, or*
(ii) *$p\mid m$ and $(m-p)(1+\sqrt{m})>p(m-1)$.*

Then there is a real $\xi\neq 0$ with $p\mid[\xi\sqrt{m}^{\,n}]$ for every $n\geq 1$. Consequently $[\xi\sqrt{m}^{\,n}]$ is composite for every sufficiently large n.

In case (i) the smallest admissible base is $m = p^2 + 1$, and in case (ii) it is the smallest multiple of p satisfying the displayed inequality ($12, 20, 30, \ldots$ for $p = 3, 4, 5$). At $p = 2$ case (i) covers every odd $m \geq 5$ and case (ii) every even non-square $m \geq 6$, so Theorem C contains all of Theorem B except the two cells $m = 3$ and $m = 2$; the first is Theorem A and the second is Proposition 2.3. Theorem C is the designed-ξ counterpart of the recent designed-α results of [HIKK26], where it is the algebraic number rather than the multiplier that is constructed.

1.3. **Method.** The mechanism is not Diophantine approximation but the arithmetic of Cantor sets, and the reason is a triviality: $\sqrt{m}^{\,2} = m$ is an integer. Splitting n into even and odd values turns the single sequence $([\xi\sqrt{m}^{\,n}])_{n\geq 1}$ into the two sequences $[u\, m^s]$ and $[(u/\sqrt{m})m^s]$, $s \geq 0$, where $u = m\xi$ — both in the *same* base m. By the digit-parity lemma of Section 2, asking that all these integral parts be even is asking that u and $u/\sqrt{m}$ both lie in

$$2\mathbb{Z}_{\geq 0} + K_m, \qquad K_m = \Big\{ \textstyle\sum_{i\geq 1} a_i m^{-i} : a_i \in \{0, 2, \ldots, e_m\} \Big\},$$

K_m being a self-similar Cantor set — the middle-thirds set when $m = 3$ (Figure 1). So the problem becomes: does $(2\mathbb{Z} + K_m) \cap \sqrt{m}\,(2\mathbb{Z} + K_m)$ contain a positive real? That is one linear equation

$$x - \sqrt{m}\, y = v, \qquad x, y \in K_m, \tag{1.2}$$

in two restricted-digit reals, with v ranging over a lattice of spacing 2. It is solvable because $K_m - \sqrt{m}\, K_m$ is a genuine interval, of length $M_m(1+\sqrt{m}) > 2$, so the lattice meets its interior.

Two remarks on the shape of this argument. First, it was natural to expect the coupling in (1.2) to be where rigidity phenomena for $\times m$-invariant sets would bite. They do not: both sides of the intersection are structured in the *same* base m, and $\sqrt{m}$ enters only as an affine parameter, not as a second multiplicatively independent scale. Same-base affine intersections are the classical playground of thickness and sum-set arguments, where such intersections are large rather than rigid. Second, the interval statement $K_m + \sqrt{m}\, K_m = [0, M_m(1 + \sqrt{m})]$ is proved here by an explicit covering induction (Section 4) which at $m = 3$ is exactly Utz's 1951 theorem [Utz51] on the distance set of the Cantor discontinuum, statement and proof; Section 4.1 explains why for even m no thickness theorem applies and the covering induction is doing work that 1951 does not do.

The one point that needs care is that the digit expansions produced by the greedy must be *canonical*: in base 3 the stream $2, 2, 2, \ldots$ represents 1, whose canonical expansion carries an odd digit, and the parity lemma then fails. Section 5 rules this out. For even m it is automatic, for odd $m \geq 5$ it follows from a single inequality, and for $m = 3$ — the case of Problem 3 — it requires a separate argument.

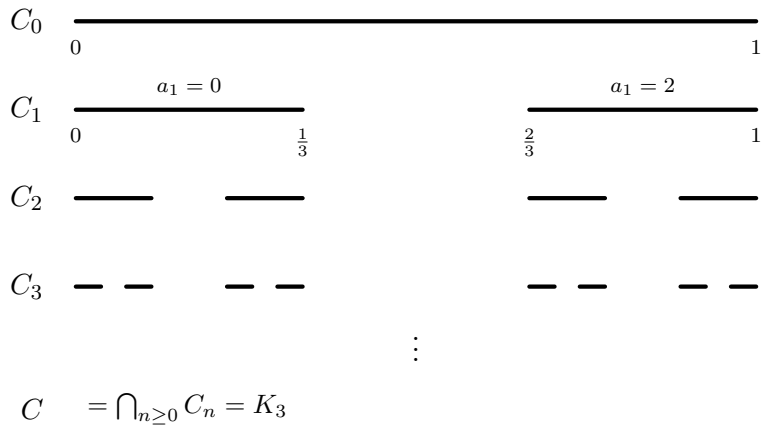

FIGURE 1. The middle-thirds set $C = K_3$, built by deleting open middle thirds; the two level-one pieces are labelled by the base-3 digit $a_1 \in \{0, 2\}$ that their points carry. For a general base m the same picture has $\frac{1}{2}e_m + 1$ children at each step, of length M_m/m and with left endpoints spaced $2/m$ apart, so the gaps have length $(2 - M_m)/m$; Section 4.1 measures them.

1.4. **What was known.** The statement $\sqrt{m} \in \mathcal{Z}$ decomposes as follows.

cells	status before this paper	reference
m a perfect square	trivial, $\xi = 2$	[Dub06EO, p. 333]
$m \geq 9$	known, since $\sqrt{m} \geq 3$	[Dub06EO, Thm 1(i)], [Tij72]
$m \in \{5, 6, 7, 8\}$	asserted without proof	[Dub06EO, p. 332]
$m = 3$	open	[Dub06EO, Problem 3]

The third row deserves a word. On p. 332 of [Dub06EO] one reads that $\sqrt{m} \in \mathcal{Z}$ for each integer $m \geq 4$, with no proof and no reference attached. We have not been able to locate a proof. The paper's reference list contains no candidate: its Theorem 1 does not reach the interval $(2, 3)$, since the nested-interval construction there needs a per-step window of length at least 2 and the available window is $\beta(\alpha-1)$ with $\beta < 1$; the two companion papers of the same author and year that are the natural candidates in that reference list concern $\|\alpha^n\|$ for rational α and the limit points of fractional parts of powers of Pisot numbers, and contain no statement about $\mathcal{Z}$ or about $\sqrt{m}$; and the review [Boyd07] does not mention the assertion. We therefore treat $m \in \{3, 5, 6, 7, 8\}$ as the cells requiring proof, note that a reader who knows a source for $m \in \{5, 6, 7, 8\}$ should regard those four as a reproof, and observe that in any case the uniform argument below is what makes Theorem B a single statement rather than a case list.

For the Cantor-arithmetic side the attribution is clean and old. That $C + C = [0, 2]$ for the middle-thirds set C goes back to Steinhaus [Ste17], and was rediscovered by Randolph [Ran40] in the equivalent form that the distance

set of C is $[0,1]$; [Paw13] recounts why the first paper went unnoticed. The form we need, that $y - \lambda x = c$ is solvable in C for $1/3 \le |\lambda| \le 3$ and $0 \le c \le 1$, is Utz [Utz51], recorded as an equivalence by Pawłowicz [Paw13]. The general thickness criterion is Newhouse's, and [CHM02, Cor. 4.5] show it sharp for central Cantor sets with ratios of dissection a and a^s, s irrational. A modern account of the chain is [ART19]. What Section 4 adds to [Utz51] is the even-m half, where the thickness hypothesis fails (Section 4.1).

1.5. **Formal verification.** Theorems A, B and C are all complete in the Lean 4 proof assistant, in the directory `SZ/` of the Github repository `https://github.com/rwst/Square-Roots/` (namespace `SZ`), and depend on nothing beyond Lean's three standard axioms. The digit lemma of Section 2 is formalized for arbitrary base m; the reduction of Section 3, the covering induction of Section 4 and the tail hygiene of Section 5 for an arbitrary base m, modulus p and digit alphabet; and then the cells of Theorem B one by one — $m = 2$ (Proposition 2.3), the perfect squares, $m = 3$ (Theorem A), the four cells $m \in \{5,6,7,8\}$, and $m \ge 9$ — followed by both branches of Theorem C and the four (p,m) pairs of its table. Theorem 9.1, with its branching analysis, is verified as well. The covering induction is *proved* there rather than cited to [Utz51], so the formal capstones carry no literature axiom. What is *not* formalized — the thickness computation of Section 4.1 and all of Section 8 — is flagged as such in Appendix A, which gives the Lean identifier of every formalized statement, records where the Lean form differs from the printed one, and describes five independent numerical cross-checks.

2. Digit parity

Fix an integer $m \ge 2$. For a sequence $d = (d_i)_{i \ge 1}$ with $d_i \in \{0, 1, \dots, m-1\}$ write

$$\langle d \rangle := \sum_{i \ge 1} d_i m^{-i} \in [0,1], \qquad \langle d \rangle_s := \sum_{i=1}^{s} d_i m^{-i}.$$

Every $t \in [0,1]$ is $\langle d \rangle$ for some such d, uniquely except at the m-adic rationals, where exactly two expansions exist; the *canonical* one is the one that is not eventually equal to $m-1$.

Lemma 2.1 (digit parity)**.** *Let $N \in \mathbb{Z}$, let d be as above, and let $s \ge 0$. Suppose there is $i_0 > s$ with $d_{i_0} \le m-2$. Then*

$$\left\lceil (N + \langle d \rangle) m^s \right\rceil = N m^s + \sum_{i=1}^{s} d_i m^{s-i}. \tag{2.1}$$

If moreover N and every d_i are even, this integer is even.

Proof. Put $Z = Nm^s + \sum_{i \le s} d_i m^{s-i}$, an integer, and note $Z = (N + \langle d \rangle_s) m^s$. The tail satisfies

$$0 \le \langle d \rangle - \langle d \rangle_s = \sum_{i > s} d_i m^{-i} \le (m-1) \sum_{i > s} m^{-i} - m^{-i_0} = m^{-s} - m^{-i_0} < m^{-s},$$

the subtraction being legitimate because $d_{i_0} \le m-2$ makes the i_0-th term smaller by at least m^{-i_0} than its maximum. Multiplying by m^s gives $Z \le (N+\langle d\rangle)m^s < Z+1$, which is (2.1). For the parity claim, Nm^s is even and each $d_i m^{s-i}$ is even. □

Remark 2.2. The hypothesis cannot be dropped. In base 3 take $d_i = 2$ for all i, so $\langle d\rangle = 1$; then $[(N+\langle d\rangle)3^s] = (N+1)3^s$ is odd for every even N and every s. This is the sole reason the construction below needs Section 5: the greedy has to be prevented from producing a stream that is eventually constant at the top digit. For *even* m the difficulty evaporates, because then the largest even digit is $m-2$ and the hypothesis of Lemma 2.1 holds at $i_0 = s+1$ for free.

We record the alphabet once. For $m \ge 2$ let

$$e_m = \begin{cases} m-1, & m \text{ odd}, \\ m-2, & m \text{ even}, \end{cases} \qquad \mathcal{D}_m = \{0, 2, \dots, e_m\}, \qquad M_m = \frac{e_m}{m-1},$$

and let $K_m = \{\langle a\rangle : a_i \in \mathcal{D}_m \text{ for all } i\}$, a self-similar Cantor set with $K_m \subseteq [0, M_m]$ and $\max K_m = M_m$. Thus $M_m = 1$ for odd m and $K_3 = C$ is the middle-thirds set. Finally put

$$U = U_m := M_m(1+\sqrt{m}). \tag{2.2}$$

The following degenerate case is the whole of the $\mathcal{S}$ side of Theorem B.

Proposition 2.3. *$\mathcal{D}_2 = \{0\}$, hence $K_2 = \{0\}$. Moreover, if $t \in \mathbb{R}$ satisfies $[t\,2^k]$ even for every $k \ge 0$, then $t \in 2\mathbb{Z}$. Consequently $\sqrt{2} \in \mathcal{S}$.*

Proof. The first assertion is the definition, $e_2 = 0$. For the second, from $[2u] = 2[u] + [2\{u\}]$ with $[2\{u\}] \in \{0,1\}$ we get, taking $u = t2^k$, that $[2\{t2^k\}]$ is even, hence zero, hence $\{t2^k\} < 1/2$ and $\{t2^{k+1}\} = 2\{t2^k\}$ for every $k \ge 0$. So $\{t2^k\} = 2^k\{t\} < 1/2$ for all k, forcing $\{t\} = 0$; and $[t] = t$ is even.

Now suppose $\xi \ne 0$ has $[\xi\sqrt{2}^{\,n}]$ even for all $n \ge 1$. Put $t_1 = 2\xi$ and $t_2 = \sqrt{2}\,\xi$, both nonzero. Taking $n = 2s$ with $s \ge 1$ gives $[t_1 2^{s-1}]$ even for all $s \ge 1$, and taking $n = 2s+1$ with $s \ge 0$ gives $[t_2 2^s]$ even for all $s \ge 0$. By the previous paragraph $t_1, t_2 \in 2\mathbb{Z} \setminus \{0\}$, whence $\sqrt{2} = t_1/t_2 \in \mathbb{Q}$, a contradiction. □

Remark 2.4. This is the mechanism inside [Dub06EO, Thm 2(i)], isolated: the obstruction at $m = 2$ is that the restricted-digit set collapses to a point, so there is no Cantor set to intersect. The dichotomy $\dim K_2 = 0$ versus $\dim K_3 = \log 2/\log 3 > 0$ is the entire difference between $\sqrt{2} \in \mathcal{S}$ and $\sqrt{3} \in \mathcal{Z}$.

3. The reduction

Fix an integer $m \geq 3$ that is not a perfect square, and an integer $p \geq 2$ with $p \mid m$ or $p \mid m-1$; for Theorems A and B take $p = 2$. Let

$$e = p\Big\lfloor \frac{m-1}{p} \Big\rfloor, \qquad \mathcal{D} = \{0, p, 2p, \dots, e\}, \qquad M = \frac{e}{m-1},$$

$$K = \big\{ \langle a \rangle : a_i \in \mathcal{D} \text{ for all } i \big\},$$

and $U = M(1+\sqrt{m})$ as in (2.2); for $p = 2$ these are $e_m, \mathcal{D}_m, M_m, K_m$.

Lemma 3.1 (telescoping)**.** *Let $t > 0$ have $[t] \in p\mathbb{Z}$ and all canonical base-m digits of $\{t\}$ in $\mathcal{D}$. If $p \mid m$ or $p \mid m-1$, then $p \mid [t\, m^k]$ for every $k \geq 0$.*

Proof. Write $[tm^k] = m[tm^{k-1}] + d_k$ with $d_k \in \mathcal{D}$ the k-th digit, valid for $k \geq 1$ by the argument of Lemma 2.1. If $p \mid m$ this gives $[tm^k] \equiv d_k \equiv 0$; if $p \mid m-1$ it gives $[tm^k] \equiv [tm^{k-1}] + d_k \equiv [tm^{k-1}]$, and one induces from $[t] \equiv 0$. □

For $p = 2$ this is Lemma 2.1, and the hypothesis is vacuous since $2 \mid m$ or $2 \mid m-1$ always. Set

$$E := \{ t > 0 : [t\, m^k] \in p\mathbb{Z} \text{ for all } k \geq 0 \} \;\supseteq\; p\mathbb{Z}_{\geq 0} + K,$$

the inclusion by Lemma 3.1 whenever the digits used are canonical.

Proposition 3.2 (reduction)**.** *Let $j \geq 0$ and $k_0 \geq 1$ be integers and suppose $x, y'' \in K$ satisfy*

$$x + \sqrt{m}\, y'' = \sigma, \qquad \sigma := (pk_0 + M)\sqrt{m} - pj, \tag{3.1}$$

with the digit streams of x and of $y := M - y''$ both canonical. Put $u = pj + x$ and $\xi = u/m$. Then $\xi > 0$ and $p \mid [\xi \sqrt{m}^{\,n}]$ for every $n \geq 1$.

Proof. From (3.1), $u = pj + x = (pk_0 + M)\sqrt{m} - \sqrt{m}\, y'' = \sqrt{m}\,(pk_0 + y)$, so

$$u = pj + x = \sqrt{m}\,(pk_0 + y). \tag{3.2}$$

Note that the digitwise complement $a_i \mapsto e - a_i$ carries K to $M - K = K$, so $y = M - y'' \in K$; its digits are $e - b_i \in \mathcal{D}$, where b is the stream of y''.

Let $n \geq 1$. If $n = 2s$ with $s \geq 1$ then $\xi\sqrt{m}^{\,n} = \xi m^s = u\, m^{s-1} = (pj+x)m^{s-1}$, and Lemma 3.1 applies to $t = pj + x$. If $n = 2s+1$ with $s \geq 0$ then by (3.2)

$$\xi\sqrt{m}^{\,n} = \frac{u}{m}\, m^s \sqrt{m} = \frac{u}{\sqrt{m}}\, m^s = (pk_0 + y)\, m^s,$$

and Lemma 3.1 applies to $t = pk_0 + y$. Finally $\xi > 0$ because $u = \sqrt{m}\,(pk_0 + y) \geq \sqrt{m}\, pk_0 > 0$. □

It remains to (a) produce a translate, i.e. a pair (j, k_0) with $\sigma \in (0, U)$, and (b) solve (3.1) with canonical streams. Part (a) is immediate.

Lemma 3.3 (translate)**.** *If $U > p$ then for every $k_0 \geq 1$ there is an integer $j \geq 1$ with $0 < \sigma < U$, where σ is as in* (3.1)*.*

Proof. As j runs through $\mathbb{Z}$, σ runs through a decreasing arithmetic progression of step $p < U$, so some value lies in the interval $(0, U)$ of length U; equality with an endpoint is impossible, since $\sigma = 0$ would give $(pk_0+M)\sqrt{m} = pj$ and $\sigma = U$ would give $pk_0\sqrt{m} = M+pj$, both contradicting the irrationality of $\sqrt{m}$. The j realising it satisfies $pj > (pk_0+M)\sqrt{m} - U \geq p\sqrt{m} - M > 0$, using $M \leq 1 < p\sqrt{m}$, so $j \geq 1$. □

4. The covering induction

Part (b) is the heart of the matter. Keep the notation of Section 3.

Theorem 4.1 (cover lemma)**.** *Suppose*

$$p \leq U \qquad \text{and} \qquad p\sqrt{m} - e \leq U. \tag{4.1}$$

Then $K + \sqrt{m}\,K = [0, U]$.

Proof. The inclusion $\subseteq$ is clear from $\max K = M$ and $M + \sqrt{m}\,M = U$. For $\supseteq$, fix $\sigma \in [0, U]$ and run a greedy. Put $w_0 = m\sigma \in [0, mU]$. We claim that for every $w \in [0, mU]$ there is an *offset* $c = a + b\sqrt{m}$ with $a, b \in \mathcal{D}$ and

$$0 \leq w - c \leq U. \tag{4.2}$$

Granting the claim, choose such a c at each step, set $a_{k+1} = a$, $b_{k+1} = b$ and $w_{k+1} = m(w_k - c) \in [0, mU]$. An immediate induction gives

$$\sigma - \langle a\rangle_k - \sqrt{m}\,\langle b\rangle_k = \frac{w_k}{m^{k+1}} \in \big[0,\, Um^{-k}\big],$$

so letting $k \to \infty$ yields $\langle a\rangle + \sqrt{m}\langle b\rangle = \sigma$ with $a_i, b_i \in \mathcal{D}$, as required.

For the claim we pin the choice down, since the same rule is used again in Section 5: let b be the largest element of $\mathcal{D}$ with $b\sqrt{m} \leq w$ (it exists, $b = 0$ qualifies), put $v := w - b\sqrt{m} \geq 0$, and let a be the least non-negative multiple of p with $v - a \leq U$; in closed form,

$$b = \min\Big(e,\ p\Big\lfloor \frac{w}{p\sqrt{m}} \Big\rfloor\Big), \qquad a = p\max\Big(0,\ \Big\lceil \frac{v-U}{p} \Big\rceil\Big). \tag{4.3}$$

If $v \leq U$ then $a = 0$ and $w - c = v \in [0, U]$. Otherwise a is the least multiple of p that is at least $v - U$, so $v - U \leq a < v - U + p$ and hence $0 \leq U - p < w - c = v - a \leq U$, using the first condition of (4.1). So (4.2) holds as soon as $a \in \mathcal{D}$, and this is where the second condition enters. If $b < e$ then $(b + p)\sqrt{m} > w$ by maximality of b, so $v < p\sqrt{m}$ and $a < p\sqrt{m} - U + p \leq e + p$. If $b = e$ then, since $e(1 + \sqrt{m}) = (m - 1)U$,

$$v = w - e\sqrt{m} \leq mU - e\sqrt{m} = mU - (m-1)U + e = U + e,$$

so again $a < e + p$. As a and e are multiples of p, $a \leq e$ in both cases. □

Remark 4.2 (Utz). At $p = 2$, $m = 3$ one has $e = 2$, $M = 1$, $U = 1 + \sqrt{3}$, and the four offsets are 0, 2, $2\sqrt{3}$, $2 + 2\sqrt{3}$; the two conditions (4.1) read $2 \leq 1 + \sqrt{3}$ and $2\sqrt{3} - 2 \leq 1 + \sqrt{3}$, and case (iii) of the proof is the exact identity $2 + 2\sqrt{3} + U = 3U$. Theorem 4.1 then says $C + \sqrt{3}\,C = [0, 1 + \sqrt{3}]$, which after the substitutions $\lambda = 1/\sqrt{3}$, $c = 4/\sqrt{3} - 2$ is Utz's theorem

[Utz51]: for $0 \le c \le 1$ and $1/3 \le |\lambda| \le 3$ there are $x, y \in C$ with $y - \lambda x = c$. Utz's proof is the four-surviving-squares induction, which is the greedy above in digit coordinates, the four squares being the four digit pairs in $\{0,2\}^2$. Pawłowicz [Paw13] records the statement as an equivalence. We reprove it because we need the version with a general alphabet, and because the formalization is not to depend on a citation.

4.1. **Thickness, and what is new at even m.** The classical route to statements like Theorem 4.1 is Newhouse's gap lemma [New79]: if two Cantor sets have thicknesses whose product is at least 1, their sum contains an interval, and if in addition no translate of either is contained in a gap of the other, the sum *is* an interval; see [CHM02, §4] and [ART19]. Here the *gaps* of a Cantor set K are the bounded connected components of its complement, the *bridge* at an endpoint u of a gap G is the maximal interval with endpoint u containing no point of a gap of length at least $|G|$, and the *thickness* is $\tau(K) = \inf\{|B|/|G| : (B, G) \text{ a bridge/gap pair}\}$; this is the definition of [CHM02, §4], following [PT93, Ch. 4]. For $p = 2$ the thickness of K_m is computable at once from $K_m = \bigcup_{a \in \mathcal{D}_m} (a/m + K_m/m)$: the $\frac{1}{2}e_m + 1$ children have length M_m/m and left endpoints spaced $2/m$ apart, so every gap has length $(2 - M_m)/m$ and every bridge length M_m/m, whence $\tau(K_m) = M_m/(2 - M_m)$, that is

$$\tau(K_m) = \begin{cases} 1, & m \text{ odd}, \\ \dfrac{m-2}{m}, & m \text{ even}. \end{cases} \tag{4.4}$$

Thickness is unchanged by scaling, which multiplies every bridge and every gap by the same factor, so $\tau(\sqrt{m}\, K_m) = \tau(K_m)$ and the gap lemma applies precisely when $\tau(K_m)^2 \ge 1$, that is precisely for odd m — and then only as the boundary case $\tau = 1$, which is where the classical $m = 3$ result sits. For even m we have $\tau(K_m)^2 = ((m-2)/m)^2 < 1$ and *no* thickness theorem applies, yet the conclusion of Theorem 4.1 still holds. The reason is visible in the proof: the covering is done by the $(\frac{1}{2}e_m + 1)^2$ joint offsets $a + b\sqrt{m}$, not by the two-piece data that thickness records. This is the part of Theorem 4.1 that is not in [Utz51].

5. Tail hygiene

Theorem 4.1 produces *some* admissible digit streams; by Remark 2.2 we need streams that are not eventually constant at the top digit e. Concretely, Proposition 3.2 requires

$$a_i \le e - p \ \text{ for infinitely many } i, \qquad b_i \ge p \ \text{ for infinitely many } i, \tag{5.1}$$

the second condition because the digits of $y = M - y''$ are $e - b_i$.

If $e \le m - 2$ — equivalently, if $p \nmid m - 1$, and in particular for $p = 2$ and m even — then *every* digit satisfies the hypothesis of Lemma 2.1 and (5.1) is not needed at all. The rest of this section deals with the remaining case $e = m - 1$.

5.1. **The greedy, pinned down.** From now on the greedy runs with the rule (4.3) pinned down in the proof of Theorem 4.1: the largest available b, then the least admissible a. Write the state as w_k, with $w_0 = m\sigma$ and $w_{k+1} = m(w_k - c_{k+1})$, where $c_{k+1} = a_{k+1} + b_{k+1}\sqrt{m}$ is the offset selected at w_k.

Two consequences of (4.3) are what make the two halves of (5.1) independent, and both are immediate from the formulas. First, since b is the largest multiple of p with $b\sqrt{m} \le w$,

$$b_{k+1} = 0 \iff w_k < p\sqrt{m}. \tag{5.2}$$

Second, writing $v = w - b\sqrt{m}$ as in that proof, $a = 0$ when $v \le U$, and $v - U \le a < v - U + p$ when $v > U$.

Remark 5.1. The rule is the one it looks like: among the offsets admissible at w it takes the one with a smallest and, among those, b largest. Indeed an admissible (a', b') has $b'\sqrt{m} \le w$, hence $b' \le b$; if $b' = b$ then $a' \ge v - U$, and a is the least multiple of p with that property, so $a' \ge a$; and if $b' \le b - p$ then $a' \ge v + p\sqrt{m} - U$, which is $> v - U + p > a$ when $v > U$, while $a' \ge 0 = a$ when $v \le U$. We record this because it is why the rule was chosen, but nothing below uses it: the two displayed facts are all that Sections 5.3 and 5.4 need.

5.2. **The state is never a trap fixed point.** On a subinterval of $[0, mU]$ where a single offset c is forced, the dynamics is the affine map $w \mapsto m(w-c)$, expanding by m, with the unique fixed point $mc/(m-1)$. The next two lemmas say that the orbit never sits at such a point, and that an expanding map cannot confine an orbit anywhere else.

Lemma 5.2 (growth)**.** *Write $w_k = P_k + Q_k\sqrt{m}$ with $P_k, Q_k \in \frac{1}{m-1}\mathbb{Z}$. Then $Q_k \ge pk_0 m^{k+1}$ for every $k \ge 0$. In particular $Q_k > m \ge mM$, so w_k is irrational, $w_k \ne 0$, and $w_k \ne \rho + \varrho\sqrt{m}$ for any rationals ρ, ϱ with $\varrho \le m$.*

Proof. From $w_0 = m\sigma$ we read $Q_0 = m(pk_0 + M)$, and $Q_{k+1} = m(Q_k - b_{k+1})$. Dividing by m^{k+1} and telescoping,

$$\frac{Q_k}{m^k} = Q_0 - \sum_{i=1}^{k} \frac{b_i}{m^{i-1}} \;\ge\; Q_0 - e\sum_{i\ge 0} m^{-i} = m(pk_0 + M) - \frac{em}{m-1} = pk_0 m,$$

since $e = (m-1)M$. Hence $Q_k \ge pk_0 m^{k+1} \ge 2m > m$, and $m \ge mM$ as $M \le 1$. □

Lemma 5.3 (expansion)**.** *Let $K \ge 0$ and $c \in \mathbb{R}$, and suppose $w_{k+1} = m(w_k - c)$ for all $k \ge K$ while the w_k, $k \ge K$, remain in a set of diameter $D < \infty$. Then $w_K = mc/(m-1)$.*

Proof. Put $\epsilon_k = w_k - mc/(m-1)$; then $\epsilon_{k+1} = m\epsilon_k$, so $\epsilon_{K+n} = m^n \epsilon_K$ and $|\epsilon_{K+n}| \le D$ for all n. As $m \ge 3$, this forces $\epsilon_K = 0$. □

Every fixed point that will occur below has the form $mc/(m-1)$ with $c = a + b\sqrt{m}$, $b \le e$, so its $\sqrt{m}$-coefficient is $mb/(m-1) \le mM \le m$; by Lemma 5.2 no w_k equals it. It is the *growth* in Lemma 5.2, not mere irrationality, that is needed: at $m = 3$ one of the two fixed points is $3+3\sqrt{3}$, which is irrational.

5.3. **The uniform case.**

Proposition 5.4. *Assume $e = m-1$, so $M = 1$ and $U = 1+\sqrt{m}$, and assume $\sqrt{m} \ge p$. Then the greedy of Section* 5.1 *produces streams satisfying* (5.1).

Proof. Both halves rest on the hypothesis $\sqrt{m} \ge p$, in the two forms

$$(p-1)(\sqrt{m}+1) \le e \qquad \text{and} \qquad m(U-p) \ge p\sqrt{m}. \tag{5.3}$$

The first is $(p-1)(\sqrt{m}+1) \le (\sqrt{m}-1)(\sqrt{m}+1) = m-1 = e$. For the second, $U - p = \sqrt{m}+1-p$, so dividing by $\sqrt{m}$ turns the claim into $m - p \ge \sqrt{m}\,(p-1)$; and $\sqrt{m} \ge p$ gives both $m \ge p\sqrt{m}$ and $p \le \sqrt{m}$, whence $m - p \ge p\sqrt{m} - p \ge p\sqrt{m} - \sqrt{m} = \sqrt{m}\,(p-1)$.

The a-side. We claim that $a = e$ forces $b = e$. If $b < e$ then $(b+p)\sqrt{m} > w$ by maximality of b, so $v < p\sqrt{m}$; hence either $v \le U$ and $a = 0 < e$, or $v > U$ and

$$a < v - U + p < p\sqrt{m} - U + p = (p-1)(\sqrt{m}+1) \le e$$

by (5.3), using $U = 1+\sqrt{m}$. Either way $a \ne e$. So if $a_k = e$ for all $k \ge N$ then $b_k = e$ for all $k \ge N$ as well, the offset is the constant $c = e(1+\sqrt{m}) = (m-1)U$, and from step N on the dynamics is the single map $w \mapsto m(w-(m-1)U)$. The orbit stays in $[0, mU]$, so Lemma 5.3 puts it at the fixed point $m(m-1)U/(m-1) = mU$, whose $\sqrt{m}$-coefficient is $mM \le m$ — contradicting Lemma 5.2.

The b-side. Suppose $b_k = 0$ for all $k \ge N$; by (5.2) this says $w_k < p\sqrt{m}$ for all $k \ge N$. If $w_k > U$ for some $k \ge N$, then $v = w_k > U$, so $a < w_k - U + p$ and

$$w_{k+1} = m(w_k - a) > m(U-p) \ge p\sqrt{m}$$

by (5.3), contradicting $w_{k+1} < p\sqrt{m}$. Hence $w_k \le U$ for all $k \ge N$; then $v = w_k \le U$ gives $a = 0$, so $w_{k+1} = mw_k$ and $w_{N+n} = m^n w_N$ for all n. But $w_N > 0$ by Lemma 5.2, so $m^n w_N \to \infty$, contradicting $w_{N+n} < p\sqrt{m}$. □

For $p = 2$ the hypothesis $\sqrt{m} \ge p$ reads $m \ge 4$, so Proposition 5.4 covers every odd $m \ge 5$. Together with the free case $e \le m-2$ this leaves exactly one base.

5.4. **The base $m = 3$.** Here $p = 2$, $e = 2$, $M = 1$, $U = 1+\sqrt{3}$, $\mathcal{D} = \{0, 2\}$, and $\sqrt{3} < 2$: both inequalities of Proposition 5.4 fail. The selection rule of

Section 5.1 reduces to four regions,

$$\begin{array}{lll} (1)\quad w \in [0, U] & \longrightarrow & (a,b) = (0,0), \\ (2)\quad w \in (U, 2\sqrt{3}) & \longrightarrow & (a,b) = (2,0), \\ (3)\quad w \in [2\sqrt{3},\, 2\sqrt{3}+U] & \longrightarrow & (a,b) = (0,2), \\ (4)\quad w \in (2\sqrt{3}+U,\, 3U] & \longrightarrow & (a,b) = (2,2), \end{array} \tag{5.4}$$

so that the rule selects $a = 2$ exactly on $A := A_1 \cup A_2$, where $A_1 = (U, 2\sqrt{3})$ and $A_2 = (2\sqrt{3}+U, 3U] = (3U-2, 3U]$, and selects $b = 0$ exactly on $[0, 2\sqrt{3})$, as (5.2) says. Numerically $U = 2.7320\ldots$, $2\sqrt{3} = 3.4641\ldots$, $2\sqrt{3}+U = 6.1961\ldots$, $3U = 8.1961\ldots$

Proposition 5.5. *At $m = 3$ the greedy of Section 5.1 started from $w_0 = 3\sigma$ with $\sigma = 3\sqrt{3} - 4$ produces streams satisfying (5.1).*

Proof. The a-side. On A_1 the offset 2 is forced and $w \mapsto 3w-6$ maps A_1 into $(3\sqrt{3}-3, 6\sqrt{3}-6)$, an interval disjoint from A_2 because $6\sqrt{3}-6 < 1+3\sqrt{3}$, i.e. $3\sqrt{3} < 7$. So inside A the component A_1 is absorbing, and an orbit eventually contained in A is eventually contained in a single component. On A_1, Lemma 5.3 with $c = 2$ and $D = 2\sqrt{3} - U < 1$ forces $w = 3$; on A_2, with $c = 2 + 2\sqrt{3}$ and $D = 2$, it forces $w = 3 + 3\sqrt{3}$. Both are excluded by Lemma 5.2, the first because it is rational, the second because its $\sqrt{3}$-coefficient is $3 \le m$.

The b-side. Suppose $b_k = 0$, that is $w_k < 2\sqrt{3}$, for all $k \ge N$. If $w_k \le U$, region (1) applies and $w_{k+1} = 3w_k < 2\sqrt{3}$, so $w_k < 2\sqrt{3}/3$. If $w_k > U$, region (2) applies and $w_{k+1} = 3w_k - 6 < 2\sqrt{3}$, so $w_k < 2 + 2\sqrt{3}/3$. Thus every w_k, $k \ge N$, lies in $[0, 2\sqrt{3}/3) \cup (U, 2 + 2\sqrt{3}/3)$. The second zone is absorbing: from $w_k \in (U, 2 + 2\sqrt{3}/3)$ we get $w_{k+1} = 3w_k - 6 > 3U - 6 = 3\sqrt{3} - 3 > 2\sqrt{3}/3$, so w_{k+1} is not in the first zone, hence is in the second. If the orbit ever enters it, Lemma 5.3 with $c = 2$ and $D = 2 + 2\sqrt{3}/3 - U < 1$ forces $w = 3$, excluded. Otherwise $w_k \le U$ for all $k \ge N$ and $w_{N+n} = 3^n w_N \to \infty$ with $w_N > 0$, contradicting $w_{N+n} < 2\sqrt{3}$. □

Remark 5.6. The two zones $(U, 2\sqrt{3})$ and $(U, 2+2\sqrt{3}/3)$ overlap but are not equal; the a-argument and the b-argument are genuinely different, and each uses the selection rule in an essential way. Note also that the fixed point 3 of $w \mapsto 3w - 6$ lies in both zones. An earlier version of this argument proceeded by alternating the two escape goals; the observation that the rule of Section 5.1 makes $a_k = 2$ hold exactly on A and $b_k = 0$ exactly on $[0, 2\sqrt{3})$ removes the need for any scheduling, and is what makes the formalization short.

6. Theorems A and B

Proof of Theorem A. Take $m = 3$, $p = 2$, $k_0 = 1$, $j = 2$, so that $U = 1 + \sqrt{3} > 2$ and $\sigma = (2+1)\sqrt{3} - 4 = 3\sqrt{3} - 4 = 1.1961\ldots \in (0, U)$, as Lemma 3.3 guarantees. Conditions (4.1) hold (Remark 4.2), so the greedy of Section 5.1 runs forever and, by Theorem 4.1, produces $x = \langle a \rangle$ and

$y'' = \langle b \rangle$ in $K_3 = C$ with $x + \sqrt{3}\, y'' = \sigma$. By Proposition 5.5 the streams satisfy (5.1), so Proposition 3.2 applies with $u = 4 + x$ and $\xi = u/3$. The numerical value is computed in Appendix A. □

Proof of Theorem B. The case $m = 2$ is Proposition 2.3. If m is a perfect square then $\sqrt{m}$ is an integer ≥ 2 and $\xi = 2$ works. Otherwise let $m \geq 3$ be a non-square and take $p = 2$. Then $U > 2$: for odd m, $U = 1 + \sqrt{m} > 2$; for even m, $U > 2$ reads $(m-2)(1+\sqrt{m}) > 2(m-1)$, and

$$(m-2)(1+\sqrt{m})-2(m-1) = \sqrt{m}\,(m-2-\sqrt{m}) = \sqrt{m}\,(\sqrt{m}-2)(\sqrt{m}+1) > 0$$

because $m \geq 6 > 4$. Conditions (4.1) hold as well: $2 \leq U$ is immediate from the previous sentence, and $2\sqrt{m} - e \leq U$ reads $\sqrt{m} \leq m$ for odd m, while for even m one has $2\sqrt{m} - e = 2\sqrt{m} - m + 2 \leq 2 \leq U$ because $m \geq 4$. Lemma 3.3 supplies a translate, Theorem 4.1 supplies the streams, and (5.1) holds: for even m it is not needed (Remark 2.2), for odd $m \geq 5$ it is Proposition 5.4, and for $m = 3$ it is Proposition 5.5. Proposition 3.2 finishes. □

Corollary 6.1. $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8} \in \mathcal{Z}$, *so* $\mathcal{Z}$ *meets* $(2, 3)$.

Since $[3, +\infty) \subset \mathcal{Z}$, any element of $\mathcal{S}$ greater than 2 would have to lie in $(2, 3)$; and every element of $\mathcal{S}$ exhibited in [Dub06EO] lies in $(1, 2)$, as the review [Boyd07] records. So Corollary 6.1 does not answer

Problem 5 ([Dub06EO, p. 335])**.** Is there an element of $\mathcal{S}$ greater than 2?

but it adds four explicit points on the other side to the only interval where the answer can live. They are of a kind not previously available there: within [Dub06EO] the elements of $\mathcal{Z} \cap (2, 3)$ are the rationals $3 - 2/q$, $q \geq 3$, of Theorem 1(ii), and the Pisot and Salem numbers supplied by Theorem 1(iii),(iv), whereas $\sqrt{5}, \sqrt{6}, \sqrt{7}, \sqrt{8}$ are irrational and, having a conjugate of modulus > 1, are neither Pisot nor Salem.

6.1. **Explicit witnesses.** Theorem B is an existence statement, but in the cells that are new one can be concrete. The table lists, for the selection rule of Section 5.1 and $k_0 = 1$, the least admissible j, the resulting $\xi = (2j + x)/m$ to 25 exact decimals, and the first six integral parts. All entries were computed in exact rational arithmetic with a rigorous m^{-N} tail, and the integral parts were checked to be even and unambiguously determined for $n \leq 200$.

m	j	ξ	$[\xi\sqrt{m}^{\,n}]$, $n = 1, \ldots, 6$
3	2	1.34160899796112665163098 94 …	2, 4, 6, 12, 20, 36
5	2	0.89600024609041133371844 90 …	2, 4, 10, 22, 50, 112
6	3	1.00184334482904155982300 06 …	2, 6, 14, 36, 88, 216
7	3	0.86392614172012302854596 22 …	2, 6, 16, 42, 112, 296
8	3	0.81794086107404517969876 35 …	2, 6, 18, 52, 148, 418

Note that the translate is not uniform in m: $j = 2$ serves $m = 3$ and $m = 5$ but not $m \geq 6$, where the first admissible translate is $j = 3$.

Remark 6.2 (the witness is not unique). The solution set of (3.1) in $K \times K$ is itself a Cantor set: the tree of admissible digit pairs branches at every node (Theorem 9.1), at an empirical rate of about $4/3$. A witness is therefore pinned only by a selection rule, and different rules give different ξ. The values above belong to the rule of Section 5.1; a first-fit rule preferring to alternate digits gives, at $m = 3$, the witness $1.341665814942779748832194\ldots$, which agrees with ours on the first six integral parts and differs at $n = 7$ (64 against 62). We quote the rule of Section 5.1 because it is the one for which Section 5 is short, and the one that is formally verified.

7. Divisibility by p

Proof of Theorem C. In case (i), $p \mid m-1$ gives $e = m-1$, $M = 1$, $U = 1+\sqrt{m}$; from $m \geq p^2$ we get $\sqrt{m} \geq p$, hence $U = 1+\sqrt{m} > p$, which is the first condition of (4.1), and $p\sqrt{m} \leq \sqrt{m}\,\sqrt{m} = m \leq m+\sqrt{m}$, which on substituting $e = m-1$ and $U = 1+\sqrt{m}$ is the second. In case (ii), $p \mid m$ gives $e = m-p$ and $M = (m-p)/(m-1)$, and the hypothesis is exactly $U > p$, which is the first condition of (4.1). For the second we first note that $U > p$ forces $m > p^2$ here. Indeed write $m = kp$ with $k \geq 1$ and suppose $m \leq p^2$, so $k \leq p$ and $\sqrt{m} \leq p$; then

$$\begin{aligned}(m-p)(1+\sqrt{m}) &= p(k-1)\bigl(1+\sqrt{kp}\bigr)\\ &\leq p(k-1)(1+p) = p(kp+k-p-1) \leq p(kp-1),\end{aligned}$$

the last step because $k \leq p$, and this says $U \leq p$. So $p < \sqrt{m}$, hence $p\sqrt{m} < m$ and $p\sqrt{m} - e = p\sqrt{m} - m + p < p < U$. Lemma 3.3 supplies a translate and Theorem 4.1 the streams. For (5.1): in case (ii) $e = m-p \leq m-2$ and the condition is not needed; in case (i) it is Proposition 5.4. Proposition 3.2 then gives ξ. Finally $[\xi\sqrt{m}^{\,n}] \to \infty$, so it exceeds p for all large n, and being a multiple of p larger than p it is composite. □

In case (i) the constraint is only $m \geq p^2$, and the smallest admissible base is $m = p^2+1$, which is $\equiv 1 \pmod p$ and never a square. The following are computed exactly, for $k_0 = 1$ and the least admissible j, with the integral parts verified divisible by p and unambiguously determined for $n \leq 200$.

p	m	j	ξ	$[\xi\sqrt{m}^{\,n}]$, $n = 1,\ldots,6$
3	10	3	$0.96006939603936630600\ldots$	$3, 9, 30, 96, 303, 960$
3	12	4	$1.02289606837742417779\ldots$	$3, 12, 42, 147, 510, 1767$
4	17	4	$0.97053751681712857019\ldots$	$4, 16, 68, 280, 1156, 4768$
5	26	5	$0.98376108969646594364\ldots$	$5, 25, 130, 665, 3390, 17290$

Each of these four pairs is also an instance of Theorem C in Lean: the hypotheses of the branch in question are checked there for that (p, m) and the conclusion instantiated, so the rows are certified instances of the theorem and not merely output of a program. The numerical columns are the fifth cross-check of Appendix A.2.

Remark 7.1 (the threshold). The covering induction and the translate need only $U > p$, which on branch (i) reads $m > (p-1)^2$ and is first met at $m = p^2 - p + 1$ — the previous candidate $\equiv 1 \pmod p$ being $(p-1)^2$ itself, a perfect square, where $U = p$ exactly. Thus for $(p, m) = (3, 7)$, say, the construction runs and produces $\xi = 1.29588921264222346083\ldots$ with $[\xi\sqrt{7}^n] = 3, 9, 24, 63, 168, 444, \ldots$, verified divisible by 3 for $n \le 200$; what is missing between $p^2 - p + 1$ and p^2 is only the tail hygiene (5.1), since $e = m - 1$ there and Proposition 5.4 needs $\sqrt{m} \ge p$. At $p = 2$ that gap is the single base $m = 3$, and Proposition 5.5 closes it. We have not tried to close it for $p \ge 3$.

Remark 7.2. Theorem C is a statement about a designed multiplier ξ for a fixed simple $\alpha = \sqrt{m}$. It is complementary to [HIKK26], where $\xi = 1$ throughout and the algebraic number α is designed — there, quadratic, cubic and biquadratic Pisot units μ with $[\mu^n]$ composite, obtained from trace congruences. Neither construction subsumes the other.

8. Limits of the method

It is natural to ask what the same idea does for $\alpha = m^{1/q}$ with $q \ge 3$. Writing $t_r = \xi\alpha^r = 2j_r + x_r$ with $x_r \in K_m$ for $r = 0, \ldots, q-1$ — we take $p = 2$ here, a general p only replacing the alphabet size $\lceil m/2 \rceil$ below by $|\mathcal{D}|$ — the constraints $t_{r+1} = \alpha t_r$ become $q - 1$ linear equations

$$x_{r+1} - \alpha x_r = v_r, \qquad r = 0, \ldots, q-2,$$

and the greedy carries a residual *vector*. A step appends a digit tuple $(a_0, \ldots, a_{q-1}) \in \mathcal{D}^q$, so the natural necessary condition is a volume count,

$$\lceil m/2 \rceil^q \;\ge\; m^{q-1}, \tag{8.1}$$

which for instance $5^{1/3}$ passes ($27 \ge 25$) while $3^{1/3}$ fails ($8 < 9$). The condition is very far from sufficient, and the reason is geometric. The residual vectors do not fill a product box $\prod_r [-\alpha M, M]$ but the *zonotope*

$$\mathcal{Z}_q = \big\{(\tau_1 - \alpha\tau_0, \ldots, \tau_{q-1} - \alpha\tau_{q-2}) : \tau_r \in [0, M]\big\},$$

the image of the τ-cube; the two coincide exactly when $q = 2$, which is precisely why the argument of Section 4 works there. Replacing the box by $\mathcal{Z}_q$ and testing whether the $\lceil m/2 \rceil^q$ translates cover $m\,\mathcal{Z}_q$ on a deterministic grid in exact arithmetic gives, with a $q = 2$ control that reports no uncovered point:

(m, q)	budget (8.1)	grid points of $m\mathcal{Z}_q$ uncovered
$(5, 3)$	$27 \ge 25$	$430/2701 = 15.9\%$
$(7, 3)$	$64 \ge 49$	$316/2729 = 11.6\%$
$(13, 4)$	$2401 \ge 2197$	$302/1341 = 22.5\%$

and the corresponding greedy, started from a non-degenerate translate, dies after 0, 24 and 0 steps respectively. (A trap worth naming: started from the all-zero digit vector the greedy runs forever, but that is the solution $\xi = 0$,

excluded by definition.) A grid can refute a cover but not certify one, and refutation is the direction used here.

The envelope of this method is therefore $q = 2$, and beyond that exponent it does not merely thin out: it never approaches the range where the problem is interesting. Indeed (8.1) reads $\alpha \geq 2$ for even m, since $\lceil m/2 \rceil = m/2$, and

$$\alpha \;\geq\; \frac{2m}{m+1} = 2 - \frac{2}{m+1}$$

for odd m, since $\lceil m/2 \rceil = (m+1)/2$ and the count is $(m+1)/2 \geq m^{1-1/q} = m/\alpha$. So an $\alpha < 2$ surviving the count at some $q \geq 3$ has m odd: $m = 3$ is impossible, as $3^{1/q} \leq 3^{1/3} < 3/2$; $m = 5$ forces $q = 3$, as $5^{1/4} < 5/3$; and $m \geq 7$ gives $\alpha \geq 7/4$. The smallest survivor is therefore $5^{1/3}$ itself. As q grows the survivors are the odd m in a window of width $O(q)$ below 2^q — $5, 7$ at $q = 3$; $13, 15$ at $q = 4$; $27, 29, 31$ at $q = 5$ — so they accumulate at 2 from below and never come near $3/2$. At the three tested above the cover then fails geometrically as well, and the membership of $5^{1/3} = 1.70997\ldots$ or of $3^{1/3} = 1.44224\ldots$ is not decided here in either direction. Since $3^{1/3}$ fails even (8.1), it inherits from $\sqrt{3}$ the role of the sharpest undecided case below 2; we return to it as Question 9.4.

9. Remarks and questions

Why this was not harder. The construction is an afternoon's work from parts that are seventy-five and fifty years old: Utz's covering induction [Utz51], Tijdeman's idea of maintaining a nested interval [Tij72], and the elementary digit lemma. What kept the two apart is a reading of the coupling (1.2) as a place where rigidity for $\times m$-invariant sets ought to intervene. It does not, and the reason is structural rather than technical: rigidity phenomena live where $\times p$- and $\times q$-invariant structures with $\log p / \log q \notin \mathbb{Q}$ meet, whereas both sides of (1.2) are structured in the same base m and $\sqrt{m}$ enters only as an affine parameter. The dimension arithmetic agrees with the soft expectation: $2 \dim K_3 = 2 \log 2 / \log 3 \approx 1.26 > 1$.

Witnesses in $\mathbb{Q}(\sqrt{3})$. The witness pinned by a selection rule is digitally defined, and its own arithmetic nature is out of reach: its base-3 digit stream has full factor complexity $p(n) = 2^n$ as far as one can measure (Appendix A.2), so it is neither automatic nor of linear complexity and no transcendence criterion of Adamczewski–Bugeaud type applies to it. Tijdeman's witnesses are of the same character, whereas the witnesses for quadratic Pisot numbers are algebraic [Ste26]. What can be settled is the nature of *some* witness, because the solution set of (3.1) is large. Whether $\sqrt{3}$ admits a witness in $\mathbb{Q}(\sqrt{3})$ we do not know, and we note that no conjugate argument can settle it: $|\bar{\xi}\sqrt{3}^n|$ grows, since $\sqrt{3}$ is neither Pisot nor Salem, whereas every trace design in this literature needs a decaying conjugate. An exhaustive search over $\xi = (a + b\sqrt{3})/c$ with $c \leq 40$ and $|a|, |b| \leq 6c$ found a longest all-even prefix of length 39, at $(a, b, c) = (-205, -25, 39)$, with the lengths distributed geometrically and no near-miss family.

Theorem 9.1. *The set $W = \{\xi > 0 : [\xi\sqrt{3}^n] \in 2\mathbb{Z} \text{ for all } n \geq 1\}$ has cardinality $2^{\aleph_0}$, and already $W \cap [4/3, 5/3]$ does. Hence all but countably many witnesses for $\sqrt{3}$ are transcendental; in particular Problem 3 has a transcendental witness.*

Proof. Keep $m = 3$, $p = 2$, $k_0 = 1$, $j = 2$, $U = 1 + \sqrt{3}$ and $w_0 = 3\sigma = 9\sqrt{3} - 12$ from the proof of Theorem A, and call $w \in [0, 3U]$ *reachable* if it arises from w_0 by finitely many admissible steps $w \mapsto 3(w - c)$, where $c \in \{0, 2, 2\sqrt{3}, 2 + 2\sqrt{3}\}$ and $0 \leq w - c \leq U$.

Where the greedy has a choice. The four admissibility intervals $[c, c + U]$ are

$$[0, 1 + \sqrt{3}], \quad [2, 3 + \sqrt{3}], \quad [2\sqrt{3}, 1 + 3\sqrt{3}], \quad [2 + 2\sqrt{3}, 3 + 3\sqrt{3}],$$

increasing in both endpoints. Consecutive ones meet, as $2 \leq 1 + \sqrt{3}$, $2\sqrt{3} \leq 3 + \sqrt{3}$ and $2 + 2\sqrt{3} \leq 1 + 3\sqrt{3}$; non-consecutive ones do not, as $1 + \sqrt{3} < 2\sqrt{3}$ and $3 + \sqrt{3} < 2 + 2\sqrt{3}$; and their union is $[0, 3U]$ by Theorem 4.1. So each $w \in [0, 3U]$ admits exactly one offset except on

$$O = [2, 1 + \sqrt{3}] \cup [2\sqrt{3}, 3 + \sqrt{3}] \cup [2 + 2\sqrt{3}, 1 + 3\sqrt{3}],$$

where it admits exactly two, and there the two differ in their a-part.

The forced play cannot run forever. Off O the offset is unique; let T be the resulting map on $Z = [0, 3U] \setminus O$, whose components are

$$Z_1 = [0, 2), \qquad Z_2 = (1 + \sqrt{3},\, 2\sqrt{3}),$$
$$Z_3 = (3 + \sqrt{3},\, 2 + 2\sqrt{3}), \qquad Z_4 = (1 + 3\sqrt{3},\, 3U],$$

carrying the offsets $0, 2, 2\sqrt{3}, 2 + 2\sqrt{3}$ and having images

$$[0, 6), \quad (3\sqrt{3} - 3, 6\sqrt{3} - 6), \quad (9 - 3\sqrt{3}, 6), \quad (3\sqrt{3} - 3, 3U].$$

Since $\sqrt{3} < 2$, $4\sqrt{3} > 6$, $3\sqrt{3} > 5$ and $5\sqrt{3} < 9$, comparison of endpoints gives

$$T(Z_2) \cap Z = Z_2, \quad T(Z_3) \cap Z = Z_3, \quad T(Z_1) \cap Z_4 = \emptyset, \quad T(Z_4) \cap Z_1 = \emptyset.$$

So an orbit that stays in Z is eventually confined to a single component: if it ever meets Z_2 or Z_3 it stays there, and otherwise it lies in $Z_1 \cup Z_4$ forever, with no transition available between the two. Lemma 5.3 then places it at that component's fixed point — one of 0, 3, $3\sqrt{3}$, $3 + 3\sqrt{3}$, whose $\sqrt{m}$-coefficients are $0, 0, 3, 3$. But the telescoping in the proof of Lemma 5.2 uses only $b_i \leq e$, so it is valid along every admissible run and bounds the $\sqrt{3}$-coefficient of a reachable state below by $2 \cdot 3^{k+1} \geq 6$. Hence the forced orbit of a reachable state meets O after finitely many steps, and there it splits into two reachable states whose a-digits differ.

Keeping the tails canonical. The proof of Proposition 5.5 uses only the region table (5.4) and Lemma 5.2, both available from any reachable state; so the rule of Section 5.1, started at any reachable state, emits $a_i = 0$ infinitely often and $b_i = 2$ infinitely often. Build a binary tree of runs: at a node with state w, follow the rule until it has emitted both an a-digit 0 and

a b-digit 2, continue with the rule until the state next lies in O, and there take both offsets. Both waits are finite, by this paragraph and the previous one.

Conclusion. An infinite path in the tree is an admissible run satisfying (5.1), so Theorem 4.1 and Proposition 3.2 convert it into a witness $\xi = (4 + \langle a \rangle)/3 \in [4/3, 5/3]$. Two paths that diverge at a branch point give streams a differing at that index, hence distinct $\langle a \rangle \in K_3$ and distinct ξ. There are $2^{\aleph_0}$ paths and only countably many algebraic numbers. □

Remark 9.2. The same comparison of endpoints determines the survivors completely: the $w \in Z$ whose entire forced orbit avoids O are

$$\{0\} \cup \{3^{1-k},\, 3^{1-k}\sqrt{3} : k \geq 0\} \quad \text{together with its reflection in } w \mapsto 3U - w,$$

a countable set. Eight of its members lie in $\mathbb{Z}[\sqrt{3}]$: the four fixed points $0, 3, 3\sqrt{3}, 3U$, the two survivors 1 and $\sqrt{3}$, which reach 3 and $3\sqrt{3}$ in one step, and the reflections $2 + 3\sqrt{3}$ and $3 + 2\sqrt{3}$ of those two. What excludes them all at once is not their arithmetic but their size: every survivor has $\sqrt{3}$-coefficient at most 3, and a reachable state has $\sqrt{3}$-coefficient at least 6 by Lemma 5.2. The proof above does not need the list, and uses only the growth bound; we record it because it locates the obstruction to a quantitative statement, which is not the size of the survivor set but the absence of a uniform bound on how long a play can be forced.

Question 9.3. Does the set of witnesses for $\sqrt{3}$ — the set of $\xi > 0$ with all $[\xi\sqrt{3}^{\,n}]$ even — have positive Hausdorff dimension?

The naive count suggests ≈ 0.26, matching $2 \dim K_3 - 1$ and the observed branching rate of the tree (Remark 6.2).

Question 9.4. Does $3^{1/3} = 1.44224\ldots$ belong to $\mathcal{S}$ or to $\mathcal{Z}$?

Question 9.5. What is the Lebesgue measure of $\mathcal{Z} \cap (1, 3)$?

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Appendix A. Formalization and cross-checks

The Lean 4 development lives in the directory `SZ/` of the Github repository `https://github.com/rwst/Square-Roots/`, namespace `SZ`; the tables below name the module without the `SZ.` prefix. "std3" abbreviates Lean's three standard axioms (propositional extensionality, choice, quotient soundness); every declaration listed depends on those and nothing else — no literature axiom, no `native_decide`, no `sorry`. The name `MahlerZ` is the set $\mathcal{Z}$ of (1.1) and `S` is $\mathcal{S}$; the declarations of `Slice` and `SliceHygiene` live in the nested namespace `SZ.Slice`.

A.1. **Statement index.**

statement	Lean identifier	module
(1.1)	`MahlerZ`	`Defs`
Lemma 2.1	`floor_add_digitReal_mul_pow`	`DigitParity`
Lemma 2.1, parity	`even_floor_add_digitReal_mul_pow`	`DigitParity`
Lemma 3.1	`dvd_floor_add_digitReal_mul_pow`	`DigitParity`
$K_2 = \{0\}$	`digitReal_two_eq_zero`	`DigitParity`
Prop. 2.3, rigidity	`eq_intCast_of_even_floors`	`SqrtTwo`
Prop. 2.3	`sqrtTwo_mem_S`	`SqrtTwo`
$M - K = K$	`digitReal_compl`	`DigitParity`
Thm 4.1, any (m, e)	`pick_adm`	`Slice`
(3.1) solved, any (m, e)	`digitReal_translate`	`Slice`
$M - K = K$, any e	`digitReal_compl'`	`Slice`
Lemma 3.3	`exists_translate`	`Slice`
Prop. 3.2, any p	`dvd_floor_of_hygiene`	`Slice`
ibid. at $p = 2$	`mem_MahlerZ_of_hygiene`	`Slice`
Rem. 2.2, $e \leq m - 2$	`dvd_floor_of_e_le`	`Slice`
ibid. at $p = 2$	`mem_MahlerZ_of_e_le`	`Slice`
Lemma 5.2, uniform	`stt_snd_lower`	`SliceHygiene`
Lemma 5.3, any $r > 1$	`eq_fixed_of_trapped`	`SliceHygiene`
(5.3)	`form_a`, `form_b`	`SliceHygiene`
Prop. 5.4, any p	`dvd_floor_of_uniform`	`SliceHygiene`
ibid. at $p = 2$	`mem_MahlerZ_of_uniform`	`SliceHygiene`
$\sqrt{5}, \sqrt{7} \in \mathcal{Z}$	`sqrtFive/Seven_mem_MahlerZ`	`Cells`
$\sqrt{6}, \sqrt{8} \in \mathcal{Z}$	`sqrtSix/Eight_mem_MahlerZ`	`Cells`
Cor. 6.1	`exists_mem_MahlerZ_Ioo_two_three`	`Cells`
Theorem B	`sqrt_natCast_mem_S_iff`	`Cells`
Thm C, branch (i)	`sliceOne`	`ThmC`
Thm C, branch (ii)	`sliceTwo`	`ThmC`
$U > p \Rightarrow m > p^2$	`sq_lt_of_window`	`ThmC`
Theorem C	`exists_dvd_floor_sqrt`	`ThmC`
Thm B, $m \geq 4$, by C	`sqrt_mem_MahlerZ_of_four_le`	`ThmC`
ibid., composite	`exists_eventually_composite_sqrt`	`ThmC`
table of §7, row 1	`exists_three_dvd_floor_sqrt_ten`	`ThmC`
rows 2–4	`exists_three/four/five_dvd_...`	`ThmC`

statement	Lean identifier	module
Thm 4.1 at $m = 3$	`pick_residual_mem`	`CoverGame`
$w_k \in [0, mU]$	`W_mem`	`CoverGame`
Lemma 5.2	`coverState_snd_lower`	`CoverGame`
Lemma 5.3	`eq_fixed_of_trapped`	`CoverGame`
Prop. 5.5, a-side	`exists_digitA_eq_zero`	`CoverGame`
Prop. 5.5, b-side	`exists_digitB_eq_two`	`CoverGame`
(3.1) solved	`digitReal_translate`	`CoverGame`
(3.2)	`coupling`	`SqrtThree`
Theorem A	`sqrtThree_mem_MahlerZ`	`SqrtThree`
$\sqrt{3} \notin \mathcal{S}$	`sqrtThree_notMem_S`	`SqrtThree`
$[\xi\sqrt{3}] = 2$	`floor_witXi_sqrtThree`	`SqrtThree`
$[3\xi] = 4$	`floor_witXi_three`	`SqrtThree`
Thm B, m a square	`natCast_mem_MahlerZ`	`Defs`
Thm B, $m \geq 9$	`mem_MahlerZ_of_three_le`	`Tijdeman`
plays of the game	`IsRun`, `runState`	`CoverTree`
Lemma 5.2, any play	`runState_snd_lower`	`CoverTree`
Prop. 5.5, any start	`gExists_fst_eq_zero`	`CoverTree`
ibid.	`gExists_snd_eq_two`	`CoverTree`
O; Step 2 of 9.1	`InO`, `gExists_inO`	`CoverTree`
two offsets on O	`branchDigit_adm`	`CoverTree`
the tree	`node`, `treeDigit`	`CoverTree`
Step 4 of 9.1	`tree_exists_branching`	`CoverTree`
(5.1) for every play	`tree_exists_fst_eq_zero`	`CoverTree`
every play is a witness	`run_even_floor`	`CoverTree`
distinct plays, distinct ξ	`runXi_treeDigit_injective`	`CoverTree`
Thm 9.1	`continuum_le_mk_witnessSet`	`CoverTree`
ibid.	`exists_transcendental_witness_sqrtThree`	`CoverTree`

All entries above are `std3`; the rows of Lemma 2.1, `digitReal_compl` and `digitReal_two_eq_zero` are proved for a general base m, the `Slice` and `SliceHygiene` rows and the general `ThmC` rows for an arbitrary base m, modulus p and digit alphabet, and the rest — the cells, the witnesses, and the four table rows of Section 7 — at the base and modulus named. Not formalized, and flagged as such here: the thickness computation of Section 4.1 and all of Section 8. Neither is used anywhere: no proof in this paper refers to the thickness of K_m.

The formalization of Theorem A follows the text with two deviations. First, Lemma 2.1 is formalized in the sharper form (2.1) — the integral part is computed exactly, not merely up to parity — which makes the parity corollary a one-line divisibility argument and makes the two entries $[\xi\sqrt{3}] = 2$, $[3\xi] = 4$ instances of it at $s = 0$. Second, the real $\langle d \rangle$ is defined as the supremum of the partial sums rather than as a sum of a series, which keeps every estimate order-theoretic; the limit description is recovered by a single lemma where it is needed. A by-product of the first deviation: in that sharper form Lemma 3.1 needs *no* hypothesis relating p to m, since $[t\, m^s] = N m^s + \sum_{i<s} d_i m^{s-1-i}$ exhibits every summand as a multiple of p on the nose. The hypothesis $p \mid m$ or $p \mid m - 1$ is what the telescoping induction of Section 3 needs, not what the statement needs; in Theorem C the two divisibilities do real work elsewhere, in fixing e.

Five statements are formalized in a form that differs from the printed one, in each case the form the paper's own proof establishes and uses. (a) Theorem 4.1 appears as the admissibility claim of its proof (`pick_adm`) together with the resulting solution of (3.1) in K (`digitReal_translate`), not as the set identity $K+\sqrt{m}\,K=[0,U]$: the set K is never introduced in Lean, only its digit streams, and the inclusion $\subseteq$ is the half nothing uses. (b) Proposition 3.2 is stated for the streams the greedy produces rather than for an arbitrary pair (x,y'') solving (3.1). (c) Lemma 3.3 is proved under $p\le U$ rather than $U>p$: at $U=p$ the interval $\big((T-U)/p,\,T/p\big)$, $T=(pk_0+M)\sqrt{m}$, has length exactly 1, but both endpoints are irrational, so it still contains an integer — strictness at the left end comes from the irrationality of $\sqrt{m}$, not from the length. It is formalized at all because Theorem C ranges over infinitely many m and so cannot exhibit a j by hand; the realising value is $j=\lceil (T-U)/p\rceil$, which is the j the fifth cross-check reproduces. (d) The word *composite* in Theorem C is rendered as an explicit factorization with both factors >1. (e) In the proof of Theorem B the cells $m\ge 9$ take the shorter route through $[3,\infty)\subset\mathcal{Z}$ (`mem_MahlerZ_of_three_le`) rather than the uniform greedy of the printed proof. The printed proof is formalized too, as `sqrt_mem_MahlerZ_of_four_le`: Theorem C at $p=2$ puts every non-square $m\ge 4$ in $\mathcal{Z}$ by the greedy, branch (i) taking the odd bases and branch (ii) the even ones. Either route closes the cell.

The individual cells of Theorem B still supply their own j, for which $\sigma\in(0,U)$ is a rational inequality in $\sqrt{m}$, verified at $m=3$ and at the four cells. The remark of Section 5.1 — that (4.3) selects the smallest admissible a — is neither used nor formalized; the fourth cross-check below computes both descriptions independently and verifies that they select the same offset at every step.

A.2. **Numerical cross-checks.** Five independent programs bear on the statements of this paper; they share no code with the Lean development or with each other, and are included in `SZ/checks/` of the repository named above. The first re-implements the selection rule of Section 5.1 at $m=3$ using the integer comparator $P^2\lessgtr 3Q^2$ in $\mathbb{Z}[\sqrt{3}]$; it asserts the invariant of Theorem 4.1 at each of 2000 steps, reproduces 40 decimals of ξ identically at truncation depths 400, 600 and 800 with the 3^{-N} tail carried rigorously, and certifies that $[\xi\sqrt{3}^{\,n}]$ is even and unambiguously determined for $n=1,\dots,1000$; it also reports 1214 digits $a_i=0$ and 1233 digits $b_i=2$ among the first 2000, with longest constant runs 7 and 8, which is (5.1) in visible form; it also carries out the exhaustive probe of $\xi=(a+b\sqrt{3})/c$ with $c\le 40$, $|a|,|b|\le 6c$ and $n\le 60$ quoted in Section 9, whose record is a prefix of length 39 at $(a,b,c)=(-205,-25,39)$. The second runs the general-(p,m) greedy of Sections 3–5 in exact rational arithmetic and produced the two tables of Sections 6.1 and 7, together with the verification that the gap conditions (4.1) hold at every (p,m) with $p\le 8$, $m<200$ for which $U>p$. A third explores the admissible tree in $\mathbb{Z}[\sqrt{3}]$: it confirms that the construction in

the proof of Theorem 9.1 produces a full binary tree to depth 10 with every segment closing within 12 steps, that each of the survivors listed there does fail to branch, and that 20 000 random states reach a branch point within 11 steps. The same program measures the factor complexity of the stream a of Theorem A: among the first 60 000 digits every word of length $n \le 11$ occurs, so $p(n) = 2^n$ there, and the shortfalls at $n = 12, 13, 14$ are what the observed frequency $\Pr[a_i = 0] \approx 0.624$ predicts, namely 64, 543 and 2933 missing words against the 39, 422 and 2656 observed. A fourth runs the generic cover game of Sections 3–5 at the four cells $m \in \{5, 6, 7, 8\}$ in exact integer arithmetic in $\mathbb{Z}[\sqrt{m}]$: it verifies the gap conditions and the translate, asserts the cover invariant and the digit alphabet at each of 2000 steps per cell, computes the closed-form rule and the rule of Section 5.1 independently and checks that they select the same offset at every step, reproduces all 25 published decimals of the four witnesses of Section 6.1, and certifies that $[\xi\sqrt{m}^{\,n}]$ is even and unambiguously determined for $n = 1, \dots, 200$. The fifth does the same for Theorem C, in exact arithmetic in $\mathbb{Q} + \mathbb{Q}\sqrt{m}$: for each row of the table of Section 7 it checks the branch hypotheses, rederives $m > p^2$ on branch (ii), checks the two gap conditions (4.1), computes the translate of Lemma 3.3 and finds the j printed in the table, asserts the cover invariant and the alphabet $\{0, p, \dots, e\}$ at each of 2000 steps, reproduces all 20 published decimals of ξ and the six printed integral parts, and certifies that $[\xi\sqrt{m}^{\,n}]$ is divisible by p and unambiguously determined for $n = 1, \dots, 200$; the row $(p, m) = (3, 7)$ of Remark 7.1 is carried through the same checks, minus the hygiene hypothesis it fails.

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