MOMENT MAPS AND EQUIVARIANT COHOMOLOGY IN TORIC GEOMETRY

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Abstract. Toric varieties serve as a rich interface between algebraic geometry, symplectic geometry, and combinatorics. Their structure is deeply tied to combinatorial objects such as polytopes and fans, which encode their topology and geometry. This thesis explores the construction of toric varieties via moment maps, symplectic reduction, and Geometric Invariant Theory, providing a comprehensive framework for understanding these spaces. The central focus is the application of equivariant cohomology, an enrichment of ordinary cohomology, which captures additional symmetries and enriches the topological study of toric varieties. We utilize combinatorial techniques, including localization formulas, shellings, and the Białynicki-Birula decomposition, to analyze the topology and cohomological invariants of these varieties.

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1. Introduction

Toric varieties are a fundamental class of spaces in algebraic geometry, characterized by their combinatorial and geometric simplicity. They arise naturally in several mathematical disciplines, including symplectic geometry, representation theory, and mathematical physics, see [4] and [16]. They also provide, as Fulton writes in [7], "a remarkably fertile testing ground for general theories."

In this work, we begin by describing several polytopal constructions that highlight the interplay between algebraic and symplectic geometry. We then introduce the concept of fans, which provide an equivariant atlas of charts for abstract toric varieties. These combinatorial tools offer a powerful framework for understanding the topology of toric varieties, enabling precise calculations and interpretations. Using these combinatorial insights, we investigate the equivariant cohomology of toric varieties.

As a refinement of ordinary cohomology, equivariant cohomology captures additional symmetries and provides a richer topological invariant. A unique feature of equivariant cohomology is the localization package for equivariantly formal spaces, developed by Atiyah-Bott and Berline-Vergne. We study the localization formula and the GKM conditions, which are central tools in the analysis of toric varieties. In particular, we introduce the Bialynicki-Birula decomposition, an algebraic analogue of Morse theory, and demonstrate that smooth projective toric varieties are equivariantly formal.

2. Constructing toric varieties

We describe multiple equivalent constructions of a smooth toric variety starting from the data of a polytope P subject to certain conditions. Each of the constructions will yield us a space $X_i(P)$ which will all be equivariantly diffeomorphic to each other as smooth manifolds.

2.1. **Delzant polytopes.**

Let V be a real vector space of dimension n and let $V_{\mathbb{Z}}$ be a full dimensional lattice inside V. Given N linear functionals $a_i \in V^*$ and N integers λ_i the set

$$P = \{v \in V \mid a_i(v) + \lambda_i \geq 0 \text{ for all } i\}$$

is called a *polyhedron*. If all of the a_i preserve $V_{\mathbb{Z}}$ then the edges of P point in lattice directions and P is a *rational polyhedron*. It is called a *rational polytope* if it is bounded. P will always denote a rational polytope.

A polytope P has *facets*

$$F_i = \{v \in P \mid a_i(v) + \lambda_i = 0\}$$

and *faces*, which are intersections of facets. The *vertices* of P are the 0-dimensional faces of P and the *edges* of P are the 1-dimensional faces of P. We say P is:

- *simple* if exactly n edges meet at each vertex
- *smooth* if the edges meeting at each vertex form a lattice basis for $V_{\mathbb{Z}}$
- *Delzant* if P is simple and smooth

Example 2.1. The right triangle $P = \{(x, y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x + y \leq 1\}$ is a Delzant polytope. The square pyramyd is not a Delzant polytope because the apex is not simple.

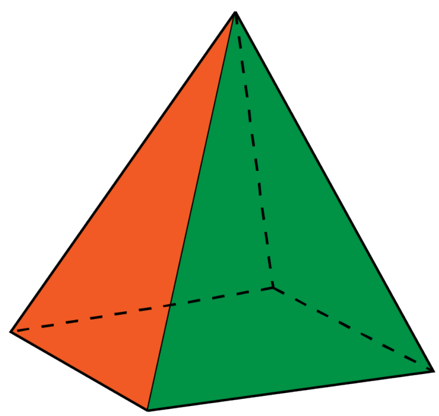

FIGURE 1. Not a Delzant polytope

Delzant polytopes are the building blocks of toric geometry. They are in bijection with smooth projective toric varieties equipped with a very ample line bundle, as we will see in Theorem 2.5.

2.2. **Moment maps.** A symplectic manifold is a pair (M, ω) where M is a smooth manifold and ω is a closed nondegenerate 2-form. This means that

$$d\omega = 0 \quad \text{and} \quad \omega|_p : T_pM \times T_pM \to \mathbb{R} \text{ is nondegenerate}$$

for all $p \in M$. The nondegeneracy of ω allows us to pair vector fields with 1-forms. We say that a vector field X is *Hamiltonian* if the corresponding 1-form $\iota_X\omega = \omega(X, \cdot)$ is exact, in which case it is equal to dH for some smooth function H. The function H is called a *Hamiltonian* of X.

Given a Lie group G acting on M by symplectomorphisms, the Lie algebra $\mathfrak{g}$ acts on M by symplectic vector fields. This linearized action of $\mathfrak{g}$ is given by the expression

$$X_\zeta(m) = \frac{d}{dt}\Big|_{t=0} \exp(t\zeta) \cdot m$$

where we interpret the given expression via parallel transport along the flow of ζ.

Remark 2.2. In general, whenever we have a Lie group G acting on a manifold M, we get a linearized action of $\mathfrak{g}$ on $\Gamma(E)$ for any vector bundle E over M. For the trivial line bundle $E = M \times \mathbb{R}$ we have $\Gamma(E) = C^\infty(M)$ and the linearized action of $\mathfrak{g}$ on $C^\infty(M)$ is given by the Lie derivative of the function along the vector field.

Example 2.3. Consider $G = \mathrm{SL}(2, \mathbb{C})$ acting on $\mathbb{P}^1$ by linear fractional transformations. Explicitly we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [z_0 : z_1] = [az_0 + bz_1 : cz_0 + dz_1]$$

The Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ is generated by the matrices

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Writing down the exponential, we find that

$$\exp(tE) = I + tE \quad \text{since } E^2 = 0 \implies \exp(tE) \cdot [z_0 : z_1] = [z_0 + tz_1 : z_1]$$

On an affine chart, the action of E is given by $z \mapsto z + t$ and we compute

$$\frac{d}{dt}\Big|_{t=0} z + t = 1 \implies X_E(z) = \frac{\partial}{\partial z}$$

Similarly we compute

$$\exp(tH) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \implies \exp(tH) \cdot [z_0 : z_1] = [e^t z_0 : e^{-t} z_1]$$

which looks like $z \mapsto e^{2t} z$ on an affine chart, which gives us

$$\frac{d}{dt}\Big|_{t=0} e^{2t} z = 2z \implies X_H(z) = 2z \frac{\partial}{\partial z}$$

Finally we compute

$$\exp(tF) = 1 + tF \implies \exp(tF) \cdot [z_0 : z_1] = [z_0 : z_1 + tz_0]$$

On an affine chart this transformation looks like $z \mapsto z/(1 + tz)$ and we compute

$$\frac{d}{dt}\Big|_{t=0} \frac{z}{1 + tz} = -z^2 \implies X_F(z) = -z^2 \frac{\partial}{\partial z}$$

In particular we get a map $\mathfrak{sl}(2, \mathbb{C}) \to \Gamma(T\mathbb{P}^1)$ given by

$$E \mapsto \frac{\partial}{\partial z}$$
$$H \mapsto 2z \frac{\partial}{\partial z}$$
$$F \mapsto -z^2 \frac{\partial}{\partial z}$$

which gives coordinates for the action of $\mathfrak{sl}(2,\mathbb{C})$ on $\Gamma(T\mathbb{P}^1)$. Recall that $T\mathbb{P}^1 \cong \mathcal{O}(2)$ and $\dim\Gamma(T\mathbb{P}^1) = 3$. In particular, $\mathfrak{sl}(2,\mathbb{C})$ acts on $\Gamma(T\mathbb{P}^1)$ by the standard representation. In general, this isomorphism can be thought of as happening in degree 1. This map extends to a graded isomorphism between the enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ and the algebra of differential operators $\mathcal{D}_{\mathrm{hol}}(\mathbb{P}^1)$ on $\mathbb{P}^1$. This is a shadow of the Beilinson-Bernstein localization theorem (see [12]).

Let M be a symplectic manifold. We say that the action of T on M is *weakly Hamiltonian* if for every $\zeta \in \mathfrak{t}$ the corresponding vector field X_ζ is Hamiltonian, i.e. $\iota_{X_\zeta}\omega = dH_\zeta$ for some smooth function H_ζ. The H_ζ is determined only up to a constant, so choose the map $\mathfrak{g} \to C^\infty(M)$ given by $\zeta \mapsto H_\zeta$ to be linear. If the map can be chosen to be equivariant with respect to the adjoint action of T on $\mathfrak{t}$, then the action of T is called *Hamiltonian*. In this case, there is a map $\mu : M \to \mathfrak{t}^*$ called the *moment map* defined by

$$H_\zeta(m) := \langle \mu(m), \zeta \rangle$$

If the action of T is Hamiltonian, then the moment map is T-equivariant and unique up to the addition of a constant. Since T is abelian, the adjoint action is trivial, and equivariance requirement reads that the moment map $\mu : M \to \mathfrak{t}^*$ is T-invariant. We are now prepared to state a foundational result in the classification of toric symplectic manifolds. For a complete discussion, see section 5.5 of [15].

Theorem 2.4 (Atiyah-Guillemin-Sternberg Convexity Theorem)**.** *Let M be a compact connected symplectic manifold with a Hamiltonian T-action. Then the image of the moment map is a convex polytope in $\mathfrak{t}^*$ whose vertices are the image of the fixed points of the T-action.*

We say M is a *toric symplectic manifold* in the sense of Theorem 2.5 if (M,ω) is a compact connected symplectic manifold with a effective (meaning no element of T acts trivially) Hamiltonian T-action.

Theorem 2.5 (Delzant)**.** [5] *There is a correspondence between Delzant polytopes up to* $\mathrm{GL}(n,\mathbb{Z})$ *and translation, and toric symplectic manifolds up to equivariant symplectomorphism.*

Corollary 2.6. *There is a correspondence between Delzant polytopes up to* $\mathrm{GL}(n,\mathbb{Z})$ *particular choice of very ample line bundle, up to equivariant isomorphism.*

2.3. **Symplectic reduction.** We describe how to construct M as the symplectic reduction of affine space $\mathbb{C}^N$ for a particular moment map. In particular, M carries a natural symplectic form ω and a Hamiltonian T-action. This section follows [2].

Let P be a Delzant polytope. There are maps

$$\begin{aligned} \pi : \mathbb{R}^N &\to \mathbb{R}^n \\ e_i &\mapsto a_i \end{aligned}$$

and the induced map

$$\pi : \mathbb{R}^N/\mathbb{Z}^N \to \mathbb{R}^n/\mathbb{Z}^n$$

of tori, which give rise to the following short exact sequences.

$$1 \to K \to \mathbb{T}^N \to \mathbb{T}^n \to 1 \tag{2.7}$$

$$0 \to k \to \mathbb{R}^N \to \mathbb{R}^n \to 0$$

The dual of the second sequence gives

$$0 \to (\mathbb{R}^n)^* \to (\mathbb{R}^N)^* \to k^* \to 0$$

and denote the map $i^* : (\mathbb{R}^N)^* \to k^*$. Now consider $\mathbb{C}^N$ with the standard symplectic form $\omega = \sum dz_i \wedge d\bar{z}_i$ and the standard Hamiltonian torus action

$$(e^{i\theta_1}, \ldots, e^{i\theta_N}) \cdot (z_1, \ldots, z_N) = (e^{i\theta_1} z_1, \ldots, e^{i\theta_N} z_N)$$

and corresponding moment map

$$\phi : \mathbb{C}^N \to (\mathbb{R}^N)^*$$
$$\phi(z_1, \ldots, z_N) = -\pi(|z_1|^2, \ldots, |z_N|^2) + (\lambda_1, \ldots, \lambda_N)$$

The subtorus K acts on $\mathbb{C}^N$ via restriction and the restricted action is Hamiltonian. Moreover, the moment map for the action of K is given by $i^* \circ \phi : M \to k^*$.

Let $Z = (i{*}^\circ \phi)^{-1}(0)$ be the zero level set of the moment map. The following claims are all justified in [2].

Lemma 2.8. *Z is compact and K freely acts on Z.*

The following theorem tells us that the orbit space Z/K is a symplectic manifold.

Theorem 2.9. *[Marsden-Weinstein-Meyer] Let G be a compact group and let (M, ω) be a symplectic manifold with a Hamiltonian G-action. Let $i : \mu^{-1}(0) \to M$ be the inclusion of the zero level set of the moment map. Assume G acts freely on $\mu^{-1}(0)$. Then*

- *the orbit space $M_{red} = \mu^{-1}(0)/G$ is a smooth manifold*
- *$\pi : \mu^{-1}(0) \to M_{red}$ is a principal G-bundle*
- *there is a unique symplectic form ω_{red} on M_{red} such that $\pi^*\omega_{red} = i^*\omega$*

Symplectic reduction realizes one direction of Delzant's correspondence.

Proposition 2.10. *The reduced space $X_1(P) := Z/K$ is a toric symplectic manifold with moment map image P.*

2.4. **Projective GIT.** Let P be a Delzant polytope. Complexifying (2.7), we get

$$1 \to K_{\mathbb{C}} \to T^N_{\mathbb{C}} \to T^n_{\mathbb{C}} \to 1$$

Let F_i denote the facets of Δ and for any $z = (z_1, \dots, z_N) \in \mathbb{C}^n$ let $F_z := \cap_{z_i=0} F_i$. Consider the set

$$U = \{z \in \mathbb{C}^n : F_z \neq \emptyset\}$$

Then the quotient $X_2(P) = U/K_{\mathbb{C}}$ is a manifold with an action of $T^N_{\mathbb{C}}/K_{\mathbb{C}} = T^n_{\mathbb{C}}$. It is a smooth projective toric variety because it is a projective GIT quotient, as we will explain with the following theorem of Kempf-Ness.

Remark 2.11. There is a surjective map from X^{ss} to $X//G$. Two points in X^{ss} lie in the same fiber of this map if and only if the closures of their G-orbits intersect. In this case, the $K_{\mathbb{C}}$ orbits are closed. See [17] for more details.

Proposition 2.12. *Let $M \subset \mathbb{CP}^n$ be a smooth projective toric variety embedded by a line bundle. Then M is equivariantly symplectomorphic to a toric symplectic manifold.*

Proof. $\mathbb{CP}^n$ carries a natural symplectic form ω called the Fubini-Study form. Any smooth projective toric variety M embedded in projective space carries a symplectic form ω induced by pulling back the Fubini-Study form. Moreover, the action of T on M is Hamiltonian with respect to ω. □

Conversely, given a toric symplectic manifold (M, ω), we can associate a smooth projective toric variety to the moment polytope $\mu(M)$ which will be equivariantly symplectomorphic to M.

2.5. **Kempf-Ness theorem.** The Kempf-Ness theorem provides a connection between algebraic geometry and symplectic geometry. Recall that if K is a real compact group, then its complexification $G := K_{\mathbb{C}}$ is a complex Lie group which contains K and $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ is the complexification of $\mathfrak{k}$. See [11] for more details about the following theorems.

Theorem 2.13. *Complexification gives a bijection between the isomorphism classes of compact real Lie groups and complex reductive groups.*

The following theorem of Kempf-Ness establishes a relationship between the reduction of a Hamiltonian system by the real group K and the corresponding projective GIT quotient by the complex group G.

Theorem 2.14 (Kempf-Ness)**.** [14] *Let G be a complex reductive group acting on a smooth complex projective variety $X \subset \mathbb{P}^n$. Let K be a maximal compact subgroup of G and suppose K is connected and acts on X Hamiltonianly. Let $\mu : X \to \mathfrak{k}^*$ be the moment map. Then the inclusion $\mu^{-1}(0) \to X$ induces a homeomorphism*

$$\mu^{-1}(0)/K \to X//G$$

Remark 2.15. This theorem is fundamental in moduli problems, where spaces of equivalence classes of geometric objects (like vector bundles, sheaves, or varieties) can be constructed as quotients. It is widely used in understanding spaces like the moduli of vector bundles, the geometry of flag varieties, and moment polytope theory.

2.6. **Fans and abstract toric varieties.** Let T be an n-dimensional torus with character group M, and let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual lattice, with pairing denoted $\langle , \rangle$. Recall that Theorem 2.5 gives a correspondence between Delzant polytopes and smooth projective toric varieties equipped with a very ample line bundle. Forgetting the embedding, we pass to the abstract toric variety, whose combinatorics is encoded in the data of a fan.

Definition 2.16. A *fan* Σ in a real vector space N is a collection of cones σ such that

- σ is a strongly convex polyhedral cone
- if $\sigma \in \Sigma$ and τ is a face of σ, then $\tau \in \Sigma$
- the intersection of any two cones in Σ is a face of each

Given a Delzant polytope P, there is a fan Σ_P in $N_{\mathbb{R}}$ obtained by taking normal directions to the facets of P. The fan Σ_P is called the *normal fan* of P and it is a combinatorial object which encodes an equivariant atlas of charts for the toric variety $X(P)$.

Definition 2.17. A fan Σ is *complete* if the union of the cones in Σ is all of $N_{\mathbb{R}}$. A fan Σ is *nonsingular* if for each k-dimensional cone $\sigma \in \Sigma$, there exist k lattice vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ generate σ and $v_1, \ldots, v_k$ can be extended to a basis of N. A fan Σ is *projective* if there is a rational polytope P such that Σ is the normal fan of P.

As the geometric language suggests, the toric variety $X(\Sigma)$ corresponding to a fan Σ is complete if and only if the fan is complete, and $X(\Sigma)$ is smooth if and only if the fan is nonsingular. See [3] for more details.

Example 2.18. Consider the unit right triangle with corresponding normal fan

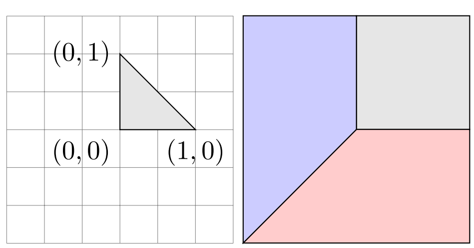

FIGURE 2. Polytope and normal fan for $\mathbb{CP}^2$

Note that the fan has three 2-dimensional cones which are filled in. These cones represent the three standard coordinate charts of $\mathbb{P}^2$ given by $x_i \neq 0$ for $i = 0, 1, 2$. The isosceles right triangle with side lenghth a corresponds to the a-th Veronese embedding of $\mathbb{P}^2$.

The data of a fan and in particular the primitive edge vectors (defined as the generators of the rays of the fan), will prove important in our discussion on equivariant cohomology. See [3] for a complete discussion.

2.7. **Cone-orbit correspondence.** Let T be an n-dimensional torus with character group M. Let $N = \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice, their pairing is denoted by $\langle \cdot, \cdot \rangle$. Let $X = X(\Sigma)$ be a smooth complete toric variety, which are in bijection with complete nonsingular fans Σ in $N_{\mathbb{R}}$.

For any convex cone $\sigma \subset N_{\mathbb{R}}$, the *dual cone* in $M_{\mathbb{R}}$ is

$$\sigma^\vee = \{u \in M_{\mathbb{R}} \mid \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma\}.$$

By intersecting with the lattice, we obtain a semigroup $\sigma^\vee \cap M$ with corresponding semigroup algebra $\mathbb{C}[\sigma^\vee \cap M]$. The toric variety X is covered by T-invariant open affine sets

$$U_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$$

The affine charts corresponding to the top-dimensional cones of Σ are enough to cover X, and the intersection of cones corresponds to the intersection of affine charts.

Each cone τ of the fan also defines a torus-invariant subvariety $V(\tau)$ of X of codimension $\dim \tau$. On open affines, the subvariety looks like

$$V(\tau) \cap U_\sigma = \mathrm{Spec}\, \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M] \hookrightarrow \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$$

and so elements of the dual lattice N can be thought of as rational functions on X.

3. Equivariant cohomology

We introduce T-equivariant cohomology and some classical results about the T-equivariant cohomology of smooth projective toric varieties.

3.1. **Basic properties.** Let T be a complex torus. The equivariant cohomology ring $H_T^*(X)$ of a T-space X is defined as the singular cohomology of the Borel construction $X \times_T ET$, where ET is a contractible space on which T acts freely. Such a space always exists and is unique up to homotopy equivalence, see chapter 1 of [8].

Example 3.1. We can identify $U(n)$ as those complex matrices preserving the standard Hermitian form on $\mathbb{C}^n$. The group $U(n)$ acts on S^{2n-1} transitively and the stabilizer of the point $(1, 0, \ldots, 0)$ is $U(n-1)$.

Therefore there is a canonical action of $U(1) = U(n)/U(n-1)$ acting as scalar matrices on S^{2n-1} inherited from the action on $U(n)$. None of these odd dimensional spheres are contractible, but S^∞ is contractible. In particular $EU(1) = S^\infty$ and

$$BU(1) = S^\infty/U(1) = \mathbb{CP}^\infty$$

Therefore

$$H^*_{U(1)}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[t]$$

where t is the first Chern class of the tautological line bundle and $\deg t = 2$. The classifying space $B\mathbb{C}^*$ of an algebraic torus is homotopy equivalent to that of its maximal compact subgroup, so

$$H^*_{\mathbb{C}^*}(pt) = H^*(\mathbb{CP}^\infty) = \mathbb{Z}[t]$$

as well.

Example 3.2. In general, $H^*_T(\mathrm{pt})$ identifies with the representation ring of T.

$$BT \cong \prod_{\operatorname{rank} T} \mathbb{CP}^\infty$$
$$H^*_T(pt) = H^*(BT) = \mathbb{Z}[t_1, \dots, t_{\operatorname{rank} T}]$$

Given a representation V of T, we can form a vector bundle on the classifying space whose total Chern class is equal to the class of V in the representation ring, see [13].

All subsets, maps, and vector bundles will be taken to be equivariant. Then equivariant cohomology has the following key properties:

(1) functoriality;

(2) a ring structure;

(3) excision;

(4) the Mayer-Vietoris sequence;

(5) the Künneth formula;

(6) the Leray spectral sequence;

(7) for smooth orientable X, Poincaré duality; and

(8) existence of Chern classes,

Let $\Lambda = H^*_T(\mathrm{pt})$. The ring $H^*_T(X)$ is a module over Λ via the map $X \to \mathrm{pt}$.

3.2. **Invariant curves.** This section explores some key topological ideas underpinning equivariant localization. Theorem 3.3 provides the local structure near orbits and simplifies the analysis of tangent spaces at fixed points. See chapters 5 and 7 in [8] for a detailed discussion of the results in this section.

Theorem 3.3 (Slice theorem)**.** *Let X be a nonsingular complex algebraic variety.*

(1) *Suppose K is a compact Lie group acting on X, with an orbit $O = K{\cdot}x \subseteq X$. Then there is a K-invariant open neighborhood $U \subseteq X$ of O which is equivariantly isomorphic to an open neighborhood of the zero section in the normal bundle $N_{O/X}$.*

(2) *Suppose X is affine, and G is a reductive group acting on X, with a closed orbit $O = G \cdot x$. Then there is a G-equivariant étale neighborhood $U \to X$ of O which is equivariantly isomorphic to an étale neighborhood of the zero section of the normal bundle $N_{O/X}$.*

Lemma 3.4. *Let G be a connected reductive linear algebraic group (or compact connected Lie group) acting on a nonsingular algebraic variety X, with a fixed point $p \in X^G$. The point p is isolated if and only if the trivial representation does not occur in T_pX.*

When G is a torus and $\dim X = n$, the lemma reads that $p \in X^T$ is isolated if and only of $c_n^T(T_pX) \neq 0$.

Proof. By the slice theorem, we can reduce to the case where $X = V$ is a representation of G, and $p = 0$ is the origin. In this case, for any representation V of a connected group, the origin $0 \in V$ is an isolated fixed point if and only if V contains no copy of the trivial representation. □

The T-invariant curves in a variety X are important invariants. In particular, they determine the image of the restriction homomorphism

$$\iota^* : H_T^*X \to H_T^*X^T.$$

First, we introduce notation and basic facts about such curves. Suppose T acts on $\mathbb{P}^1$ by distinct characters χ_1 and χ_2, so the fixed points are $0 = [1,0]$ and $\infty = [0,1]$. Writing $\chi = \chi_2 - \chi_1$, we have

$$T_0\mathbb{P}^1 = \mathbb{C}_\chi \quad \text{and} \quad T_\infty\mathbb{P}^1 = \mathbb{C}_{-\chi}.$$

More generally, if T acts on a nonsingular curve C with $C^T = \{p, q\}$, then there is an equivariant isomorphism $C \cong \mathbb{P}^1$ sending p to 0 and q to ∞. The action of T on the open set $\mathbb{C}^* \subset \mathbb{P}^1$ determines up to sign $\pm\chi$ the *character of T acting on C*.

Proposition 3.5. *Let T act on an n-dimensional nonsingular algebraic variety X, and let $p \in X^T$ be an isolated fixed point, so the tangent weights $\chi_1, \dots, \chi_n$ on T_pX are all nonzero.*

(1) *If no two characters at p are parallel, then there are finitely many T-curves in X through p. In fact, there are n such curves, all nonsingular at p, with characters $\chi_1, \dots, \chi_n$.*

(2) *If two characters have the same direction, then there are infinitely many T-curves through p.*

(3) *If two characters have opposite directions, then there are infinitely many T-curves through any T-invariant neighborhood of p.*

Understanding the tangent weights at fixed points is crucial for the study of equivariant cohomology. The following theorem is a key result in this direction.

Theorem 3.6 (Localization Theorem)**.** *Consider a d-dimensional nonsingular variety X with finitely many fixed points. Let*

$$c = \prod_{p \in X^T} c_d^T(T_pX) \in \Lambda,$$

and let $S \subseteq \Lambda$ be a multiplicative set containing c (which is nonzero, since all fixed points are isolated). Assume there are $m \leq \#X^T$ classes in H_T^*X restricting to a basis of H^*X.

Then $m = \#X^T$, the homomorphisms

$$S^{-1}H_T^*X \xrightarrow{S^{-1}l^*} S^{-1}H_T^*X^T \quad \text{and} \quad S^{-1}H_T^*X^T \xrightarrow{S^{-1}l_*} S^{-1}H_T^*X$$

are isomorphisms, and $l^* : H_T^*X \to H_T^*X^T$ is injective.

3.3. **Localization formula.** At its core, the equivariant localization formula, introduced by Atiyah-Bott and Berline-Vergne, arises from the principle that, under certain conditions, integrals over a compact space with a torus action can be "localized" to the fixed points of the action. This idea can be traced back to the stationary phase approximation in physics, where integrals are approximated by contributions from critical points. In the equivariant cohomology setting, the fixed points of the torus action play a similar role. We refer to [8] for a detailed discussion.

Theorem 3.7 (Atiyah-Bott, Berline-Vergne)**.** *Let X be a d-dimensional nonsingular compact algebraic variety with finitely many fixed points. Then*

$$\rho_*(u) = \sum_{p \in X^T} \frac{u|_p}{c_d^T(T_pX)}$$

for any class $u \in H_T^*X$.

Example 3.8. Consider $T = \mathbb{C}^*$ acting on $\mathbb{P}^2$ by the characters $0, t, 2t$. The fixed points are the usual coordinate points p_1, p_2, p_3. For $u \in H_T^*\mathbb{P}^2$, let $u_i = u|_{p_i}$. Near p_1 say, we have coordinates $y/x, z/x$ and therefore the weights of the T-action on $T_{p_1}\mathbb{P}^2$ are $t, 2t$. As a whole, the integration formula says

$$\rho_*(u) = \frac{u_1}{2t^2} + \frac{u_2}{-t^2} + \frac{u_3}{2t^2} = \frac{u_1 - 2u_2 + u_3}{2t^2}.$$

This must be a class in $\Lambda = \mathbb{Z}[t]$, so the integration formula implies a *divisibility condition* relating the restrictions to the three fixed points: $2t^2$ must divide the polynomial $u_1 - 2u_2 + u_3$.

When computing via localization, it is often convenient to represent the fixed points of X as the vertices of a graph, with edges connecting vertices when the corresponding fixed points are connected by a T-invariant curve. This graph is called the *moment graph* of X, and coincides precisely with the image of the invariant curves in X under the moment map.

The image of a class under the restriction

$$H_T^*X \hookrightarrow H_T^*X^T$$

is given by labeling the vertices of the moment graph with characters. See [10] for detailed discussion of these moment graphs, also known as GKM graphs due to work by Goresky, Kottwitz, and MacPherson [9].

Theorem 3.9 (GKM)**.** *Let X be a nonsingular variety with X^T finite, and assume H_T^*X is free over Λ. Suppose that for each $p \in X^T$, the weights on T_pX are relatively prime. Then a tuple*

$$(u_p)_{p\in X^T} \in H_T^*X^T$$

lies in the image of $\iota^* : H_T^*X \to H_T^*X^T$ if and only if for each T-curve $C_{pq} \cong \mathbb{P}^1$ connecting distinct points $p, q \in X^T$, the difference $u_p - u_q$ is divisible by the character $\pm\chi_{pq}$ of C_{pq}.

3.4. **Bialynicki-Birula decomposition.** The Bialynicki-Birula decomposition is a generalization of the Morse theory for torus actions. For $\mathbb{C}^*$-varieties with finitely many fixed points, it implies a particularly nice algebraic notion of equivariant formality, and in particular the equivariant cohomology of X is a free module over the equivariant cohomology of a point. Equivariant formality also implies the GKM condition.

Definition 3.10. A T-space X is called **equivariantly formal** if the Leray spectral sequence associated to the fibration $X \to X \times_T ET \to BT$ collapses at the E_2-page.

By definition, we have that if X is equivariantly formal, then

$$H_T^*(X) \cong \Lambda \otimes H^*(X).$$

When X is equivariantly formal, the ordinary cohomology can be recovered from equivariant cohomology as the quotient

$$H^*(X) = \frac{H_T^*(X)}{\Lambda \cdot H_T^*(X)}$$

which in effect simply sets each $t_i = 0$. In particular, $H_T^*(X)$ is a free module over Λ. Many varieties of interest are equivariantly formal, including:

(1) a smooth complex projective variety (with respect to any linear algebraic T-action);

(2) a variety whose ordinary cohomology vanishes in odd degree (with respect to any T-action);

(3) a compact symplectic manifold with a Hamiltonian T-action, where T is a compact torus.

See chapter 7 in [8] for a good discussion of these facts.

We now introduce the Bialynicki-Birula decomposition. Suppose that $\mathbb{C}^*$ acts on a smooth projective variety X with finitely many fixed points $p_1, \dots, p_k$. Then each $T_{p_i}X$ is a representation of $\mathbb{C}^*$, and so we can decompose into weight spaces

$$T_{p_i}X = \bigoplus_{\lambda \in \mathbb{C}} V_\lambda$$

where $V_\lambda = \{v \in T_{p_i}X \mid t \cdot v = t^\lambda v\}$. Note that $\lambda \neq 0$ because the fixed point set is isolated.

Define the attracting set

$$C_i = \{x \in X \mid \lim_{t \to 0} t \cdot x = p_i\}$$

Theorem 3.11 (Bialynicki-Birula)**.** [1] *There exists a filtration of X by closed subschemes*

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 = \emptyset$$

such that each $X_i \backslash X_{i-1}$ is a disjoint union of affine spaces called cells, the attracting sets. In particular, there are $\#X^T$ of them, and the closure of each cell is a union of cells.

The attracting sets give rise to a *stratification* of X into locally closed subvarieties, so that the closure of each cell is a union of cells. General results about stratified spaces (see chapter 1 of [6]) imply that:

Corollary 3.12. *Let X be a smooth projective $\mathbb{C}^*$-variety with finitely many fixed points. Then*

(1) $H_{2i+1}(X) = 0$ *for all i;*

(2) $H_{2i}(X)$ *is a $\mathbb{Z}$-module freely generateed by the classes of the closures of the i-dimensional cells.*

This implies such varieties are always equivariantly formal.

Corollary 3.13. *Let a torus T act on a nonsingular complete variety X, with finitely many fixed points. Then X is equivariantly formal with integral coefficients. In particular:*

(1) $H_T^*X \to H^*X$ *is surjective, with kernel generated by the kernel of $\Lambda_T \to \mathbb{Z}$; and*

(2) $H_T^*X \to H_T^*X^T$ *is injective, and becomes an isomorphism after inverting finitely many characters in Λ_T.*

Proof. The classes of the attracting sets form a basis of H_T^*X and H^*X over Λ and $\mathbb{Z}$ respectively, so X is equivariantly formal. Injectivity of the restriction homomorphism comes from the diagram:

$$\begin{tikzcd} H_T^*X \arrow[r, "\iota^*"] \arrow[d] & H_T^*X^T \arrow[d] \\ S^{-1}H_T^*X \arrow[r, "\sim"] & S^{-1}H_T^*X^T \end{tikzcd}$$

for a suitable multiplicative set $S \subseteq \Lambda$, where the vertical arrows are injective since H_T^*X and $H_T^*X^T$ are free over Λ, and the bottom arrow is an isomorphism by the localization theorem. □

3.5. **Shellings.** In this section, we apply the setup of the Bialynicki-Birula decomposition to the context of toric varieties.

Let $X = X(\Sigma)$ be projective, with P a polytope whose normal fan is Σ. Choosing a general vector $v \in N_{\mathbb{R}}$ we obtain an ordering of the vertices $u_1, \dots, u_s$ of P by the order of the inner products $\langle v, u_i\rangle$. Geometrically, we are choosing a 1-parameter subgroup of T which acts on X. The corresponding sub-moment map turns out to be a perfect Morse-Bott function on X.

By the polytope-fan correspondence, we get an ordering of the maximal cones $\sigma_1, \dots, \sigma_s$ of Σ. For $1 \le i \le s$ let

$$\tau_i = \bigcap_{j>i, \dim(\sigma_j \cap \sigma_i) = n-1} \sigma_j \cap \sigma_i$$

so that $\tau_1 = \{0\}, \tau_s = \sigma_s$ and $\tau_p \subset \tau_q$ implies $p \le q$. Such an ordering of cones is called a *shelling* of Σ.

Proposition 3.14. *A shelling gives a cellular decomposition of X, with the closures of the cells being $V(\tau_i)$. In particular, the classes*

$$\alpha_i = [V(\tau_i)] \in H^{2(n-\dim \tau_i)}(X)$$

form an additive $\mathbb{Z}$-basis of $H^*(X)$. Moreover, the corresponding equivariant cohomology classes

$$\alpha_i^T = [V(\tau_i)]^T \in H_T^{2(n-\dim \tau_i)}(X)$$

form an additive $\mathbb{Z}$-basis of $H_T^*(X)$.

This proposition follows from the Bialynicki-Birula decomposition, as the $V(\tau_i)$ are precisely the closures of the attracting sets of the chosen $\mathbb{C}^*$-action on X. We will demonstrate this in a particularly pleasant example.

Example 3.15 (Morse theory on $\mathbb{CP}^2$). Classically, recall that the Chow ring (or cohomology ring) of $\mathbb{CP}^2$ is

$$H^*(\mathbb{CP}^2) = \mathbb{Z}[0] \oplus \mathbb{Z}[\mathbb{P}^1] \oplus \mathbb{Z}[\mathbb{P}^2]$$

Consider the standard action of T^2 on $\mathbb{CP}^2$ and consider the 1-parameter subgroup acting by $t \cdot [x : y : z] = [tx : t^2y : z]$. Consider the following moment image, whose edges are labeled by the weights of the action on the tangent space at the fixed points:

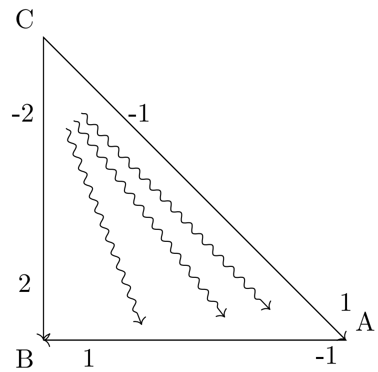

FIGURE 3. Flow lines from choice of subgroup.

Based on our choice of 1-parameter subgroup, we have the following decomposition:

$$\mathbb{CP}^2 = C \coprod (\mathbb{P}^1 \backslash C) \coprod \mathbb{P}^2 \backslash \mathbb{P}^1$$

One sees this decomposition by considering the attracting sets of the action for each fixed point. For C the attracting set is C itself, for A the attracting set is the line $\mathbb{P}^1$, all the points of $\mathbb{P}^1$ except C are attracted to A, and for B the attracting set is $\mathbb{P}^2 \backslash \mathbb{P}^1$, where the $\mathbb{P}^1$ is the T-invariant curve which joins the fixed points A and C. These cells are all affine spaces, and their closures are precisely $V(\tau_i) = \mathbb{P}^i$.

If X is not projective, then one can subdivide cones and produce a refinement Σ' of Σ so that the corresponding map

$$\pi : X(\Sigma') \to X(\Sigma)$$

is a surjective birational T-equivariant morphism and $X(\Sigma')$ is smooth and projective. The composition

$$\pi_* \circ \pi^* : H^*(X(\Sigma)) \to H^*(X(\Sigma))$$

is the identity on $H^*(X(\Sigma))$ and on $H^*_T(X(\Sigma))$ and therefore π^* is injective and π_* is surjective.

Assembling the results of the previous sections, we obtain the following proposition.

Proposition 3.16. *For any complete smooth toric variety X, the cohomology ring $H^*(X)$ is generated by the classes $[V(\tau_i)]$ of the closures of the attracting sets as $\mathbb{Z}$-module, and the equivariant cohomology ring $H^*_T(X)$ is generated by the classes $[V(\tau_i)]^T$ as a module over Λ.*

3.6. **Danilov's theorem.** Let $D_1, \dots, D_d$ be the T-invariant divisors, $D_i = V(\rho_i)$ for rays $\rho_1, \dots, \rho_d$ of Σ. Let $v_i \in N$ be the minimal generator of the ray ρ_i. For $u \in M$, the element $e^u \in \mathbb{C}[M]$ determines a rational function on X. The corresponding divisor is

$$\mathrm{div}(e^u) = \sum_i \langle u, v_i \rangle D_i.$$

Equivariantly, e^u is a rational section of the line bundle L_u with character u, so we have a relation

$$u = c_1^T(L_u) = [\mathrm{div}(e^u)]^T = \sum_i \langle u, v_i \rangle [D_i]^T$$

in $H_T^2 X$.

Moreover given distinct rays $\rho_{i_1}, \dots, \rho_{i_r}$, we have

$$[D_{i_1}]^T \cdots [D_{i_r}]^T = [V(\tau)]^T$$

if the rays span a cone τ of Σ, zero otherwise. Let $X_1, \dots, X_d$ be variables, one for each ray of the fan Σ. Consider the following ideals in $\mathbb{Z}[X] = \mathbb{Z}[X_1, \dots, X_d]$

- I is generated by all monomials $X_{i_1} \cdots X_{i_r}$, such that the corresponding rays $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone.
- J is generated by all elements $\sum \langle u, v_i \rangle X_i$, ranging over all $u \in M$.

The ring $\mathbb{Z}[X]/I$ is called the *Stanley-Reisner ring* of Σ.

We have a homomorphism

$$\mathbb{Z}[X]/(I + J) \to H_T^* X,$$

given by $X_i \mapsto [D_i]$. Indeed, we have seen that I and J map to zero, so the homomorphism is well-defined. It is surjective, because

$$[V(\tau)] = [D_{i_1}] \cdots [D_{i_r}]$$

where $\rho_{i_1}, \dots, \rho_{i_r}$ are the rays spanning τ. In fact, it is an isomorphism, and one deduces this from the corresponding equivariant statement.

In equivariant cohomology, we have two ideals in $\Lambda[X] = \Lambda[X_1, \dots, X_d]$:

- I' has the same generators as I, all monomials $X_{i_1} \cdots X_{i_r}$, such that the corresponding rays $\rho_{i_1}, \dots, \rho_{i_r}$ do not span a cone.
- J' is generated by elements $u - \sum \langle u, v_i \rangle X_i$, ranging over all $u \in M$ (or a basis for M).

We have a homomorphism

$$\Lambda[X]/(I' + J') \to H_T^* X,$$

by $X_i \mapsto [D_i]^T$. Again, we have seen that I' and J' map to zero, so the homomorphism is well-defined; it is surjective for similar reasons. We will see that it is an isomorphism.

Theorem 3.17 (Danilov)**.** *For any complete smooth toric variety $X = X(\Sigma)$, we have isomorphisms of cohomology rings*

$$H^*X \cong \mathbb{Z}[X]/(I+J) \quad \textit{and} \quad H^*_TX \cong \Lambda[X]/(I'+J').$$

This also identifies the equivariant cohomology of $X = X(\Sigma)$ with the Stanley-Reisner ring of Σ because the canonical homomorphism

$$\mathbb{Z}[X]/I \to \Lambda[X]/(I'+J')$$

is an isomorphism.

Proof. We sketch a proof of Danilov's theorem using the GKM relations, see [8]. For any cone $\tau \subseteq N_{\mathbb{R}}$, one has the sublattice $N_\tau \subseteq N$ spanned by τ, with corresponding quotient lattice $M \to M_\tau$. For $\gamma \subseteq \tau$, there is a corresponding projection $M_\tau \to M_\gamma$. We will write $f \mapsto f|_\gamma$ for the corresponding map $\mathrm{Sym}^* M_\tau \to \mathrm{Sym}^* M_\gamma$. For any rational polyhedral fan Σ in N, the ring of *piecewise polynomial functions* with respect to Σ is

$$PP^*(\Sigma) = \left\{(f_\tau)_{\tau\in\Sigma} \mid f_\tau \in \mathrm{Sym}^* M_\tau, \text{ and } f_\tau|_\gamma = f_\gamma \text{ for all } \gamma \subseteq \tau\right\}.$$

When Σ is a complete fan, $PP^*(\Sigma)$ is the ring of continuous functions on $N_{\mathbb{R}}$ which are given by polynomials in $\Lambda = \mathrm{Sym}^* M$ on each maximal cone σ. Then there are canonical isomorphisms

$$\mathbb{Z}[X]/I \cong PP^*(\Sigma) \cong \{(f_\sigma)_{\dim\sigma=n} \mid f_\sigma|_\tau = f_{\sigma'}|_\tau \text{ if } \tau \text{ is a facet of } \sigma \text{ and } \sigma'\}$$

The first map is an isomorphism because of the following observation. The correspondence takes X_i, which indexes the T-invariant divisor D_i, to a piecewise polynomial function f_σ which takes the value $c_1(N_{D_i}X|_p)$ where p is the fixed point corresponding to σ, i.e. to each cone σ we associate the character of the normal bundle of the T-invariant divisor restricted to the fixed point corresponding to σ. Because X is smooth, this correspondence is surjective cone by cone, and so the map is surjective. The kernel certainly contains I because the intersection of the T-invariant divisors is empty if the corresponding rays do not span a cone. The kernel is precisely I. Indeed, observe that the kernel cannot contain a polynomial with multiple terms, because if the corresponding piecewise polynomial function is zero, then the function is zero cone by cone. For a fixed cone σ, only n divisors $D_{i_1}, \dots, D_{i_n}$ contribute to the value of f_σ and all others are zero. Therefore we see that for each cone, we get an equation of the form

$$\sum_{a_1,\dots,a_n\geq 0} h_{a_1,\dots,a_n} c_1(N_{D_{i_1}}X|_p)^{a_1} \cdots c_1(N_{D_{i_n}}X|_p)^{a_n} = 0$$

where the h are integers. Since X is smooth, the $c_1(N_{D_i}X|_p)$ together form a basis of the tangent space T_pX and this is enough to see that all the hs are zero. Therefore the kernel contains only monomials. This implies that we took a product of X_i where for every σ, the value $c_1(N_{D_i}X|_p)$ is zero for some X_i in the product, i.e. the corresponding rays do not span a cone.

The second map is an isomorphism because of the following observation. Suppose $u = 0$ defines the common facet $\tau = \sigma \cap \sigma'$. Then $V(\tau)$ has character u, and the relation $f_\sigma|_\tau = f_{\sigma'}|_\tau$ is the same as requiring that u divide the difference $f_\sigma - f_{\sigma'}$.

By computing the characters on the T-invariant curves $V(\tau)$, we can identify this ring with the subring of

$$H_T^* X^T = \bigoplus_{\dim \sigma = n} \Lambda$$

defined by the GKM conditions. It follows that

$$\mathbb{Z}[X]/I \cong PP^*(\Sigma) \cong H_T^* X$$

The ordinary cohomology ring is obtained by setting all $t_i = 0$, so

$$H^* X \cong \mathbb{Z}[X]/(I + J)$$

as desired. $\square$

4. Appendix A: Vector bundles and connections

We provide a brief introduction to connections on vector bundles.

4.1. **Parallel transport.**

One way to think about a connection is to consider parallel transport. You want to be able to differentiate sections of a vector bundle along paths. When we are dealing with functions, we can form the directional derivative

$$ds(x)X = \lim_{t \to 0} \frac{s(\gamma(t)) - s(\gamma(0))}{t}$$

for any smooth path γ representing the tangent vector $X \in T_x M$ and this expression gives us a linear map $ds(x) : T_x M \to E_x$.

However if E is a nontrivial bundle, then this expression does not make sense because the summands live in different fibers. There is unfortunately no natural way to compare vectors in different fibers. Therefore we need to introduce additional structure to be able to compare these fibers.

For a general vector bundle $E \to M$, we want to associate, to a path γ in M, a smooth family of parallel transport isomorphisms $P_\gamma^t : E_{\gamma(0)} \to E_{\gamma(t)}$ such that

- $P_\gamma^0 = \mathrm{id}$
- $P_{\gamma_1 \cdot \gamma_2}^t = P_{\gamma_2}^t \circ P_{\gamma_1}^t$

for any paths γ_1, γ_2 and $t \in \mathbb{R}$.

Such a choice would allow us to define the directional (”covariant”) derivative of a section s along a path γ as before. We shuold require that

- The directional derivative depends only on s and $X \in T_x M$, not the particular choice of γ.
- The map $\nabla s(x) : T_x M \to E_x$ is $\mathbb{C}$-linear.

This will give us the richest notion of a connection on a vector bundle.

4.2. **Connections.** In this section, we consider M a real manifold and $\pi : E \to M$ complex vector bundle. Let $\mathcal{A}^i(E) = \Omega^i(M) \otimes E$ denote the sheaf of smooth i-forms with values in E.

Definition 4.1. A **connection** on E is a $\mathbb{C}$-linear map of sheaves $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ satisfying the Leibniz rule

$$\nabla(fs) = df \otimes s + f\nabla s$$

We can interpret this definition in the sense of parallel transport. Given a section $s \in \mathcal{A}^0(E)$, we can differentiate it along a path γ to get another section of E, i.e. $\nabla : \Gamma(E) \to \Gamma(\mathrm{Hom}(TM, E))$.

Theorem 4.2. *The space of all connections $\mathcal{A}(E)$ is an affine space modelled on $\mathcal{A}^1(\mathrm{End}\, E)$. In particular*

- *$\mathcal{A}(E)$ is nonempty*
- *For any two connections ∇_1, ∇_2 the difference $\nabla_1 - \nabla_2$ is a global section of $\mathcal{A}^1(\mathrm{End}\, E)$.*
- *$(\nabla + a)s := \nabla s + as$ is a connection whenever ∇ is a connection and $a \in \mathcal{A}^1(\mathrm{End}\, E)$.*

The idea of a connection generalizes the exterior differential to sections of general vector bundles. However, a connection need not satisfy $\nabla^2 = 0$ in general. The obstruction for a connection define a differential is measured by its curvature. We explain this now.

4.3. **Curvature.** A connection $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ induces "differentials"

$$\nabla : \mathcal{A}^i(E) \to \mathcal{A}^{i+1}(E)$$

given by the formula

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^i \alpha \wedge \nabla s$$

Definition 4.3. The **curvature** F_∇ of a connection ∇ is the composition

$$F_\nabla := \nabla^2 : \mathcal{A}^0(E) \to \mathcal{A}^2(E)$$

In particular F_∇ is a global section of $\mathcal{A}^2(\mathrm{End}\, E)$. This is because the curvature homomorphism is $\mathcal{A}^0$-linear.

Example 4.4. Consider the connections on the trivial bundle $M \times \mathbb{C}^r$. If $\nabla = d$ is the trivial connection then $F_\nabla = 0$.

Any other connection is of the form $\nabla = d + A$ where A is a matrix of 1-forms. For a section s we compute

$$\begin{aligned} F_\nabla(s) &= (d+A)(d+A)(s) \\ &= d^2 s + dAs + Ads + AAs \\ &= d(A)s + A \wedge As \end{aligned}$$

and therefore

$$F_\nabla = dA + A \wedge A$$

For line bundles we get that $F_\nabla = dA$ is an ordinary 2-form.

5. Appendix B: Prequantization

Manifolds equipped with integral symplectic forms admit prequantization line bundles in the following sense [2].

Theorem 5.1. *Let (M, ω) be a symplectic manifold. Suppose that $[\omega]$ is integral. Then there exists a "prequantization" line bundle $\mathcal{L} \to M$ with $c_1(\mathcal{L}) = [\omega]$ and a Hermitian connection α whose corresponding curvature form is ω. Moreover $\mathcal{L}$ is unique up to isomorphism.*

Let P be a Delzant polytope. Complexifying (2.7) and passing to the dual of the Lie algebras, we get

$$1 \to K_{\mathbb{C}} \to T^N_{\mathbb{C}} \to T^n_{\mathbb{C}} \to 1$$
$$0 \to (\mathbb{R}^n)^* \to (\mathbb{R}^N)^* \to k^* \to 0$$

where k^* is the dual of the Lie algebra of $K_{\mathbb{C}}$. Let F_i denote the facets of Δ and for any $z = (z_1, \ldots, z_N) \in \mathbb{C}^n$ let $F_z := \cap_{z_i=0} F_i$. Consider the set

$$U = \{z \in \mathbb{C}^n : F_z \neq \emptyset\}$$

Recall that we defined the toric symplectic manifold $X_2(P) = U/K_{\mathbb{C}}$.

Proposition 5.2. *The line bundle $\mathcal{L} = U \times_{K_C} \mathbb{C}$ where $K_{\mathbb{C}}$ acts on $\mathbb{C}$ with weight $\nu = i^*(-\lambda)$ is a prequantization line bundle for $M = U/K_{\mathbb{C}}$.*

Symplectic reduction realizes a Kahler form on the reduced space, in particular M and $\mathcal{L}$ actually carry complex structures. The following theorem is about the space of holomorphic sections of $\mathcal{L}$.

Theorem 5.3. *Let M be a toric symplectic manifold with moment polytope P. Let $\mathcal{L}$ be the prequantization line bundle for M. Then we have*

$$\dim H^0(M, \mathcal{L}) = \#(\textit{integer points in } P)$$

Proof. A holomorphic section of $\mathcal{L}$ over M corresponds to a $K_{\mathbb{C}}$-equivariant holomorphic function $f : U \to \mathbb{C}$. Such f extends to all of $\mathbb{C}^N$ because of Hartog's theorem (A holomorphic function on $\mathbb{C}^N$ for $N > 1$ canont have an isolated singularity and therefore cannot have a singularity on a submanifold of codimension ≥ 2).

Write such a function as its Taylor series so that

$$f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha z^\alpha$$

Consider the equivariance one term at a time. Thinking about the monomial $f(z) = z^I$ we see that

$$\begin{aligned} &f(k \cdot z) = f(i(k) \cdot z) = (i(k) \cdot z)^I = i(k)^I z^I = k^{i^*(I)} z^I \\ &k \cdot f(z) = k^\nu z^I \end{aligned}$$

and therefore a basis for the space of equivariant functions $f : U \to \mathbb{C}$ is given by

$$\{z^I \mid i^*(I) = \nu, I \in \mathbb{N}^n\}$$

and the set of such I is

$$\mathbb{Z}_+^n \cap (i^*)^{-1}(\nu)$$

monomials corresponding to lattice points in P. $\square$

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> Doing math is akin to unfolding a melody; its first sounds are usually a gift from someone else.
>
> Alexander Beilinson

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