

EVERY T-PERFECT GRAPH IS 50-COLORABLE

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ABSTRACT. A graph is t -perfect if its stable set polytope is described by non-negativity, edge, and odd-cycle inequalities. In the 1990s, Bruce Shepherd asked whether every t -perfect graph is 4-colorable. Chudnovsky, Cook, Davies, Oum and Tan proved that t -perfect graphs are 199053-colorable. Novaković and Weltge lowered the bound to 186 by refining the construction of arithmetic ropes in that proof. We prove that every t -perfect graph is 50-colorable. A Lean formalization of the 50-color theorem accompanies this paper.

1. INTRODUCTION

Use of AI tools. The proof of the 50-color theorem was found by GPT (OpenAI) through multiple rounds of questions from the author. Claude (Anthropic) drafted this manuscript from research notes generated by GPT. The author gave Claude detailed instructions and requested many revisions, and GPT also helped revise the manuscript. GPT (OpenAI) generated the accompanying Lean proof. The author does not plan to submit this version to a journal.

For a graph G , the *stable set polytope* $\text{STAB}(G)$ is the convex hull in $\mathbb{R}^{V(G)}$ of the incidence vectors $\mathbf{1}_S$ of stable sets S . For $x \in \mathbb{R}^{V(G)}$ and $U \subseteq V(G)$, write $x(U) = \sum_{u \in U} x_u$, and let $\mathbf{1}$ denote the all-ones vector. Every point of $\text{STAB}(G)$ satisfies the edge and odd-cycle inequalities, so $\text{STAB}(G) \subseteq \text{TSTAB}(G)$, where

$$(1.1) \quad \text{TSTAB}(G) = \left\{ x \in [0, 1]^{V(G)} : x_u + x_v \leq 1 \ (uv \in E(G)), \right. \\ \left. x(V(C)) \leq \frac{|V(C)|-1}{2} \ (C \text{ an odd cycle of } G) \right\}.$$

Following Chvátal [4], a graph G is *t-perfect*¹ if $\text{STAB}(G) = \text{TSTAB}(G)$. Bipartite graphs are t -perfect, and t -perfection is closed under taking induced subgraphs and, more generally, t -minors [5].

The vector $\frac{1}{3}\mathbf{1}$ lies in $\text{TSTAB}(G)$ for every graph G , so in a t -perfect graph it is a convex combination of stable sets, and the fractional chromatic number is at most three. Shepherd asked whether $\text{STAB}(G)$ then has the integer decomposition property, which would make every t -perfect graph 3-colorable. This fails: Laurent and Seymour found a 4-chromatic t -perfect graph (see Schrijver [9, p. 1207]), the complement of the line graph of the triangular prism, and Benchetrit [1, 2] found a second one, the complement of the line graph of the 5-wheel. Shepherd's question whether every t -perfect graph is 4-colorable remains open [6, Problem 8.14]. Even the existence of *some* finite bound was open until Chudnovsky, Cook, Davies, Oum and Tan [3] proved that t -perfect graphs are 199053-colorable, a bound that Novaković and Weltge [8] lowered to 186 by refining the proof. We prove a new upper bound of 50 for the chromatic number of t -perfect graphs.

Theorem 1.1. *Every t -perfect graph is 50-colorable.*

Theorem 1.1 is proved in Sections 2–8.

The proofs in [3, 8] use only two properties of t -perfect graphs. The first is the stable-set result in Lemma 2.3: a t -perfect graph has a stable set meeting every shortest odd cycle. Deleting such a set raises the odd girth by at least two and lowers the chromatic number by at most one, so after six deletions we may assume the odd girth is at least 15 at the cost of six colors. The second property is that t -perfect graphs contain no odd wheel as a t -minor. The task is

¹The letter t stands for the French word *trou* (“hole”).

therefore to find an odd wheel t-minor in every graph of odd girth at least 15 whose chromatic number is large.

For this, [3] introduced *arithmetic ropes*: a cyclic chain of anchors $q_1, \dots, q_5$, consecutive anchors joined by pairs of induced paths whose parities can be chosen so that any three anchors lie on an odd induced cycle cut into three odd arcs. A rope inside a single breadth-first level yields an odd wheel t-minor. Novaković and Weltge [8, Sections 2–3] introduced *quasi-arithmetic ropes*, which retain the needed odd-cycle property without requiring an even-path alternative in every segment. Under their stronger odd-girth assumption, their construction reduces the chromatic-number requirement from the exponential dependence on the number of segments in [3, Theorem 4.1] to a linear one [8, Theorem 2.3]. Parity is the whole difficulty: all paths of a rope must be induced, must avoid each other, and must realize a prescribed pattern of parities.

A rope is grown one segment at a time inside a connected set C of large chromatic number, using a set B of vertices that cover C (a breadth-first level and the next one). Each segment ends at a new anchor, and the construction passes to a smaller set $C' \subseteq C$ that is kept clear of everything built so far. Throwing away part of C is what costs chromatic number. Two halvings (a level of a level) precede the construction, and a total cost is charged for the segments. If a component of a level inside a level must have chromatic number at least T , the argument gives $\chi \leq 4T + 2$ (Section 3). In [8] one needs $T = 46$, because a segment with a path of prescribed parity costs 5 and a segment with two paths of opposite parity costs 11, giving the bound 186. Here $T = 12$, giving 50; this works because of the following changes.

One palette for a block and its ball. To keep control of a retained set one must discard a stable block of vertices that would otherwise produce chords, and a ball around the current anchor. Rather than paying for each separately, Lemma 4.2 colors the block, the radius-six ball around one of its vertices, and the part already colored with one common palette. Removing a block together with its ball costs a single color, provided the vertices of the block are pairwise joined by even walks of length at most four, which the canonical grading guarantees.

Forward correction of parity. A path of prescribed parity is obtained by first building a path of arbitrary parity to a vertex $\hat{q}$ of the earliest block, and, if the parity is wrong, stepping *forward* to a neighbor inside the retained set (Proposition 6.2). The earlier method moved backward to a parent and had to discard more.

Route parity and active vertices. Two paths of opposite parity are produced by a single edge whose endpoints have equal *route parity*, the parity of the canonical path from the current anchor (Lemma 6.5). Vertices without an earlier equal-parity neighbor induce a bipartite graph, so discarding them costs two colors, and the joint palette lemma costs one more (Proposition 6.3).

Direct closing. Instead of a fifth extension, the last segment is obtained directly from one equal-parity edge among the vertices still available, and is joined back to the first anchor through the lower levels (Proposition 7.4). It needs a potential of 3.

The numbers 1, 3, 1, 3 for the four extensions, 3 for the closing, and 1 for the initial ball add up to $T = 12$ (Table 1). Proposition 2.7, stated by Novaković and Weltge [8, Proposition 2.2] using the proof of [3], shows that a quasi-arithmetic rope in a level yields an odd wheel t-minor. The stable-grading and extension framework of [3, Section 4], as used and refined for quasi-arithmetic ropes in [8, Sections 3–5], is recalled and re-proved in the form needed in Sections 5–7; Appendix A lists three places where our formulation differs from theirs.

We also use the classical facts on t-minors stated in Lemma 2.2 and the stable-set result stated in Lemma 2.3. The remaining steps are proved here. Section 2 collects notation and the facts about t-perfect graphs used in the proof. Section 3 gives the reduction and color accounting; Section 4 proves the two coloring lemmas; Section 5 develops the canonical grading; Section 6 proves the extension steps; Section 7 chains them into a rope and proves the closing lemma; and Section 8 completes the proof. Appendix A compares our proof with that of Novaković and Weltge [8].

2. PRELIMINARIES

2.1. Graphs and distances. All graphs are finite and simple. For $X \subseteq V(G)$ we write $G[X]$ for the induced subgraph and $\chi(X) = \chi(G[X])$ when the ambient graph is understood. A path or cycle has the length of its number of edges, and the *parity* of a path is the parity of its length. A path is *induced* if it is an induced subgraph. The *odd girth* $\text{og}(G)$ is the length of a shortest odd cycle, and $\text{og}(G) = \infty$ if G is bipartite. A closed walk of odd length contains an odd cycle of at most the same length; we call this the *odd closed walk principle*, and use it repeatedly to bound $\text{og}(G)$ from above.

Distances are always measured in the ambient graph G , even when a vertex set is displayed through an induced subgraph. Write $N(v)$ for the open neighborhood of v , and for $A \subseteq V(G)$ write $N(A) = \{v \in V(G) : v \text{ has a neighbor in } A\}$. For nonempty vertex sets A, B put $\text{dist}(A, B) = \min\{\text{dist}(a, b) : a \in A, b \in B\}$; a vertex in either argument is identified with its singleton set. For a vertex v and an integer $j \geq 0$, write $N^j(v)$ for the set of vertices at distance exactly j from v and $N^j[v]$ for the set of vertices at distance at most j . Two sets $B, C \subseteq V(G)$ are *disjoint with B covering C* , written “ B covers C ”, if $B \cap C = \emptyset$ and every vertex of C has a neighbor in B . A set X is *anticomplete* to a set Y if no vertex of X has a neighbor in Y .

Observation 2.1 ([3, Observation 2.3]). If $\text{og}(G) \geq 2k + 3$ and $v \in V(G)$, then the vertices of $N^k[v]$ at even distance from v form a stable set, and so do those at odd distance. In particular $\chi(N^k[v]) \leq 2$.

Proof. Adjacent vertices have distances from v differing by at most one. An edge xy with x, y in the ball and $\text{dist}(v, x) = \text{dist}(v, y) = j \leq k$ together with shortest paths from x and y to v is a closed walk of length $2j + 1 \leq 2k + 1$, so $\text{og}(G) \leq 2k + 1$. $\square$

2.2. t-perfect graphs and odd wheels. A *t-contraction* contracts all edges incident with a vertex whose neighborhood is stable. A *t-minor* of G is a graph obtained from G by vertex deletions and *t*-contractions. An *odd wheel* is a graph consisting of an odd cycle and one further vertex adjacent to all vertices of the cycle.

The closure of *t*-perfect graphs under taking *t*-minors is due to Gerards and Shepherd [5]. The fact that no odd wheel is *t*-perfect is folklore; see [3, Lemma 2.2].

Lemma 2.2. *Every t-minor of a t-perfect graph is t-perfect, and no odd wheel is t-perfect. Consequently, a t-perfect graph has no odd wheel as a t-minor.*

Lemma 2.3 (Marcus [7, Proposition 3.6]; see also [3, Lemma 3.4]). *Let G be a t-perfect graph that has an odd cycle, and let $2\ell + 1 = \text{og}(G)$. Then G has a stable set S that meets every odd cycle of length $2\ell + 1$. Hence $G - S$ is t-perfect and has odd girth at least $2\ell + 3$.*

Corollary 2.4. *Let G be a t-perfect graph. There is an induced subgraph H of G with $\text{og}(H) \geq 15$ and $\chi(H) \geq \chi(G) - 6$.*

Proof. Apply Lemma 2.3 repeatedly: the odd girth of the remaining graph increases by at least two each time, or the graph becomes bipartite. Starting from odd girth at least 3, we need at most six deletions to reach odd girth at least 15. Deleting a stable set lowers the chromatic number by at most one. $\square$

2.3. Quasi-arithmetic ropes. Arithmetic ropes were introduced in [3]; Novaković and Weltge [8] introduced quasi-arithmetic ropes to improve the coloring bound. All indices are taken modulo 5.

Definition 2.5. Let G be a graph and $X \subseteq V(G)$. A *quasi-5-arithmetic rope in X* consists of vertices $q_1, \dots, q_5$ and, for each i , two $q_i q_{i+1}$ -paths $Q_{i,1}, Q_{i,2}$ (which are allowed to coincide) such that

- (a) all vertices of all paths belong to X ;
- (b) $Q_{i,1}$ has odd length for every i ;

- (c) for every i , the path $Q_{i,2}$ or the path $Q_{i+1,2}$ has even length;
- (d) for every $(h_1, \dots, h_5) \in \{1, 2\}^5$ the union $Q_{1,h_1} \cup \dots \cup Q_{5,h_5}$ is an induced cycle;
- (e) $\text{dist}_G(q_i, q_j) \geq 6$ for all $i \neq j$.

The distances in (e) are measured in G , not in $G[X]$.

Remark 2.6. Condition (c) is exactly what makes an odd cycle available through any three anchors that is cut into three odd arcs. Three anchors cut the five segments into three arcs of 1, 1, 3 or of 1, 2, 2 consecutive segments. An arc of one or three segments is odd if all its segments use their paths $Q_{i,1}$, which are odd. An arc of two consecutive segments $i, i+1$ has odd length when one of $Q_{i,2}, Q_{i+1,2}$, which exists by (c), is even and the other segment uses its odd path. The three arcs are then odd, so the cycle is odd; by (d) it is induced; see Figure 1.

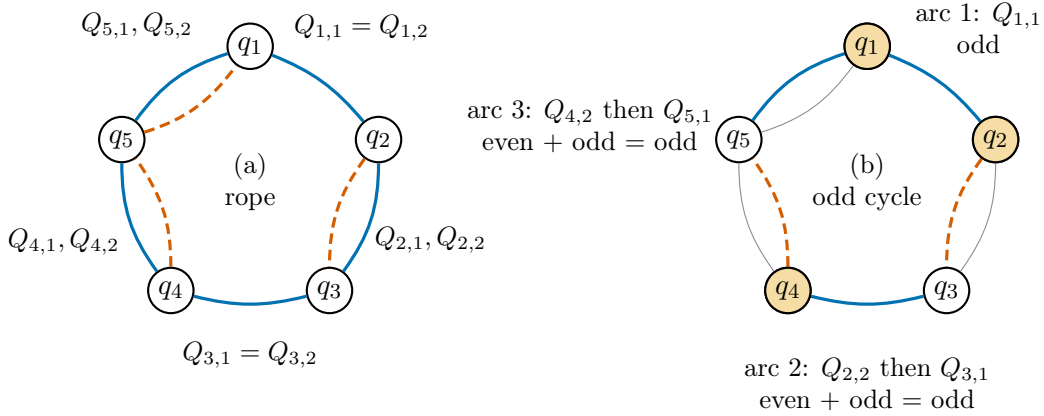


FIGURE 1. (a) A quasi-5-arithmetic rope (Definition 2.5), schematically. Solid blue paths are the odd paths $Q_{i,1}$, dashed vermilion paths the even paths $Q_{i,2}$; for $i = 1, 3$ the second path coincides with the first. Consecutive segments never both lack an even alternative, which is condition (c). (b) Choosing $Q_{1,1}, Q_{2,2}, Q_{3,1}, Q_{4,2}, Q_{5,1}$ gives an induced cycle through q_1, q_2, q_4 (shaded) cut into three odd arcs of one, two and two segments, hence an odd cycle (Remark 2.6). Unused alternatives are gray.

Proposition 2.7 ([8, Proposition 2.2]). *Let G be a graph of odd girth at least 7, let $u \in V(G)$ and $t \geq 4$. If G contains a quasi-5-arithmetic rope in $N^t(u)$, then G has an odd wheel as a t -minor.*

The formulation in Proposition 2.7 follows from the proof in [3], as explained in [8]. That proof is written for arithmetic ropes in which every segment has both an odd and an even path and the anchors are at distance at least 5; as noted there, that proof uses only the existence, for every three anchors, of an odd induced cycle cut by them into three odd arcs, and this is Remark 2.6.

3. REDUCTION AND BOOKKEEPING

This section reduces Theorem 1.1 to the existence of a rope in a graph of odd girth at least 15 and large chromatic number, and records how the constants fit together. The two statements below are proved in Section 8, using the results of Sections 4–7.

Proposition 3.1. *If $\text{og}(G) \geq 15$ and $\chi(G) \geq 45$, then G has an odd wheel as a t -minor.*

Proposition 3.2. *If $\text{og}(G) \geq 15$ and $X \subseteq V(G)$ satisfies $\chi(X) \geq 23$, then G contains a quasi-5-arithmetic rope in X .*

Proof of Theorem 1.1 from Proposition 3.1. Let G be t-perfect and suppose $\chi(G) \geq 51$. By Corollary 2.4 there is an induced subgraph H with $\text{og}(H) \geq 15$ and $\chi(H) \geq 51 - 6 = 45$. By Proposition 3.1, H has an odd wheel t-minor, which is also a t-minor of G , contradicting Lemma 2.2. Hence $\chi(G) \leq 50$. $\square$

Halving. We use the following standard device twice: a connected graph of chromatic number m contains a breadth-first level of chromatic number at least $\lceil m/2 \rceil$.

Lemma 3.3. *Let G' be a connected graph, $v \in V(G')$, and $L_i = N^i(v)$ computed in G' . Then L_i is anticomplete to L_j whenever $|i - j| \geq 2$, and $\chi(G') \leq 2 \max_i \chi(L_i)$. If $\text{og}(G') \geq 15$, then L_i is stable for $i \leq 6$.*

Proof. Adjacent vertices lie in levels whose indices differ by at most one, which gives the first claim. Consequently the union of the even levels induces the disjoint union of the graphs $G'[L_{2i}]$, which are pairwise anticomplete, and similarly for the odd levels; so $\chi(G') \leq \max_i \chi(L_{2i}) + \max_i \chi(L_{2i+1})$. An edge inside L_i and shortest paths to v form a closed walk of odd length at most $2i + 1$, so $\text{og}(G') \leq 2i + 1$; for $\text{og}(G') \geq 15$ this excludes $i \leq 6$. $\square$

Proof of Proposition 3.1 from Proposition 3.2. Let G have $\text{og}(G) \geq 15$ and $\chi(G) \geq 45$, and let G' be a connected component of G of maximum chromatic number, with a vertex v and levels $L_i = N^i(v)$. By Lemma 3.3 some level L_t has $\chi(L_t) \geq \lceil 45/2 \rceil = 23$, and $t \geq 7$ because the levels up to 6 are stable. Since G' is a component of G , the level L_t is the sphere $N^t(v)$ of G as well. Proposition 3.2 with $X = L_t$ gives a quasi-5-arithmetic rope in $N^t(v)$, and Proposition 2.7 gives an odd wheel t-minor. $\square$

The accounting. Proposition 3.2 is proved in Section 8 by a second halving, which gives a component C_* of a breadth-first level of a component of $G[X]$ with $\chi(C_*) \geq 12$, and then by building the rope from C_* , using the preceding level to cover C_* and the levels below it to return to the first anchor. At every stage of the construction we hold a triple (B, C, q) consisting of a covering set B , a retained set $C \subseteq C_*$ covered by B , and the most recent anchor q . The quantity that has to be preserved is the *potential*

$$(3.1) \quad \Phi(C, q) = \chi(C \setminus N^5[q]),$$

the chromatic number of the part of C at distance at least six from the anchor. This is the part from which the next segment can be grown without touching q . By Observation 2.1, discarding the ball $N^5[q]$ costs one color (Lemma 4.1), and this cost is charged once, at the start; every extension step then delivers its output with the next ball already discarded. The five segments of the rope are built as follows (Section 8): an odd path, a pair of paths of opposite parity, an odd path, a pair of paths of opposite parity, and finally a closing pair of paths of opposite parity, which joins q_5 to q_1 through the lower levels. This pattern satisfies condition (c) of Definition 2.5. The extension of Proposition 6.2 lowers the potential by at most 1, that of Proposition 6.3 by at most 3, and the closing step of Proposition 7.4 needs a potential of at least 3. Table 1 traces the potential through these steps.

Remark 3.4. In general, if a component of a level of $G[X]$ with chromatic number at least T suffices to build a rope, the argument above gives the bound $4T + 2$ for t-perfect graphs. For the argument of [8] one has $T = 46$ and $4T + 2 = 186$; here $T = 12$.

4. TWO COLORING LEMMAS

Building a rope forces us, again and again, to cut a set of large chromatic number down to a smaller set that stays clear of the part already built. Two kinds of cuts occur: removing a ball around the current anchor, and removing a stable block of the grading of Section 5 together with its immediate surroundings. Both are paid for in colors, and in the argument of [8] each costs two or more. The lemmas of this section show that both costs can be made small by *reusing colors*: a vertex set that lies only at distance at most $k - 1$ from the ball's center never sees the far part, so it may share colors with it.

Lemma 4.1 (Ball). *Let $k \geq 0$, $C \subseteq V(G)$ and $q \in V(G)$, and suppose that the vertices of $N^k[q]$ at even distance from q , and those at odd distance, form two stable sets. Put $F = C \setminus N^k[q]$. Then $\chi(C) \leq \max\{2, \chi(F) + 1\}$. In particular, if $\chi(C) \geq 3$ then $\chi(F) \geq \chi(C) - 1$.*

Proof. Let I be the set of vertices of $C \cap N^k[q]$ whose distance from q has parity opposite to that of k , and J the other vertices of $C \cap N^k[q]$. Both are stable by hypothesis. Vertices of I have distance at most $k - 1$ from q , so I is anticomplete to F , whose vertices have distance at least $k + 1$. If $F = \emptyset$ then $C = I \cup J$ is 2-colorable. Otherwise take an optimal coloring of F , give all vertices of I one of its colors, and give J one new color. $\square$

By Observation 2.1, the hypothesis of Lemma 4.1 holds whenever $\text{og}(G) \geq 2k + 3$. The following lemma combines a ball and a stable set in a single palette.

Lemma 4.2 (Joint block and ball). *Let $k \geq 5$ and $c \geq 2$ be integers and suppose $\text{og}(G) \geq 2k + 3$. Let H be a connected induced subgraph of G and let $W \subseteq V(H)$ be a non-empty stable set such that any two vertices of W are joined in G by an even walk of length at most four. If $\chi(H) \geq c + 1$, then there exist a component K of $H - W$ and a vertex $\hat{q} \in W$ with a neighbor in K such that*

$$\chi(K \setminus N^k[\hat{q}]) \geq c.$$

Proof. Every component K of $H - W$ has a neighbor in W , since H is connected and W is non-empty (and $H - W \neq \emptyset$ because $\chi(H) \geq 3 > \chi(W)$). For each component choose a vertex $q_K \in W$ with a neighbor in K , and suppose for a contradiction that $K \setminus N^k[q_K]$ admits a proper coloring with colors $1, \dots, c - 1$, for every component K . We build a proper c -coloring of H , which contradicts $\chi(H) \geq c + 1$.

Step 1: the neighbors of W in K are at odd distance at most five from q_K . Let $z \in K$ have a neighbor $w' \in W$. An even walk of length at most four from q_K to w' , followed by the edge $w'z$, is an odd walk of length at most five from q_K to z . Thus $d := \text{dist}(q_K, z) \leq 5$. If d were even, then $d \leq 4$, and a shortest $q_K z$ -path together with the odd walk would form an odd closed walk of length at most $4 + 5 = 9$, which is impossible as $\text{og}(G) \geq 2k + 3 \geq 13$. So d is odd and, as $k \geq 5$, $z \in N^k[q_K]$.

Step 2: coloring. Fix a component K . Let A be the set of vertices of $K \cap N^k[q_K]$ whose distance from q_K has the same parity as k , and let A' be the set of the remaining vertices of $K \cap N^k[q_K]$, which have distance at most $k - 1$. By Observation 2.1, A and A' are stable. Color A with color c and A' with color 1. Color the far part $K \setminus N^k[q_K]$ with colors $1, \dots, c - 1$ as supposed. This is proper on K : A uses a color absent from the far part, and A' is anticomplete to the far part, because its vertices are at distance at most $k - 1$ from q_K , so it may share color 1 with it; and $c \neq 1$ separates A from A' .

Step 3: coloring W . By Step 1 the neighbors of W in K are at odd distance from q_K . If k is odd, they lie in A and have color c , so color all of W with 1. If k is even, they lie in A' and have color 1, so color all of W with c . In either case this choice is the same for all components, since the neighbors of W in every component receive the same color. Distinct components of $H - W$ are anticomplete and W is stable, so the coloring of H is proper. $\square$

Remark 4.3. Lemma 4.2 is where the saving in the extension steps comes from. Passing from a set H with $\chi(H) \geq c + 1$ to a far part with $\chi \geq c$ removes both the stable block W and a ball of radius k , which are of unrelated shape, at the price of one color. The even-walk hypothesis is what makes the neighbors of W in K all lie in a single parity class of a single ball; see Figure 2.

5. THE CANONICAL GRADING

A rope is built one segment at a time, each segment inside the part of the graph that earlier segments have left untouched. To extend from an anchor q , we need a vertex $\hat{q}$ that can be reached from q by an induced path of known parity, together with a connected set of large chromatic number beyond it that has no edges to this path. The stable-grading method for

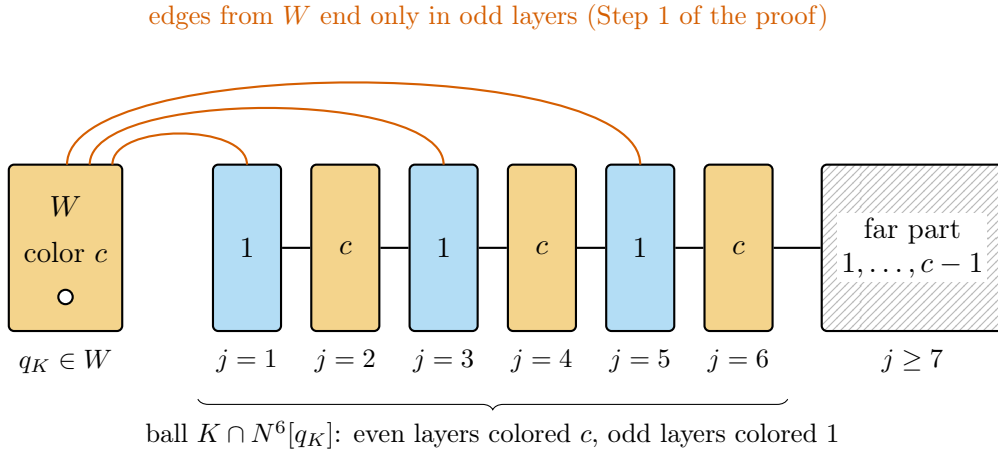


FIGURE 2. The coloring in the proof of Lemma 4.2 for $k = 6$ (even), schematically. The layers of K at distance $j = 1, \dots, 6$ from $q_K \in W$ are stable (Observation 2.1); even layers get color c and odd layers color 1. Odd layers are anticomplete to the far part $K \setminus N^6[q_K]$, which keeps colors $1, \dots, c-1$, so color 1 may be reused there; layer 6 meets the far part, but color c does not occur in it. Every neighbor of W in K is at odd distance at most five, hence has color 1, so W receives color c . For odd k the roles of 1 and c are interchanged as in the proof.

rope extensions appears in [3, Section 4]. Novaković and Weltge [8, Section 4] use the same method with a levelwise grading for quasi-arithmetic ropes. Here we give every vertex far from q a canonical route and order the vertices in a *stable grading* so that later vertices stay away from the route of an earlier vertex. Taking $\hat{q}$ in the earliest block and the retained set among later vertices then gives the separation that an induced concatenation requires. This section recalls the construction in the form needed here, with the parity of the canonical route made part of the definition (see Appendix A for a comparison with the proof of Novaković and Weltge).

Setting 5.1 (Extension setting). Let G be a graph with $\text{og}(G) \geq 15$. An *extension setting* is a triple (B, C, q) with $B, C \subseteq V(G)$ and $q \in V(G) \setminus C$ such that B covers C and $G[C \cup \{q\}]$ is connected.

5.1. Levels, plain vertices and keyed vertices. Fix an extension setting (B, C, q) and put

$$F = C \setminus N^5[q],$$

so that $\Phi(C, q) = \chi(F)$ in the notation of (3.1); we assume $F \neq \emptyset$.

For $i \geq 0$ let M_i be the set of vertices at distance exactly i from q in the *internal* graph $G[C \cup \{q\}]$, and call $\text{depth}(v) = i$ for $v \in M_i$ the *depth* of v . Write d for the largest depth, and $\mathcal{M}_{\leq t} = M_0 \cup \dots \cup M_t$. Adjacent vertices have depths differing by at most one. Internal distances are at least G -distances, so every $z \in F$ has $\text{depth}(z) \geq \text{dist}(q, z) \geq 6$; in particular $d \geq 6$. List the vertices of $\mathcal{M}_{\leq d-1}$ as $m_1, m_2, \dots$ by non-decreasing depth, breaking ties in any fixed way. Thus if $\text{depth}(m_i) < \text{depth}(m_j)$ then $i < j$.

For $6 \leq t \leq d$ let

$$B_0^t = \{b \in B : b \text{ has no neighbor in } \mathcal{M}_{\leq t-3}\}, \quad B_+^t = B \setminus B_0^t.$$

Clearly $B_0^6 \supseteq B_0^7 \supseteq \dots \supseteq B_0^d$.

Definition 5.2 (Plain and keyed vertices). Let $z \in F$ and $t = \text{depth}(z)$. Call z *plain* if every neighbor of z in B belongs to B_0^t , and *keyed* otherwise.

- If z is plain, then z has a neighbor in M_{t-1} and none in $\mathcal{M}_{\leq t-2}$. Let $\pi(z)$ be the least index i such that m_i is adjacent to z ; the *parent* $m_{\pi(z)}$ lies in M_{t-1} .
- If z is keyed, then some neighbor of z in B has a neighbor in $\mathcal{M}_{\leq t-3}$. Let $\alpha(z)$ be the least index i such that m_i is adjacent to some vertex of $B \cap N(z)$, and call $\text{key}(z) = m_{\alpha(z)}$ the

key of z and $r(z) = \text{depth}(\text{key}(z)) \leq t - 3$ its *key depth*. Fix a *witness* $b(z) \in B \cap N(z)$ adjacent to $\text{key}(z)$. By minimality of $\alpha(z)$:

(5.1) no vertex of $B \cap N(z)$ is adjacent to any m_i with $i < \alpha(z)$.

Every vertex of F is plain or keyed; that B covers C is used in Lemma 6.4(a). The *route* of a vertex $z \in F$ is the path $R(z)$ from q to z defined as follows. If z is plain, $R(z)$ is a shortest path in $G[C \cup \{q\}]$ from q to $m_{\pi(z)}$, followed by z ; its length is $\text{depth}(z)$. If z is keyed, $R(z)$ is a shortest path in $G[C \cup \{q\}]$ from q to $\text{key}(z)$, followed by $b(z)$ and z ; its length is $r(z) + 2$. The *route parity* $p(z) \in \{0, 1\}$ is the parity of the length of $R(z)$, that is, $p(z) \equiv \text{depth}(z)$ if z is plain and $p(z) \equiv r(z)$ if z is keyed. For a keyed z , the shortest path from q to $\text{key}(z)$ is its *prefix*. Figure 3 shows one route of each kind.

Lemma 5.3 (Routes). *For every $z \in F$ the route $R(z)$ is an induced path of G .*

- (a) *If z is plain with $t = \text{depth}(z)$, all vertices of $R(z)$ other than z lie in $\mathcal{M}_{\leq t-1}$ and have index at most $\pi(z)$, and z has no neighbor in $\mathcal{M}_{\leq t-2}$.*
- (b) *If z is keyed, the prefix consists of vertices m_i with $i \leq \alpha(z)$ and depth at most $r(z)$; the witness $b(z)$ has the key as its only neighbor on the prefix, z has no neighbor on the prefix, $\text{dist}(q, b(z)) \geq 5$, and $r(z) \geq 4$.*

Proof. A shortest path of $G[C \cup \{q\}]$ is induced in G and has exactly one vertex at each depth $0, 1, \dots$ up to its end. (a) The vertex z has depth t and is adjacent to no vertex of depth at most $t - 2$, so its only possible neighbor on the path ending at its parent is that parent, which is the vertex of depth $t - 1$. The vertices of depth smaller than $t - 1$ have smaller index than every vertex of depth $t - 1$. (b) Since $\text{dist}(q, z) \geq 6$ and $b(z) \sim z$, we have $\text{dist}(q, b(z)) \geq 5$, so $b(z) \neq q$; also $b(z) \notin C$ since $B \cap C = \emptyset$; thus $b(z) \notin \{q\} \cup C$, and it is not on the prefix. The prefix vertices have index at most $\alpha(z)$ and depth at most $r(z)$; by (5.1), $b(z)$ is adjacent to none of those of index below $\alpha(z)$, hence only to the key. Finally $\text{depth}(z) \geq r(z) + 3$ exceeds the depth of every prefix vertex by at least three. Since $b(z)$ is adjacent to the key, $\text{dist}(q, b(z)) \leq r(z) + 1$, so $r(z) \geq 4$. $\square$

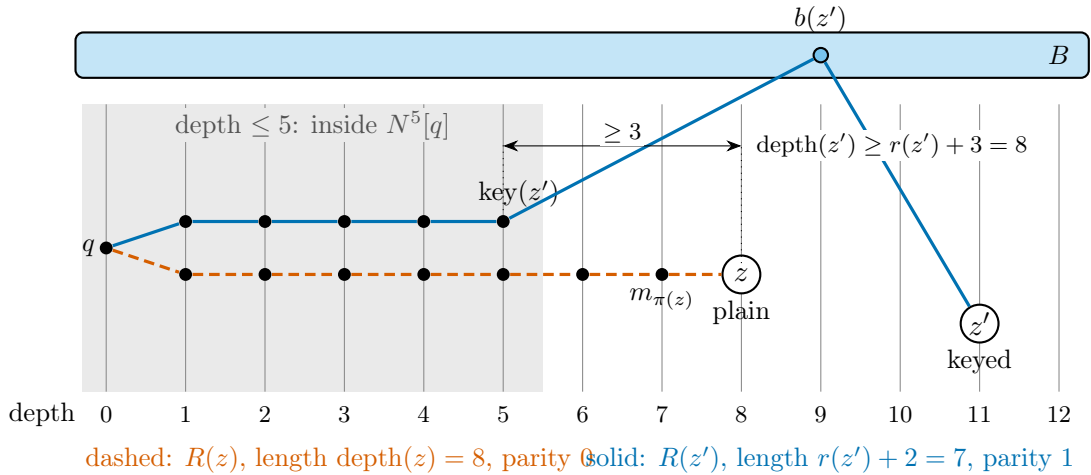


FIGURE 3. Routes from q in an extension setting (Section 5), with depth horizontal; proportions are schematic, the stated lengths are exact. A plain vertex z of depth $t = 8$ is reached along a shortest path of $G[C \cup \{q\}]$ to its parent $m_{\pi(z)}$; its route has length t and parity $t \bmod 2$ (dashed). A keyed vertex z' is reached along a shortest path to its key, at depth $r(z') = 5$, and then across its witness $b(z') \in B$; its route has length $r(z') + 2$ and parity $r(z') \bmod 2$ (solid). Since $\text{depth}(z') \geq r(z') + 3$, the vertex z' has no neighbor on the prefix (Lemma 5.3).

5.2. The grading. The *stage* of $z \in F$ is $\sigma(z) = \text{depth}(z)$ if z is plain and $\sigma(z) = r(z) + 3 \geq 7$ if z is keyed (recall $r(z) \geq 4$ by Lemma 5.3(b)). In both cases $\text{depth}(z) \geq \sigma(z)$, and for keyed vertices σ is a non-decreasing function of the key index $\alpha(z)$. A *block* is a class of one of the following equivalence relations: two keyed vertices are equivalent if they have the same key index α ; two plain vertices are equivalent if they have the same depth and the same parent index π . Order the blocks by stage; among the blocks of one stage, the keyed blocks come first, by increasing α , and then the plain blocks, by increasing π . We say z is *earlier* than z' (and z' is *later* than z) if the block of z strictly precedes the block of z' . Vertices of the same block are neither earlier nor later than one another.

Lemma 5.4 (Blocks). *Every block is stable, and any two vertices of a block are joined in G by an even walk of length at most four.*

Proof. Let z, z' be distinct plain vertices of one block, with common parent m . Then $z - m - z'$ is an even walk of length two, and $z \not\sim z'$ because zmz' would be a triangle. Let z, z' be distinct keyed vertices of one block, with common key m and witnesses $b = b(z), b' = b(z')$, both adjacent to m . Then $z - b - m - b' - z'$ is an even walk of length four. If $z \sim z'$, then either $b = b'$, giving a triangle $zz'b$, or $zbmb'z'$ is a closed walk of length five; both contradict $\text{og}(G) \geq 15$. $\square$

The point of the order is the following.

Lemma 5.5 (Later vertices). *Let $z \in F$ and let Z be the set of vertices of F that are later than z .*

- (i) *Suppose z is plain with $t = \text{depth}(z)$ and parent index π . Every vertex of Z has depth at least t and is adjacent to none of $m_1, \dots, m_\pi$, and every neighbor in B of a vertex of Z lies in B_0^t .*
- (ii) *Suppose z is keyed with key index α and key depth r . Every vertex of Z has depth at least $r + 3$ and is adjacent to none of $m_1, \dots, m_\alpha$, and no neighbor in B of a vertex of Z is adjacent to any of $m_1, \dots, m_\alpha$.*

Proof. (i) Let $z' \in Z$. *Case 1:* z' is plain of depth t with $\pi(z') > \pi$. By definition of $\pi(z')$, z' is adjacent to none of $m_1, \dots, m_{\pi(z')-1}$, which include $m_1, \dots, m_\pi$, and its B -neighbors lie in B_0^t because z' is plain. *Case 2:* $\sigma(z') > t$. Then $\text{depth}(z') \geq \sigma(z') \geq t + 1$, whereas $m_1, \dots, m_\pi$ have depth at most $t - 1$, so z' is adjacent to none of them. If z' is plain of depth $t' > t$, its B -neighbors lie in $B_0^{t'} \subseteq B_0^t$. If z' is keyed, then $\sigma(z') = r(z') + 3 > t$, so $r(z') \geq t - 2$. By (5.1) applied to z' , every neighbor in B of z' has all its neighbors among the m_i at depth at least $r(z') \geq t - 2$, hence has no neighbor in $\mathcal{M}_{\leq t-3}$, that is, lies in B_0^t .

(ii) Put $\sigma = r + 3$ and let $z' \in Z$. By the order of blocks, z' is keyed with $\sigma(z') = \sigma$ and $\alpha(z') > \alpha$, or keyed with $\sigma(z') > \sigma$, or plain with $\sigma(z') \geq \sigma$. In all cases $\text{depth}(z') \geq \sigma(z') \geq \sigma$. Since $m_1, \dots, m_\alpha$ have depth at most $r \leq \sigma - 3$, z' is adjacent to none of them. If z' is keyed with $\sigma(z') = \sigma$, then $\alpha(z') > \alpha$; if $\sigma(z') > \sigma$ then $r(z') > r$, so again $\alpha(z') > \alpha$ because the vertices m_i are listed by depth. By (5.1) for z' , no B -neighbor of z' is adjacent to $m_1, \dots, m_\alpha$. If z' is plain of depth $t' \geq \sigma \geq r + 3$, its B -neighbors lie in $B_0^{t'}$ and so have no neighbor in $\mathcal{M}_{\leq t'-3} \supseteq \mathcal{M}_{\leq r} \ni m_1, \dots, m_\alpha$. $\square$

Remark 5.6. Lemmas 5.3, 5.4 and 5.5 are the properties of the grading of [8, Section 4] that the extension steps use. Stated this way, a vertex later than the end z of a route has no edge to the route except possibly to z , and none of its covering vertices touches the part of the route that lies at distance at least three from z (the part nearer to q); this is what lets a connected set of later vertices be retained after the route has been used.

6. EXTENSION STEPS

An extension step takes the current triple (B, C, q) and produces one or two induced paths from the anchor q to a new anchor q' , together with a smaller retained set $C' \subseteq C$ and a smaller covering set $B' \subseteq B$, in such a way that the paths can be concatenated with those already built

and the new triple (B', C', q') can be extended again. The paths must not touch C' except at q' , and B' must not touch them except near q' ; these are the separation requirements that make a union of paths induced. The cost of a step is the loss of potential $\Phi(C, q) = \chi(C \setminus N^5[q])$.

Definition 6.1 (Extension). Let (B, C, q) be an extension setting. An *extension* of (B, C, q) is a tuple $(B', C', q'; \mathcal{Q})$ with $B' \subseteq B$, $C' \subseteq C$, $q' \in C \setminus C'$, and $\mathcal{Q}$ a set of one or two induced q - q' paths in $G[(C \setminus C') \cup (B \setminus B') \cup \{q\}]$, such that, for every $Q \in \mathcal{Q}$:

- (E1) $G[C' \cup \{q'\}]$ is connected and B' covers C' ;
- (E2) B' is anticomplete to $V(Q) \setminus N^3[q']$;
- (E3) $V(Q) \subseteq C \cup \{q\} \cup ((B \cap N^2[q']) \setminus N^4[q])$;
- (E4) C' is anticomplete to $V(Q) \setminus \{q'\}$;
- (E5) $\text{dist}(q, C' \cup \{q'\}) \geq 6$;
- (E6) $q \notin B'$.

It is a *prescribed-parity extension* if $\mathcal{Q}$ consists of a single path, and an *opposite-parity extension* if $\mathcal{Q} = \{Q_1, Q_2\}$ with Q_1 odd and Q_2 even.

This extension framework originates in [3, Lemma 4.4]; the properties here follow its quasi-arithmetic-rope version in [8, Lemma 3.2], except that the requirement on the chromatic number of C' is replaced by a requirement on the potential, and that (E6) is added. Figure 4 sketches conditions (E1)–(E5). In words, (E1) keeps C' connected, attached to its new anchor and covered; (E2) and (E4) say that the paths are seen by the unused covering vertices only near q' and by C' only at q' ; (E3) says that a path uses covering vertices only near the new anchor and far from the old one; (E5) separates anchors; and (E6) records that the old anchor, which may itself be a covering vertex, is not kept for later use.

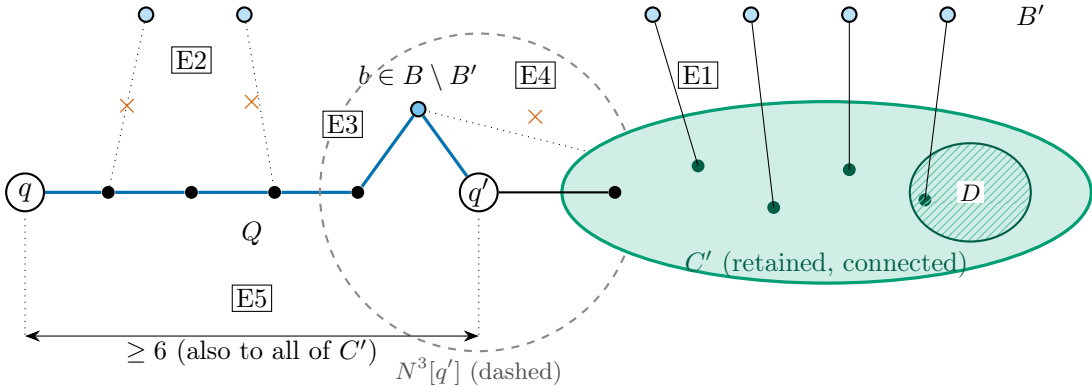


FIGURE 4. The separation pattern of an extension (Definition 6.1), schematically; one path Q is shown, ending at a keyed vertex q' through a covering vertex b . **E1**: $C' \cup \{q'\}$ is connected and B' covers C' . **E2**: B' has no edge to Q outside $N^3[q']$ (crossed dotted edges). **E3**: the only covering vertices on Q lie within distance two of q' and beyond distance four of q . **E4**: C' has no edge to Q except at q' (crossed dotted edge to b). **E5**: q is at distance at least six from $C' \cup \{q'\}$. (E6), $q \notin B'$, is not drawn. The set $D \subseteq C'$ is the part far from q' that keeps the potential: $D \cap N^5[q'] = \emptyset$.

Proposition 6.2 (Prescribed parity, cost 1). *Let $c \geq 2$ and $\delta \in \{0, 1\}$, and let (B, C, q) be an extension setting with $\Phi(C, q) \geq c + 1$. Then (B, C, q) has a prescribed-parity extension $(B', C', q'; \{Q\})$ in which Q has parity δ and $\Phi(C', q') \geq c$.*

Proposition 6.3 (Opposite parity, cost 3). *Let $c \geq 2$ and let (B, C, q) be an extension setting with $\Phi(C, q) \geq c + 3$. Then (B, C, q) has an opposite-parity extension $(B', C', q'; \{Q_1, Q_2\})$ with $\Phi(C', q') \geq c$.*

Both propositions use the notation of Section 5 for a fixed extension setting (B, C, q) and $F = C \setminus N^5[q]$. Their earliest-block selections adapt the stable-grading argument of [3, Lemmas 4.2–4.3], also used in [8, Section 4]; the activity criterion in Proposition 6.3 is based on route parity. For $z \in F$ put

$$B_z = \begin{cases} B_0^t & \text{if } z \text{ is plain of depth } t, \\ B \setminus N(\{m_1, \dots, m_{\alpha(z)}\}) & \text{if } z \text{ is keyed} \end{cases},$$

In both cases B_z is the set of vertices of B with no neighbor in a set $P(z)$ of initial vertices of the list $m_1, m_2, \dots$, namely $P(z) = \mathcal{M}_{\leq t-3}$ for plain z and $P(z) = \{m_1, \dots, m_{\alpha(z)}\}$ for keyed z .

Lemma 6.4 (Route to a vertex). *Let $z \in F$ and let Z be the set of vertices of F later than z .*

- (a) *Every neighbor in B of a vertex of Z lies in B_z ; in particular B_z covers Z .*
- (b) *Z is anticomplete to $V(R(z)) \setminus \{z\}$.*
- (c) *B_z is anticomplete to $V(R(z)) \setminus N^2[z]$.*
- (d) *$V(R(z)) \cap Z = \emptyset$, and $V(R(z)) \setminus (C \cup \{q\})$ is empty if z is plain and equals $\{b(z)\}$ if z is keyed, where $b(z) \in (N(z) \cap B) \setminus (B_z \cup N^4[q])$.*
- (e) *If $z' \in F$ is later than z , then $B_{z'} \subseteq B_z$.*
- (f) *$q \notin B_z$.*

Proof. (a) is the last assertion of Lemma 5.5 in each case, and $B_z \cap Z = \emptyset$ since $Z \subseteq C$; each vertex of Z has a neighbor in B because B covers C .

(b) If z is plain with parent index π , the vertices of $R(z)$ other than z have index at most π (Lemma 5.3(a)), and Z is anticomplete to $m_1, \dots, m_\pi$ by Lemma 5.5(i). If z is keyed, the prefix lies in $\{m_1, \dots, m_\alpha\}$, to which Z is anticomplete by Lemma 5.5(ii); and a vertex of Z adjacent to $b(z)$ would have a B -neighbor adjacent to the key m_α , which Lemma 5.5(ii) excludes.

(c) If z is plain of depth t , the vertex of $R(z)$ at depth $j < t$ is at distance $t - j$ from z along the route, so a vertex outside $N^2[z]$ has depth at most $t - 3$ and B_0^t is anticomplete to it by definition. If z is keyed, $V(R(z)) \setminus N^2[z]$ is contained in the prefix, which lies in $P(z)$.

(d) If z is plain of depth t , the vertices of $R(z)$ other than z have depth at most $t - 1$, while the vertices of Z have depth at least t by Lemma 5.5(i). If z is keyed, the prefix has depth at most $r(z)$, the vertices of Z have depth at least $r(z) + 3$ by Lemma 5.5(ii), and $b(z) \notin C$. Also $z \notin Z$. For keyed z the vertex $b(z)$ is adjacent to $\text{key}(z) \in P(z)$, hence $b(z) \notin B_z$, and Lemma 5.3(b) gives $b(z) \in N(z)$ and $b(z) \notin N^4[q]$.

(e) It suffices to show $P(z) \subseteq P(z')$. Suppose z is plain of depth t . If z' is plain, its depth is at least t and $P(z') = \mathcal{M}_{\leq t'-3} \supseteq \mathcal{M}_{\leq t-3}$. If z' is keyed, then $\sigma(z') > t$ (keyed blocks of stage t precede plain blocks of stage t) gives $r(z') \geq t - 2$, so all vertices of depth at most $t - 3$ come before $\text{key}(z')$ in the list. Suppose z is keyed. If z' is keyed, then $\alpha(z') > \alpha(z)$. If z' is plain, its depth is at least $\sigma(z) \geq r(z) + 3$ and $P(z') \supseteq \mathcal{M}_{\leq r(z)} \ni m_1, \dots, m_{\alpha(z)}$.

(f) If $q \in B$, then q has a neighbor in M_1 , because $C \neq \emptyset$ and $G[C \cup \{q\}]$ is connected, and $M_1 \subseteq P(z)$: for plain z since the depth of z is at least 6, and for keyed z since the key has depth at least 4, as $\text{dist}(q, b(z)) \geq 5$ and $b(z)$ is adjacent to the key. So $q \notin B_z$. $\square$

6.1. Prescribed parity.

Proof of Proposition 6.2. Figure 5 illustrates the construction. We have $\chi(F) \geq c + 1$. Choose a component H of $G[F]$ with $\chi(H) \geq c + 1$, let β be the earliest block that meets $V(H)$, and put $W = V(H) \cap \beta$. By Lemma 5.4, W is a non-empty stable set whose vertices are pairwise joined by even walks of length at most four, and every vertex of $H - W$ is later than every vertex of W . Lemma 4.2 with $k = 6$ (note $\text{og}(G) \geq 15 = 2k + 3$) provides a vertex $\hat{q} \in W$ and a component X of $H - W$ with a neighbor of $\hat{q}$, such that $\chi(X \setminus N^6[\hat{q}]) \geq c$. Let D be the vertex set of a component of $G[X \setminus N^6[\hat{q}]]$ of maximum chromatic number, so $G[D]$ is connected and $\chi(D) \geq c$. Let Z be the set of vertices of F later than $\hat{q}$; then $X \subseteq Z$.

Step 1: a path to $\hat{q}$. Let $Q_0 = R(\hat{q})$ and $B' = B_{\hat{q}}$. By Lemma 6.4, B' covers X ; X is anticomplete to $V(Q_0) \setminus \{\hat{q}\}$; B' is anticomplete to $V(Q_0) \setminus N^2[\hat{q}]$; $V(Q_0) \cap X = \emptyset$; and the only vertex of Q_0 outside $C \cup \{q\}$ is, if $\hat{q}$ is keyed, a vertex $b(\hat{q}) \in N(\hat{q}) \setminus (B' \cup N^4[q])$.

Step 2: the retained set. Let T be a shortest path from $\hat{q}$ to D in $G[X \cup \{\hat{q}\}]$. Since $\hat{q} \notin X$ has a neighbor in X , and $D \cap N^6[\hat{q}] = \emptyset$, the path T has length at least seven, and its vertices other than $\hat{q}$ lie in X . Let $X_0 = D \cup (V(T) \setminus \{\hat{q}\}) \subseteq X$. Then $G[X_0]$ is connected, as D is connected and contains the last vertex of T , and the only neighbor of $\hat{q}$ in X_0 is the vertex w following $\hat{q}$ on T , since T is shortest and D is far from $\hat{q}$.

Step 3: fixing the parity. If Q_0 has parity δ , put $q' = \hat{q}$, $C' = X_0$ and $Q = Q_0$. Otherwise put $q' = w$, $C' = X_0 \setminus \{w\}$ and let Q be Q_0 followed by the edge $\hat{q}w$. In the second case Q is an induced path because $w \in X$ is anticomplete to $V(Q_0) \setminus \{\hat{q}\}$, and its parity differs from that of Q_0 , hence equals δ .

Step 4: verification. The path Q lies in $G[(C \setminus C') \cup (B \setminus B') \cup \{q\}]$ by Step 1 and since $C' \subseteq X$ is disjoint from $V(Q_0)$ and from w (removed from C' in the second case). (E1): $G[C' \cup \{q'\}]$ is connected in the first case because $X_0 \cup \{\hat{q}\}$ induces a connected graph containing T , and in the second because $C' \cup \{w\} = X_0$. B' covers $C' \subseteq X$. (E2): in the first case B' is anticomplete to $V(Q) \setminus N^2[\hat{q}] \supseteq V(Q) \setminus N^3[\hat{q}]$; in the second case $N^2[\hat{q}] \subseteq N^3[w]$, so $V(Q) \setminus N^3[w] \subseteq V(Q_0) \setminus N^2[\hat{q}]$. (E3): $V(Q) \subseteq C \cup \{q\} \cup \{b(\hat{q})\}$ and $b(\hat{q}) \in N(\hat{q}) \subseteq N^2[q']$ lies outside $N^4[q]$. (E4): $C' \subseteq X$ is anticomplete to $V(Q_0) \setminus \{\hat{q}\}$; in the second case it is also anticomplete to $\hat{q}$ by Step 2. (E5): $C' \cup \{q'\} \subseteq F$. (E6): $q \notin B' = B_{\hat{q}}$ by Lemma 6.4(f). Finally $D \subseteq C'$ (a vertex of D is not adjacent to $\hat{q}$, in particular $w \notin D$), and $\text{dist}(q', \hat{q}) \leq 1$ shows $D \cap N^5[q'] = \emptyset$. Hence $\Phi(C', q') \geq \chi(D) \geq c$. $\square$

6.2. Opposite parity. The cheapest way found to produce two paths of opposite parity is to use a single edge whose two ends have the same route parity. Call an edge uv of $G[F]$ *allowed* if $p(u) = p(v)$ (see Figure 6). Adjacent vertices lie in distinct blocks, so one of them is earlier than the other.

Lemma 6.5 (Allowed edge). *Let uv be an allowed edge, and let $U \subseteq F$ be a non-empty set with $G[U]$ connected, all of whose vertices are later than both u and v . Suppose that v has a neighbor in U and that u is anticomplete to U . Then (B, C, q) has an opposite-parity extension $(B', C', q'; \{Q_1, Q_2\})$ with $C' = U$ and $q' = v$.*

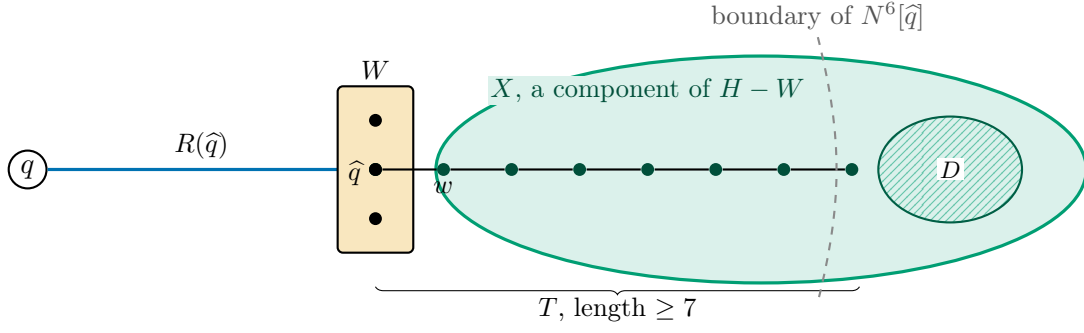
Proof. Let $w_0 \in \{u, v\}$ be the later of u and v , and put $B' = B_{w_0}$. By Lemma 6.4(e), $B' \subseteq B_u \cap B_v$; and U is contained in the set of vertices later than u and in the set later than v . We construct two induced paths Q_1 (odd) and Q_2 (even) from q to v , each of which is a union of a route, possibly truncated, and at most one more edge uv , and then check (E1)–(E6) uniformly. The paths are built in the following cases; t always denotes a depth, and $\text{depth}(x)$ for adjacent vertices differs by at most one.

Case PP: u, v plain. Then $p(u) = p(v)$ and $|\text{depth}(u) - \text{depth}(v)| \leq 1$ give $\text{depth}(u) = \text{depth}(v) = t$. Let $S_a = R(v)$, of length t , and let S_b be $R(u)$ followed by the edge uv , of length $t + 1$. The path S_b is induced: v has no neighbor of depth at most $t - 2$ and is not adjacent to the parent of u , as v, u and that parent would form a triangle.

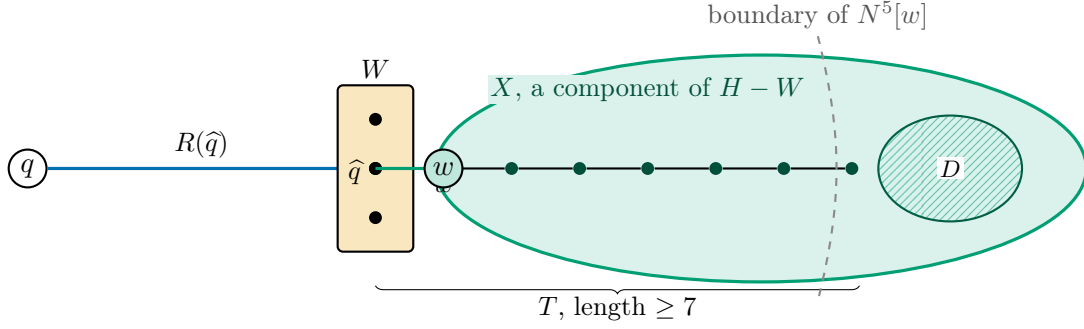
Case KK: u, v keyed. Here $r(u) \equiv r(v) \pmod{2}$. Let $S_a = R(v)$, of length $r(v) + 2$, and let S_b be $R(u)$ followed by uv , of length $r(u) + 3$. Then S_b is induced: v is not adjacent to $b(u)$ (triangle), and $\text{depth}(v) \geq \text{depth}(u) - 1 \geq r(u) + 2$ exceeds the depth of every prefix vertex of $R(u)$ by at least two.

Case PK: one of u, v plain and the other keyed. Let x be the plain vertex, of depth t , and y the keyed one, with key depth $r = r(y)$. As $\text{depth}(y) \leq t + 1$ and $\text{depth}(y) \geq r + 3$ we have $r \leq t - 2$, and $p(x) = p(y)$ gives $r \equiv t \pmod{2}$. Thus $r = t - 2$ or $r \leq t - 4$.

Subcase $r = t - 2$. Then $\text{depth}(y) = t + 1$, so $\sigma(y) = t + 1 > \sigma(x)$ and y is later than x . If $v = y$ let $S_a = R(y)$, of length t , and let S_b be $R(x)$ followed by xy , of length $t + 1$. If $v = x$ let $S_a = R(x)$, of length t , and let S_b be $R(y)$ followed by yx , of length $t + 1$. Both S_b are induced:



(a) $R(\hat{q})$ has parity δ : $q' = \hat{q}$, $Q = R(\hat{q})$, $C' = X_0$ (thick path T without $\hat{q}$, together with D).



(b) wrong parity: extend $R(\hat{q})$ by the edge $\hat{q}w$, so $q' = w$, $C' = X_0 \setminus \{w\}$; still $D \cap N^5[q'] = \emptyset$.

FIGURE 5. Prescribed parity by moving forward (proof of Proposition 6.2), schematically. H is a component of $G[F]$ with $\chi(H) \geq c + 1$ and W its earliest occupied block; Lemma 4.2 gives $\hat{q} \in W$, a component X of $H - W$ adjacent to $\hat{q}$, and a connected set $D \subseteq X \setminus N^6[\hat{q}]$ with $\chi(D) \geq c$. The path T from $\hat{q}$ to D in $G[X \cup \{\hat{q}\}]$ has length at least 7, and $X_0 = D \cup (V(T) \setminus \{\hat{q}\})$. If the parity of $R(\hat{q})$ is wrong, the new anchor is the neighbor w of $\hat{q}$ on T : it is later than $\hat{q}$, stays inside the retained component, and $\text{dist}(w, \hat{q}) = 1$ keeps D outside $N^5[w]$, which preserves the potential $\chi(D) \geq c$.

y has depth $t + 1$ and is not adjacent to the vertices of $R(x)$ of depth at most $t - 1$; x has depth t , so is not adjacent to the prefix of $R(y)$ (depth at most $t - 2$), and is not adjacent to $b(y)$ (triangle).

Subcase $r \leq t - 4$. Then $\sigma(y) \leq t - 1 < \sigma(x)$, so x is later than y . If $v = x$, let $S_a = R(x)$, of length t , and let S_b be $R(y)$ followed by yx , of length $r + 3 \equiv t + 1$; it is induced since x has depth t , the prefix has depth at most $t - 4$, and $xb(y) \notin E$ (triangle). If $v = y$, let $S_a = R(y)$, of length $r + 2 \equiv t$. The path $R(x)$ followed by xy is the candidate for S_b , of length $t + 1$; we examine the neighbors of y on $R(x) \setminus \{x\}$. They have depth at least $\text{depth}(y) - 1 \geq t - 2$, so they are the vertex x^- of depth $t - 2$ or the parent x^* of x at depth $t - 1$. Here $y \neq x^*$, because $x^* = m_{\pi(x)}$ has no neighbor in U by Lemma 5.5(i), whereas y has one; hence $y \not\sim x^*$, as y, x, x^* would form a triangle. If y is adjacent to x^- , then $\text{depth}(y) = t - 1$; let S_b be the part of $R(x)$ from q to x^- followed by the edge x^-y , of length $t - 1$. Otherwise let $S_b = R(x)$ followed by xy . In each alternative S_b is an induced path from q to y whose length has parity opposite to that of S_a .

In all cases S_a and S_b are induced q - v paths of opposite parity; let Q_1 be the odd and Q_2 the even one. Every vertex of $Q_1 \cup Q_2$ lies on $R(u)$ or on $R(v)$.

Verification. Let Z_u, Z_v be the sets of vertices later than u and than v ; then $U \subseteq Z_u \cap Z_v$ and $u, v \notin U$. (E1): $G[U]$ is connected and v has a neighbor in U , so $G[U \cup \{v\}]$ is connected;

$B' = B_{w_0}$ covers $U \subseteq Z_{w_0}$ by Lemma 6.4(a). (E2): let $x \in V(Q_i) \setminus N^3[v]$. Since $u \sim v$ we have $N^2[u] \subseteq N^3[v]$ and $N^2[v] \subseteq N^3[v]$, so $x \notin N^2[u] \cup N^2[v]$, and x lies on $R(u)$ or $R(v)$. By Lemma 6.4(c) B_u and B_v are anticomplete to $V(R(u)) \setminus N^2[u]$, respectively $V(R(v)) \setminus N^2[v]$, and $B' \subseteq B_u \cap B_v$. (E3): the vertices of the paths outside $C \cup \{q\}$ are among $b(u), b(v)$; these lie in $N(u)$ and $N(v)$, hence in $N^2[v]$, and outside $N^4[q]$ by Lemma 6.4(d). (E4): U is anticomplete to $V(R(u)) \setminus \{u\}$ and $V(R(v)) \setminus \{v\}$ by Lemma 6.4(b), and to u by hypothesis. (E5): $U \cup \{v\} \subseteq F$. (E6): $q \notin B' = B_{w_0}$ by Lemma 6.4(f). The paths lie in $G[(C \setminus U) \cup (B \setminus B') \cup \{q\}]$: the routes avoid U and $u, v \notin U$ (Lemma 6.4(d)), and $b(u), b(v) \notin B'$ because $b(u)$ is adjacent to $\text{key}(u) \in P(u) \subseteq P(w_0)$ and similarly for v . $\square$

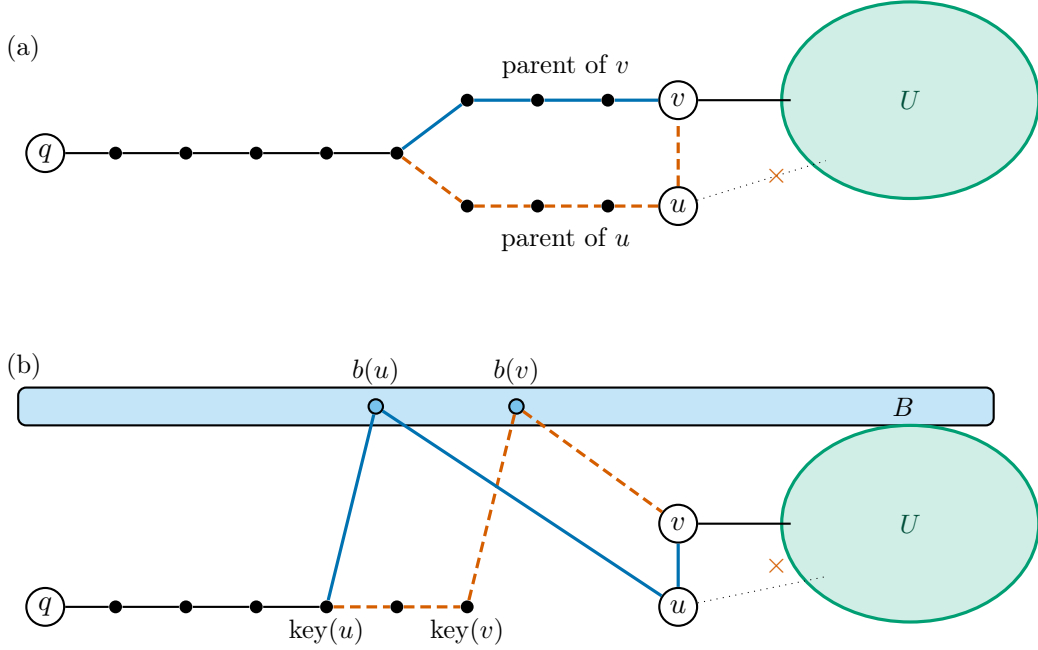


FIGURE 6. Two paths of opposite parity from one allowed edge uv , $p(u) = p(v)$ (Lemma 6.5); the new anchor is $q' = v$. U is later than both u and v , v has a neighbor in U and u has none (crossed dotted edge). Solid paths are odd and dashed paths even; the shared initial part is drawn in black, and the numbers are an example. (a) Both plain, of depth 9: $R(v)$ has length 9, and $R(u)$ followed by uv has length 10. (b) Both keyed, with $r(u) = 4 < r(v) = 6$: $R(u)$ followed by uv has length $4 + 3 = 7$, and $R(v)$ has length $6 + 2 = 8$. Because the two routes have the same parity, appending uv to $R(u)$ gives a path of parity opposite to $R(v)$. The mixed cases, one endpoint plain and one keyed, are similar (Lemma 6.5, case PK). Proportions are schematic.

Proof of Proposition 6.3. Call a vertex of F *active* if it has an earlier neighbor in F with the same route parity, and let A be the set of active vertices. If two adjacent vertices of $F \setminus A$ had equal route parity, the later of them would be active; hence p is a proper 2-coloring of $G[F \setminus A]$, and $\chi(A) \geq \chi(F) - 2 \geq c + 1$. Choose a component H of $G[A]$ with $\chi(H) \geq c + 1$, let β be the earliest block meeting $V(H)$ and $W = V(H) \cap \beta$. As in the proof of Proposition 6.2, Lemma 4.2 with $k = 6$ gives $w \in W$ and a component K of $H - W$ with a neighbor of w such that $\chi(K \setminus N^6[w]) \geq c$; let D be the vertex set of a component of $G[K \setminus N^6[w]]$ of maximum chromatic number, so $\chi(D) \geq c$ and $G[D]$ is connected. Every vertex of K is later than w . Since w is active, it has an earlier neighbor $u \in F$ with $p(u) = p(w)$, so uw is an allowed edge, and both u and w are earlier than every vertex of K .

Let K_1 be the component of $G[K \setminus N(w)]$ that contains D . Since $G[K]$ is connected and w has a neighbor in K , we have $V(K_1) \neq V(K)$, and some vertex of $K \setminus V(K_1)$ has a neighbor in K_1 ; by maximality of the component K_1 of $G[K \setminus N(w)]$, this vertex z lies in $N(w)$. If u has a

neighbor in K_1 , put $U = V(K_1)$; then u plays the role of the endpoint with a neighbor in U , and w is anticomplete to U . Otherwise, put $U = V(K_1) \cup \{z\}$; then U induces a connected graph, w has the neighbor z in U , and u is anticomplete to U , using that u has no neighbor in K_1 and $uz \notin E$ because G is triangle-free. In both cases $U \subseteq K$ consists of vertices later than u and w , and $D \subseteq U$. Lemma 6.5, applied to the allowed edge uw with the endpoint that has a neighbor in U playing the role of v , gives an opposite-parity extension $(B', U, q'; \{Q_1, Q_2\})$ with $q' = u$ in the first case and $q' = w$ in the second. In both cases $\text{dist}(q', w) \leq 1$ and $D \cap N^6[w] = \emptyset$, so $D \cap N^5[q'] = \emptyset$ and $\Phi(U, q') \geq \chi(D) \geq c$. $\square$

7. CHAINS OF EXTENSIONS AND THE CLOSING STEP

The partial-rope construction below adapts the broken arithmetic ropes of [3, Section 4], in the quasi-arithmetic form used by Novaković and Weltge [8, Section 3]. Four extension steps produce four segments of a rope and leave a retained set of potential at least three. The fifth segment is not another extension: it must return to the first anchor q_1 , which lies in the covering level. It is obtained directly from one more allowed edge, by joining the ends of two paths of opposite parity to q_1 through the levels below the covering level. This section sets up the global framework, shows that extensions can be concatenated, and proves the closing step.

7.1. Partial ropes.

Setting 7.1 (Rope setting). Let G be a graph with $\text{og}(G) \geq 15$ and let $X \subseteq V(G)$ be such that $G[X]$ is connected. Fix a root $o \in X$ and let L_i be the set of vertices at distance exactly i from o in $G[X]$ (the *levels* of $G[X]$). Fix $s \geq 6$ and the vertex set C_* of a connected component of $G[L_{s+1}]$, put $B_* = L_s$, and let $q_1 \in B_*$ have a neighbor in C_* .

Then B_* covers C_* , because every vertex of L_{s+1} has a neighbor in L_s , and $G[C_* \cup \{q_1\}]$ is connected; so (B_*, C_*, q_1) is an extension setting. By Lemma 3.3 applied to $G[X]$, the level L_i is anticomplete to L_j for $|i - j| \geq 2$. The graph $G[X]$ is used only to define levels; *distances are in G* .

Definition 7.2 (Partial rope). In the rope setting, a *partial rope with ρ segments*, $0 \leq \rho \leq 4$, consists of vertices $q_1, \dots, q_{\rho+1}$ (with the given q_1), paths $Q_{i,1}, Q_{i,2}$ for $1 \leq i \leq \rho$, and sets $B_\rho \subseteq B_*$, $C_\rho \subseteq C_*$ such that, writing *old paths* for the paths $Q_{i,h}$:

- (P0) $q_2, \dots, q_{\rho+1} \in C_* \setminus C_\rho$; each $Q_{i,h}$ is an induced $q_i q_{i+1}$ -path; $Q_{i,1}$ is odd and $Q_{i,2}$ is even or equal to $Q_{i,1}$; for every $(h_1, \dots, h_\rho)$ the paths $Q_{1,h_1}, \dots, Q_{\rho,h_\rho}$ concatenate to an induced $q_1 q_{\rho+1}$ -path; when $\rho = 0$, this means the one-vertex path at q_1 ;
- (P1) B_ρ covers C_ρ and $G[C_\rho \cup \{q_{\rho+1}\}]$ is connected;
- (P2) C_ρ is anticomplete to $\bigcup_{i,h} V(Q_{i,h}) \setminus \{q_{\rho+1}\}$, and $\text{dist}(q_i, C_\rho \cup \{q_{\rho+1}\}) \geq 6$ for $i \leq \rho$;
- (P3) $\text{dist}(q_i, q_j) \geq 6$ for $1 \leq i < j \leq \rho + 1$;
- (P4) B_ρ is anticomplete to $V(Q_{i,h}) \setminus N^3[q_{i+1}]$ for $i \leq \rho$;
- (P5) $V(Q_{i,h}) \subseteq C_* \cup \{q_1\} \cup (B_* \cap N^2[q_{i+1}])$ for $i \leq \rho$;
- (P6) no vertex of an old path other than $q_{\rho+1}$ belongs to $B_\rho \cup C_\rho$ (for $\rho \geq 1$, also $q_{\rho+1} \notin B_\rho$).

With $\rho = 0$, $B_0 = B_*$ and $C_0 = C_*$, there are no paths, and (P1) holds; Figure 7 shows a partial rope with four segments. Condition (P6) is a convenient way to record that new paths meet old ones only at the junction; for $\rho \geq 1$ its last clause is automatic, as $q_{\rho+1} \in C_*$ and $B_* \cap C_* = \emptyset$, and condition (E6) is needed only at the first step, to keep $q_1 \in B_*$ out of B_1 .

Lemma 7.3 (Chaining). *Let $0 \leq \rho \leq 3$ and let a partial rope with ρ segments be given. Let $(B', C', q'; \mathcal{Q})$ be an extension of the extension setting $(B_\rho, C_\rho, q_{\rho+1})$, where either $\mathcal{Q} = \{Q\}$ with Q odd, or $\mathcal{Q} = \{Q_1, Q_2\}$ with Q_1 odd and Q_2 even. Put $q_{\rho+2} = q'$, $B_{\rho+1} = B'$, $C_{\rho+1} = C'$, and let $(Q_{\rho+1,1}, Q_{\rho+1,2})$ be (Q, Q) or (Q_1, Q_2) respectively. Then this is a partial rope with $\rho + 1$ segments.*

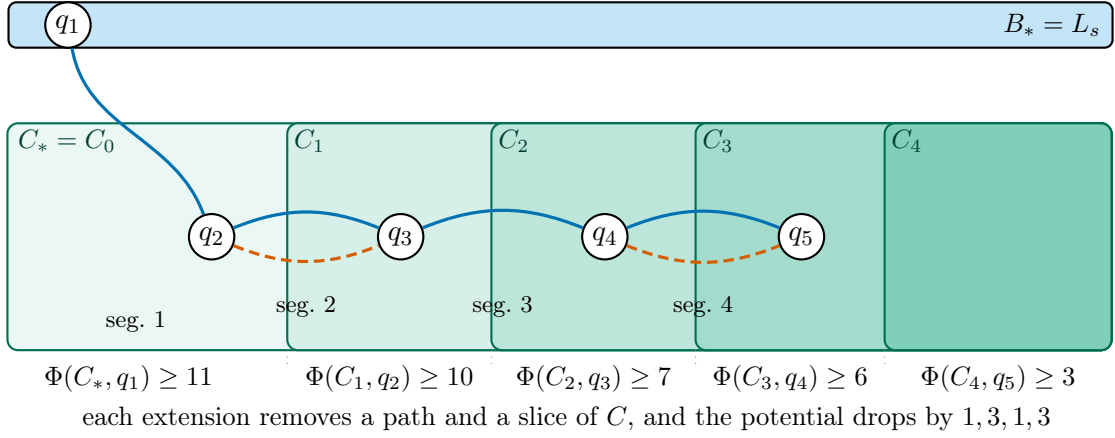


FIGURE 7. A partial rope with four segments in the rope setting (Definition 7.2), schematically. The anchor q_1 lies in the covering level B_* ; $q_2, \dots, q_5$ lie in C_* , each outside the next retained set: $q_{\rho+1} \in C_{\rho-1} \setminus C_\rho$. The retained sets decrease, $C_* \supseteq C_1 \supseteq \dots \supseteq C_4$, and the paths $Q_{\rho,h}$ lie in the removed slices, together with covering vertices that are not kept in B_ρ . Solid paths are odd and dashed paths even: segments 1 and 3 carry one odd path, segments 2 and 4 two paths of opposite parity. The numbers are the guaranteed potential (Table 1).

Proof. Write $q = q_{\rho+1}$, $B = B_\rho$, $C = C_\rho$; the paths $Q \in \mathcal{Q}$ are the *new paths*.

Disjointness. A new path lies in $G[(C \setminus C') \cup (B \setminus B') \cup \{q\}]$. By (P6) no vertex of an old path other than q lies in $B \cup C$; hence a new path meets the old paths only in q .

Induced concatenation. Let P_0 be a union as in (P0), an induced path ending in q , and let Q be a new path. Suppose that an edge xy joins $x \in V(P_0) \setminus \{q\}$ and $y \in V(Q) \setminus \{q\}$, where $x \in V(Q_{i,h})$ with $i \leq \rho$. If $y \in C$, then C is not anticomplete to x , contrary to (P2). Hence $y \notin C$, and (E3) gives $y \in (B \cap N^2[q']) \setminus N^4[q]$; in particular $y \notin \{q, q'\}$, so y has two neighbors y^-, y^+ on Q . As $y \in B$, (P4) yields $x \in N^3[q_{i+1}]$, hence $\text{dist}(y, q_{i+1}) \leq 4$. If $i = \rho$ this contradicts $y \notin N^4[q]$. Let $i < \rho$. We claim that y has a neighbor $c \in C \cap V(Q)$; then $\text{dist}(q_{i+1}, c) \leq 5$, contradicting $\text{dist}(q_{i+1}, C) \geq 6$ from (P2). Suppose the claim fails. Then $y^-, y^+ \in \{q\} \cup (B \cap N^2[q'])$, and neither is q since $y \notin N^4[q]$. So both lie in $B \cap N^2[q']$, and they are not adjacent to q' simultaneously, as q' has only one neighbor on the induced path Q . Say $y^- \not\sim q'$. Also $y \not\sim q'$, since $q' \in C \cap V(Q)$. Since $y, y^- \in B$ are different from $q' \in C$, we get $\text{dist}(y, q') = \text{dist}(y^-, q') = 2$. Let a be a common neighbor of y and q' , and a^- one of y^- and q' ; they are different, since G has no triangle. Then $yy^-a^-q'a$ is a closed walk of length five, which is impossible. Thus $P_0 \cup Q$ is an induced path, and (P0) holds for $\rho + 1$ segments.

(P1)–(P6). (P1) is (E1). (P2): $C' \subseteq C$ is anticomplete to the old paths except q by (P2), and to $V(Q) \setminus \{q'\} \ni q$ by (E4); moreover $\text{dist}(q_i, C' \cup \{q'\}) \geq 6$ for $i \leq \rho$ since $C' \cup \{q'\} \subseteq C$, and $\text{dist}(q, C' \cup \{q'\}) \geq 6$ is (E5). (P3): $\text{dist}(q', q_i) \geq 6$ for $i \leq \rho$ since $q' \in C$, and $\text{dist}(q', q) \geq 6$ by (E5). (P4): $B' \subseteq B$, and for new paths this is (E2). (P5): by (E3), $V(Q) \subseteq C \cup \{q\} \cup (B \cap N^2[q']) \subseteq C_* \cup \{q_1\} \cup (B_* \cap N^2[q_{\rho+2}])$, as $q \in C_* \cup \{q_1\}$. (P6): old vertices other than q avoid $B \cup C \supseteq B' \cup C'$; and $q \notin C'$ because $q \notin C$, and $q \notin B'$ by (E6). For the new paths, vertices other than q lie in $(C \setminus C') \cup (B \setminus B')$, hence avoid $B' \cup C'$. Finally $q_{\rho+2} = q' \in C \setminus C' \subseteq C_* \setminus C_{\rho+1}$. $\square$

7.2. Closing. The return through levels below the covering level follows the closing constructions in [3, proof of Theorem 4.1] and [8, proof of Theorem 2.3]. Here we obtain the last pair of paths directly, without a fifth extension.

Proposition 7.4 (Closing). *In the rope setting, let a partial rope with four segments be given in which $Q_{2,2}$ and $Q_{4,2}$ are even, and suppose that $\Phi(C_4, q_5) \geq 3$. Then there are induced q_5q_1 -paths $Q_{5,1}$ (odd) and $Q_{5,2}$ (even) such that $q_1, \dots, q_5$ and the paths $Q_{i,h}$, $1 \leq i \leq 5$, form a quasi-5-arithmetic rope in X .*

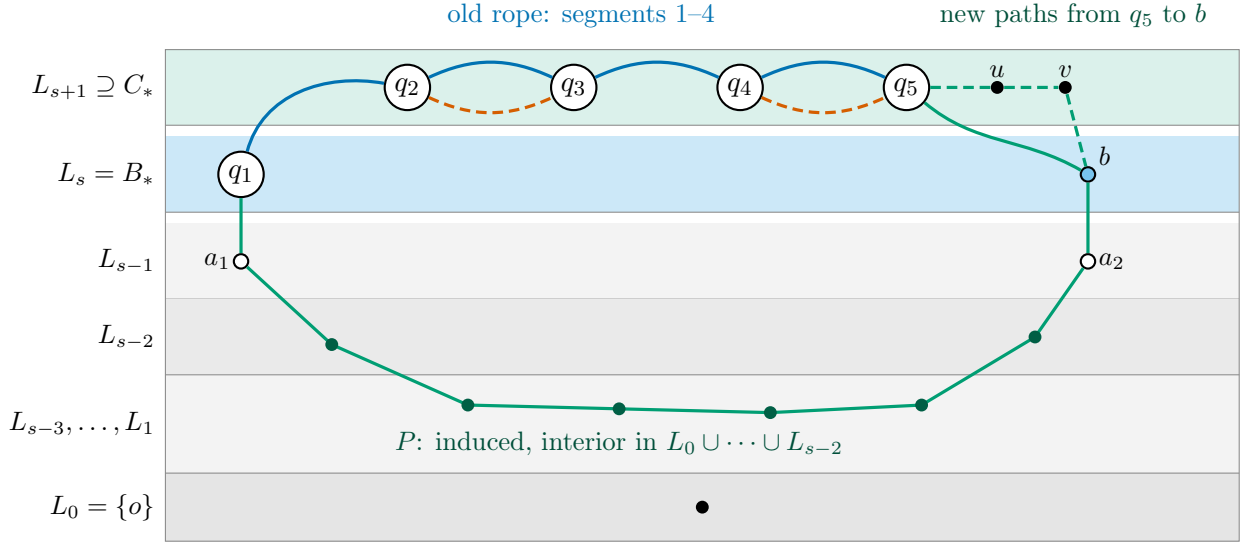


FIGURE 8. The closing step (Proposition 7.4) in the rope setting, with the levels of $G[X]$ drawn horizontally and $L_0 = \{o\}$ at the bottom; schematic. Blue: the old rope in $C_* \cup B_*$ (Definition 7.2); solid paths are odd and dashed paths even. Green: the two new paths $Q_{5,1}, Q_{5,2}$ from q_5 to q_1 . They differ only in the part from q_5 to the covering vertex b (two paths S_1, S_2 of opposite parity, one of which may pass through the allowed edge uv), and share the return segment $b a_2 P a_1 q_1$ through the lower levels; the odd one of the two is $Q_{5,1}$. The anchors $q_1, \dots, q_5$ are pairwise at distance at least 6. The return segment avoids the old rope because the old rope lies in $L_s \cup L_{s+1}$ and the interior of P lies in $L_0 \cup \dots \cup L_{s-2}$.

Proof. Figure 8 shows the configuration. Write $q = q_5$, $B = B_4$, $C = C_4$; then (B, C, q) is an extension setting, and we use the notation of Section 5 for it. Let $F = C \setminus N^5[q]$, so $\chi(F) \geq 3$. Recall that $q \in C_* \subseteq L_{s+1}$, $C \subseteq L_{s+1}$ and $B \subseteq L_s$.

Step 1: an allowed edge. If no edge of $G[F]$ had endpoints of equal route parity, p would be a proper 2-coloring of $G[F]$. So there is an edge uv of $G[F]$ with $p(u) = p(v)$. In particular $\text{dist}(u, q), \text{dist}(v, q) \geq 6$, and $\text{dist}(u, q_i), \text{dist}(v, q_i) \geq 6$ for $i \leq 4$ by (P2).

Step 2: two paths of opposite parity to a covering vertex. We construct a vertex $b \in B$, a set $\mathbb{B} \subseteq B \cap (N(u) \cup N(v))$ with $b \in \mathbb{B}$ and $|\mathbb{B}| \leq 2$, and two induced q - b paths S_1, S_2 of opposite parity whose vertices outside $\mathbb{B}$ lie in $C \cup \{q\}$. We also choose $a_2 \in L_{s-1} \cap N(b)$ that has no neighbor in $\mathbb{B} \setminus \{b\}$. (A vertex of L_{s-1} has no neighbor in $C \cup \{q\} \subseteq L_{s+1}$, so $\mathbb{B}$ is the only concern.) The cases are those of Lemma 6.5; we use the routes and the notation there.

Case PP: u, v plain, of the same depth t . Let b be a neighbor of v in B ; since v is plain, $b \in B_0^t$, so b has no neighbor of depth at most $t - 3$. Let $S'_1 = R(v)$, of length t , and let S'_2 be $R(u)$ followed by v , of length $t + 1$; both are induced (Lemma 6.5, Case PP). Append b to each and truncate at the first neighbor of b along the path. Write v^- for the vertex of $R(v)$ of depth $t - 2$ and v^* for the parent of v , and similarly u^-, u^* . On S'_1 the neighbors of b lie among v^- and v , since $b \not\sim v^*$ (triangle b, v, v^*). So the resulting induced path S_1 has length $t - 1$ or $t + 1$. On S'_2 , the vertex b is not adjacent to u^- (the 5-cycle b, u^-, u^*, u, v), nor to u (triangle b, u, v); so the first neighbor of b on S'_2 is u^* or v , and the resulting induced path S_2 has length t or $t + 2$. Here $\mathbb{B} = \{b\}$ and $a_2 \in L_{s-1} \cap N(b)$ is arbitrary.

Case KK: u, v keyed. Relabel so that $\alpha(u) < \alpha(v)$; they differ because adjacent vertices lie in different blocks. Put $b = b(v)$. Let S_1 be the prefix of v followed by b , of length $r(v) + 1$, and let S'_2 be the prefix of u followed by $b(u)$, u , v and b , of length $r(u) + 4$. As $p(u) = p(v)$ means $r(u) \equiv r(v) \pmod{2}$, the lengths have opposite parity. S_1 is induced by Lemma 5.3. In S'_2 the vertices u, v have no neighbor on the prefix of u (their depths are at least $r(u) + 3$ and $r(u) + 2$, the prefix has depth at most $r(u)$); b has no neighbor on that prefix, whose vertices have index at most $\alpha(u) < \alpha(v)$, by (5.1) for v ; and $b(u)v, bu$ are non-edges and $b(u) \neq b$, as

each would create a triangle. So the only possible chord of S'_2 is $b(u)b$. If it is present, replace $b(u), u, v, b$ by $b(u), b$, which shortens the path by two and preserves its parity. Let S_2 be the resulting path. Here $\mathbb{B} = \{b(u), b\}$; the vertex $b(u)$ stays on S_2 also if S'_2 was shortened. Choose $a_2 \in L_{s-1} \cap N(b)$. Then $a_2 \not\sim b(u)$: otherwise either $b(u) \sim b$ and $a_2, b(u), b$ is a triangle, or $a_2, b(u), u, v, b$ is a 5-cycle.

Case PK, $r = t - 2$. Let x be the plain vertex, of depth t , and y the keyed vertex, with key depth $t - 2$ and depth $t + 1$; put $b = b(y)$. Let S_1 be the prefix of y followed by b , of length $t - 1$, and let S'_2 be $R(x)$ followed by y and b , of length $t + 2$. Let x^- and x^* be the vertices of $R(x)$ of depth $t - 2$ and $t - 1$ (x^* is the parent of x). The vertex b has no neighbor on $R(x)$ of depth at most $t - 3$, since such vertices have index below $\alpha(y)$ and (5.1) applies; $b \not\sim x^-$, since b, x^-, x^*, x, y would be a 5-cycle; and $b \not\sim x$ by the triangle b, x, y . Truncating S'_2 at the first neighbor of b , which is x^* or y , gives an induced path S_2 of length t or $t + 2$. Here $\mathbb{B} = \{b\}$ and $a_2 \in L_{s-1} \cap N(b)$ is arbitrary.

Case PK, $r \leq t - 4$. Let x be plain of depth t , let $b \in B_0^t$ be a neighbor of x , and let y be the keyed vertex. Let S'_1 be $R(x)$ followed by b , truncated at the first neighbor of b . Since $b \in B_0^t$, the neighbors of b on $R(x)$ lie at depth at least $t - 2$, and $b \not\sim x^*$ (triangle b, x, x^*), where x^* is the parent of x ; so the first neighbor of b is the vertex x^- of depth $t - 2$ or x , and S_1 has length $t - 1$ or $t + 1$. Let S'_2 be the prefix of y followed by $b(y), y, x$ and b , of length $r + 4 \equiv t$. Neither x nor y has a neighbor on the prefix, which has depth at most $r \leq t - 4$; nor has b , since $b \in B_0^t$; and $b(y)x$ and by are non-edges (triangles). The only possible chord is $b(y)b$; if present, replace $b(y), y, x, b$ by $b(y), b$. Let S_2 be the result; then $\mathbb{B} = \{b(y), b\}$, since $b(y)$ stays on S_2 also after the shortcut. Choose $a_2 \in L_{s-1} \cap N(b)$; then $a_2 \not\sim b(y)$, since otherwise $a_2, b(y), b$ is a triangle or $a_2, b(y), y, x, b$ is a 5-cycle.

In every case S_1, S_2 are induced q - b paths of opposite parity, and $\mathbb{B} \subseteq B \cap (N(u) \cup N(v))$.

Step 3: the return path. Let $a_1 \in L_{s-1}$ be a neighbor of q_1 , and let P be an induced $a_2 a_1$ -path in $G[\{a_1, a_2\} \cup L_0 \cup \dots \cup L_{s-2}]$; it exists because this graph is connected. For $j = 1, 2$ let Q'_j be S_j followed by P (starting with the edge ba_2) and by the edge $a_1 q_1$. The two walks have opposite parity, as they share the return segment; let $Q_{5,1}$ be the odd and $Q_{5,2}$ the even one.

Step 4: distances. Every vertex of $\mathbb{B}$ is adjacent to u or v , so $\text{dist}(\mathbb{B}, q_i) \geq 5$ for $1 \leq i \leq 5$. Consequently

$$(7.1) \quad \text{dist}(a_2, q_i) \geq 4 \quad (i \leq 5), \quad \text{dist}(a_1, q_i) \geq 5 \quad (2 \leq i \leq 5),$$

and $\text{dist}(q_1, u), \text{dist}(q_1, v) \geq 6$. In particular, since $a_2 \sim b$ and $\text{dist}(q_1, b) \geq 5$, we have $a_1 \neq a_2$, $a_1 \not\sim a_2$ and $a_2 \not\sim q_1$; and a_1 is adjacent to no vertex of $\mathbb{B}$, because $a_1 \sim q_1$ and $\text{dist}(q_1, \mathbb{B}) \geq 5$.

Step 5: the new paths avoid the old rope. By (P5), the vertices of the old paths lie in $C_* \cup B_* \subseteq L_{s+1} \cup L_s$.

- (a) *Vertices of C .* The vertices of C on $Q_{5,j}$ lie in $C = C_4$ (note $q \notin C$); they are anticomplete to the old rope except q , and have distance at least 6 from $q_1, \dots, q_4$, by (P2). In particular they are not on the old paths, by (P6).
- (b) *Vertices of $\mathbb{B}$.* If $b'' \in \mathbb{B}$ were adjacent to a vertex x of an old path $Q_{i,h}$, then $x \in N^3[q_{i+1}]$ by (P4), so $\text{dist}(b'', q_{i+1}) \leq 4$, contradicting $\text{dist}(\mathbb{B}, q_{i+1}) \geq 5$. If b'' were on an old path, then by (P5) $b'' = q_1$ or $b'' \in N^2[q_{i+1}]$, again contradicting the same bound.
- (c) *The vertex a_2 .* It lies in L_{s-1} , so its neighbors on old paths lie in $B_* = L_s$, and by (P5) such a neighbor y_0 is q_1 or lies in $N^2[q_{i+1}]$, which gives $\text{dist}(a_2, q_1) \leq 1$ or $\text{dist}(a_2, q_{i+1}) \leq 3$, contrary to (7.1).
- (d) *The vertex a_1 .* Likewise a neighbor of a_1 on an old path other than q_1 would lie in $N^2[q_{i+1}]$ for some $i \leq 4$, giving $\text{dist}(a_1, q_{i+1}) \leq 3$, contrary to (7.1).
- (e) *The path P .* Its internal vertices lie in $L_0 \cup \dots \cup L_{s-2}$ and have no neighbor in $L_s \cup L_{s+1}$, which contains the old paths, $\mathbb{B}, C \cup \{q\}$ and q_1 .

- (f) *The vertex q_1 .* It is at distance at least 6 from the vertices of $C \cup \{q\}$ on $S_1 \cup S_2$ and at distance at least 5 from $\mathbb{B}$, it is not adjacent to a_2 or to the internal vertices of P (levels), and its only neighbor on $Q_{5,j}$ is a_1 .

By these facts and Step 2, each $Q_{5,j}$ is an induced q_5q_1 -path, internally disjoint from the old paths and anticomplete to them except for the edges at q_5 and q_1 .

Step 6: the rope. For $(h_1, \dots, h_5) \in \{1, 2\}^5$, the union $Q_{1,h_1} \cup \dots \cup Q_{4,h_4}$ is an induced q_1q_5 -path by (P0); Q_{5,h_5} is an induced q_5q_1 -path with the properties of Step 5; and $q_1 \not\sim q_5$ by (P3). So the union is an induced cycle, which is (d) of Definition 2.5. Condition (a): the old paths lie in $C_* \cup \{q_1\} \cup B_* \subseteq X$ by (P5); the new vertices $C, \mathbb{B} \subseteq X$, and a_1, a_2 and P lie in $L_0 \cup \dots \cup L_{s-1} \subseteq X$. Condition (b) holds by (P0) and the choice of $Q_{5,1}$. Condition (c): $Q_{2,2}$, $Q_{4,2}$ and $Q_{5,2}$ are even, and every pair of cyclically consecutive indices contains one of 2, 4, 5. Condition (e) holds by (P3) and $\text{dist}(q_1, q_5) \geq 6$. $\square$

8. COMPLETION OF THE PROOF

It remains to prove Proposition 3.2. The proof halves once more, inside a level of $G[X]$, and then performs four extension steps and the closing step in turn, keeping track of the potential as in Table 1.

Proof of Proposition 3.2. Let $\text{og}(G) \geq 15$ and $\chi(X) \geq 23$. Let $Y \subseteq X$ be the vertex set of a connected component of $G[X]$ with $\chi(Y) = \chi(X)$, fix a root $o \in Y$, and let L_i denote the set of vertices at distance exactly i from o in $G[Y]$. By Lemma 3.3 some level L_j has $\chi(L_j) \geq \lceil 23/2 \rceil = 12$, and $j \geq 7$ since $L_0, \dots, L_6$ are stable. Put $s = j - 1 \geq 6$. The components of $G[L_{s+1}]$ are pairwise anticomplete, so one of them, with vertex set C_* , has $\chi(C_*) \geq 12$. Let $B_* = L_s$ and let $q_1 \in B_*$ have a neighbor in C_* . This is a rope setting (Setting 7.1, with Y in the role of X), and (B_*, C_*, q_1) is an extension setting.

Since $\text{og}(G) \geq 13$, Observation 2.1 and Lemma 4.1 with $k = 5$ give

$$\Phi(C_*, q_1) = \chi(C_* \setminus N^5[q_1]) \geq \chi(C_*) - 1 \geq 11.$$

We start from the partial rope with no segment, $B_0 = B_*$, $C_0 = C_*$, and apply Lemma 7.3 four times, with extensions supplied as follows (each time the potential hypothesis is met by the output of the previous step; see Table 1).

- (1) Proposition 6.2 with $c = 10$ and $\delta = 1$ gives a prescribed-parity extension with an odd path and potential at least 10.
- (2) Proposition 6.3 with $c = 7$ gives an opposite-parity extension and potential at least 7.
- (3) Proposition 6.2 with $c = 6$ and $\delta = 1$ gives an odd path and potential at least 6.
- (4) Proposition 6.3 with $c = 3$ gives an opposite-parity extension and potential at least 3.

We obtain a partial rope with four segments in which $Q_{1,2} = Q_{1,1}$ and $Q_{3,2} = Q_{3,1}$ are odd, $Q_{2,2}$ and $Q_{4,2}$ are even, and $\Phi(C_4, q_5) \geq 3$. Proposition 7.4 yields a quasi-5-arithmetic rope in $Y \subseteq X$. $\square$

Stage	Operation	Needed before	Guaranteed after
start	$\chi(C_*) \geq 12$; ball (Lemma 4.1)	—	≥ 11
segment 1	prescribed, $c = 10$ (Prop. 6.2)	≥ 11	≥ 10
segment 2	opposite, $c = 7$ (Prop. 6.3)	≥ 10	≥ 7
segment 3	prescribed, $c = 6$ (Prop. 6.2)	≥ 7	≥ 6
segment 4	opposite, $c = 3$ (Prop. 6.3)	≥ 6	≥ 3
segment 5	closing (Prop. 7.4)	≥ 3	—

TABLE 1. The potential $\Phi(C, q) = \chi(C \setminus N^5[q])$ along the construction of the rope.

The proof of Theorem 1.1 is now complete: Proposition 3.2 gives Proposition 3.1, which gives Theorem 1.1.

8.1. Further improvements. In terms of the threshold T of Section 3, the bound is $4T + 2$: after at most six deletions a graph of chromatic number $4T + 3$ would have chromatic number at least $4T - 3$, which suffices for an odd wheel t -minor; here $4T - 3 = 2(2T - 1) - 1$ reflects the two halvings and $T = 1 + (1 + 3 + 1 + 3) + 3 = 12$ is the initial ball allowance, the four extension costs, and the closing threshold. A bound strictly below 50 for the general class would therefore need a further saving of one color, in T or in the number of deletions or halvings.

We have not found a way to avoid the inner breadth-first halving. Using the lower levels of the outer breadth-first search for the closure would make the closing paths compete with the connector that produces the odd wheel in Proposition 2.7, and keeping only the marked vertices in a single level is insufficient. We therefore claim no bound below 50 for the general class.

APPENDIX A. COMPARISON WITH THE PROOF OF NOVAKOVIĆ AND WELTGE

The extension and stable-grading framework comes from [3, Section 4]. The levelwise sets B_0^t, B_1^t, B_2^t , the ordering $m_1, m_2, \dots$ and the grading used in Section 5 follow Novaković and Weltge [8, Lemma 3.2] (and [8, Lemma 3.3], which uses the same grading). This appendix records three differences between our formulation and theirs; none changes the geometric goals of the extension steps.

Parity classes of keyed vertices. In [8], for $6 \leq t \leq d$ the set B_+^t is split into B_1^t and B_2^t according to the parity of the distance from q to b in $G[\mathcal{M}_{\leq t-3} \cup \{b\}]$, and the class C_i^t consists of the vertices of $M_t \setminus N^5[q]$ with a neighbor in B_i^t . A vertex with neighbors in both B_1^t and B_2^t then lies in both C_1^t and C_2^t . Here a keyed vertex z is attached to a single key $\text{key}(z)$ of minimum index, with a witness $b(z)$. By (5.1) the distance from q to $b(z)$ in $G[\mathcal{M}_{\leq t-3} \cup \{b(z)\}]$ is $r(z) + 1$, so $b(z) \in B_i^t$ for $i = 1$ if $r(z)$ is even and $i = 2$ if $r(z)$ is odd; the route parity $p(z) \equiv r(z)$ of Section 5 is thus the class index (shifted by one). This refines the existential classes into a partition with the same union, and the stable grading depends only on $B_1^t \cup B_2^t = B_+^t$, hence is unchanged. Whenever an argument compares two vertices that lie in the same class, the routes of the two vertices have, by definition, the same parity. The one-path construction of [8, Section 5] is treated in the same way.

Domain of the grading. Every block used here is a subset of $F = C \setminus N^5[q]$. The two initial blocks of [8] that split $C \cap N^5[q]$ into two stable sets play no role, since the potential $\Phi(C, q)$ already discards the ball (Lemma 4.1). The prescribed-parity step of [8, Lemma 3.3] works with $C \setminus N^6[q]$; our construction (Proposition 6.2) stays inside F , because the parity correction moves forward from $\hat{q}$ to a vertex w of the retained component rather than backward to a parent, so the extra radius is not needed.

Invariants for the separation of the retained set. In the second case of the proof of [8, Lemma 3.2], the retained set is separated from the paths by the fact that the two endpoints are earlier than every vertex of the retained set. A block of the grading can contain vertices of much greater depth than the vertices of an earlier block, so the depth of a retained vertex is not bounded below by the depth of the earlier endpoint alone. What the grading provides, and what is used in Lemma 5.5, is the following: if a block is keyed by a vertex m_α at depth r , then every later vertex has depth at least $r + 3$ (stage σ), is adjacent to none of $m_1, \dots, m_\alpha$, and has no B -neighbor adjacent to any of them; if a block is plain of depth t with parent index π , every later vertex has depth at least t , is adjacent to none of $m_1, \dots, m_\pi$, and has all its B -neighbors in B_0^t . This is exactly the information needed to conclude that the unused covering set B' covers the retained set and that the retained set misses the canonical prefix and the chosen covering parents.

FORMALIZATION

The companion Lean files formalize the statement that every finite t -perfect graph is 50-colorable.

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