

COMPLEX AND HYPERCOMPLEX ITERATIVE METHODS

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SEATTLE UNIVERISITY

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- 1 Iteration
- 2 Complex Iterative Methods
- 3 Quaternions
- 4 Quaternion Iterative Methods

DEFINITIONS

Definition (Iterative Method)

A sequence where each value in the sequence is defined by the previous value.

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A system that enacts rules on a set of variables to produce a state.

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A sequence where each value in the sequence is defined by the previous value.

Definition (Dynamical System)

A system that enacts rules on a set of variables to produce a state.

Definition (Complex Dynamics)

The study of dynamical systems defined by complex functions.

ITERATION

Definition (Function Iteration)

$$f^0 := I$$

$$f^{k+1} := f \circ f^k$$

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Example

Given $f(x) = x + 1$,

$$f^0(x) = x$$

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$$f^2(x) = (x + 1) + 1$$

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Given $f(x) = x + 1$,

$$f^0(x) = x$$

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$$f^2(x) = (x + 1) + 1$$

$$f^3(x) = ((x + 1) + 1) + 1$$

⋮

COMPLEX ITERATIVE METHODS

Definition (Complex Numbers)

$$i^2 = -1$$

$$\{a + bi : a, b \in \mathbb{R}\} \in \mathbb{C}$$

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Addition

Let $a, b, x, y \in \mathbb{R}$,

$$(a + bi) + (x + yi)$$

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$$(a + bi) * (x + yi)$$

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Let $a, b, x, y \in \mathbb{R}$,

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Let $a, b, x, y \in \mathbb{R}$,

$$\begin{aligned}(a + bi) * (x + yi) &= ax + ayi + bxi + byi^2 \\ &= (ax - by) + (ay + bx)i\end{aligned}$$

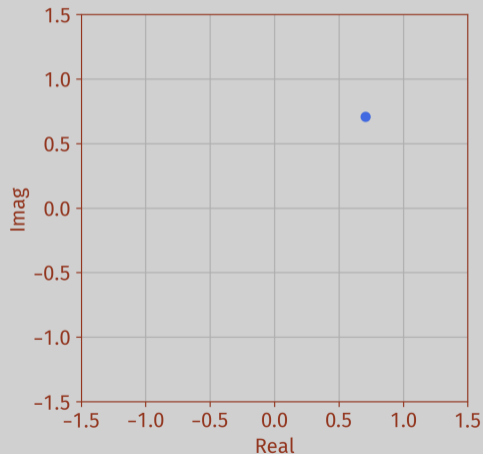
COMPLEX ITERATION

Rules

$$f(z) = z^2$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$



COMPLEX ITERATION

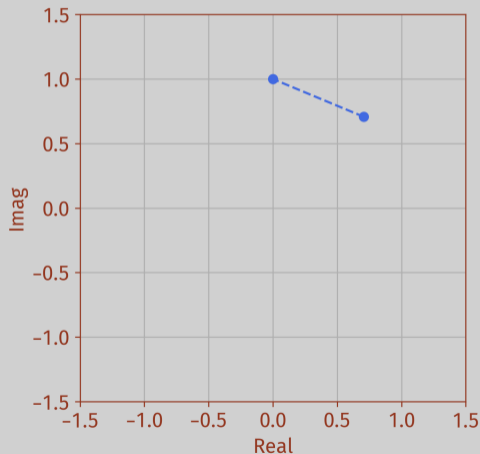
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$$\blacksquare f^1(z) = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)^2 = \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)^2 i = i$$



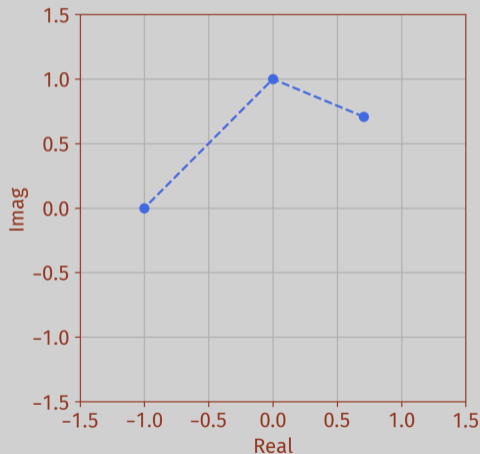
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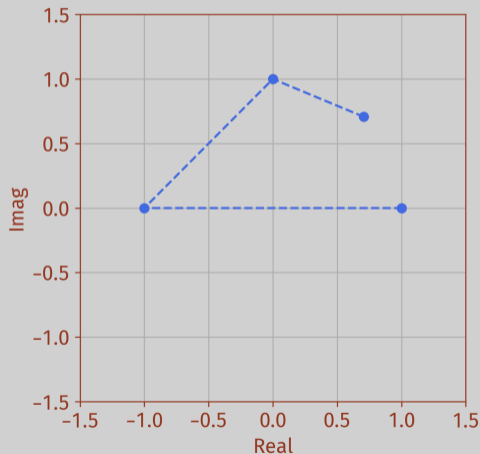
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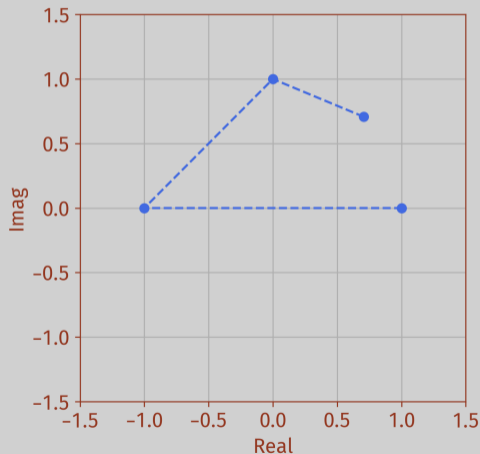
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- $f^4(z) = f^5(z) = f^6(z) = 1$



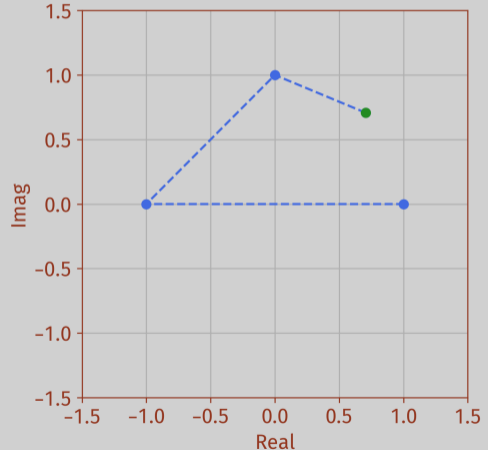
COMPLEX ITERATION

Rules

$$f(z) = z^2 - \frac{1}{10} - \frac{1}{10}i$$

$$z_0 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

■ $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$



COMPLEX ITERATION

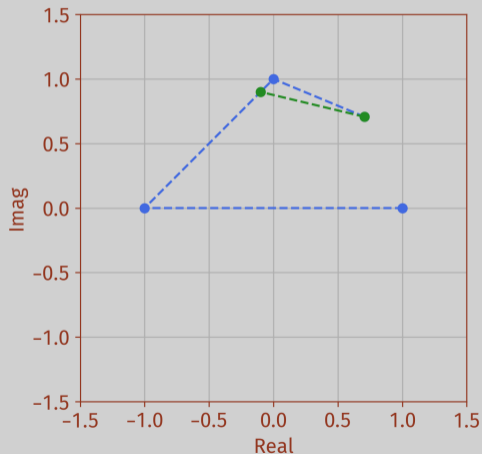
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$$\blacksquare f^1(z) = -\frac{1}{10} + \left(1 - \frac{1}{10}\right)i = -0.1 + 0.9i$$



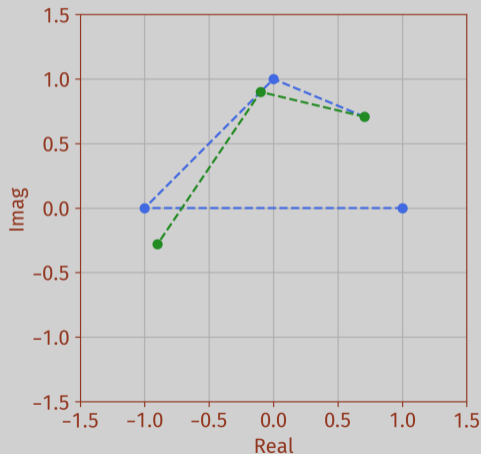
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- $f^0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$
- $f^1(z) = -0.1 + 0.9i$
- $f^2(z) = -0.9 - 0.28i$



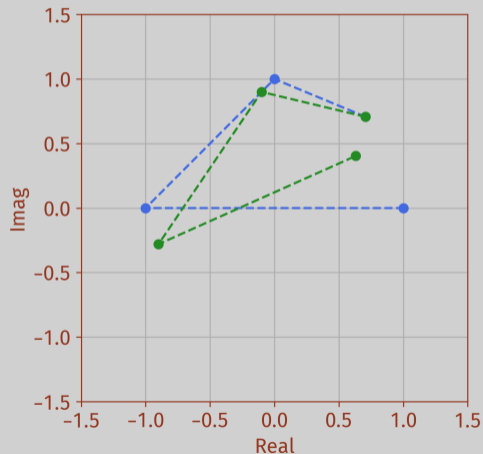
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- $f^3(z) = 0.6316 + 0.404i$



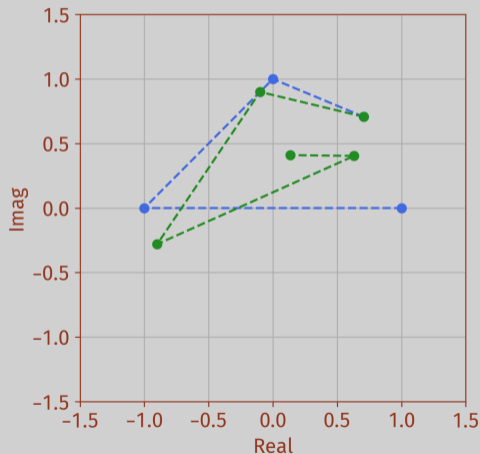
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- $f^4(z) \approx 0.13570256 + 0.4103328i$



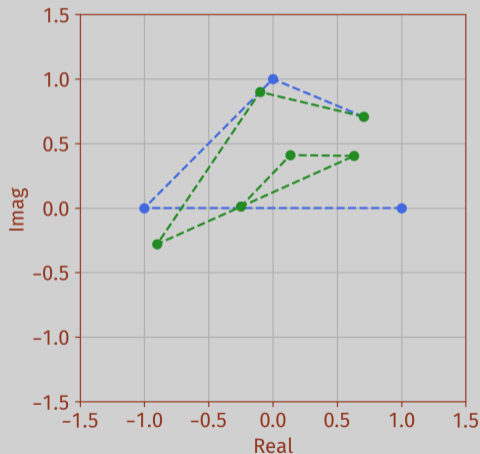
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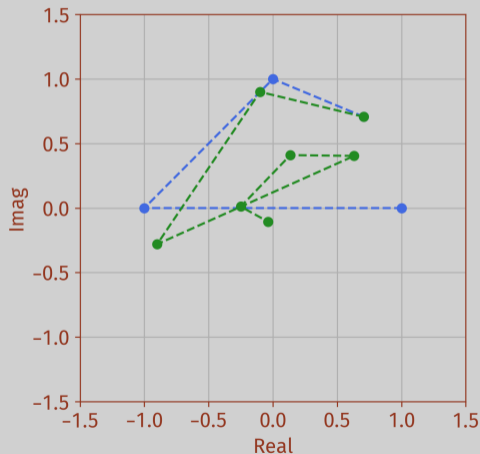
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- $f^5(z) \approx -0.24995782 + 0.01136642i$
- $f^6(z) \approx -0.03765028 - 0.10568225i$



GROUP ACTIVITY

Easier

$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

Harder

$$f(z) = z^2 - 0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

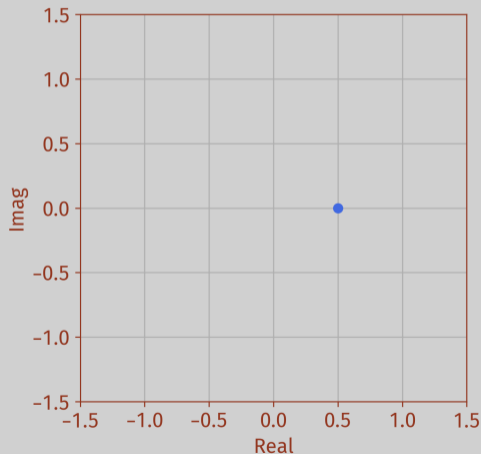
GROUP ACTIVITY (EASIER)

Rules

$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

■ $f^0(z) = 0.5$



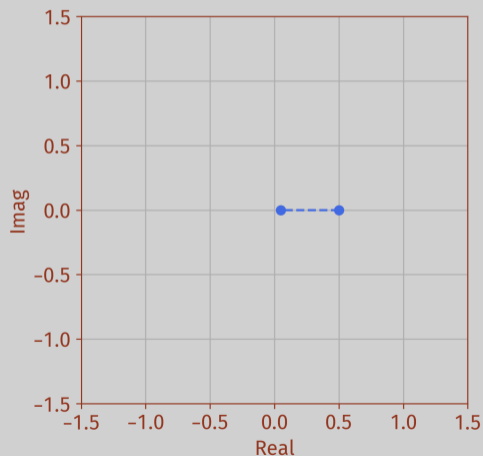
GROUP ACTIVITY (EASIER)

Rules

$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

- $f^0(z) = 0.5$
- $f^1(z) = 0.05$



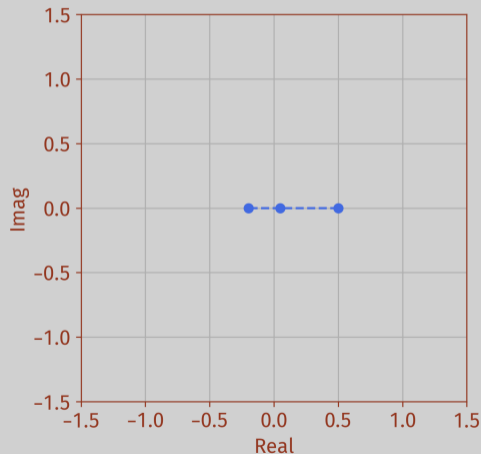
GROUP ACTIVITY (EASIER)

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$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

- $f^0(z) = 0.5$
- $f^1(z) = 0.05$
- $f^2(z) = -0.1975$



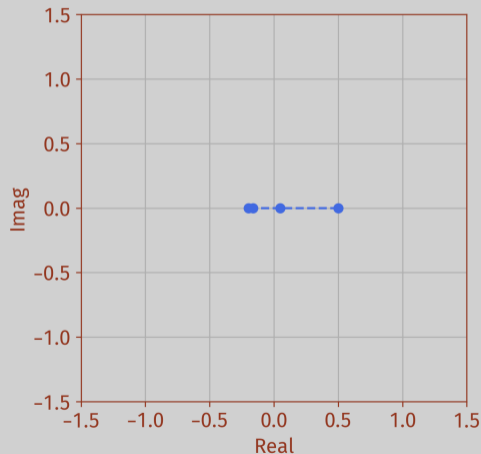
GROUP ACTIVITY (EASIER)

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$$f(z) = z^2 - 0.2 + 0i$$

$$z_0 = 0.5 + 0i$$

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- $f^3(z) = -0.16099375$



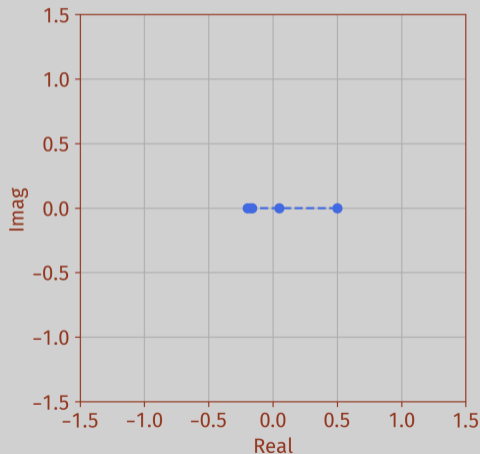
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- $f^3(z) = -0.16099375$
- $f^4(z) \approx -0.1740810125$



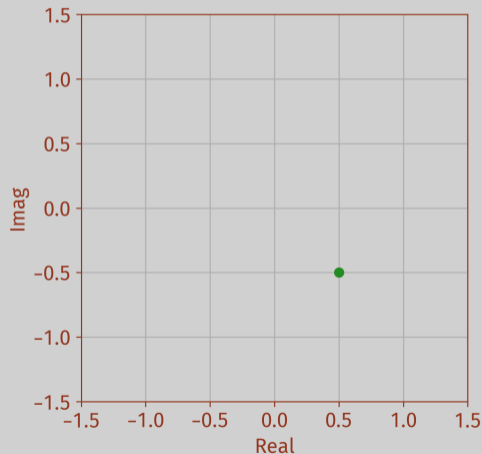
GROUP ACTIVITY (HARDER)

Rules

$$f(z) = z^2 - 0.2 + 0.4i$$

$$z_0 = 0.5 - 0.5i$$

■ $f^0(z) = 0.5 - 0.5i$



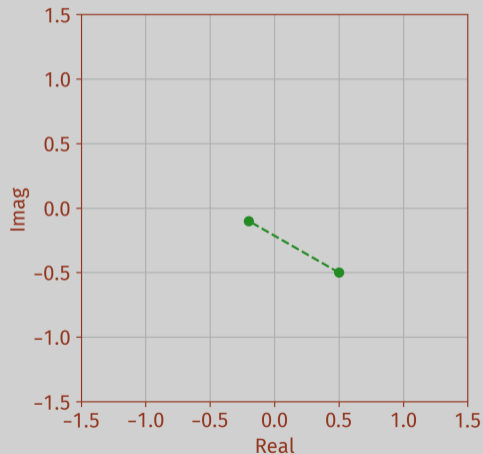
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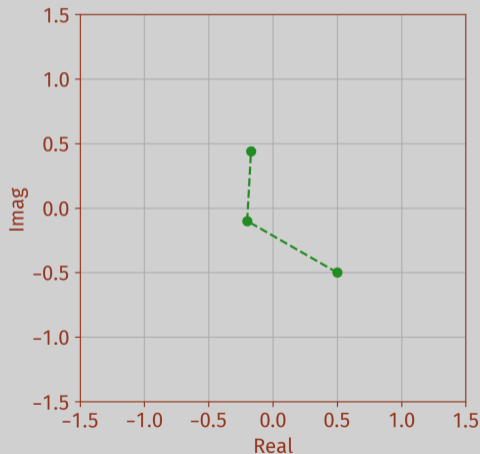
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- $f^2(z) = -0.17 + 0.44i$



GROUP ACTIVITY (HARDER)

Rules

$$f(z) = z^2 - 0.2 + 0.4i$$

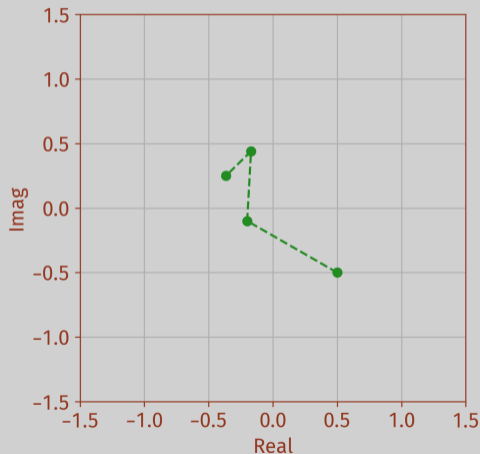
$$z_0 = 0.5 - 0.5i$$

$$\blacksquare f^0(z) = 0.5 - 0.5i$$

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$$\blacksquare f^2(z) = -0.17 + 0.44i$$

$$\blacksquare f^3(z) = -0.3647 + 0.2504i$$



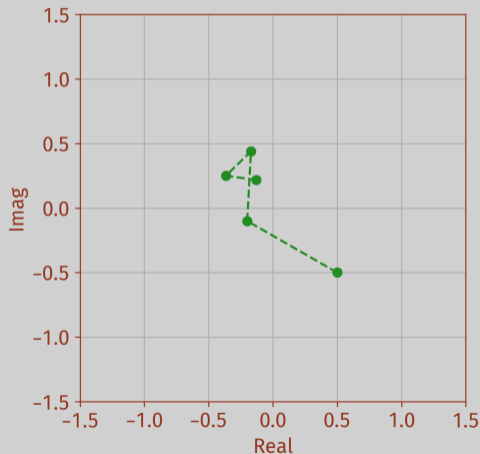
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- $f^2(z) = -0.17 + 0.44i$
- $f^3(z) = -0.3647 + 0.2504i$
- $f^4(z) = -0.12969407 + 0.21735824i$



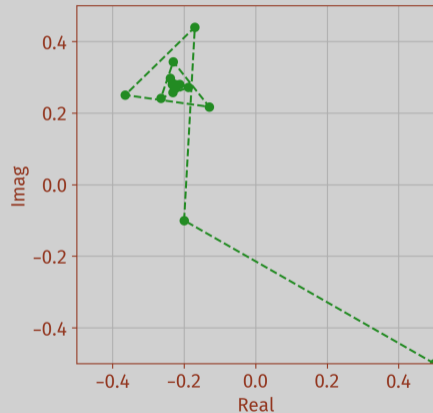
Iteration (Python)

```
1 N = 128
2 B = 4
3 c = complex(-0.2, 0.4)
4 def iterate(z):
5     for n in range(N):
6         z = z*z + c
7         if abs(z) > B*B: break
8     return n
```

IMPLEMENTATION

Iteration (Python)

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```



Complex Julia Set Example

Defined by iterative function in complex space

- $f_c(z) = z^2 - 0.675 - 0.112i$
- $K_c = \{z_0 \in \mathbb{C} : |f_c^k(z_0)| > B \text{ as } k \rightarrow \infty\}$

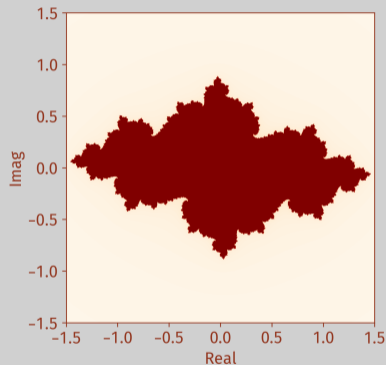


Figure: $f(z) = z^2 - 0.675 - 0.112i$

QUATERNIONS

QUATERNIONS HISTORY



Figure: Quaternion plaque on Brougham Bridge, Dublin
© Wikipedia - Cone83



Figure: Portrait of Sir William Rowan Hamilton
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QUATERNIONS

Definition (Quaternion)

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$\{d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$$

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■ $\mathbf{i}^2 = \mathbf{ijk}$

$$\mathbf{i}^{-1}\mathbf{i}^2 = \mathbf{i}^{-1}\mathbf{ijk}$$

$$\mathbf{i} = \mathbf{jk}$$

Definition (Quaternion)

$$i^2 = j^2 = k^2 = ijk = -1$$

$$\{d + ai + bj + ck : a, b, c, d \in \mathbb{R}\} \in \mathbb{H}$$

■ $i^2 = ijk$

$$i^{-1}i^2 = i^{-1}ijk$$

$$i = jk$$

■ $k^2 = ijk$

$$k^2k^{-1} = ijkk^{-1}$$

$$k = ij$$

■ $j = ki$

Definition (Quaternion)

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$$i = jk$$

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$$k^2k^{-1} = ijkk^{-1}$$

$$k = ij$$

$$\blacksquare j = ki$$

$$\blacksquare i = jk$$

$$ji = jjk$$

$$ji = j^2k$$

$$ji = -k$$

$$-k = ji$$

Definition (Quaternion)

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$$k = ij$$

$$\blacksquare j = ki$$

$$\blacksquare i = jk$$

$$ji = jkj$$

$$ji = j^2k$$

$$ji = -k$$

$$-k = ji$$

$$\blacksquare -i = kj$$

$$\blacksquare -j = ik$$

Let,

$$i^2 = j^2 = k^2 = ijk = -1$$

$$p = d + ai + bj + ck$$

$$q = w + xi + yj + zk$$

$$\begin{aligned} p * q &= dw + dxi + dyj + dzk \\ &+ awi + axi^2 + ayij + azik \\ &+ bwj + bxji + byj^2 + bzjk \\ &+ cwk + cxki + cykj + czk^2 \end{aligned}$$

Let,

$$i^2 = j^2 = k^2 = ijk = -1$$

$$p = d + ai + bj + ck$$

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$$\begin{aligned} p * q &= dw + dxi + dyj + dzk \\ &+ awi + axi^2 + ayij + azik \\ &+ bwj + bxji + byj^2 + bzjk \\ &+ cwk + cxki + cykj + czk^2 \\ &= dw - ax - by - cz \\ &+ dxi + awi + bzi - cyi \\ &+ dyj - azj + bwj + cxj \\ &+ dzk + ayk - bxk + cwk \end{aligned}$$

Let,

$$i^2 = j^2 = k^2 = ijk = -1$$

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$$\begin{aligned} p * q &= dw - ax - by - cz \\ &+ dxi + awi + bzi - cyi \\ &+ dyj - azj + bwj + cxj \\ &+ dzk + ayk - bxk + cwk \end{aligned}$$

Let,

$$i^2 = j^2 = k^2 = ijk = -1$$

$$p = d + ai + bj + ck$$

$$q = w + xi + yj + zk$$

$$\begin{aligned} p * q &= dw - (ax + by + cz) \\ &+ dxi + awi + bzi - cyi \\ &+ dyj - azj + bwj + cxj \\ &+ dzk + ayk - b xk + cwk \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &+ dx\mathbf{i} + aw\mathbf{i} + bz\mathbf{i} - cy\mathbf{i} \\ &+ dy\mathbf{j} - az\mathbf{j} + bw\mathbf{j} + cx\mathbf{j} \\ &+ dz\mathbf{k} + ay\mathbf{k} - bx\mathbf{k} + cw\mathbf{k} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$p * q = dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ + \begin{pmatrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \\ &\quad + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + \left(d \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \dots \right) \cdot \left\langle \begin{matrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{matrix} \right\rangle \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\quad + \begin{pmatrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{pmatrix} \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \\ &= dw - \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \left(d \begin{pmatrix} x \\ y \\ z \end{pmatrix} + w \begin{pmatrix} a \\ b \\ c \end{pmatrix} \dots \right) \cdot \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \end{aligned}$$

Let,

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

$$p = d + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$\begin{aligned} p * q &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \\ &\quad + \left\langle \begin{matrix} dx + aw + bz - cy \\ dy - az + bw + cx \\ dz + ay - bx + cw \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{matrix} \right\rangle \\ &= dw - \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \cdot \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + \left(d \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle + w \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle + \left\langle \begin{matrix} a \\ b \\ c \end{matrix} \right\rangle \times \left\langle \begin{matrix} x \\ y \\ z \end{matrix} \right\rangle \right) \cdot \left\langle \begin{matrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{matrix} \right\rangle \end{aligned}$$

QUATERNION ITERATIVE METHODS

Quaternion Multiplication

```
1 def q_mult(p, q):  
2     r = Quat(  
3         p.w*q.w - p.x*q.x - p.y*q.y - p.z*q.z,  
4         p.w*q.x + p.x*q.w + p.y*q.z - p.z*q.y,  
5         p.w*q.y - p.x*q.z + p.y*q.w + p.y*q.x,  
6         p.w*q.z + p.x*q.y - p.y*q.x + p.z*q.w  
7     )  
8     return r
```

Quaternion Square

```
1 def q_square(q):  
2     r = Quat(  
3         q.w*q.w - q.x*q.x - q.y*q.y - q.z*q.z,  
4         2*q.w*q.x,  
5         2*q.w*q.y,  
6         2*q.w*q.z  
7     )  
8     return r
```

Quaternion Add

```
1 def q_add(p, q):  
2     r = Quat(  
3         p.w + q.w,  
4         p.x + q.x,  
5         p.y + q.y,  
6         p.z + q.z  
7     )  
8     return r
```

Quaternion Add

```
1 def q_add(p, q):  
2     r = Quat(  
3         p.w + q.w,  
4         p.x + q.x,  
5         p.y + q.y,  
6         p.z + q.z  
7     )  
8     return r
```

Quaternion Magnitude

```
1 def q_abs(q):  
2     return q.w*q.w +  
3         q.x*q.x +  
4         q.y*q.y +  
5         q.z*q.z
```

IMPLEMENTATION

Iteration

```
1 N = 12
2 B = 16
3 q = Quat(-0.2, 0.4, -0.4, -0.4)
4 def iterate(z):
5     for n in range(N):
6         z = q_add(q_square(z), q)
7         if q_abs(z) > B*B: break
8     return n
```

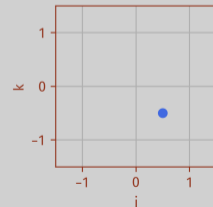
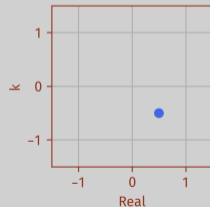
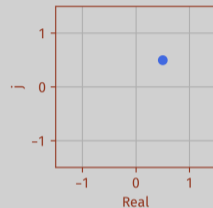
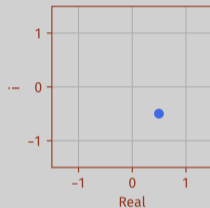
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

■ $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$



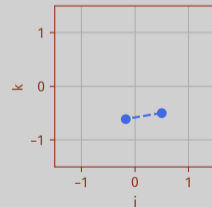
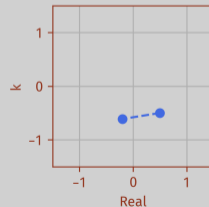
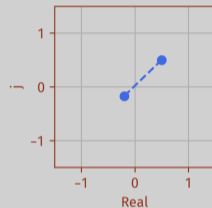
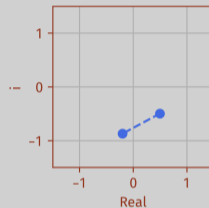
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$



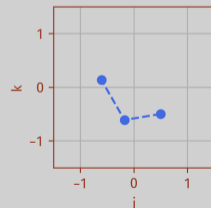
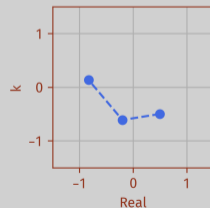
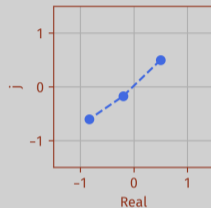
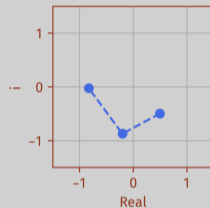
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$



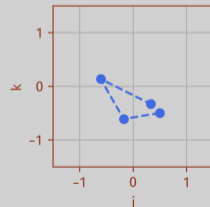
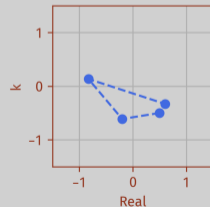
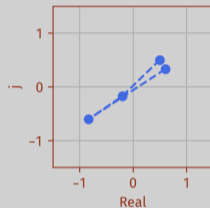
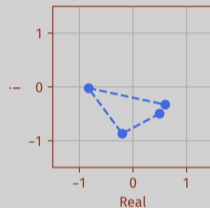
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$
- $f^3(z) \approx 0.6066 - 0.333i + 0.330j - 0.333k$



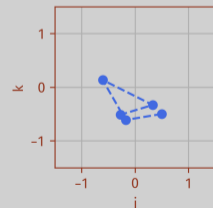
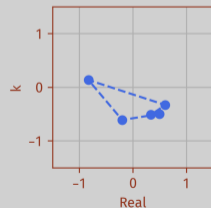
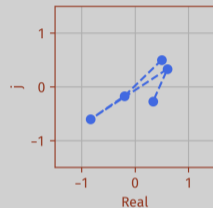
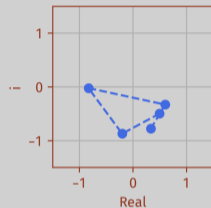
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$
- $f^3(z) \approx 0.6066 - 0.333i + 0.330j - 0.333k$
- $f^4(z) \approx 0.336 - 0.779i - 0.274j - 0.515k$



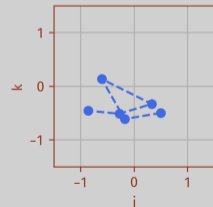
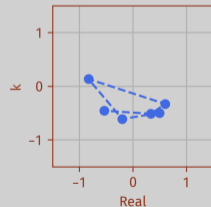
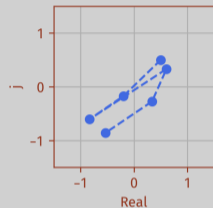
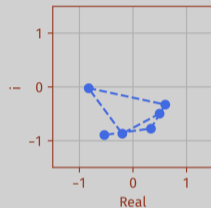
ITERATION

Rules

$$f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$$

$$z_0 = 0.5 - 0.5i + 0.5j - 0.5k$$

- $f^0(z) = 0.5 - 0.5i + 0.5j - 0.5k$
- $f^1(z) = -0.2 - 0.875i - 0.175j - 0.612k$
- $f^2(z) = -0.831 - 0.025i - 0.605j + 0.133k$
- $f^3(z) \approx 0.6066 - 0.333i + 0.330j - 0.333k$
- $f^4(z) \approx 0.336 - 0.779i - 0.274j - 0.515k$
- $f^5(z) \approx -0.535 - 0.899i - 0.860j - 0.458k$



PLOTTING

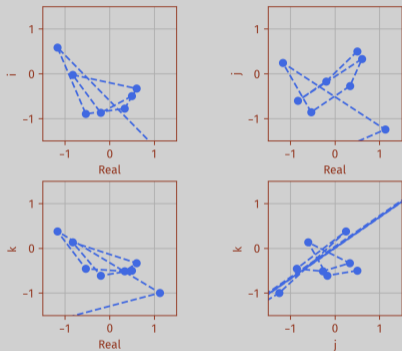


Figure: $f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$
 $f^{12}(0.5 - 0.5i + 0.5j - 0.5k)$

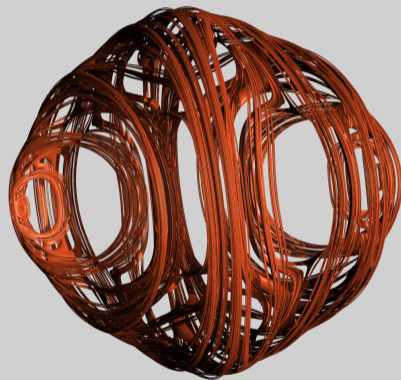


Figure: $f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$
Axis: Real, i, j

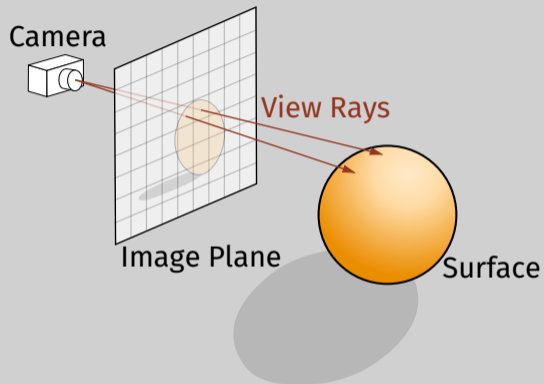


Figure: Ray Tracing Diagram

© ⓘ ⓘ Wikipedia - Henrik with modifications

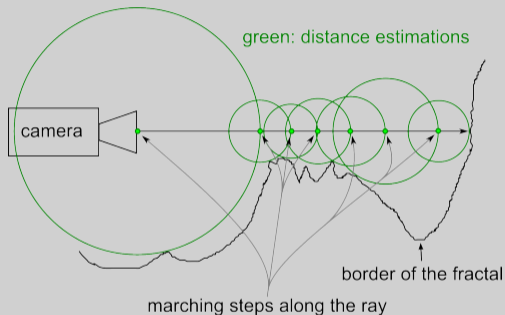


Figure: Ray Marching
©Adam Celarek

Theorem (Distance Estimation)

Let $f(z) = z^m + q$ where $q \in \mathbb{H}$, $m \in \mathbb{Z}^+$ be the iterative function of a Julia fractal. Then, the distance, δ , to the fractal can be approximated by

$$a \frac{z_n}{z'_n} < \delta$$

where $a \in \mathbb{R}$ is some constant coefficient.

NORMAL ESTIMATION

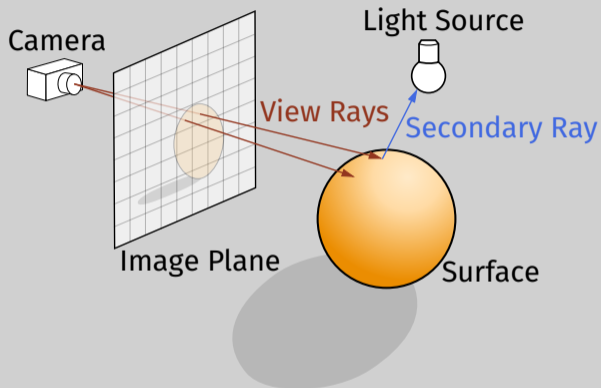


Figure: Ray Tracing Diagram

© ⓘ ⓘ Wikipedia - Henrik with modifications

Quaternion Julia Set Example

- Defined by iterative function in 4D Quaternion space

Figure: $f(z) = z^2 + 0.3 - 0.375i - 0.675j - 0.112k$

Axis: Real, i, j

N = 1-11

- 1 Iteration
- 2 Complex Iterative Methods
- 3 Quaternions
- 4 Quaternion Iterative Methods

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Accessed: 2023-02-10.

QUESTIONS?

<https://github.com/scrifulufugus/senior-synthesis>

