Useful recurrence relations for multidimensional volumes and monomial integrals*

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June 12, 2020

This article gives closed formulas and recurrence expressions for many n-dimensional volumes and monomial integrals. The recurrence expressions are often much simpler, more instructive, and better suited for numerical computation.

n-dimensional unit cube

$$C_n = \{(x_1, \dots, x_n) : -1 \le x_i \le 1\}$$

- Volume.

$$|C_n| = 2^n = \begin{cases} 1 & \text{if } n = 0 \\ |C_{n-1}| \times 2 & \text{otherwise} \end{cases} \tag{1}$$

- Monomial integration.

$$\begin{aligned} I_{k_1,\dots,k_n} &= \int_{C_n} x_1^{k_1} \cdots x_n^{k_n} \\ &= \prod_{i=1}^{n} \frac{1 + (-1)^{k_i}}{k_i + 1} = \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ |C_n| & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times \frac{k_{i_0}-1}{k_{i_0}+1} & \text{if } k_{i_0} > 0 \end{cases} \end{aligned} \tag{2}$$

n-dimensional unit simplex

$$T_n = \left\{ (x_1, \dots, x_n) : x_i \ge 0, \sum_{i=1}^{n} x_i \le 1 \right\}$$

*The LaTeX sources of this article are on `https://github.com/nschloe/useful-recurrence-relations`

- Volume.

$$|T_n| = \frac{1}{n!} = \begin{cases} 1 & \text{if } n = 0 \\ |T_{n-1}| \times \frac{1}{n} & \text{otherwise} \end{cases} \tag{3}$$

- Monomial integration.

$$\begin{aligned} I_{k_1,\dots,k_n} &= \int_{T_n} x_1^{k_1} \cdots x_n^{k_n} \\ &= \frac{\prod_{i=1}^n \Gamma(k_i + 1)}{\Gamma\left(n + 1 + \sum_{i=1}^n k_i\right)} && (4) \\ &= \begin{cases} |T_n| & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-1,\dots,k_n} \times \frac{k_{i_0}}{n+\sum_{i=1}^n k_i} & \text{if } k_{i_0} > 0 \end{cases} && (5) \end{aligned}$$

Remark. *Note that both numerator and denominator in expression* (4) *will assume very large values even for polynomials of moderate degree. This can lead to difficulties when evaluating the expression on a computer; the registers will overflow. A common countermeasure is to use the log-gamma function,*

$$\frac{\prod_{i=1}^n \Gamma(k_i)}{\Gamma\left(\sum_{i=1}^n k_i\right)} = \exp\left(\sum_{i=1}^n \ln\Gamma(k_i) - \ln\Gamma\left(\sum_{i=1}^n k_i\right)\right),$$

but a simpler and arguably more elegant solution is to use the recurrence (5)*. This holds true for all such expressions in this note.*

n-dimensional unit sphere

$$U_n = \left\{ (x_1, \dots, x_n) : \sum_{i=1}^n x_i^2 = 1 \right\}$$

See also [2].

- Volume.

$$|U_n| = \frac{n\sqrt{\pi}^n}{\Gamma(\frac{n}{2} + 1)} = \begin{cases} 2 & \text{if } n = 1 \\ 2\pi & \text{if } n = 2 \\ |U_{n-2}| \times \frac{2\pi}{n-2} & \text{otherwise} \end{cases} \tag{6}$$

- Monomial integral [1].

$$\begin{aligned} I_{k_1,\dots,k_n} &= \int_{U_n} x_1^{k_1} \cdots x_n^{k_n} \\ &= \frac{2\prod_{i=1}^n \Gamma\left(\frac{k_i+1}{2}\right)}{\Gamma\left(\sum_{i=1}^n \frac{k_i+1}{2}\right)} \qquad (7) \\ &= \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ |U_n| & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times \frac{k_{i_0}-1}{n-2+\sum_{i=1}^n k_i} & \text{if } k_{i_0} > 0 \end{cases} \qquad (8) \end{aligned}$$

n-dimensional unit ball

$$S_n = \left\{(x_1, \dots, x_n) : \sum_{i=1}^n x_i^2 \le 1\right\}$$

- Volume.

$$|S_n| = \frac{\sqrt{\pi}^n}{\Gamma(\frac{n}{2}+1)} = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ |S_{n-2}| \times \frac{2\pi}{n} & \text{otherwise} \end{cases} \qquad (9)$$

- Monomial integral [1].

$$\begin{aligned} I_{k_1,\dots,k_n} &= \int_{S_n} x_1^{k_1} \cdots x_n^{k_n} \\ &= \frac{2^{n+p}}{n+p}|S_n| = \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ |S_n| & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times \frac{k_{i_0}-1}{n+p} & \text{if } k_{i_0} > 0 \end{cases} \qquad (10) \end{aligned}$$

with $p = \sum_{i=1}^n k_i$.

n-dimensional unit ball with Gegenbauer weight

$\lambda > -1$. (Compare with (9) for $\lambda = 0$.) See A.1 for a proof.

- Volume.

$$\begin{aligned} |G_n^\lambda| &= \int_{S^n} \left(1 - \sum_{i=1}^n x_i^2\right)^\lambda \\ &= \frac{\Gamma(1+\lambda)\sqrt{\pi}^n}{\Gamma\left(1+\lambda+\frac{n}{2}\right)} = \begin{cases} 1 & \text{for } n = 0 \\ B\left(\lambda+1, \frac{1}{2}\right) & \text{for } n = 1 \\ |G_{n-2}^\lambda| \times \frac{2\pi}{2\lambda+n} & \text{otherwise} \end{cases} \qquad (11) \end{aligned}$$

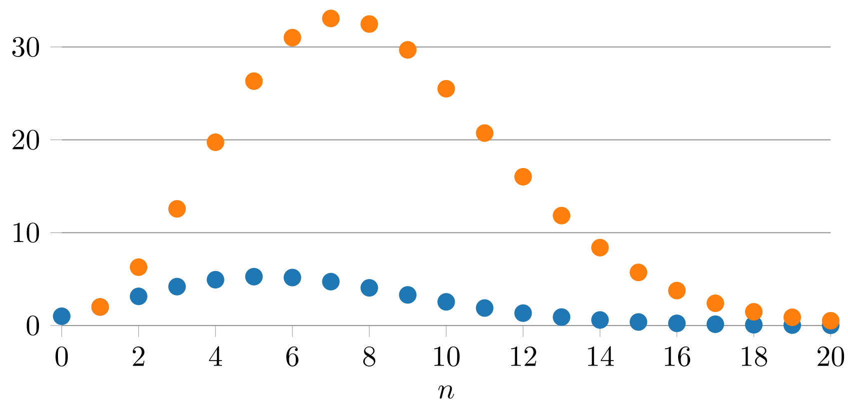

Figure 1: The volumes of the n-dimensional ball (and sphere) mysteriously peak at 5 (and 7, respectively). The recurrence relations (6) and (9) make it obvious why: The factor $\frac{2\pi}{n}$ $(\frac{2\pi}{n-2})$ becomes smaller than 1.

- Monomial integration.

$$\begin{aligned} I_{k_1,\dots,k_n} &= \int_{S^n} x_1^{k_1}\cdots x_n^{k_n}\left(1-\sum_{i=1}^n x_i^2\right)^\lambda \\ &= \frac{\Gamma(1+\lambda)\prod_{i=1}^n \Gamma\left(\frac{k_i+1}{2}\right)}{\Gamma\left(1+\lambda+\sum_{i=1}^n \frac{k_i+1}{2}\right)} &(12)\\ &= \begin{cases} 0 & \text{if any } k_i \text{ is odd}\\ |G_n^\lambda| & \text{if all } k_i = 0\\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times \frac{k_{i_0}-1}{2\lambda+n+\sum_{i=1}^n k_i} & \text{if } k_{i_0} > 0 \end{cases} &(13) \end{aligned}$$

n-dimensional unit ball with Chebyshev-1 weight

Gegenbauer with $\lambda = -\frac{1}{2}$.

- Volume.

$$\begin{aligned} |G_n^{-1/2}| &= \int_{S^n} \frac{1}{\sqrt{1-\sum_{i=1}^n x_i^2}} \\ &= \frac{\sqrt{\pi}^{n+1}}{\Gamma\left(\frac{n+1}{2}\right)} = \begin{cases} 1 & \text{if } n = 0\\ \pi & \text{if } n = 1\\ |G_{n-2}^{-1/2}| \times \frac{2\pi}{n-1} & \text{otherwise} \end{cases} &(14) \end{aligned}$$

- Monomial integration.

$$
\begin{aligned}
I_{k_1,\dots,k_n} &= \int_{S^n} \frac{x_1^{k_1}\cdots x_n^{k_n}}{\sqrt{1-\sum_{i=1}^n x_i^2}} \\
&= \frac{\sqrt{\pi}\prod_{i=1}^n \Gamma\left(\frac{k_i+1}{2}\right)}{\Gamma\left(\frac{1}{2}+\sum_{i=1}^n \frac{k_i+1}{2}\right)} && (15) \\
&= \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ |G_n^{-1/2}| & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times \frac{k_{i_0}-1}{n-1+\sum_{i=1}^n k_i} & \text{if } k_{i_0} > 0 \end{cases} && (16)
\end{aligned}
$$

n-dimensional unit ball with Chebyshev-2 weight

Gegenbauer with $\lambda = +\frac{1}{2}$.

- Volume.

$$
\begin{aligned}
|G_n^{+1/2}| &= \int_{S^n} \sqrt{1-\sum_{i=1}^n x_i^2} \\
&= \frac{\sqrt{\pi}^{n+1}}{2\Gamma\left(\frac{n+3}{2}\right)} = \begin{cases} 1 & \text{if } n = 0 \\ \frac{\pi}{2} & \text{if } n = 1 \\ |G_{n-2}^{+1/2}| \times \frac{2\pi}{n+1} & \text{otherwise} \end{cases} && (17)
\end{aligned}
$$

- Monomial integration.

$$
\begin{aligned}
I_{k_1,\dots,k_n} &= \int_{S^n} x_1^{k_1}\cdots x_n^{k_n}\sqrt{1-\sum_{i=1}^n x_i^2} \\
&= \frac{\sqrt{\pi}\prod_{i=1}^n \Gamma\left(\frac{k_i+1}{2}\right)}{2\Gamma\left(\frac{3}{2}+\sum_{i=1}^n \frac{k_i+1}{2}\right)} && (18) \\
&= \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ |G_n^{+1/2}| & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times \frac{k_{i_0}-1}{n+1+\sum_{i=1}^n k_i} & \text{if } k_{i_0} > 0 \end{cases} && (19)
\end{aligned}
$$

n-dimensional generalized Laguerre volume

$\alpha > -1$. See A.2 for a proof.

- Volume.

$$\begin{aligned} V_n &= \int_{\mathbb{R}^n} \left(\sqrt{x_1^2+\cdots+x_n^2}\right)^\alpha \exp\left(-\sqrt{x_1^2+\cdots+x_n^2}\right) \\ &= \frac{2\sqrt{\pi}^n\Gamma(n+\alpha)}{\Gamma(\frac{n}{2})} = \begin{cases} 2\Gamma(1+\alpha) & \text{if } n=1 \\ 2\pi\Gamma(2+\alpha) & \text{if } n=2 \\ V_{n-2}\times\frac{2\pi(n+\alpha-1)(n+\alpha-2)}{n-2} & \text{otherwise} \end{cases} \end{aligned} \tag{20}$$

- Monomial integration.

$$\begin{aligned} I_{k_1,\ldots,k_n} &= \int_{\mathbb{R}^n} x_1^{k_1}\cdots x_n^{k_n} \left(\sqrt{x_1^2+\cdots+x_n^2}\right)^\alpha \exp\left(-\sqrt{x_1^2+\cdots+x_n^2}\right) \\ &= \frac{2\Gamma\left(\alpha+n+\sum_{i=1}^n k_i\right)\left(\prod_{i=1}^n\Gamma\left(\frac{k_i+1}{2}\right)\right)}{\Gamma\left(\sum_{i=1}^n\frac{k_i+1}{2}\right)} \end{aligned} \tag{21}$$

$$= \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ V_n & \text{if all } k_i=0 \\ I_{k_1,\ldots,k_{i_0}-2,\ldots,k_n}\times\frac{(\alpha+n+p-1)(\alpha+n+p-2)(k_{i_0}-1)}{n+p-2} & \text{if } k_{i_0}>0 \end{cases} \tag{22}$$

with $p = \sum_{k=1}^n k_i$.

n-dimensional Hermite (physicists')

- Volume.

$$\begin{aligned} V_n &= \int_{\mathbb{R}^n} \exp\left(-(x_1^2+\cdots+x_n^2)\right) \\ &= \sqrt{\pi}^n = \begin{cases} 1 & \text{if } n=0 \\ \sqrt{\pi} & \text{if } n=1 \\ V_{n-2}\times\pi & \text{otherwise} \end{cases} \end{aligned} \tag{23}$$

- Monomial integration.

$$\begin{aligned} I_{k_1,\ldots,k_n} &= \int_{\mathbb{R}^n} x_1^{k_1}\cdots x_n^{k_n}\exp(-(x_1^2+\cdots+x_n^2)) \\ &= \prod_{i=1}^n \frac{(-1)^{k_i}+1}{2}\times\Gamma\left(\frac{k_i+1}{2}\right) \end{aligned} \tag{24}$$

$$= \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ V_n & \text{if all } k_i=0 \\ I_{k_1,\ldots,k_{i_0}-2,\ldots,k_n}\times\frac{k_{i_0}-1}{2} & \text{if } k_{i_0}>0 \end{cases} \tag{25}$$

n-dimensional Hermite (probabilists')

- Volume.

$$V_n = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2}(x_1^2 + \cdots + x_n^2)\right) = 1 \tag{26}$$

- Monomial integration.

$$\begin{aligned} I_{k_1,\dots,k_n} &= \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} x_1^{k_1} \cdots x_n^{k_n} \exp\left(-\frac{1}{2}(x_1^2 + \cdots + x_n^2)\right) \\ &= \prod_{i=1}^n \frac{(-1)^{k_i} + 1}{2} \times \frac{2^{\frac{k_i+1}{2}}}{\sqrt{2\pi}} \Gamma\left(\frac{k_i+1}{2}\right) \qquad (27) \\ &= \begin{cases} 0 & \text{if any } k_i \text{ is odd} \\ V_n & \text{if all } k_i = 0 \\ I_{k_1,\dots,k_{i_0}-2,\dots,k_n} \times (k_{i_0} - 1) & \text{if } k_{i_0} > 0 \end{cases} \qquad (28) \end{aligned}$$

A. Some proofs

A.1. Gegenbauer

Proof.

$$\begin{aligned} \int_{\mathbb{R}^n} x^{k_1} \cdots x^{k_n} \left(1 - \sum_{i=1}^n x_i^2\right)^\lambda dx &= \int_{S_n} \int_0^1 r^{n-1} r^{\sum k_i} (1-r^2)^\lambda x'^{k_1} \cdots x_n'^{k_n}\, dr\, d\sigma(x') \\ &= \int_0^1 r^{n-1} r^{\sum k_i} (1-r^2)^\lambda\, dr \times \int_{S_n} x'^{k_1} \cdots x_n'^{k_n}\, d\sigma(x') \end{aligned}$$

with $x_i' = x_i/r$. The one-dimensional integral in r can be evaluated explicitly such that, with the spherical integral taken from (7),

$$\begin{aligned} I_{k_1,\dots,k_n} &= \frac{\Gamma\left(\frac{n+\sum k_i}{2}\right)\Gamma(1+\lambda)}{2\Gamma\left(\frac{n+\sum k_i}{2} + \lambda + 1\right)} \times \frac{2\prod_{i=1}^n \Gamma\left(\frac{k_i+1}{2}\right)}{\Gamma\left(\sum_{i=1}^n \frac{k_i+1}{2}\right)} \\ &= \frac{\Gamma(1+\lambda)\prod_{i=1}^n \Gamma\left(\frac{k_i+1}{2}\right)}{\Gamma\left(\sum \frac{k_i+1}{2} + \lambda + 1\right)}. \end{aligned}$$

□

A.2. Generalized Laguerre

Proof.

$$\begin{aligned}\int_{\mathbb{R}^n} x^{k_1} \cdots x^{k_n} r^{\alpha} \exp(-r)\, dx &= \int_{S_n} \int_0^{\infty} r^{n-1} r^{\sum k_i} r^{\alpha} \exp(-r) x'^{k_1} \cdots x_n'^{k_n}\, dr\, d\sigma(x') \\ &= \int_0^{\infty} r^{n-1} r^{\sum k_i} r^{\alpha} \exp(-r)\, dr \times \int_{S_n} x'^{k_1} \cdots x_n'^{k_n}\, d\sigma(x')\end{aligned}$$

with $x_i' = x_i/r$. The one-dimensional integral in r can be evaluated explicitly such that, with the spherical integral taken from (7),

$$I_{k_1,\ldots,k_n} = \Gamma\left(\alpha + n + \sum k_i\right) \times \frac{2 \prod_{i=1}^{n} \Gamma\left(\frac{k_i+1}{2}\right)}{\Gamma\left(\sum_{i=1}^{n} \frac{k_i+1}{2}\right)}.$$

□

References

[1] Gerald B. Folland. How to integrate a polynomial over a sphere. *The American Mathematical Monthly*, 108(5):446–448, May 2001.

[2] Michael Hartl. The tau manifesto, 2010. URL:https://tauday.com/tau-manifesto.