COMPLETION OF (QUASI-)EXCELLENT LOCAL DOMAINS OF CHARACTERISTIC p

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1. Summary

The following result is due to Susan Loepp. Part *(ii)* is [Loe03, Theorem 9] and part *(i)* follows from the same proof.

Theorem 1.1. *Let T be a Noetherian complete local ring flat over $\mathbf{Z}$. Then the following hold.*

(i) T is the completion of a quasi-excellent local domain if and only if T is reduced.
(ii) T is the completion of an excellent local domain if and only if T is reduced and equidimensional.

In this note, we discuss the characteristic p case. In the rest of this note, all rings contain $\mathbf{F}_p$ where p is a prime number.

Definition 1.2. Let K be a field. We say K is *F-maximal* if a basis of the vector space $\Omega_{K/\mathbf{F}_p}$ has the same cardinality as K.

We will prove

Theorem 1.3. *Let T be a Noetherian complete local ring. Assume that $\kappa(P) = T_P/PT_P$ is F-maximal for all $P \in \operatorname{Min}(T)$. Then the following hold.*

(i) T is the completion of a quasi-excellent local domain if and only if T is reduced.
(ii) T is the completion of a excellent local domain if and only if T is reduced and equidimensional.

Let T be a reduced Noetherian complete local ring and let k be a coefficient field of T. Let k' be the field obtained from k by adding $|k|^{\aleph_0}$ variables. Then $T' := T\widehat{\otimes}_k k'$ is a complete local T-algebra so that $T \to T'$ is regular [Stacks, Tag 07PM], in particular T' is reduced. By Cohen–Gabber [GO08, Théorème 7.1] $\kappa(P')$ is separable over an isomorphic copy of k' for all $P' \in \operatorname{Min}(T')$, so $\kappa(P')$ is F-maximal, as $|T'/P'| = |T'| = |k'| = |k|^{\aleph_0}$, cf. [Loe03, Proof of Lemma 3].

For every $P \in \operatorname{Min}(T)$, let T_1 be the normalization of T/P, which is a finite T-algebra [Stacks, Tag 0335]. Let k_1 be the residue field of T_1. As $k_1 \otimes_k k'$ is a field we see $T_1 \otimes_T T'$ is local, and it is normal as $T \to T'$ is regular. Therefore $T_1 \otimes_T T'$ is a local domain. This discussion tells us if T is equidimensional, so is T'. We see

Corollary 1.4. *Let (T, M) be a reduced Noetherian complete local ring. Then there exists a flat local map $T \to T'$ of Noetherian complete local rings so that the following hold.*

(i) *MT' is the maximal ideal of T'.*
(ii) *The residue field of T' is purely transcendental over that of T.*
(iii) *T' is the completion of a quasi-excellent domain A'.*

If T is also equidimensional, then A' is excellent.

In contrast with the characteristic $(0, 0)$ or $(0, p)$ case, we have

Theorem 1.5. *Let A be a Nagata local ring with F-finite residue field. Then A is F-finite, in particular excellent.*

This follows from the proof of [Sey80, Corollaire 1.1.2] where $I = 0$.

As a quasi-excellent ring is Nagata [Stacks, Tag 07QV] we see if A is a quasi-excellent local domain with F-finite residue field, then A is excellent. By [Stacks, Tag 0AW6] $A^\wedge$ is equidimensional. Therefore $k[[x, y, z]]/(x, y) \cap (z)$ is the completion of a quasi-excellent domain when $k = \mathbf{Q}$ or a purely transcendental extension of $\mathbf{F}_p$ of transcendence degree $\kappa^{\aleph_0}$ for any infinite cardinal κ, but not when $k = \mathbf{F}_p$.

2. Proof of Theorem 1.3

Similar to SQA-subrings [Loe03, p.223, Definition] we have

Definition 2.1. Let (T, M) be a Noetherian complete local ring. We say a local subring $(R, M \cap R)$ of T is a *small separably Q-avoiding subring* (abbv. SSQA-subring) if the following hold.

(1) $|R| < |T|$.
(2) $Q \cap R = 0$ for all $Q \in \mathrm{Ass}(T)$.
(3) The field extension $\kappa(Q)/\kappa(0)$ is separable for all $Q \in \mathrm{Ass}(T)$.

Here $\kappa(0)$ is just the fraction field of R.

Lemma 2.2. *Let (T, M) be a reduced Noetherian complete local ring of dimension at least 1. Assume that $\kappa(Q)$ is F-maximal for all $Q \in \mathrm{Ass}(T)$.*

Let J be an ideal of T such that $J \not\subseteq Q$ for all $Q \in \mathrm{Ass}(T)$. Let R be an SSQA-subring of T with fraction field K and let $u + J \in T/J$.

Then there exists an infinite SSQA-subring S of T such that $R \subseteq S \subseteq T$ and $u + J$ is in the image of the map $S \to T/J$. Moreover, if $u \in J$, then $J \cap S \neq 0$.

Proof. Take, by prime avoidance, an element $a \in J$ so that $a \notin Q$ for all $Q \in \mathrm{Ass}(T)$. For each $Q \in \mathrm{Ass}(T)$, we will find an element $b_Q \in T$ so that $b_Q \notin Q$, $b_Q \in P$ for all $P \in \mathrm{Ass}(T) \setminus \{Q\}$, and that $S = R[u + a \sum_Q b_Q]_{M \cap R[u + a \sum_Q b_Q]}$ is the desired subring.

Let $Q \in \operatorname{Ass}(T)$. From the Leibniz Rule $\mathrm{d}(xy) = x\mathrm{d}y + y\mathrm{d}x$ we see

$$\{\mathrm{d}x \mid x \in P \text{ for all } P \in \operatorname{Ass}(T) \setminus \{Q\}\}$$

generates $\Omega_{\kappa(Q)/\mathbf{F}_p}$ as a $\kappa(Q)$-vector space. Let V_Q be the subspace of $\Omega_{\kappa(Q)/\mathbf{F}_p}$ generated by $\{\mathrm{d}x \mid x \in K\} \cup \{\mathrm{d}u, \mathrm{d}a\}$. As $\kappa(Q)$ is F-maximal and as $|R| < |T| = |\kappa(Q)|$ we see $\dim V_Q < |T| = \dim \Omega_{\kappa(Q)/\mathbf{F}_p}$. Thus there exists an element $b_Q \in T$ so that $b_Q \notin Q$, $b_Q \in P$ for all $P \in \operatorname{Ass}(T) \setminus \{Q\}$, $u + ab_Q \in \kappa(Q)$ is transcendental over K, and $\mathrm{d}b_Q \notin V_Q$.

Write $v = u + a\sum_Q b_Q$, $S = R[v]_{M \cap R[v]}$. Then $v \in \kappa(Q)$ is transcendental over K for all Q, so S is a SQA-subring (see [Loe03, Proof of Lemma 5]) with fraction field $L = K(v)$. In $\Omega_{\kappa(Q)/\mathbf{F}_p}$ we have

$$\mathrm{d}v = \mathrm{d}u + b_Q\mathrm{d}a + a\mathrm{d}b_Q.$$

As $a \notin Q$ and as $\mathrm{d}b_Q \notin V_Q$, we see $\mathrm{d}v$ is not in the subspace of $\Omega_{\kappa(Q)/\mathbf{F}_p}$ generated by $\{\mathrm{d}x \mid x \in K\}$. As $\kappa(Q)/K$ is separable, so is $\kappa(Q)/L$, as separability can be detected with the module of differentials [Stacks, Tag 031X]. Therefore S is a SSQA-subring. As $v \neq 0$, we have $J \cap S \neq 0$ if $u \in J$. □

Note that in the situation described in the first paragraph of [Loe03, Proof of Lemma 6], the extension $R \to S$ is birational, therefore S is a SSQA-subring if R is. In the second paragraph, similar to the proof of Lemma 2.2 we can find the element $t \in T$ such that $x_1 + Q$ is transcendental over K, the fraction field of R, for all $Q \in \operatorname{Ass}(T)$, and that $\mathrm{d}x_1$ is not in the subspace of of $\Omega_{\kappa(Q)/\mathbf{F}_p}$ generated by $\{\mathrm{d}x \mid x \in K\}$. Therefore we have

Lemma 2.3. *Let (T, M) be a reduced Noetherian complete local ring of dimension at least 1. Assume that $\kappa(Q)$ is F-maximal for all $Q \in \operatorname{Ass}(T)$.*

Let R be an SSQA-subring of T. Let I be a finitely generated ideal of R and $c \in IT \cap R$. Then there exists an SSQA-subring S of T such that $R \subseteq S \subseteq T$ and $c \in IS$.

Note that if F is a field and $(F_\alpha)_\alpha$ is a filtered family of subfields so that F/F_α is separable for all α, then $F/\bigcup_\alpha F_\alpha$ is separable (cf. [Stacks, Tag 031X]). Therefore the proof of Lemma 7 (resp. Lemma 8) in [Loe03] works verbatim with Lemmas 5 and 6 replaced Lemmas 2.2 and 2.3, proving Lemma 7 with SQA replaced by SSQA (resp. Lemma 8 with the additional conclusion that $\kappa(Q)$ is separable over the fraction field of A for all $Q \in \operatorname{Ass}(T)$), with the additional assumption that $\kappa(Q)$ is F-maximal for all $Q \in \operatorname{Ass}(T)$.

Finally, as separability is the same as geometric regularity for field extensions [Stacks, Tag 0322], the proof of [Loe03, Theorem 9] shows our Theorem 1.3.

References

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