

ON FLENNER LOCAL BERTINI PAPER

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Kazuma Shimomoto had a discussion with me on Flenner's local Bertini paper [Fle77], especially about the proof of Lemma 4.5. The main point of concern was that, in the proof, there was the implicit assumption $x \in \mathfrak{p}$, which may not hold in the situation of concern in Satz 4.6. This is settled by Lemma 1.2 of this document, which may be well-known to the expected reader in the 70s; they use complete modules of differentials way more often than we do.

In any case, after looking at this subject for a short while, I realized there is an extension (Lemma 1.3) of Lemma 1.2, which leads to a different proof of Flenner's Satz 4.6.

1. INDEPENDENCE IN MODULE OF DIFFERENTIALS

Theorem 1.1. *Let k be a field. Let A be a complete Noetherian local domain containing k as a coefficient field and let $d_{A/k} : A \rightarrow \widehat{\Omega}_{A/k}$ be the complete module of differentials. Let $y_1, y_2, \dots, y_d$ a system of parameters of A . Then the following hold.*

- (i) *If A is regular and y is a regular system of parameters, then $\widehat{\Omega}_{A/k}$ is a free module with a basis $d_{A/k}(y_1), d_{A/k}(y_2), \dots, d_{A/k}(y_d)$.*
- (ii) *Let E be the fraction field of A . Assume E is separable over $k((y_1, y_2, \dots, y_d))$ (e.g. if $\text{char } k = 0$). Then $d_{A/k}(y_1), d_{A/k}(y_2), \dots, d_{A/k}(y_d)$ is a basis of the vector space $\widehat{\Omega}_{A/k} \otimes_A E$.*

Proof. See the statement and proof of [EGA 0_{IV}, Proposition 21.9.5]. $\square$

Lemma 1.2. *Let k be a field of characteristic 0. Let A be a complete Noetherian local domain containing k as a coefficient field and let $d_{A/k} : A \rightarrow \widehat{\Omega}_{A/k}$ be the complete module of differentials. Let $y \in \mathfrak{m}_A$ be nonzero. Then $d_{A/k}(y)$ is nonzero in $\widehat{\Omega}_{A/k} \otimes_A E$, where E is the fraction field of A .*

Proof. We may extend y to a system of parameters $y = y_1, y_2, \dots, y_d$ of A where $d = \dim A$. Let $P = k[[y_1, y_2, \dots, y_d]]$ be a power series subring of A over which A is finite. Since $\text{char } k = 0$, the fraction field E of A is separable over that of P . Therefore Theorem 1.1 applies and we see $d_{A/k}(y_1) \neq 0 \in \widehat{\Omega}_{A/k} \otimes_A E$. $\square$

We can extend Lemma 1.2 to the case of two elements.

Lemma 1.3. *Let k be a field of characteristic 0. Let A be a complete Noetherian local domain containing k as a coefficient field and let $d_{A/k} : A \rightarrow \widehat{\Omega}_{A/k}$ be the complete module of differentials. Let $y_1, y_2 \in \mathfrak{m}_A$ be such that the complete local subring $k[[y_1, y_2]]$ of A ¹ has dimension 2. Then $d_{A/k}(y_1)$ and $d_{A/k}(y_2)$ are linearly independent in $\widehat{\Omega}_{A/k} \otimes_A E$, where E is the fraction field of A .*

Proof. By assumption $y_1, y_2 \neq 0$, so we can find $z_2, \dots, z_d$ such that y_1, z and y_2, z are systems of parameters of A . Let $R = k[[y_1, y_2]]$, regular of dimension 2, and let $P = R[[Z_2, \dots, Z_d]]$ be a power series ring of $d - 1$ variables over R . Then $\widehat{\Omega}_{P/k}$ is a free P -module with basis $d(y_1), d(y_2), d(z_2), \dots, d(z_d)$ (Theorem 1.1).

Let $\varphi : P \rightarrow A$ be the R -algebra map sending Z_j to z_j and let A_0 be the image. It is clear that $y_1, z_2, \dots, z_d$ is a system of parameters of A_0 , as both A and A_0 are finite over the regular subring $k[[y_1, z_2, \dots, z_d]]$. By Theorem 1.1 we therefore have $\widehat{\Omega}_{A_0/k} \otimes_{A_0} E = \widehat{\Omega}_{A/k} \otimes_A E$. Let E_0 be the

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¹That is, the image of the map $k[[Y_1, Y_2]] \rightarrow A$ from a power series ring of two variables sending Y_i to y_i .

fraction field of A_0 . Since the canonical base change map $\widehat{\Omega}_{A_0/k} \otimes_{A_0} E_0 \rightarrow \widehat{\Omega}_{A_0/k} \otimes_{A_0} E$ preserves linear independence, we may replace A by A_0 to assume φ is surjective. Then $\ker(\varphi)$ is generated by a single element $G \in \mathfrak{m}$ as P is a UFD, where $\mathfrak{m}$ is the maximal ideal of P . We see

$$\widehat{\Omega}_{A/k} \otimes_A E = (E d(y_1) \oplus E d(y_2) \oplus E d(z_2) \oplus \dots \oplus E d(z_d)) / E d(G).$$

We need to show $d(y_1)$ and $d(y_2)$ are linearly independent, *i.e.*, it is not possible that $\frac{\partial G}{\partial Z_j} = 0 \in E$ for all j , or that it is not possible that G divides $\frac{\partial G}{\partial Z_j}$ in P for all j .

Let G_{init} be the nonzero homogeneous component of G of the smallest degree δ . We separate two cases:

Case 1. Neither y_1 nor y_2 appears in G_{init} . As k is infinite, after a linear change of variables $Z'_j = \sum \gamma_{jk} Z_k$, we may assume that Z_2^δ appears in G_{init} . Then $G = Qu$, where Q is a Weierstrass polynomial in the variable Z_2 and $u \in P^\times$. Therefore A is finite over the subring $k[[y_1, y_2, z_3, \dots, z_d]]$, so $y_1, y_2, z_3, \dots, z_d$ is a system of parameters of A and $d(y_1), d(y_2), d(z_3), \dots, d(z_d)$ is a basis of $\widehat{\Omega}_{A/k} \otimes_A E$ (Theorem 1.1), as desired.

Case 2. Either y_1 or y_2 appears in G_{init} . We may assume y_2 appears in G_{init} by symmetry. As k is infinite, after a linear change of variables $y'_1 = y_1 + \beta_1 y_2, Z'_j = Z_j + \gamma_j y_2$, we may assume that y_2^δ appears in G_{init} . Then $G = Qu$, where Q is a Weierstrass polynomial in the variable y_2 and $u \in P^\times$. We may then assume

$$G = Q = y_2^\delta + a_{\delta-1}(y_1, \underline{Z}) y_2^{\delta-1} + \dots + a_0(y_1, \underline{Z}),$$

where a_ℓ are in the maximal ideal $\mathfrak{m}_1$ of $P_1 = k[[y_1, \underline{Z}]]$, in fact $a_\ell \in \mathfrak{m}_1^{\delta-\ell}$ as $G \in \mathfrak{m}^\delta$. Consequently

$$\frac{\partial G}{\partial Z_j} = \frac{\partial a_{\delta-1}}{\partial Z_j} y_2^{\delta-1} + \dots + \frac{\partial a_0}{\partial Z_j}.$$

If G divides $\frac{\partial G}{\partial Z_j}$ in P for some j , we claim that in fact $\frac{\partial G}{\partial Z_j} = 0 \in P$. If not, write $GH = \frac{\partial G}{\partial Z_j}$, let H_{init} be the nonzero homogeneous component of H of the smallest degree ρ , and let $y_2^\alpha m$ be the monomial in H_{init} with the largest possible α (and m a monomial in P_1). The coefficient of $y_2^{\delta+\alpha} m$ in $GH = \frac{\partial G}{\partial Z_j}$ is 0 as y_2 does not appear in $\frac{\partial a_\ell}{\partial Z_j} \in P_1$. We now note that $(a_{\delta-1} y_2^{\delta-1} + \dots + a_0)H$ does not contribute to this coefficient. This is because a monomial of $a_{\delta-1} y_2^{\delta-1} + \dots + a_0$ is of the form $y_2^\ell n$ where $n \in \mathfrak{m}_1^{\delta-\ell}$ where $\ell < \delta$. To make $y_2^{\delta+\alpha} m$ you need a monomial of the form $y_2^{\alpha+\delta-\ell} p$ with $pn = m$, whose degree is $\alpha + \delta - \ell + \deg p = \alpha + \delta - \ell + \deg m - \deg n \leq \alpha + \delta - \ell + \rho - \alpha - (\delta - \ell) = \rho$, hence cannot appear in H by the maximality of α . Hence the coefficient of $y_2^{\delta+\alpha} m$ in GH is $1 \neq 0$, a contradiction.

At this point, we see that if $d(y_1)$ and $d(y_2)$ are not linearly independent in $\widehat{\Omega}_{A/k} \otimes_A E$, then $\frac{\partial G}{\partial Z_j} = 0 \in P$ for all j . As $\text{char } k = 0$, this means that $G \in k[[y_1, y_2]]$, contradicting the fact $R \rightarrow A$ is injective, as desired. $\square$

Remark 1.4. One might wish to extend Lemmas 1.2 and 1.3 to more elements. In a more conceptual form, this is asking if we can prove the following statement.

(†) *Let $R \rightarrow A$ be an injective local map of complete Noetherian local domains inducing an isomorphism of residue fields. Let E be the fraction field of A . Let k be a coefficient field of R . Then the canonical map $\widehat{\Omega}_{R/k} \otimes_R E \rightarrow \widehat{\Omega}_{A/k} \otimes_A E$ is injective.*

Lemmas 1.2 and 1.3 are essentially the cases where $\dim R = 1$ and $\dim R = 2$, respectively. Note that if $R \rightarrow A$ is flat then the statement is clear from Theorem 1.1.

However, such an extension is impossible. The following counterexample was communicated to us by Linquan Ma, who got it from chatGPT. Since $1, t, \exp(t)$ are algebraically independent over $\mathbf{C}$, using homogeneity we see the subring $\mathbf{C}[[s, st, s \exp(t)]]$ of $\mathbf{C}[[s, t]]$ is of dimension 3. In fact,

for the same reason, you can find Noetherian complete local $\mathbf{C}$ -subalgebras of $\mathbf{C}[[s, t]]$ of arbitrary dimension. Hence $(\dagger)$ is false in general when $\dim R \geq 3$.

2. FLENNER'S SATZ 4.6

With the help of previous lemmas, let us give a new proof of Flenner's Satz 4.6 (modulo his Satz 1.6), which does not use his Lemma 4.5 [Fle77].

Theorem 2.1. *Let A be a complete Noetherian local ring containing a coefficient field k of characteristic 0. Let $d : A \rightarrow \widehat{\Omega}_{A/k}$ be the complete module of differentials. Let $x_1, \dots, x_n \in \mathfrak{m}_A$ where $n \geq 1$.*

Let $L = k(Y_1, \dots, Y_n)$ be the field of rational functions generated by variables $Y_1, \dots, Y_n$, and let $B = A \widehat{\otimes}_k L$, so $\widehat{\Omega}_{B/L} = \widehat{\Omega}_{A/k} \otimes_A B$.

Let U (resp. V) be the open subset of $\text{Spec } A$ (resp. $\text{Spec } B$) that is the union of the principal open subsets defined by $x_1, \dots, x_n$. Then for every $\mathfrak{q} \in V$ we have

(i) *The element $d(Y_1x_1 + \dots + Y_nx_n)$ is nonzero in $\widehat{\Omega}_{B/L} \otimes_B \kappa(\mathfrak{q})$.*

(ii) *The element $d(x_1 + Y_2x_2 + \dots + Y_nx_n)$ is nonzero in $\widehat{\Omega}_{B/L} \otimes_B \kappa(\mathfrak{q})$.*

Proof. Since L is also generated by $Y_1, Y_1Y_2, \dots, Y_1Y_n$ and $Y_1, Y_1^{-1}Y_2, \dots, Y_1^{-1}Y_n$ it is clear that for every fixed n (i) and (ii) are equivalent if proved for all $\mathfrak{q}$. When $n = 1$, (ii) follows from Lemma 1.2 applied to $B/\mathfrak{q}$, so we may assume $n \geq 2$.

We perform induction on n . The base case $n = 2$ will be dealt later. Assume (ii) is true for some $n \geq 2$ and let us show (i) for $n + 1$. We have $Y_1x_1 + \dots + Y_{n+1}x_{n+1} = (Y_1x_1 + Y_2x_2) + Y_3x_3 + \dots + Y_{n+1}x_{n+1}$. Therefore $d(Y_1x_1 + \dots + Y_{n+1}x_{n+1})$ is nonzero in $\widehat{\Omega}_{B/L} \otimes_B \kappa(\mathfrak{q})$ for all $\mathfrak{q} \in D(Y_1x_1 + Y_2x_2) \cup D(x_3) \cup \dots \cup D(x_{n+1})$, in particular for all $\mathfrak{q} \in D(x_3)$, as $n \geq 2$. By symmetry we have $d(Y_1x_1 + \dots + Y_{n+1}x_{n+1})$ is nonzero in $\widehat{\Omega}_{B/L} \otimes_B \kappa(\mathfrak{q})$ for all $\mathfrak{q} \in V$, as desired.

It remains to show (i) for $n = 2$. We may assume A is an integral domain and $\mathfrak{q} \cap A = 0$. Denote by E the fraction field of A . Replacing A by its normalization we may also assume A is normal. Note that x_1, x_2 are not both zero as $\mathfrak{q} \in V$, so the complete subring $R = k[[x_1, x_2]]$ satisfies $1 \leq \dim R \leq 2$. We separate two cases:

Case 1. $\dim R = 2$. By Lemma 1.3, $d(x_1)$ and $d(x_2)$ are linearly independent in $W := \widehat{\Omega}_{A/k} \otimes_A E$, hence so in $\widehat{\Omega}_{B/L} \otimes_B \kappa(\mathfrak{q}) = W \otimes_E \kappa(\mathfrak{q})$, as desired.

Case 2. $\dim R = 1$. Since A is normal, R is then contained in a complete local subring S of A of the form $k[[\pi]]$ where π is a nonzero element in the maximal ideal of A . We may then write $x_1 = u_1\pi^{a_1}$ and $x_2 = u_2\pi^{a_2}$ where $a_1, a_2 \in \mathbf{Z}_{>0}$ and $u_1, u_2 \in S^\times$. Since $\text{char } k = 0$ we see there exist $v_i \in S^\times$ such that $d(x_i) = v_i\pi^{a_i-1}d(\pi)$, so $d(Y_1x_1 + Y_2x_2) = (Y_1v_1\pi^{a_1-1} + Y_2v_2\pi^{a_2-1})d(\pi)$. Since the coefficients of v_i are in k we have $Y_1v_1\pi^{a_1-1} + Y_2v_2\pi^{a_2-1} \neq 0 \in L[[\pi]]$. Note that $L[[\pi]]$ is a subring of $B/\mathfrak{q}$, as if not, then π is nilpotent in $B/\mathfrak{q}$, contradicting $\mathfrak{q} \cap A = 0$. By Lemma 1.2, we even have $d(Y_1x_1 + Y_2x_2) = (Y_1v_1\pi^{a_1-1} + Y_2v_2\pi^{a_2-1})d(\pi)$ is nonzero in $\widehat{\Omega}_{(B/\mathfrak{q})/L} \otimes_{B/\mathfrak{q}} \kappa(\mathfrak{q})$, as desired. $\square$

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