

NOTES (OCTOBER 13)

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Made from handwritten notes by myself with chatGPT and editing.

REVIEW OF LAST MONTH

Let $I = (f_1, \dots, f_n)$ be a finitely generated ideal of a ring A . Write

$$R\Gamma_I A = [A \longrightarrow \prod_i A_{f_i} \longrightarrow \cdots \longrightarrow A_{f_1 \cdots f_n}],$$

where the displayed complex is in cohomological degrees $[0, n]$, in fact has projective amplitude $[0, n]$. $R\Gamma_I(A)$ can also be described as the cofiber of the canonical map $A \rightarrow R\Gamma(U, \mathcal{O}_U)$, where U is the complement of $V(I)$. Hence $R\Gamma_I(A)$ and the functors below depends only on $\sqrt{I}$. For $M \in D(A)$, set

$$R\Gamma_I M = R\Gamma_I A \otimes_A^{\mathbf{L}} M, \quad \widehat{M}^I = R\mathrm{Hom}_A(R\Gamma_I A, M).$$

Both functors $\widehat{(-)}^I$ and $R\Gamma_I(-)$ are idempotent, their essential image denoted $D_{I^\infty\text{-tors}}(A)$ and $D_{I\text{-comp}}(A)$ respectively.

Theorem 1. For $X, Y \in D(A)$,

$$R\mathrm{Hom}_A(\widehat{X}^I, \widehat{Y}^I) = R\mathrm{Hom}_A(R\Gamma_I X, R\Gamma_I Y).$$

Lemma 1.1. For $X, Y \in D(A)$, (the ideal I omitted from the completion)

$$R\mathrm{Hom}_A(X, Y)^\wedge = R\mathrm{Hom}_A(X, \widehat{Y}) = R\mathrm{Hom}_A(\widehat{X}, \widehat{Y}),$$

Consequently, there is an equivalence of categories

$$D_{I\text{-comp}}(A) \xrightleftharpoons[(-)^\wedge]{R\Gamma_I(-)} D_{I^\infty\text{-tors}}(A).$$

Theorem 2. For $M \in D(A)$,

$$M \in D_{I\text{-comp}} \iff H^j(M) \in D_{I\text{-comp}}(A) \text{ for every } j.$$

Corollary 2.1. $D_{I\text{-comp}}(A) \cap D^\heartsuit(A)$ is an abelian subcategory.

Nakayama. Let $X \in D_{I\text{-comp}}(A)$, $f \in I$, and

$$C := \mathrm{cofib}(X \xrightarrow{\times f} X).$$

(i) If $C = 0$, then $X = 0$.

(ii) If $A \in D_{I\text{-comp}}(A)$ and $X \in D^-(A)$, then $C \in D_{\mathrm{coh}}^-(A)$ implies $X \in D_{\mathrm{coh}}^-(A)$.

To prove (ii) you resolve X from the right using Theorem 2 and (i), cf. [Lyu25, Lemma 5.4].

MATLIS DUALITY

In the remainder of today's note, let $(A, \mathfrak{m}, k)$ be a Noetherian complete local ring (in particular, $\mathfrak{m}$ is finitely generated). We omit $\mathfrak{m}$ from the notation when it is obvious (e.g. D_{comp} means $D_{\mathfrak{m}\text{-comp}}$).

We give the axiomatic description of the injective hull of k (cf. [Stacks, Tag 08Y3]).

Theorem 3. There exists an A -module E such that

- (i) E is injective;
- (ii) $\mathrm{Hom}_A(k, E) \simeq k$;
- (iii) E is $\mathfrak{m}^\infty$ -torsion (cf. [Stacks, Tag 08XW]).

Note that (i) and (ii) imply $R\mathrm{Hom}_A(k, E) \simeq k$. We will deal with uniqueness later (next time).

Define

$$(-)^\vee := R\mathrm{Hom}_A(-, E)$$

so for all $M \in D(A)$ we have a canonical map $\mu_M : M \longrightarrow M^{\vee\vee}$.

Observation. The canonical map $\mu_k : k \rightarrow k^{\vee\vee}$ is an isomorphism. Indeed, $\mu_M : M \rightarrow M^{\vee\vee}$ is nonzero for any $M \in D^\heartsuit(A)$ as soon as $R\mathrm{Hom}_A(M, E) \neq 0$, since $\mu_M(x) = (\varphi \mapsto \varphi(x))$ is the evaluation map. But any nonzero map between two isomorphic copies of k is an isomorphism.

Theorem 4. For any $M \in D_{\mathrm{coh}}(A)$, the canonical map

$$\mu_M : M \longrightarrow R\mathrm{Hom}_A(R\mathrm{Hom}_A(M, E), E)$$

is an isomorphism.

Proof. Since E is injective, we may assume M is bounded. Induct on $d = \dim \mathrm{Supp} M$. If $d = 0$, M is a successive extension of shifts of k , so we win by the observation above. Assume $d > 0$ and the theorem is true for smaller d , take $f \in \mathfrak{m}$ such that

$$\dim(V(f) \cap \mathrm{Supp} M) < d.$$

Consider the distinguished triangle which defines C

$$M \xrightarrow{\times f} M \longrightarrow C \longrightarrow +1.$$

Applying double duality gives a morphism of distinguished triangles

$$\begin{array}{ccccccc} M & \xrightarrow{\times f} & M & \longrightarrow & C & \longrightarrow & +1 \\ \downarrow \mu_M & & \downarrow \mu_M & & \downarrow \mu_C & & \\ M^{\vee\vee} & \xrightarrow{\times f} & M^{\vee\vee} & \longrightarrow & C^{\vee\vee} & \longrightarrow & +1. \end{array}$$

We know $\mathrm{Supp} C \subseteq V(f) \cap \mathrm{Supp} M$. By the induction hypothesis, μ_C is an isomorphism. Thus $\times f$ is an isomorphism on $\mathrm{cofib}(\mu_M)$. However, $M \in D_{\mathrm{comp}}(A)$ as A is complete, and $M^{\vee\vee} \in D_{\mathrm{comp}}(A)$: both $R\mathrm{Hom}_A(M, E)$ and E are $\mathfrak{m}^\infty$ -torsion, as $M \in D_{\mathrm{coh}}^b(A)$, so Theorem 1 and Lemma 1.1 tell us $M^{\vee\vee} \in D_{\mathrm{comp}}(A)$. Hence $\mathrm{cofib}(\mu_M) \in D_{\mathrm{comp}}(A)$. By Nakayama, $\mathrm{cofib}(\mu_M) = 0$. $\square$

Corollary 4.1. $\mathrm{Hom}_A(E, E) = A$.

Remark. There is an equivalence of abelian categories (Matlis duality)

$$\{\text{Noetherian (i.e. finitely generated) } A\text{-modules}\} \xrightleftharpoons[(-)^\vee]{(-)^\vee} \{\text{Artinian } A\text{-modules}\}.$$

We have proved part of it ($\mathrm{id} = (-)^{\vee\vee}$ on the category on the left hand side). The rest of the proof is left to the interested reader (not trivial).

LOCAL DUALITY

Definition. The normalized dualizing complex is

$$\omega_A^\bullet := \widehat{E},$$

and we set, for every $M \in D(A)$

$$\mathbb{D}(M) := R\mathrm{Hom}_A(M, \omega_A^\bullet) \in D_{\mathrm{comp}}(A) \text{ (cf. Lemma 1.1).}$$

Again uniqueness will be dealt with the next time.

Theorem 5 (the local duality theorem, [Stacks, Tag 0A84]). For $M \in D(A)$,

$$\mathbb{D}(M) \simeq R\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}M, E) \in D_{\mathrm{comp}}(A).$$

In particular, $\mathbb{D}(k) \simeq k$.

Corollary 5.1. If $M \in D^{[a,b]}(A)$, then $\mathbb{D}(M) \in D^{[-b-\dim A, -a]}(A)$. If $M \in D^\heartsuit(A)$, then $\mathbb{D}(M) \in D^{[-\dim A, 0]}(A)$. (This is because $\mathfrak{m}$ is the radical of an ideal generated by $\dim A$ elements, hence $R\Gamma_{\mathfrak{m}}(-)$ has cohomological dimension $\dim A$.)

In particular, $\omega_A^\bullet \in D^{[-\dim A, 0]}(A)$.

Proof of Theorem 5.

$$\begin{aligned} \mathbb{D}(M) &= R\mathrm{Hom}_A(M, \omega_A^\bullet) \\ &= R\mathrm{Hom}_A(\widehat{M}, \widehat{E}) && \text{(Lemma 1.1 and Definition)} \\ &= R\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}M, R\Gamma_{\mathfrak{m}}E) && \text{(Theorem 1)} \\ &= R\mathrm{Hom}_A(R\Gamma_{\mathfrak{m}}M, E) && (E \in D_{\mathrm{tors}}(A)). \end{aligned}$$

□

Theorem 6. For every $M \in D_{\mathrm{coh}}(A)$, the canonical map

$$\gamma_M : M \longrightarrow \mathbb{D}(\mathbb{D}(M))$$

is an isomorphism.

Proof. The proof is the same as for μ_M . Indeed, by Corollary 5.1, we may assume $M \in D_{\mathrm{coh}}^b(A)$. Using $\mathbb{D}(\mathbb{D}(M)) \in D_{\mathrm{comp}}(A)$, we can reduce to the case $M = k$. Then Theorem 1 tells us for $X \in D_{\mathrm{comp}} \cap D_{\mathrm{tors}}$ we have $\mathbb{D}(X) = X^\vee$. Therefore

$$\mathbb{D}(\mathbb{D}(k)) = \mathbb{D}(k^\vee) = k^{\vee\vee},$$

and γ_k corresponds to μ_k . □

Theorem 7. If $M \in D_{\mathrm{coh}}(A)$, then $\mathbb{D}(M) \in D_{\mathrm{coh}}(A)$ (in particular, $\omega_A^\bullet \in D_{\mathrm{coh}}(A)$).

Proof. By Corollary 5.1, it suffices to consider the case $M \in D_{\mathrm{coh}}^b(A)$. Again, induct on $d = \dim \mathrm{Supp} M$. If $d = 0$, we may assume $M = k$, and $\mathbb{D}(k) \simeq k$. If $d > 0$, take $f \in \mathfrak{m}$ with $\dim(V(f) \cap \mathrm{Supp} M) < d$ and consider the triangle

$$M \xrightarrow{\times f} M \longrightarrow C \longrightarrow +1$$

which defines C . Dualizing gives

$$\mathbb{D}(C) \longrightarrow \mathbb{D}(M) \xrightarrow{\times f} \mathbb{D}(M) \longrightarrow +1.$$

By induction, $\mathbb{D}(C) \in D_{\mathrm{coh}}^b(A)$, and $\mathbb{D}(M) \in D_{\mathrm{comp}}^b(A)$ (Theorem 5 and Corollary 5.1). By Nakayama, $\mathbb{D}(M) \in D_{\mathrm{coh}}(A)$. □

REFERENCES

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