Game-theoretic Foundations of Multi-agent Systems

Lecture 2: Preferences and Utilities

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Outline

1. Agent Preferences
2. von Neumann–Morgenstern Rationality
3. von Neumann–Morgenstern Utilities
4. Uncertainty and Risk Attitudes

Outcomes and Lotteries

- Let $O = \{o_1, \ldots, o_K\}$ be set of mutually exclusive outcomes

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  - Let $A = 0.2o_1 + 0.8o_2$ and $B = 0.4o_2 + 0.6o_3$
  - $C = 0.5A + 0.5B$ is a compound lottery:

$$C = 0.5(0.2o_1 + 0.8o_2) + 0.5(0.4o_2 + 0.6o_3) = 0.1o_1 + 0.6o_2 + 0.3o_3$$

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  - $A \succ B$ means agent strictly prefers A to B
  - $A \succeq B$ means agent weakly prefers A to B
  - $A \sim B$ means agent is indifferent between A and B ($A \succeq B$ and $B \succeq A$)

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Axioms of von Neumann–Morgenstern (VNM) Rationality

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4. Continuity
   - For all lotteries A, B, and C, if $A \succeq B \succeq C$, then $\exists p \in [0, 1]$ such that $B \sim pA + (1-p)C$

Auxiliary Axioms

Lemma

Given VNM axioms, for any pair of lotteries A and B with $A \succ B$, we have

- *Betweenness: for $p \in (0, 1)$, $A \succ pA + (1 - p)B \succ B$*

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  Proof sketch

  - *By independence, $A = pA + (1 - p)A \succ pA + (1 - p)B \succ pB + (1 - p)B = B$*

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  Proof sketch

  - *Define $\delta = q/p$*
  - *By betweenness, $A \succ pA + (1-p)B \succ B$*
  - *Apply betweenness to second part with δ:*
    $pA + (1-p)B \succ \delta[pA + (1-p)B] + (1-\delta)B = qA + (1-q)B \succ B$

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von Neumann-Morgenstern Utility Theorem

Theorem (von Neumann and Morgenstern, 1944)

For any VNM-rational agent, there exists a function u that maps each lottery A to a real number $u(A)$ such that

- $u(A) = u\left(\sum p_k o_k\right) = \sum p_k u(o_k)$ *(expected utility)*
- $u(A) \geq u(B) \iff A \succeq B$,

Such a function is called von Neumann-Morgenstern (VNM) utility.

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    - Replace o_k by $u(o_k)\overline{o} + (1 - u(o_k))\underline{o}$ (by independence)

$$A = \sum p'_k o_k \sim \left(\sum p'_k u(o_k)\right) \overline{o} + \left(1 - \sum p'_k u(o_k)\right) \underline{o}$$

    - This is a lottery on $\overline{o}$ and $\underline{o}$
    - By the definition of u, we conclude

$$u(A) = u\left(\sum p'_k o_k\right) = \sum p'_k u(o_k)$$

von Neumann-Morgenstern Utility (Proof Sketch)

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  - $A \sim u(A)\overline{o} + (1 - u(A))\underline{o}$ and $B \sim u(B)\overline{o} + (1 - u(B))\underline{o}$
  - If $u(A) = u(B)$, then A and B define identical lotteries
  - If $u(A) > u(B)$, then by monotonicity, we have

$$A \sim u(A)\overline{o} + (1 - u(A))\underline{o} \succ u(B)\overline{o} + (1 - u(B))\underline{o} \sim B$$

- **Part III**. Show $A \succeq B \Longrightarrow u(A) \geq u(B)$

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- **Part III**. Show $A \succeq B \implies u(A) \geq u(B)$
  - If $u(A) < u(B)$, then by (Part II), $B \succ A$
  - By completeness, this is a contradiction

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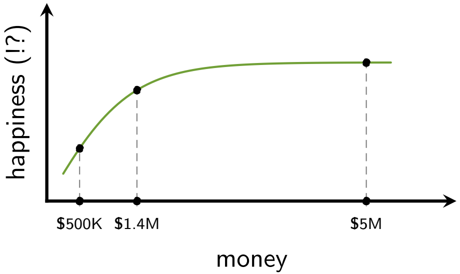
Example

- More money makes people happier (?) but with diminishing marginal returns!



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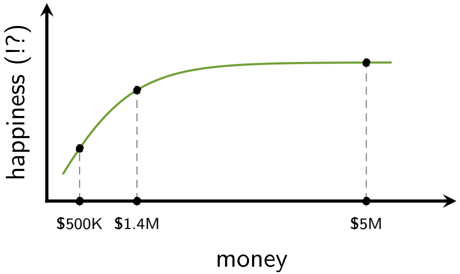

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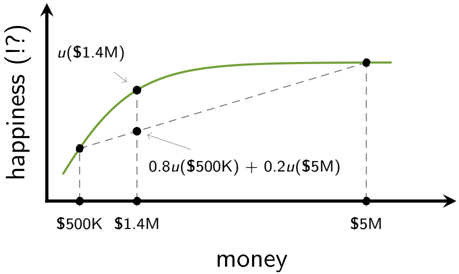

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- For a risk-seeking investor, $u(A) > u(z)$

Are You a Risk-taker or Risk-seeker?

- Which one do you prefer?
  - Lottery A: \$50 with prob 0.1 and \$0 otherwise
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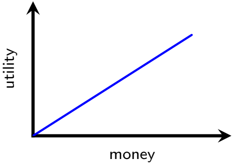
- How about these?
  - Lottery A: \$5,000,000 with prob 0.1 and \$0 otherwise
  - Lottery B: \$500,000 with prob 1

Risk Attitudes (revisited)

- Blue has constant marginal utility $\longrightarrow$ risk-neutral



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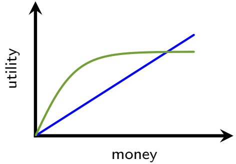

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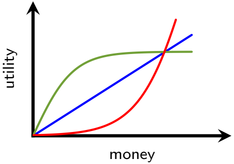

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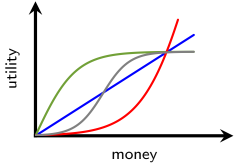
- Gray neither risk-averse nor risk-seeking



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