Game-theoretic Foundations of Multi-agent Systems

Lecture 2: Preferences and Utilities

Seyed Majid Zahedi

UNIVERSITY OF WATERLOO

Outline

Outcomes and Lotteries

- Let $O = \{o_1, \ldots, o_K\}$ be set of mutually exclusive outcomes
- Lottery A describes a probability distribution over outcomes
- We write $A = \sum p_k o_k$ to indicate that $o_k \in O$ happens with probability p_k
  - $\sum p_k = 1$
  - E.g., $A = 0.75 o_1 + 0.25 o_2$ means $P(o_1) = 0.75$ and $P(o_2) = 0.25$
- Compound lottery is a lottery defined based on other lotteries
  - Suppose $O = \{o_1, o_2, o_3\}$
  - Let $A = 0.2 o_1 + 0.8 o_2$ and $B = 0.4 o_2 + 0.6 o_3$
  - $C = 0.5A + 0.5B$ is a compound lottery:

$$C = 0.5(0.2 o_1 + 0.8 o_2) + 0.5(0.4 o_2 + 0.6 o_3) = 0.1 o_1 + 0.6 o_2 + 0.3 o_3$$

Ordinal Preferences

- We define preference relation over lotteries as follows
  - $A \succ B$ means agent strictly prefers A to B
  - $A \succeq B$ means agent weakly prefers A to B
  - $A \sim B$ means agent is indifferent between A and B ($A \succeq B$ and $B \succeq A$)

Outline

Axioms of von Neumann–Morgenstern (VNM) Rationality

1. Completeness
   - For all lotteries A and B, either $A \succ B$ or $B \succ A$ or $A \sim B$
2. Transitivity
   - For all lotteries A, B, and C, if $A \succeq B$ and $B \succeq C$, then $A \succeq C$
3. Independence of irrelevant alternatives
   - For all lotteries A, B, and C, and $p \in [0, 1]$, then $A \succeq B \iff pA + (1 - p)C \succeq pB + (1 - p)C$
4. Continuity
   - For all lotteries A, B, and C, if $A \succeq B \succeq C$, then $\exists p \in [0, 1]$ such that $B \sim pA + (1 - p)C$

Auxiliary Axioms

Lemma

Given VNM axioms, for any pair of lotteries A and B with $A \succ B$, we have

- *Betweenness: for $p \in (0,1)$, $A \succ pA + (1-p)B \succ B$*

  Proof sketch

  - *By independence, $A = pA + (1-p)A \succ pA + (1-p)B \succ pB + (1-p)B = B$*

- *Monotonicity: for any $p, q \in [0,1]$, if $p > q$, then $pA + (1-p)B \succeq qA + (1-q)B$*

  Proof sketch

  - *Define $\delta = q/p$*
  - *By betweenness, $A \succ pA + (1-p)B \succ B$*
  - *Apply betweenness to second part with δ:*
    $pA + (1-p)B \succ \delta[pA + (1-p)B] + (1-\delta)B = qA + (1-q)B \succ B$

Outline

von Neumann-Morgenstern Utility Theorem

Theorem (von Neumann and Morgenstern, 1944)

For any VNM-rational agent, there exists a function u that maps each lottery A to a real number $u(A)$ such that

- $u(A) = u\left(\sum p_k o_k\right) = \sum p_k u(o_k)$ *(expected utility)*
- $u(A) \geq u(B) \iff A \succeq B$,

Such a function is called von Neumann-Morgenstern (VNM) utility.

von Neumann-Morgenstern Utility (Proof Sketch)

- If agent is indifferent between all outcomes, then set $u(o) = 0$ for all outcomes o
- Otherwise, there must be most-preferred and least-preferred outcomes, $\overline{o}$ and $\underline{o}$
- Set $u(o_k)$ to be p_k such that $o_k \sim p_k \overline{o} + (1 - p_k)\underline{o}$ (by continuity)
- **Part I**. Show $u\left(\sum p'_k o_k\right) = \sum p'_k u(o_k)$
  - Replace o_k by $u(o_k)\overline{o} + (1 - u(o_k))\underline{o}$ (by independence)

$$A = \sum p'_k o_k \sim \left(\sum p'_k u(o_k)\right) \overline{o} + \left(1 - \sum p'_k u(o_k)\right) \underline{o}$$

  - This is a lottery on $\overline{o}$ and $\underline{o}$
  - By the definition of u, we conclude

$$u(A) = u\left(\sum p'_k o_k\right) = \sum p'_k u(o_k)$$

von Neumann-Morgenstern Utility (Proof Sketch)

- **Part II**. Show $u(A) \geq u(B) \implies A \succeq B$
  - $A \sim u(A)\overline{o} + (1 - u(A))\underline{o}$ and $B \sim u(B)\overline{o} + (1 - u(B))\underline{o}$
  - If $u(A) = u(B)$, then A and B define identical lotteries
  - If $u(A) > u(B)$, then by monotonicity, we have

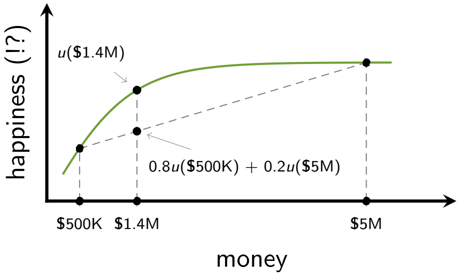
$$A \sim u(A)\overline{o} + (1 - u(A))\underline{o} \succ u(B)\overline{o} + (1 - u(B))\underline{o} \sim B$$

- **Part III**. Show $A \succeq B \implies u(A) \geq u(B)$
  - If $u(A) < u(B)$, then by (Part II), $B \succ A$
  - By completeness, this is a contradiction

Outline

Example

- More money makes people happier (?) but with diminishing marginal returns!



- Based on this utility function, which one is more preferred?
  - \$500K with probability 0.8, and \$5M with probability 0.2
  - \$1.4M with probability 1

Risk Attitudes

- Let u be utility of an investor
- Lottery A pays \$$x$ with probability p and \$$y$ with probability $(1-p)$
- By utility theorem, $u(A) = pu(x) + (1-p)u(y)$
- Let $z = \$(px + (1-p)y)$
- For a risk-neutral investor, $u(A) = u(z)$
- For a risk-averse investor, $u(A) < u(z)$
- For a risk-seeking investor, $u(A) > u(z)$

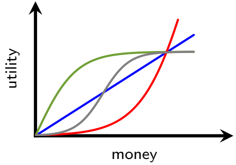
Are You a Risk-taker or Risk-seeker?

- Which one do you prefer?
  - Lottery A: \$50 with prob 0.1 and \$0 otherwise
  - Lottery B: \$5 with prob 1

- How about these?
  - Lottery A: \$5,000,000 with prob 0.1 and \$0 otherwise
  - Lottery B: \$500,000 with prob 1

Risk Attitudes (revisited)

- Blue has constant marginal utility $\longrightarrow$ risk-neutral
- Green has decreasing marginal utility $\longrightarrow$ risk-averse
- Red has increasing marginal utility $\longrightarrow$ risk-seeking
- Gray neither risk-averse nor risk-seeking



Acknowledgment

- This lecture is a slightly modified version of ones prepared by
  - Asu Ozdaglar [MIT 6.254]
  - Vincent Conitzer [Duke CPS 590.4]