Game-theoretic Foundations of Multi-agent Systems

Lecture 3: Games in Normal Form

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Outline

1. Normal-form Games: Definition, Notations, and Examples
2. Dominant Strategy Equilibrium
3. Nash Equilibrium
4. Price of Anarchy
5. Minmax Theorem
6. Rationalizability
7. Correlated Equilibrium

Normal-form Games

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- Such games are referred to as strategic-form games or normal-form games

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- Agent i has a utility function u_i that maps outcomes to real numbers

Some Notations

- $a_{-i} = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n)$ is an action profile of all agents except i
- $A_{-i} = \prod_{j \neq i} A_j$ is set of action profiles of all agents except i
- $a = (a_i, a_{-i}) \in A$ is another way of denoting an action profile (or an outcome)

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- Each cell indexed by row r and column c contains a pair, (p, q), where $p = u_1(r, c)$ and $q = u_2(r, c)$.

Example: Matching Pennies

- Each agent has a penny and independently chooses to display either heads or tails
- Agents compare their pennies
- If they are the same, agent 2 pockets both, otherwise agent 1 pockets them

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- Zero-sum game: Utility of one agent is negative of utility of other agent

Example: Rock, Paper, Scissors Game

- Three-strategy generalization of the matching-pennies game

	Rock	Paper	Scissors
Rock	$0, 0$	$-1, 1$	$1, -1$
Paper	$1, -1$	$0, 0$	$-1, 1$
Scissors	$-1, 1$	$1, -1$	$0, 0$

Example: Coordination Game

- Two drivers driving towards each other in a country with no traffic rules
- Drivers must independently decide whether to drive on the left or on the right
- If drivers choose the same side (left or right) they have some high utility, and otherwise they have a low utility

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- Team game: For all outcomes s, and any pair of agents i and j, it is the case that $u_i(a) = u_j(a)$ (also known as common-payoff game or pure-coordination game)

Example: Cournot Competition

- Two firms producing a homogeneous good for the same market
- Action of each firm is the amount of good it produces ($a_i \in [0, \infty]$)
- Utility of each firm is its total revenue minus its total cost

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  - $p(\cdot)$ is the price function that maps total production to a price
  - c is a unit cost
  - E.g., $p(x) = \max(0, 2 - x)$ and $c = 1$

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- Pure strategy is a mixed strategy that puts probability 1 on a single action
- Support of mixed strategy s_i is set of pure strategies, a_i, such that $s_i(a_i) > 0$
- Expected utility of agent i for a (mixed) strategy profile $s = (s_1, \ldots, s_n)$ is

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in N} s_j(a_j)$$

Example

			Agent 2	
		R ($\frac{2}{3}$)	P (0)	S ($\frac{1}{3}$)
	R ($\frac{1}{3}$)	$0, 0$	$-1, 1$	$1, -1$
Agent 1	P ($\frac{2}{3}$)	$1, -1$	$0, 0$	$-1, 1$
	S (0)	$-1, 1$	$1, -1$	$0, 0$

- $u_1 = 2/9 \times 0 + 1/9 \times 1 + 4/9 \times 1 - 2/9 \times 1 = 1/3$
- $u_2 = 2/9 \times 0 - 1/9 \times 1 - 4/9 \times 1 + 2/9 \times 1 = -1/3$

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- $s = (s_1, \ldots, s_n)$ is dominant strategy equilibrium if s_i is dominant strategy for all i

Example: Prisoner's Dilemma

- Two prisoners suspected of a crime are taken to separate interrogation rooms
- Each can either confess to the crime or deny it

	D	C
D	$-2, -2$	$-4, -1$
C	$-1, -4$	$-3, -3$

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- The dilemma: (D,D) is better for both prisoners, but they won't play it!

Iterated Elimination of Strictly Dominated Strategies

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	L	C	R
U	3, 1	0, 2	0, 0
M	1, 2	1, 1	5, 0
D	0, 1	4, 2	0, 0

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- If the procedure ends with a single strategy for each agent, then the game is said to be dominance solvable

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  - Because agents do not know what strategies others will play
- However, we can leverage the idea of best response to define what is arguably the most central notion in game theory, the Nash equilibrium

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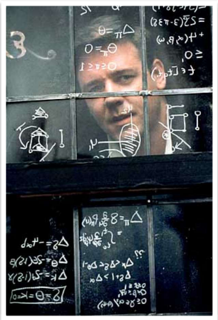

Fig. A Beautiful Mind, from s.hdnux.com

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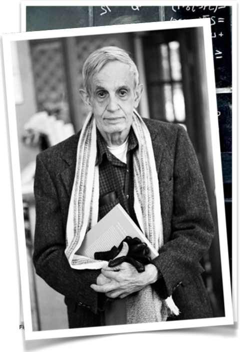

John Forbes Nash Jr.
1928-2015

Example: Battle of Sexes

- Husband and wife wish to meet this evening, but have a choice between two events to attend: football or opera
- Husband would prefer to go to football, wife would prefer opera
- Both would prefer to go to the same event rather than different ones

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- Are these the only Nash equilibria?

Example: Battle of Sexes (cont.)

	F (p)	O ($1-p$)
F	$2,1$	$0,0$
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- In general, it is tricky to compute mixed-strategy equilibria (will discuss this later)

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$$u_H(F) = u_H(O) \Rightarrow 2 \times p = (1-p) \Rightarrow p = 1/3$$

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- You can show that the unique mixed-strategy NE is $\{(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})\}$

Example: Cournot Competition

- $u_i(a_1, a_2) = a_i \max(0, 2 - a_1 - a_2) - a_i$
- Using first order optimality conditions, we have

$$
\begin{aligned}
BR_i(a_{-i}) &= \operatorname*{argmax}_{a_i \geq 0} a_i(2 - a_i - a_{-i}) - a_i \\
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\end{aligned}
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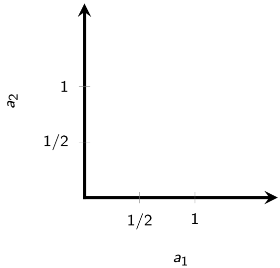

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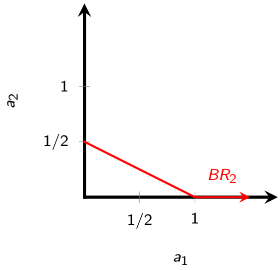

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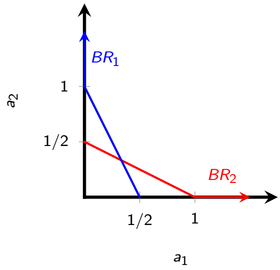

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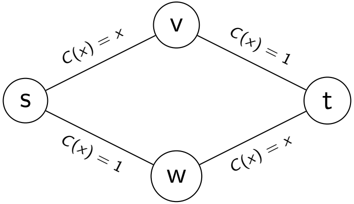
  - ...
- Equilibrium selection is a difficult problem

Outline

Braess's Paradox



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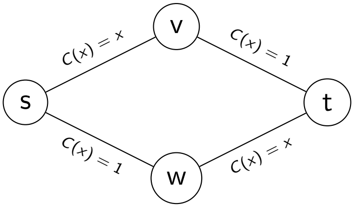

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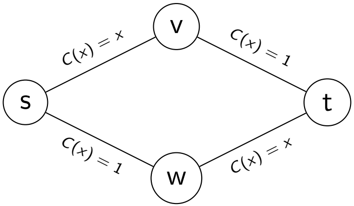
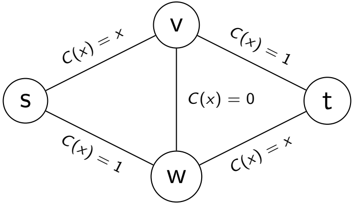


- Suppose there are $2k$ drivers commuting from s to t
- $C(x)$ indicates travel time in hours for x fraction of drivers
- k drivers going through v, and k going through w is NE (why?)

Braess's Paradox (cont.)



- Suppose we install a teleportation device allowing instant travel from v to w
- What is new NE?
- What is optimal commute time?
- Price of anarchy: ratio between (worst) NE performance and optimal performance
  - Ratio between 2 and 3/2 in Braess's Paradox

Outline

Maxmin Strategy

- Maxmin strategy for agent i is

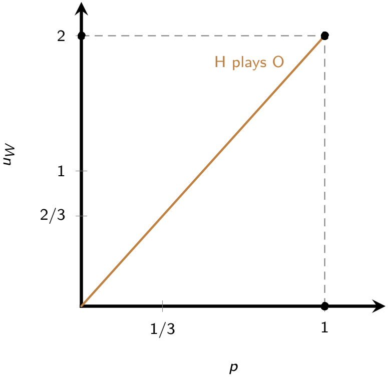
$$\operatorname*{argmax}_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- Maxmin value for agent i is

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- If i plays maxmin strategy and others play arbitrarily, i still receives expected payoff of at least their maxmin value

Example: Battle of Sexes



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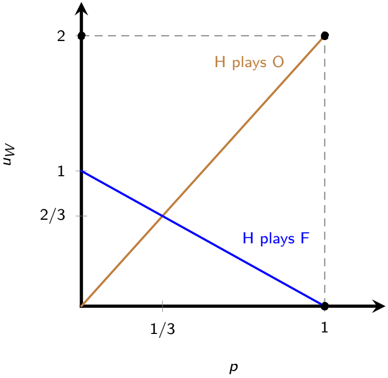

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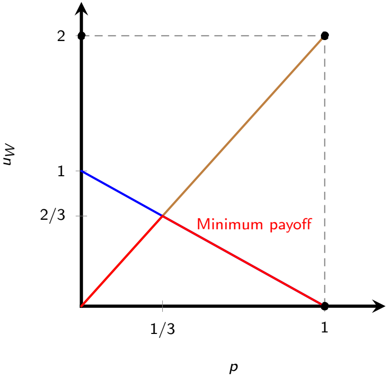

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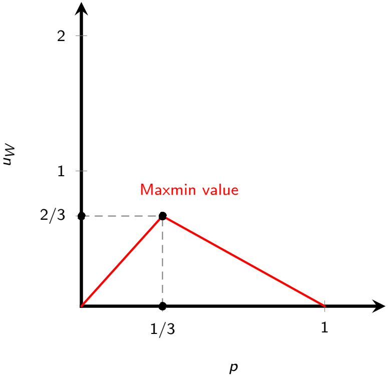

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- Minmax strategy against i keeps maximum payoff of agent i at minimum
- Agents' maxmin value is always less than or equal to their minmax value (try to show this!)

Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any Nash equilibrium[1], each agent receives a payoff that is equal to both their maxmin value and their minmax value

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- Minimax theorem does not hold with pure strategies only (example?)

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		Agent 2	
		Left	Right
Agent 1	Up	$20, -20$	$0, 0$
	Down	$0, 0$	$10, -10$

- What is maximin value of agent 1 with and without mixed strategies?

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- What is NE of this game?

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- In this game, all pure strategies are rationalizable

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- To find rationalizable strategies:
  - In 2-player games, use iterated elimination of strictly dominated strategies
  - In n-player games, iterated elimination of never-best response strategies
    - Eliminate strategies that are not optimal against any belief about others' strategies

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- Only rationalizable action is playing 1

Outline

Example: Battle of Sexes

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		Football	Opera
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- Unique mixed strategy NE yields each agent expected payoff of $2/3$

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- Can they both do better by coordinating?
- Agents can observe random coin flip and condition their strategies on its outcome

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- Similar argument can be made when they see tails

Example: Battle of Sexes (cont.)

- Suppose there is publicly observable fair coin
- If it is heads/tails, they both get recommendation to go to football/opera
- If they see heads, they believe that the other one goes to football
- Going to football is best response, agents have no incentive to deviate
- Similar argument can be made when they see tails
- Expected utilities for this play of game increases to $(1.5, 1.5)$

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- $\pi(r_i)$ represents marginal probability for $R_i = r_i$
- Given r_i, agent i can use conditional probability to form beliefs others' signals

$$\pi(r_{-i}|r_i) = \frac{\pi(r_i, r_{-i})}{\sum_{r'_{-i} \in A_{-i}} \pi(r_i, r'_{-i})}$$

Correlated Equilibrium: Formal Definition

- Correlated equilibrium of finite game is joint probability distribution $\pi \in \Delta(A)$ such that if R is random variable distributed according to π, then for all $i, r_i \in A_i$ with $\pi(r_i) > 0$, and $r_i' \in A_i$

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- No agent can benefit by deviating from their recommendation, assuming that other agents follow their recommendations

Example: Game of Chicken

		Driver 2	
		Dare	Yield
Driver 1	D	$-5, -5$	$1, -1$
	Y	$-1, 1$	$0, 0$

- (D,Y) and (Y,D) are strict pure-strategy NE

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- Assume Driver 1 yields w.p. p and Driver 2 yields w.p. q
- Using mixed equilibrium characterization, we have

$$p - 5 \times (1 - p) = -(1 - p) \implies p = 4/5$$

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- Mixed-strategy NE utilities are $(-0.2, -0.2)$, people die with probability 0.04

Example: Game of Chicken (cont.)

- Is this correlated equilibrium?

		D2: D	D2: Y
D1	D	$-5, -5$ 0%	$1, -1$ 40%
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Example: Game of Chicken (cont.)

- Is this correlated equilibrium?
- Suppose D1 gets Y recommendation

D2

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- Following the recommendation is better
- If D1 gets D recommendation, D2 must yield
- Following recommendation is again better
- Similar analysis works for D2
- Expected utilizes are $(0, 0)$, so nobody dies!

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- Joint distribution $\pi \in \Delta(A)$ is correlated equilibrium of finite game iff

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  - So it can be factored and canceled
  - If $\pi(r_i) = 0$, LHS of (1) is zero regardless of i and r_i', so equation always holds

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