Game-theoretic
Foundations of Multi-agent Systems

Lecture 4: Computing Solution Concepts of Normal-form Games

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Outline

1. Brief Overview of (Mixed Integer) Linear Programming
2. Dominated Strategies
3. Minmax and Maxmin Strategies
4. Nash Equilibrium
5. Correlated NE

Example: Reproduction of Two Paintings

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- Painting 1 sells for \$30

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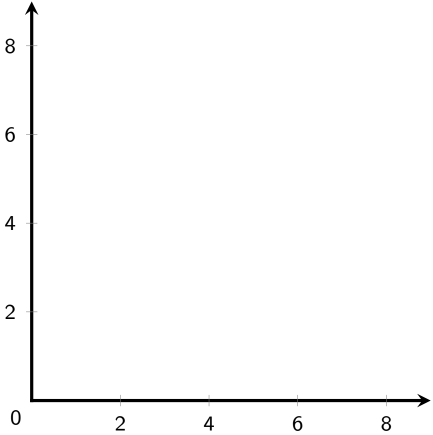

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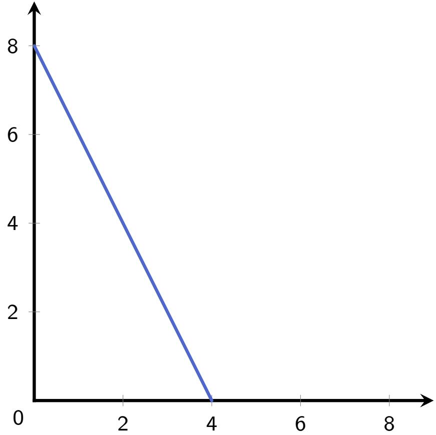

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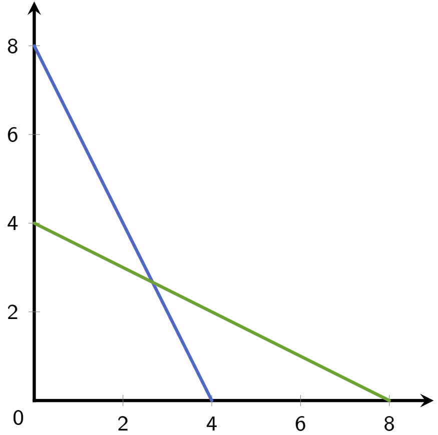

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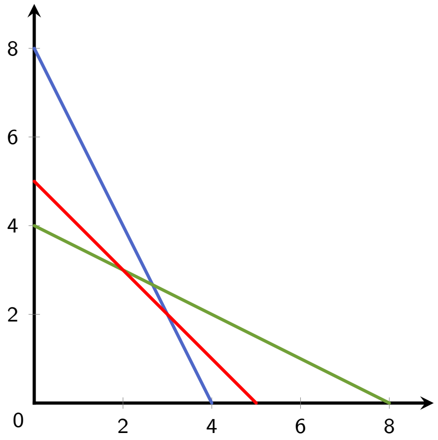

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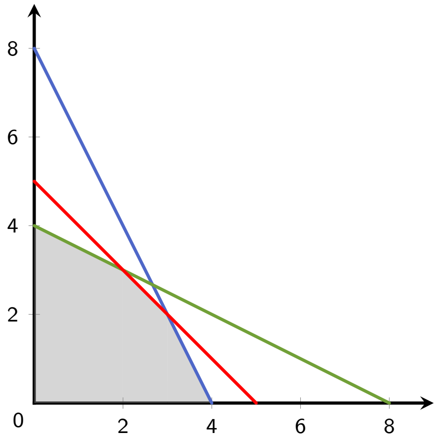

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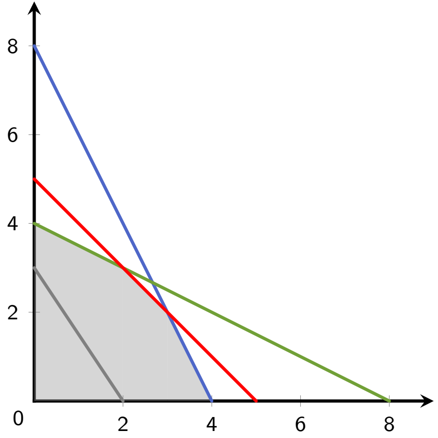

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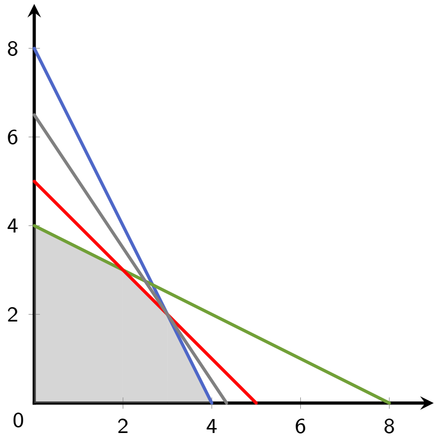

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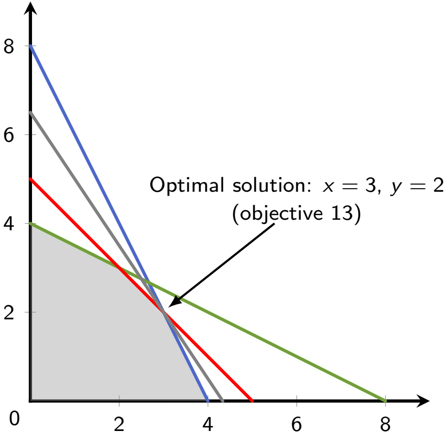

Modified LP

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- Objective $= 7.5 + 5 = 12.5$

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- Optimal solution: x = 2.5, y = 2.5
- Objective = 7.5 + 5 = 12.5
- Can we sell half paintings?

Integer Linear Program

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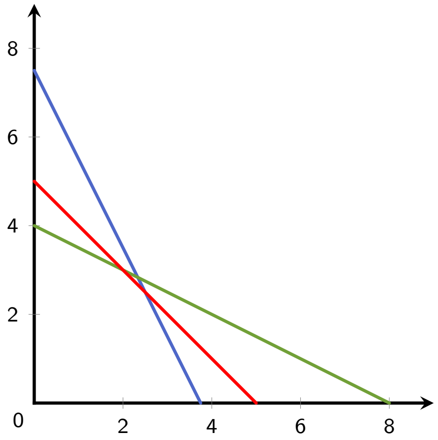

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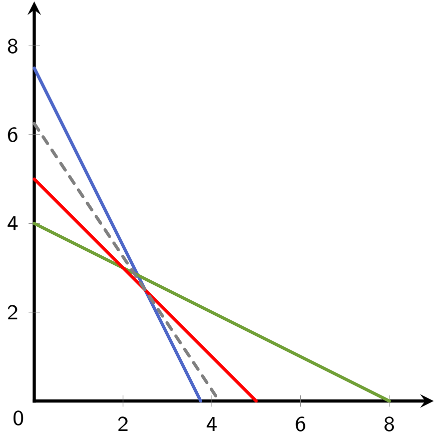

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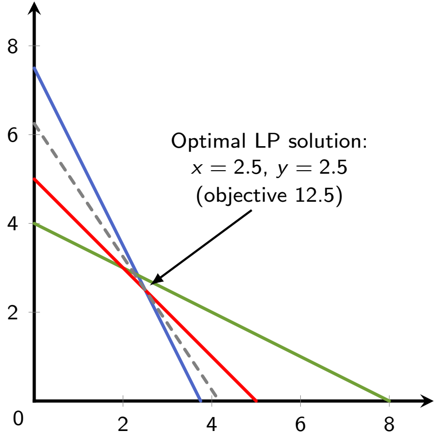

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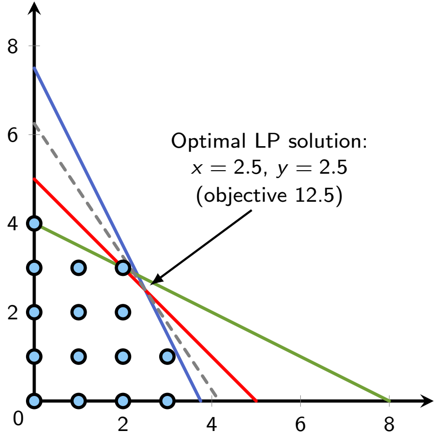

Integer Linear Program

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\begin{aligned}
\text{max.} \quad & 3x + 2y \\
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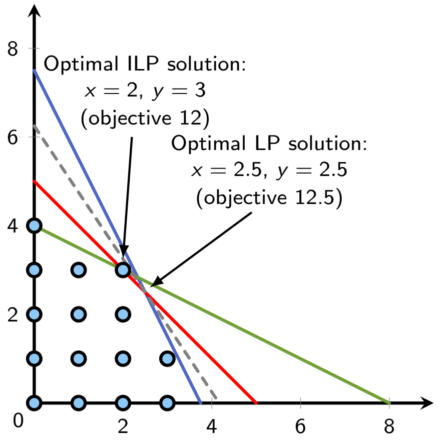

Mixed Integer Linear Program

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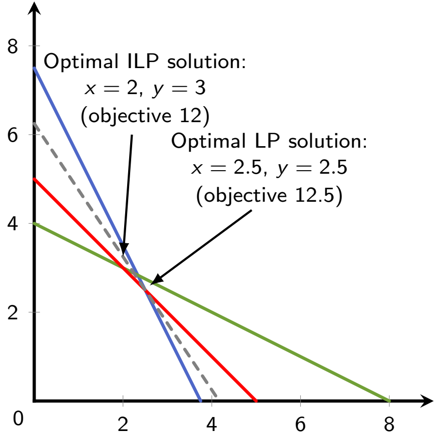

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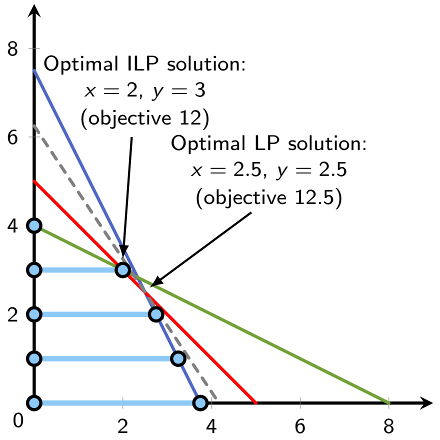

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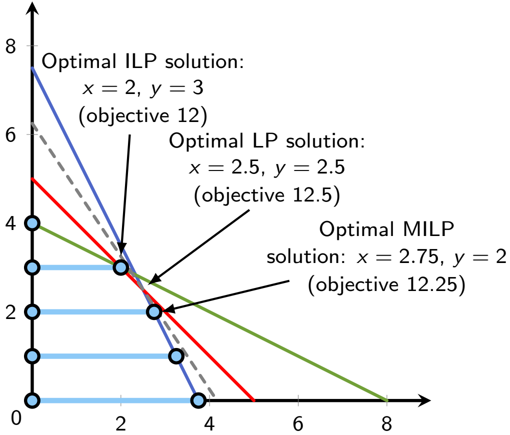

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- LP relaxation of (M)ILP: remove integrality constraints
  - Gives upper bound on MILP ($\sim$ admissible heuristic)

Exercise I: Knapsack-type Problem

- We arrive in room full of precious objects
- Can carry only 30kg out of the room
- Can carry only 20 liters out of the room
- Want to maximize our total value
- Unit of object A: 16kg, 3 liters, sells for \$11 (3 units available)
- Unit of object B: 4kg, 4 liters, sells for \$4 (4 units available)
- Unit of object C: 6kg, 3 liters, sells for \$9 (1 unit available)
- What should we take?

Exercise II: Cellphones (Set Cover)

- We want to have a working phone in every continent (besides Antarctica)
- But we want to have as few phones as possible
- Phone A works in NA, SA, Af
- Phone B works in E, Af, As
- Phone C works in NA, Au, E
- Phone D works in SA, As, E
- Phone E works in Af, As, Au
- Phone F works in NA, E

Exercise III: Hot-dog Stands

- We have two hot-dog stands to be placed in somewhere along beach
- We know where groups of people who like hot dogs are
- We also know how far each group is willing to walk
- Where do we put our stands to maximize # hot dogs sold? (price is fixed)

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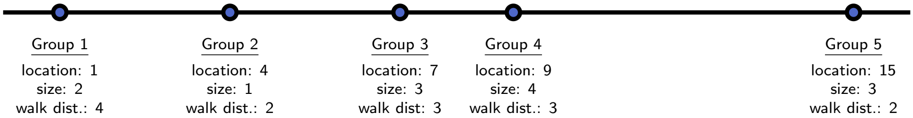

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Recall: Strict Dominance

a_i strictly dominates s_i if $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i})\ \forall s_{-i} \in S_{-i}$

Dominance by Pure Strategy

Algorithm 1: Determine whether s_i is strictly dominated by any pure strategy

```
for all a_i ∈ A_i where a_i ≠ s_i do
    dom ← true;
    forall a_{-i} ∈ A_{-i} do
        if u_i(s_i, a_{-i}) ≥ u_i(a_i, a_{-i}) then
            dom ← false;
            break;
    if dom = true then
        return true;
return false;
```

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  - Suppose a_i strictly dominates s_i for all a_{-i}
  - Then, there is no s_{-i} for which $u_i(a_i, s_{-i}) \geq u_i(s_i, s_{-i})$
  - This holds because of the linearity of expectation

Weak Dominance by Mixed Strategy

- Checking if strategy s_i is weakly dominated by any mixed strategy

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\begin{aligned}
\text{s.t.} \quad & \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) && \forall a_{-i} \in A_{-i} \\
& \sum_{a_i \in A_i} p_{a_i} = 1 \\
& p_{a_i} \geq 0, && \forall a_i \in A_i
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- If optimal solution is strictly positive, then s_i is weakly dominated by $\{p_{a_i}\}$

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$$\text{s.t.} \qquad \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq u_i(s_i, a_{-i}) + \epsilon \quad \forall a_{-i} \in A_{-i}$$

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& p_{a_i} \geq 0, \qquad \forall a_i \in A_i
\end{aligned}
$$

- If optimal solution is strictly positive, then s_i is strictly dominated by $\{p_{a_i}\}$

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  - Sequence of eliminations may determine which solution we get (if any)

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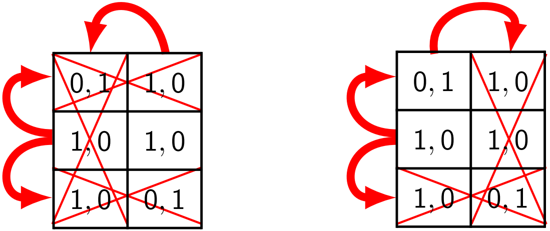

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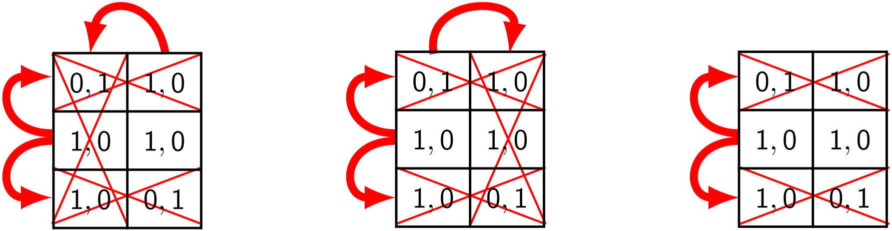

- Iterated strict dominance is path-independent:
  - Elimination process will always terminate at the same point

Computational Questions for Iterated Dominance

- Is there some elimination path under which s_i is eliminated?

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  - Due to path-independence
  - Check if any strategy is dominated, remove it, repeat
  - With or without dominance by mixed strategies

Computational Questions for Iterated Dominance

- Is there some elimination path under which s_i is eliminated?
- Is there maximally reduced game where each agent has exactly 1 action?
- For strict dominance, both can be solved in polynomial time
  - Due to path-independence
  - Check if any strategy is dominated, remove it, repeat
  - With or without dominance by mixed strategies
- For weak dominance, both questions are NP-hard[1]
  - Even when all utilities are 0 or 1
  - With or without dominance by mixed strategies

[1] [Conitzer, Sandholm 05] and weaker version proved by [Gilboa, Kalai, Zemel 93]

Outline

1. Brief Overview of (Mixed Integer) Linear Programming
2. Dominated Strategies
3. Minmax and Maxmin Strategies
4. Nash Equilibrium
5. Correlated NE

Recall: Minmax and Maxmin

- Maxmin strategy for agent i (maxmin value for agent i)

$$\underset{s_i}{\operatorname{argmax}} \min_{s_{-i}} u_i(s_i, s_{-i})$$

Recall: Minmax and Maxmin

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$$\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$$

- Minmax strategy against agent i (minmax value for agent i)

$$\arg\min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

Maxmin Strategy and Value

- Finding maxmin strategy of agent i

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$$\text{s.t.} \qquad \sum_{a_i \in A_i} p_{a_i} u_i(a_i, a_{-i}) \geq U_i, \quad \forall a_{-i} \in A_{-i}$$

- Given p_{a_i}, first constraint ensures that U_i is less than any achievable expected utility for any pure strategies of opponents

Maxmin Strategy and Value

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- Given p_{a_i}, first constraint ensures that U_i is less than any achievable expected utility for any pure strategies of opponents
- Objective of this LP, U_i, is maxmin value of agent i

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NE of Two-player, Zero-sum Games

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- NE is expressed as LP $\Rightarrow$ NE can be computed in polynomial time

Maxmin Strategy for General-sum Games

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		Agent 2: Left	Agent 2: Right
Agent 1	Up	$0, 0$	$3, 1$
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  - If A2 was trying to hurt A1, she would play Left, so A1 should play Down

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Agent 1	Up	$0, 0$	$3, 1$
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  - If A2 was trying to hurt A1, she would play Left, so A1 should play Down
  - In reality, A2 will play Right (strictly dominant), so A1 should play Up

Hardness of Computing NE for General-sum Games

- Complexity was open for long time
  - "together with factoring [. . .] the most important concrete open question on the boundary of P today" [Papadimitriou STOC'01]

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- Sequence of papers showed that computing any NE is PPAD-complete (even in 2-player games) [Daskalakis, Goldberg, Papadimitriou 2006; Chen, Deng 2006]
- All known algorithms require exponential time (in worst case)

Hardness of Computing NE for General-Sum Games (cont.)

- What about computing NE with specific property?
  - NE that is not Pareto-dominated
  - NE that maximizes expected social welfare (i.e., sum of all agents' utilities)
  - NE that maximizes expected utility of given agent
  - NE that maximizes expected utility of worst-off player
  - NE in which given pure strategy is played with positive probability
  - NE in which given pure strategy is played with zero probability
  - ...

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  - NE that maximizes expected utility of worst-off player
  - NE in which given pure strategy is played with positive probability
  - NE in which given pure strategy is played with zero probability
  - ...
- All of these are NP-hard (and the optimization questions are inapproximable assuming P != NP), even in 2-player games

  [Gilboa, Zemel 89; Conitzer & Sandholm IJCAI-03/GEB-08]

Search-based Approaches (for Two-player Games)

- We can use LP, if we know support X_i of each player i's mixed strategy

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\begin{aligned}
\text{find} \quad & (U_1, U_2) \\
\text{s.t.} \quad & p_{a_i} \geq 0, && \forall i, a_i \in A_i \\
& \sum_{a_i \in A_i} p_{a_i} = 1, && \forall i \\
& p_{a_i} = 0, && \forall i, a_i \in A_i / X_i \\
& \sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) = U_i, && \forall i, a_i \in X_i \\
& \sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) \leq U_i, && \forall i, a_i \in A_i / X_i
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$$

- Thus, we can search over possible supports, which is basic idea underlying methods in [Dickhaut & Kaplan 91; Porter, Nudelman, Shoham AAAI04/GEB08]

NE using MILP (for Two-player Games)

[Sandholm, Gilpin, Conitzer AAAI05]

$$
\begin{aligned}
\text{max.} \quad & \text{whatever you like (e.g., social welfare)} & \\
\text{s.t.} \quad & p_{a_i} \geq 0, & \forall i, a_i \in A_i \\
& \sum_{a_i \in A_i} p_{a_i} = 1, & \forall i \\
& \sum_{a_{-i} \in A_{-i}} p_{a_{-i}} u_i(a_i, a_{-i}) = u_{a_i}, & \forall i, a_i \in A_i \\
& u_{a_i} \leq u_i, & \forall i, a_i \in A_i \\
& p_{a_i} \leq b_{a_i}, & \forall i, a_i \in A_i \\
& u_i - u_{a_i} \leq M(1 - b_{a_i}), & \forall i, a_i \in A_i \\
& b_{a_i} \in \{0, 1\}, & \forall i, a_i \in A_i
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& b_{a_i} \in \{0, 1\}, & \forall i, a_i \in A_i
\end{aligned}
$$

- b_{a_i} indicates whether a_i is in support of i's mixed strategy, and M is large number

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Correlated Equilibrium (N-player Games!)

- Variables are now p_a for all action profiles a (i.e., outcome)

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\begin{aligned}
\text{max.} \quad & \text{whatever you like (e.g., social welfare)} \\
\text{s.t.} \quad & \sum_{a_{-i} \in A_{-i}} p_a u_i(a) \geq \sum_{a_{-i} \in A_{-i}} p_a u_i(t_i, a_{-i}) \quad \forall i, a_i, t_i \in A_i \\
& \sum_{a \in A} p_a = 1 \\
& p_a \geq 0, \qquad \forall a \in A
\end{aligned}
$$

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  - Vincent Conitzer [Duke CPS 590.4]
- Xiaoliang Zhou helped with importing slides from PowerPoint to LaTeX