Game-theoretic Foundations of Multi-agent Systems

Lecture 7: Stochastic Games

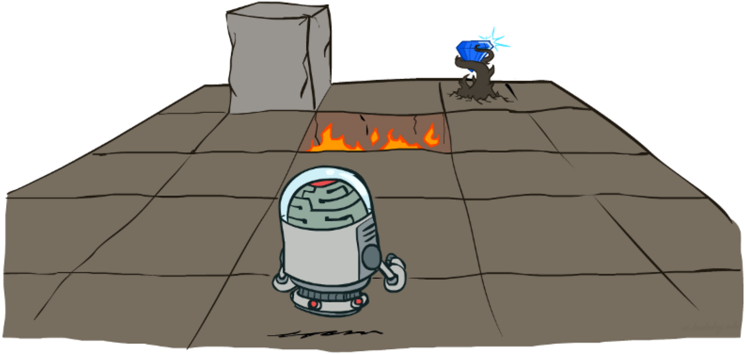
Seyed Majid Zahedi

UNIVERSITY OF WATERLOO

Outline

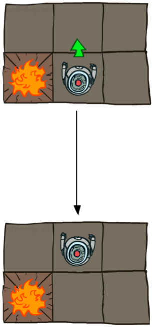
1. Markov Decision Processes
2. Definition of Stochastic Games
3. Strategies and Equilibria in Stochastic Games

Motivation: Non-deterministic Search in Grid World



Grid World Actions

Deterministic



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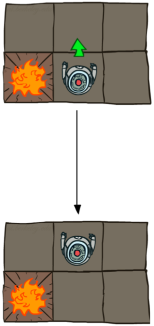
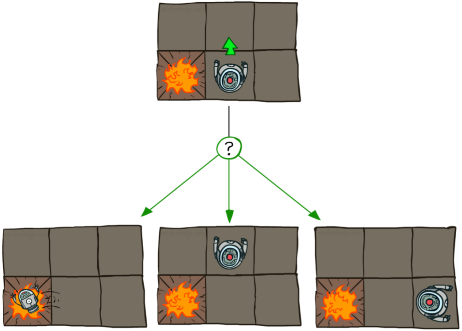
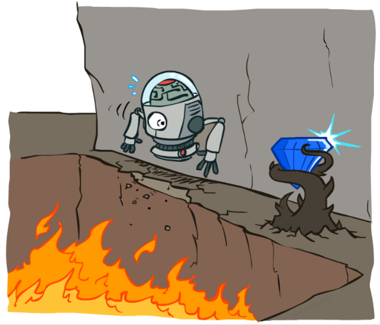


Stochastic



A Grid World Instance

- Agent lives in a grid
- Walls block agent's path
- Actions do not always go as planned
  - 80% of time, action "North" takes us north
  - 10% of time, "North" takes us west; 10% east
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- Agent receives rewards at each step
- Goal is to maximize sum of rewards

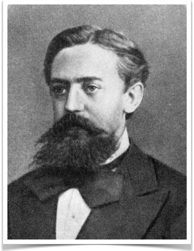


Change in Notation

- So far, we have used s to denote strategy profile
- In this lecture, we use π for strategy
- We use s to denote state

Markov Property

- Given present state, future and past are independent



Andrey Markov (1856-1922)

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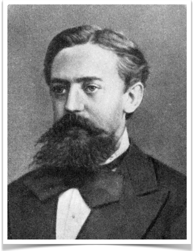

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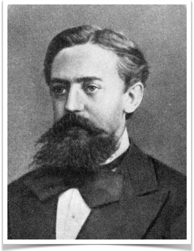

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$$P(S_{t+1} = s \mid S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1} = a_{t-1}, \ldots, S_0 = s_0) =$$
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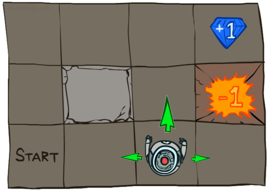
Markov Decision Processes: Formal Definition

- S is set of states and A is set of actions



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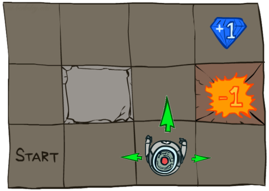

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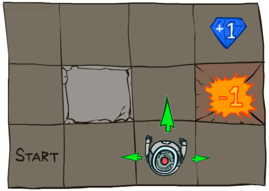

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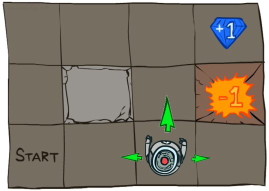
- There are two ways to aggregate rewards
  - Limit-average reward: $\lim_{T\to\infty} \sum_{t=1}^{T} r_t / T$
  - Future-discounted reward: $\sum_{t=1}^{\infty} \delta^{t-1} r_t$



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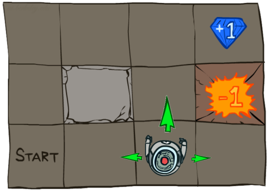

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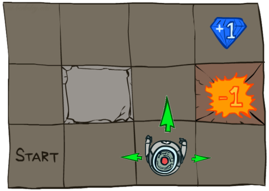
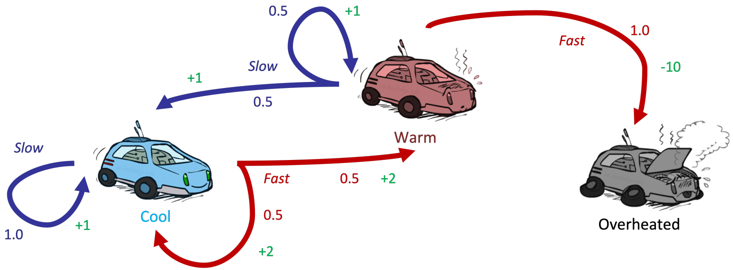
    (we will focus on this)



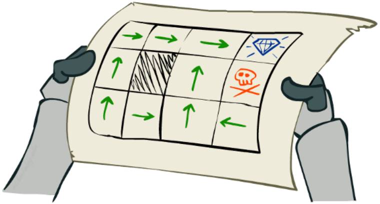
Example: Racing

- Autonomous car wants to travel far, quickly
- There are 3 states: Cool, Warm, Overheated, and 2 actions: Slow, Fast
- Going faster gets double reward



Policies

- A (stationary, deterministic) policy $\pi : S \mapsto A$ gives action for each state
- An optimal policy is one that maximizes expected utility if followed



Values of States

- Value function: $V^\pi(s)$ specifies value of following π starting in s

$$V^\pi(s) = \mathbb{E}\left[\sum_{t=1}^{\infty} \delta^{t-1} r_t(s_t, \pi(s_t), s_{t+1}) \mid s_0 = s\right]$$

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- These two are related to each other by

$$Q^\pi(s, a) = \sum_{s'} p(s, a, s') \left(r(s, a, s') + \delta V^\pi(s')\right)$$

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Policy Evaluation

Initialize $V_0^\pi(s) \leftarrow 0$ for all states s;
for $t = 1 \dots T$ **do**
  for *each state* s **do**
    $V_t^\pi(s) \leftarrow \sum_{s'} p(s, \pi(s), s') \left(r(s, \pi(s), s') + \delta V_{t-1}^\pi(s')\right)$

- How many iterations should we have (what should T be)?

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- How many iterations should we have (what should T be)?
- Repeat until values do not change much:

$$\max_{s \in S} |V_t^\pi(s) - V_{t-1}^\pi(s)| < \epsilon$$

Solving MDP: Bellman Equations

$$Q^*(s,a) = \sum_{s'} p(s,a,s')\left(r(s,a,s') + \delta V^*(s')\right)$$

$$V^*(s) = \max_{a \in A} \; Q^*(s,a)$$

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- Bellman equations characterize the optimal values
- Value iteration computes them
- Value iteration is just a fixed-point solution method

Policy Extraction

- Given V^*, we can compute optimal policy as follows:

$$\pi^*(s) = \operatorname*{argmax}_a \sum_{s'} p(s,a,s') \left(r(s,a,s') + \delta V^*(s') \right)$$

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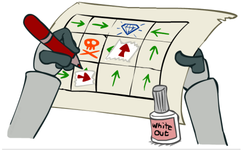
- Takeaway: actions are easier to select from Q-values than values!

Problems with Value Iteration

- It is slow - $O(S^2A)$
- The max at each state rarely changes
- The policy often converges long before the values

Policy Iteration

- (I) Policy evaluation: Calculate values for some fixed policy



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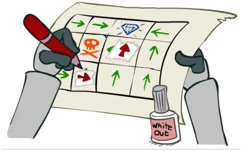

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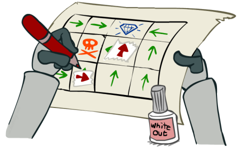
- Repeat steps until policy converges



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  - New policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

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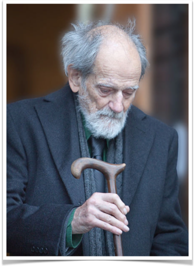

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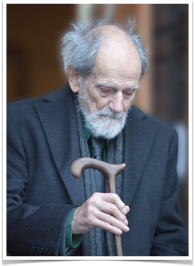

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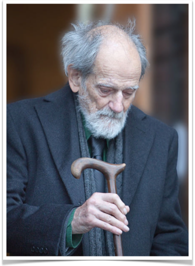

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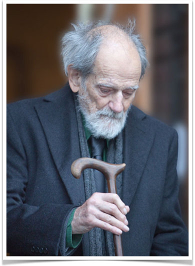

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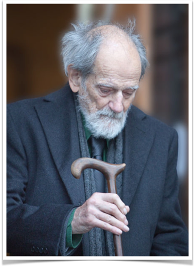

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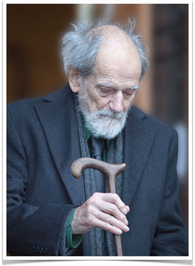

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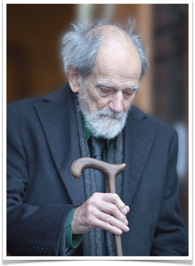
- Single-state stochastic game = (infinitely) repeated game
- Single-agent stochastic game = MDP



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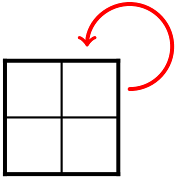
Repeated Games vs Stochastic Games

Repeated Games



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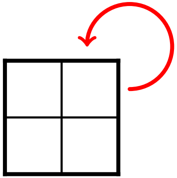

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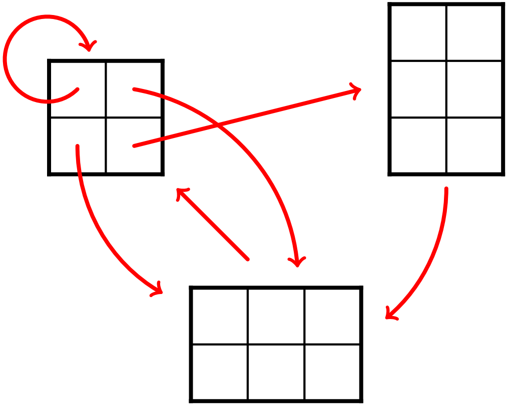

Stochastic Games: Formal Definition[1]

- S is finite set of stage games
- N is finite set of n agents
- A_i is finite set of actions available to agent i
- $p : S \times A \times S \mapsto [0, 1]$ is transition probability function
  - $p(s, a, s')$ is probability of going from s to s' after action profile a
- $r_i : S \times A \mapsto \mathbb{R}$ is real-valued utility function for agent i
  - $r_i(s, a)$ is agent i's utility at state s for action profile a

[1] Note that this definition assume actions available to agents are the same across different stage games. Changing this assumption leads to a more involved notation.

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2. Definition of Stochastic Games

3. Strategies and Equilibria in Stochastic Games

Stochastic Games: Strategies

- Let $h_t = (s_0, a_0, s_1, a_1, \ldots, a_{t-1}, s_t)$ denote history of t stages
- Let H_t be set of all possible histories of this length
- Set of all deterministic strategies for agent i is

$$\prod_{t, H_t} A_i$$

- Agents' strategies can consist of any mixture over deterministic strategies
- However, there are several restricted classes of strategies

Behavioral, Markov, and Stationary Strategies

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  - For each t, distribution over actions depends only on current state
- Stationary strategy π_i is a Markov strategy s.t. $\pi_i(h_{t_1}, a_i) = \pi_i(h'_{t_2}, a_i)$ if $s_{t_1} = s'_{t_2}$
  - s_{t_1} and s'_{t_2} are final states of h_{t_1} and h'_{t_2}, respectively
  - This removes possible dependence on time t

Markov-perfect Equilibrium (MPE)

- Strategy π is MPE if it is Markov strategy and is NE regardless of starting state

$$V_i^{\pi}(s) \geq V_i^{(\pi_i', \pi_{-i})}(s) \quad \forall i, s, \pi_i'$$

- MPE is similar to subgame-perfect equilibrium in perfect-information games

- Every n-player, general-sum, discounted-reward stochastic game has MPE

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- E.g., 2-player, zero-sum, discounted-reward stochastic games

Shapley Algorithm: Finding MPE in 2-player Zero-sum Games

Initialize $V_0(s)$ arbitrarily for all s; ▷ Agent 1's utility for being in s

repeat *until* $V(s)$ *converges for all* s

  for *each state* s **do**

    Compute matrix game $G(s, V_{t-1})$:

    $u(s,a) = r(s,a) + \delta \sum_{s'} p(s,a,s') V_{t-1}(s')$

  for *each state* s **do**

    $V_t(s) \leftarrow \max_{\pi_1} \min_{\pi_2} u_1(s, \pi_1, \pi_2)$

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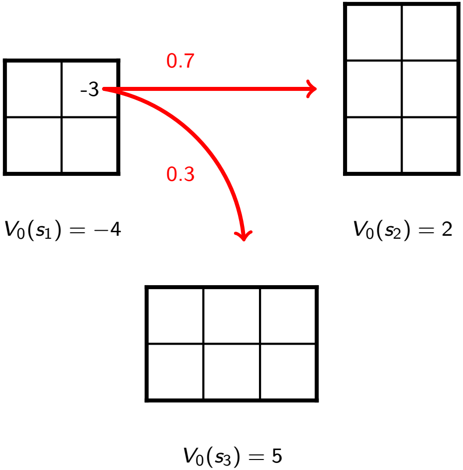
- Shapley's algorithm is extension of value iteration to stochastic games

Shapley Algorithm: Example



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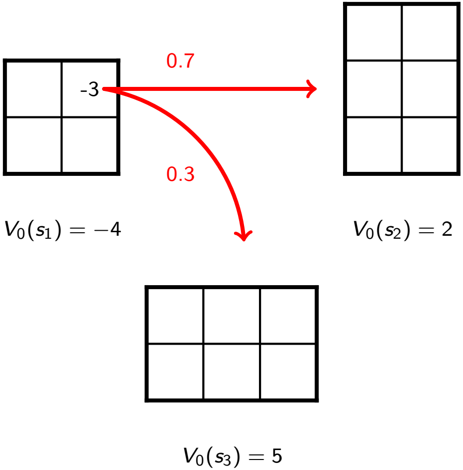

$$-3 + \delta\,(0.7 \times 2 + 0.3 \times 5)$$
$$= -3 + 2.9\,\delta$$

	-3 + 2.9δ

$G(s_1, V_0)$

Pollatschek & Avi-Itzhak Algorithm (Extension of Policy Iteration)

Initialize $V(s)$ arbitrarily for all s; ▷ Agent 1's utility for being in s
repeat *until* $\pi_1(s)$ *and* $\pi_2(s)$ *converge for all* s
  for *each state* s **do**
    Compute matrix game $G(s, V)$ as in Shapley's algorithm;
    $\pi_1(s) \leftarrow$ maxmin strategy of Agent 1 in $G(s, V)$;
    $\pi_2(s) \leftarrow$ minmax strategy agents Agent 1 in $G(s, V)$;
  Calculate $V(s)$ with policy evaluation for π_1 and π_2

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