Game-theoretic Foundations of Multi-agent Systems

Lecture 9: Learning in Games

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Outline

Single-agent vs Muli-agent Learning

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- Different learning rules lead to different dynamical system
- Simple learning rules can lead to complex global behaviors of system

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- E.g., in game of Chicken, if your opponent is learning your strategy to play best response, then optimal strategy is to always dare

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- Note that in GT, optimal strategy is replaced by best response (and equilibrium)

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- Rationality: Settle on best response to opponent's strategy whenever opponent settles on stationary strategy
  - Opponent adopts same mixed strategy each time, regardless of the past
- No regret: Yield payoff that is no less than payoff agent could have obtained by playing any pure strategy against any set of opponents (details later!)

Outline

Nash Equilibrium

- Nash equilibrium (NE): No agent wins from unilateral deviation

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  - Strict NE is necessarily a pure-strategy NE (why?)

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  - Agents who did not change have no better strategy in the new circumstance
  - Agent who made a small unilateral change will return immediately to NE

Nash Equilibrium Beyond Two-player Zero-sum Games

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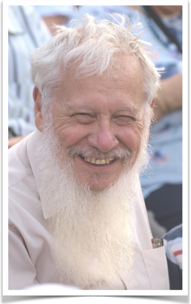
- NE is hard to compute even in two-player general-sum games
- Equilibrium selection is challenging (coordination without communication)

Correlated Equilibrium (CE)

- CE is notion of rationality proposed by Aumann[1]



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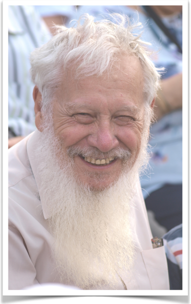

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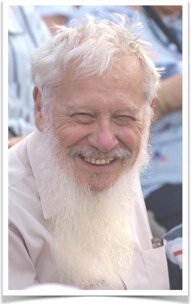

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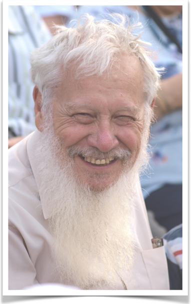

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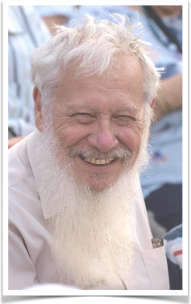

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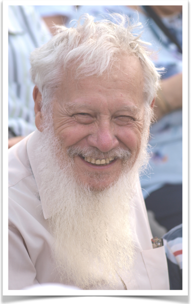
- CE does not suffer from equilibrium selection
- And, it enables better social welfare



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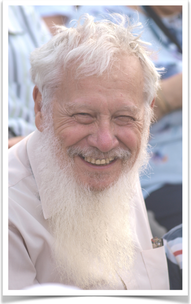
- CE arises naturally as empirical frequency of play by independent learners (details later!)



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Correlated Equilibrium CE (cont.)

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for all i and a_i'

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- Coarse correlated equilibrium could occasionally recommend really bad actions!

Coarse Correlated Equilibrium: Example

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- But, if recommendation is C, it is not best response to play C (why?)
- Therefore, π is not correlated equilibrium

Equilibrium Notions for Normal-form Games

- Dominant strategy equilibria (DSE)
- Pure strategy Nash equilibria (PSNE)
- Mixed strategy Nash equilibria (MSNE)
- Correlated equilibria (CE)
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- In two-player zero-sum games, CE $=$ CCE $=$ NE

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Fictitious Play: Introduction

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- In its current use, FP is misnomer, since each play of the game actually occurs

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- Agents assume that their opponent is using stationary mixed strategy
- Agents update their beliefs about this strategy at each step according to:

$$\mu_i^t(a_{-i}) = \frac{\eta_i^t(a_{-i})}{\sum_{a'_{-i}} \eta_i^t(a'_{-i})}$$

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- Agents best-respond to their beliefs about opponent' strategy

$$a_i^{t+1} = \underset{a_i}{\operatorname{argmax}}\ u_i(a_i, \mu_i^t)$$

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- Agents do not learn true model that generates empirical frequencies
- In other words, they do not learn how their opponent is actually playing the game

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- (I) If sequence converges to a^*, then a^* is pure-strategy NE of G
- (II) If for some t, $a^t = a^*$, where a^* is strict NE of G, then $a^\tau = a^*$ for all $\tau > t$

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- Since a_i^* maximizes both terms, it follows that it is played at $t+1$

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- This is possible because $\mu_i^t(a_{-i}) \to s_{-i}^*(a_{-i})$ by assumption

Proof (cont.)

- Then, for any $t \geq T$, we have:

$$u_i(a_i, \mu_i^t) \;=\; \sum_{a_{-i}} u_i(a_i, a_{-i}) \mu_i^t(a_{-i})$$

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- So after sufficiently large t, a_i is never played
- This implies that as $t \to \infty$, $\mu_i^t(a_i) \to 0$, which contradicts with $s_i^*(a_i) > 0$

Example: Matching Pennies

- Consider the matching-pennies game

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Round	1's η	2's η	1's action	2's action
1	(1.5, 2)	(2, 1.5)	T	T

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- FP continues as deterministic cycle, time average converges to unique NE

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  - G is potential game (more on this later!)

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- Shapley showed that in modified rock-scissors-paper game, FP does not converge

	Rock	Paper	Scissors
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Paper	$1, 0$	$0, 0$	$0, 1$
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- Convergence of fictitious play can not be guaranteed in general
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- This game has unique NE: each agent mixes uniformly
- Suppose $\eta_1^1 = (1, 0, 0)$ and $\eta_2^1 = (0, 1, 0)$
- Shapley showed that play cycles among 6 (off-diagonal) profiles with periods of ever-increasing length, thus non-convergence

Smooth Fictitious Play (SFP)

- Instead of best-responding to beliefs, agents respond randomly, but somewhat proportional to their expected utility

$$s_i^t(a_i \mid \mu_i^t) = \frac{\exp(u_i(a_i, \mu_i^t)/\gamma)}{\sum_{a_i'} \exp(u_i(a_i', \mu_i^t)/\gamma)}$$

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- Soft-max policy respects best replies, but leaves room for exploration
- If all agents use SFP with sufficiently small γ_i, empirical play converges to ϵ-CCE

Outline

Best-response Dynamics (BRD): Introduction

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- In arbitrary order, agents take turns updating their action
- Agent update their action only if doing so can improve their utility
- This is repeated until no agents wants to update their action

---

Initialize $a = (a_1, \ldots, a_n)$ to be arbitrary action profile;
while *there exists* i *such that* $a_i \notin \operatorname{argmax}_{a \in A_i} u_i(a, a_{-i})$ **do**
  Let a'_i be such that $u_i(a'_i, a_{-i}) > u(a)$;
  Set $a_i \leftarrow a'_i$;
return a

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  - No: Consider matching pennies/Rock Paper Scissors

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- $c_i(a) = \sum_{j \in a_i} \ell_j(n_j(a))$ is total cost of agent
- Agents minimize their total cost (instead of maximizing their total utility)

BRD in Congestion Games

- Consider potential function $\phi : A \to \mathbb{R}$:

$$\phi(a) = \sum_{j=1}^{m} \sum_{k=1}^{n_j(a)} \ell_j(k)$$

(Note: not social welfare)

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- Well... We know it must have decreased agent i's cost:

$$\begin{aligned} \Delta c_i &\equiv c_i(b_i, a_{-i}) - c_i(a_i, a_{-i}) \\ &= \sum_{j \in b_i \setminus a_i} \ell_j(n_j(a) + 1) - \sum_{j \in a_i \setminus b_i} \ell_j(n_j(s)) < 0 \end{aligned}$$

BRD in Congestion Games (cont.)

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- Which proves the existence of pure strategy Nash equilibria!

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- n clients $i \in N$ schedule jobs of size $w_i > 0$ on m identical servers M

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- For each server $j \in M$, load $\ell_j(a) = \sum_{i:a_i=j} w_i$
- Cost of client i is load of server that i chooses : $c_i(a) = \ell_{a_i}(a)$

Load Balancing Games on Identical Servers: Discussion

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- Load balancing games on identical servers have pure strategy NE

BRD in Load Balancing Games on Identical Servers

- Consider potential function ϕ as:

$$\phi(a) = \frac{1}{2} \sum_{j=1}^{m} \ell_j(a)^2$$

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Note: $\Delta c_i \neq \Delta\phi$

Potential Games

- $\phi : A \to \mathbb{R}_{\geq 0}$ is exact potential function for game G if for all a, i,a_i, and b_i:

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- BRD is guaranteed to converge in game G iff G has ordinal potential function

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  - Start at arbitrary vertex a, and then traverse arbitrary outgoing edges

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- Its true! $\phi(a) \geq \phi(b) + 1$. (why?)

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- However, we get advice

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- We (algorithm) want to aggregate predictions, to make our own prediction p_A^t

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- We (algorithm) want to aggregate predictions, to make our own prediction p_A^t
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Expert Advice

- There are N experts who make predictions in T rounds
- At each round t, each expert i makes prediction $p_i^t \in \{U, D\}$
- Expertise is self proclaimed — no promise experts know what they're talking about
- We (algorithm) want to aggregate predictions, to make our own prediction p_A^t
- We learn true outcome o^t at the end of each round
- If we predicted incorrectly (i.e. $p_A^t \neq o^t$), then we made a mistake

Expert Advice (cont.)

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- Goal is to after a while do (almost) as well as best expert in hindsight
- To make things easy, we assume for now that there is one perfect expert
- Perfect expert never makes mistakes (but we don't know who the expert is)
- Can we find strategy that is guaranteed to make at most $\log(N)$ mistakes?

The Halving Algorithm

Let $S^1 \leftarrow \{1, \ldots, N\}$ be set of all experts;
for $t = 1$ *to* T **do**
  Predict with majority vote;
  Observe the true outcome o^t;
  Eliminate all experts that made a mistake: $S^{t+1} = \{i \in S^t \mid p_i^t = o^t\}$;

The Halving Algorithm: Analysis

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- Can we find a way to make not too many more than OPT mistakes?

The Iterated Halving Algorithm

Let $S^1 \leftarrow \{1, \ldots, N\}$ be the set of all experts;
for $t = 1$ *to* T **do**
    if $|S^t| = 0$ **then**
        Reset: Set $S^t \leftarrow \{1, \ldots, N\}$
    Predict with majority vote;
    Eliminate all experts that made a mistake: $S^{t+1} = \{i \in S^t \mid p_i^t = o^t\}$;

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- How about just downweight experts who make mistakes?

The Weighted Majority Algorithm

Set weights $w_i^1 \leftarrow 1$ for all experts i;
for $t = 1$ *to* T **do**
  Predict with weighted majority vote;
  Down-weight experts who made mistakes: (i.e., if $p_i^t \neq o^t$, set $w_i^{t+1} \leftarrow w_i^t/2$)

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- Algorithm makes at most $2.4\,(\mathrm{OPT} + \log(N))$ mistakes
- $\log(N)$ is constant, so ratio of mistakes to OPT is 2.4 in limit – not great, but not bad

What Do We Want in an Algorithm?

- Make only $1\times$ as many mistakes as OPT in limit, rather than $2.4\times$
- Handle N distinct actions (separate action for each expert), not just up and down
- Handle arbitrary costs in $[0, 1]$ per expert per round, not just right and wrong

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- Total loss of algorithm is $L_A^T = \sum_{t=1}^T \ell_A^t$
- Goal is to obtain loss "not much worse" than that of the best expert: $\min_i L_i^T$

Multiplicative Weights (MW) Algorithm (a.w.a. Hedge Algorithm)

Set weights $w_i^1 \leftarrow 1$ for all experts i;
for $t = 1$ *to* T **do**
    Let $W^t = \sum_{i=1}^N w_i^t$;
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- Can be used with alternative update: $w_i^{t+1} \leftarrow w_i^t \cdot (1 - \epsilon \ell_i^t)$

Multiplicative Weights Algorithm: Discussion

- For any sequence of losses, and any expert k:

$$\frac{1}{T}\ \mathbb{E}[L_{MW}^{T}] \leq \frac{1}{T}\ L_k^T + \epsilon + \frac{\ln(N)}{\epsilon \cdot T}$$

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- So we could us it to play games (experts $\leftrightarrow$ actions and losses $\leftrightarrow$ costs)

Recall: Minimax Theorem (John von Neumann, 1928)

In any finite, two-player, zero-sum game, in any NE, each agent receives a payoff that is equal to both their maxmin value and their minmax value

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$$

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  - A2 uses MW algorithm: at round t, $s_2^t(a_2) = w_{a_2}^t / W^t$
  - A1 plays best response to A2's strategy: $s_1^t = \text{argmax}_{s_1} u_1(s_1, s_2^t)$

Simple Proof for Minimax Theorem (cont.)

- For A2's MW algorithm, we have:

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[u_1(a_1^t, a_2^t)] \leq \frac{1}{T}\min_{a_2}\sum_{t=1}^{T}u_1(a_1^t, a_2) + 2\sqrt{\frac{\log n}{T}}$$

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- Let $\bar{s}_1$ be mixed strategy that puts weight $1/T$ on each action a_1^t, we have:

$$\frac{1}{T}\min_{a_2}\sum_{t=1}^{T} u_1(a_1^t, a_2) = \min_{a_2}\sum_{t=1}^{T}\frac{1}{T}u_1(a_1^t, a_2) = \min_{a_2} u_1(\bar{s}_1, a_2)$$

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- By definition, we have: $\min_{a_2} u_1(\bar{s}_1, a_2) \leq \max_{s_1}\min_{a_2} u_1(s_1, a_2) = v_2$, and so:

$$\frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[u_1(a_1^t, a_2^t)] \leq v_2 + 2\sqrt{\frac{\log n}{T}}$$

Simple Proof for Minimax Theorem (cont.)

- On the other hand, A1 best responds to A2's mixed strategy:

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- Taking T large enough leads to contradiction

External Regret

- Sequence $a^1, \dots, a^T$ has external regret of $\Delta(T)$ if for every agent i and action a_i':

$$\frac{1}{T}\sum_{t=1}^{T} u_i(a^t) \geq \frac{1}{T}\sum_{t=1}^{T} u_i(a_i', a_{-i}) - \Delta(T)$$

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- External regret measures regret to the best fixed action in hindsight

External Regret

- Sequence $a^1, \ldots, a^T$ has external regret of $\Delta(T)$ if for every agent i and action a_i':

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- Idea: Run k copies of PW, one responsible for each S_j

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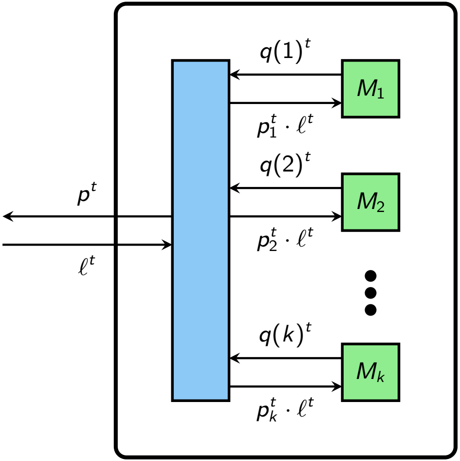
- I.e., to copy i, we report the true losses scaled by p_i^t

No-swap-regret Algorithm



No-swap-regret Algorithm: Analysis

- Expected cost of the master algorithm:

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- Goal: prove that (1) is at most (2) plus $\Delta(T) = o_T(1)$

No-swap-regret Algorithm: Analysis (cont.)

- Expected cost of M_j:

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- M_j is no-regret algorithm, so its cost is at most:

$$\frac{1}{T}\sum_{t=1}^{T} p_j^t \cdot \ell_{F(j)}^t + \Delta(T) \tag{4}$$

for any any arbitrary F

No-swap-regret Algorithm: Analysis (cont.)

- Summing inequality between (3) and (4) over all copies:

$$\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{k}\sum_{j=1}^{k} q(j)_i^t \left(p_j^t \cdot \ell_i^t\right) \leq \frac{1}{T}\sum_{t=1}^{T}\sum_{j=1}^{k} p_j^t \cdot \ell_{F(j)}^t + k \cdot \Delta(T) \tag{5}$$

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- Right-hand side is equal to (2)
- For left-hand side to be equal to (1), we need:

$$p_i^t = \sum_{j=1}^{k} p_j^t \cdot q(j)_i^t$$

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  - W.p. p_j^t we select copy j and then select expert i w.p. $q(j)_i^t$

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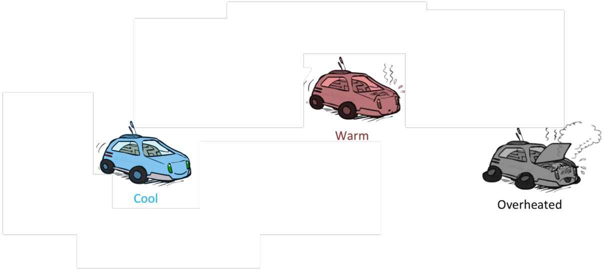
- Regret at time t for not having played a: $R^t(a) = \alpha^t(a) - \alpha^t$
- Regret matching: At time t, choose action a w.p. proportional to its regret:

$$s^t(a) = \frac{R^t(a)^+}{\sum_{a'} R^t(a')^+}$$

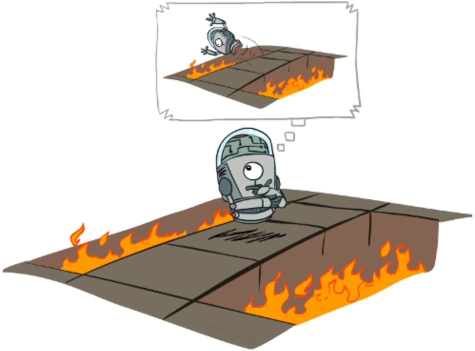
Outline

Reinforcement Learning

- Still assume MDP
  - Set of states $s \in S$
  - Set of actions $a \in A$
  - Model $p(s, a, s')$
  - Reward $r(s, a, s')$
- Still looking for policy $\pi(s)$
- New twist: we do not know p or r
- I.e. we do not know which states are good or what actions do
- Must actually try actions and states out to learn



Offline (MDPs) vs. Online (RL)



Offline solution

Online solution

Why Not Use Policy Evaluation?

- Simplified Bellman updates calculate V and Q for a fixed policy

$$V_t^{\pi}(s) \leftarrow \sum_{s'} p(s, \pi(s), s') \left(r(s, \pi(s), s') + \delta V_{t-1}^{\pi}(s') \right)$$

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- Unfortunately, we need p and r to do it!

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  - Move values toward value of whatever successor occurs: running average

$$\text{Sample of } V(s): \quad r(s,a,s') + \delta V^{\pi}(s')$$

$$\text{Update of } V(s): \quad V^{\pi}(s) \leftarrow (1-\alpha)V^{\pi}(s) + \alpha\left(r(s,a,s') + \delta V^{\pi}(s')\right)$$

$$\text{Same update}: \quad V^{\pi}(s) \leftarrow V^{\pi}(s) + \alpha\left(r(s,a,s') + \delta V^{\pi}(s') - V^{\pi}(s)\right)$$

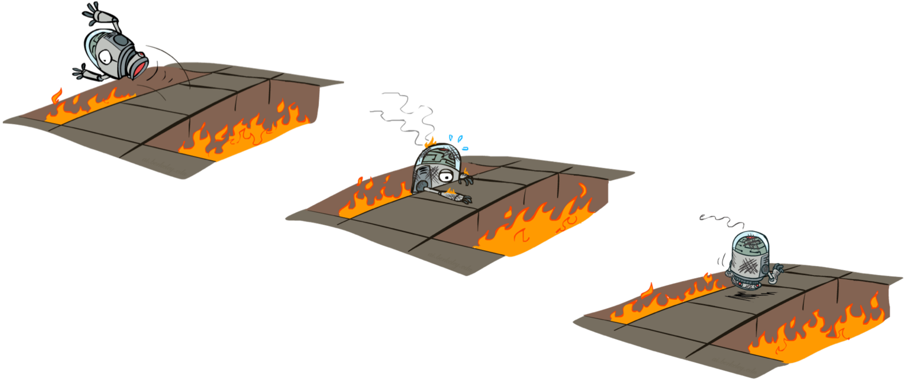
Problems with TD Value Learning

- TD value leaning is model-free way to do policy evaluation
- It mimics Bellman updates with running sample averages
- However, if we want to turn values into (new) policy, we need p and r!

$$\begin{aligned}\pi(s) &= \operatorname*{argmax}_a Q(s,a) \\ Q^{\pi}(s,a) &= \sum_{s'} p(s,a,s')\left(r(s,a,s') + \delta V(s')\right)\end{aligned}$$

- To solve this, we can learn Q-values instead of values
- This makes action selection model-free too!

Active Reinforcement Learning



Q-learning

- Q-Learning is sample-based Q-value iteration

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- We learn $Q(s,a)$ values as we go

$$\text{Sample}: \quad r(s,a,s') + \delta \max_{a' \in A} Q(s',a')$$

$$\text{Update}: \quad Q(s,a) \leftarrow (1-\alpha_t) Q(s,a) + \alpha_t \left(r(s,a,s') + \delta \max_{a' \in A} Q(s',a') \right)$$

Q-learning Algorithm

repeat *until convergence*

observe current state s;

select action a and take it (e.g., via ϵ-greedy policy);

observe next state s' and reward $r(s, a, s')$;

$Q_{t+1}(s, a) \leftarrow (1 - \alpha_t)Q_t(s, a) + \alpha_t \left(r(s, a, s') + \delta V_t(s')\right)$;

$V_{t+1}(s) \leftarrow \max_a Q_t(s, a)$;

- ϵ-greedy: W.p. ϵ, act randomly, w.p. $(1 - \epsilon)$ act according to Q_t

Q-learning Properties

- Q-learning converges to optimal policy – even if agent acts sub-optimally!
- This is called off-policy learning
- There are some caveats
  - We have to explore enough
  - We have to eventually make the learning rate small enough
  - But we should not decrease it too quickly
  - Q-learning converges if $\sum_0^\infty \alpha_t = \infty$ and $\sum_0^\infty \alpha_t^2 < \infty$
  - Basically, in the limit, it does not matter how you select actions (!)

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- Bad news: No convergence guarantee if both agents are learning (e.g., self play)!

Minimax-Q

- Littman[4] extended Q-learning algorithm to zero-sum stochastic games

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- Main idea is to modify Q-function to consider actions of opponent

$$Q_{i,t+1}(s_t, a_t) = (1 - \alpha_t) Q_{i,t}(s_t, a_t) + \alpha_t \left(r_i(s_t, a_t) + \delta V_{i,t}(s_{t+1}) \right)$$

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- Since game is zero sum, we can have

$$V_{i,t}(s) = \max_{\pi_i} \min_{a_{-i}} Q_{i,t}(s, \pi_i, a_{-i})$$

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Minimax-Q Algorithm

repeat *until convergence*

observe current state s;

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- $V_{t+1}(s) \leftarrow \min_{a_{-i}} \sum_{a_i} \pi(s, a_i)Q_{i,t}(s, a_i, a_{-i})$;

Minimax-Q Algorithm: Discussion

- It guarantees agents payoff at least equal to that of their maxmin strategy
- In zero-sum games, minimax-Q converges to the value of the game in self play
- It no longer satisfies no-regret property
- If opponent plays sub-optimally, minimax-Q does not exploit it in most games

Nash-Q

- Hu and Wellman[5] extended minimax-Q to general-sum games
- Algorithm is structurally identical to minimax-Q
- Extension requires that each agent maintains values for all other agents
- LP to find maxmin value is replaced with quadratic programming to find NE
- Nash-Q makes number of very limiting assumptions (e.g., uniqueness of NE)

[5] Hu, J, and Wellman, M. P. "Multiagent reinforcement learning: theoretical framework and an algorithm." 1998

Recall: Stochastic Games Model

- Focus on stationary Markov strategies (a mixed strategy per state)

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- $\pi = (\pi_1, \ldots, \pi_n)$ denotes strategy profile of all agents
- Expected utility (value) function of agent i is

$$v_i(s, \pi) := \mathbb{E}_{a_k \sim \pi(s_k)} \left[\sum_{k=0}^{\infty} \delta^k r_i(s_k, a_k) \mid s_0 = s \right]$$

Equilibrium Characterization

- Equilibrium value function is defined using one-stage deviation principle (multi-agent extension of Bellman's equation) as

$$v_i(s, \pi^*) = \max_{\pi_i} \mathbb{E}_{a \sim (\pi_i, \pi^*_{-i}(s))} \left[r_i(s, a) + \delta \sum_{s' \in S} p(s, a, s') v_i(s', \pi^*) \right]$$

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$$Q_i(s, a, \pi^*) = r_i(s, a) + \delta \sum_{s' \in S} p(s, a, s') v_i(s', \pi^*)$$

- Recursion is then defined as

$$v_i(s, \pi^*) = \max_{\pi_i} \mathbb{E}_{a \sim (\pi_i, \pi^*_{-i}(s))} [Q_i(s, a, \pi^*)]$$

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- Consider learning dynamic that combines FP with value-function (or Q-function) iteration

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- Agents then choose best response action in auxiliary game given their beliefs (where payoffs are given by Q-function estimates)
- Key challenge is that payoffs or value functions in these auxiliary games are non-stationary (unlike repeated play of stage games)

FP for Model-based Learning: Model

- At time t, i's belief on $-i$'s strategy is μ_i^t and on own Q-function is

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where $v_i^t(s') = \max_{a_i} Q_i^t(s', a_i, \mu_i^t(s))$

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- Beliefs on Q-functions are updated at slower rate than beliefs on opponent strategies
- This postulate agents' choices to be more dynamic than changes in their preferences
- Q-functions in auxiliary games can be viewed as slowly evolving agent preferences
- This enables weakening the dependence between evolving strategies and Q-functions

Convergence of Two-timescale Learning Framework

- If each state is visited infinitely many times
- And, if $\lim_{k\to\infty} \alpha_k = \lim_{k\to\infty} \beta_k = 0$ and $\sum_k \alpha_k = \sum_k \beta_k = \infty$
- And, if $\lim_{k\to\infty} \beta_k/\alpha_k = 0$ (two-timescale learning: $\beta_k \to 0$ faster than $\alpha_k \to 0$)
- Then Q and μ converge to NE value and strategy in zero-sum stochastic games
- They also converge to NE value for single-controller stochastic games

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