A/B Testing in Auctions

Wednesday, May 3, 2017 7:12 AM

▷ Last time:

- Inferring Value distribution from bids
- We showed uniform convergence estimators with rates $O\left(\frac{1}{n^{1/8}}\right)$ for CDF of value distribution.
- Optimal is $O\left(\left(\frac{\log n}{n}\right)^{R/(2R+3)}\right)$ if PDF has $R+1$ continuous bounded derivatives.
- Even for $R=1 \Rightarrow n^{-1/5}$ is a very slow rate: if you want an $\varepsilon = 0.01$ error, you need: $\frac{1}{\varepsilon^5} = 10^{10}$ samples!
- Too impractical.

▷ What if all we want is to understand revenue comparisons between two auctions? i.e. A/B testing auctions for revenue.

▷ We will see today that for a large class of auctions, the latter can be done at a $O\left(\frac{1}{\sqrt{N}}\right)$ rate.

▷ <u>Position Auctions:</u> 1.

n bidders

$w_1 \geq w_2 \geq w_3 \geq w_m$ m positions.

Allocate to bidders in decreasing order of bids. If bidder gets slot j he gets an allocation of w_j

▷ Equivalent: Randomization over k-unit auctions.

- k-unit auction:
  - n bidders, k units.
  - Allocate to highest k bidders.

- A position auction w/ $(w_1, \ldots, w_m)$ is equivalent in allocation to running

w.p. $\bar{w}_k = w_k - w_{k+1}$ a k-unit auction.

$$\bar{w}_0 = 1 - w_1$$

- Pf) If you are the j-th highest bidder then in position auction you get w_j

In randomized k-unit auction:

- You are allocated if $k \geq j$, i.e.

$$Pr(\text{alloc}) = \sum_{j=k}^{m} \bar{w}_j = \sum_{j=k}^{m} (w_j - w_{j+1}) = w_k$$

$\boxed{}$

- So we will look at the class of auctions that are defined as randomizations over k-unit highest-bids-win auctions.
- Payment Rule: All-Pay: pay what you bid no matter what allocation you get

(similar analysis can be done for First Price but open question for second-price)

- Value distribution is symmetric, with continuous F on support $[0, \bar{v}]$.

Lemma) Any such auction has a unique equilibrium which is symmetric.

▷ So expected allocation function if you have value v in a k-unit auction is

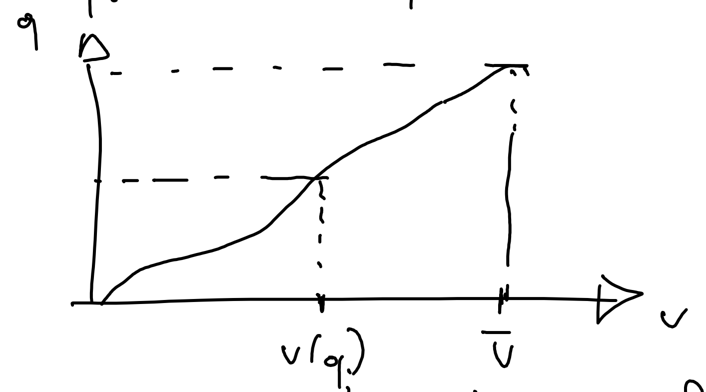
$$x_k(v) = P(\leq k-1 \text{ other bidders above you})$$
$$= \sum_{t=1}^{k-1} \binom{n-1}{t} (1-F(v))^t \, F(v)^{n-1-t}.$$

▷ Because we don't know $F(.)$ we will transform Myerson's theory in what is known as the quantile space:

- Each player has a quantile $q_i = F(v_i)$ i.e. his private parameter is the probability ...

that a random sample from F falls below his value.

- Then $v(q_i) = F^{-1}(q_i)$



- Quantiles are distributed uniformly in $[0,1]$. since:

$$P(Q(V) \le q) = P(V \le v(q))$$
$$= F(v(q)) = q$$

- Now if you have quantile q your expected allocation is:

$$x_k(q) = \sum_{t=0}^{k-1} \binom{n-1}{t} (1-q)^t q^{n-1-t}$$

which is a known function.

- Utility now becomes:

$$u(q) = v(q)\, x(q) - b(q).$$

Where $b(.)$ is equilibrium bid function.

▷ Part 1 : Myerson's theory in quantile space

- Myersons Lemma: For any mechanism

$$\mathbb{E}[\text{Rev}] = \mathbb{E}\Big[\sum_i \phi_i(v)\, x_i(v)\Big]$$

Since we are in symmetric setting, suffices to look at $\mathbb{E}[P]$ = revenue contribution of player:

$$E[P] = \mathbb{E}[\phi(v)\, x(v)]$$

$$\phi(v) = v - \frac{1-F(v)}{f(v)}$$

- Quantile space transformation:

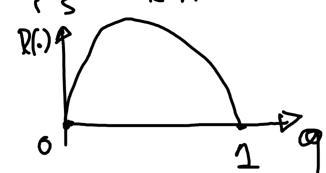
$$\phi(v(q)) = v(q) - (1-q)\, v'(q)$$

Reason: $1 - F(v(q)) = 1-q$

$$v'(q) = \left(F^{-1}(q)\right)' = \frac{1}{f(v(q))}$$

Observe: $\phi(q) = -\left(v(q)(1-q)\right)' = R'(q)$

where: $R(q) = v(q)(1-q)$ is known as the "Revenue function"



Finally: $\mathbb{E}[\phi(v)\, x(v)] = \int_0^{\bar{v}} \phi(v)\, x(v)\, f(v)\, dv$

$q = F(v) \Rightarrow dq = f(v)\,dv$

$$= \int_0^1 \phi(v(q))\, x(v(q))\, dq$$

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$-R'(q) \uparrow$ in q $\quad = -\int_0^1 R'(q)\, x(q)\, dq$

$\Rightarrow R'(q) \downarrow$ in q

$\Rightarrow R''(q) \leq 0$ $\quad = -\mathbb{E}[R'(q)\, x(q)]$.

$\Rightarrow R$ is concave.

<u>Alternative</u>: Integration by parts:

$$-\mathbb{E}[R'(q)x(q)] = -[R(q)x(q)]_0^1 + \mathbb{E}[R(q)x'(q)]$$

$$= +\mathbb{E}[R(q)\, x'(q)]$$

$$= +\mathbb{E}[v(q)(1-q)\, x'(q)]$$

So P is a weigthed integral of $v(q)$.

- We still don't know $v(q)$ but at least now we can use the value inversion trick we did in the FPA.

<u>Aproach</u>

- We need to relate observed bids in auction A w/ underlying values.

$$\underbrace{v(q)}_{\text{unknown}} = f(\underbrace{b_A(q)}_{\text{(observed via samples and can be estimated)}})$$

Think of $b_A(\cdot)$ as CDF of bid distr. Actually it is the $G_A^{-1}(\cdot)$.

- Then we will plug these values in the revenue expression

$$P_B = E_q\left[v(q)\,(1-q)\,x_B'(q) \right]$$

$$= E_q\left[f\big(b_A(q)\big)\,(1-q)\,x_B'(q) \right]$$

- Then we have a direct connection between observed $b_A(\cdot)$ (via samples) and P_B.
- Hopefully above expression will be robust to estimation errors in $b_A(\cdot)$.

▷ <u>Bid inversion for All-pay auction</u>

$u_A(z; q)$ = Expected utility if I have quantile q and bid as if I have quantile z

$$= v(q)\,x_A(z) - b_A(z).$$

By FOC:

$$\left.\frac{\partial u(z;q)}{\partial z}\right|_{z=q} = 0 \Rightarrow v(q)\,x_A'(q) - b_A'(q) = 0$$

$$\Rightarrow \boxed{v(q) = \frac{b_A'(q)}{x_A'(q)}}$$

value inversion

([illegible] vs density)

value inversion
(think of $b_A'(\cdot)$ as density of bid distribution)

▷ So now let's plug it in:

$$P_B = \mathbb{E}\left[v(q)\,(1-q)\,x_B'(q) \right]$$

$$= \mathbb{E}\left[\frac{b_A'(q)}{x_A'(q)}\,(1-q)\,x_B'(q) \right]$$

$$= \mathbb{E}\Big[\underbrace{b_A'(q)}_{\substack{\text{derivative}\\ \text{of bid}\\ \text{function.}\\ \text{Hard to estimate}\\ \text{Can we replace}\\ \text{with } b(\cdot)}} \; \underbrace{(1-q)\,\frac{x_B'(q)}{x_A'(q)}}_{\substack{\text{known function}\\ Z(q)}} \Big]$$

(integration by parts)

$$= \Big[b_A(q)\,Z(q) \Big]_0^1 - \mathbb{E}\left[b_A(q)\,Z'(q) \right]$$

$$= \underbrace{- \mathbb{E}\left[Z'(q)\,b_A(q) \right]}_{\text{Success!}}$$

▷ Since $b_A(\cdot) = G_A^{-1}(\cdot)$ we know

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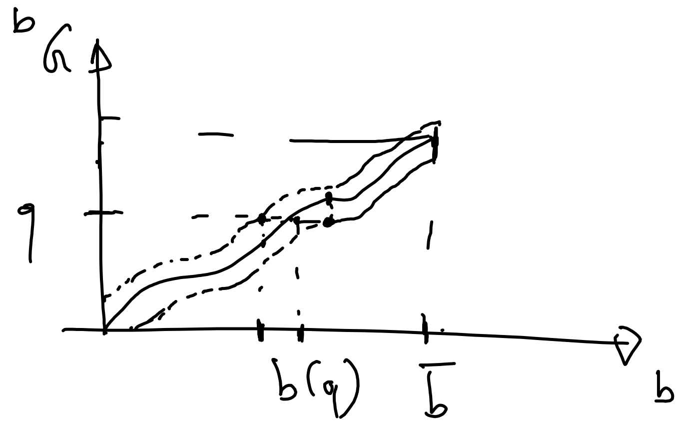
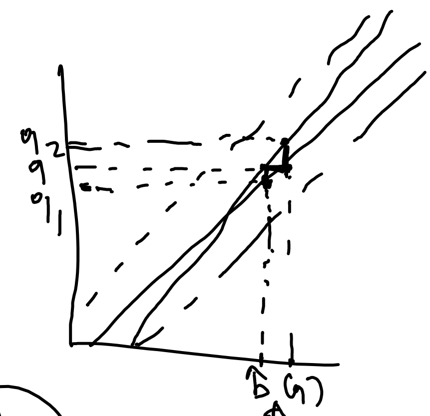
by DKW: w.p. $1-\delta$

$$\sup \left| \hat{G}_A(b) - G(b) \right| \le \sqrt{\frac{\log 1/\delta}{N}}$$

N ← number of samples.

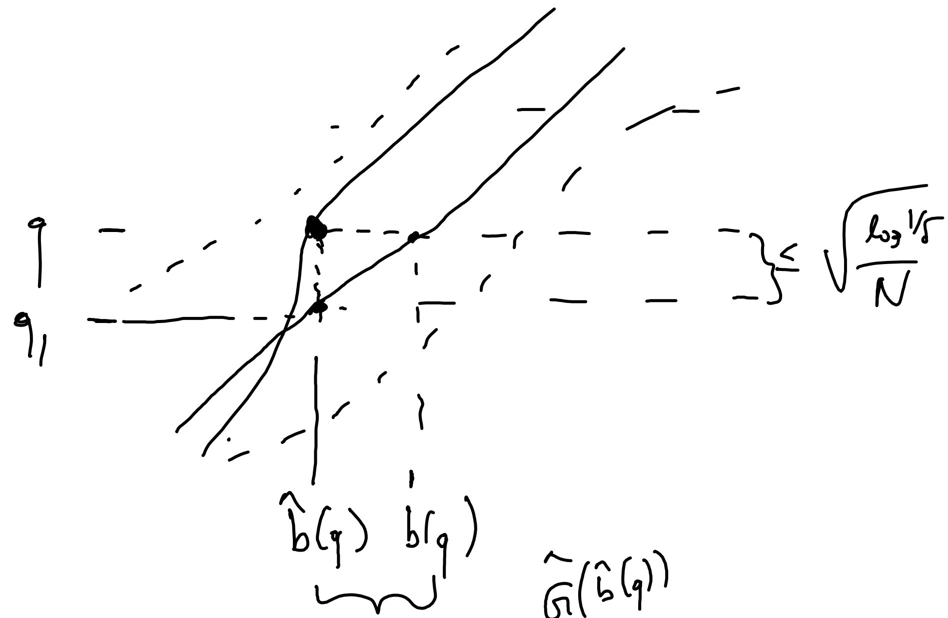




For any $q \in [0,1]$:

$$\sup_q \left| \hat{b}_A(q) - b_A(q) \right| \le \sup_q b'(q) \cdot \sup_b \left| \hat{G}_A(b) - G(b) \right|$$

Pf]



$\tilde G(\hat b(q))$ $\quad$ $G(\hat b(q))$

$$\left| \hat b(q) - b(q) \right| \le b'(q) \left| q - q_1 \right|$$

$$\le b'(q) \sqrt{\frac{\log 1/\delta}{N}}$$

$\hat b(q) = b(q_1)$

▷ So :

$$\left| \hat P_B - P_B \right| = \left| \mathbb{E}\left[z'(q) \left(\hat b_A(q) - b_A(q) \right) \right] \right|$$

$$\le \mathbb{E}\left[|z'(q)| \right] \sup_q \left| \hat b_A(q) - b_A(q) \right|$$

$$\le \mathbb{E}\left[|z'(q)| \right] \left(\sup_q b'_A(q) \right) \sqrt{\frac{\log 1/\delta}{N}}$$

$$b'(q) = v(q)\, x'_A(q) \le \bar v\, x'_A(q) \le n \cdot \bar v$$

▷ All remains to show $\mathbb{E}_q\left[\,|Z'(q)|\,\right]$ is bounded by a constant.

$$Z(q) = (1-q)\,\frac{x_B'(q)}{x_A'(q)} = \sum_{k=1}^{m} w_k^B \underbrace{(1-q)\,\frac{x_k'(q)}{x_A'(q)}}_{Z_k}$$

Suffices $\mathbb{E}_q\left[\,|Z_k'(q)|\,\right] \le ct.$

$$\frac{1}{Z_k(q)} = \sum_{\tilde{k}} w_{\tilde{k}}^A \,\frac{x_{\tilde{k}}'(q)}{(1-q)\,x_k'(q)} = \sum_{\tilde{k}} w_{\tilde{k}}^A\,\alpha_{k\tilde{k}} \underbrace{(1-q)^{\tilde{k}-k-1}\, q^{k-\tilde{k}}}_{\text{convex}}$$

$\Rightarrow \dfrac{1}{Z_k(q)}$ has unique min $\Rightarrow Z_k(q)$ unique max

$$\Rightarrow \mathbb{E}\left[\,|Z_k'(q)|\,\right] \le \left(Z_k(q^*) - Z_k(0)\right) + \left(Z_k(q^*) - Z_k(1)\right)$$

$$\le 2 Z_k(q^*) \le 2 \sup_q Z_k(q)$$

$$\Rightarrow \left|\hat{P}_B - P_B\right| \le O\left(\sqrt{\frac{\log 1/\delta}{N}} \cdot n\bar{V} \cdot m \left\{\sup_{k,q} \frac{x_k'(q)}{x_A'(q)}\right\}\right)$$

if x_A has at least ε of each k-unit

duction: $x_A'(q) = \sum_k w_k\, x_k'(q)$

$$\ge \varepsilon\, x_k'(q) \quad \forall k.$$

$$\geq \varepsilon \, x_k'(q) \quad \forall k.$$

$$\Rightarrow \quad |\hat{P}_B - P_B| \leq O\left(\sqrt{\frac{\log 1/\delta}{N}} \quad n \cdot \bar{v} \cdot m \, \frac{1}{\varepsilon}\right)$$