MS&E 233
Game Theory, Data Science and AI
Lecture 11

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(by courtesy) Computer Science and Electrical Engineering

Institute for Computational and Mathematical Engineering

Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- Basics of extensive-form games
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games, equilibria and online learning (T)
- Online learning in general games
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Basic Auctions and Learning to bid in auctions (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- **Optimal auctions and mechanisms (T)**
- Simple vs optimal mechanisms (T)
- *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

6
- Optimizing mechanisms from samples (T)
- Online optimization of auctions and mechanisms (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Sum: Vickrey-Clarke-Groves (VCG)

A universal welfare maximizing auction/mechanism!

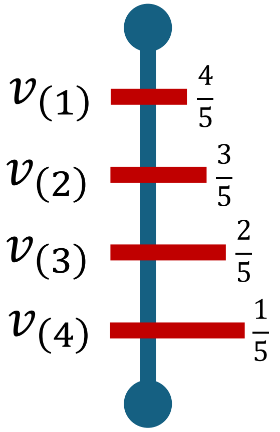
For any mechanism design setting, it guarantees that:

1. It is dominant strategy truthful
2. It always chooses the welfare maximizing outcome/allocation
3. All bidders have non-negative utility
4. All payments are non-negative

For special case of single-item auction = Second-Price Auction

What if we want to maximize revenue?

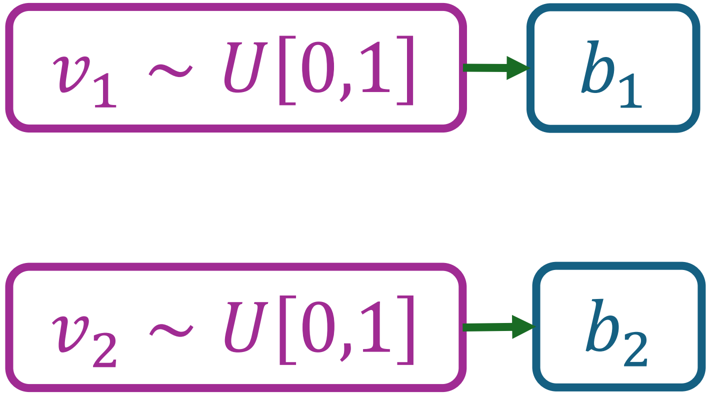
Let's go back to basics: Single-Item Auction

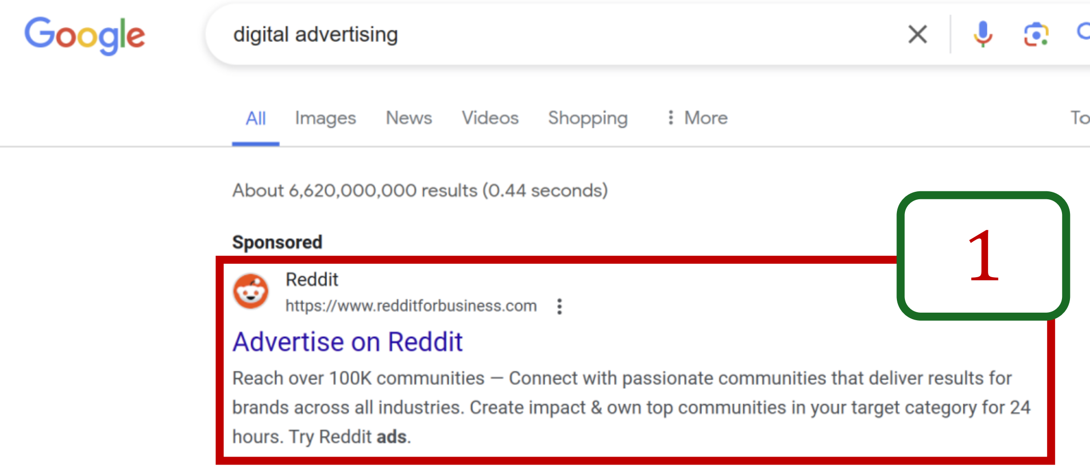


- How much revenue does the second-price auction achieve?

$$\text{Rev} = E\left[v_{(2)}\right] = E[\min(v_1, v_2)] = 1/3$$

- Can we do better?





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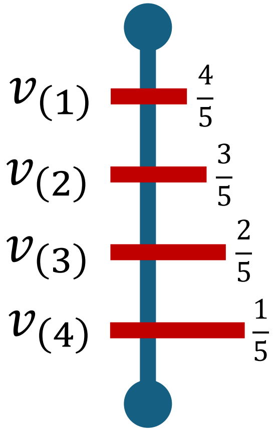
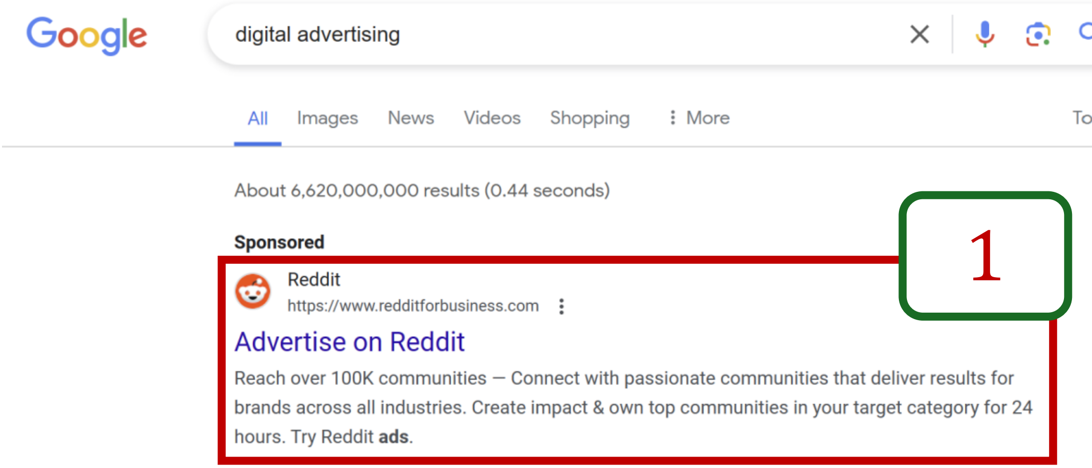


- What if we only had one bidder?

$$\text{Rev} = E\left[v_{(2)}\right] = 0$$

- Can we do better?

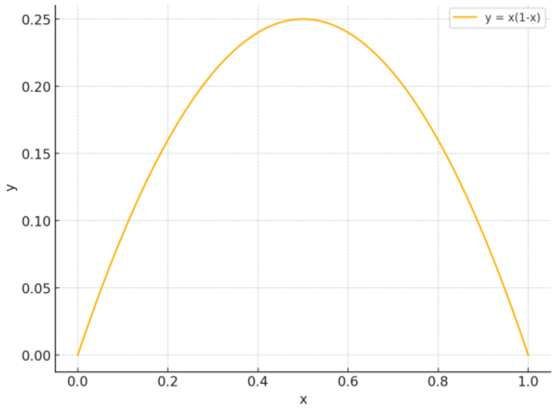


What if we post a reserve price?

Let's go back to basics: Single-Item Auction

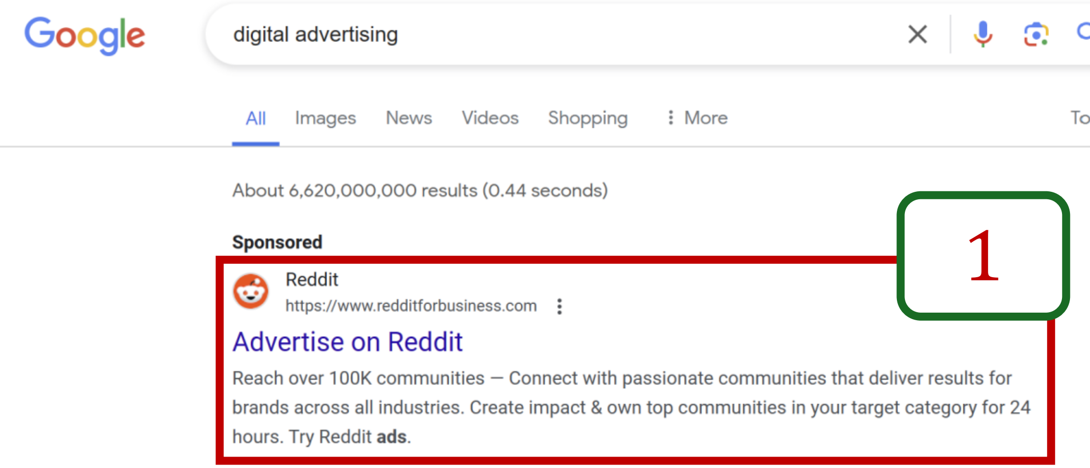
- **Auctioneer:** "If you bid less than r, I will not accept your bid and not show any ad on the page! If you win you must pay r."

$$\text{Rev(r)} = E[\text{r } 1(\text{v} \geq \text{r})] = r\,(1 - r) \Rightarrow \text{Rev}(1/2) = 1/4$$



- Is the auction truthful?
- Is the auction efficient?

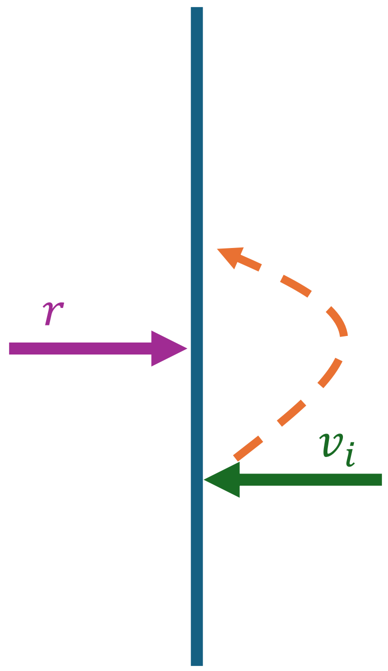
$v_1 \sim U[0,1] \rightarrow b_1$



Truthfulness of Mechanism

Suppose I bid my value. Would I want to deviate?

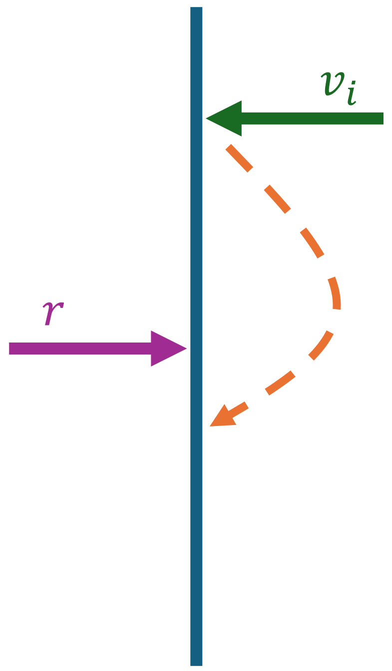
- **Case 1.** My value is below reserve price
- Only way to change anything is bid above
- But then I get negative utility as I pay more than value



Truthfulness of Mechanism

Suppose I bid my value. Would I want to deviate?

- **Case 2.** My value is above reserve price
- I get non-negative utility
- Only way to change anything is bid below
- But then I get zero utility as I lose

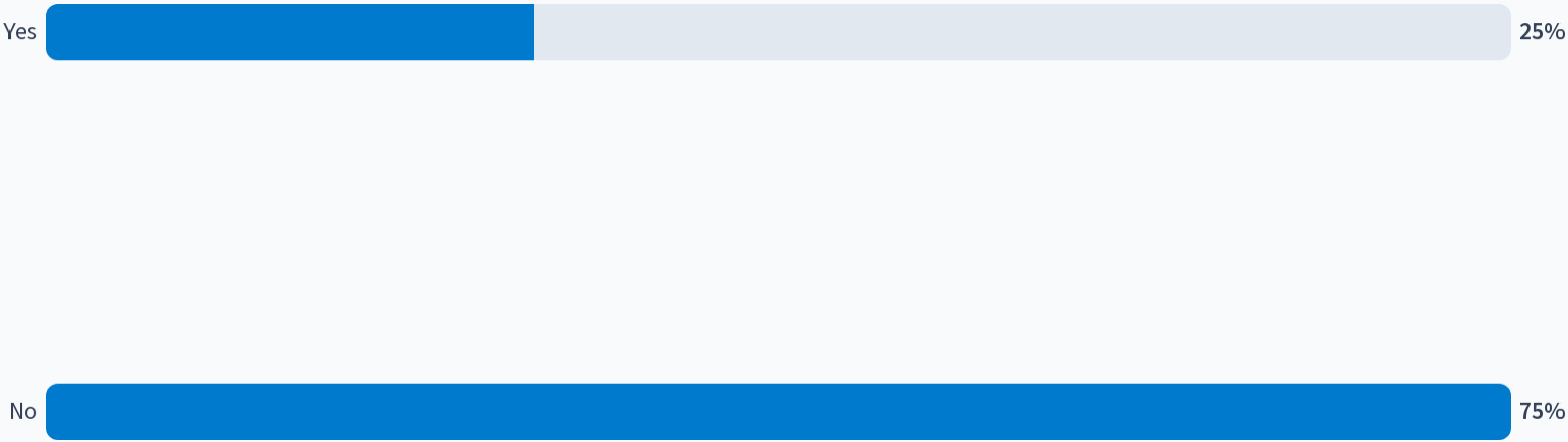


Is the mechanism efficient?

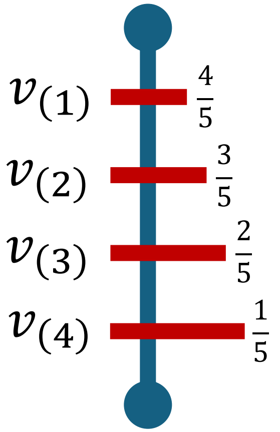
Yes

No

Is the mechanism efficient?



Let's go back to basics: Single-Item Auction



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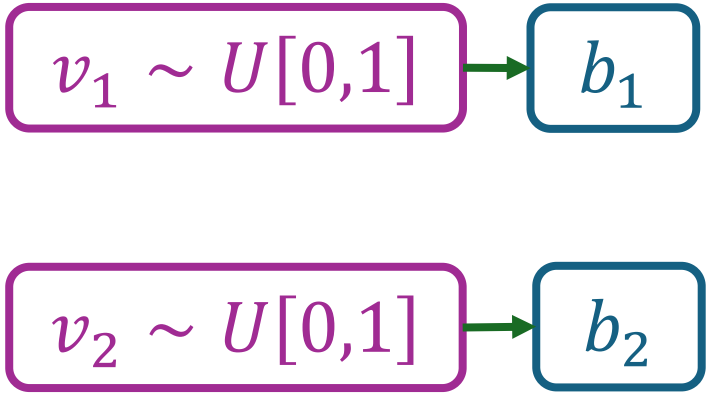
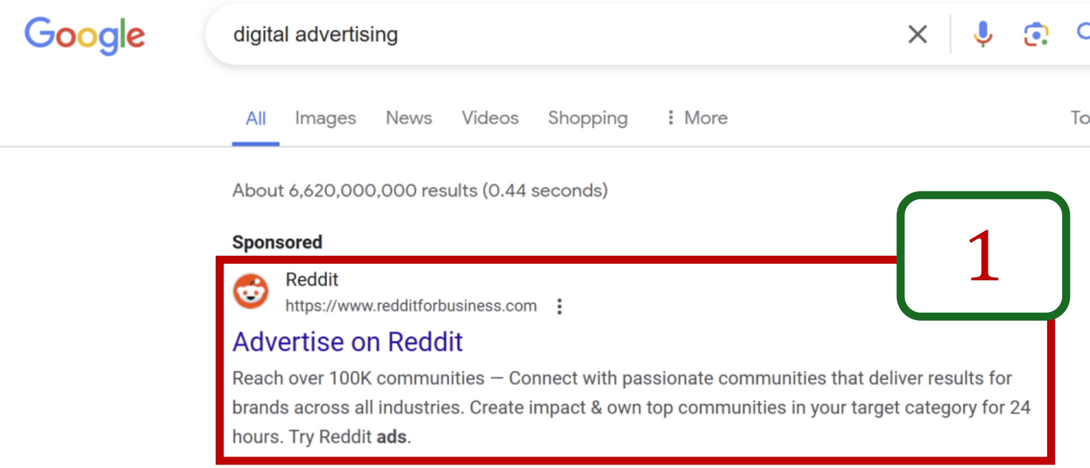


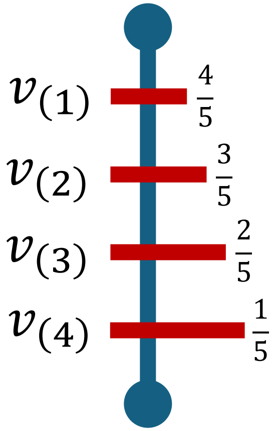


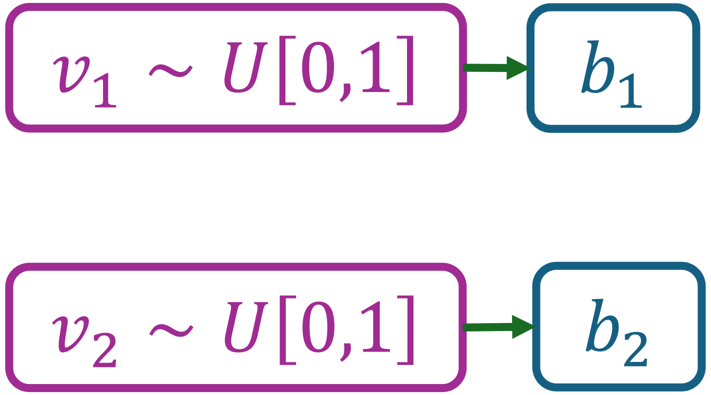
Let's go back to basics: Single-Item Auction

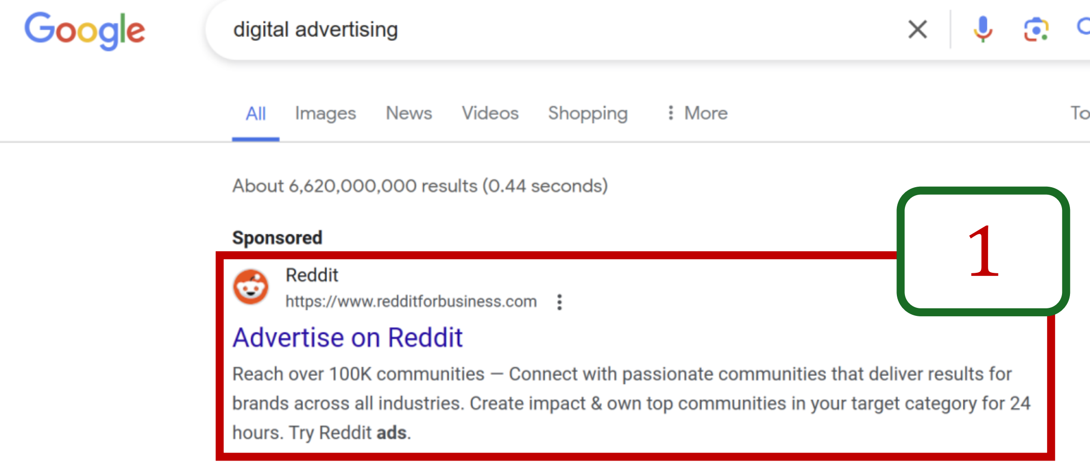
- **Auctioneer:** "If you bid less than r, I will not accept your bid! If you win you must pay $\max(b_2, r)$."

$$\mathrm{Rev}(1/2) = E\left[\max\left(v_{(2)}, r\right) 1\left(v_{(1)} \geq r\right)\right] = 5/12$$

- Can we do better?







How do we optimize over all possible mechanisms!

Single-Parameter Settings

- Each bidder has some value v_i for being allocated
- Bidders submit a reported value b_i (without loss of generality)
- Mechanism decides on an allocation $x \in X \subseteq \{0,1\}^n$
- Mechanism fixes a probabilistic allocation rule:

$$x(b) \in \Delta(X)$$

- **First question.** Given an allocation rule, when can we find a payment rule p so that the overall mechanism is truthful?
- If we can find such a payment, we will say that x is implementable

Some Shorthand Notation

- Let's fix bidder i and what other bidders bid b_{-i}
- For simplicity of notation, we drop index i and b_{-i}

- What properties does the function

$$x(v) \equiv x_i(v, b_{-i})$$

need to satisfy, so that x is implementable?

- Can we find a truthful payment function

$$p(v) \equiv p(v, b_{-i})$$

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?

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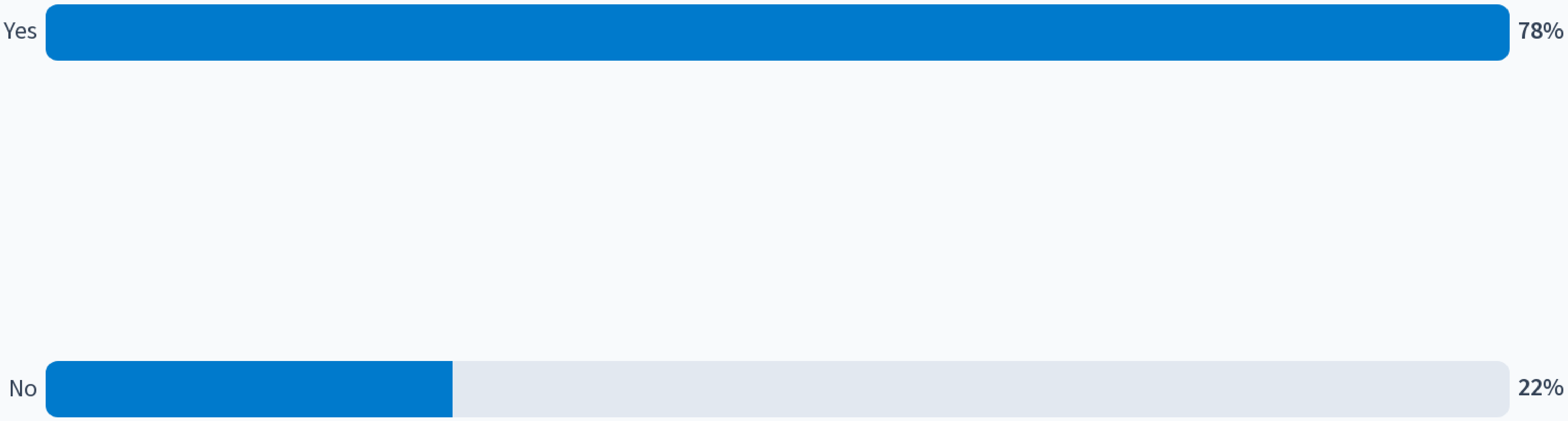
Yes

No

Is it possible to construct a mechanism that always allocates to the second highest value player and is dominant strategy truthful?



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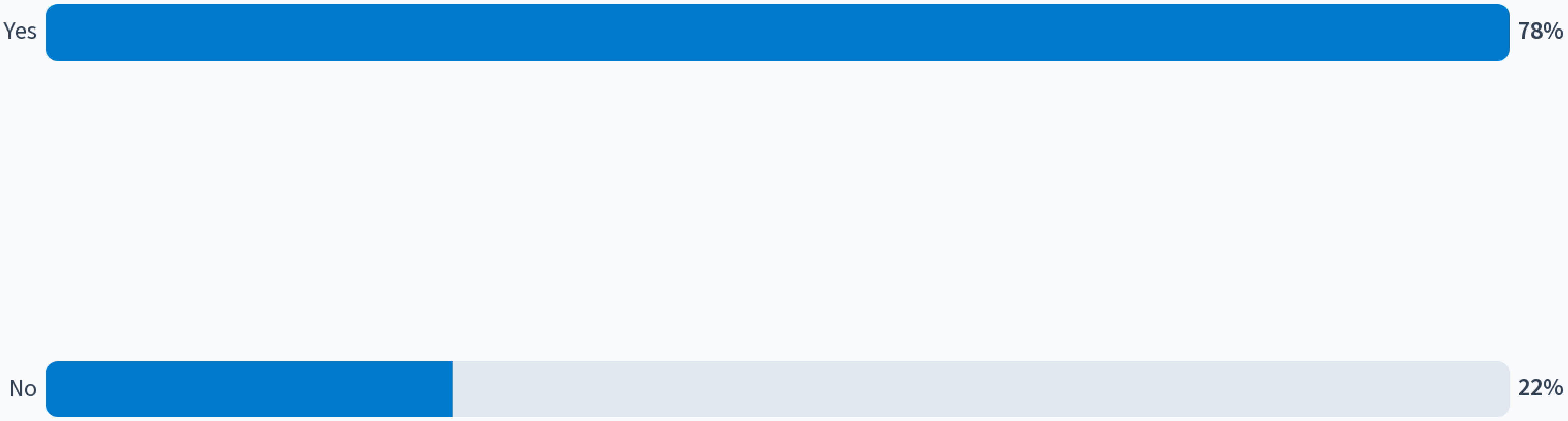
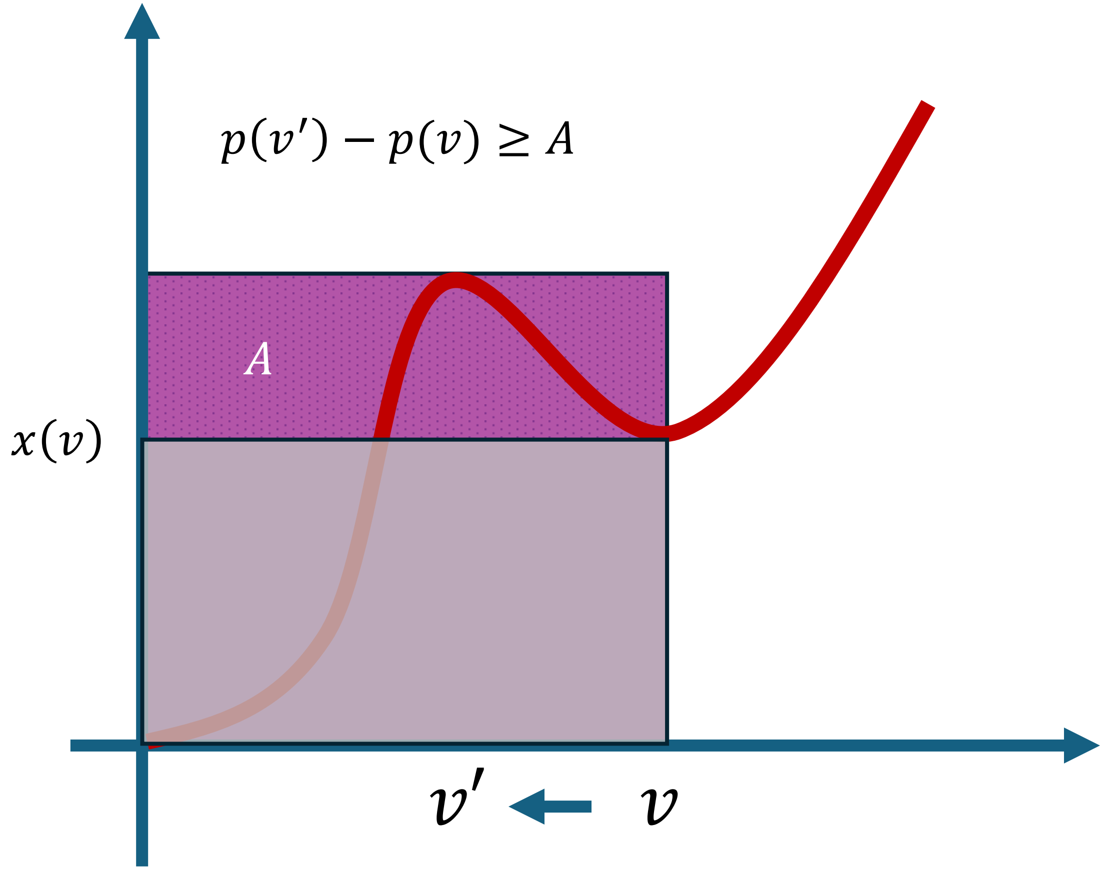


Suppose it is possible

- Suppose that we both bid truthfully
- Suppose that I am the highest value bidder
- No matter what the payment rule is, I can always reduce my bid to the second highest bid minus ϵ
- By doing so, I am paying at most the second highest bid and I am winning deterministically

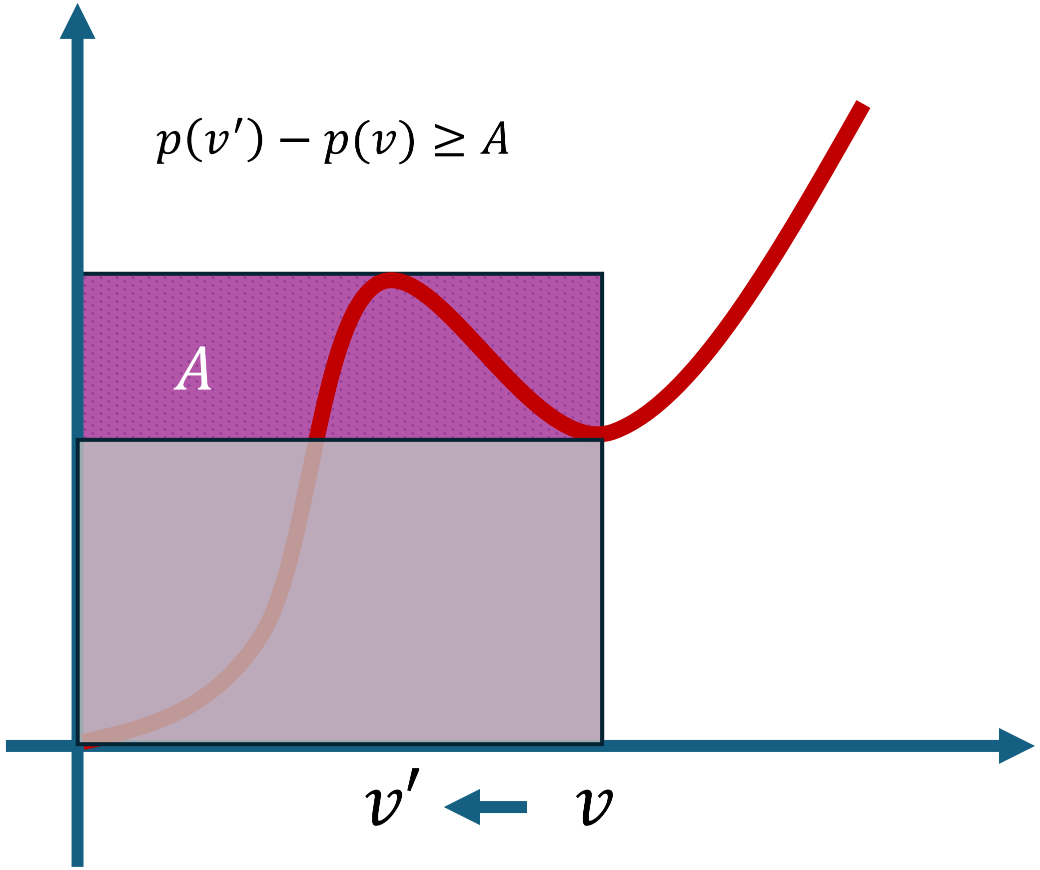
Implementable Rules are Monotone

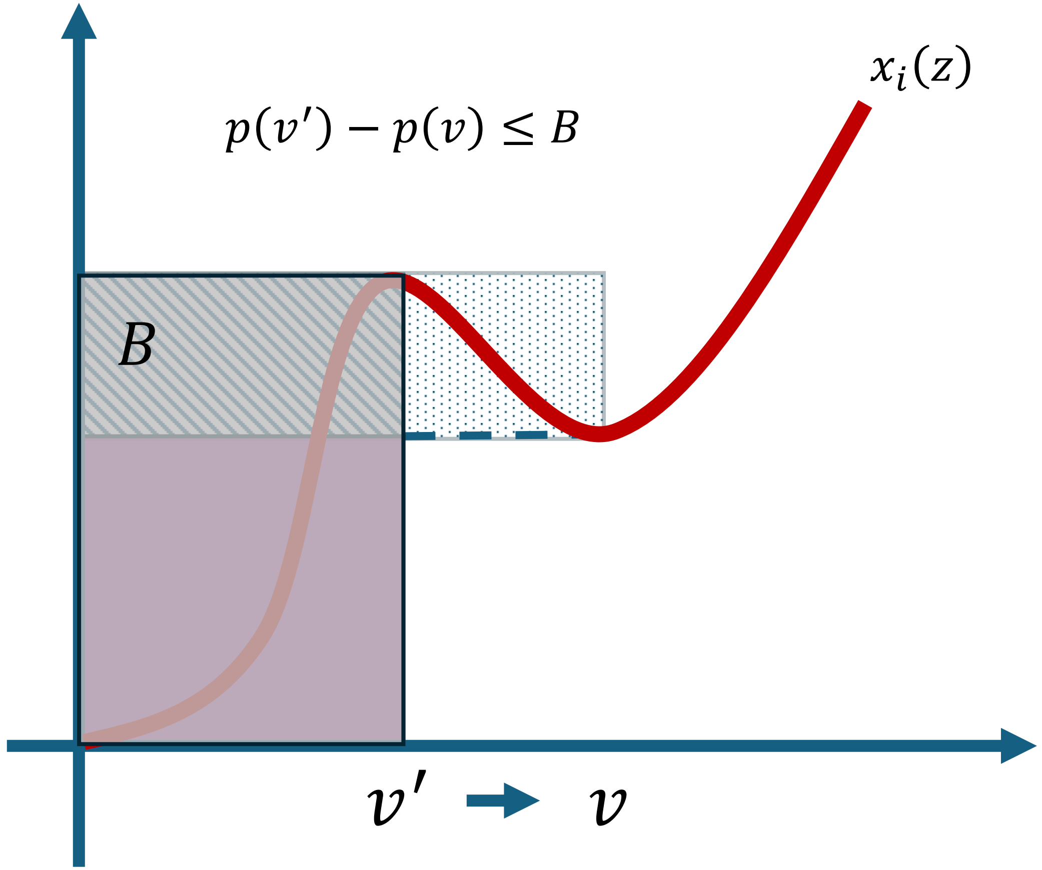
$$v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$$



Implementable Rules are Monotone

$$v \cdot x(v) - p(v) \geq v \cdot x(v') - p(v')$$
$$v' \cdot x(v') - p(v') \geq v' \cdot x(v) - p(v)$$



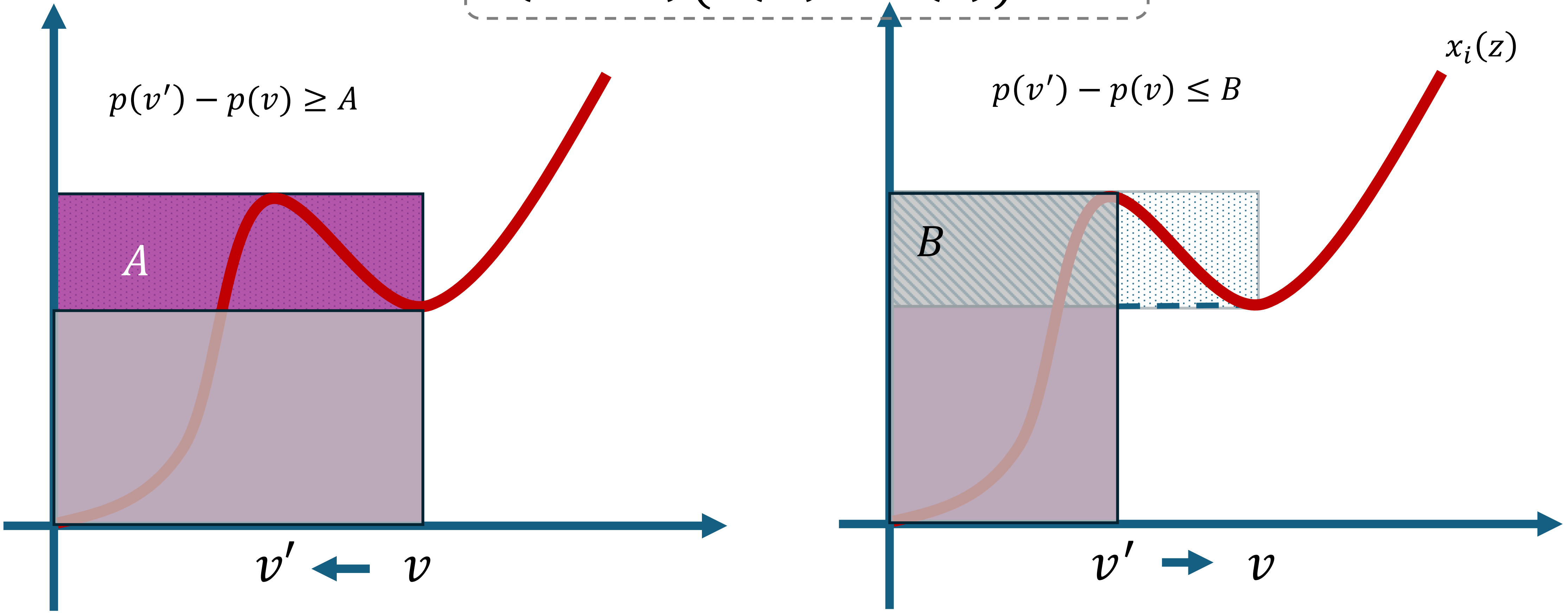


Implementable Rules are Monotone

$$A = v\left(x(v') - x(v)\right) \leq v'\left(x(v') - x(v)\right) = B$$

$$(v' - v)\left(x(v') - x(v)\right) \geq 0$$

Non-decreasing function



Any implementable allocation rule must be monotone!

"If not allocated with value v, I should not be allocated if I report a lower value!"

Uniqueness of Payment Rule

- I should not want to deviate locally up or down infinitesimally

$$u(v) \geq v \cdot x(v+\epsilon) - p(v+\epsilon) = u(v+\epsilon) - \epsilon \cdot x(v+\epsilon)$$
$$u(v) \geq v \cdot x(v-\epsilon) - p(v-\epsilon) = u(v-\epsilon) + \epsilon \cdot x(v-\epsilon)$$

- Dividing over by ϵ, restricts the rate of change of utility

$$\frac{u(v+\epsilon) - u(v)}{\epsilon} \leq x(v+\epsilon)$$
$$\frac{u(v) - u(v-\epsilon)}{\epsilon} \geq x(v-\epsilon)$$

- If u was differentiable, then taking the limit of the above as $\epsilon \to 0$

$$x(v) \leq u'(v) \leq x(v) \Rightarrow u'(v) = x(v) \Rightarrow u(v) - u(0) = \int_0^v x(z)\, dz$$

Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z)\, dz$$

Discontinuity of Allocation Rule

- Even though allocation rule can be discontinuous, because it is monotone, it is Riemann integrable

$$\int_0^v x(z)\,dz = \lim_{\epsilon \to 0} \sum_k x(z+\epsilon)\cdot\epsilon \geq \lim_{\epsilon \to 0} \sum_k u(z+\epsilon) - u(z) = u(v) - u(0)$$

$$\int_0^v x(z)\,dz = \lim_{\epsilon \to 0} \sum_k x(z-\epsilon)\cdot\epsilon \leq \lim_{\epsilon \to 0} \sum_k u(z) - u(z-\epsilon) = u(v) - u(0)$$

Under any truthful payment rule

$$u(v) = u(0) + \int_0^v x(z)\, dz$$

What does that imply about payments

- Since utility is value minus payment

$$v\,x(v) - p(v) = -p(0) + \int_0^v x(z)\,dz$$

- Non-Negative-Transfers (NNT). We never have negative payments

$$p(0) \geq 0$$

- Individually Rational (IR). We never give bidders negative utility

$$p(0) \leq 0$$

- Thus, payment at 0 should be zero!

Under any truthful payment rule that satisfies NNT and IR

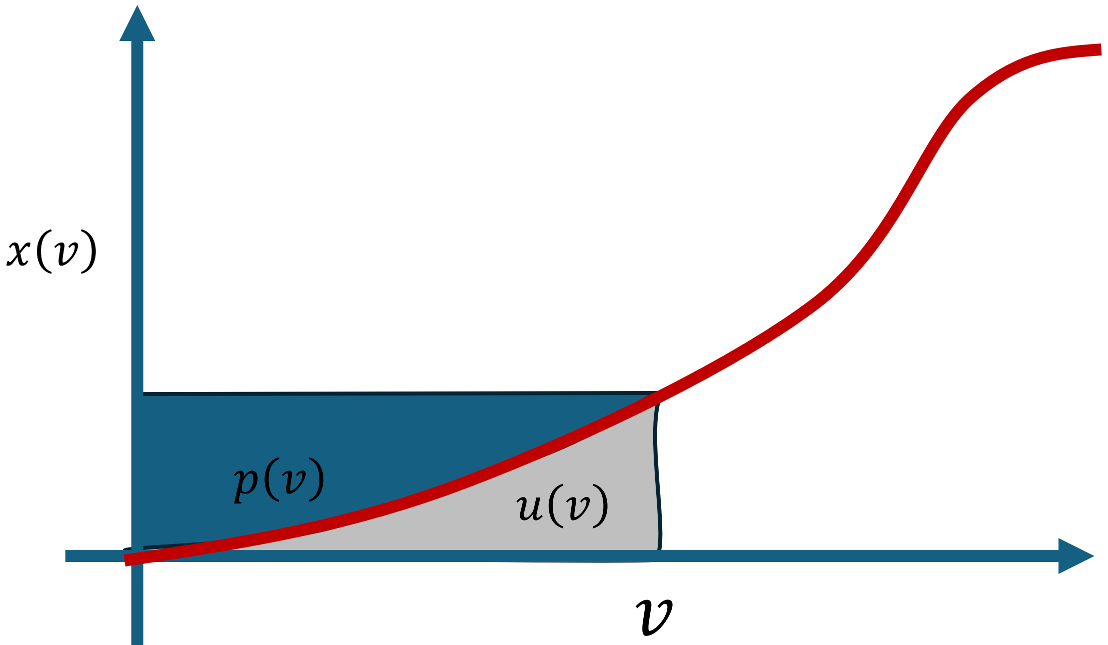
$$p(v) = v \cdot x(v) - \int_0^v x(z)\, dz$$

Given an allocation rule, the payment is uniquely determined!

Visualizing Utility

- Under any truthful payment rule with IR and NNT

$$u(v) = \int_0^v x(z)\,dz \qquad\qquad p(v) = v \cdot x(v) - \int_0^v x(z)\,dz$$



Visualizing Utility

- Under any truthful payment rule with IR and NNT

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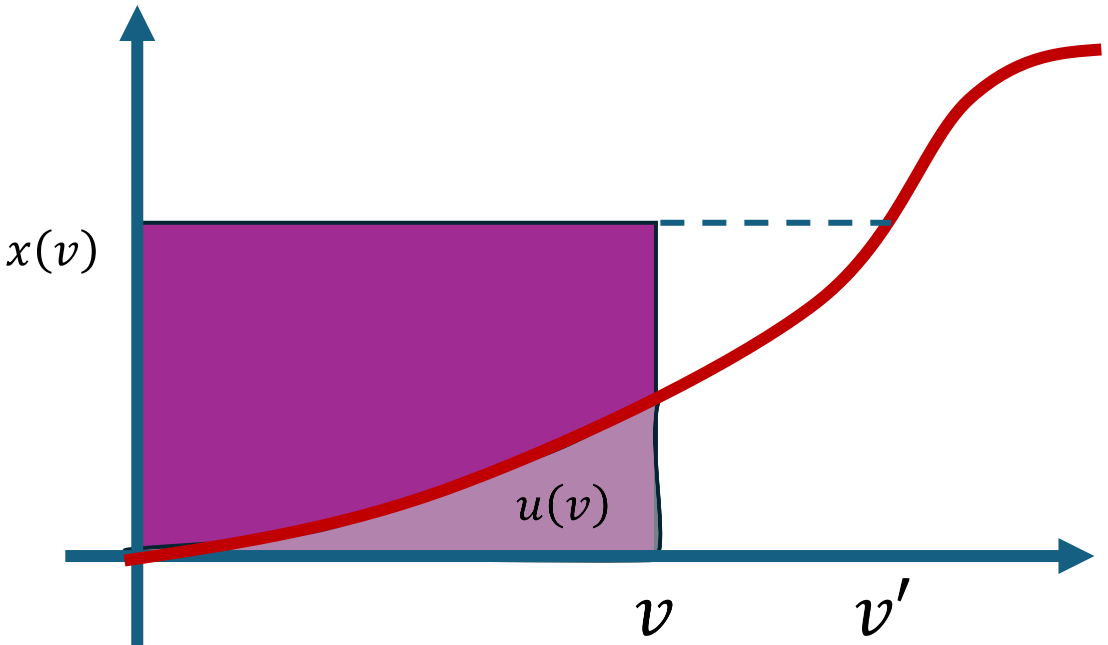

Visualizing Utility

- Under any truthful payment rule with IR and NNT

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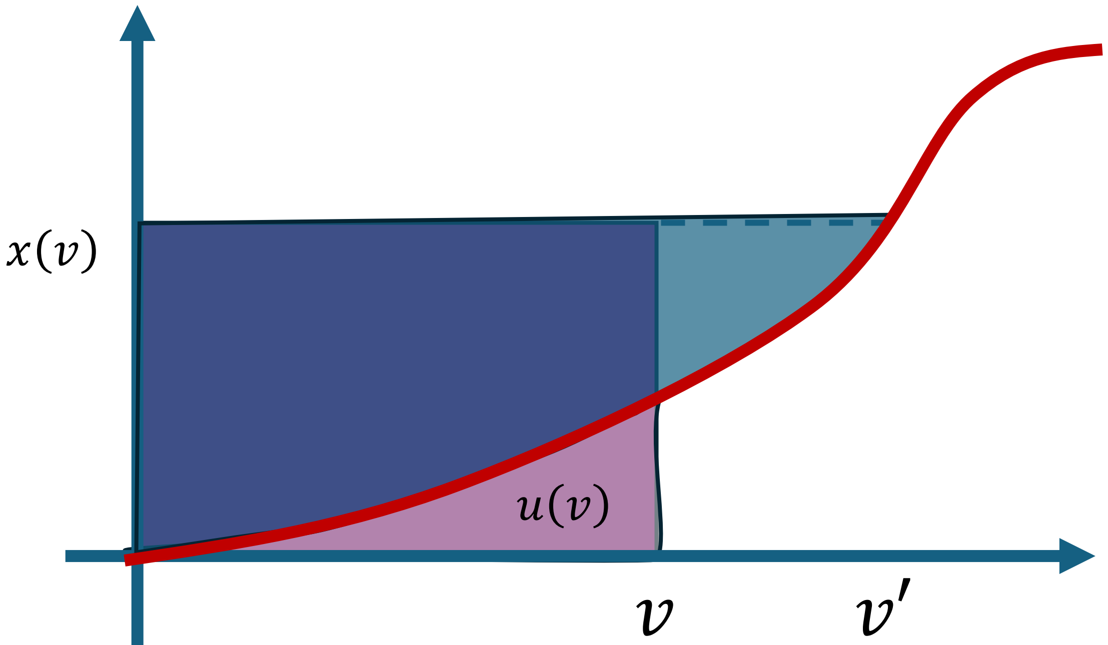

Visualizing Utility

- Under any truthful payment rule with IR and NNT

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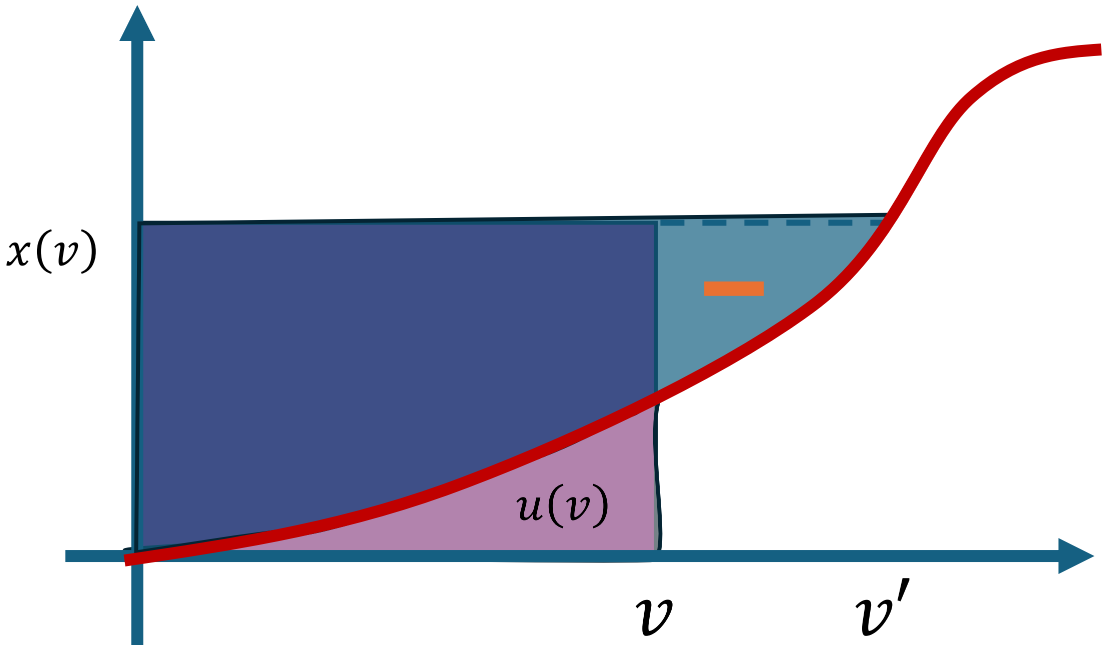
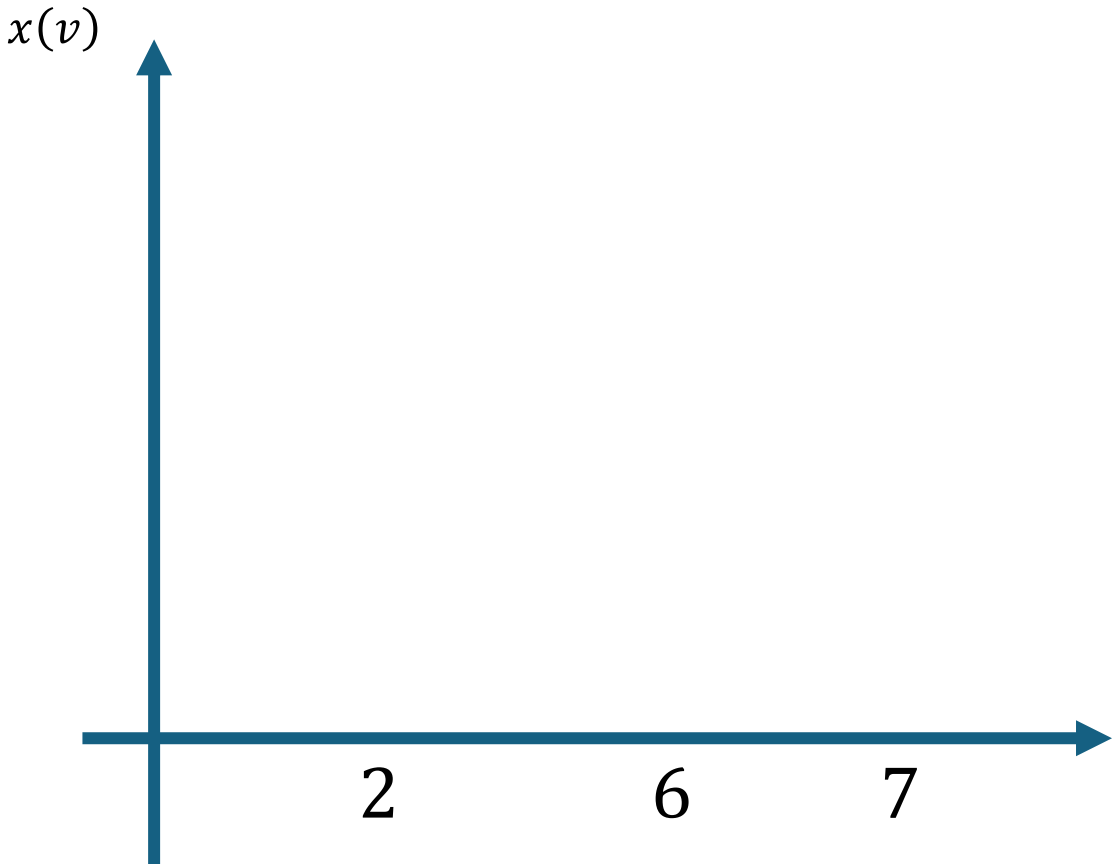


Back to Sponsored Search

Applying this to Sponsored Search



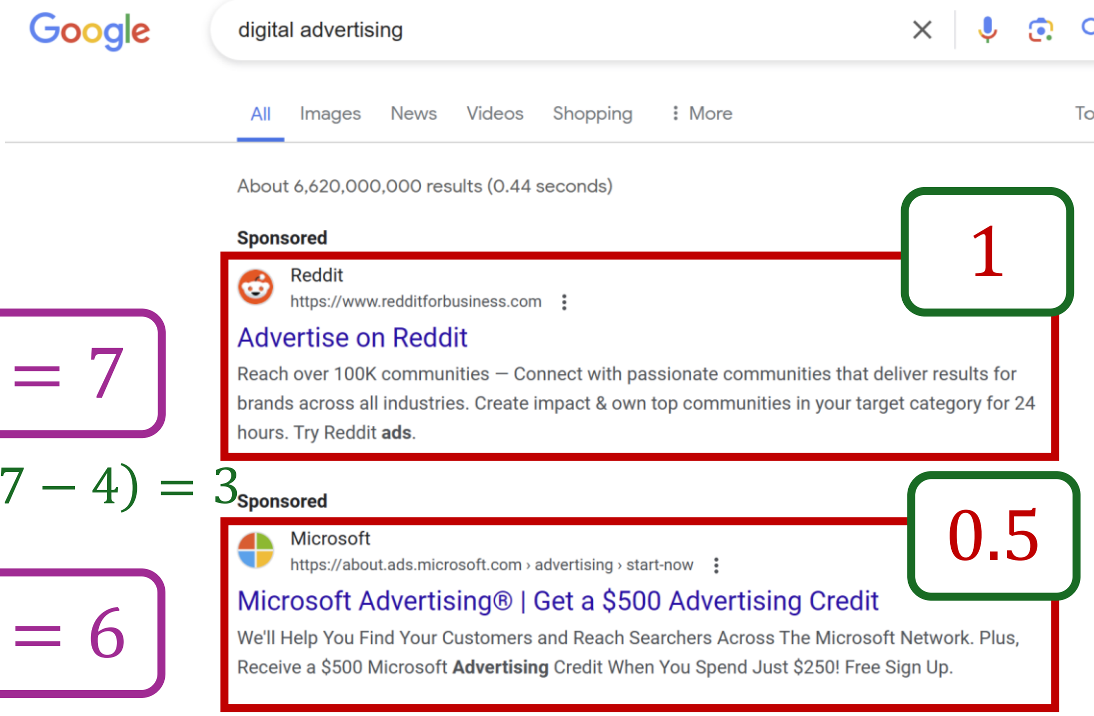
$v_1 = 7$

$u_1 = 1 \cdot (7 - 4) = 3$

$v_2 = 6$

$u_2 = .5 \cdot (6 - 2) = 2$

$v_3 = 2$



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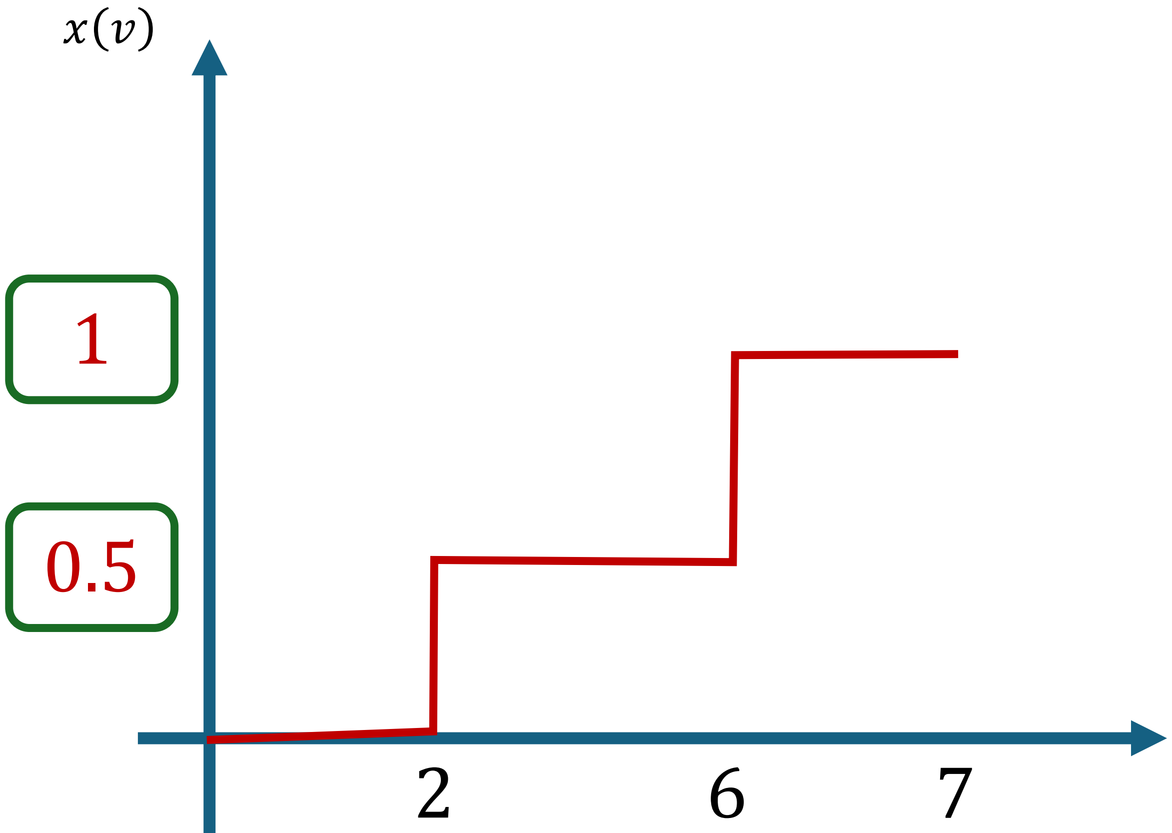

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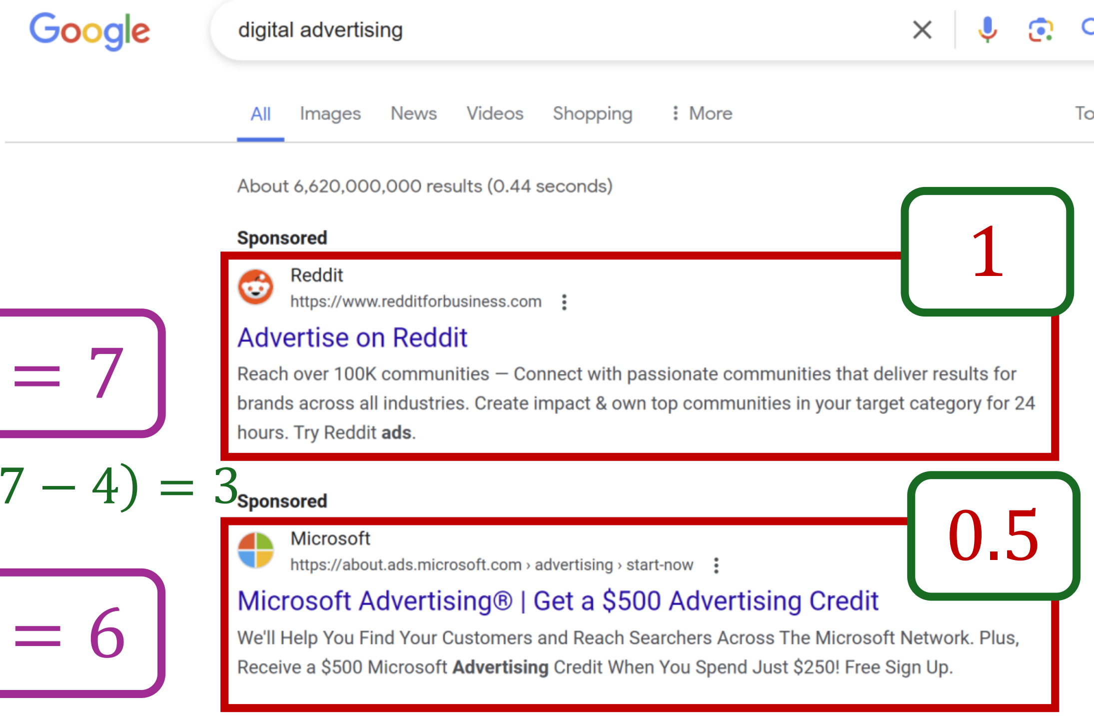
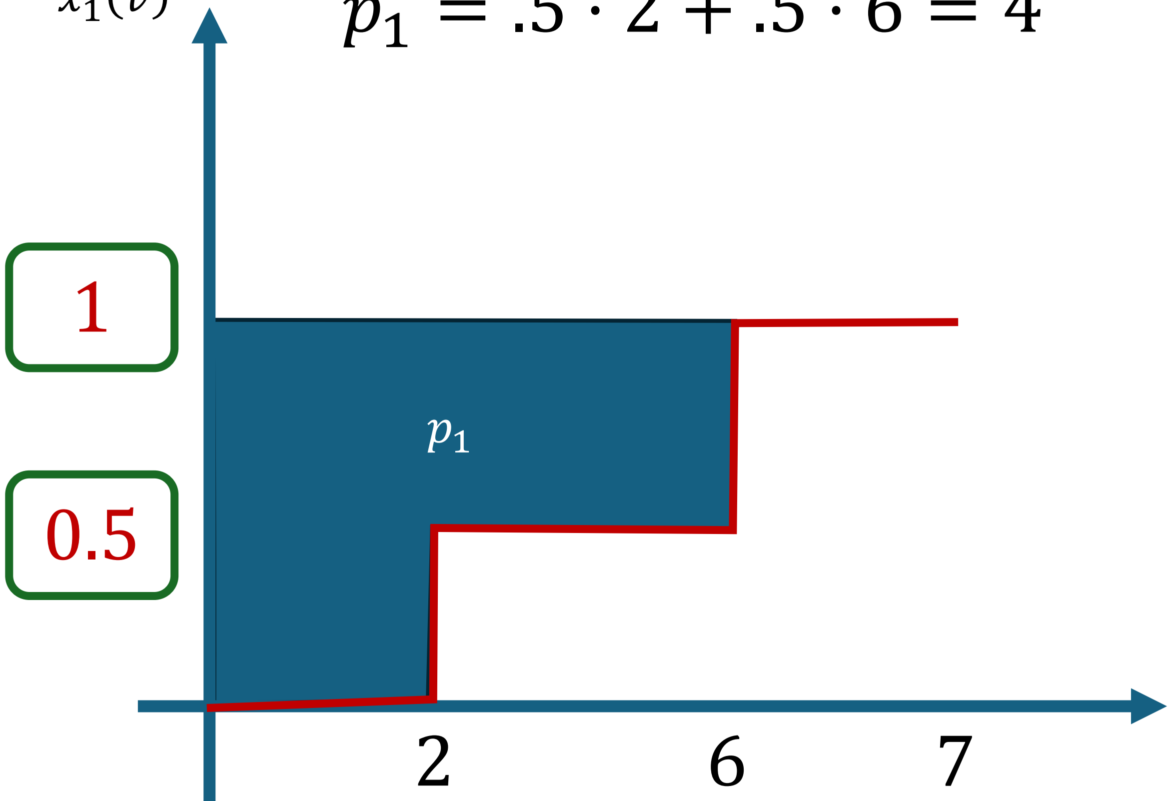


Applying this to Sponsored Search

$$p_1 = .5 \cdot 2 + .5 \cdot 6 = 4$$



$v_1 = 7$

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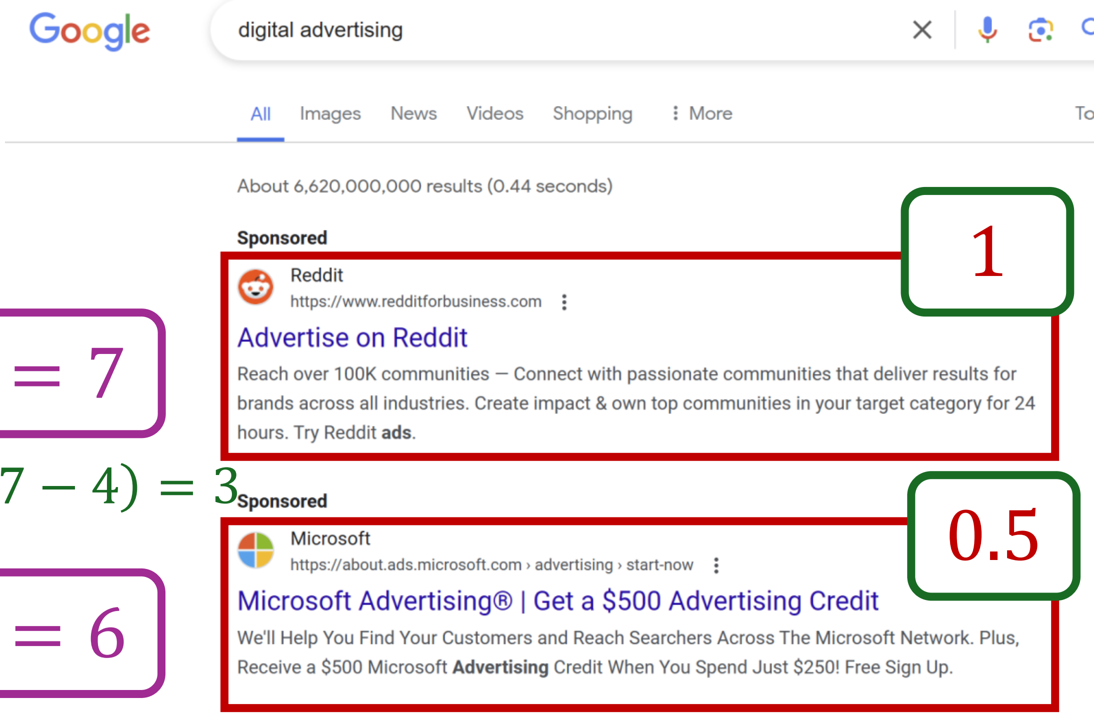

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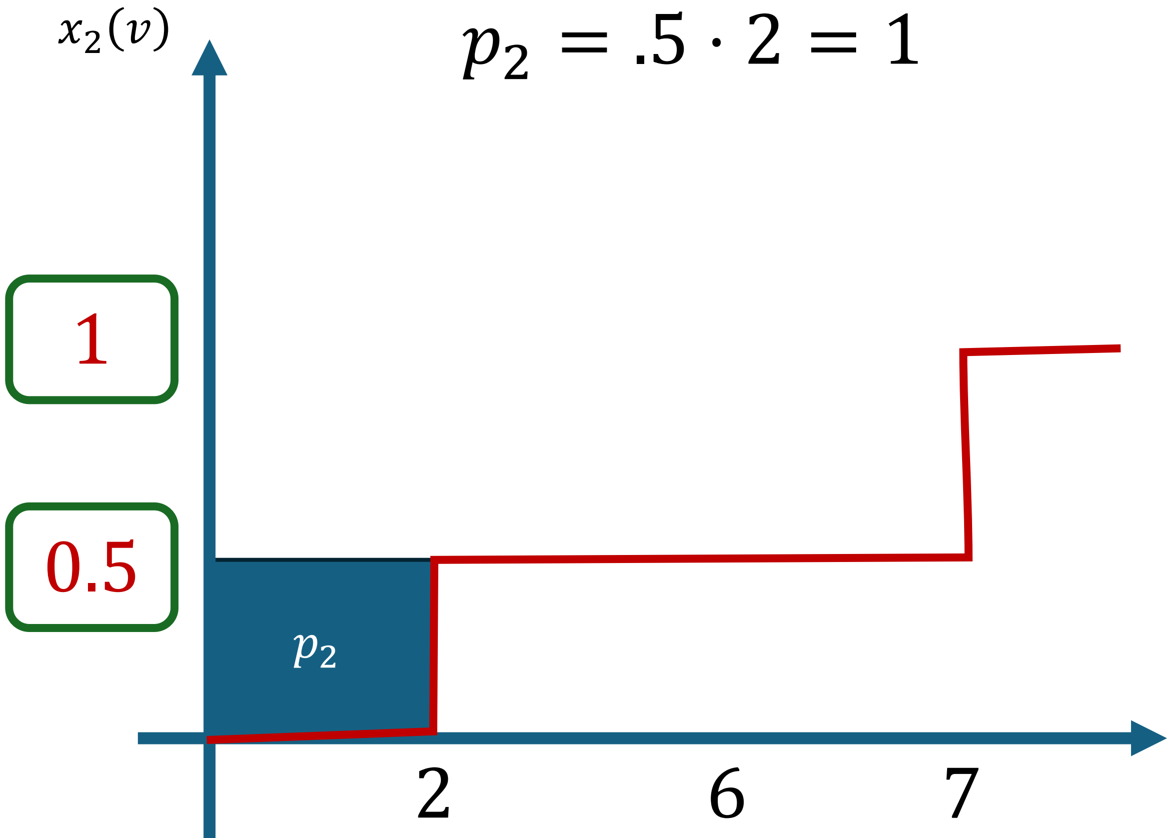


$$p_2 = .5 \cdot 2 = 1$$

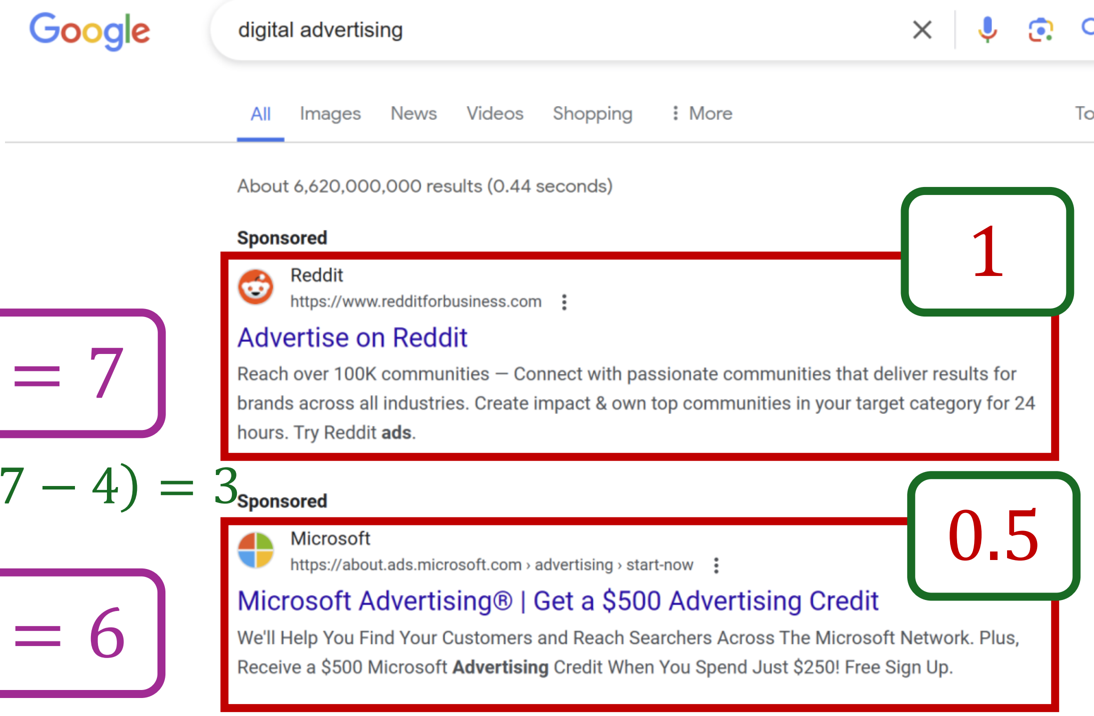
$$v_1 = 7$$

$$u_1 = 1 \cdot (7 - 4) = 3$$

$$v_2 = 6$$

$$u_2 = .5 \cdot (6 - 2) = 2$$

$$v_3 = 2$$



Applying this to Sponsored Search

With an arbitrary number of slots, payment of bidder in slot j is:

$$p_{(j)} = \sum_{\ell=j}^{k} (a_\ell - a_{\ell+1}) \cdot b_{(\ell)}$$

where $b_{(\ell)}$ is the bid of the player allocated in slot ℓ

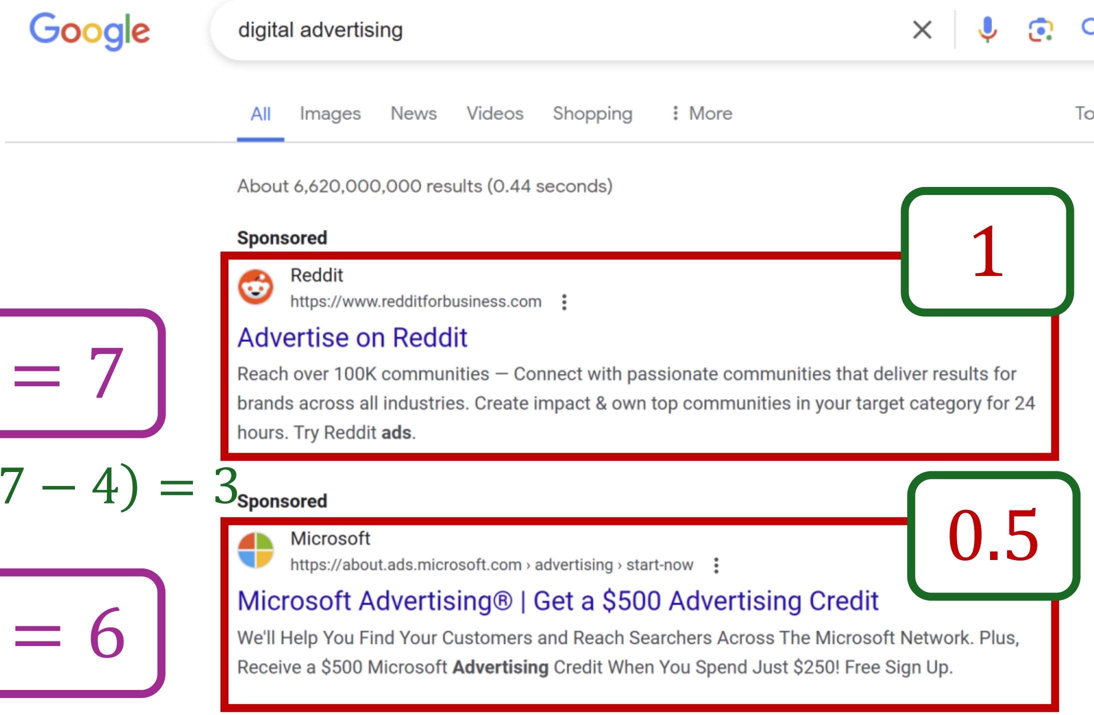
$v_1 = 7$

$u_1 = 1 \cdot (7 - 4) = 3$

$v_2 = 6$

$u_2 = .5 \cdot (6 - 2) = 2$

$v_3 = 2$



Optimizing over allocation rules

Myerson's Theorem

- Let x, p be any DSIC mechanism
- Suppose each value $v_i \sim F_i$ independently and let $v = (v_1, \dots, v_n)$
$$E[p_i(v)] = E[x_i(v) \cdot \phi_i(v_i)]$$
  where $\phi_i(v_i)$ is bidder i's "virtual value".
- Letting F_i the CDF and f_i the density:
$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$
- Assuming $\phi_i(v_i)$ is monotone non-decreasing, then the optimal DSIC mechanism is the mechanism that allocates to the highest virtual value bidder (or none if highest virtual value is negative)

Back to Uniform Example

- If $v_i \sim U[0,1]$ then $F(v) = v$ and $f(v) = 1$
- Virtual value simplifies to

$$\phi_i(v_i) = v_i - (1 - v_i) = 2v_i - 1$$

- We should allocate to the highest virtual value player, as long as the highest virtual value is non-negative

$$v_i \geq 1/2$$

- Since all virtual value functions are the same, allocating to the highest virtual value is the same as allocating to the highest value
- Simply: Second Price with a reserve price of 1/2!

Myerson's Theorem

- Consider the revenue contribution of a single bidder i and drop other bids and index from notation

$$E[p(v)] = E\left[v\, x(v) - \int_0^v x(z)dz\right] = E\left[v\, \hat{x}(v) - \int_0^v \hat{x}(z)\, dz\right]$$

- Allocation $\hat{x}(z)$ is the expected allocation over other bidder values

$$\hat{x}(z) = E_{v_{-i}}[x(z, v_{-i})]$$

- We can do an exchange of the integrals:

$$E\left[\int_0^v \hat{x}(z)\, dz\right] = \int_{v=0}^{\infty} \int_{z=0}^{v} \hat{x}(z)\, dz\, f(v) dv = \int_{z=0}^{\infty} \hat{x}(z) \int_{v=z}^{\infty} f(v)\, dv\, dz$$

$$= \int_{z=0}^{\infty} \hat{x}(z)\big(1 - F(z)\big)dz = E\left[\hat{x}(v)\, \frac{1 - F(v)}{f(v)}\right]$$

Myerson’s Theorem (cont’d)

- Consider the revenue contribution of a single bidder i and drop other bids and index from notation

$$E[p(v)] = E\left[\hat{x}(v)\left(v - \hat{x}(v)\frac{1 - F(v)}{f(v)}\right)\right] = E[\hat{x}(v)\,\phi(v)]$$

- Re-introducing the bidder index:

$$E[p_i(v)] = E[\hat{x}_i(v_i) \cdot \phi_i(v_i)] = E[x_i(v) \cdot \phi_i(v_i)]$$

- Summing across bidders we get:

$$\sum_i E[p_i(v)] = \sum_i E[x_i(v) \cdot \phi_i(v_i)] = E\left[\sum_i x(v) \cdot \phi_i(v_i)\right]$$

Myerson's Optimal Auction. The optimal mechanism is the mechanism that maximizes virtual welfare (and charges the corresponding payments that make this truthful)

$$x(v) = \text{argmax}_{x \in X} \sum_i x \cdot \phi_i(v_i)\,, \qquad p_i(v) = v_i x_i(v) - \int_0^{v_i} x_i(z, v_{-i})\, dz$$

$$\text{Rev} = E\left[\max_{x \in X} \sum_i x \cdot \phi_i(v_i)\right]$$

Appendix: Deriving the Optimal Reserve

- Bidders are symmetric. Revenue is twice the revenue we collect from each bidder

$$\begin{aligned}
\mathrm{Rev}_1(r) &= E[\max(v_2, r)\, 1(v_1 \geq \max(v_2, r))] \\
&= E\big[v_2 \mid v_2 \in [r, v_1]\big] \Pr(v_2 \in [r, v_1] | v_1 \geq r) \Pr(v_1 \geq r) + r \Pr(v_2 \leq r) \Pr(v_1 \geq r) \\
&= \int_r^1 \frac{v + r}{2} (v - r) dv \; + r^2 (1 - r) \\
&= \int_r^1 \frac{v^2 - r^2}{2} dv + r^2 (1 - r) \\
&= \left(\frac{1 - \mathrm{r}^3}{6} - \frac{\mathrm{r}^2}{2} (1 - \mathrm{r}) + \mathrm{r}^2 (1 - r) \right) \\
&= \frac{1 - \mathrm{r}^3}{6} + \frac{r^2 (1 - r)}{2} = \frac{1 - r^3 + 3r^2 - 3r^3}{6} = \frac{1 + 3r^2 - 4r^3}{6}
\end{aligned}$$

- The first order condition

$$\big(\mathrm{Rev}_1(r)\big)' = r(1 - 2r) = 0 \Rightarrow r = 1/2$$