MS&E 233
Game Theory, Data Science and AI
Lecture 5

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Management Science and Engineering

(by courtesy) Computer Science and Electrical Engineering

Institute for Computational and Mathematical Engineering

Computational Game Theory for Complex Games

1
- Basics of game theory and zero-sum games (T)
- Basics of online learning theory (T)
- Solving zero-sum games via online learning (T)
- *HW1: implement simple algorithms to solve zero-sum games*
- Applications to ML and AI (T+A)
- *HW2: implement boosting as solving a zero-sum game*

2
- **Basics of extensive-form games**
- Solving extensive-form games via online learning (T)
- *HW3: implement agents to solve very simple variants of poker*

3
- General games and equilibria (T)
- Online learning in general games, multi-agent RL (T+A)
- *HW4: implement no-regret algorithms that converge to correlated equilibria in general games*

Data Science for Auctions and Mechanisms

4
- Basics and applications of auction theory (T+A)
- Learning to bid in auctions via online learning (T)
- *HW5: implement bandit algorithms to bid in ad auctions*

5
- Optimal auctions and mechanisms (T)
- Simple vs optimal mechanisms (T)
- *HW6: calculate equilibria in simple auctions, implement simple and optimal auctions, analyze revenue empirically*

6
- Optimizing mechanisms from samples (T)
- Online optimization of auctions and mechanisms (T)
- *HW7: implement procedures to learn approximately optimal auctions from historical samples and in an online manner*

Further Topics

7
- Econometrics in games and auctions (T+A)
- A/B testing in markets (T+A)
- *HW8: implement procedure to estimate values from bids in an auction, empirically analyze inaccuracy of A/B tests in markets*

Guest Lectures

- Mechanism Design for LLMs, Renato Paes Leme, Google Research
- Auto-bidding in Sponsored Search Auctions, Kshipra Bhawalkar, Google Research

Extensive Form Games

History and Progress

Historical Challenge in Game Theory and AI

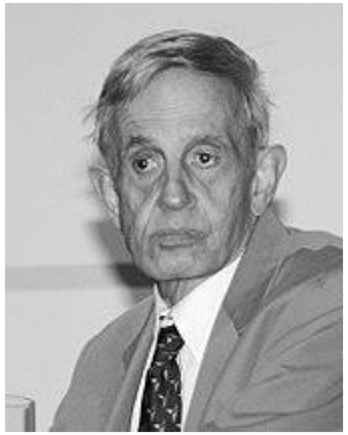
Nash1950

294 JOHN NASH

A Three-Man Poker Game

As an example of the application of our theory to a more or less realistic case we include the simplified poker game given below. The rules are as follows:

(a) The deck is large, with equally many *high* and *low* cards, and a hand consists of one card.
(b) Two chips are used to ante, open, or call.
(c) The players play in rotation and the game ends after all have passed or after one player has opened and the others have had a chance to call.
(d) If no one bets the antes are retrieved.
(e) Otherwise the pot is divided equally among the highest hands which have bet.

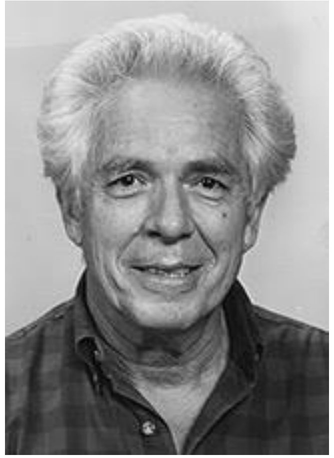


Kuhn1950

A SIMPLIFIED TWO-PERSON POKER*

H. W. Kuhn[1]

A fascinating problem for the game theoretician is posed by the common card game, Poker. While generally regarded as partaking of psychological aspects (such as bluffing) which supposedly render it inaccessible to mathematical treatment, it is evident that Poker falls within the general theory of games as elaborated by von Neumann and Morgenstern [1]. Relevant probability problems have been considered by Borel and Ville [2] and several variants are examined by von Neumann [1] and by Bellman and Blackwell [3].



Waterman1970

ARTIFICIAL INTELLIGENCE 121

Generalization Learning Techniques for Automating the Learning of Heuristics[1]

D. A. Waterman

Carnegie-Mellon University, Pittsburgh, Pennsylvania

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Superhuman AI for heads-up no-limit poker: Libratus beats top professionals

NOAM BROWN AND TUOMAS SANDHOLM Authors Info & Affiliations

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RESEARCH ARTICLE | MACHINE LEARNING

Mastering the game of Stratego with model-free multiagent reinforcement learning

JULIEN PEROLAT, BART DE VYLDER, DANIEL HENNES, EUGENE TARASSOV, [...], AND KARL TUYLS +29 authors Authors Info & Affiliations

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REPORT

A general reinforcement learning algorithm that masters chess, shogi, and Go through self-play

DAVID SILVER, THOMAS HUBERT, JULIAN SCHRITTWIESER, IOANNIS ANTONOGLOU, [...], AND DEMIS HASSABIS +8 authors Authors Info & Affiliations

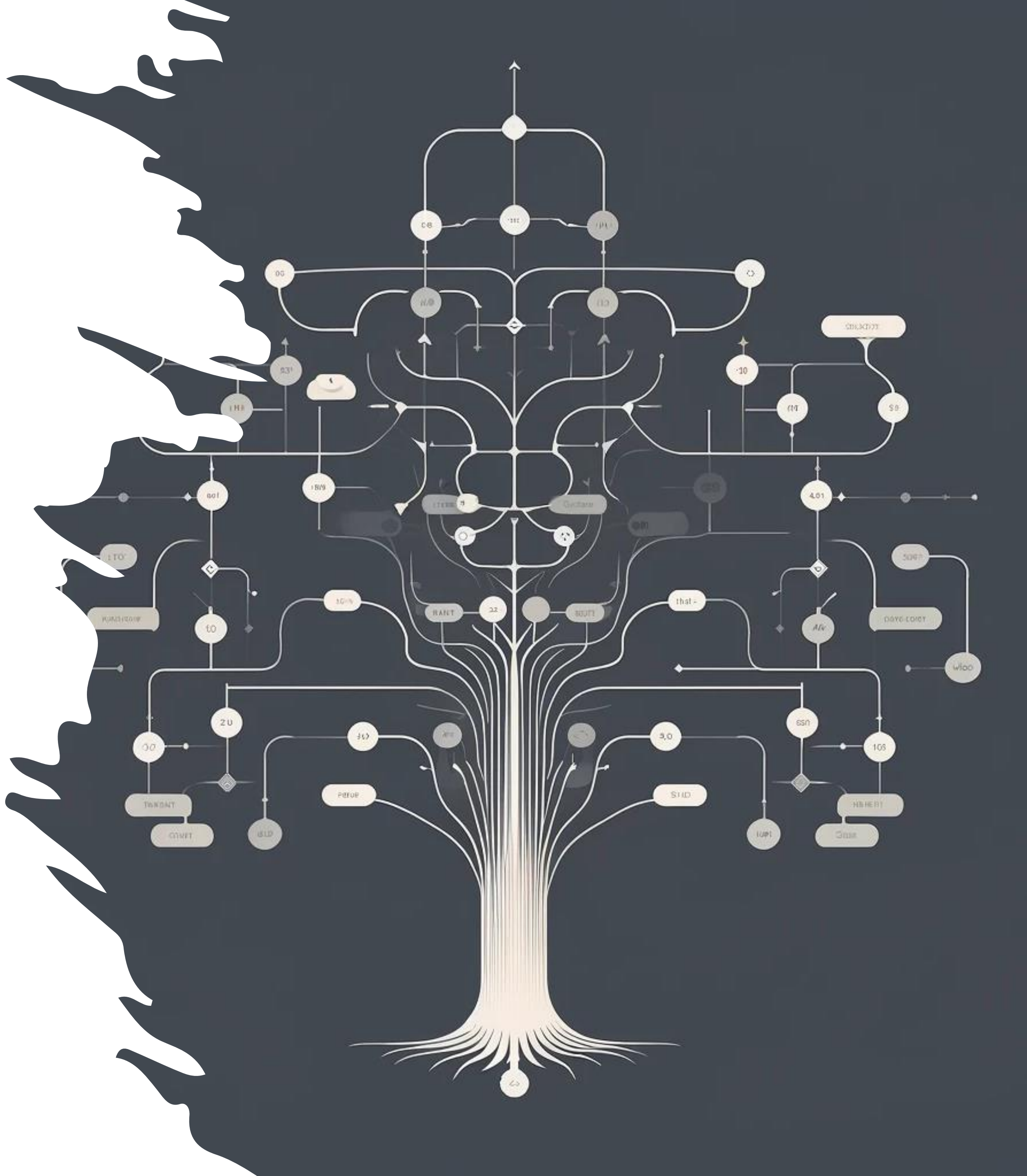
Key Elements to Success

New approaches to approximate the "continuation value of the game" via deep learning and other domain specific techniques

Scalable algorithmic methods to compute approximate Nash equilibria of zero-sum games via learning dynamics

Extensive Form Games

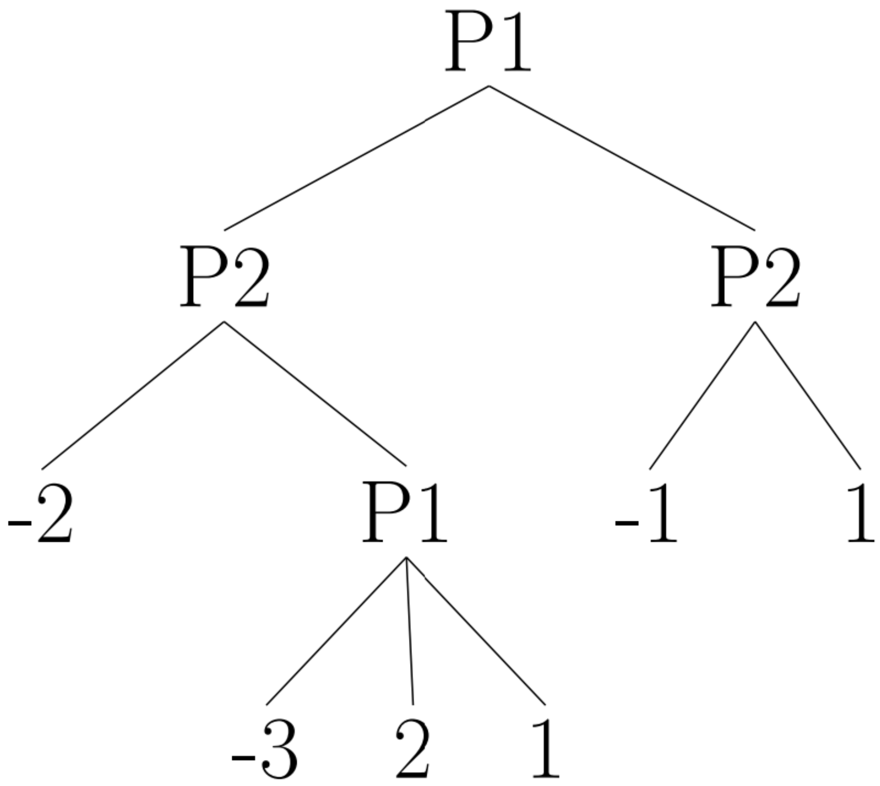
The Basics



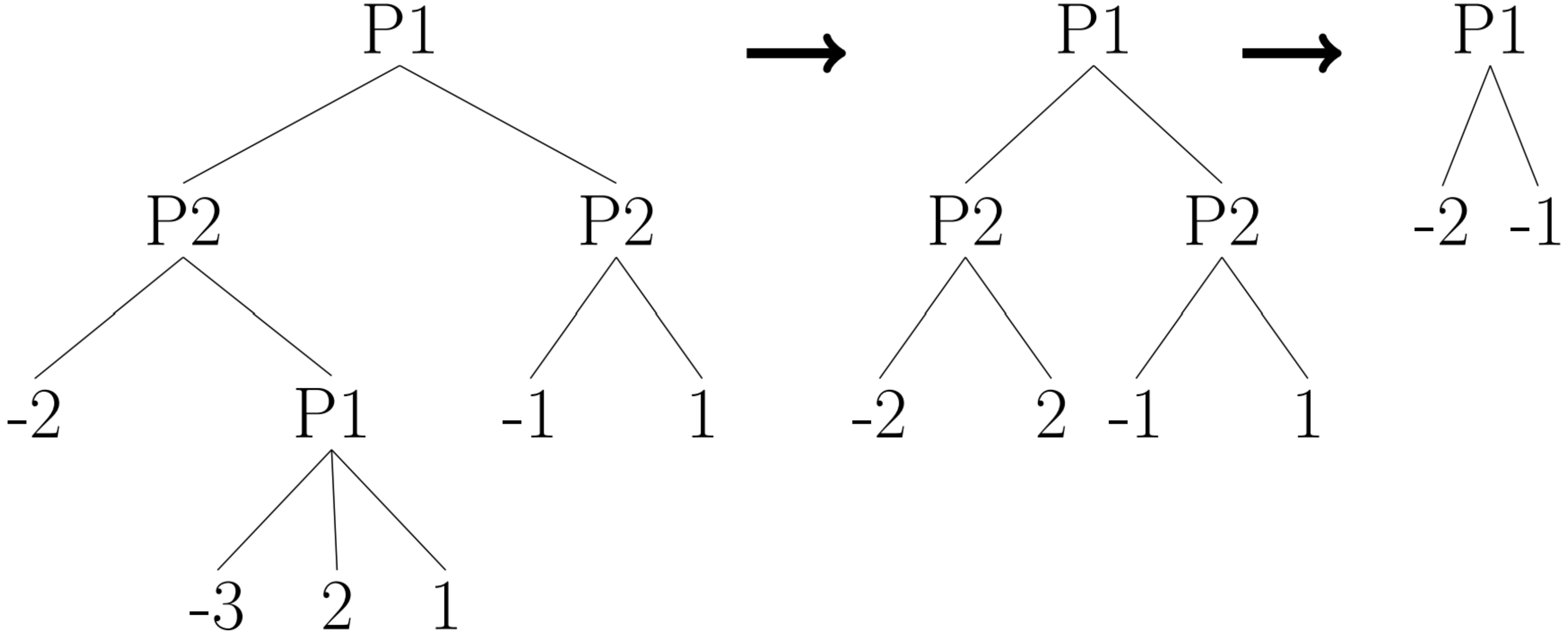
Perfect Information Games

- Players take turns in choosing actions
- All actions are publicly observable
- The “state” of the game is publicly observable
- Some sequence of actions lead to terminal states
- Each player receives some utility/loss at a terminal state
- In zero-sum games: utility of player 1 equals loss of player 2

Tree Representation



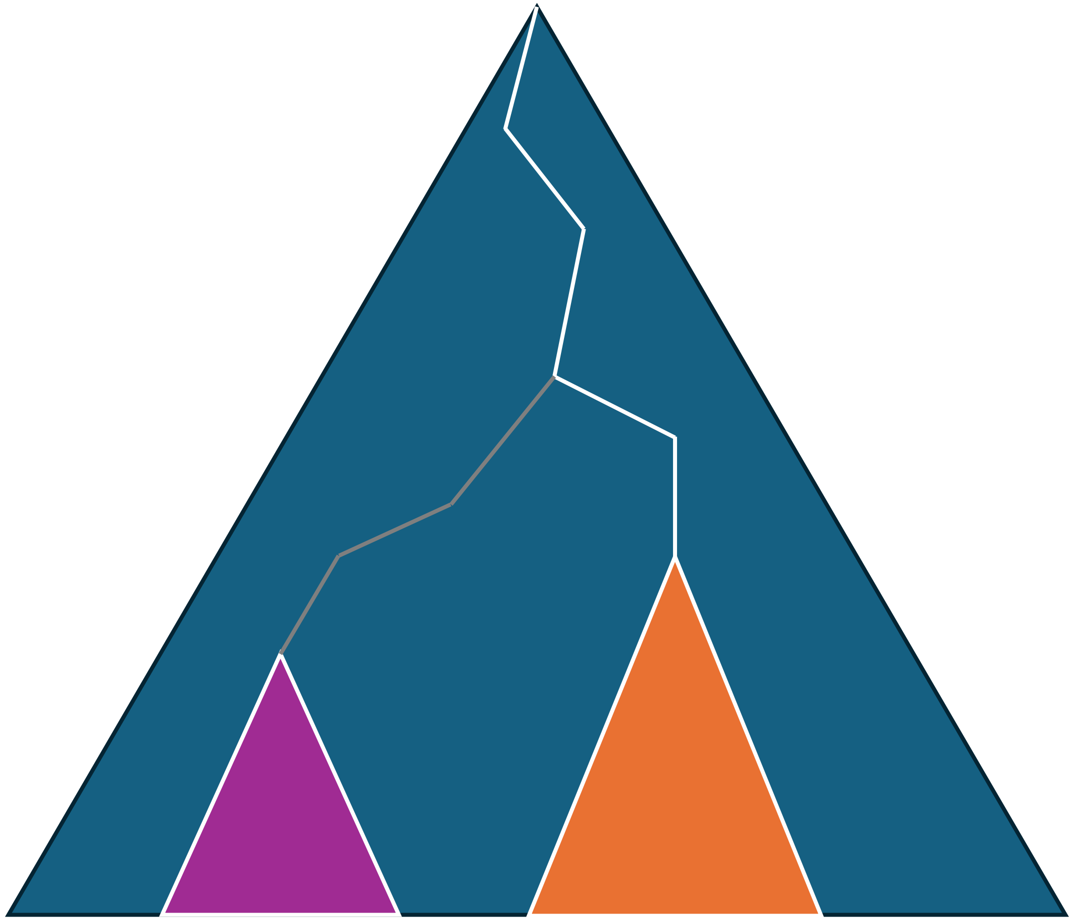
Solving Games via Backwards Induction



Imperfect Information Games

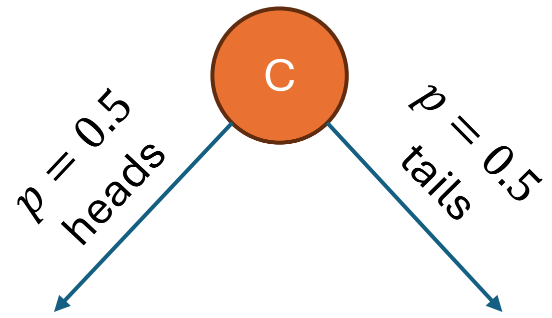
Players don’t have perfect knowledge about the “state” of the game

Why are Imperfect Information Games Hard



The optimal strategy in the orange sub-tree can depend on how we play and what happens in the purple sub-tree

A Simple Game



Rules of the game

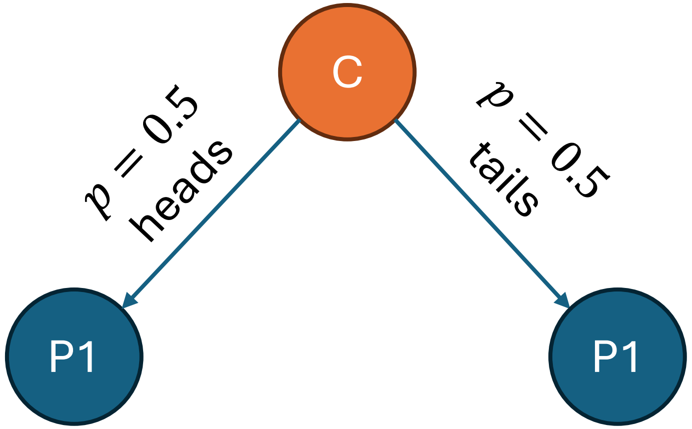
- Nature (chance) flips a coin

Credits: Superhuman AI for heads-up no-limit poker: Libratus beats top professionals (youtube.com)

A Simple Game

Rules of the game

- Nature (chance) flips a coin
- Player one sees the outcome of the coin



A Simple Game

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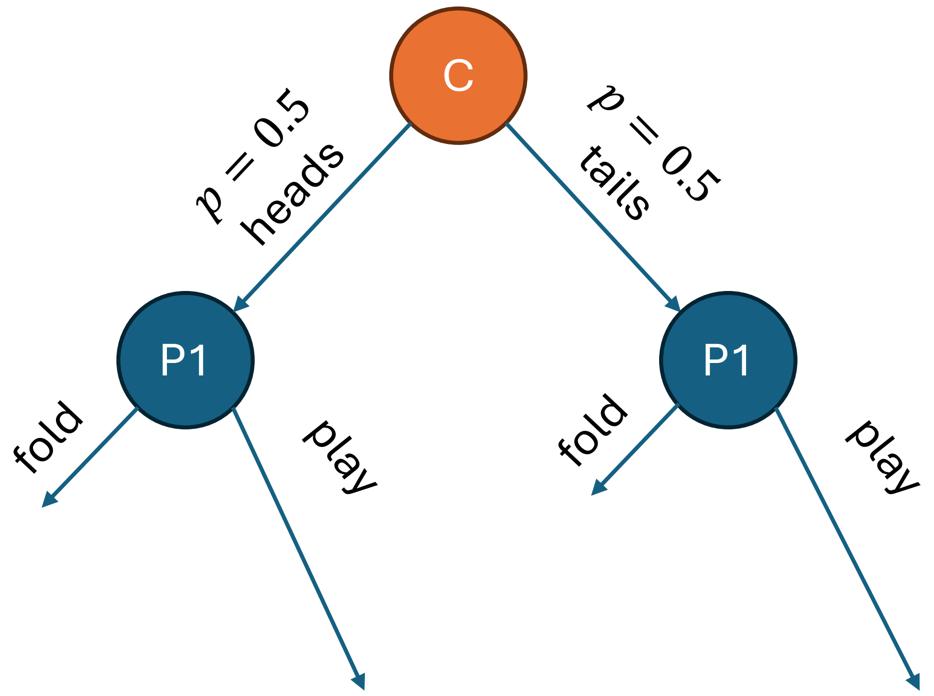
- Player one chooses whether to fold or play



A Simple Game

Rules of the game

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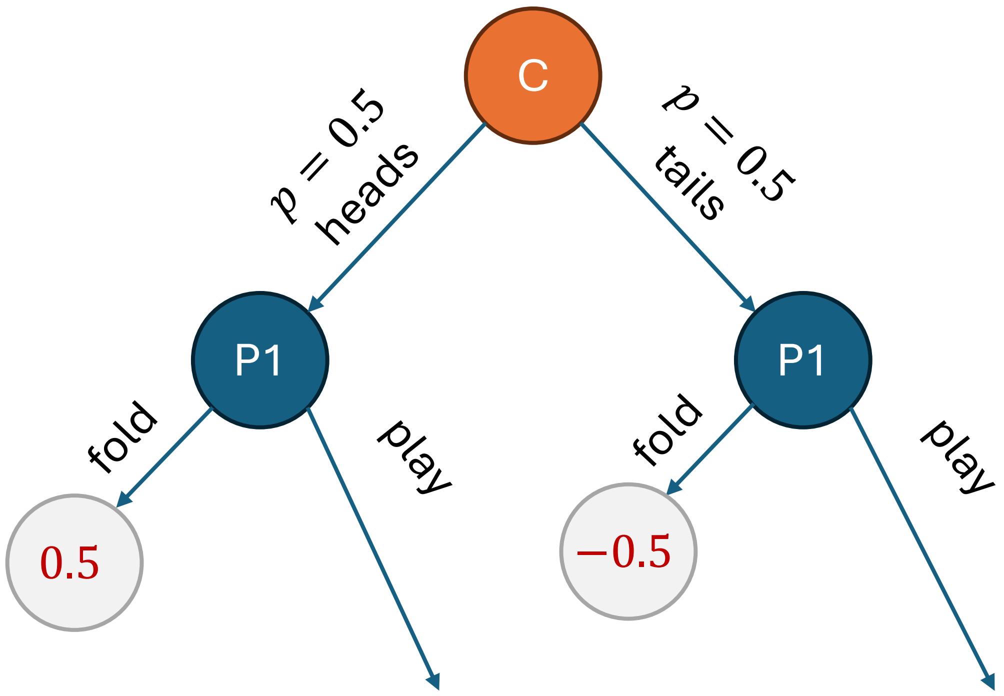
- If they fold with heads they win 0.5 if they fold with tails they lose 0.5



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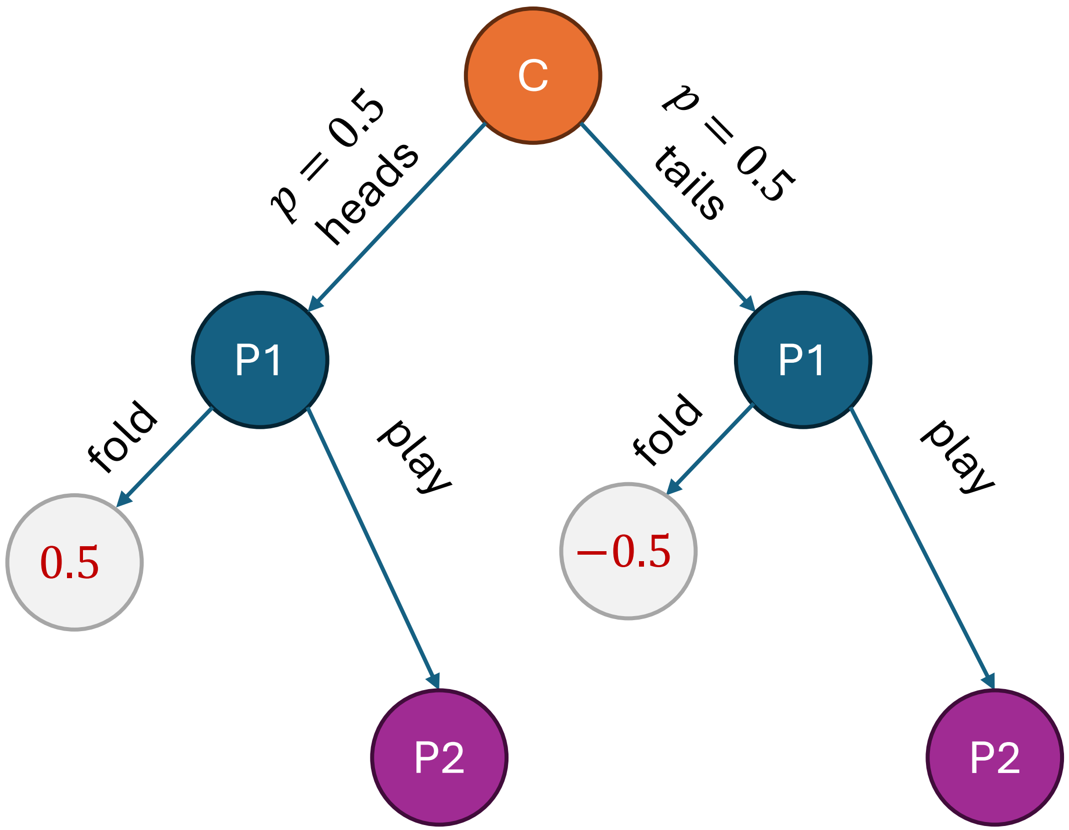
- If they play then it is player two turn



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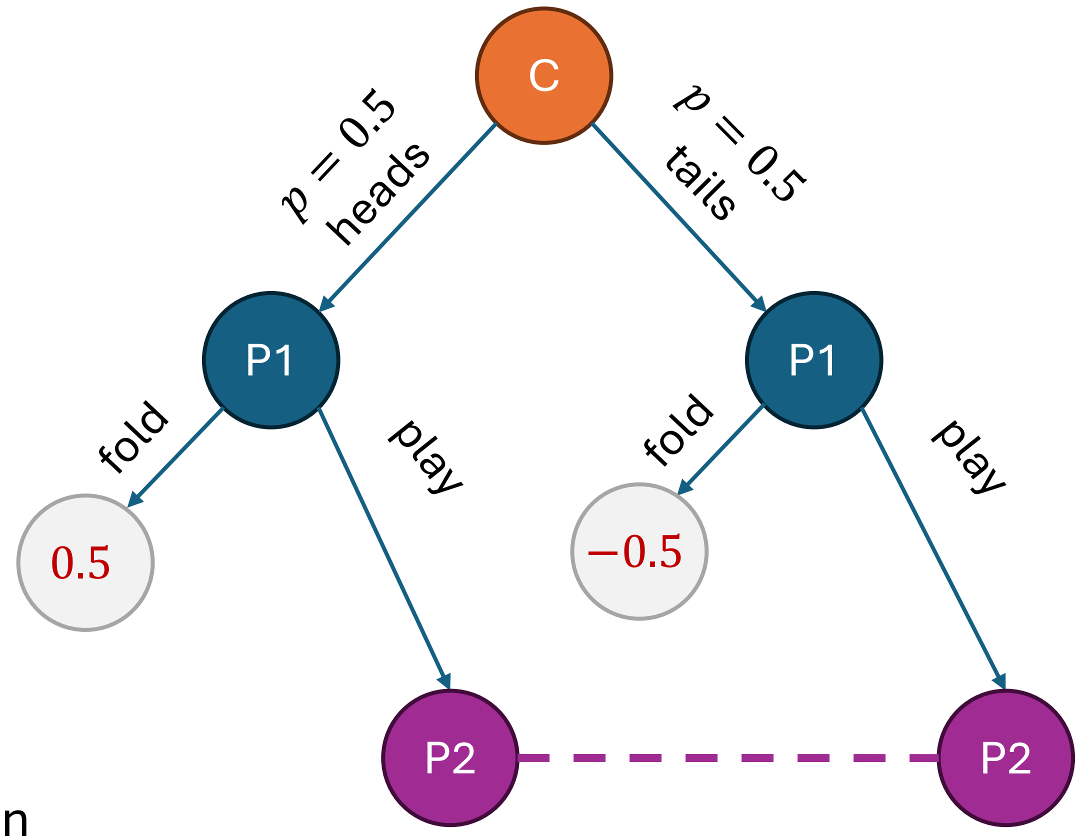
- If they play then it is player two turn
- Player two doesn't see the outcome of the coin



Credits: Superhuman AI for heads-up no-limit poker: Libratus beats top professionals (youtube.com)

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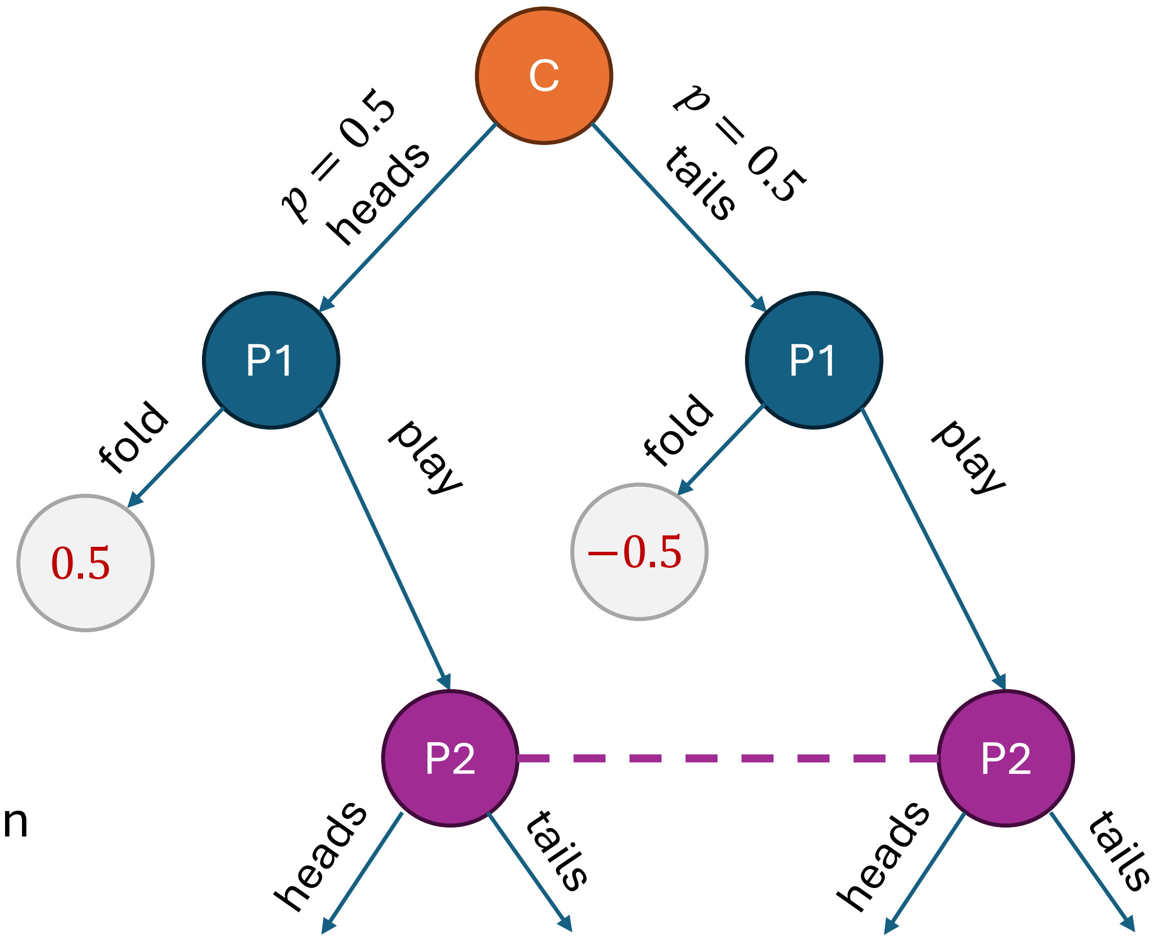
- They choose either heads or tails



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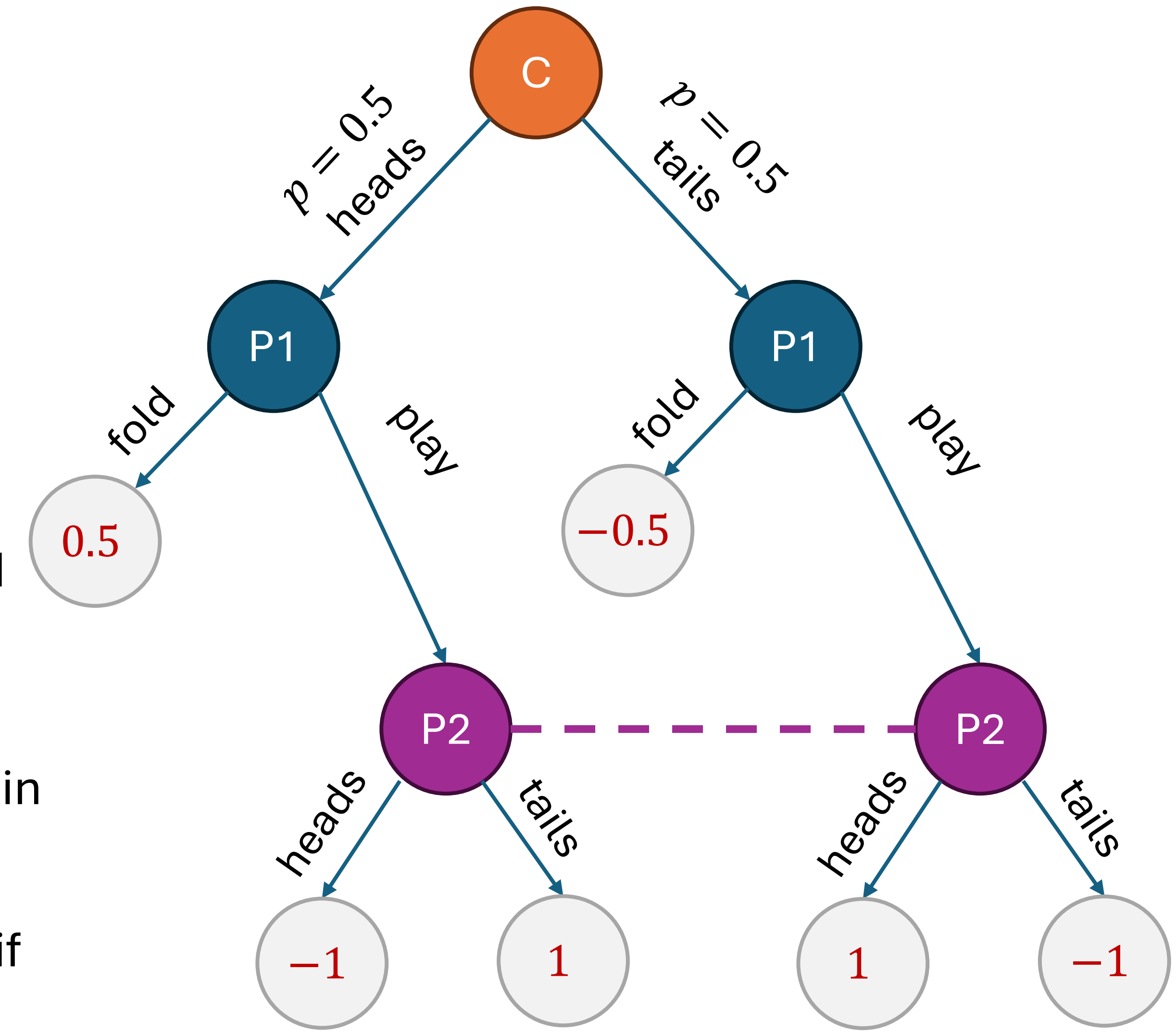
- If they fold with heads they win 0.5 if they fold with tails they lose 0.5
- If they play then it is player two turn
- Player two doesn't see the outcome of the coin
- They choose either heads or tails
- If they match the coin they win 1 (P1 loses 1) if they don't match they lose 1 (P1 wins 1)



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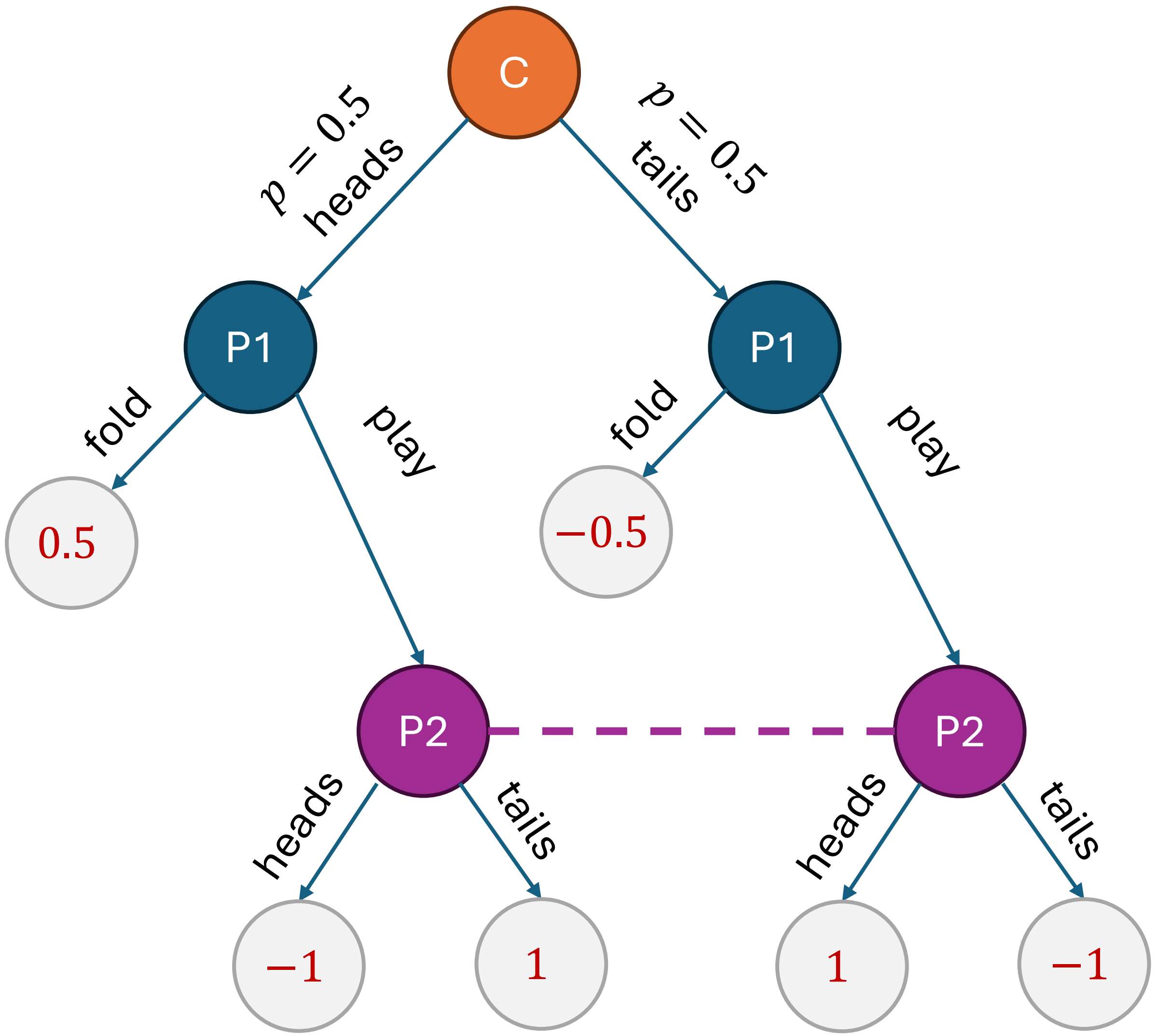
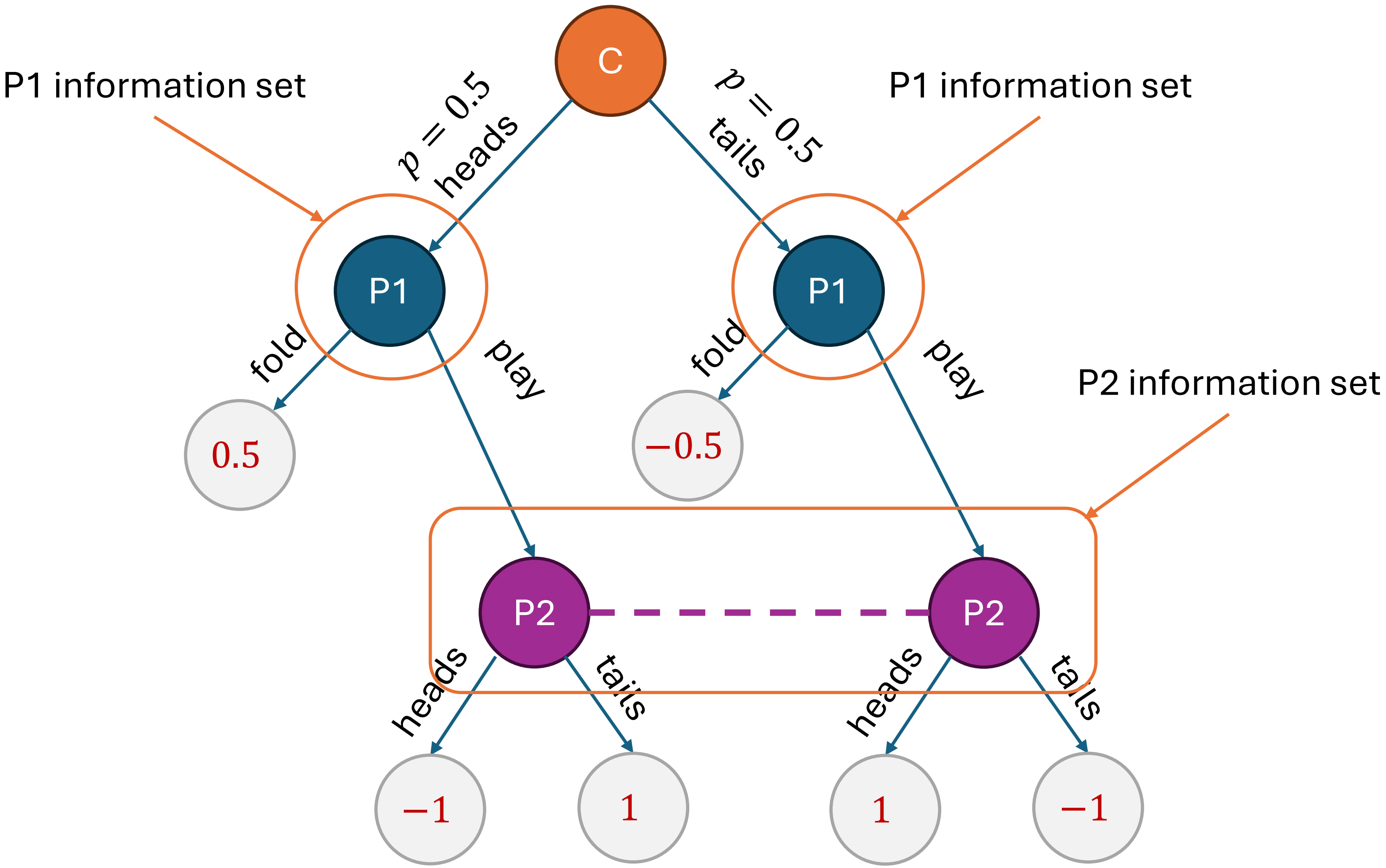
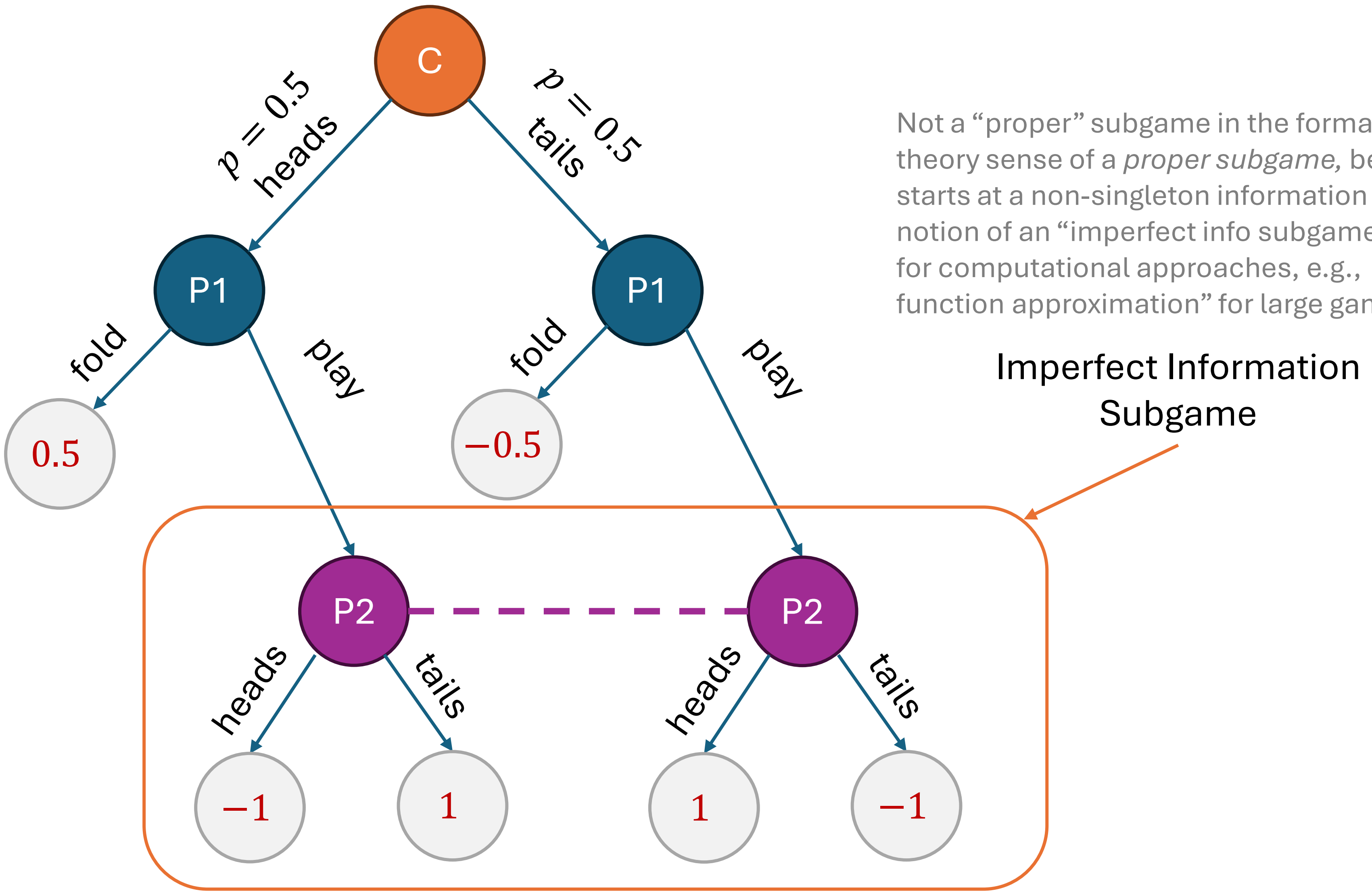
A Simple Game



A Simple Game

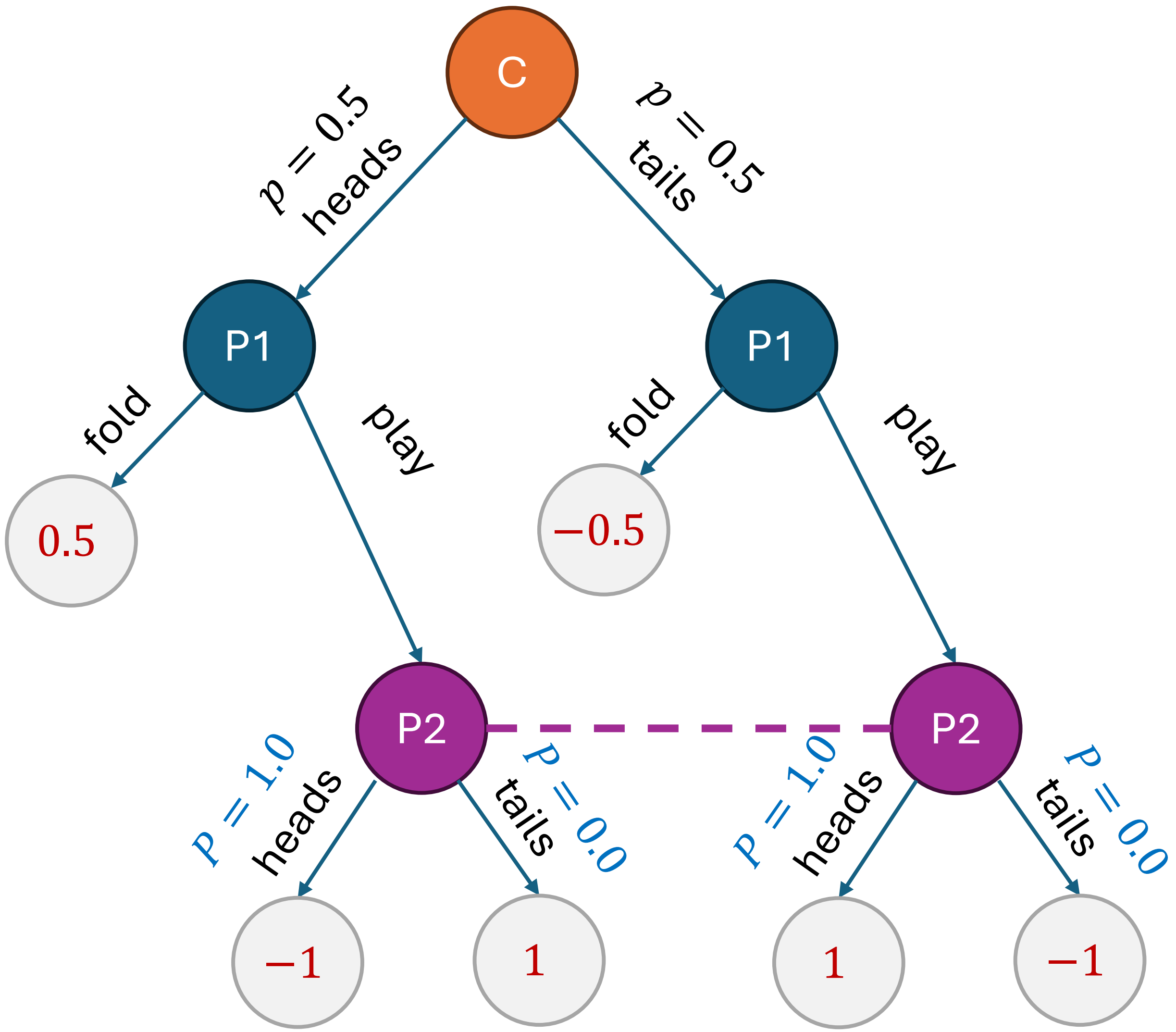


A Simple Game



Not a "proper" subgame in the formal game theory sense of a *proper subgame,* because it starts at a non-singleton information set. This notion of an "imperfect info subgame" is useful for computational approaches, e.g., "value function approximation" for large games.

How should P2 play in the sub-game?



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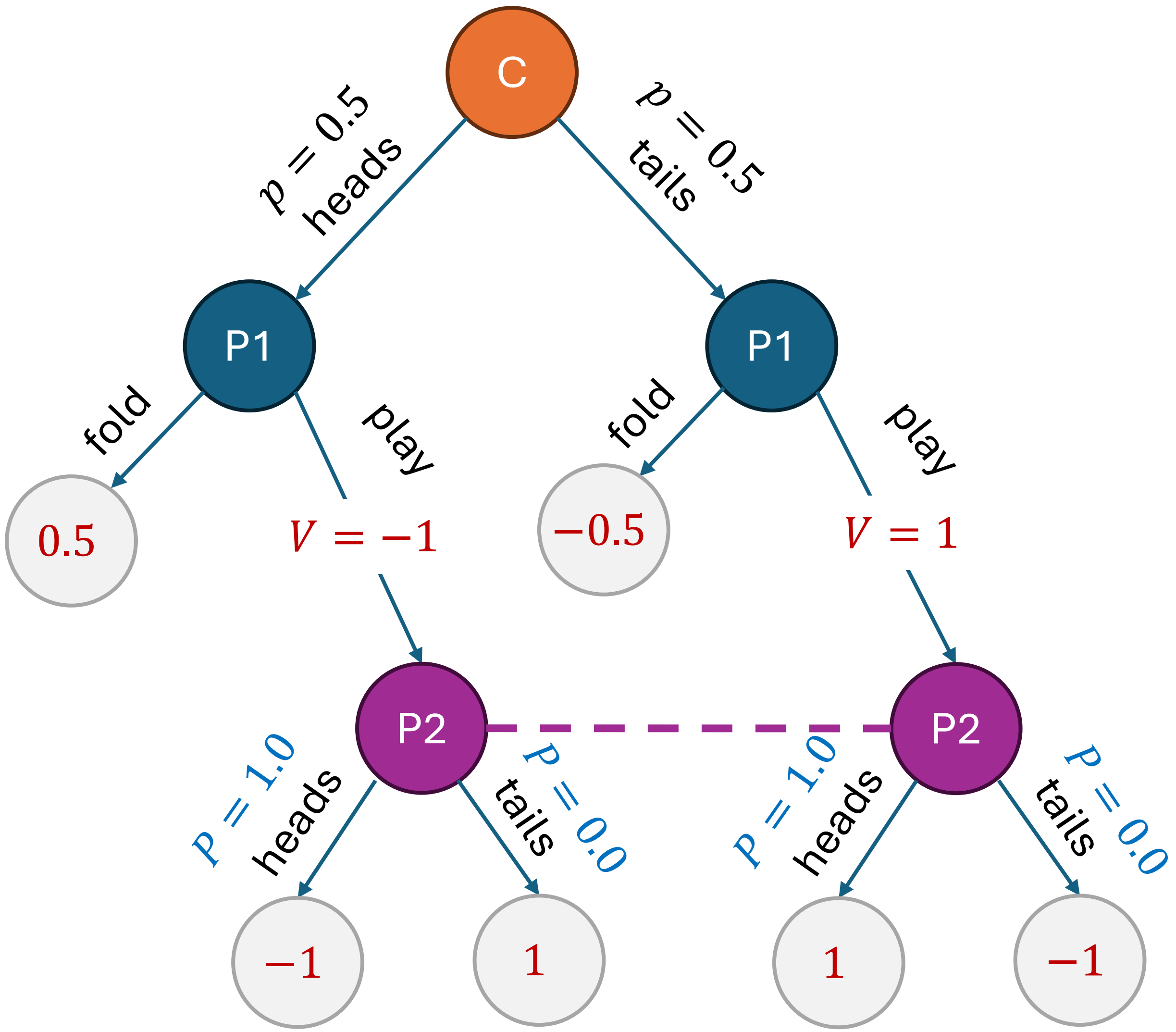
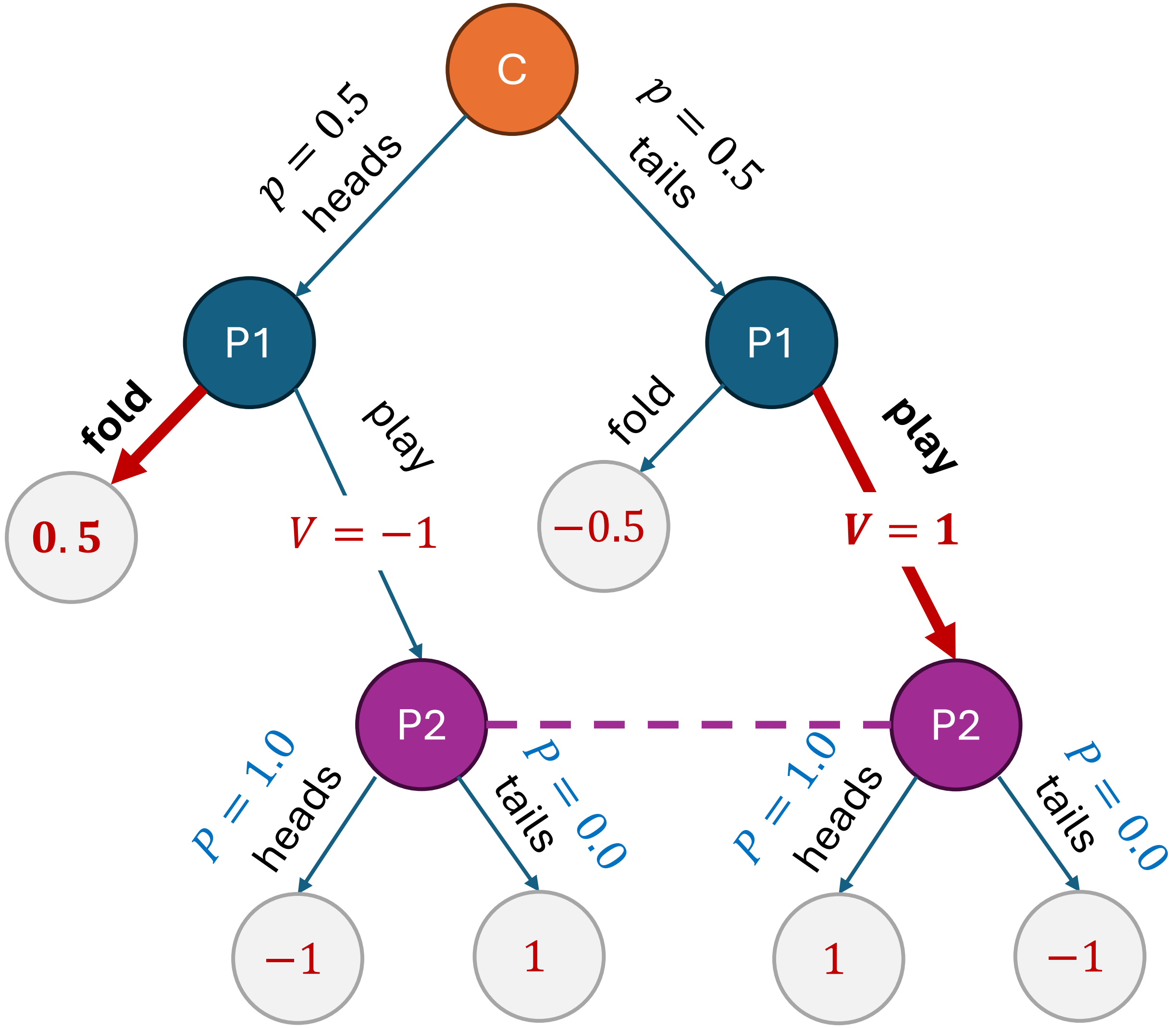


How should P2 play in the sub-game?

$E[V] = .75$



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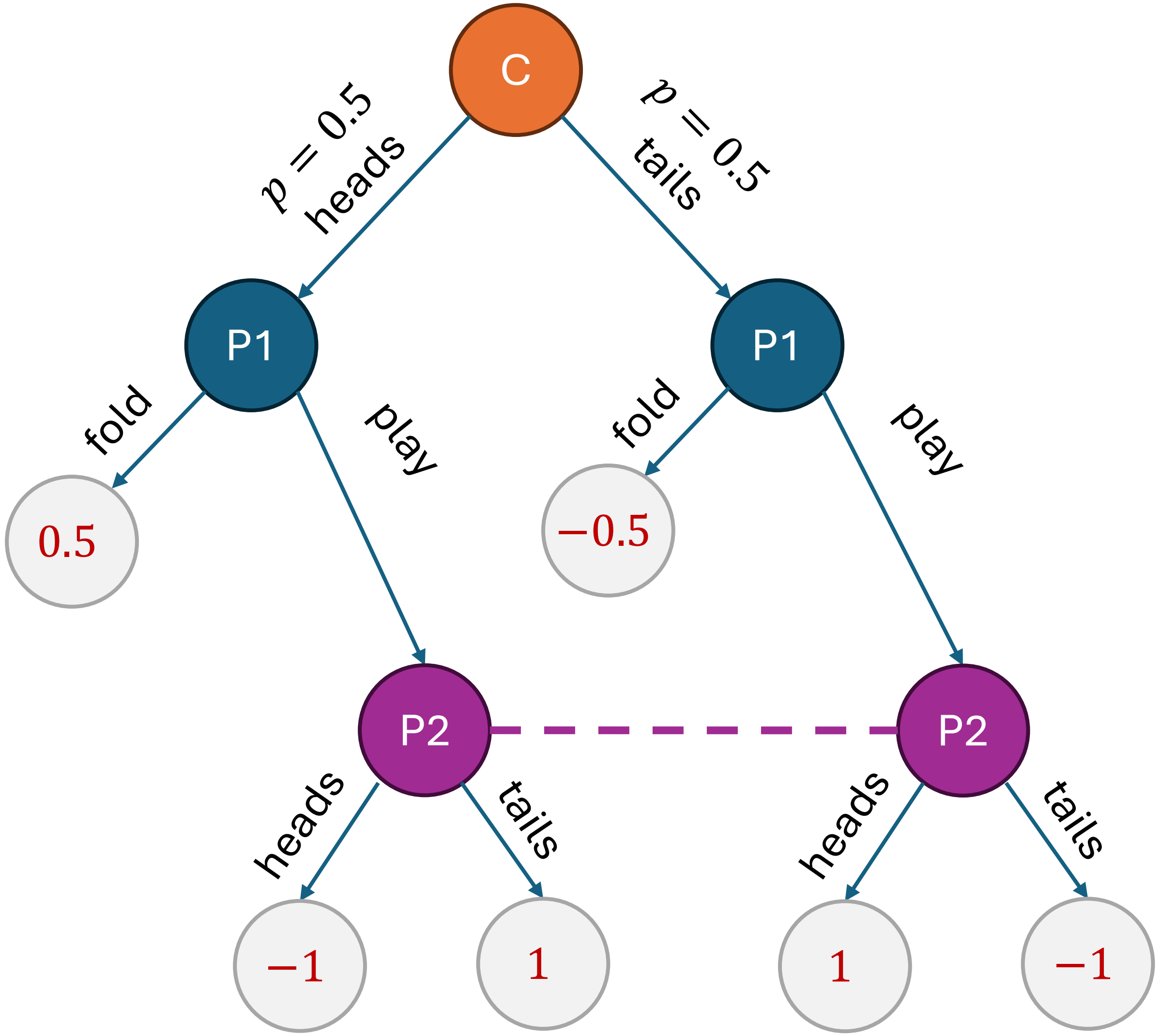

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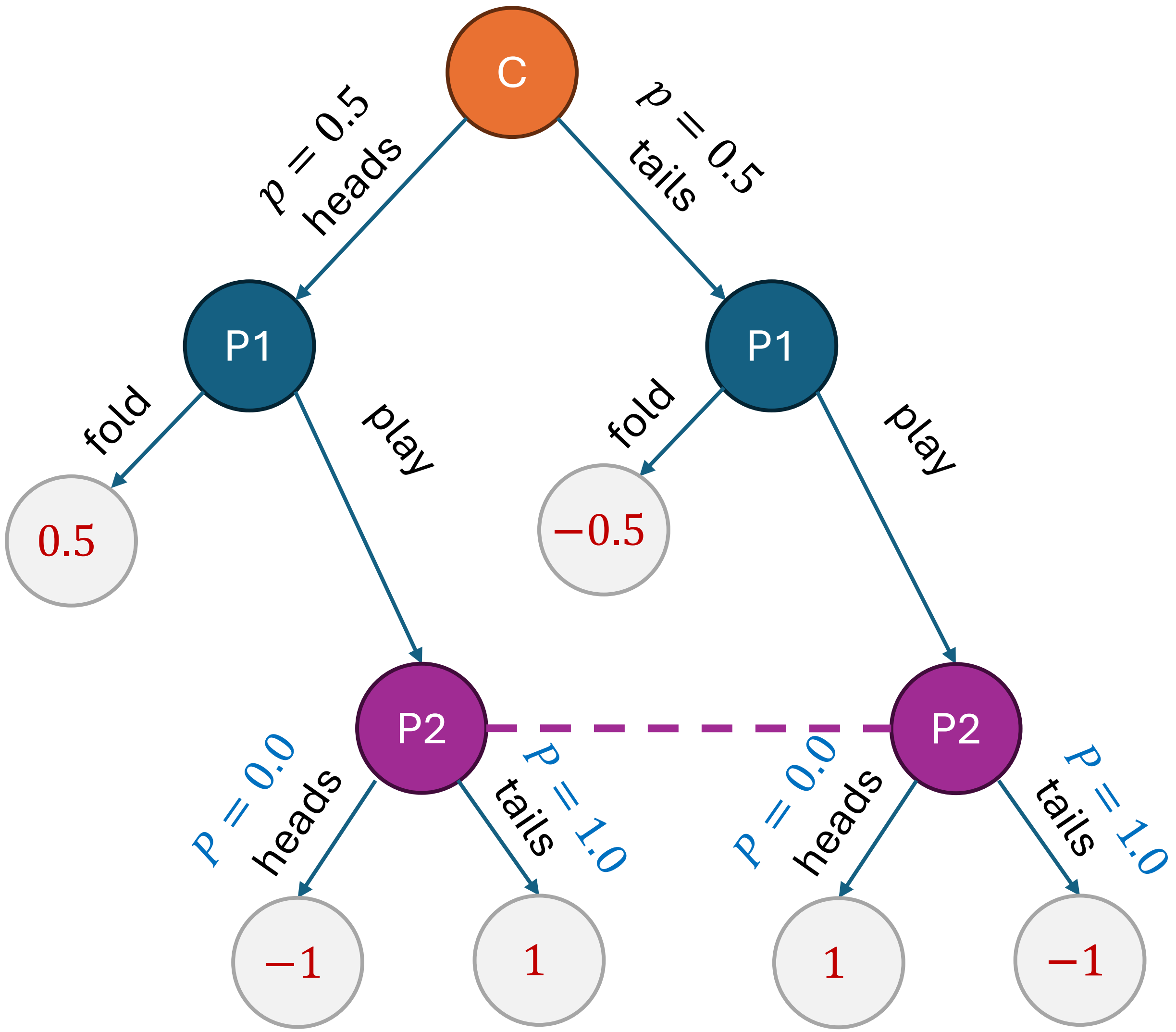

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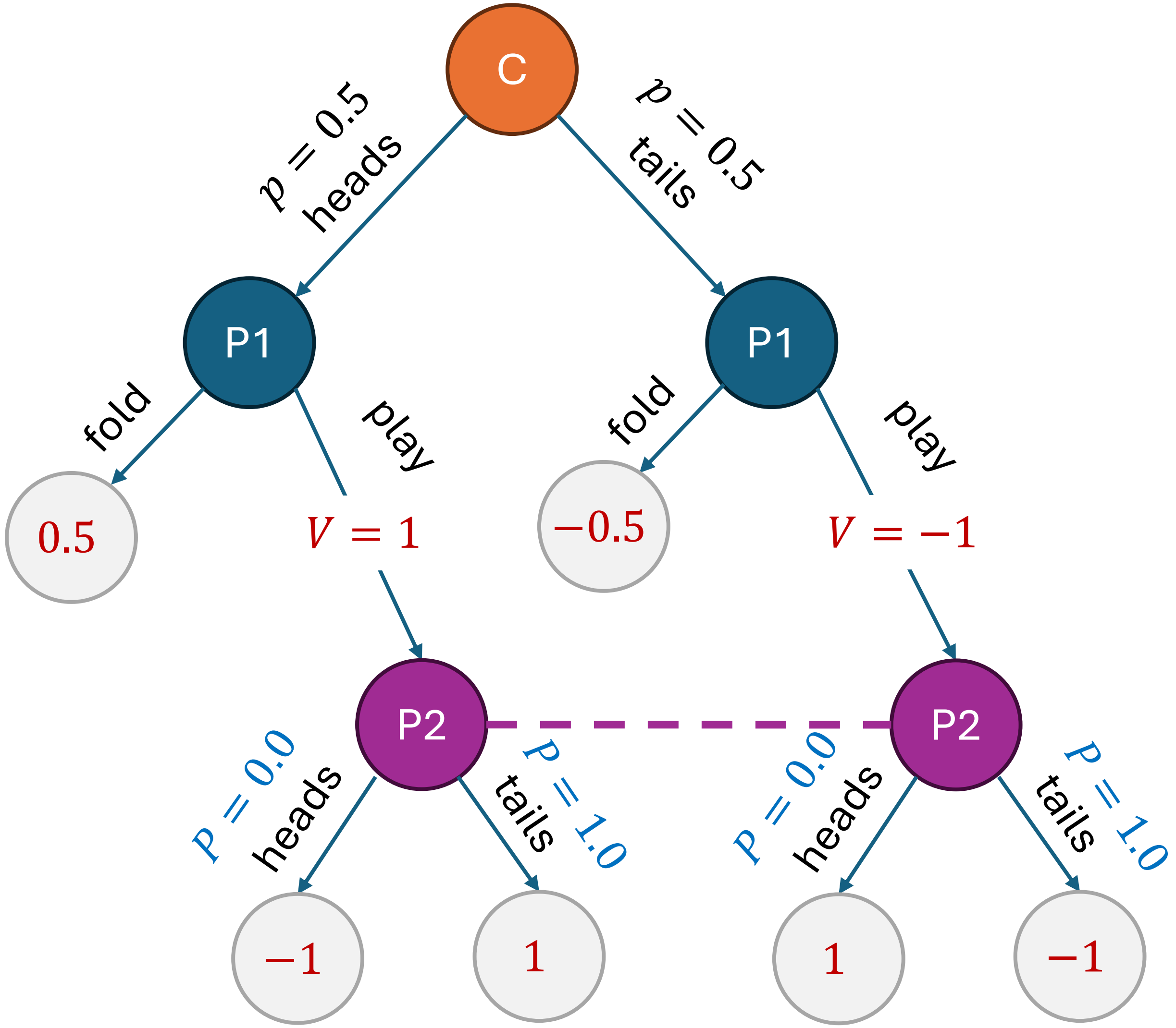
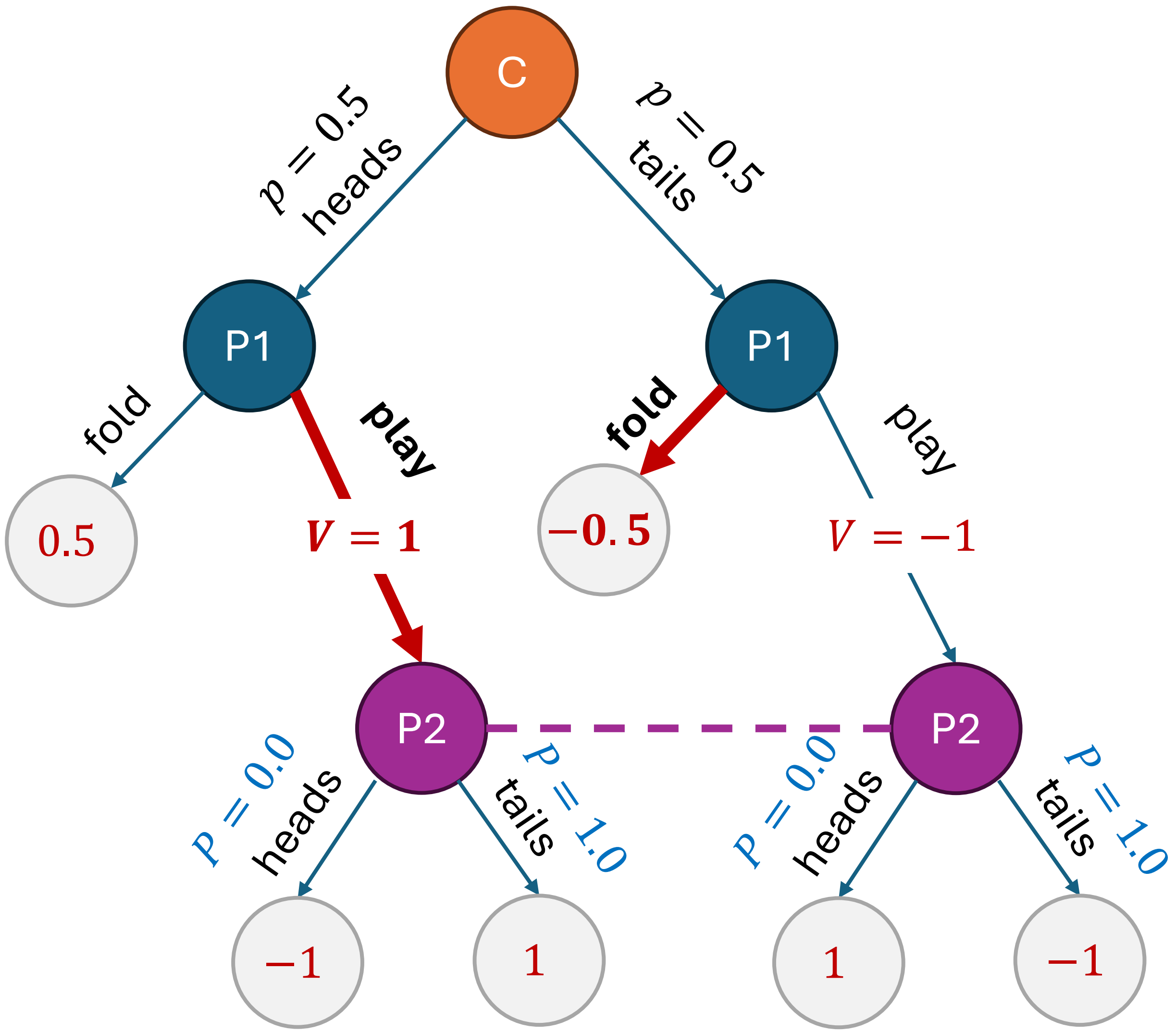
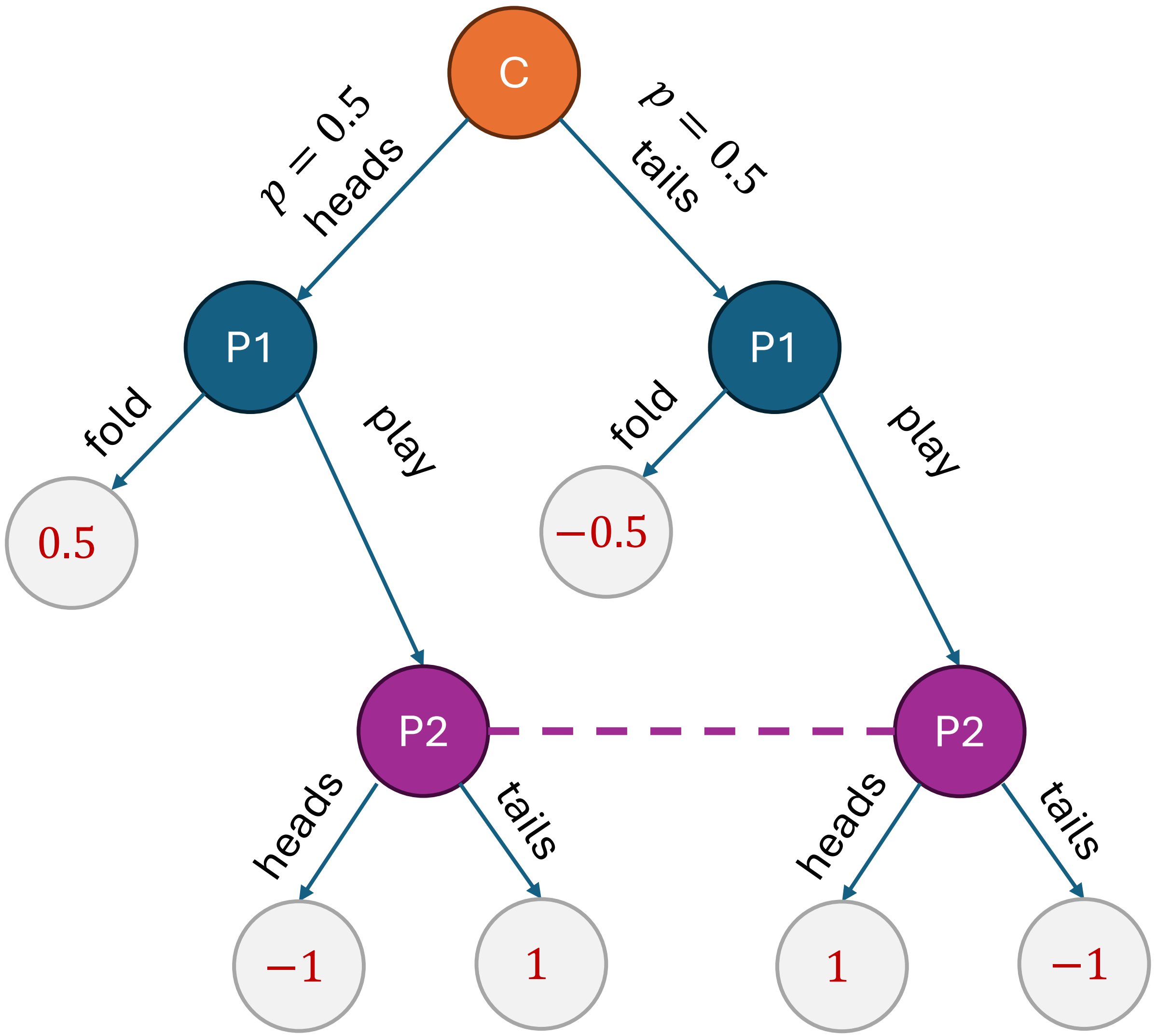


How should P2 play in the sub-game?

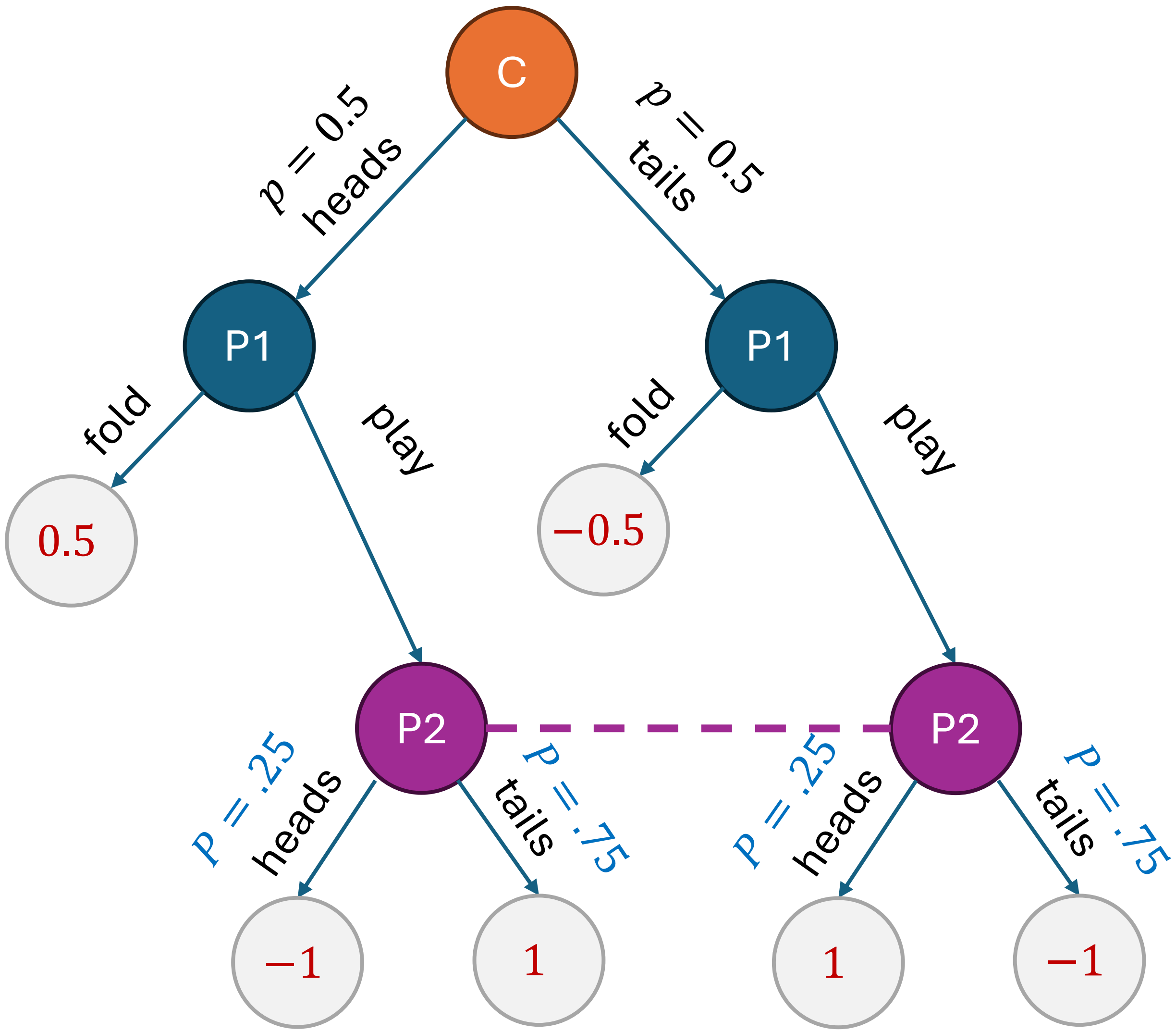
$\boldsymbol{E[V] = .25}$



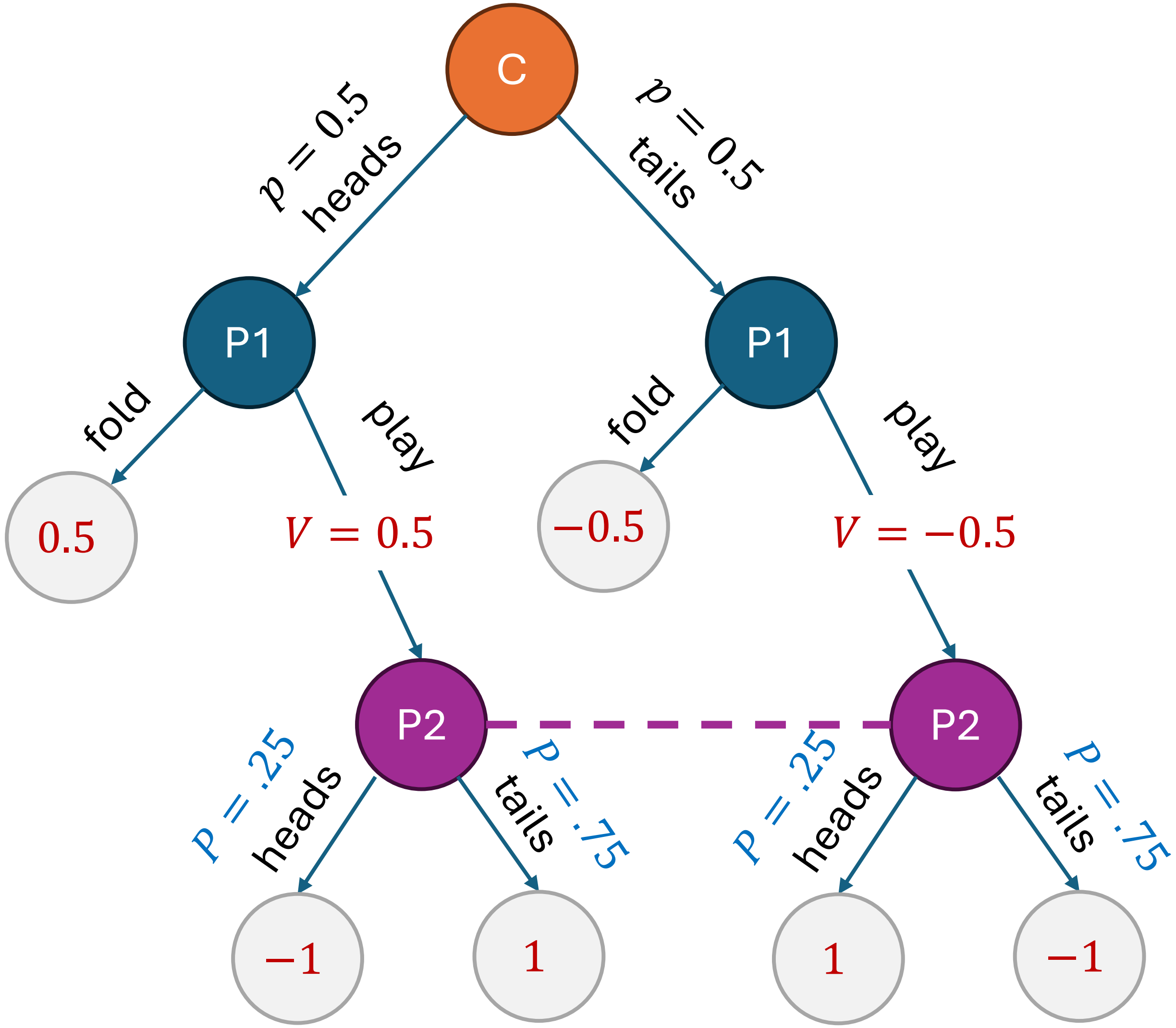
How should P2 play in the sub-game?



How should P2 play in the sub-game?

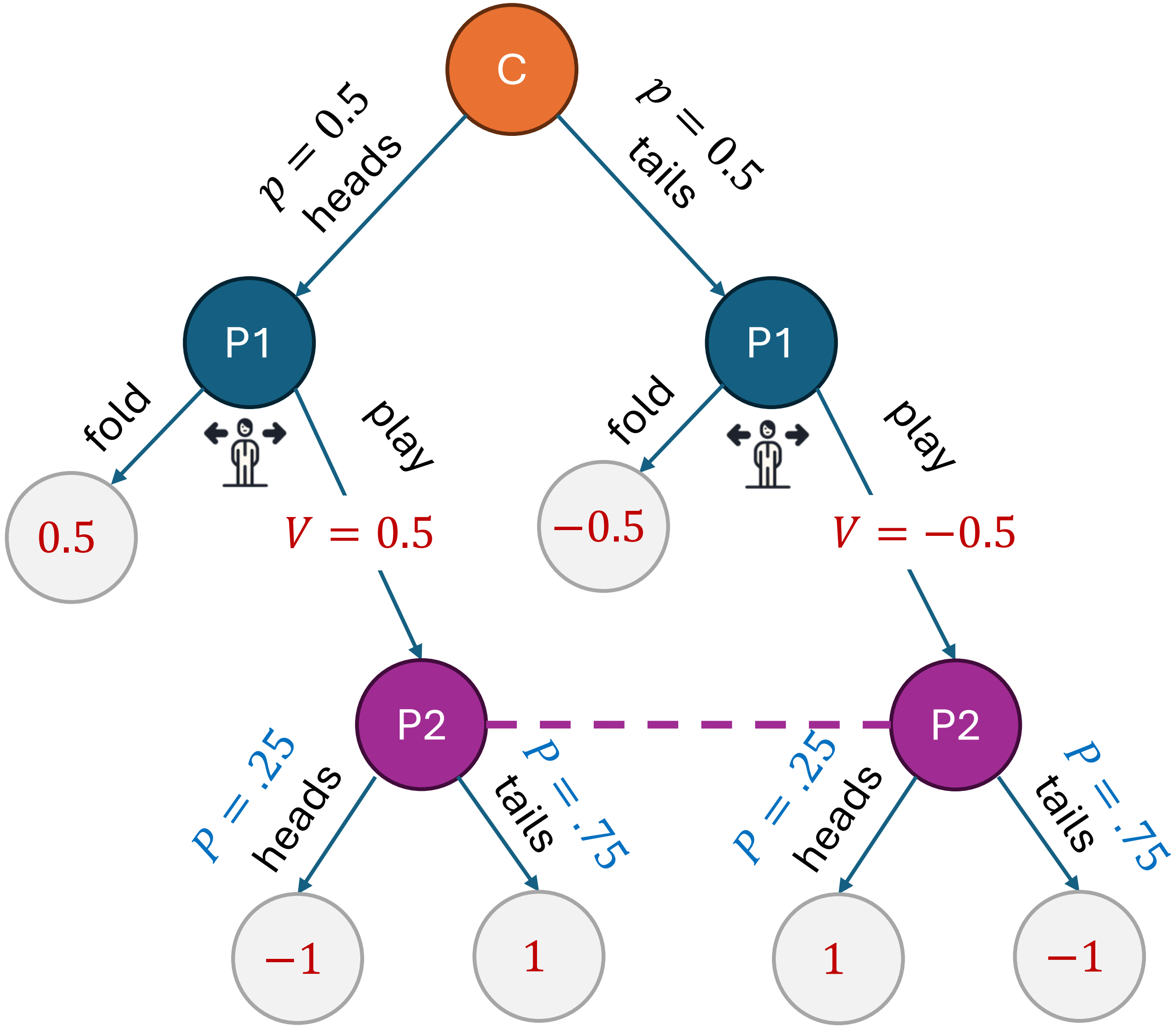


How should P2 play in the sub-game?

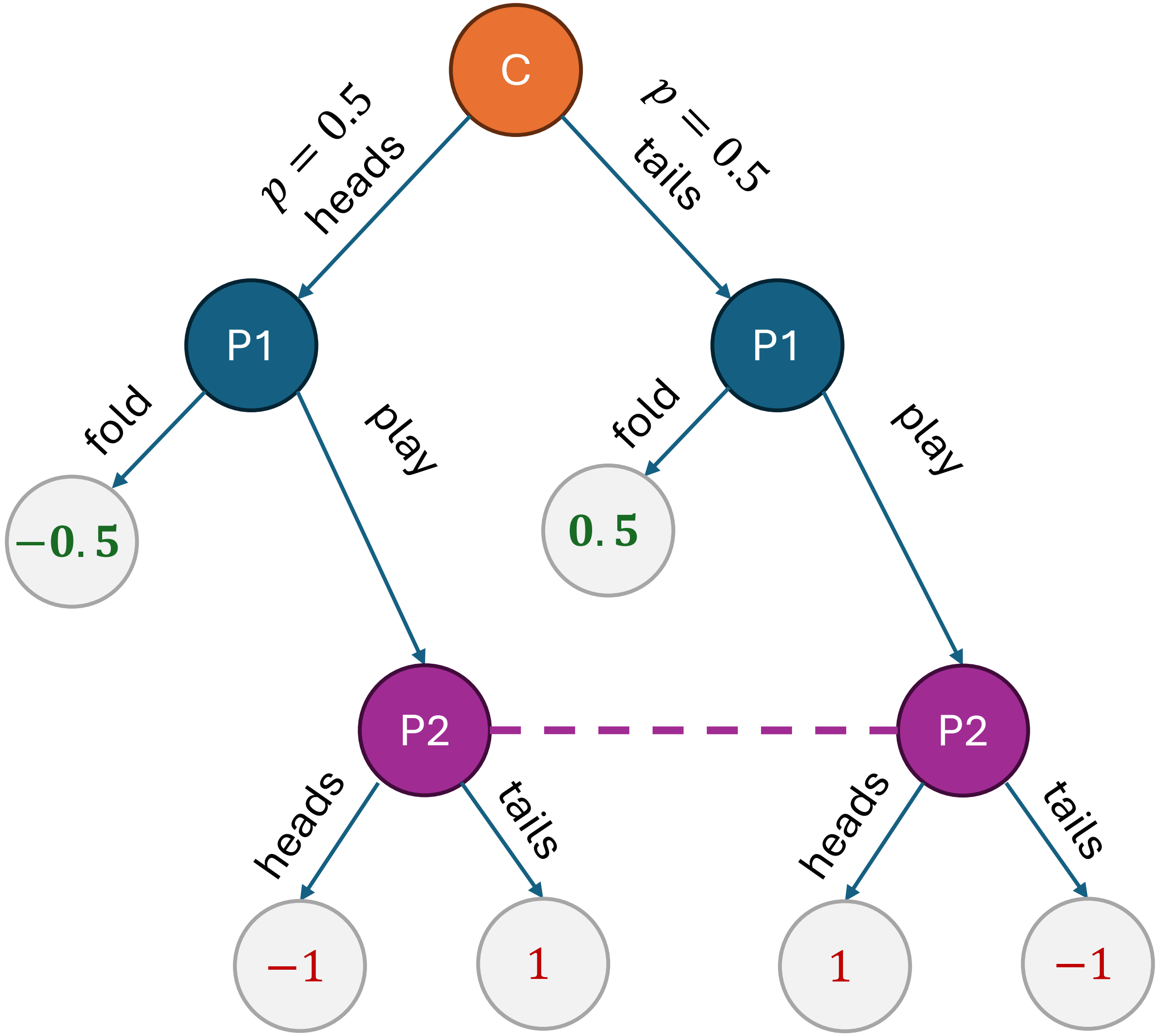


How should P2 play in the sub-game?

$\boldsymbol{E[V] = 0}$



What if we change the value of the fold?



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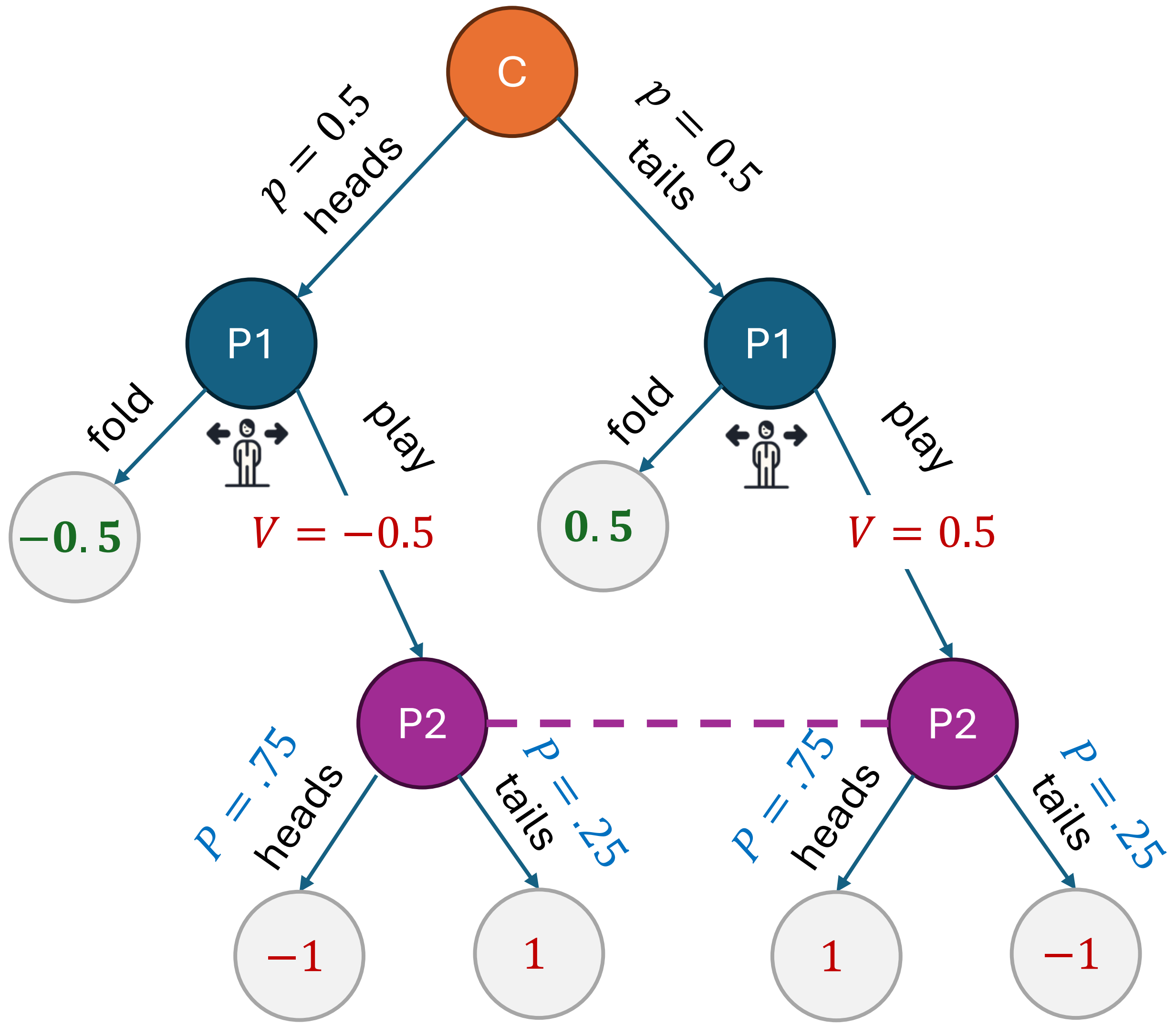
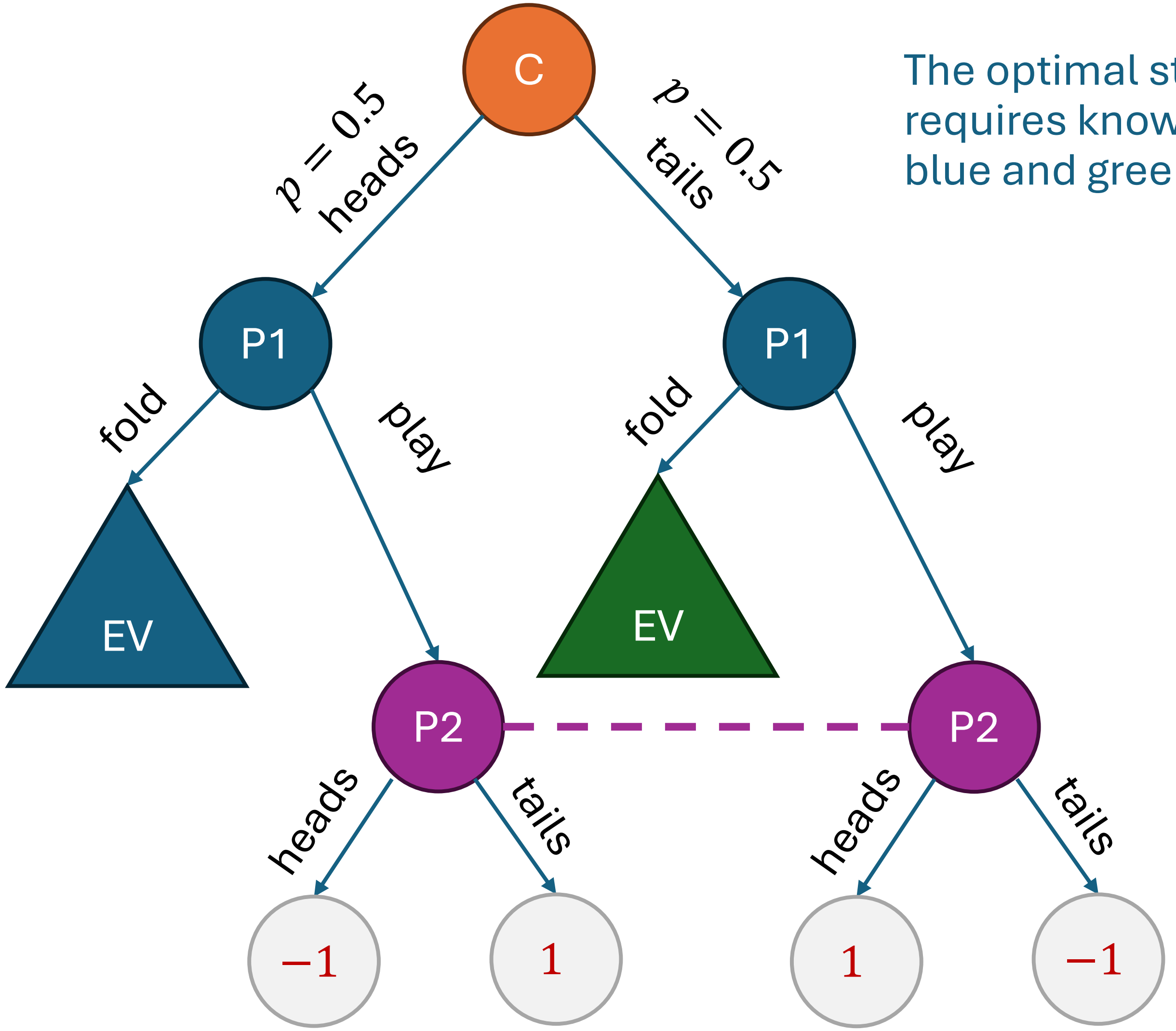


What if we change the value of the fold?



The optimal strategy in the “play” sub-game, requires knowledge of what happens in the blue and green “fold” sub-games

The Elements of an Imperfect Information Game Tree

Tree Representation and Information Sets

- **Nodes.** Each node in the tree is a decision point for some player
- **Information sets (infoset).** Nodes that belong to player i are partitioned into information sets $I \in \mathcal{J}_i$, with indices $\mathcal{J}_i = \{j_1, \dots, j_{K_i}\}$
- Player does not know which node in the information set is chosen
- Must use the same strategy on all nodes in information set
- Each infoset $j \in \mathcal{J}_i$ has a set of actions A_j that the player can take
- **Leaf nodes Z.** The set of terminal states. Player 1 gains utility $u(z)$
- **Chance nodes.** Chance or Nature moves with a fixed distribution

Perfect Recall

- Players remember all the past actions they took
- For each infoset I, there is unique “parent” (infoset, action) pair
- For every node in $I \in \mathcal{I}_i$, the parent pair I', a' was the last infoset visited and action taken by player i before reaching I
- Let p_j the last action player took before reaching infoset indexed j

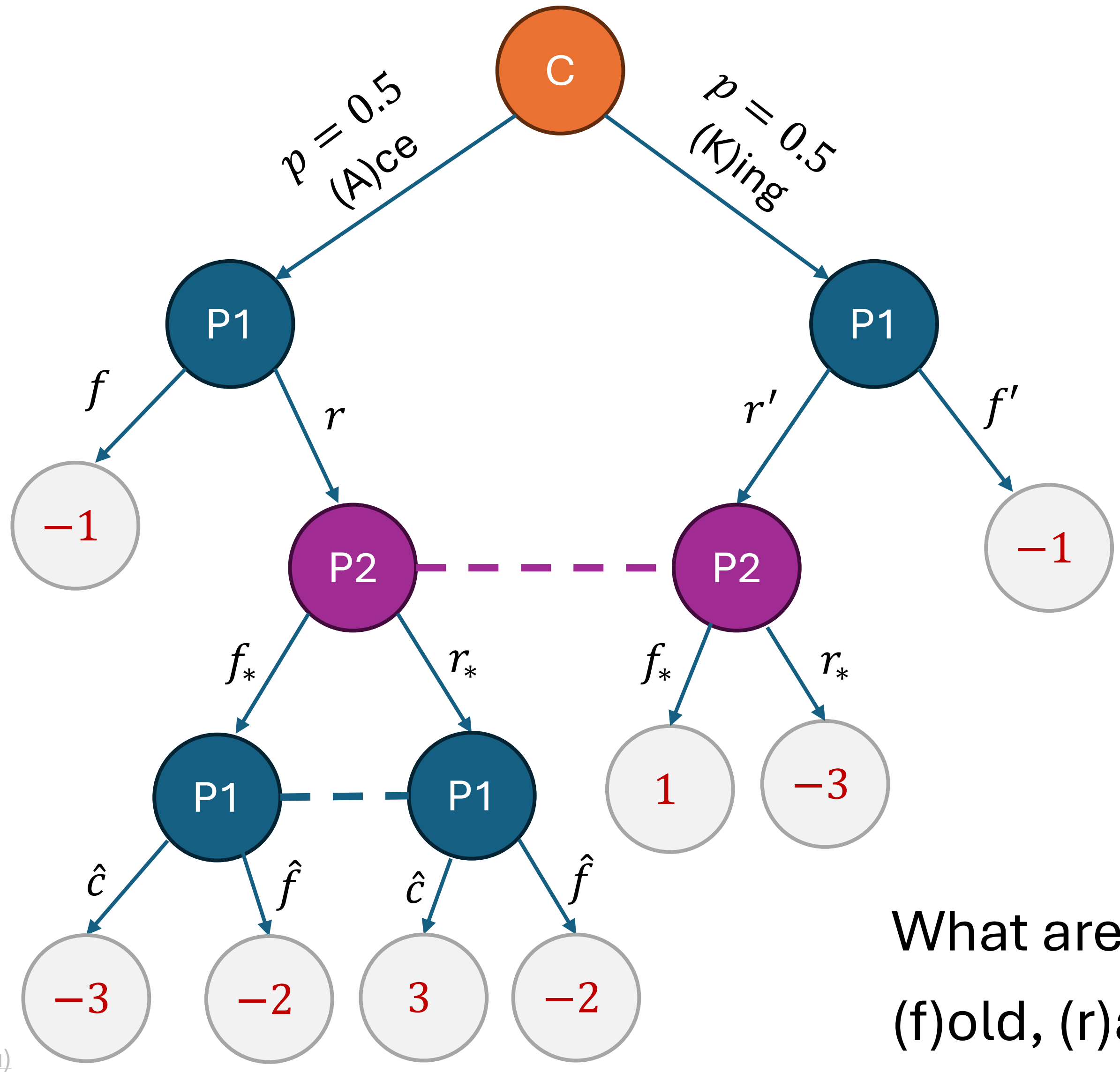
Strategic Form Representation

- **Mixed strategy.** A distribution over pure strategies
- **Behavioral strategy.** A set of distributions over actions at each information set

Kuhn's Theorem. For every mixed strategy there is an equivalent behavioral strategy that against all profiles of strategies of opponents induces the same distribution over terminal nodes

We will only be talking about behavioral strategies hereafter

A Simple "Weird" Poker Game



What are the rules of the game?

(f)old, (r)aise, (c)heck

Credits: main_ai_games_markets.pdf (columbia.edu)

Why is Nash Equilibrium a Good Idea?

- In zero-sum games where no player has an a-priori competitive advantage, Nash Equilibrium guarantees no loss in expectation
- It is a “safe” strategy no matter what the opponent does!

Game Representations Convenient for Computing a Nash Equilibrium

Computing Nash Equilibrium

- We know how to compute equilibria of static zero-sum games
- Can we view the extensive form zero-sum game also as min-max

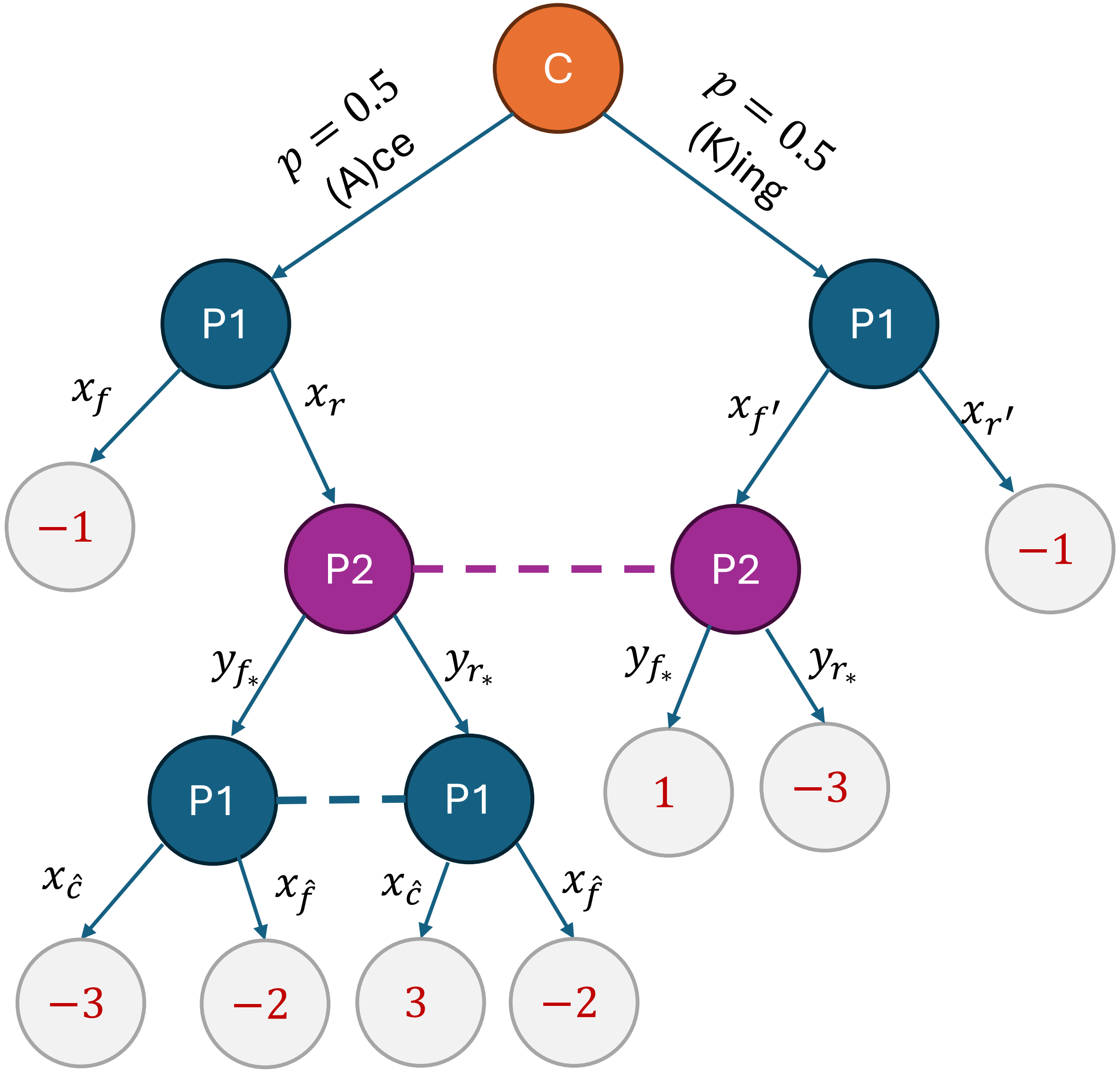
$$\max_{x \in X} \min_{y \in Y} x^\top A y$$

- What does x and y encode?

- What if $x = \left(x^j\right)_{j \in \mathcal{J}_1}$, where x^j is mixed strategy at infoset $j \in \mathcal{J}_1$
- What if $y = \left(y^j\right)_{j \in \mathcal{J}_2}$, where y^j is mixed strategy at infoset $j \in \mathcal{J}_2$

Behavioral Strategy Representation

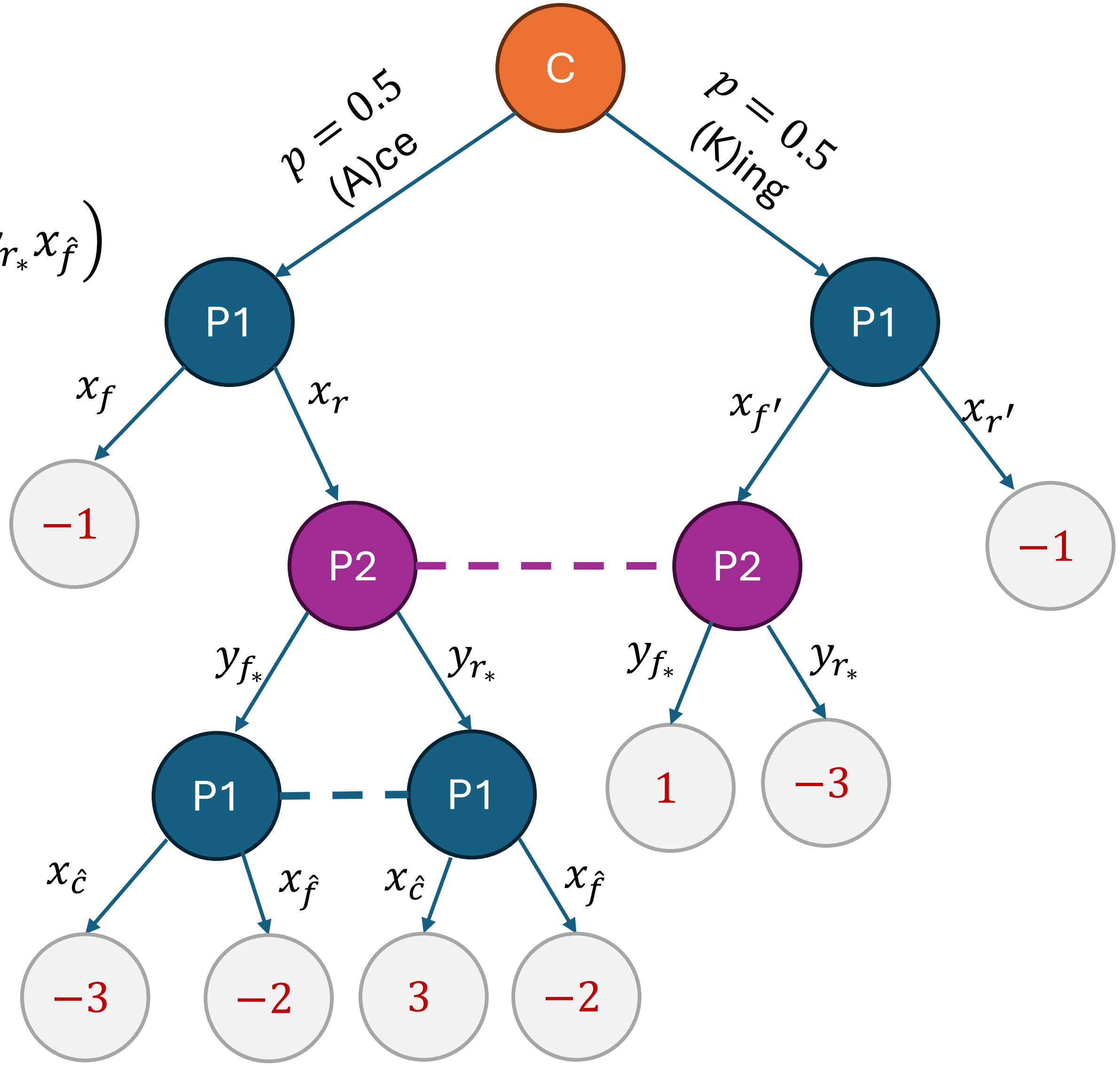
What is the expected payoff of x?



Behavioral Strategy Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-x_f - 3\, x_r y_{f_*} x_{\hat{c}} - 2\, x_r y_{f_*} x_{\hat{f}} + 3 x_r y_{r_*} x_{\hat{c}} + 2\, x_r y_{r_*} x_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(x_{f'} y_{f_*} - 3\, x_{f'} y_{r_*} - x_{r'}\right)$$



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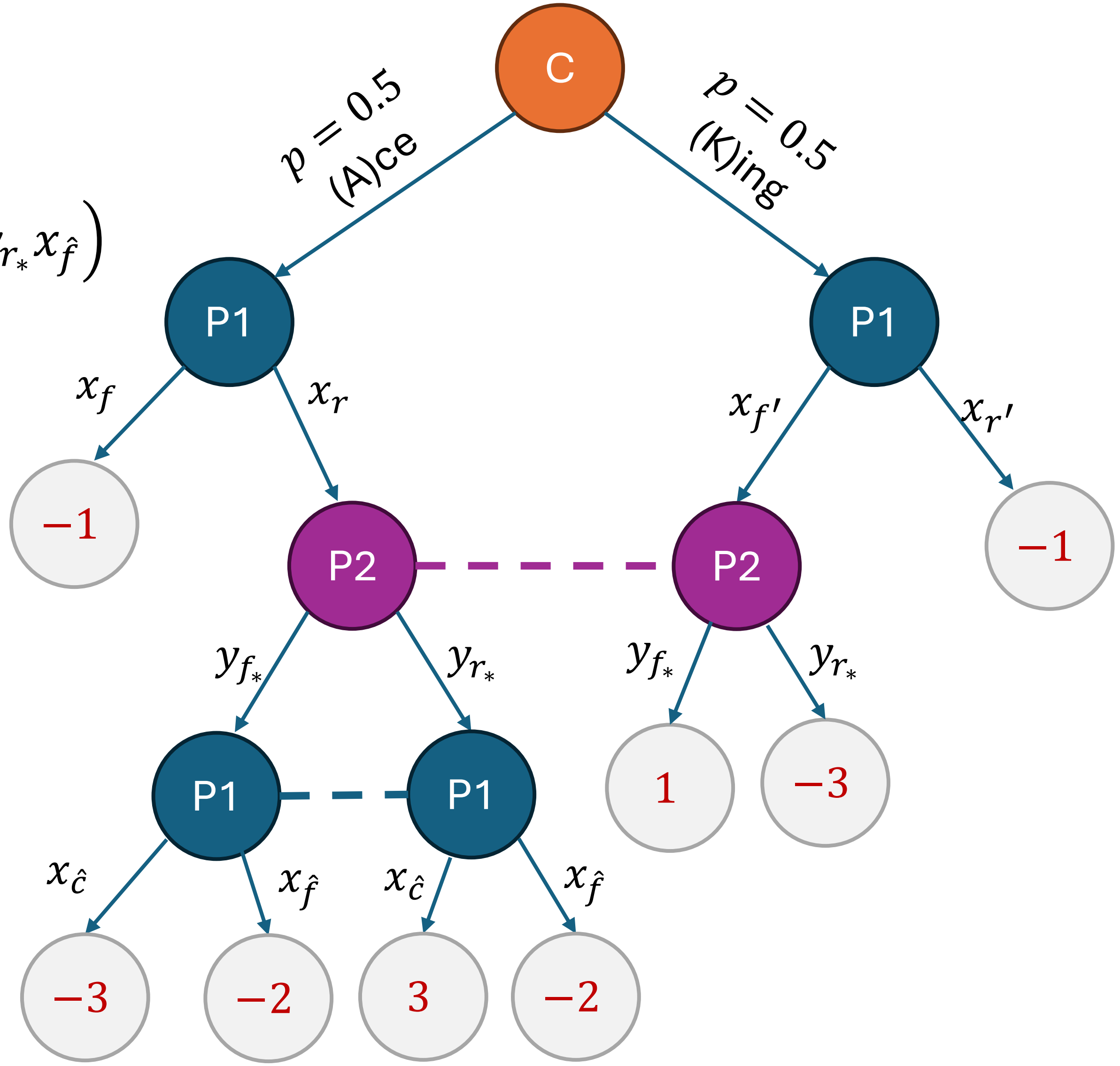
$$+ \frac{1}{2}\left(x_{f'} y_{f_*} - 3\, x_{f'} y_{r_*} - x_{r'}\right)$$

Is it of the form $x^\top A y$?



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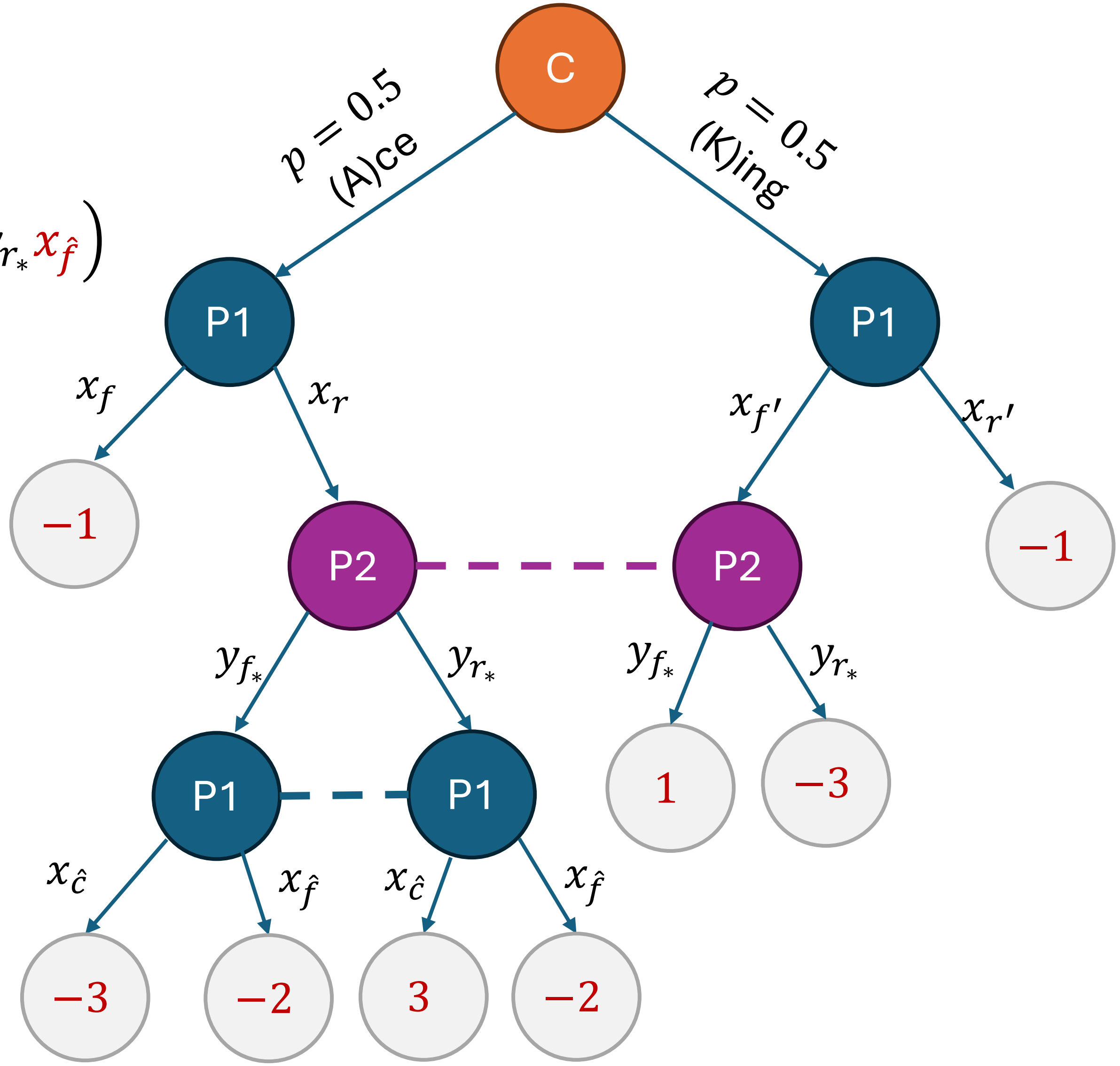


Behavioral Strategy Representation

What is the expected payoff of x?

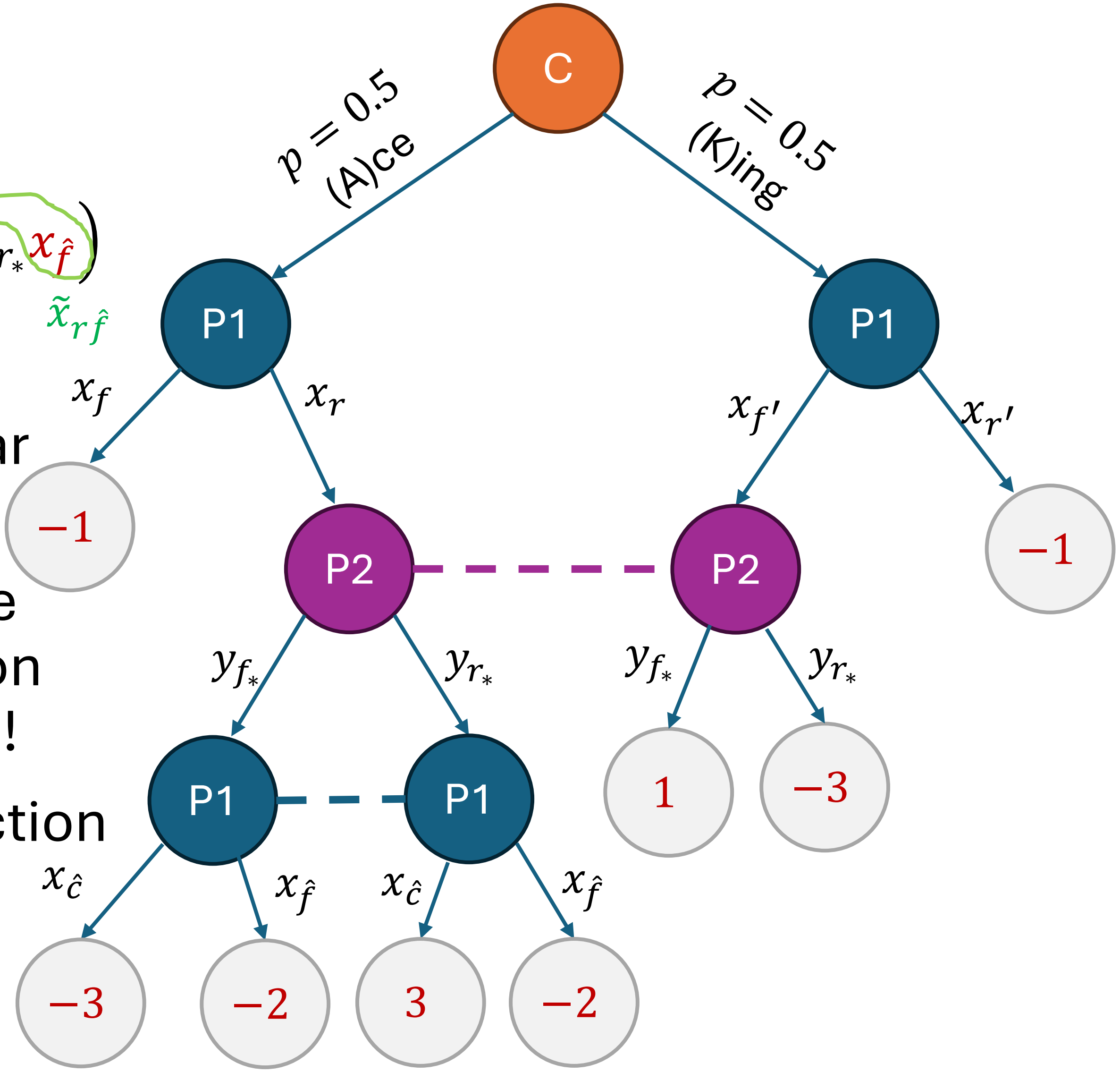
$$\frac{1}{2}\Big(-x_f - 3\,\underbrace{x_r y_{f_*} x_{\hat{c}}}_{\tilde{x}_{r\hat{c}}} - 2\,\underbrace{x_r y_{f_*} x_{\hat{f}}}_{\tilde{x}_{r\hat{f}}} + 3\,\underbrace{x_r y_{r_*} x_{\hat{c}}}_{\tilde{x}_{r\hat{c}}} + 2\,\underbrace{x_r y_{r_*} x_{\hat{f}}}_{\tilde{x}_{r\hat{f}}}\Big)$$
$$+\frac{1}{2}\left(x_{f'} y_{f_*} - 3\, x_{f'} y_{r_*} - x_{r'}\right)$$

IDEA. Group together products that appear into new "variables"

New variables represent the product of the probabilities of the actions chosen by P1 on the path to the last action in the sequence!

We will annotate them just with the last action

$$\tilde{x}_{r\hat{c}} \equiv \tilde{x}_{\hat{c}}, \qquad \tilde{x}_{r\hat{f}} \equiv \tilde{x}_{\hat{f}}$$



Sequence Form Representation

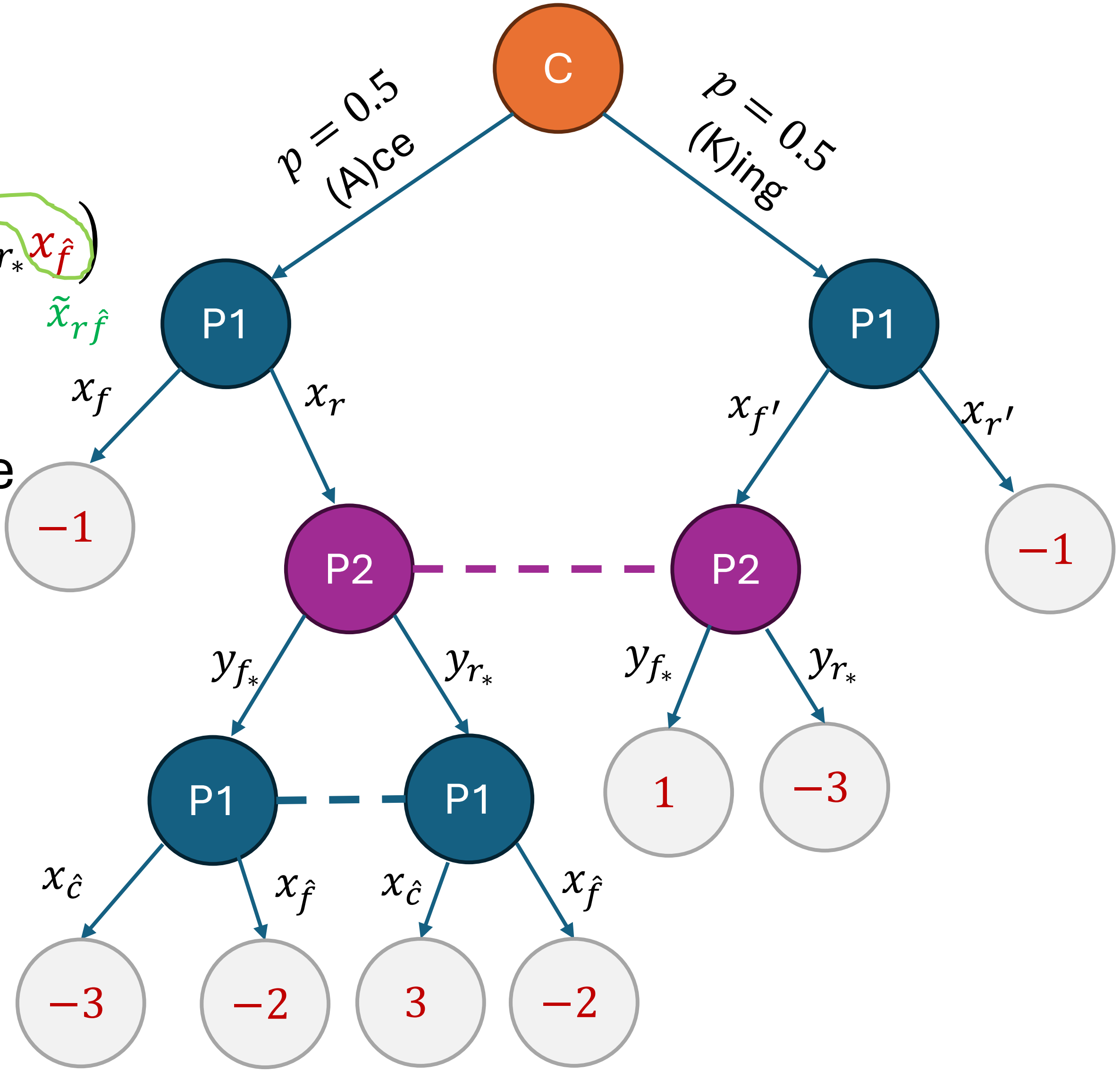
What is the expected payoff of x?

$$\frac{1}{2}\Big(-x_f - 3\,\underbrace{x_r y_{f_*} x_{\hat{c}}}_{\tilde{x}_{r\hat{c}}} - 2\,\underbrace{x_r y_{f_*} x_{\hat{f}}}_{\tilde{x}_{r\hat{f}}} + 3\,\underbrace{x_r y_{r_*} x_{\hat{c}}}_{\tilde{x}_{r\hat{c}}} + 2\,\underbrace{x_r y_{r_*} x_{\hat{f}}}_{\tilde{x}_{r\hat{f}}}\Big)$$
$$+\frac{1}{2}\Big(x_{f'} y_{f_*} - 3\,x_{f'} y_{r_*} - x_{r'}\Big)$$

Sequence form strategies. We can define these new variables $\tilde{x}_{\mathrm{a}}$ for all actions of P1

$\tilde{x}_{\mathrm{a}}$: represents product of probabilities of all actions of P1 on the path to a

$$\tilde{x}_f, \tilde{x}_r, \tilde{x}_{f'}, \tilde{x}_{r'}, \tilde{x}_{\hat{c}}, \tilde{x}_{\hat{f}}$$



Sequence Form Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-x_f - 3\, x_r y_{f_*} x_{\hat{c}} - 2\, x_r y_{f_*} x_{\hat{f}} + 3\, x_r y_{r_*} x_{\hat{c}} + 2\, x_r y_{r_*} x_{\hat{f}}\right)$$

$\tilde{x}_{r\hat{c}}$ $\tilde{x}_{r\hat{f}}$ $\tilde{x}_{r\hat{c}}$ $\tilde{x}_{r\hat{f}}$

$$+\frac{1}{2}\left(x_{f'} y_{f_*} - 3\, x_{f'} y_{r_*} - x_{r'}\right)$$

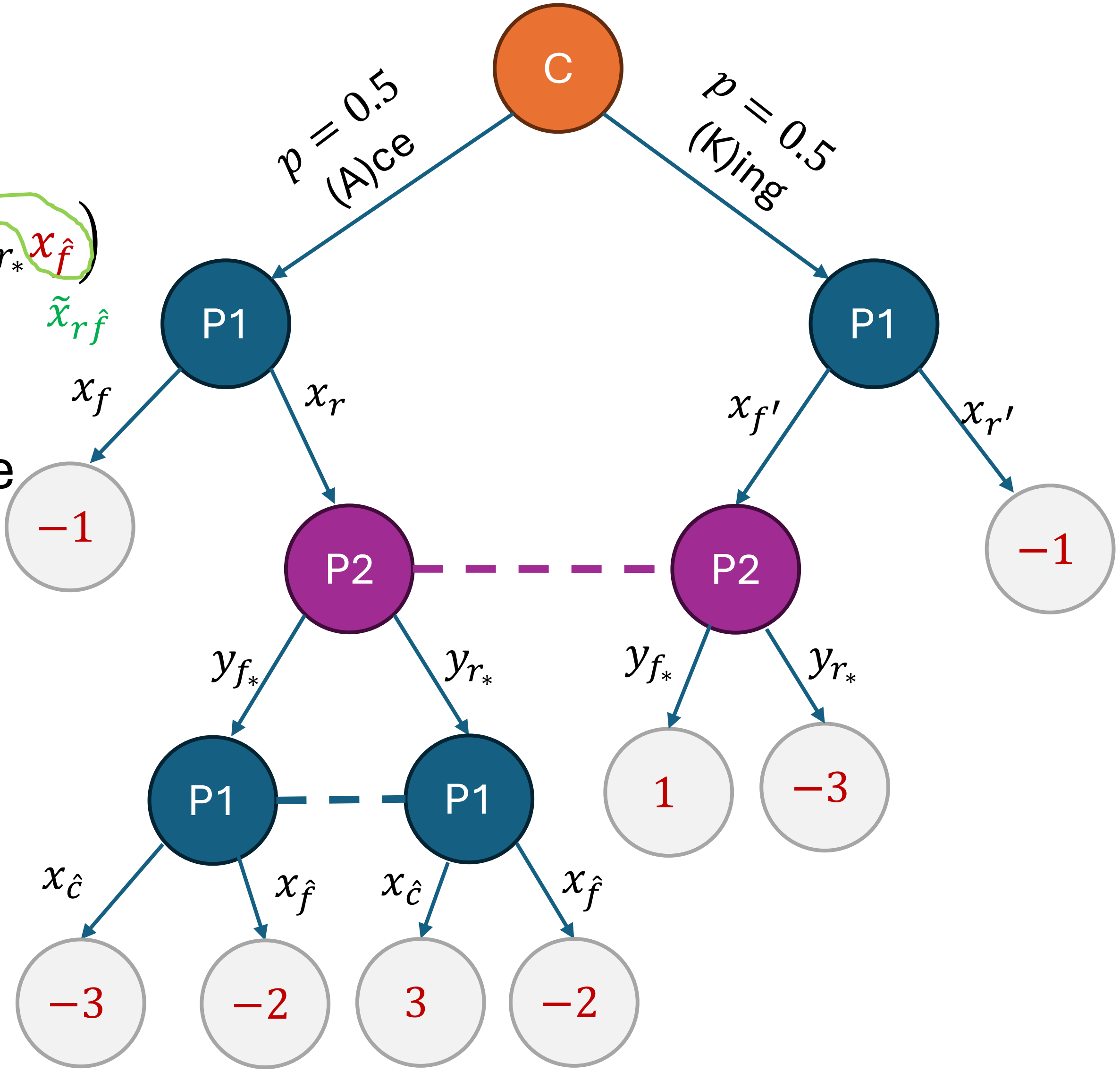
Sequence form strategies. We can define these new variables $\tilde{x}_{\mathrm{a}}$ for all actions of P1

$\tilde{x}_{\mathrm{a}}$: represents product of probabilities of all actions of P1 on the path to a

$$\tilde{x}_f, \tilde{x}_r, \tilde{x}_{f'}, \tilde{x}_{r'}, \tilde{x}_{\hat{c}}, \tilde{x}_{\hat{f}}$$

$\tilde{y}_{\mathrm{a}}$: represents product of probabilities of all actions of P2 on the path to a

$$\tilde{y}_{f_*}, \tilde{y}_{r_*}$$



Sequence Form Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-\tilde{x}_f - 3\,y_{f_*}\tilde{x}_{\hat{c}} - 2\,y_{f_*}\tilde{x}_{\hat{f}} + 3y_{r_*}\tilde{x}_{\hat{c}} + 2\,y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\,\tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

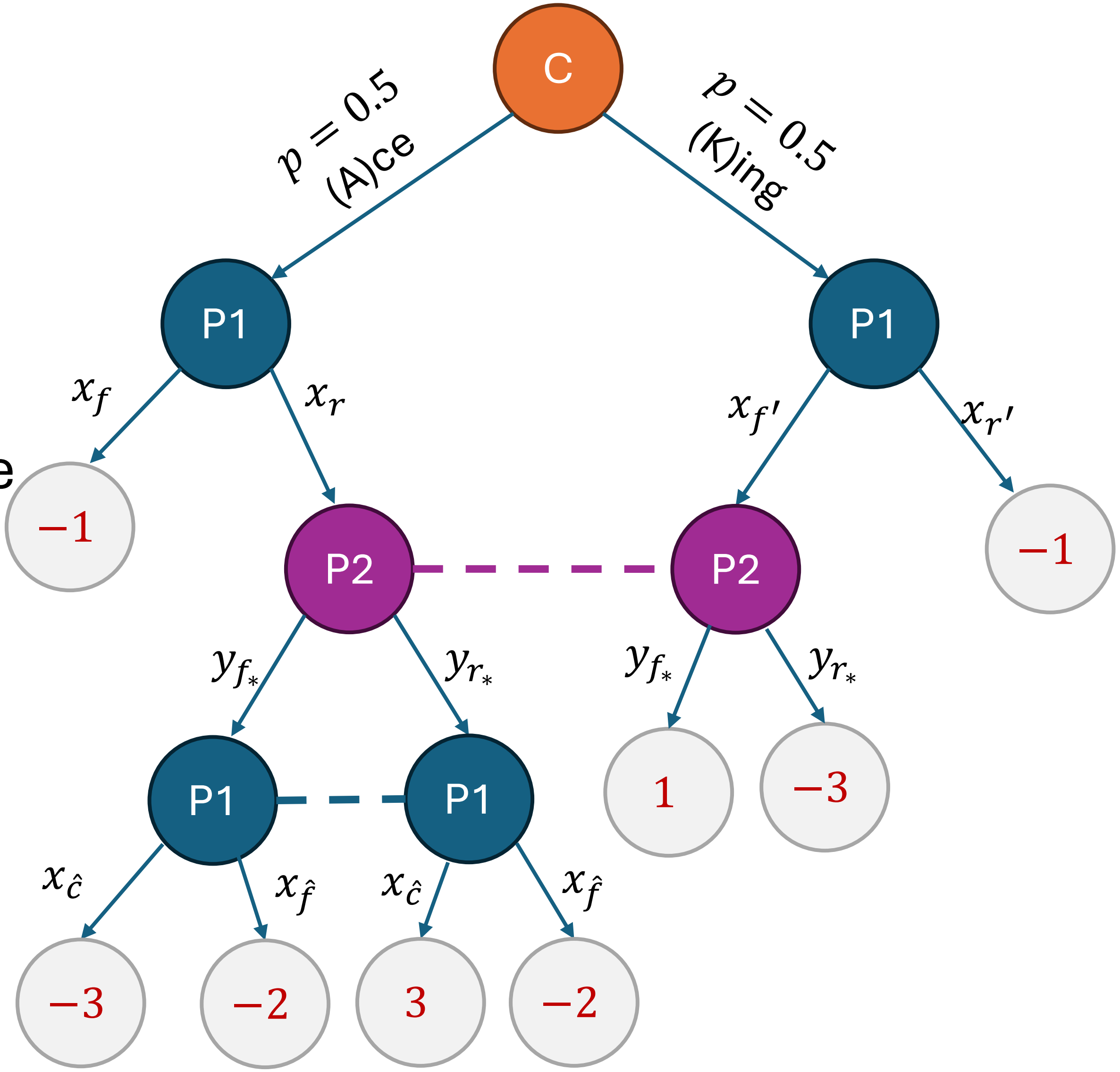
Sequence form strategies. We can define these new variables $\tilde{x}_a$ for all actions of P1

$\tilde{x}_a$: represents product of probabilities of all actions of P1 on the path to a

$$\tilde{x}_f, \tilde{x}_r, \tilde{x}_{f'}, \tilde{x}_{r'}, \tilde{x}_{\hat{c}}, \tilde{x}_{\hat{f}}$$

$\tilde{y}_a$: represents product of probabilities of all actions of P2 on the path to a

$$\tilde{y}_{f_*}, \tilde{y}_{r_*}$$



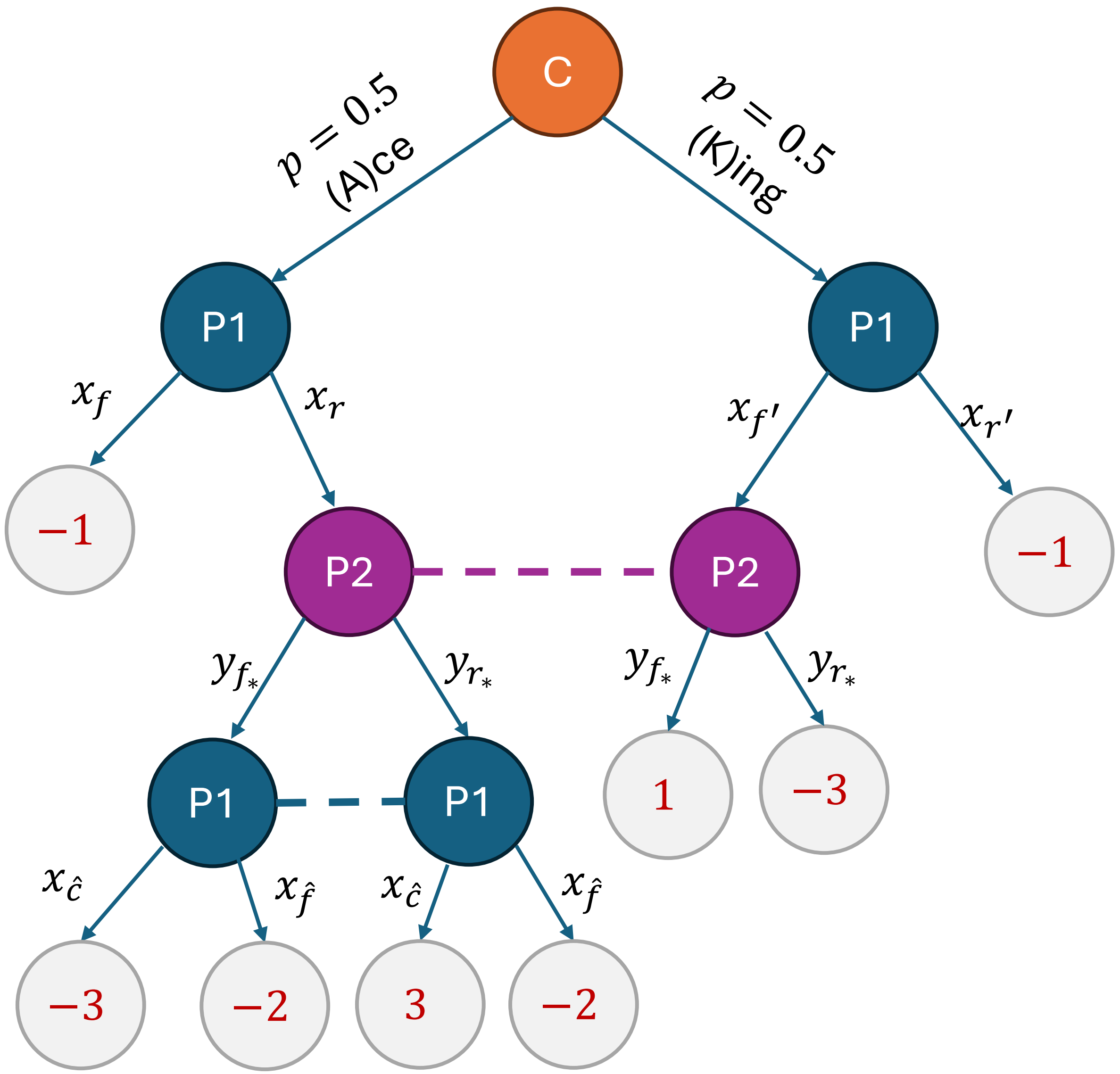
Sequence Form Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3 y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\, \tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

Observation. This is of the form $x^\top A y$.

What is the dimension of A?



Sequence Form Representation

What is the expected payoff of x?

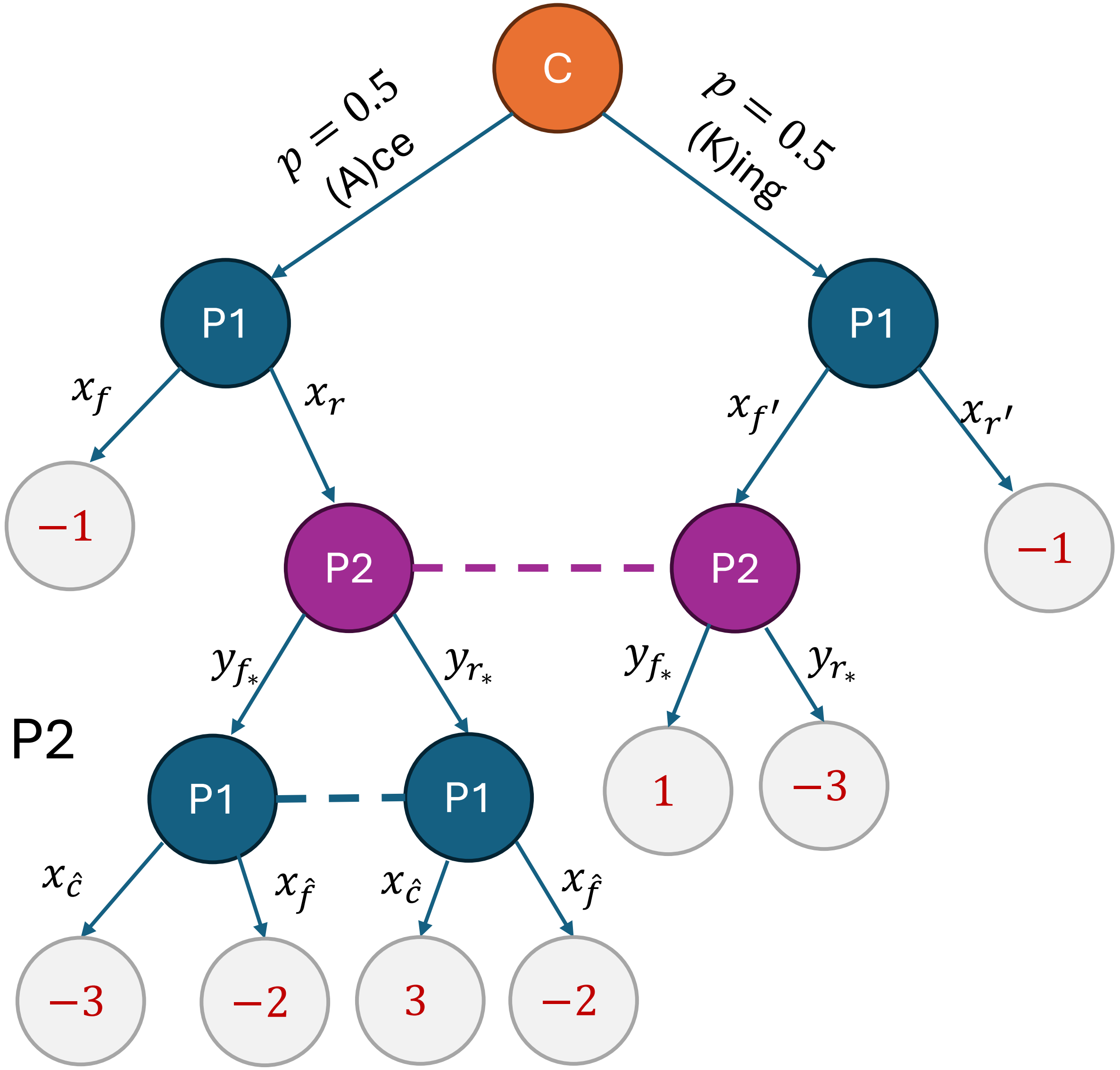
$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3 y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\,\tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

Observation. This is of the form $x^\top A y$.

What is the dimension of A?

One row for each possible action a of P

One column for each possible action a' of P2



Sequence Form Representation

What is the expected payoff of x?

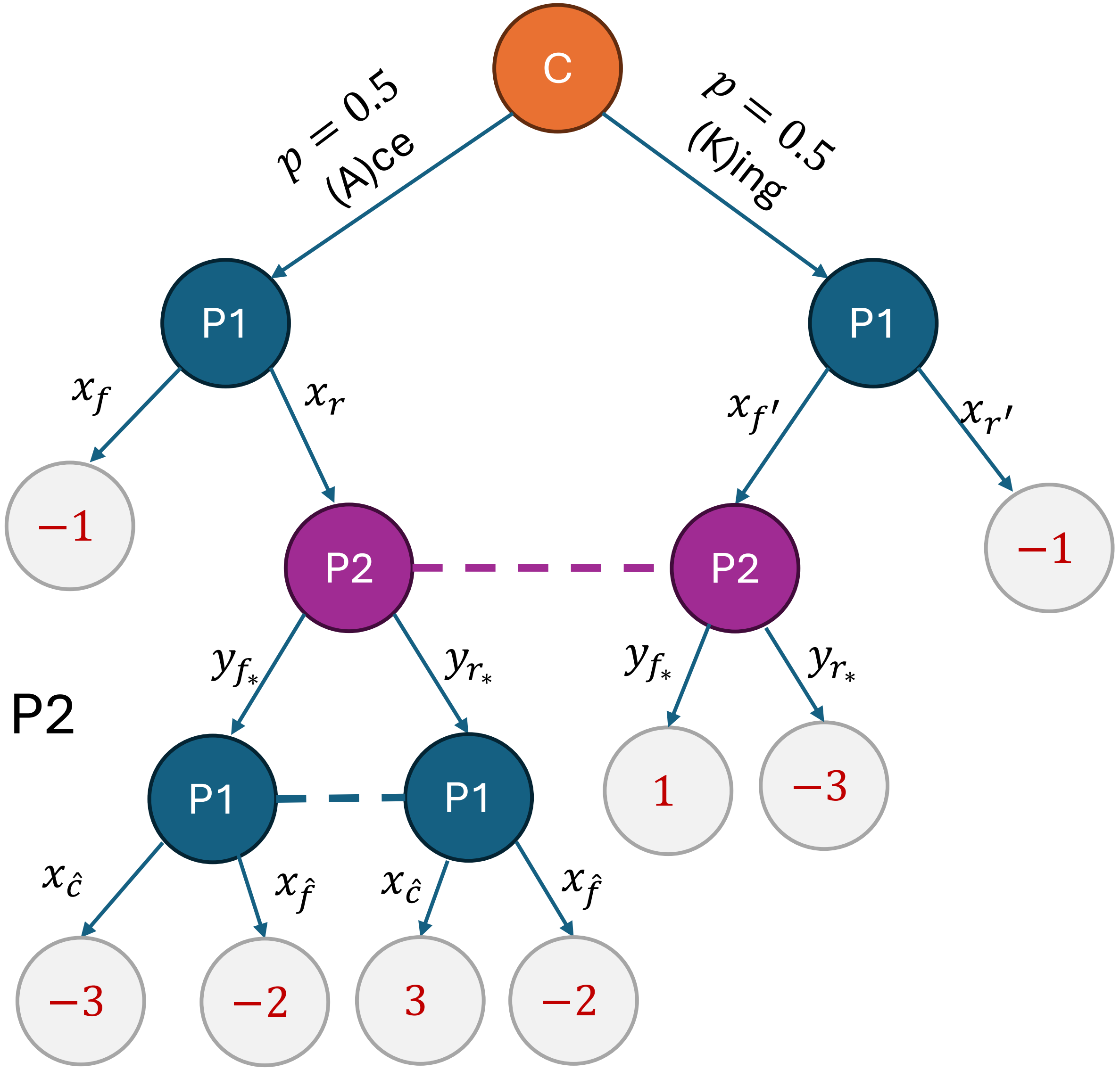
$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\,\tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

Observation. This is of the form $x^\top A y$.

What is the dimension of A?

One row for each possible action a of P
One column for each possible action a' of P2

What is the value $A_{a,a'}$?



Sequence Form Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\,\tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

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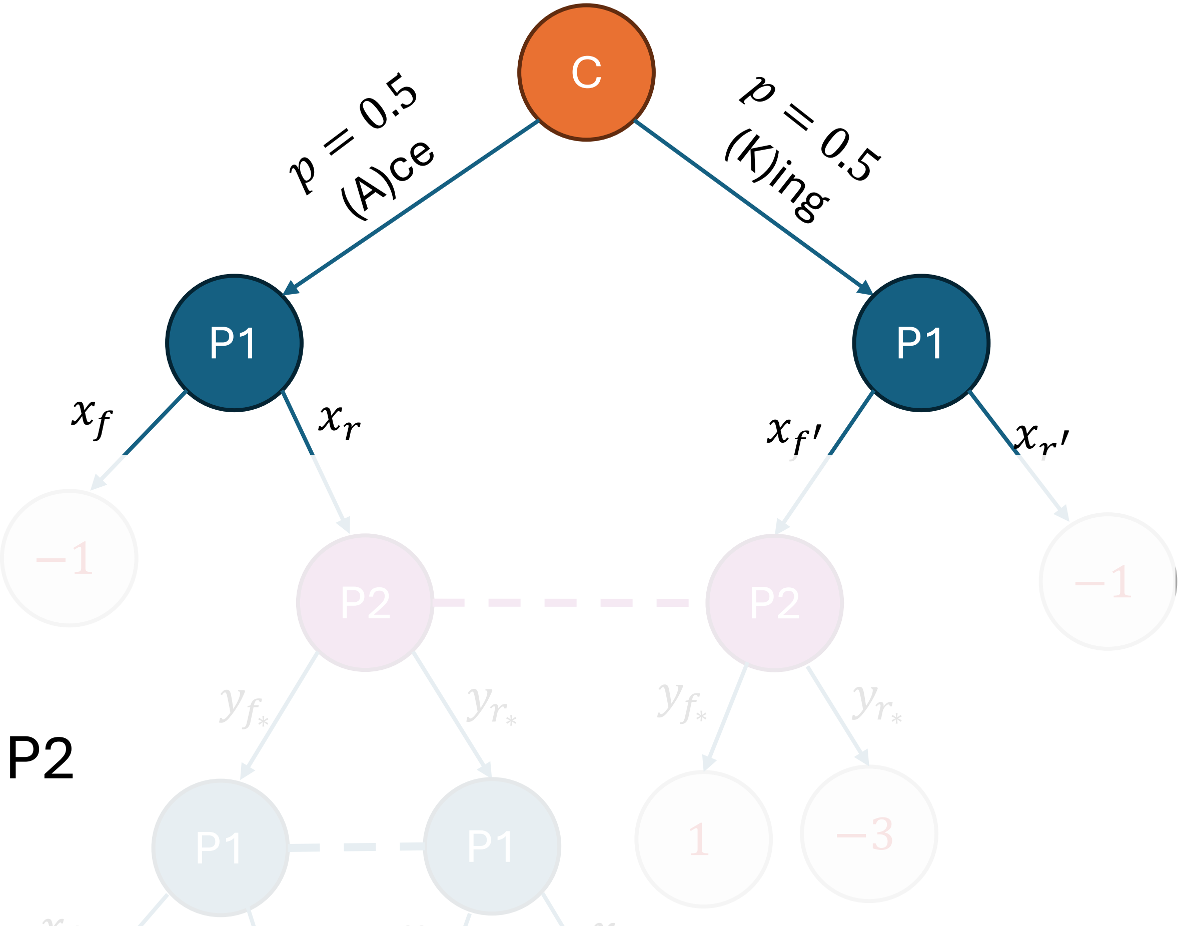


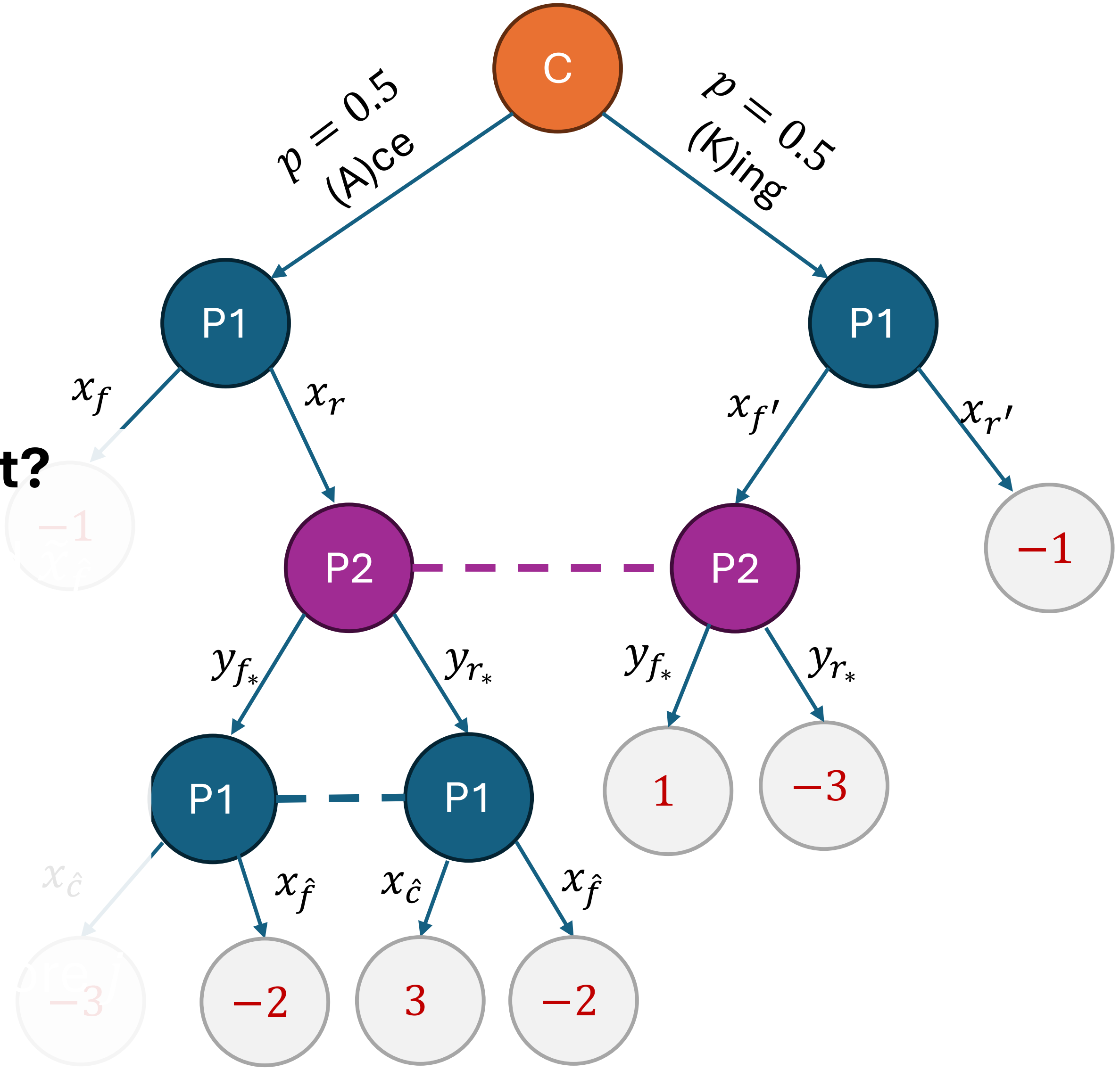
If there exists a terminal node, such that a was the last action chosen by P1 and a' was the last action chosen by P2 then it is the value of the leaf multiplied by all "chance" probabilities on the path to the leaf

Sequence Form Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3 y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+ \frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\, \tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

What constraints does $\tilde{x}$ need to respect?



Sequence Form Representation

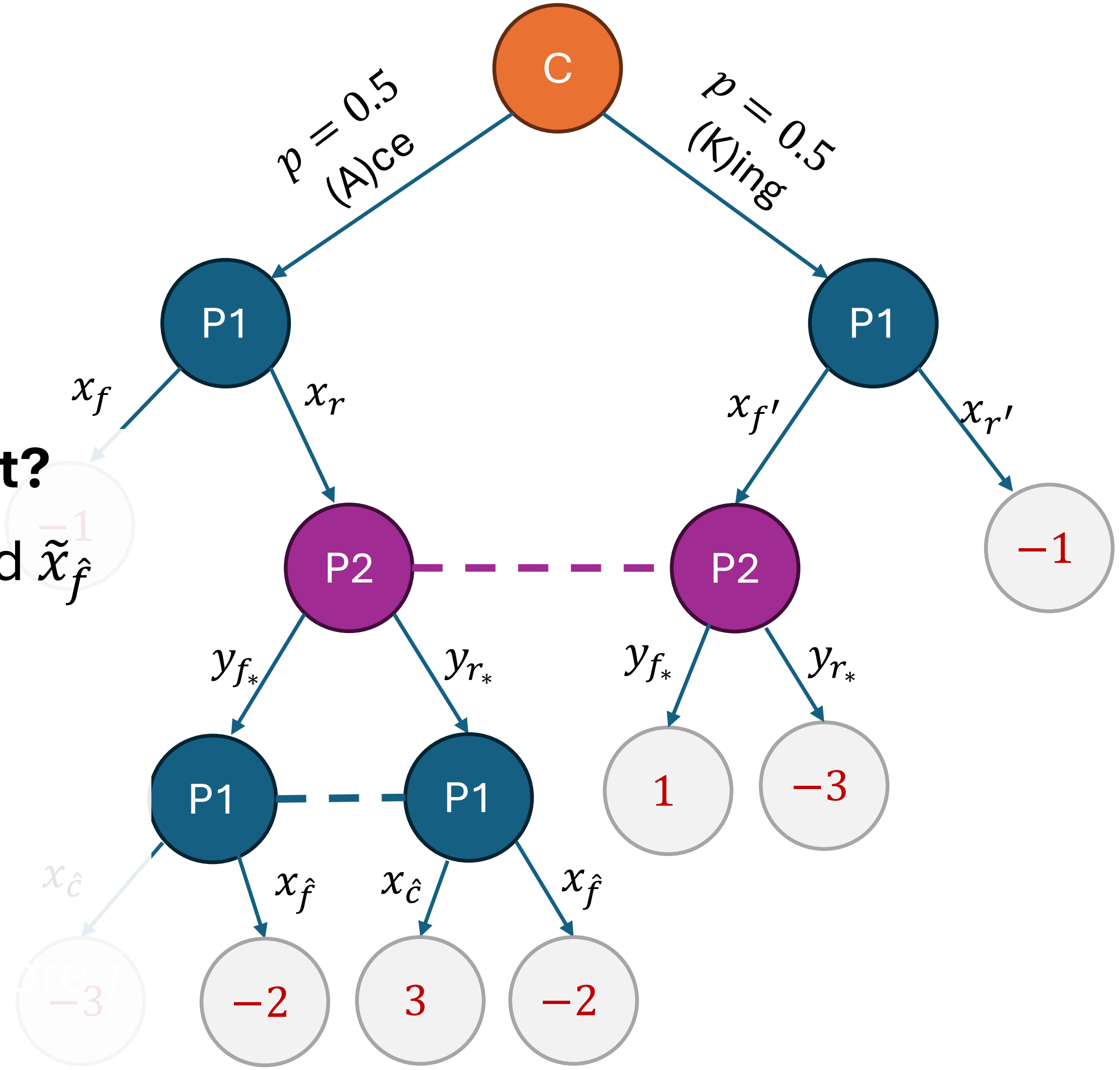
What is the expected payoff of x?

$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\, \tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

What constraints does $\tilde{x}$ need to respect?

Since $\tilde{x}_{\hat{c}}$ is supposed to represent $x_r x_{\hat{c}}$ and $\tilde{x}_{\hat{f}}$ is supposed to represent $x_r x_{\hat{f}}$

$$\tilde{x}_{\hat{c}} + \tilde{x}_{\hat{f}} = x_r\left(x_{\hat{c}} + x_{\hat{f}}\right) = x_r$$



Sequence Form Representation

What is the expected payoff of x?

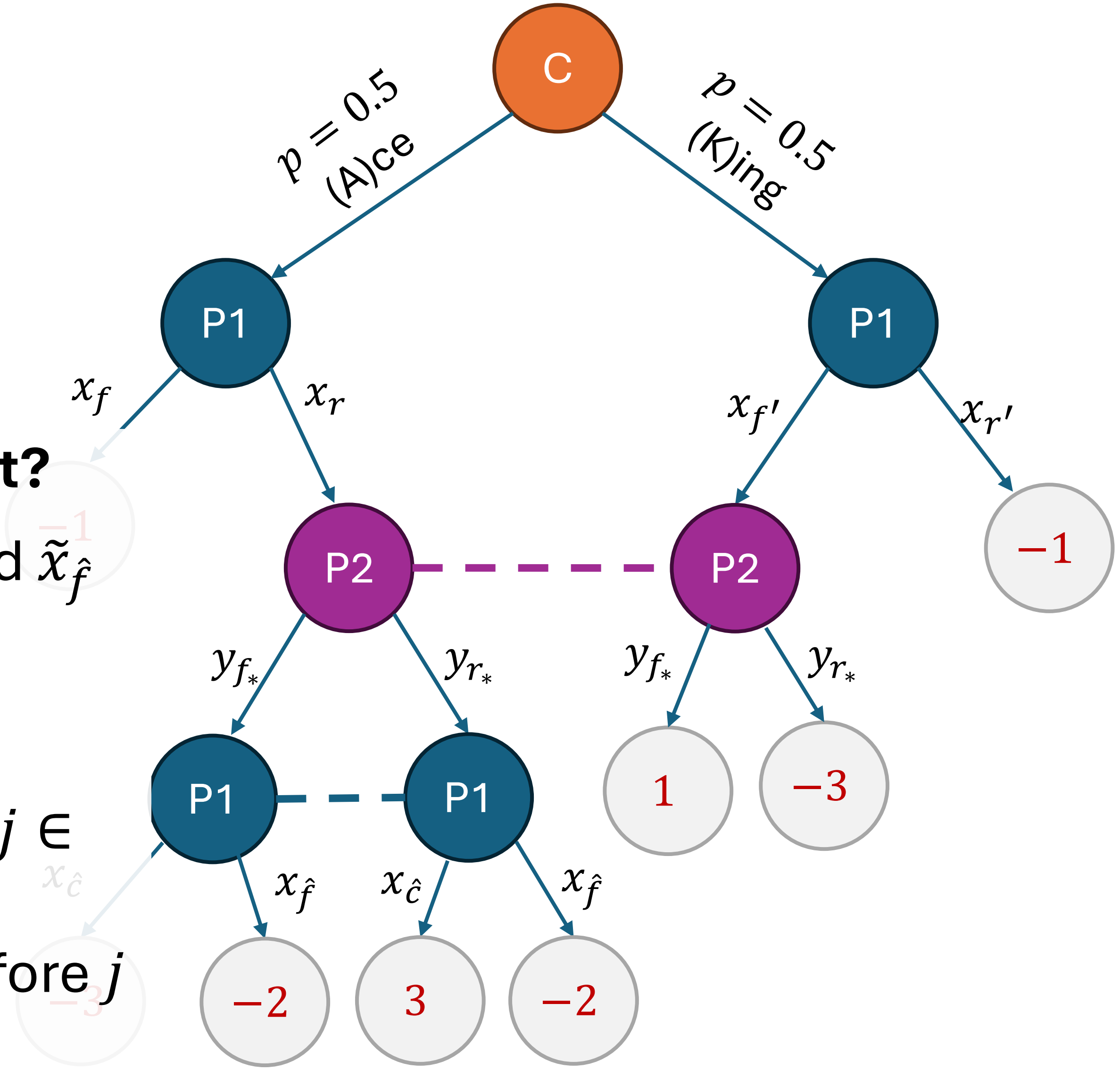
$$\frac{1}{2}\left(-\tilde{x}_f - 3\, y_{f_*}\tilde{x}_{\hat{c}} - 2\, y_{f_*}\tilde{x}_{\hat{f}} + 3y_{r_*}\tilde{x}_{\hat{c}} + 2\, y_{r_*}\tilde{x}_{\hat{f}}\right)$$
$$+\frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\, \tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

What constraints does $\tilde{x}$ need to respect?

Since $\tilde{x}_{\hat{c}}$ is supposed to represent $x_r x_{\hat{c}}$ and $\tilde{x}_{\hat{f}}$ is supposed to represent $x_r x_{\hat{f}}$

$$\tilde{x}_{\hat{c}} + \tilde{x}_{\hat{f}} = x_r\left(x_{\hat{c}} + x_{\hat{f}}\right) = x_r$$

The sum of $\tilde{x}_a$ for all actions at an info set $j \in \mathcal{J}_i$ must be matching $\tilde{x}_{p_j}$, i.e., variable associated with the last action chosen before j



Sequence Form Representation

What is the expected payoff of x?

$$\frac{1}{2}\left(-\tilde{x}_f - 3\,y_{f_*}\tilde{x}_{\hat{c}} - 2\,y_{f_*}\tilde{x}_{\hat{f}} + 3y_{r_*}\tilde{x}_{\hat{c}} + 2\,y_{r_*}\tilde{x}_{\hat{f}}\right) + \frac{1}{2}\left(\tilde{x}_{f'}\tilde{y}_{f_*} - 3\,\tilde{x}_{f'}\tilde{y}_{r_*} - \tilde{x}_{r'}\right)$$

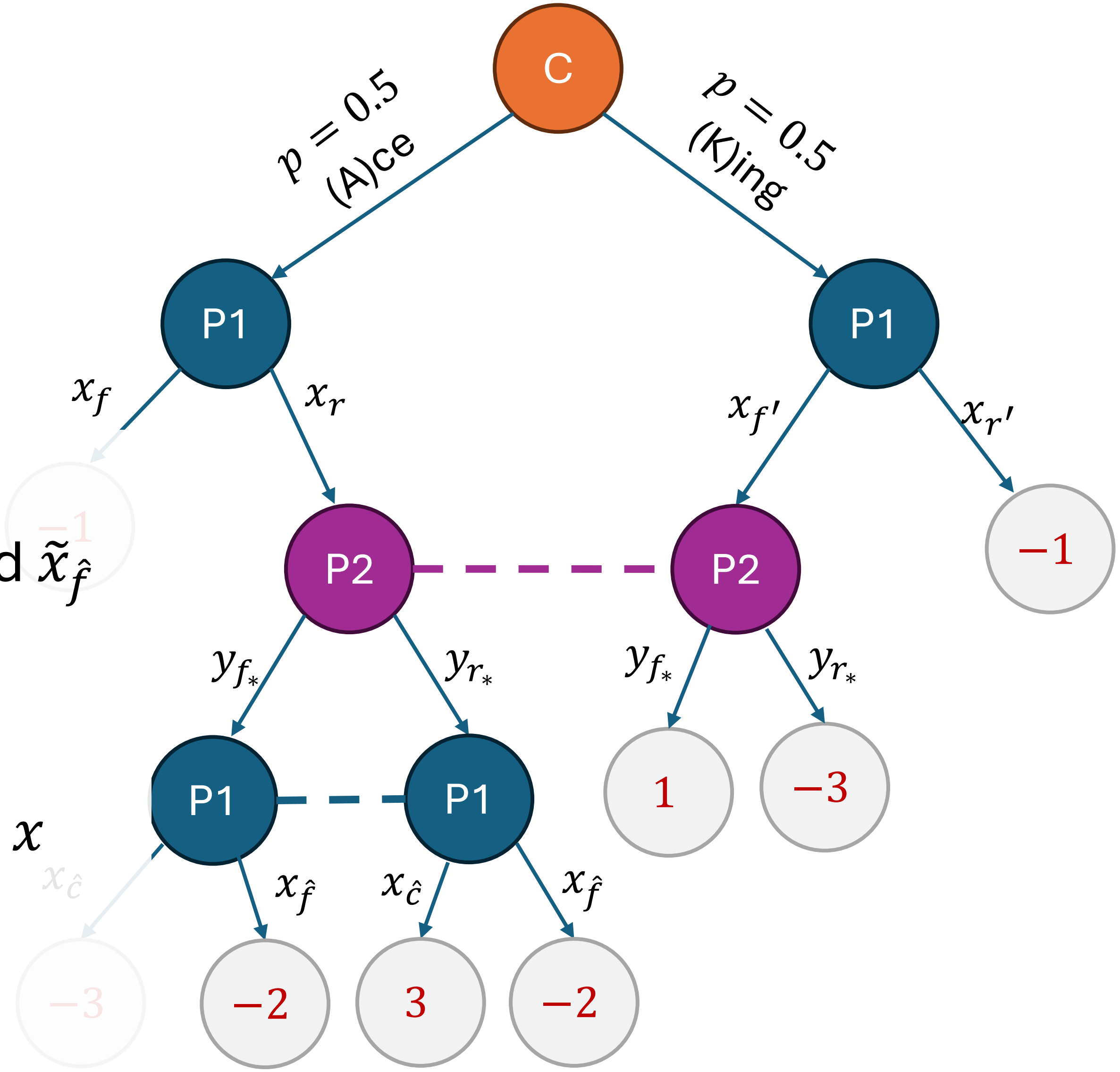
Are these all?

Since $\tilde{x}_{\hat{c}}$ is supposed to represent $x_r x_{\hat{c}}$ and $\tilde{x}_{\hat{f}}$ is supposed to represent $x_r x_{\hat{f}}$

$$\tilde{x}_{\hat{c}} + \tilde{x}_{\hat{f}} = x_r\left(x_{\hat{c}} + x_{\hat{f}}\right) = x_r = \tilde{x}_r$$

For every, $\tilde{x}$, we can find a valid behavioral x

$$\frac{\tilde{x}_c}{\tilde{x}_r} = x_{\hat{c}}, \qquad \frac{\tilde{x}_{\hat{f}}}{\tilde{x}_r} = x_{\hat{f}}$$



Recap: Sequence Form Representation

- The strategies of the player can be represented as $\tilde{x} \in X, \tilde{y} \in Y$
- $\tilde{x}_{\mathrm{a}}$: product of probabilities of all actions of P1 on the path to *a*
- $\tilde{y}_{\mathrm{a}}$: product of probabilities of all actions of P2 on the path to *a*

$$X \coloneqq \left\{ \forall j \in \mathcal{J}_1 \colon \sum_{a \in A_j} \tilde{x}_a = \tilde{x}_{p_j} \right\}, \qquad Y \coloneqq \left\{ \forall j \in \mathcal{J}_2 \colon \sum_{a \in A_j} \tilde{y}_a = \tilde{y}_{p_j} \right\}$$

- The payoff to P1 under sequence strategies $\tilde{x} \in X, \tilde{y} \in Y$ is

$$\tilde{x}^\top A \tilde{y}$$

- $A_{a,a'} =$ **if** a was the last action of P1 and a' the last action of P2 before some leaf z, **then** payoff to P1 at z times product of chance probabilities on path to z **else** zero

Recap: From Sequence to Behavioral

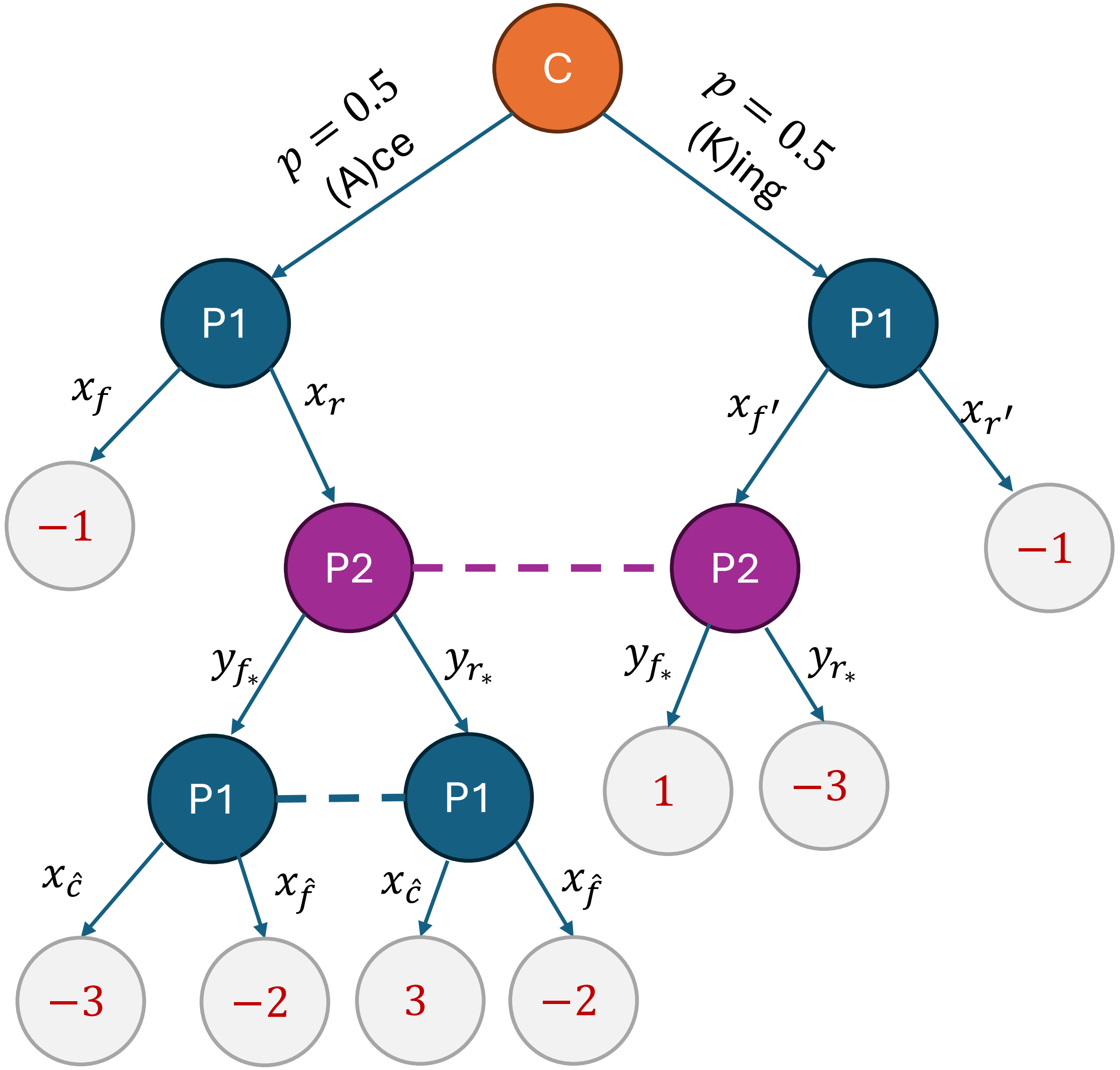
- Every sequence form strategy $\tilde{x}$ can be transformed into a behavioral form strategy as (recursively bottom up):
$$\forall a \in A_j\colon\ x_a = \frac{\tilde{x}_a}{\tilde{x}_{p_j}}$$
  if info-set is un-reachable, i.e. $\tilde{x}_{p_j} = 0$, then use any behavioral

- Every behavioral strategy x can be transformed into a sequence form strategy as (recursively top down):
$$\forall a \in A_j\colon\ \tilde{x}_a = \tilde{x}_{p_j} \cdot x_a$$

Sequence Form Representation

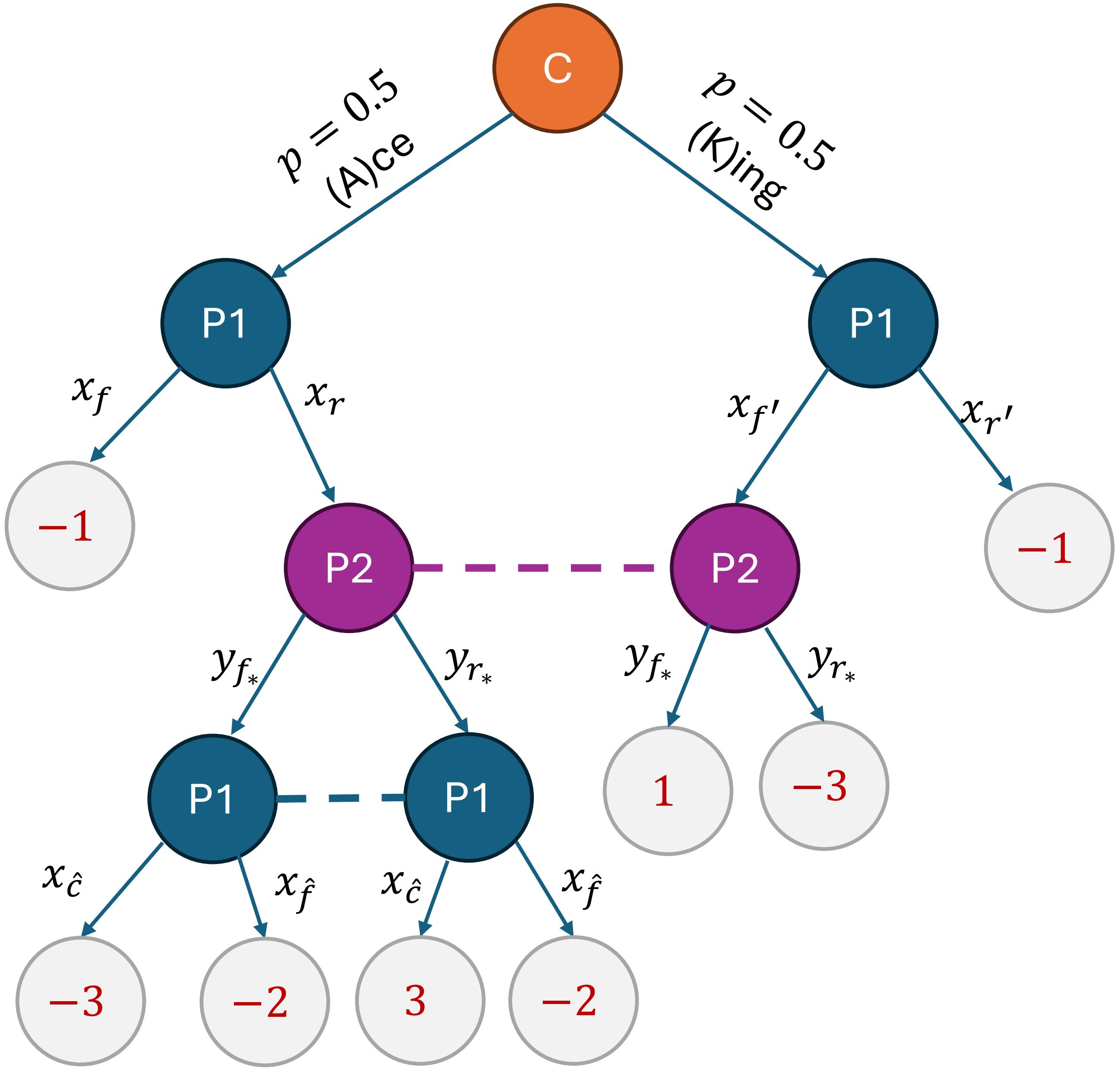
	$\emptyset$	f_*	r_*
f			
r			
f'			
r'			
$\hat{c}$			
$\hat{f}$			

Let's fill it in!



Sequence Form Representation

	$\emptyset$	f_*	r_*
$\boldsymbol{f}$	$-1/2$		
$\boldsymbol{r}$			
$\boldsymbol{f'}$		$1/2$	$-3/2$
$\boldsymbol{r'}$	$-1/2$		
$\hat{\boldsymbol{c}}$		$-3/2$	$3/2$
$\hat{\boldsymbol{f}}$		$-2/2$	$-2/2$



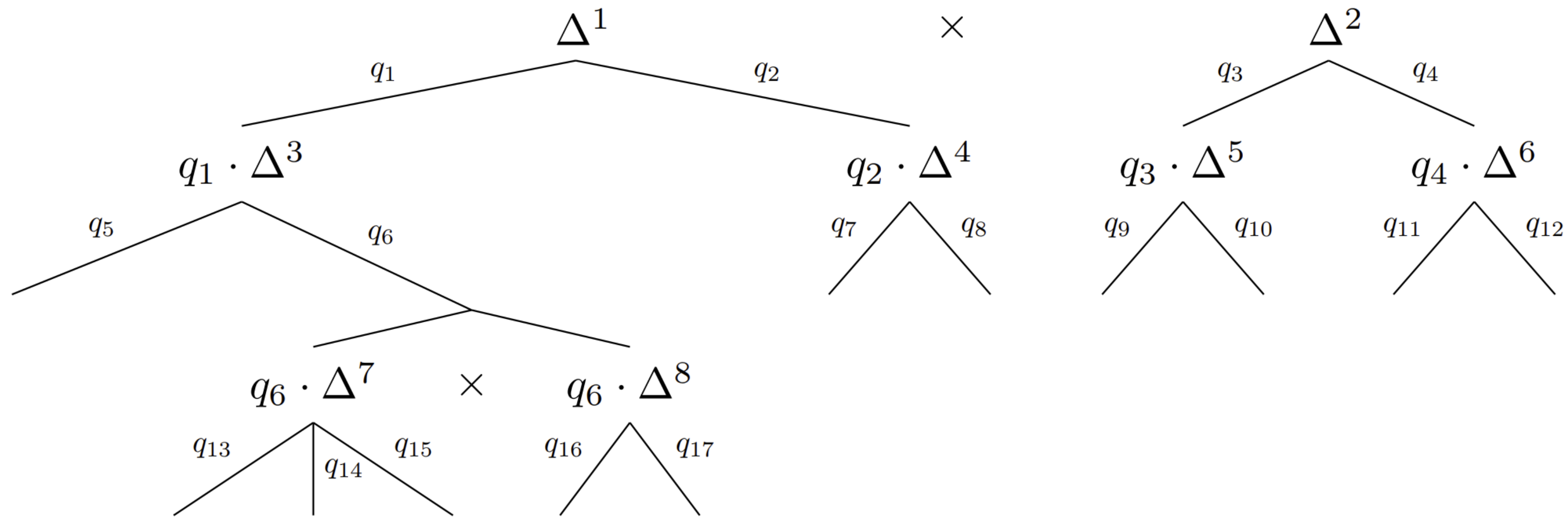
TreePlex Representation of Strategy Space

The strategy space of each player is a set of interconnected "scaled" simplices

$$\forall j \in \mathcal{J}_1 \colon \sum_{a \in A_j} \tilde{x}_a = \tilde{x}_{p_j}$$

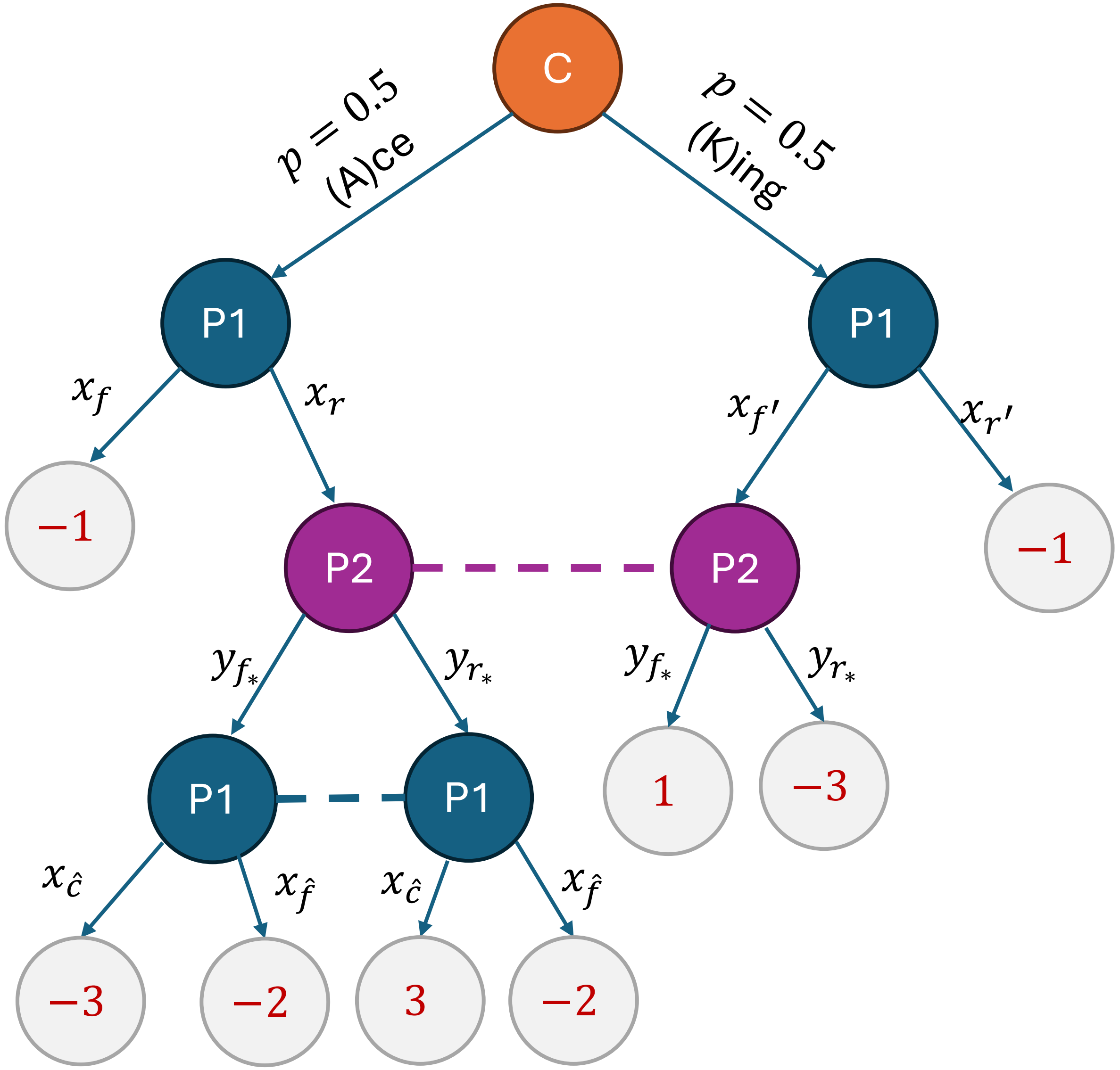
To generate $\tilde{x}_a$

- Generate an element of the simplex (i.e. a behavioral strategy x_a)
- Scale all its coordinates by $\tilde{x}_{p_j}$, i.e. $\tilde{x}_a = \tilde{x}_{p_j} \cdot x_a$

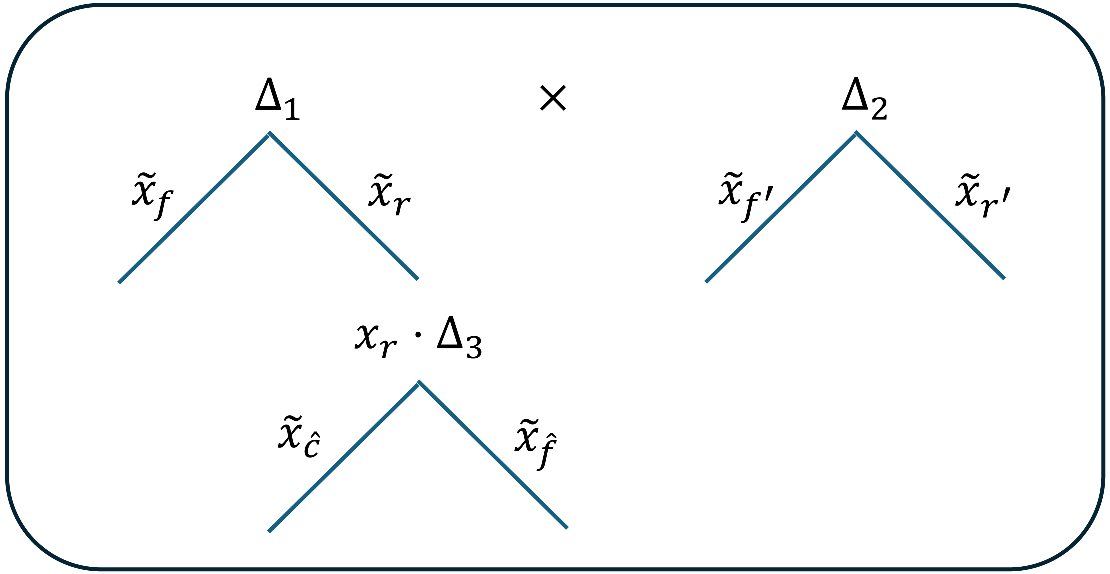


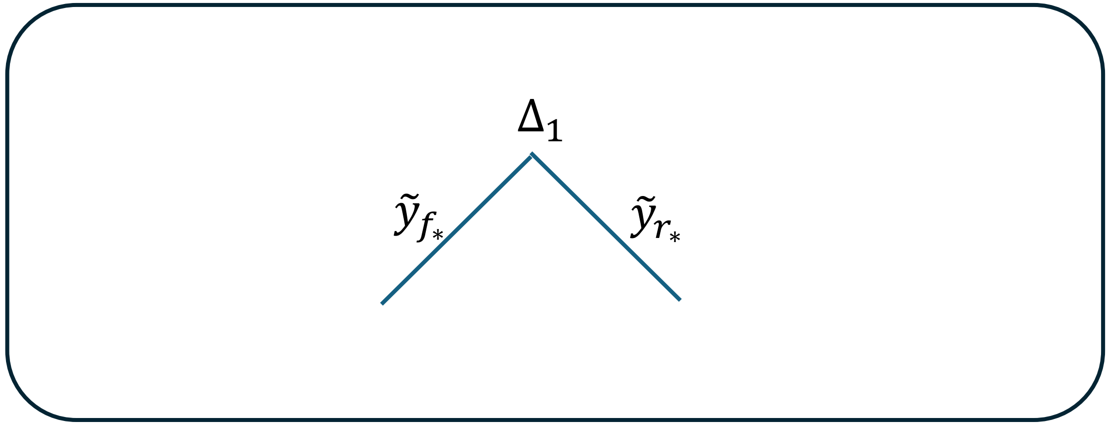
TreePlex Representation

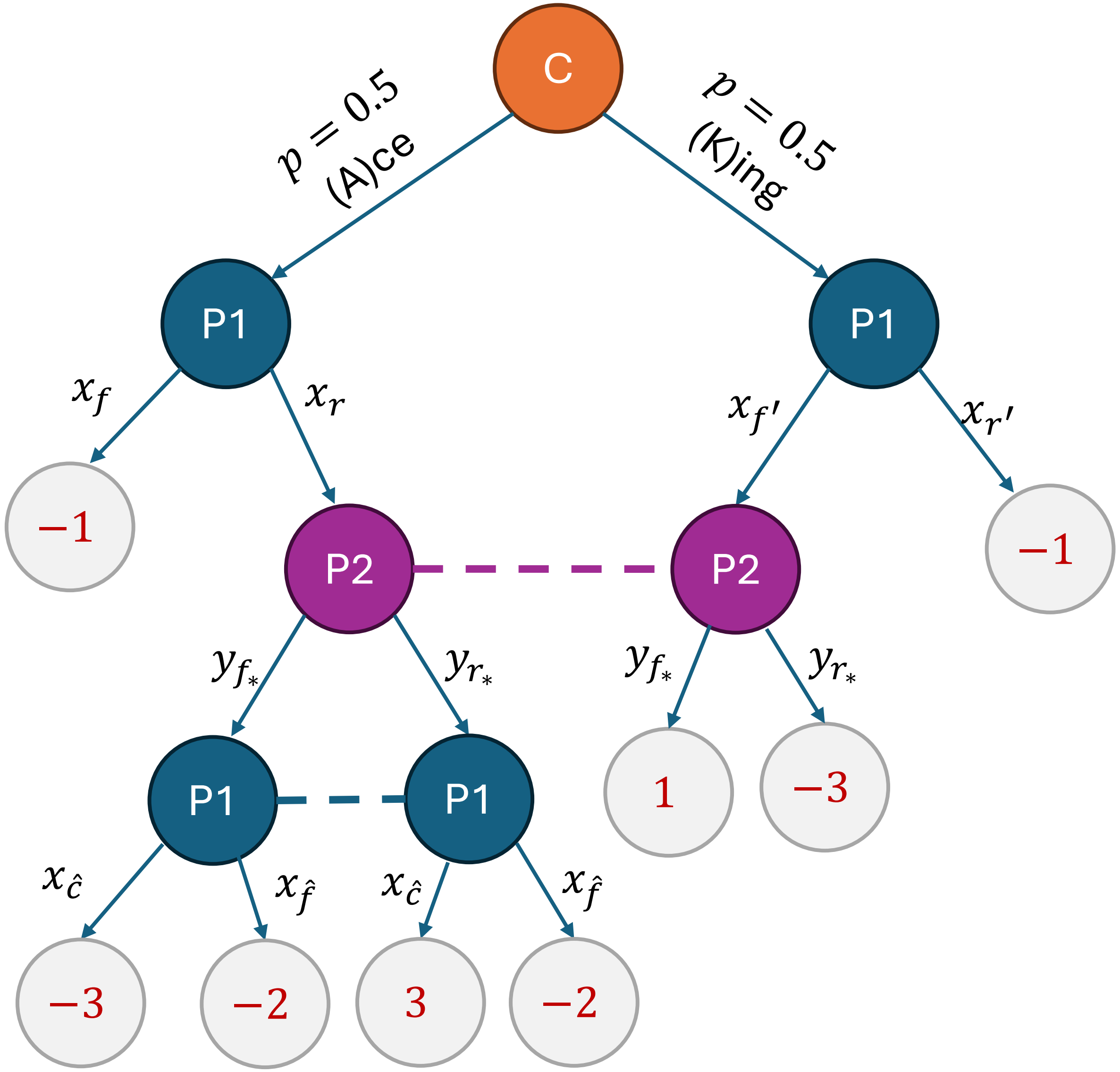
Let's fill it in!



TreePlex Representation







Solving Extensive Form Games via No-Regret Learning

No-Regret Learning in Sequence Form

- We have successfully turned imperfect information extensive form zero-sum games into a familiar object

$$\max_{\tilde{x} \in X} \min_{\tilde{y} \in Y} \tilde{x}^\top A \tilde{y}$$

- X, Y are convex sets, i.e., sequence-form strategies

- We can invoke minimax theorem to prove existence of equilibria
- We can calculate equilibria via LP duality
- We can calculate equilibria via no-regret learning!

Recap from Lecture 2: Regret of FTRL

$$\text{(FTRL)} \qquad x_t = \operatorname*{argmin}_{x \in X} \boxed{\sum_{\tau < t} \langle x, \ell_\tau \rangle} + \boxed{\frac{1}{\eta} \mathcal{R}(x)}$$

1-strongly convex function of x that stabilizes the minimizer

Historical performance of always choosing strategy x

Theorem. Assuming the loss function at each period

$$f_t(x) = \langle x, \ell_t \rangle$$

is L-Lipschitz with respect to some norm $\|\cdot\|$ and the regularizer is 1-strongly convex with respect to the same norm then

$$\text{Regret} - \text{FTRL}(T) \leq \boxed{\eta L} + \boxed{\frac{1}{\eta T}\left(\max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x)\right)}$$

Average stability induced by regularizer

Average loss distortion caused by regularizer

Same for utilities

(FTRL) $$x_t = \operatorname*{argmax}_{x \in X} \boxed{\sum_{\tau < t} \langle x, u_\tau \rangle} - \boxed{\frac{1}{\eta} \mathcal{R}(x)}$$

1-strongly convex function of x that stabilizes the maximizer

Historical performance of always choosing strategy x

Theorem. Assuming the utility function at each period

$$f_t(x) = \langle x, u_t \rangle$$

is L-Lipschitz with respect to some norm $\|\cdot\|$ and the regularizer is 1-strongly convex with respect to the same norm then

$$\text{Regret} - \text{FTRL}(T) \leq \boxed{\eta L} + \boxed{\frac{1}{\eta T} \left(\max_{x \in X} \mathcal{R}(x) - \min_{x \in X} \mathcal{R}(x) \right)}$$

Average stability induced by regularizer

Average loss distortion caused by regularizer

Regularizer for the Treeplex Space X

$$U_{t-1} = \sum_{\tau<t} u_\tau$$

- The only thing we are missing is a good Regularizer for X
- **Desiderata.** Be strongly convex in x within X and for the optimization problem to be fast to solve

$$\tilde{x}_t = \underset{\tilde{x}\in X}{\operatorname{argmax}} \sum_{\tau<t} \langle \tilde{x}, u_\tau \rangle - \frac{1}{\eta}\mathcal{R}(\tilde{x}) = \underset{\tilde{x}\in X}{\operatorname{argmax}} \, \langle \tilde{x}, \boxed{U_{t-1}} \rangle - \frac{1}{\eta}\mathcal{R}(\tilde{x})$$

- X is no longer a “simplex”, so entropy is not a good Regularizer

Dilated Entropy

- X is a combination of *scaled simplices*, i.e., $\tilde{x} = \left(\tilde{x}^j\right)_{j \in \mathcal{J}_1}$
- $\tilde{x}^j = (\tilde{x}_a)_{a \in A_j}$: sequence-form strategies for actions in infoset $j \in \mathcal{J}_1$

$$\tilde{x}^j \in \tilde{x}_{p_j} \cdot \Delta_j \quad \Leftrightarrow \quad \tilde{x}^j / \tilde{x}_{p_j} \in \Delta_j$$

- Consider a *weighted combination of local negative entropies*

$$\mathcal{R}(\tilde{x}) \coloneqq \sum_j \beta_j \, \tilde{x}_{p_j} \, \mathrm{H}\left(\tilde{x}^j / \tilde{x}_{p_j}\right), \qquad \mathrm{H}(u) = \sum_i u_i \log(u_i)$$

Lies in a simplex Δ_j

Equivalent to the behavioral strategy x^j

Negative Entropy

- $\mathcal{R}(\tilde{x})$ is $1/M$ strongly convex w.r.t. ℓ_1 norm, where $M = \max_{\tilde{x} \in X} \|x\|_1$, for appropriate choice of β_j based on game tree structure

Solving the Optimization Problem

- Optimization problem decomposes into local simplex problems

$$\sum_{j \in \mathcal{J}_1} \left\langle \tilde{x}^j, U_{t-1}^j \right\rangle - \underbrace{\boxed{\frac{1}{\eta}\beta_j}}_{:= \frac{1}{\eta_j}} \tilde{x}_{p_j} \, \mathrm{H}\left(\frac{\tilde{x}^j}{\tilde{x}_{p_j}}\right) = \sum_{j \in \mathcal{J}_1} \tilde{x}_{p_j} \left\{ \left\langle \frac{\tilde{x}^j}{\tilde{x}_{p_j}}, U_{t-1}^j \right\rangle - \frac{1}{\eta_j} \mathrm{H}\left(\frac{\tilde{x}^j}{\tilde{x}_{p_j}}\right) \right\}$$

- Max of quantity $\frac{\tilde{x}^j}{\tilde{x}_{p_j}}$ over simplex Δ_j is independent of solution x_a for all ancestral actions

- Quantity $\frac{\tilde{x}^j}{\tilde{x}_{p_j}}$ is essentially the behavioral strategy x^j at infoset j

Solving the Optimization Problem

- Decomposes in local max over behavioral strategies x^j solved bottom up

$$V^j = \max_{x^j \in \Delta_j} \left\langle x^j, U_{t-1}^j \right\rangle - \frac{1}{\eta_j} H\left(x^j\right) \Rightarrow \begin{cases} x^j \propto \exp\left(\eta_j U_{t-1}^j\right) \\ V^j = \log \sum_{a \in A_j} \exp\left(\eta_j U_{t-1}^a\right) = \operatorname{softmax}_{\eta_j}\left(U_{t-1}^j\right) \end{cases}$$

- Value V^j multiplies x_{p_j}; when solving for x_{p_j} we need to take it into account. If $p_j \in A_k$

$$\max_{x^k \in \Delta_k} \left\langle \tilde{x}^k, U_{t-1}^k \right\rangle - \eta_k \, \tilde{x}_{p_k} \, \mathrm{H}\left(\frac{\tilde{x}^k}{\tilde{x}_{p_k}}\right) + x_{p_j} V^j + \cdots$$

- Add V^j to "cumulative utility" Q_{p_j} (initialized at U_{t-1,p_j}) associated with p_j

$$Q_{p_j} \leftarrow Q_{p_j} + V^j$$