Stability Theory and Numerical Analysis of Non-Autonomous Dynamical Systems

by
D. J. Stonier
BSc, BSc(Hons).
Submitted in Fulfillment of the Requirements
for the Degree of Doctor of Philosophy.
Deakin University.

November 10, 2005

Introduction

The development and use of cocycles for analysis of non-autonomous behaviour is a technique that has been known for several years. Initially developed as an extension to semi-group theory for studying non-autonomous behaviour, it was extensively used in analysing random dynamical systems [2, 9, 10, 12].

Many of the results regarding asymptotic behaviour developed for random dynamical systems, including the concept of cocycle attractors were successfully transferred and reinterpreted for deterministic non-autonomous systems primarily by P. Kloeden and B. Schmalfuss [20, 21, 28, 29]. The theory concerning cocycle attractors was later developed in various contexts specific to particular classes of dynamical systems [6, 7, 13], although a comprehensive understanding of cocycle attractors (redefined as pullback attractors within this thesis) and their role in the stability of non-autonomous dynamical systems was still at this stage incomplete.

It was this purpose that motivated Chapters 1-3 to define and formalise the concept of stability within non-autonomous dynamical systems. The approach taken incorporates the elements of classical asymptotic theory, and refines the notion of pullback attraction with further development towards a study of pullback stability and pullback asymptotic stability. In a comprehensive manner, it clearly establishes both pullback and forward (classical) stability theory as fundamentally unique and essential components of non-autonomous stability. Many of the introductory theorems and examples highlight the key properties and differences between pullback and forward stability. The theory also cohesively retains all the properties of classical asymptotic stability theory in an autonomous environment. These chapters are intended as a fundamental framework from which further research in the various fields of non-autonomous dynamical systems may be extended.

A preliminary version of a Lyapunov-like theory that characterises pullback attraction is created as a tool for examining non-autonomous behaviour in Chapter 5. The nature of its usefulness however is at this stage restricted to the converse theorem of asymptotic stability.

Chapter 7 introduces the theory of Loci Dynamics. A transformation is made to an alternative dynamical system where forward asymptotic (classical asymptotic) behaviour characterises pullback attraction to a particular point in the original dynamical system. This has the advantage in that certain conventional techniques for a forward analysis may be applied.

The remainder of the thesis, Chapters 4, 6 and Section 7.3, investigates the effects of perturbations and discretisations on non-autonomous dynamical systems known to possess structures that exhibit some form of stability or attraction. Chapter 4 investigates autonomous systems with semi-group attractors, that have been non-autonomously perturbed, whilst Chapter 6 observes the effects of discretisation on non-autonomous dynamical systems that exhibit properties of forward asymptotic stability. Chapter 7 explores the same problem of discretisation, but for pullback asymptotically stable systems. The theory of Loci Dynamics is used to analyse the nature of the discretisation, but establishment of results directly analogous to those discovered in Chapter 6 is shown to be unachievable. Instead a case by case analysis is provided for specific classes of dynamical systems, for which the results generate a numerical approximation of the pullback attraction in the original continuous dynamical system.

The nature of the results regarding discretisation provide a non-autonomous extension to the work initiated by A. Stuart and J. Humphries [34, 35] for the numerical approximation of semi-group attractors within autonomous systems. Of particular importance is the effect on the system's asymptotic behaviour over non-finite intervals of discretisation.

Contents

List of Figures

Terminology

The following terminology is frequently used throughout the thesis, and is listed here as a reference.

t	Actual time.
t	Elapsed time.
P	Parameter set for cocycles. Typically $P = \mathbb{R}, \mathbb{Z}$.
$\mathbb{R}^+$	Non-negative reals. $(\mathbb{R}^+ = \{x \in \mathbb{R}; x \geq 0\})$
$\mathbb{Z}^+$	Non-negative integers. $(\mathbb{Z}^+ = \{z \in \mathbb{Z}; z \geq 0\})$
$\|\| \cdot \|\|$	Usual distance metric on $\mathbb{R}^n$.
$\mathrm{dist}(\cdot, \cdot)$	Distance measure of a single point from a compact set. $(\mathrm{dist}(a, B) = \min_{b \in B} \|\|a - b\|\|)$
$H^*(A, B)$	Hausdorff Semi-Metric acting on compact sets A, B.
$\hat{\delta}$	A uniformly bounded δ-set over P. $(\hat{\delta} = \{\delta_p; \delta_p > 0, p \in P\})$
$\hat{A}$	Family of uniformly bounded sets over P. $(\hat{A} = \{A(p); p \in P\})$
$\mathcal{N}_\delta(A)$	δ-neighbourhood of the set A. $(\mathcal{N}_\delta(A) = \{x; \mathrm{dist}(x, A) < \delta\})$
$\hat{\mathcal{N}}_{\delta,\hat{A}}$	δ-neighbourhood of the family $\hat{A}$. $(\hat{\mathcal{N}}_{\delta,\hat{A}} = \{\mathcal{N}_\delta(A(p)); p \in P\})$
$\hat{\mathcal{N}}_{\hat{\delta},\hat{A}}$	δ-neighbourhood system of the family $\hat{A}$. $(\hat{\mathcal{N}}_{\hat{\delta},\hat{A}} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\})$

Abbreviations

ADE	Autonomous Ordinary Differential Equation.
NDE	Non-Autonomous Ordinary Differential Equation.
ODE	Ordinary Differential Equation.
LDS	Linear Non-Autonomous Dynamical System [$\dot{x} = f(t)x$].
SDS	Separable Non-Autonomous Dynamical System [$\dot{x} = f(t)g(x)$].

Chapter 1

Dynamical Systems and Stability Theory

1.1 The Dynamical System

A dynamical system typically has three defining features. These are:

- *Phase* or *state space* X. Elements of this space represent possible states of the system at any given time.
- *Time*, which may be discrete or continuous. Solutions must exist for future times, but some systems may also be reversible. For discrete systems, the time set is represented by $\mathbb{Z}^+/\mathbb{Z}$, and for continuous systems by $\mathbb{R}^+/\mathbb{R}$.

  It is often important to distinguish between the actual time and the time elapsed since the system was initialised. Differential equations typically refers to t as the actual time, whereas dynamical systems theory utilises the same t to represent the elapsed time. Throughout the thesis we will use $\mathtt{t}$ for the actual time, and t for the elapsed time to avoid confusion with the different conventions.
- *Evolution of the System.* As a general rule the system behaves in a fashion that evolves with time and allows us to uniquely determine the state of

the system at each moment t from its state at any previous moment. If it is a reversible process, then we may also determine the state of the system preceding a given initial state and moment in time.

1.1.1 The State Space

In many dynamical systems the state space, X, is represented by a measure or topological space, or a space possessing the structure of a smooth manifold. Most of the following work will involve dynamical systems of ordinary differential equations where the state is typically an element of topological space, usually some subset of Euclidean Space. When this is the case we will use the notation E for the state space, where E is open and $E \subset \mathbb{R}^d$. Any case which does not follow a similar approach will be given due attention.

1.1.2 Evolution of the System

The evolution of the state is characterised by a family of mappings or transformations, which satisfy an evolution property on the state space. For an *autonomous system*, the mapping is invariant with respect to the initial time, and so depends solely on the value of the initial state, mapping the state space into itself. In these systems the evolution property is typically a group or semi-group property. Mappings for *non-autonomous* dynamical systems, however, generally depend on both initial state and the initial time, and in general, feature much more complicated behaviour. In particular a semi-group property no longer holds, but a similar cocycle property is introduced to characterise the system's evolution.

1.2 Autonomous Dynamical Systems

The predominant characteristic of an autonomous dynamical system is its dependence solely on the time elapsed and not on the current value of the time itself. As a result the behaviour of the system is simplified in comparison to that of a non-autonomous system, and any attracting objects or stability properties it may possess are generally invariant with respect to a time parameter.

1.2.1 Semi-group Representation

The definition of a semi-group when applied to a collection of mappings can be used to describe the evolution of autonomous dynamical systems on a state space E as outlined below.

Definition 1.2.1 (Semi-group Representation)**.** *A family of mappings* $\{S_t, t \in \mathbb{T}\}$ *with* $S_t : E \mapsto E$ *for each* $t \in \mathbb{T}$ *is called a* **semi-group** *on* E *if*

$$(i) \quad S_0 = id, \qquad \textit{Identity Property} \tag{1.1}$$

$$(ii) \quad S_{t+\tau} = S_t \circ S_\tau, \qquad \textit{Semi-group Property} \tag{1.2}$$

for all $t, \tau \in \mathbb{T}$*, where* id *is the identity mapping.*

Elements of $\mathbb{T}$ are representative of the time set used. Usually $\mathbb{T} = \mathbb{R}^+$ (continuous dynamical systems), or $\mathbb{T} = \mathbb{Z}$ (discrete systems).

One can consider the evolution of an initial state by following the mapping as it traces the trajectory followed by the state with increasing time. For continuous systems this is often called the **flow** of the dynamical system for that initial state. It is also useful to consider the flow of nonempty subsets of the state space for fixed periods of time. In this way, the dynamical system as a whole can be more easily observed, noting features such as basins of attraction, cycles, etc.

Example 1.2.1. Autonomous differential equations generate dynamical systems which may be represented by a semi-group mapping. Consider the ODE

$$\dot{x} = -x, \tag{1.3}$$

where $x \in \mathbb{R}$. The ODE generates an autonomous dynamical system with solution

$$x(\mathsf{t}) = x_0 e^{-(\mathsf{t}-t_0)},$$

to the initial value problem defined by $x(t_0) = x_0$. Solutions are dependent only on the time elapsed $t = (\mathsf{t} - t_0)$, and not the initial time. In fact any ODE with $\dot{x} = f(x)$ under assumptions guaranteeing uniqueness and extendability (notably continuity and Lipschitz continuity of f with respect to x - refer to any well versed book on ordinary differential equations, [24],[26]), generates an autonomous dynamical system.

The solution may also be written in the form $\{S_t, t \in \mathbb{R}^+\}$ using the semi-group representation above. In this case the state space is simply the Euclidean Space, $\mathbb{R}^d$, and the mapping is defined by

$$\begin{aligned} S_0(x_0) &= x_0, \\ S_t(x_0) &= x(t + t_0), \\ &= x_0 e^{-t}, \end{aligned} \tag{1.4}$$

for each $t \in \mathbb{R}^+$ and $x_0 \in \mathbb{R}^d$. (Note that t in this representation indicates the time elapsed and not the actual time).

$\square$

Being autonomous by definition, the behaviour of a semi-group generating dynamical system is determined largely by any properties or structures it may possess as the elapsed time is allowed to become infinite, its *asymptotic* behaviour. There is a host of literature on autonomous dynamical systems and their asymptotic behaviour. The presence of various structures such as limit cycles, attractors, stable and asymptotically stable sets often clearly describe the asymptotic nature of the overall system. Elements of Lyapunov theory ([42]) can also predetermine the existence and location of some of these features. For completeness, a few of the definitions for such structures and Lyapunov results for autonomous dynamical systems using the semi-group representation follow.

1.2.2 Asymptotic Behaviour: Stability

Firstly, we will briefly introduce a few distance terminologies that will be used extensively in the following definitions and examples.

For consistency, lower case letters will be used to denote point elements of the space, (e.g. $x, y \in E$), and upper case letters for nonempty subsets of the space (e.g. $A, A_0 \subset E$).

$||\cdot||$ represents the usual metric on the space, E, representing the distance between two points.

The distance of a point $x \in E$ from a compact set A is defined as

$$\operatorname{dist}(x, A) = \min_{a \in A} ||x - a||.$$

The δ-neighbourhood of a set A is given by

$$\mathcal{N}_\delta(A) = \{x; \operatorname{dist}(x, A) < \delta, \delta > 0\}.$$

The **Hausdorff Separation** $H^*(A, B)$ of nonempty compact subsets $A, B \subset E$, is defined as

$$H^*(A, B) = \max_{a \in A} \operatorname{dist}(a, B) = \max_{a \in A} \min_{b \in B} ||a - b||,$$

The quantity $H(A, B) = \max\{H^*(A, B), H^*(B, A)\}$ then satisfies the properties for a metric and is called the Hausdorff metric on the space $\mathcal{H}(\mathbb{R}^d)$ of nonempty compact subsets of $\mathbb{R}^d$. $H^*(A, B)$ is often called the Hausdorff semi-metric on $\mathcal{H}(\mathbb{R}^d)$. It is a measure of the difference between two sets A and B (refer to Figure 1.1).

The following concepts of stability, asymptotic stability and attraction assist in understanding asymptotic behaviour within autonomous dynamical systems. They are extendable to non-autonomous systems for which they characterise stability only in part (refer to Section 1.3). A comprehensive analysis of stability for non-autonomous dynamical systems is covered in Chapter 2. Utilising the semi-group notation introduced previously, *stable* and *asymptotically stable* sets are defined below.

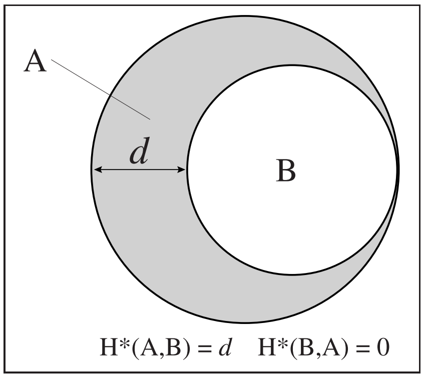

Figure 1.1: Hausdorff Semi-Metric

Definition 1.2.2 (Stable Set)**.** *A nonempty compact subset $A_0 \subset E$, (E open and $E \subset \mathbb{R}^d$) is* **stable** *under the semi-group mapping $\{S_t : t \in \mathbb{R}^+\}$, if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that*

$$H^*(S_t(\mathcal{N}_\delta(A_0)), A_0) < \epsilon, \qquad \forall t \geq 0. \tag{1.5}$$

Definition 1.2.3 (Asymptotically Stable Set)**.** *A nonempty compact subset $A_0 \subset E$, (E open and $E \subset \mathbb{R}^d$) is* **asymptotically stable** *if it is* stable, *and in addition, if there exists a $\delta > 0$, so that for each $\epsilon > 0$, there is a $T = T(\epsilon) > 0$ such that for every $x_0 \in \mathcal{N}_\delta(A_0)$*

$$H^*\left(S_t(x_0), A_0\right) < \epsilon, \qquad \forall t \geq T(\epsilon). \tag{1.6}$$

If any ϵ-neighbourhood of A_0 can be reached in finite time by every bounded subset of $\mathbb{R}^d$, then A_0 is said to be **globally asymptotically stable**.

A nonempty set B is said to be **positively invariant** if

$$S_t(B) \subseteq B, \qquad \forall t > 0.$$

Note that stable and asymptotically stable sets are positively invariant (see [34]). Additionally, a non-empty set B is said to be **S-invariant** if

$$S_t(B) = B, \qquad \forall t > 0.$$

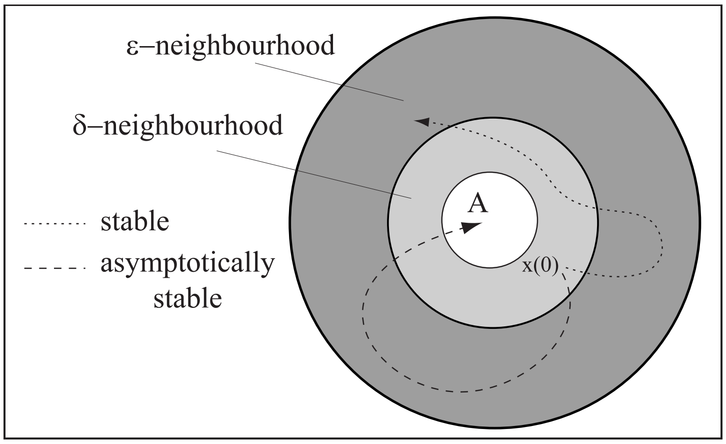

Figure 1.2: Concepts of Stability

1.2.3 Asymptotic Behaviour: Attractors

The asymptotic behaviour of a semi-group generating dynamical system is predominately determined by its limit sets and their attracting properties. Attractivity within the system is characterised by the approach of a point or set within a finite time to a neighbourhood of the attracting set. Formally, a set A is said to **attract** another set B if for every ϵ-neighbourhood of A, there exists a $T(\epsilon, A, B)$ such that $S_t(B) \subset \mathcal{N}_\epsilon(A)$ for all $t > T$. For example, an asymptotically stable set attracts an open neighbourhood of itself. As a result, it is often referred to as an *attracting set*. The definition of an attractor extends the idea of an asymptotically stable set by requiring it to be the minimal and invariant attracting set.

Definition 1.2.4 (Semi-Group Attractor)**.**
A nonempty compact and bounded subset $A_0 \subset E$, (E open and $E \subset \mathbb{R}^d$) is called an **attractor** *of a semi-group $\{S_t; t \in \mathbb{R}^+\}$ on E if*

$$S_t\left(A_0\right) = A_0, \qquad \textit{for each} \quad t \in \mathbb{R}^+, \qquad \textit{(Invariance Property)} \tag{1.7}$$

and if there exists a $\delta > 0$ such that

$$\lim_{t\to\infty} H^*\left(S_t(\mathcal{N}_\delta(A_0)), A_0\right) = 0. \qquad \textit{(Attraction Property)} \tag{1.8}$$

If the attractor attracts every bounded set of $\mathbb{R}^d$ as well as a neighbourhood of itself, it is said to be a **global attractor**.

Theorem 1.2.1. *A semi-group attractor is asymptotically stable.*

Proof: By definition, it is automatic that an attractor attracts a neighbourhood of itself. Therefore it is only required to show stability. [34] provides a complete proof.

□

On the other hand, an asymptotically stable set need not necessarily be an attractor. This is illustrated with the simple example given below.

Example 1.2.2. Consider again the dynamical system arising from the differential equation

$$\dot{x} = -x,$$

Solutions are given by

$$S_t(x) = x_0 e^{-t}.$$

Clearly the origin is an attractor, and also an asymptotically stable set as it attracts every open neighbourhood of itself. Now, consider the set $B = [-b, b]$ for some bounded $b > 0$. It is nonempty, compact, and attracts an open neighbourhood of itself, and so is asymptotically stable. However $S_t(B) = [-be^{-t}, be^{-t}]$ is a strict subset of B, and thus fails to comply with the property of invariance.

□

The property of invariance ensures that attractors are approached asymptotically, whereas asymptotically stable sets may be penetrated in finite time. The relationship between an asymptotically stable set and an attractor can be formally connected using the concept of limit sets.

Definition 1.2.5 (ω-limit Sets)**.** *Given a dynamical system on the state space E, $E \subset \mathbb{R}^d$, with semi-group representation $\{S_t; t \in \mathbb{R}^+\}$, the ω-**limit set** of a set $B \subset E$ is defined as*

$$\omega(B) = \{x \in E\,;\, \exists (t_i, x_i) \in \mathbb{R}^+ \times B, t_i \to \infty, S_{t_i}(x_i) \to x \;\; as \;\; i \to \infty\} \quad (1.9)$$

It is worth noting that in general

$$\omega(B) = \bigcup_{b \in B} \omega(b),$$

does not always hold, as can be seen in the following example.

Example 1.2.3. Consider the Bernoulli equation

$$\dot{x} - x + x^3 = 0.$$

Solutions using a semi-group representation are given by

$$S_t(x) = \frac{x}{[x^2 + (1 - x^2)\exp^{-2t}]^{\frac{1}{2}}}.$$

Refer to Figure (1.3). To contrast the difference between limit points and

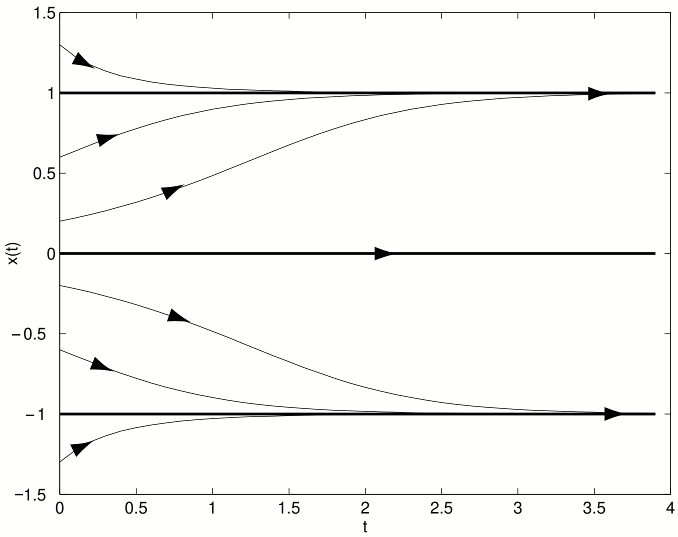

Figure 1.3: ω-Limit Sets

limit sets, consider any bounded interval $B = [-b, b]$, with $b > 0$. We have $S_t(B) = [-S_t(b), S_t(b)]$. Hence, $\omega(B) = [-1, 1]$. In particular, note that the set $[-1, 1]$ is invariant under the semi-group mapping (that is, $S_t([-1, 1]) = [-1, 1]$ for any $t > 0$).

In contrast however, if we consider the limit of any individual state $b \in B$, we find that it approaches one of three distinct limits - $\{-1\}, \{0\}$ or $\{1\}$.

Thus if we consider as a limit for B the set consisting of the union of limit points generated by individual states within B, we find that it is the point set $\{-1, 0, 1\}$.

Note that true to the definition of a limit set it is always possible to find sequences $\{x_k; x_k \in [-1, 1]\}$, $\{t_k; t_k \to \infty\}$ for any $x \in [-1, 1]$ so that $S_{t_k}(x_k) \to x$.

□

Theorem 1.2.2. *The omega (or positive limit set) set of an asymptotically stable set is an attractor.*

Proof: It can be easily shown that ω-limit sets are S-invariant, that is, $S_t(\omega(B)) = \omega(B)$ for all $t > 0$. Refer to [3]. So the proof for the above theorem involves showing that the ω-limit set attracts a neighbourhood of itself. This follows upon consideration of the neighbourhood defined by the asymptotic property of the original set, and showing that it is attracted by the ω-limit set. A detailed outline of the proof is given in Theorem 6.4 of [34].

□

Using ω-limit sets, the attraction property of an attractor, (1.8) may be equivalently written

$$H^* \left(\omega(\mathcal{N}_\delta(A_0)), A_0\right) = 0.$$

The existence of an attractor however, is often difficult to determine from its neighbourhoods, but can be more easily found with the identification of *absorbing sets.*

Definition 1.2.6 (Absorbing Sets)**.** *A nonempty compact and bounded subset $B \subset E$, is called an* **absorbing set** *for a semi-group $\{S_t, t \in \mathbb{R}^+\}$ on E if there exists a $\delta > 0$, and a $T = T(\delta) > 0$ such that*

$$S_t(\mathcal{N}_\delta(B)) \subseteq B, \qquad \forall t \geq T(\delta). \tag{1.10}$$

By definition, it is immediate that an absorbing set is asymptotically stable. Absorbing sets are also often referred to as *attracting sets*. It is often simpler to find an absorbing set within a dynamical system, than to find the actual attractor. Once the absorbing set B is found, Theorem 1.2.2 can be utilised to construct the attractor A_0 as the ω-limit set of the absorbing structure.

$$A_0 = \omega(B). \tag{1.11}$$

An alternative, yet equivalent construction for the attractor using absorbing sets is given by the well known result below.

Theorem 1.2.3.

$$A_0 = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_{(t)}(B)} \tag{1.12}$$

Proof:

i) $\omega(B) \subseteq \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_{(t)}(B)}$: Let $y \in \omega(B)$ Then there exists a sequence $\{x_n, t_n\}$ with $x_n \in B$ such that $S_{t_n}(x_n) \to y$. Now for every value of τ, $\exists N = \min\{n \in \mathbb{Z} : t_n > \tau\}$ so that $S_{t_n}(x_n) \in \overline{\bigcup_{t \geq \tau} S_{(t)}(B)}$ $\forall n > N$. As this set is closed and bounded, and thus compact, the limit y is also contained therein for every τ. Hence $y \in \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_{(t)}(B)}$.

ii) $\bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_{(t)}(B)} \subseteq \omega(B)$: Consider $y \in \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S_{(t)}(B)}$. For all values of τ, we have $y \in \overline{\bigcup_{t \geq \tau} S_{(t)}(B)}$. Now for any given $\epsilon > 0$ we can find a $z \in \bigcup_{t \geq \tau} S_{(t)}(B)$, such that $\mathrm{dist}(z, y) < \epsilon$. Then there exists a $x \in B$ and $t > \tau$ such that $S_t(x) = z$. Now consider the sequence $\epsilon_n \to 0$ and set $\tau_n = \max(t_{n-1}, n)$ so that $\tau_n \to \infty$. Then for each ϵ_n, there exists a $x_n \in B$ and $t_n > \tau_n$ such that $S_{t_n}(x_n) \to y$, with $t_n \to \infty$. Hence $y \in \omega(B)$.

□

1.2.4 Lyapunov Functions - Concepts and Terminology

Often it is not convenient or even possible to construct explicitly the stable and asymptotically stable sets that characterise the stability of a dynamical system. However, several methods are available for determining stability without requiring direct calculation of these sets. One such method is the calculation of limit sets. Further, the analysis of boundedness of solutions [42], or their prolongations under certain circumstances, can also provide useful information regarding system stability.

For dynamical systems generated by ordinary differential equations, auxiliary functions such as those of Lyapunov type can also provide a convenient way to characterise the stability of an arbitrarily shaped set A_0 without requiring explicit knowledge of the solutions of the differential equation. Yoshizawa [42], and Rouche/Habets/Laloy [27] detail a fairly comprehensive summary of the various necessary and sufficient conditions involving Lyapunov functions for a compact set A_0 to possess some form of stability for both ADE and NDE's. For reference we will briefly list a few of the relevant definitions and stability theorems which involve the use of an auxiliary, Lyapunov type, function for ADE's that will be used in later chapters. The non-autonomous counterparts will be presented in a subsequent section. These definitions will be used later in the thesis.

The following theorems and definitions pertain to autonomous dynamical systems arising from ordinary differential equations of the form

$$\dot{x} = f(x), \tag{1.13}$$

where f is required to be continuous and is assumed to be locally Lipschitz on the state space E, (E open and $E \subset \mathbb{R}^d$).

Definition 1.2.7 (The Lyapunov Function)**.** *The simplest Lyapunov functions used are C^1 functions of the type*

$$V(x) : E \to \mathbb{R}, \qquad E \subset \mathbb{R}^d.$$

A Lyapunov function will be assumed to be locally Lipschitz in x on E, that is, there exists a neighbourhood of x, $\mathcal{N} \subset E$, and a constant $L > 0$ such that

$$|V(x) - V(x')| \leq L\|x - x'\|, \quad \forall x, x' \in \mathcal{N}.$$

In many cases the function V is differentiable in which case the Lipschitz condition is automatically satisfied and the rate of change of V may be calculated through the usual rules of differentiation. If this is not the situation, then the upper right hand Dini Derivative form can be employed to characterise the function's rate of change with respect to time.

Definition 1.2.8 (Upper Right Hand Dini Derivitive)**.** *The Upper Dini Right Hand Derivative of V with respect to time is defined by*

$$\begin{aligned}\overline{D}_t^+ V[x(t)] &= \overline{\lim_{h\to 0+}} \left[\frac{V(x(t+h)) - V(x(t))}{h}\right], \\ &= \overline{D}^+ V(x) f(x), \end{aligned} \tag{1.14}$$

where $x(t)$ is the solution to the differential equation (1.13).

When V is differentiable, then the Dini derivative is equivalent to the usual time derivative.

A simplified and equivalent representation for the Dini Derivative is used throughout the work by Yoshizawa [42] (where equivalence is also shown) and Kloeden [15, 22] to analyse rates of change for Lyapunov functions in autonomous systems. We shall use this representation to investigate Lyapunov stability of perturbed autonomous systems in Chapter 4. We define this function by

$$D^+_{(1.13)} V(x) = \overline{\lim_{h\to 0+}} \left[\frac{V(x + hf(x)) - V(x)}{h}\right]. \tag{1.15}$$

The subscript (1.13) refers to the dynamical system from which the trajectories $x(t)$ are calculated.

It is important to note that since both $D^+_{(1.13)} V(x)$ and the Dini Derivative are equivalent, they may always be exchanged where convenient. We will typically use the notation $D^+_{(1.13)} V(x)$ when considering the decrescence of V as it quickly and accurately displays the system (1.13) for which solutions $x(t)$ belong. This avoids confusion when considering a perturbative analysis. It is also the same notation as that adopted by Yoshizawa and Kloeden.

Finally, we define a class of monotonically increasing functions $\mathcal{K}$.

Definition 1.2.9 (Class $\mathcal{K}$)**.** *We define a function a to be of class $\mathcal{K}$, that is $a \in \mathcal{K}$, if $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, monotonically increasing function with $a(0) = 0$.*

Note that these functions are defined such that they are not necessarily strictly monotone.

1.2.5 Lyapunov Functions - Stability Theorems

The following theorems apply to dynamical systems generated by (1.13), and are extensions of those referenced in [27], and [42].

Theorem 1.2.4 (Lyapunov Stability)**.**
A nonempty compact subset A_0 of E is locally stable *if and only if there exists a Lyapunov function $V : \mathcal{N}_R(A_0), \mapsto \mathbb{R}^+$, for some $R > 0$ that satisfies the following properties for every $x \in \mathcal{N}_R(A_0)$:*

1. $V(x) = \begin{cases} 0 & \text{if } x \in A_0 \\ > 0 & \text{if } x \notin A_0 \end{cases},$
2. *$V(x)$ is locally Lipschitz in x,*
3. $D^+_{(1.13)}V(x) \leq 0, \qquad \forall x \in \mathcal{N}_R(A_0).$

The conditions for asymptotic stability are similar, except that the Dini Derivative must be restricted further so that it only ever vanishes when the state is within A_0 itself.

Theorem 1.2.5 (Lyapunov Asymptotic Stability)**.**
A nonempty compact subset A_0 of E is locally asymptotically stable *if and only if there exists a Lyapunov function $V : \mathcal{N}_R(A_0), \mapsto \mathbb{R}^+$, for some $R > 0$ that satisfies the following properties for some $c \in \mathcal{K}$ and every $x \in \mathcal{N}_R(A_0)$:*

1. $V(x) = \begin{cases} 0 & \text{if } x \in A_0 \\ > 0 & \text{if } x \notin A_0 \end{cases},$
2. *$V(x)$ is locally Lipschitz in x,*
3. $D^+_{(1.13)}V(x) \leq -c(\mathrm{dist}(x, A_0)).$

If the first condition in both theorems is replaced by an extra radial unboundedness condition, i.e.

$$V(x) \geq a(\mathrm{dist}(x, A_0)),$$

for some $a \in \mathcal{K}$, with $a(r) \to \infty$ as $r \to \infty$, and the neighbourhood $\mathcal{N}_R(A_0)$ increased to include the entire state space E, then the stability in both cases is *global*.

The motivation for the radial unboundedness condition is to ensure that the contour curves (or surfaces) $V(x) = V_\alpha$ correspond to closed curves. If the curves are not closed, it is possible for the state trajectories to drift away from the equilibrium point, even though the state keeps passing through contour curves corresponding to smaller and smaller V_α's.

Example 1.2.4. Suppose we wish to analyse stability at the origin (i.e. $A_0 = \{0\}$, $A_0 \subset \mathbb{R}^2$) of a dynamical system with a Lyapunov function of the form

$$V = [x^2/(1 + x^2)] + y^2.$$

The curves $V(x) = V_\alpha$ for $V_\alpha > 1$ are open. Refer to Figure 1.4. Note that although it may satisfy the conditions for local stability with a neighbourhood being the entire state space E, it can be seen that divergence of the state can occur whilst moving through lower and lower V_α curves. Hence using this Lyapunov function, global stability cannot be assured.

□

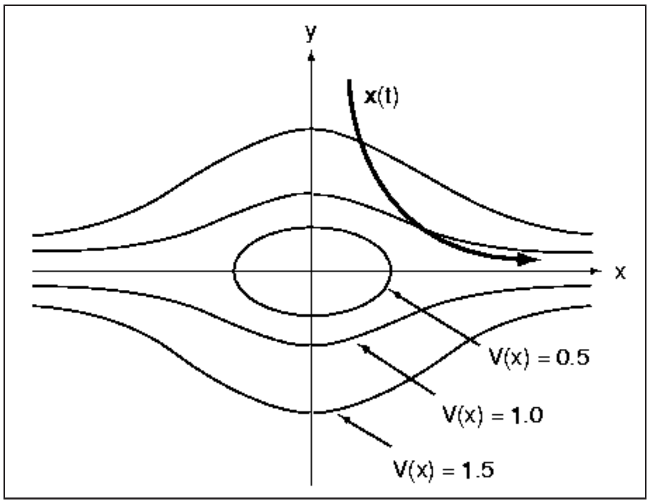

Figure 1.4: Radial Unboundedness Condition

1.3 Non-Autonomous Dynamical Systems

Dynamical systems theory has for the most part focused largely on autonomous systems for which there is a group or semi-group evolution property satisfied and attracting objects are invariant with respect to time. Non-autonomous dynamical systems however, typically possess more varied and complex behaviour, exhibiting meaningful properties that are often no longer invariant. Hence, the semi-group representation used earlier is no longer directly valid as the initial time is now just as important as the time elapsed.

Example 1.3.1. Consider the NDE

$$\dot{x} = f(\mathsf{t}, x), \tag{1.16}$$

where $\mathsf{t} \in \mathbb{R}$, and $x \in E \subset \mathbb{R}^d$. Under suitable conditions ([24],[26]) and assumptions on the state space, E, and continuity of f, there exists a unique solution $x = x(\mathsf{t}; t_0, x_0)$, to the initial value problem defined by $(t_0, x_0) \in \mathbb{R} \times E$. A **flow or cocycle property** (sometimes otherwise referred to as a 2- parameter group property or process) is also satisfied,

$$x(t + \tau + t_0; t_0, x_0) = x(t + \tau + t_0; \tau + t_0, x(\tau + t_0; t_0, x_0)), \tag{1.17}$$

for all $x_0 \in E$, and $t_0, t, \tau \in \mathbb{R}^+$ which is known as a **flow or cocycle property** (sometimes otherwise referred to as a 2- parameter group property or process).

□

Example 1.3.2. The NDE

$$\dot{x} = 2\mathsf{t}x,$$

has solutions for the initial value problem (t_0, x_0) given by

$$x(t + t_0, t_0, x_0) = x_0 e^{(t+t_0)^2 - t_0^2},$$

where t is the elapsed time, $\mathsf{t} = t + t_0$ is the actual time, and t_0 is the initial time. This then satisfies the cocycle property (1.17). This can be seen by noting that

$$\begin{aligned} x(t + \tau + t_0, t_0, x_0) &= x_0 e^{(t+\tau+t_0)^2 - t_0^2} \\ &= [x_0 e^{(\tau+t_0)^2 - t_0^2}] e^{(t+\tau+t_0)^2 - (\tau+t_0)^2} \\ &= x(t + \tau + t_0, \tau + t_0, x(\tau + t_0, t_0, x_0)). \end{aligned}$$

□

The cocycle property (1.17), is the non-autonomous counterpart of the group or semi-group evolutionary property (1.2), of an autonomous dynamical system. Essentially, it is analogous to resetting the clock.

1.3.1 Skew-Product Flow and Cocycle Representation

Several abstract formulations have evolved to serve as a non-autonomous counterpart to the semi-group representation.

Sell's skew product flows ([30],[31]) on non-autonomous differential equations (1.16), retain a semi-group property by replacing the dependence on the initial time variable with a function space, effectively converting the problem to one of an autonomous nature with an altered state space.

To do this, Sell's Skew Product flows use the function space $\mathcal{F}$, a set of functions $F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ that are continuous in both variables and Lipschitz in the second variable uniformly with respect to the first.

We then proceed to define a group of shift operators on $\mathcal{F}$ such that $\theta_t : \mathcal{F} \to \mathcal{F}$ with $\theta_0 = id$ and $\theta_t F(t_0, \cdot) = F(t + t_0, \cdot)$.

Returning to the non-autonomous problem (1.16), the product $\mathbb{R}^d \times \mathcal{F}$ of the state space with this function space provides us with an alternate space that transforms the original non-autonomous problem to an autonomous one. Solutions may then be represented using the semi-group mappings of the previous section.

Unfortunately, an attractor for such a semi-group is then a subset of the product space $\mathbb{R}^d \times \mathcal{F}$ and its meaning in terms of the original dynamics in $\mathbb{R}^d$ is not always clear or convenient. Sell's Skew Product flows are presented formally in the following definition.

Definition 1.3.1 (Skew-Product Flow Representation)**.** *Let* $\{\theta_t, t \in \mathbb{R}\}$ *act as a group of shift operators on the function space* $\mathcal{F}$ *such that* $\theta_t : \mathcal{F} \to \mathcal{F}$*, with* $\theta_0 = id$ *and* $\theta_t F(t_0, \cdot) = F(t + t_0, \cdot)$*. Finally let* $\mathcal{X} = E \times \mathcal{F}$ $(E \subseteq \mathbb{R}^d)$*, and* *define* $S_t : \mathcal{X} \to \mathcal{X}$ *by*

$$S_t(x_0, F) = (x(t + t_0; t_0, x_0), \theta_t F).$$

Then the family of mappings $\{S_t, t \in \mathbb{R}\}$ *is a continuous time semi-group on the state space* $\mathcal{X}$ *and is known as a* **skew-product flow**.

The asymptotic behaviour of semi-groups as outlined previously, apply to this semi-group also, and can be interpreted to draw some, though limited, information about the trajectories and characteristics of the non-autonomous system. For example, an attractor for such a semi-group generated by a skew product flow representation is a subset of the product space $\mathbb{R} \times \mathcal{F}$ and its meaning and conceptualisation in terms of the original state space is often not always clear or convenient, particularly if we wish to compute it numerically. However, this is of advantage if the function space is compact.

Note that the semi-group property for the skew product flow,

$$S_{t+\tau}(x_0, F) = S_t \circ S_\tau(x_0, F),$$

is also a form of the cocycle property, (1.17). In fact, the skew product flow representation is a special case of a generalised representation motivated by the characteristics of the cocycle property.

Definition 1.3.2 (Cocycle Representation). *Let* $\{\theta_t, t \in \mathbb{T}\}$ *be a group of mappings on a nonempty parameter set* P*, that is* $\theta_t : P \mapsto P$ *with* $\theta_0 = id$ *and* $\theta_t \circ \theta_\tau = \theta_{t+\tau}$ *for all* $t, \tau \in \mathbb{T}$.

A family of mappings $\{\Phi_{(t,p)}, t \in \mathbb{T}, p \in P\}$ *with* $\Phi_{(t,p)} : X \to X$ *is called a* **cocycle** *on* X *with respect to the group* $\{\theta_t, t \in \mathbb{T}\}$ *of mappings on* P *if*

$$(i) \quad \Phi_{(0,p)} = id, \qquad \textit{Identity Property} \tag{1.18}$$

$$(ii) \quad \Phi_{(t+\tau,p)} = \Phi_{(t,\theta_\tau p)} \circ \Phi_{(\tau,p)}, \qquad \textit{Cocycle Property} \tag{1.19}$$

for all $t, \tau \in \mathbf{R}^+$ *and* $p \in P$.

The set $\mathbb{T}$ represents the time element for the dynamical system, and is usually either $\mathbb{R}$ (continuous time), or $\mathbb{Z}$ (discrete time). The state space will usually be the set E, (E open and $E \subset \mathbb{R}^d$). For the following work with cocycles we will keep with the notation of using the state space E, as these will pertain to ideas presented later. However, most of these results are also applicable to other forms of the state space or even a generalised state space X, with some attention to detail. Uniqueness of the cocycle representation for a dynamical system is assumed. When used as the representation for a non-autonomous differential equation, existence and uniqueness of the cocycle follows from the usual continuity and Lipschitz conditions on f in the differential equation [24].

Note: Φ will be used throughout as a cocycle or flow representation for non-autonomous dynamical systems to distinguish it from the semi-group representation S_t introduced earlier.

Example 1.3.3. Consider again the initial value problem for Equation (1.16),

$$\dot{x} = f(\mathsf{t}, x).$$

Here $\mathbb{T} = \mathbb{R}$, and $x \in E \subset \mathbb{R}^d$. The solution $x = x(t + t_0; t_0, x_0)$ generates a cocycle $\{\Phi_{(t,t_0)}; t \in \mathbb{R}^+, t_0 \in \mathbb{R}\}$ on E

$$\Phi_{(t,t_0)}(x_0) = x(t + t_0; t_0, x_0)$$

with respect to the group $\theta_t; \mathbb{R} \mapsto \mathbb{R}$, where $\{\theta_t; t \in \mathbb{R}^+\}$ is defined by $\theta_t(t_0) = t_0 + t$.

□

The generality involved in defining the parameter space P within the cocycle definition is to allow a certain degree of flexibility when choosing a representation for the system. For such a generalised parameter set we define the following metric on P.

Definition 1.3.3 (Metric on P)**.** *We define the metric $|\cdot|$ on P by*

$$|p^* - p| = \{|t^*|; \theta_{t^*} p = p^*\}.$$

We say $p^* > p$ if there exists a $t^* > 0$ for which $\theta_{t^*} p = p^*$, and $p^* < p$ if there exists a $t^* < 0$.

The obvious and straight forward choice for the parameter set is the set of all possible initial times as illustrated in the example above. That is $P = \mathbb{R}$, with the group mapping defined by $\theta_t(t_0) = t_0 + t$.

An alternative representation for the non-autonomous dynamical system (1.16) may be obtained by representing P as a function space, $\mathcal{F}$, defined by the set of functions $\{f \in \mathcal{F}; f : \mathbb{R} \times E \mapsto E\}$, which are continuous in both variables, and Lipschitz in the second variable uniformly with respect to the first (these are the typical continuity conditions constraining the function which defines the non-autonomous differential equation (1.16)). The group mapping is defined as $\theta_t(f(\cdot, \cdot)) = f(\cdot + t, \cdot)$. For initial values x_0, and $f(t_0, x_0)$, we then have $\{\Phi_{(t,f)}; t \in \mathbb{R}^+, f \in \mathcal{F}\}$ defined by $\Phi_{(t,f)}(x_0) = x(t + t_0; t_0, x_0)$, being a cocycle on E. This construction is identical to that used for Sell's Skew Product flows except that it retains the original state space and preserves the non-autonomous aspect of the problem. While it appears less natural to represent solutions this way it is advantageous in the fact that the function space $\mathcal{F}$ can often be chosen to be a compact metric space.

The cocycle formalism introduced here provides a natural generalisation of semi-groups to non-autonomous systems with the advantage of being able to retain the original state space (this is in contrast with representations such as Sell's Skew Product flows). This is of particular advantage since attracting and stable objects may be meaningfully represented on the original state space.

Cocycles have been instrumental in developing numerical and random dynamical systems theory ([2], [12], [10], [28]) and many of the results concerning asymptotic behaviour and attractors can be usefully transferred and reworked in the context of deterministic non-autonomous dynamical systems.

1.3.2 Asymptotic Behaviour: Stability

As mentioned earlier, the classical concepts of stability, and asymptotic stability are easily transferred to non-autonomous systems. These represent the basis for a classical analysis of stability. However, they represent a limited view of stability within a non-autonomous environment, and also do not characterise attracting structures that may themselves be time-varying. These issues will be discussed in greater detail in Chapter 2.

Using the cocycle representation for a non-autonomous system, a *stable set*, and an *asymptotically stable set* are defined as follows.

Definition 1.3.4 (Stability). *A nonempty compact subset $A_0 \subset E$, (E open and $E \subset \mathbb{R}^d$) is* **stable** *under the cocycle mapping $\{\Phi_{(t,p)} : t \in \mathbb{R}^+, p \in P\}$ on E, if for any $\epsilon > 0$, and any $p \in P$, there exists a $\delta(p, \epsilon) > 0$ such that*

$$H^*(\Phi_{(t,p)}(\mathcal{N}_{\delta(p)}(A_0)), A_0) < \epsilon, \qquad \forall t \geq 0. \tag{1.20}$$

If the δ is independent of p, then the stability is referred to as uniform with respect to p. A_0 is said to be **uniformly stable**.

Definition 1.3.5 (Asymptotically Stability). *A nonempty compact subset $A_0 \subset E$, (E open and $E \subset \mathbb{R}^d$) is* **asymptotically stable** *under the cocycle mapping $\{\Phi_{(t,p)} : t \in \mathbb{R}^+, p \in P\}$ on E, if it is both* stable, *and for any $p \in P$, there exists a $\delta(p) > 0$ so that for each $x_0 \in \mathcal{N}_{\delta(p)}(A_0)$,*

$$H^*(\Phi_{(t,p)}(x_0), A_0) \to 0, \quad \text{as} \quad t \to \infty. \tag{1.21}$$

Alternatively, the attraction property (1.21) may be restated slightly differently. For each $p \in P$, there exists a $\delta(p) > 0$, so that for each $\epsilon > 0$, and $x_0 \in \mathcal{N}_{\delta(p)}(A_0)$ there is a $T(x_0, p, \epsilon)$ such that

$$\text{dist}(\Phi_{(t,p)}(x_0), A_0) < \epsilon, \qquad \forall t > T. \tag{1.22}$$

Similarly, if A_0 is uniformly stable, δ is independent of p, and $T = T(\epsilon)$ only, then A_0 is said to be **uniformly asymptotically stable** under the cocycle mapping Φ. This is the case in autonomous systems when P is a singleton set, and hence any stable objects are automatically uniform with respect to p. The above definition then reduces to the one given in (1.6).

If in the above definition, T is independent of x_0, that is $T = T(p, \epsilon)$ only, then A_0 is said to be **equi-asymptotically stable**.

We proceed with a few examples to illustrate these concepts.

Example 1.3.4. Consider the non-autonomous dynamical system generated by

$$\dot{x} = -x + e^{-\mathtt{t}}, \tag{1.23}$$

where $x \in \mathbb{R}$, and $\mathtt{t} \in \mathbb{R}$. The resulting solutions may be expressed as a cocycle (note t is the elapsed time)

$$\Phi_{(t,t_0)}(x_0) = (te^{-t_0} + x_0)e^{-t},$$

where the group shift mapping θ_t acts on the parameter space $P = \mathbb{R}$, and is defined by $\theta_t(t_0) = t + t_0$. Refer to Figure 1.5 for a graph of solutions for various values of x_0 at $t_0 = 0$. It is noted that for different values of t_0, the families of solutions follow a similar pattern as that depicted in Figure 1.5. In particular, the only major difference is in the initial point for which the solutions are strictly monotonically decreasing for all t. This occurs for all $x_0 > e^{-t_0}$.

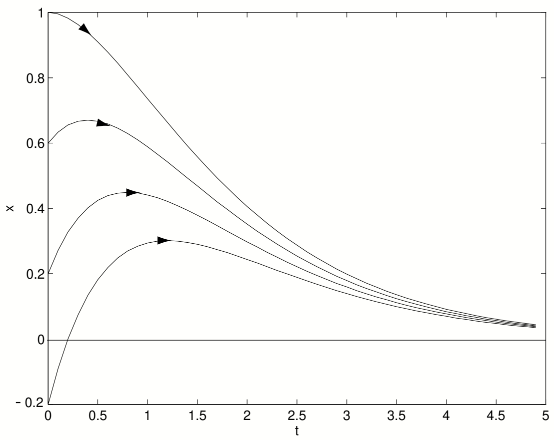

Figure 1.5: Asymptotic Convergence without Stability

Obviously, the set $A_0 = \{0\}$ is asymptotically approached. However it is not asymptotically stable as it lacks the required property for stability (all solutions in a neighbourhood of A_0 are guaranteed to travel outside a small enough epsilon neighbourhood. Take for example, $\epsilon = 0.1$.

Note, however that it is 'eventually stable'. This is often referred to as **eventual asymptotic stability**, see [27, 42].

$\square$

Example 1.3.5. Consider the non-autonomous dynamical system generated by the ordinary differential equation (where conditions for uniqueness of solutions are assumed to be satisfied)

$$\dot{x} = f(p, x),$$

where the state space $X = \mathbb{R}$. It is also known to possess an asymptotically stable set A_0.

Since A_0 is asymptotically stable, there exists a $T(x_0, p, \epsilon)$ for every $p \in P$ guaranteeing attraction of any $x_0 \in \mathcal{N}_{\delta(p)}(A_0)$ to within an ϵ-neighbourhood of A_0 in finite time. However, since the resulting solutions are unique, the only dependence T has on the initial state $x_0 \in \mathcal{N}_{\delta(p)}$, is at the boundaries of the neighbourhood. For $\mathcal{N}_{\delta(p)}(A_0) \subset \mathbb{R}$, this consists of only two points. Hence for arbitrary x_0,

$$T(x_0, p, \epsilon) = \max\{T(x_1, p, \epsilon), T(x_2, p, \epsilon)\},$$

where x_1, x_2 are the boundary points. As x_1, x_2 depend on $\delta(p)$, we can conclude $T = T(p, \epsilon)$ only. As a result, any asymptotically stable set on $\mathbb{R}$ is automatically **equi-asymptotically stable**. Note that this is not the case with higher dimensional state spaces as then the neighbourhood boundary consists of an infinite number of points, and an upper bound for the maximum may not exist.

□

1.3.3 Lyapunov Functions and Stability

The concepts and theorems of Section 1.2.4 and 1.2.5 may be extended to dynamical systems of non-autonomous differential equations (1.16). As the system is now dependent on initial times, the Lyapunov function is required to be a function of both state and time. As we are generalising to take into account more general parameter fields P, the Lyapunov function will be of the form $V = V(p, x)$. Additionally, if the function $f(p, x)$ is periodic in p, then a Lyapunov function may be chosen (or found to exist) which is also periodic in p (refer to [27]).

To investigate the rate of change of V we use the upper right hand **Dini Derivative** of $V(p, x)$ calculated with respect to time

$$\overline{D_t}^+ V(p, x) = \overline{\lim_{h \to 0^+}} \left[\frac{V(\theta_h p, \Phi_{(h,p)}(x)) - V(p, x)}{h} \right],$$

where $\Phi_{(t,p)}(x)$ is a solution to the differential equation (1.16).

The following theorems apply equally to autonomous systems and in fact reduce to those presented in Section 1.2.5 where $V(p,x)$ need only be a function of x.

Stability Theorems

Theorem 1.3.1 (Lyapunov Stability - Theorem)**.**
A nonempty compact subset A_0 of E is stable *if there exists a Lyapunov function $V : P \times \mathcal{N}_R(A_0), \mapsto \mathbb{R}^+$, for some $R > 0$ that satisfies the following properties for some $a \in \mathcal{K}$ and for every $(p,x) \in P \times \mathcal{N}_R(A_0)$:*

1. $x \in A_0, \qquad V(p,x) = 0,$
2. $a(\mathrm{dist}(x, A_0)) \leq V(p,x),$
3. $V(p,x)$ *is locally Lipschitz in x, uniformly with p.*
4. $\overline{D_t}^+ V(p,x) \leq 0.$

If in addition there is a $b \in \mathcal{K}$ so that we have $V(p,x) \leq b(\mathrm{dist}(x, A_0))$, then A_0 is locally uniformly stable.

Theorem 1.3.2 (Lyapunov Uniform Asymptotic Stability - Theorem)**.**
A nonempty compact subset A_0 of E is uniformly asymptotically stable *if there exists a Lyapunov function $V : P \times \mathcal{N}_R(A_0), \mapsto \mathbb{R}^+$, for some $R > 0$ that satisfies the following properties for some a, b and $c \in \mathcal{K}$ and for every $(p,x) \in P \times \mathcal{N}_R(A_0)$:*

1. $a(\mathrm{dist}(x, A_0)) \leq V(p,x) \leq b(\mathrm{dist}(x, A_0)),$
2. $V(p,x)$ *is locally Lipschitz in x, uniformly with p.*
3. $\overline{D_t}^+ V(p,x) \leq -c(\mathrm{dist}(x, A_0)).$

Remark 1. If the above two theorems are satisfied with $\mathcal{N}_R(A_0) = E$ and

$$a(r) \to \infty \quad \text{as} \quad r \to \infty,$$

then the stability of A_0 is *global* in each case.

Remark 2. The existence of a Lyapunov function satisfying the above three conditions, with the exception that the first is satisfied only so that

1. $a(\text{dist}(x, A_0)) \leq V(p, x), \quad V(p, 0) = 0,$

that is, there is no upper bounding $b \in \mathcal{K}$ for V, then there can be no guarantee that V implies uniform asymptotic stability, nor even asymptotic stability.

Remark 3. The last condition may be replaced by $\overline{D_t}^+ V(p, x) \leq -cV(p, x)$ and the result is still valid.

Converse Theorems

Variations of a converse theorem for Lyapunov's Theorem for uniform asymptotic stability above, exist in several forms. In many cases it is useful to know only that given an asymptotically stable set, a Lyapunov function does exist, and exhibits certain characteristics, even if the actual function is not known. We will use a converse theorem in developing various results analysing perturbations of existing systems in later chapters. For reference, it is given below.

Theorem 1.3.3 (Lyapunov Uniform Asymptotic Stability - Converse Theorem)**.**
Given a uniformly asymptotically stable subset A_0 of E there exists a Lyapunov function $V : P \times \mathcal{N}_R(A_0), \mapsto \mathbb{R}^+$, for some $R > 0$ that satisfies the following properties for some $a \in \mathcal{K}$, and some constant $c > 0$, and for every $(p, x) \in P \times \mathcal{N}_R(A_0)$:

1. $V(p, x) \geq a(\text{dist}(x, A_0)), \qquad V(p, 0) = 0,$
2. $|V(p, x) - V(p, x')| \leq h(p)||x - x'||,$
3. $\overline{D_t}^+ V(p, x) \leq -cV(p, x),$

where $h(p)$ is a continuous scalar function.

Remark 1. If the system is autonomous, or if the original function $f(p, x)$ in (1.16) satisfies a Lipschitz condition for any compact set K on E, then the function $h(p)$ may be simply represented by a constant.

1.4 Further Asymptotic Characteristics

Although the stability of many non-autonomous systems may be examined with the use of the definitions and theorems in the previous section, a wide class of non-autonomous systems possess structures (particularly time varying structures) with attracting characteristics that go beyond the scope of these definitions and theorems to describe and characterise. As a result several new concepts for convergence and stability, and attracting structures are needed.

We begin with an introductory example to illustrate these issues and introduce the concept of pullback attraction as an additional method to form a more comprehensive analysis of the system's stability.

1.4.1 Non-Autonomous Asymptotic Behaviour

Example 1.4.1. Consider the autonomous system on $\mathbb{R}$

$$\dot{x} = -x,$$

which has global attractor $A_0 = \{0\}$, and the non-autonomously perturbed system,

$$\dot{x} = -x + \sin \mathsf{t}.$$

Solutions are defined explicitly for $t \geq 0$ (t is elapsed time) and are expressed using a cocycle representation by

$$\Phi_{(t,t_0)}(x_0) = \frac{1}{2}\left(\sin(t+t_0) - \cos(t+t_0)\right) + \left(x(t_0) - \frac{1}{2}\left(\sin t_0 - \cos t_0\right)\right) e^{-t}, \tag{1.24}$$

where the group shift mapping $\theta_t : \mathbb{R} \to \mathbb{R}$ is defined by $\theta_t(t_0) = t_0 + t$.

□

We wish to analyse the system's asymptotic behaviour, and more importantly identify any objects or structures which have attracting properties.

The usual, and most obvious way to formulate asymptotic behaviour is to consider the limit set of the forwards trajectory. For any t_0 the ω-limit set for the cocycle (1.24) here is

$$\omega_(t_0, \mathbb{R}) = [-1/\sqrt{2}, 1/\sqrt{2}].$$

However, this set is not Φ-invariant, $(\Phi_{(t,t_0)}([-1/\sqrt{2}, 1/\sqrt{2}]) \neq [-1/\sqrt{2}, 1/\sqrt{2}])$, which is a fundamental property of any attracting object. Limit sets in non-autonomous systems have been investigated in [13, 40], however they have the disadvantage that the resulting ω-limit sets are generally not invariant (as in this case) with respect to Φ. It may be too restrictive searching for a single constant set which is invariant under Φ. This directs us to search for a family of sets that may be Φ-invariant.

Definition 1.4.1 (Φ-Invariance).
The family of nonempty sets $\hat{A} = \{A(\tau); \tau \in \mathbb{R}\}$ *is* **invariant** *under* Φ *(or equivalently* **Φ-invariant***) if*

$$\Phi_{(t,t_0)}(A(t_0)) = A(t + t_0), \qquad \forall t \in \mathbb{R}^+, \quad t_0 \in \mathbb{R}.$$

An 'intuitive' look at the long term behaviour of solutions Φ as $t \to \infty$ elucidates a transient component which vanishes exponentially, and a steady state component that is time varying. Thus solutions are attracted exponentially to the steady state solution. To characterise this, let us define a family of sets $\hat{A} = \{A(\tau); \tau \in \mathbb{R}\}$ as described above, where

$$A(\tau) = \frac{1}{2}\left(\sin\tau - \cos\tau\right). \tag{1.25}$$

$\hat{A}$ is obviously Φ-invariant, and also possesses the usual characteristics of an attracting set as trajectories of Φ approach it as the elapsed time is increased. This can be seen by taking

$$\lim_{t\to\infty} H^*(\Phi_{(t,t_0)}(x_0), A(t + t_0)) = 0.$$

However, does this attraction hold only for large t, or can we consider attraction to an element $A(t) \in \hat{A}$ for any arbitrary time t?

The answer is yes, however it requires a subtle addition to the usual concepts of convergence. The typical forwards convergence analysis entails increasing the final time whilst the initial time remains fixed. However, to consider convergence to a set $A(t^*)$ of this attracting family, for a particular t^*, it would be reasonable to fix the final time, and start progressively earlier (pulling back the initial time) in order to finish at t^*. By taking this limit for (1.24), and any $x_0 \in \mathbb{R}$, we find

$$\lim_{t\to\infty} H^*(\Phi_{(t,t^*-t)}(x_0), A(t^*)) = 0.$$

This is referred to as **pullback attraction** (or pullback convergence), and has the advantage in that attraction to a single element of the attracting structure may be mathematically characterised with an approach involving the use of limit sets.

Pullback convergence for this particular problem is illustrated in the diagram below, showing pullback convergence of the interval $[-2, 2]$ to $\hat{A}$ at $t = 8$.

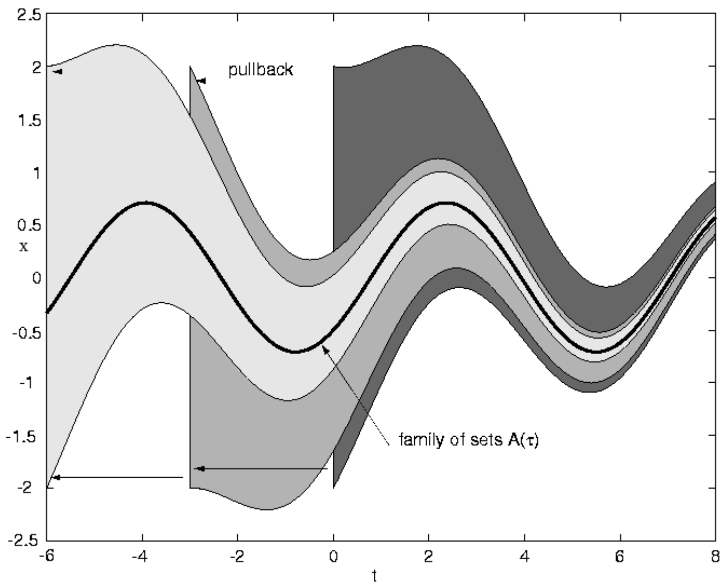

Figure 1.6: Pullback Convergence

Chapter 2

Stability of Non-Autonomous Dynamical Systems

2.1 Stability and Attraction

2.1.1 Introduction

As illustrated in the preceding example, the original concepts of stability, asymptotic stability, and the development of attracting objects that are simply a single and constant set, are restrictive and do not always provide a conclusive analysis of the behaviour and stability of a non-autonomous dynamical system. Frequently, the generic 'attracting' object for a cocycle will often consist of a family of sets rather than just a single set. Yoshizawa (see [42]) introduced briefly the concept of using a family of sets for an attracting object, however, the theory only considers forward asymptotic convergence to the attracting object. This has the disadvantage of being unable to determine attractivity except of an 'eventual' nature, and also lacks the use of a limit set theorem analogous to Theorem 1.2.3 for autonomous systems.

As well as redeveloping the notion of an attracting object as a family of sets, it should also be feasible to consider stability and asymptotic stability to a family of sets in both the forward *and* pullback sense. Analysing forward convergence within a system is certainly nothing new, and the notion of pullback convergence that was introduced has also been around for some time, though within the context of dynamical systems it is a relatively new approach. It was first used in analysing random dynamical systems, and has its value in that it does allow an extension of the limit set theorem (Theorem 1.2.3) for time-varying pullback attractors in non-autonomous systems (pullback attractors are discussed in more detail in Section 2.2). This proved in part the motivation to formalise the concepts of pullback stability and asymptotic stability in order to facilitate a more complete pullback analysis. The definitions for pullback stability and pullback asymptotic stability are newly developed here, with particular emphasis placed on a local analysis of stability. The ensuing development of attractors is also made with regards to a local analysis. This is in contrast with the global analysis for pullback attractors used by Kloeden et. al. ([20, 21].

To begin with however, we formalise our definition and notation for a family of sets within a non-autonomous system and the concept of a neighbourhood of a family of sets. These are essential in implicitly describing the nature of structures that evolve within non-autonomous systems, and have been constructed axiomatically from those for autonomous systems.

Notation 2.1.1 (Family of Sets, $\hat{\mathbf{A}}$). *We will say a collection of nonempty subsets over the parameter space P, represents a* family of sets *on E and denote it by*

$$\hat{A} = \{A(p); p \in P\}$$

The concept of a neighbourhood for a stable or attracting object is frequently used, and to extend it to a family of sets we define and use the following notations throughout the remainder of this thesis.

Notation 2.1.2 (δ-Neighbourhood of $\hat{\mathbf{A}}$). *A δ-neighbourhood, $\hat{\mathcal{N}}_{\delta,\hat{A}}$ of a family of sets $\hat{A}$ is defined as the family*

$$\hat{\mathcal{N}}_{\delta,\hat{A}} = \{\mathcal{N}_\delta(A(p)); p \in P\},$$

for some $\delta > 0$. It will be referred to as a δ-neighbourhood of $\hat{A}$.

To illustrate, consider a family of sets $\hat{A}$ that uniform attracts solutions within a local neighbourhood. Uniform attraction implies that the $\delta-neighbourhood$

of $\hat{A}$ may be chosen such that δ is constant with respect to time. This does not mean that the neighbourhood itself is fixed with respect to time! Indeed, if $\hat{A}$ is moving, then its neighbourhood must also move with it.

Rather than considering the local neighbourhood of each individual set in $\hat{A}$, we have defined above the whole family as a single entity - the δ-neighbourhood of $\hat{A}$ By using this terminology, we may conveniently discuss the properties of solutions with initial value (x_0, p_0) lying anywhere (with respect to initial state and time) within the local neighbourhood. Note that if $\hat{A}$ does not vary with time, the above terminology becomes equivalent with the usual concept of a neighbourhood for an analysis of a dynamical system's stability.

As a practical illustration, recall the attracting object $\hat{A}$ defined by (1.25) discussed in Example 1.4.1. A δ-neighbourhood with $\delta = 1.25$ (shaded region) uniformly attracts any solution with initial value lying in this neighbourhood. Refer to Figure 2.1.

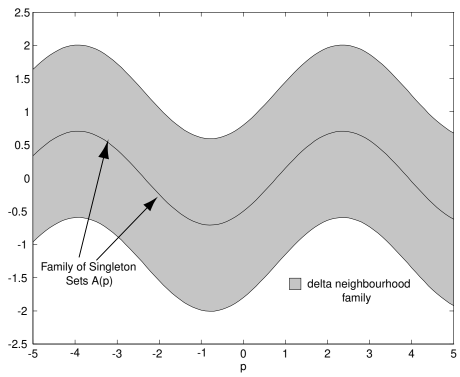

Figure 2.1: $\hat{\mathcal{N}}_{\delta,\hat{A}}$ - δ-neighbourhood of the family $\hat{A}$

The motivation for the following definition may not be so clear.

Uniform attraction involves an analysis of local neighbourhoods. For fixed sets A, a fixed local neighourhood may be represented by $\mathcal{N}_\delta(A)$. For a family of sets $\hat{A}$, a fixed local neighbourhood can be represented by $\hat{\mathcal{N}}_{\delta,\hat{A}}$ as discussed above.

An analysis of asymptotic stability (not necessarily uniform) requires an added level of complexity. For each $p \in P$ we must consider a different local neighbourhood. For fixed sets A this is typically represented by $\mathcal{N}_{\delta_p}(A)$ for each $p \in P$. For a family of sets $\hat{A}$, this corresponds to a local neighbourhood family $\hat{\mathcal{N}}_{\delta_p,\hat{A}}$ for each $p \in P$.

We collectively group the neighbourhood families defined over P as a system of families and will refer to them as follows.

Notation 2.1.3 (δ-Neighbourhood System of $\hat{\mathbf{A}}$). *A δ-neighbourhood system, $\hat{\mathcal{N}}_{\hat{\delta},\hat{A}}$ of a family $\hat{A}$ is defined as*

$$\hat{\mathcal{N}}_{\hat{\delta},\hat{A}} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\},$$

for some set of uniformly bounded δ values $\hat{\delta} = \{\delta_p; \delta_p > 0, p \in P\}$.

At an initial glance, this might not appear as intuitive or as simple as defining a varying local neighbourhood family by

$$\hat{\mathcal{N}}_{\hat{\delta},\hat{A}} = \{\mathcal{N}_{\delta_p}(A(p)); \delta_p \in \hat{\delta}, p \in P\}.$$

For a forward analysis, this definition suffices - however it does not provide a suitable construction of a neighbourhood for a pullback analysis of asymptotic stability. The reasons for this will become clear later as we investigate pullback asymptotic stability in more depth.

For convenience, any δ and ϵ sets will be assumed to be uniformly bounded with respect to p (to ensure that local neighbourhoods are indeed local) for the remainder of this work.

By convention, we will also use the hat symbol to denote a family of objects (e.g. a family of sets $\hat{A}$, a family of parameters $\hat{\delta}$ or a system $\hat{\mathcal{N}}_{\hat{\delta},\hat{A}}$) that has been collectively grouped over P.

2.1.2 Attraction

The differences between forward and pullback attraction lie in the distinction that forward attraction requires the attraction to only occur 'eventually', with no information as to the attracting characteristics of the system at any fixed time. Alternatively, pullback attraction guarantees attraction to an element at a distinct point in time. The state of the attraction thereafter is not an issue. These concepts (introduced in Example 1.4.1) are formalised below and followed by illustrative examples thereafter.

Definition 2.1.1 (Forward Attraction)**.** *A family of uniformly bounded compact sets $\hat{A} = \{A(p) : p \in P\}$ is said to* **forward attract** *another family of sets $\hat{B} = \{B(p) : p \in P\}$ from $p \in P$ if*

$$\lim_{t\to\infty} H^* \left(\Phi_{(t,p)}(B(p), A(\theta_t p))\right) = 0.$$

Also note that if $\hat{A}$ forward attracts $\hat{B}$, and $\hat{C} \subset \hat{B}$ (that is, $C(p) \subset B(p)$ for all $p \in P$), then $\hat{A}$ forward attracts $\hat{C}$.

Forward attraction of a single set, or family $\hat{B}$, to $\hat{A}$ is also referred to as *forward convergence* of $\hat{B}$ to $\hat{A}$ from $p \in P$.

Definition 2.1.2 (Pullback Attraction)**.** *A family of uniformly bounded compact sets $\hat{A} = \{A(p) : p \in P\}$ is said to* **pullback attract** *another family of uniformly bounded sets $\hat{B} = \{B(p) : p \in P\}$ at $p \in P$ if*

$$\lim_{t\to\infty} H^* \left(\Phi_{(t,\theta_{-t}(p))}(B(\theta_{-t}(p)), A(p))\right) = 0.$$

Similarly, this is also referred to as *pullback convergence* of $\hat{B}$ to $\hat{A}$ at p.

Definition 2.1.3 (Complete Attraction)**.** *If $\hat{A}$ pullback attracts $\hat{B}$ at some value of $p \in P$ and also forward attracts $\hat{B}$ from p then $\hat{A}$ is said to* **completely attract** *$\hat{B}$ at $p \in P$.*

Often we are concerned only with the attraction of single sets or of a single set B to a family $\hat{A}$, for some $p \in P$. In either case, the corresponding family can be thought of as a family of identical sets.

The definitions are also consistent with classical ideas in autonomous systems. The definition of pullback attraction (and also complete attraction) then become equivalent to the usual definition for forward attraction as the initial time is not relevant.

Example 2.1.1 (Attraction in Example 1.4.1)**.** *Consider again the perturbed autonomous system in Example 1.4.1, and consider possible attraction of the* *state to the origin. In order to analyse the behaviour at the origin we define* $\hat{A}$ *by* $A(\mathtt{t}) = 0$*, for all* $\mathtt{t}$*, and consider both forward and pullback attraction to* $\hat{A}$*.*

i) Forward Attraction: *For any single, bounded set* B*, we have*

$$\begin{aligned}
&\lim_{t\to\infty} H^*\left(\Phi_{(t,t_0)}(B), A(\theta_t(t_0))\right)\\
&\qquad = \lim_{t\to\infty}\max_{x_0\in B}\left|\frac{1}{2}[\sin(t+t_0) - \cos(t+t_0)] + [x_0 - \frac{1}{2}(\sin t_0 - \cos t_0)]\, e^{-t}\right|,\\
&\qquad = \lim_{t\to\infty}\left|\frac{1}{2}[\sin(t+t_0) - \cos(t+t_0)]\right|, \qquad \forall t_0 \in \mathbb{R}.
\end{aligned}$$

For this there exists no limiting value. Hence $\hat{A}$ *(the origin) does not forward attract any bounded set* B *for any* $t_0 \in \mathbb{R}$*.*

ii) Pullback Attraction: *For each bounded set* B*, and initial time* $t_0 \in \mathbb{R}$*,*

$$\begin{aligned}
&\lim_{t\to\infty} H^*\left(\Phi_{(t,\theta_{-t}(t_0))}(B), A(t_0)\right)\\
&\qquad = \lim_{t\to\infty}\max_{x_0\in B}\left|\frac{1}{2}[\sin(t_0) - \cos(t_0)]\right.\\
&\qquad\qquad \left. +[x_0 - \frac{1}{2}(\sin(t_0-t) - \cos(t_0-t))]\, e^{-t}\right|,\\
&\qquad = \left|\frac{1}{2}[\sin(t_0) - \cos(t_0)]\right|,\\
&\qquad = 0 \qquad\qquad \forall t_0 = \frac{\pi}{4} + n\pi \quad (n = 1, 2, ...).
\end{aligned}$$

Hence the system pullback attracts solutions to the origin only at specific times as determined by the set of discrete values given by $\{t_0; t_0 = \pi/4 + n\pi\}$ *for* $n = 1, 2, ...$

2.1.3 Stability

The following definitions are an extension of those used in classical stability analysis for non-autonomous systems (refer to Section 1.3) with further application to families of sets and with consideration of both forward and pullback analysis.

Definition 2.1.4 (Forward Stability). *A family* $\hat{A} = \{A(p); p \in P\}$ *of uniformly bounded compact subsets of* E*, is said to be* **forward stable** *with respect to the cocycle* $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ *on* E *if for any* $\epsilon > 0$*, there exists a* $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ *such that for any* $p \in P$*,*

$$H^*(\Phi_{(t,p)}(\mathcal{N}_{\delta_p}(A(p)), A(\theta_t p)) < \epsilon, \qquad \forall t \geq 0. \tag{2.1}$$

Definition 2.1.5 (Pullback Stability). *A family* $\hat{A} = \{A(p); p \in P\}$ *of uniformly bounded compact subsets of* E*, is said to be* **pullback stable** *with respect to the cocycle* $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ *on* E *if for any* $\epsilon > 0$ *there exists a* $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ *so that for any* $p \in P$*,*

$$H^*(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(A(\theta_{-t}p)), A(p)) < \epsilon, \qquad \forall t \geq 0. \tag{2.2}$$

Definition 2.1.6 (Complete Stability). *If* $\hat{A}$ *is both* pullback *and* forward stable*, then it is said to be* **completely stable**.

If the $\hat{\delta}$ in any of the above definitions are independent of the parameter p (that is, the $\delta_p = \delta$ for some $\delta > 0$ and all $p \in P$), then the respective stability of $\hat{A}$ is said to be **uniform**.

We say $\hat{A}$ is **positively invariant** under Φ if for each $p \in P$, and all $t \geq 0$,

$$\Phi_{(t,p)}(A(p)) \subseteq A(\theta_t p).$$

The property of positive invariance for stable sets in autonomous dynamical systems is also valid for both pullback and forward stable families in a non-autonomous dynamical system.

Theorem 2.1.1. *If* $\hat{A}$ *is pullback/forward/completely stable, then it is* **positively invariant** *under* Φ*.*

Proof: Assume that $\hat{A}$ is either pullback or forward stable (or completely) but is not positively invariant. Then for some $p \in P$, there exists a $t > 0$ such that

$$\Phi_{(t,p)}(A(p)) \nsubseteq A(\theta_t p).$$

Hence there exists an $a \in A(p)$ and an $\epsilon > 0$ such that

$$\mathrm{dist}(\Phi_{(t,p)}(a), A(\theta_t p)) > \epsilon. \tag{2.3}$$

i) $\hat{A}$ is pullback stable - Since $\hat{A}$ is pullback stable, there exists a $\delta_{\theta_t p}(\epsilon) > 0$ that ensures pullback stability at $\theta_t p$. Since $a \in \mathcal{N}_{\delta_{\theta_t p}}(A(p))$, then by pullback stability

$$\mathrm{dist}(\Phi_{(t,p)}(a), A(\theta_t p)) \leq H^*(\Phi_{(t,p)}(\mathcal{N}_{\delta_{\theta_t p}}(A(p)), A(\theta_t p)) < \epsilon,$$

which contradicts the initial assumption (2.3). Hence $\hat{A}$ must be positively invariant.

ii) $\hat{A}$ is forward stable - Since $\hat{A}$ is forward stable, there exists a $\delta_p > 0$ that ensures forward stability from p. Since $a \in \mathcal{N}_{\delta_p}(A(p))$, then by forward stability

$$\mathrm{dist}(\Phi_{(t,p)}(a), A(\theta_t p)) \leq H^*(\Phi_{(t,p)}(\mathcal{N}_{\delta_p}(A(p)), A(\theta_t p)) < \epsilon,$$

which similarly contradicts (2.3). Hence $\hat{A}$ is positively invariant.

The case for complete stability follows immediately from either of the above arguments for pullback or forward stability.

□

Example 2.1.2. [*Complete Stability*]
Consider the differential equation

$$\dot{x} = \frac{\cos(\mathsf{t})}{(2 + sin(\mathsf{t}))}\left[-x + arctan(\mathsf{t})\right] + \frac{1}{(1 + \mathsf{t}^2)}, \tag{2.4}$$

which has solutions given by

$$\Phi_{(t,t_0)}(x_0) = arctan(t + t_0) + \frac{(2 + sin(t_0))}{(2 + sin(t + t_0))}(x_0 - arctan(t_0)).$$

Analysing stability of solutions with initial state x_0 to the family $\hat{A} = \{A(\mathsf{t}); \mathsf{t} \in \mathbb{R}\}$ where $A(\mathsf{t}) = \arctan(\mathsf{t})$, it is easy to see that $\hat{A}$ is uniformly forward stable due to the (uniformly) bounded factor in the second term.

Alternatively, if we consider pullback analysis of solutions to a fixed and arbitrary choice of t_0,

$$\Phi_{(t,t_0-t)}(x_0) = A(t_0) + \frac{(2 + sin(t_0 - t))}{(2 + sin(t_0))}(x_0 - A(t_0 - t)).$$

Given any $\epsilon > 0$, we choose $\delta_{t_0} = (2 + sin(t_0))\epsilon/3$ so that for all $x_0 \in \mathcal{N}_{\delta_{t_0}, \hat{A}}$,

$$H^*(\Phi_{(t,t_0-t)}(x_0), A(t_0)) \le \epsilon.$$

for all $t > 0$. As a result, $\hat{A}$ is pullback stable. In fact, we may choose $\delta = \epsilon/3$ independent of t_0 and thus the pullback stability is uniform. As it is both pullback and forward stable, $\hat{A}$ is completely stable. Note, however that the stability is not asymptotic as the second term never vanishes completely in either a forward or a pullback sense. Solutions with initial values $(-1, -30)$ and $(-2, -30)$ were simulated and graphed. Refer to Figure 2.2.

□

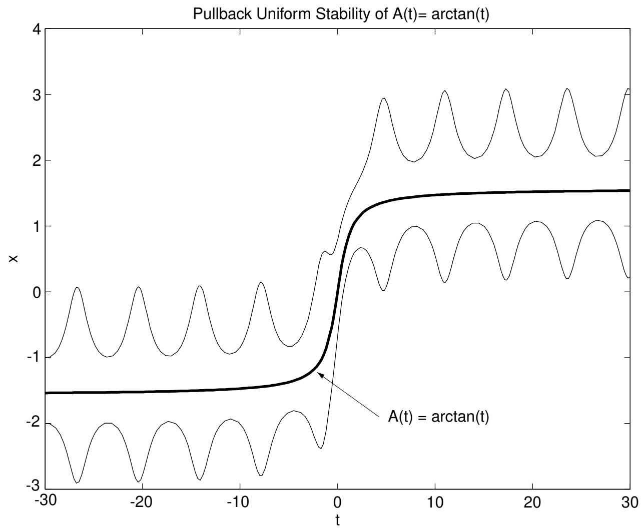

Figure 2.2: Complete Stability of $\hat{A}$ for Example 2.1.2

2.1.4 Asymptotic Stability

The concepts of classical asymptotic stability (presented in Section 1.3) may also be used in a similar manner as a basis for extending the fundamentals of asymptotic theory to families of sets and an encompassing theory inclusive of pullback analysis. This is formalised below.

Definition 2.1.7 (Forward Asymptotic Stability).
A family $\hat{A} = \{A(p); p \in P\}$ *of uniformly bounded compact subsets of* E*, is said to be* **forward asymptotically stable** *with respect to the cocycle* $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ *on* E *if it is* forward stable *and if there exists a* $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ *so that for any* $p \in P$*,* $x_0 \in \mathcal{N}_{\delta_p}(A(p))$*,*

$$\lim_{t\to\infty} H^*(\Phi_{(t,p)}(x_0), A(\theta_t p)) = 0. \tag{2.5}$$

Alternatively we may write that $\hat{A}$ is *forward asymptotically stable* if there exists a $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ so that for each $\epsilon > 0$, and $x_0 \in \mathcal{N}_{\delta_p}(A(p))$, there is a $T(x_0, p, \epsilon)$ such that

$$\mathrm{dist}(\Phi_{(t,p)}(x_0), A(\theta_t p)) < \epsilon, \qquad \forall t > T.$$

The case for *pullback asymptotic stability* however is a little more complicated. If the family $\hat{A}$ is varying with p, then the initial state x_0 may not always remain within the boundaries of $\mathcal{N}_{\delta,\hat{A}}$ as it is pulled back.

For example, consider the introductory Example 1.4.1, where $\hat{A}$ was a sinusoidal attractor family, and ignore for the moment that the attraction is global. A fixed *local* δ-neighbourhood of $\hat{A}$ will vary sinusoidally also, so an initial state may not always remain within the neighbourhood for all time values.

As a result we shall consider attraction of arbitrary sequences of initial states $\hat{x}_p = \{x(\theta_{-t}p) \in \mathcal{N}_{\delta_p}(A(\theta_{-t}p)); t \geq 0\}$ rather than a single fixed initial state x_0. For any $p \in P$, every element of the sequence is entirely contained within the δ_p-neighbourhood of $\hat{A}$. We thus define *pullback asymptotic stability* as follows.

Definition 2.1.8 (Pullback Asymptotic Stability).
A family $\hat{A} = \{A(p); p \in P\}$ *of uniformly bounded compact subsets of* E*, is said to be* **pullback asymptotically stable** *with respect to the cocycle* $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ *on* E *if it is* pullback stable *and if there exists a* $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ *so that for any* $p \in P$*,* $\hat{x}_p \in \mathcal{N}_{\delta_p,\hat{A}}$*,* $x(\theta_{-t}p) \in \hat{x}_p$*,*

$$\lim_{t\to\infty} H^*(\Phi_{(t,\theta_{-t}(p))}(x(\theta_{-t}p)), A(p)) = 0. \tag{2.6}$$

Again, it may be equivalently said that $\hat{A}$ is *pullback asymptotically stable* if there exists a $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ so that for each $\epsilon > 0$, and $\hat{x}_p \in \mathcal{N}_{\delta_p,\hat{A}}$, there is a $T(\hat{x}_p, p, \epsilon)$ such that

$$\mathrm{dist}(\Phi_{(t,p)}(x(\theta_{-t}p)), A(p)) < \epsilon, \qquad \forall t > T.$$

Remark 1 If a single δ-neighbourhood of $\hat{A}$ (that is, $\hat{\delta} = \delta$), and $T = T(\epsilon)$ can be chosen in the above definitions for forward(pullback) asymptotic stability, then $\hat{A}$ is said to be *uniformly forward asymptotically stable* (respectively pullback).

Remark 2 If in the above definitions T can be chosen so that $T = T(p, \epsilon)$ only, then $\hat{A}$ is said to be *forward(pullback) equi-asymptotically stable.*

Definition 2.1.9 (Complete Asymptotic Stability)**.** *If $\hat{A}$ is both forward and pullback asymptotically stable, then $\hat{A}$ is said to be* **completely asymptotically stable**.

The following three examples illustrate the differences between the three modes of asymptotic behaviour. Forward attraction is illustrated in Figure 2.3, pullback attraction in Figure 2.4, and complete attraction in Figure 2.5.

Example 2.1.3. [*Forward Asymptotic Stability without Pullback Attraction*] Consider the differential equation

$$\dot{x} = \frac{2\mathsf{t}}{(1+\mathsf{t}^2)}\left[-x + (\tanh(\mathsf{t}/2))\right] + \frac{2e^{-\mathsf{t}}}{(1+e^{-\mathsf{t}})^2}. \tag{2.7}$$

Defining $A(\mathsf{t}) = \tanh(\mathsf{t}/2)$ (a bipolar sigmoidal function) as the family of sets $\hat{A}$, then the solution to the ODE above with initial time t_0 and initial state x_0, is

$$\Phi_{(t,t_0)}(x_0) = A(t+t_0) + \frac{(1+(t_0)^2)}{(1+(t+t_0)^2)}(x_0 - A(t_0)).$$

Considering pullback attraction of solutions from initial state x_0 to $\hat{A}$ at a fixed element $A(t_0) \in \hat{A}$ we have

$$\Phi_{(t,t_0-t)}(x_0) = A(t_0) + \frac{(1+(t_0-t)^2)}{(1+(t_0)^2)}(x_0 - A(t_0-t)).$$

Taking the limit as $t \to \infty$ in both cases it is obvious that $\hat{A}$ forward attracts solutions, but fails to pullback attract solutions. It is also forward stable, and hence forward asymptotically stable. Several solutions for the differential equation were simulated and plotted in Figure 2.3, to display the system's pullback instability.

Note that forward convergence implies 'eventual' attraction. It cannot guarantee attraction at any other time. For example, it does not characterise the lack of attraction within this system to $A(0)$.

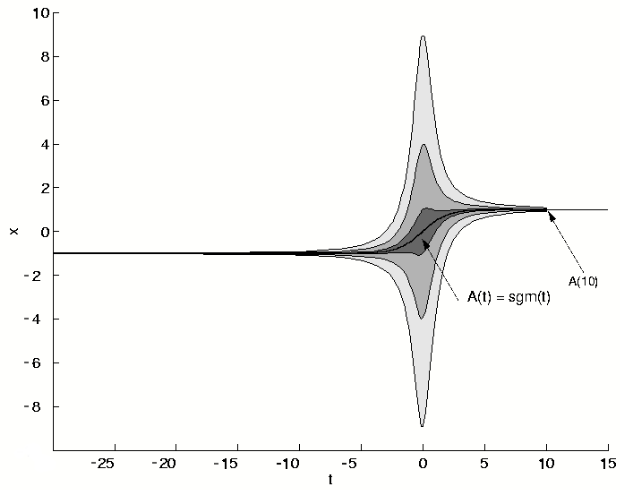

Figure 2.3: Forward Asymptotic Stability without Pullback Convergence

$\square$

Example 2.1.4. [*Pullback Asymptotic Stability without Forward Convergence*] This example is a case of a pullback asymptotically stable set which fails to exhibit any forward asymptotic behaviour. Given the differential equation

$$\dot{x} = \frac{-2\mathsf{t}}{(1+\mathsf{t}^2)}\left[-x + (\tanh(\mathsf{t}/2))\right] + \frac{2e^{-\mathsf{t}}}{(1+e^{-\mathsf{t}})^2}. \tag{2.8}$$

If we denote $A(\mathsf{t}) = \tanh(\mathsf{t}/2)$, then the solution to the ODE above with initial time t_0, and initial state x_0, is

$$\Phi_{(t,t_0)}(x_0) = A(t+t_0) + \frac{(1+(t+t_0)^2)}{(1+t_0^2)}(x_0 - A(t_0)).$$

From this it is easy to see that the family of singleton sets $\hat{A} = \{A(\mathtt{t}); \mathtt{t} \in \mathbb{R}\}$ fails to exhibit either of the usual characteristics of forward stability or asymptotic stability. Let us now consider pullback attraction of solutions from initial state x_0, to $\hat{A}$ at a fixed element $A(t_0) \in \hat{A}$

$$\Phi_{(t,t_0-t)}(x_0) = A(t_0) + \frac{(1+t_0^2)}{(1+(t_0-t)^2)}(x_0 - A(t_0-t)).$$

Taking the limit for some arbitrary $t_0, x_0 \in \mathbb{R}$ we find the second term vanishes, and hence

$$\lim_{t\to\infty} H^*(\Phi_{(t,t_0-t)}(x_0), A(t_0)) = 0.$$

Thus $\hat{A}$ is pullback attracting. It is also pullback stable. To see this, given $\epsilon > 0$, then there exists a $\delta_{t_0} = \epsilon/(1+t_0^2)$ so that for all $x_0 \in \mathcal{N}_{\delta_{t_0},\hat{A}}$, and for all $t \geq 0$,

$$H^*(\Phi_{(t,t_0-t)}(x_0), A(t_0)) \leq \epsilon$$

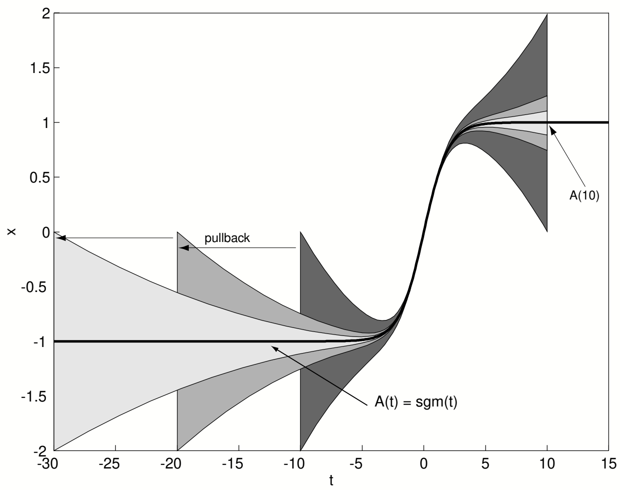

Figure 2.4: Pullback Asymptotic Stability without Forward Convergence

Evolution of solutions lying in the interval $[-2, 0]$ to $t_0 = 10$ from initial times progressively further back have been plotted, see Figure 2.4. It is easy to see that solutions become characteristically unstable once they have traversed past the switching region in the sigmoidal function (the pullback asymptotically stable set). However the interesting behaviour for this system is the fast attraction that occurs in and just preceding the switching region. It is here that pullback analysis describes the attractive behaviour to a fixed point within this region in detail and much more easily than with conventional techniques.

□

Example 2.1.5. [*Complete Asymptotic Stability*]
We take a slightly different form of the differential equation used in the preceding Example (2.8), and (2.7),

$$\dot{x} = [-x + (\tanh(\mathtt{t}/2))] + \frac{2e^{-\mathtt{t}}}{(1+e^{-\mathtt{t}})^2} \tag{2.9}$$

Again we have $A(\mathtt{t}) = \tanh(\mathtt{t}/2)$ as our family of sets under investigation, and the solution to the ODE above is given by

$$\Phi_{(t,t_0)}(x_0) = A(t+t_0) + e^{(-t)}(x_0 - A(t_0)).$$

Pulling back solutions x_0 to t_0 we have

$$\Phi_{(t,t_0-t)}(x_0) = A(t_0) + e^{-t}(x_0 - A(t_0 - t)).$$

Hence, convergence in both the forward and pullback sense occurs at an exponential rate. Together with forward and pullback stability of $\hat{A}$ (easily shown), $\hat{A}$ is thus shown to be *completely asymptotically stable*. See Figure 2.5 for a representative simulation of the system's behaviour.

□

2.1.5 Uniformity of Stability

A sufficient condition for stability or asymptotic stability of a family of sets $\hat{A} = \{A(p); p \in P\}$ to be completely stable, or completely asymptotically stable respectively, is that of uniformity. This is shown through the following lemmas.

Lemma 2.1.1. *If $\hat{A} = \{A(p); p \in P\}$ is uniformly pullback stable then it is completely stable.*

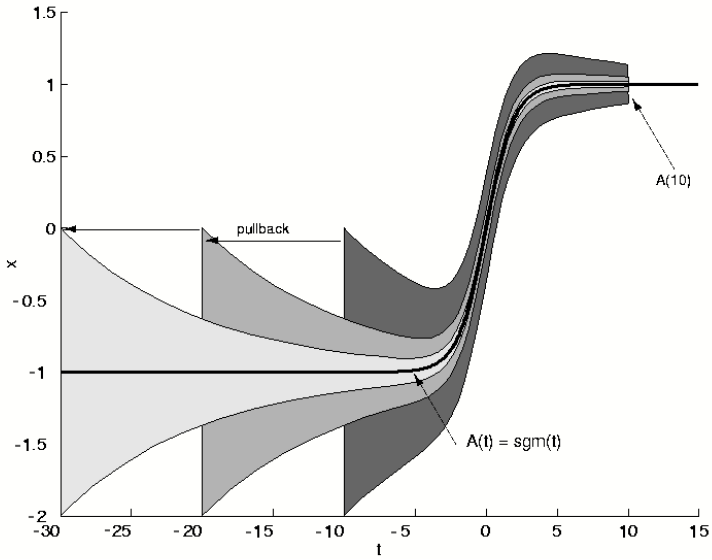

Figure 2.5: Complete Asymptotic Stability

Proof: We show by contradiction that $\hat{A}$ is uniformly forward stable, and hence by definition completely stable.

Assume $\hat{A}$ is not uniformly forwards stable, but uniformly pullback stable. Then for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ as defined for uniform pullback stability. Now, since $\hat{A}$ is not uniformly forwards stable, there exists at least one initial state within the neighbourhood system $\hat{\mathcal{N}}_{\delta,\hat{A}}$ that will escape an ϵ-neighbourhood of $\hat{A}$ at some later time. That is, there exists a $p \in P$, $t^* \in \mathbb{R}^+$ and at least one initial state $x_0 \in \mathcal{N}_\delta(A(p))$ such that

$$\mathrm{dist}(\Phi_{(t^*,p)}(x_0), A(\theta_{t^*}p)) = \epsilon.$$

However by uniform pullback stability at $\theta_{t^*}p$, it must necessarily be that

$$\mathrm{dist}(\Phi_{(t^*,p)}(x_0), A(\theta_{t^*}p)) < \epsilon.$$

This is a contradiction. Hence $\hat{A}$ must be uniformly forwards stable, and thus completely stable.

$\square$

Lemma 2.1.2. *If $\hat{A} = \{A(p); p \in P\}$ is uniformly forwards stable then it is completely stable.*

Proof: Similarly, we will show by contradiction that $\hat{A}$ is uniformly pullback stable, and hence by definition completely stable.

Assume $\hat{A}$ is not uniformly pullback stable, but uniformly forwards stable. Then for each $\epsilon > 0$, there exists a $\delta = \delta(\epsilon)$ as defined for uniform forward stability.

Now, since $\hat{A}$ is not pullback stable, there exists at least one initial sequence within the neighbourhood system $\hat{\mathcal{N}}_{\delta,\hat{A}}$ that will escape an ϵ neighbourhood of $\hat{A}$ after being pulled back from some previous time. That is, there exists a $p \in P$, $t^* \in \mathbb{R}^+$ and at least one initial sequence $\hat{x}_p$ with some initial value $x(\theta_{-t^*}p) \in \hat{x}_p$ such that

$$\text{dist}(\Phi_{(t^*,\theta_{-t^*}p)}(x(\theta_{-t^*}p)), A(p)) = \epsilon.$$

However by uniform forward stability from $\theta_{-t^*}p$,

$$\text{dist}(\Phi_{(t^*,\theta_{-t^*}p)}(x(\theta_{-t^*}p)), A(p)) < \epsilon.$$

Hence a contradiction. Consequently $\hat{A}$ must be uniformly pullback stable, and thus completely stable.

□

Lemma 2.1.3. *If $\hat{A} = \{A(p); p \in P\}$ is uniformly pullback asymptotically stable then it is completely asymptotically stable.*

Proof: $\hat{A}$ is uniformly pullback asymptotically stable. Thus, there exists a $\delta > 0$ so that for each $\epsilon > 0$ there exists a $T \in \mathbb{R}^+$ with $T = T(\epsilon)$ such that for all $p \in P$, $\hat{x}_p \in \hat{\mathcal{N}}_{\delta,\hat{A}}$ and $t > T$, we have for each $x(\theta_{-t}p) \in \hat{x}_p$,

$$\text{dist}(\Phi_{(t,\theta_{-t}p)}(x(\theta_{-t}p)), A(p)) < \epsilon.$$

Let δ be chosen as for uniform pullback asymptotic stability (above). We show that $\hat{A}$ is uniformly forward asymptotically stable by contradiction.

Assume that it is not uniformly forward asymptotically stable. That is for any $\epsilon > 0$, there exists some $p \in P$ and a sequence of values t_j, x_j with $t_j \to \infty$ as $j \to \infty$ and $x_j \in \mathcal{N}_\delta(A(p))$ such that

$$\mathrm{dist}(\Phi_{(t_j,p)}(x_j), A(\theta_{t_j}p)) > \epsilon.$$

However $\hat{A}$ is uniformly pullback asymptotically stable. Hence for all $t_j > T(\epsilon)$ ($T(\epsilon)$ as defined above),

$$\mathrm{dist}(\Phi_{(t_j,\theta_{-t_j}(\theta_{t_j}p))}(x_j), A(\theta_{t_j}p)) < \epsilon,$$

which is the required contradiction. Thus $\hat{A}$ is uniformly forwards asymptotically stable and hence completely asymptotically stable.

$\square$

Lemma 2.1.4. *If $\hat{A} = \{A(p); p \in P\}$ is uniformly forwards asymptotically stable then it is completely asymptotically stable.*

Proof: Let $\delta > 0$ be as defined for uniform forwards asymptotic stability, and assume $\hat{A}$ is not uniformly pullback asymptotically stable. Then for any $\epsilon > 0$, there exists some $p \in P$ and sequences t_j, x_j, with $t_j \to \infty$ as $j \to \infty$ and $x_j \in \mathcal{N}_\delta(A(\theta_{-t_j}p))$ such that

$$\mathrm{dist}(\Phi_{(t_j,\theta_{-t_j}p)}(x_j), A(p)) > \epsilon.$$

This leads to a contradiction since we require for uniform forwards asymptotic stability at each $\theta_{-t_j}p$ that

$$\mathrm{dist}(\Phi_{(t_j,\theta_{-t_j}p)}(x_j), A(p)) < \epsilon,$$

for all $t_j > T$, where $T = T(\epsilon)$ as defined for uniform forwards asymptotic stability. Hence $\hat{A}$ is uniformly pullback asymptotically stable and thus completely stable.

$\square$

It is often easier to determine the forwards nature of a dynamical system than its pullback characteristics. As a result, given the above lemmas, the uniform stability of $\hat{A}$ in a forwards sense is all that is required to verify complete stability of $\hat{A}$. This will be looked at in further detail upon consideration of Lyapunov functions in the following chapters.

Uniformity however, is not a necessary condition associated with complete stability or complete asymptotic stability. To see this consider Example 2.1.6 below.

Example 2.1.6. Consider the flows generated by the dynamical system defined by

$$\dot{x} = \begin{cases} -x & |x| \leq e^{-\mathsf{t}}, \\ -x + 2(x - sgn(x)e^{-\mathsf{t}}) & |x| > e^{-\mathsf{t}}. \end{cases}$$

See Figure 2.6. Here the parameter set $P = \mathbb{R}$ and the flows are distinctly different in the regions separated by Ω (defined by $|x| = e^{-\mathsf{t}}$). In the inner region, exponential attraction to the origin occurs, whilst in the outer region the dynamics may be represented in the form,

$$\frac{d}{d\mathsf{t}}(x - e^{-\mathsf{t}}) = (x - e^{-\mathsf{t}}).$$

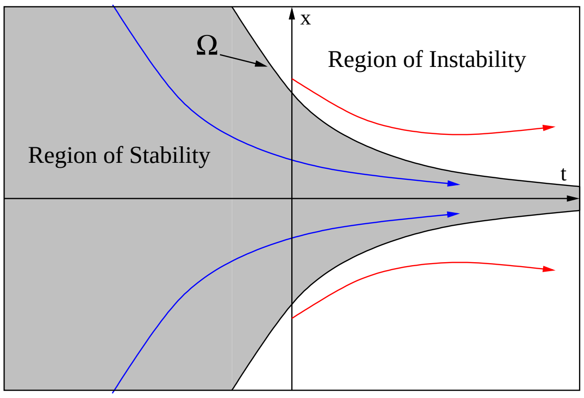

Figure 2.6: Complete Stability without Uniformity

Thus solutions diverge from the boundary at Ω ensuring instability in the outer region.

Solutions here are obviously completely stable with respect to the origin, that is they are both pullback and forward stable. However neither is uniform, since by increasing t the δ-neighbourhood must be chosen vanishingly smaller ensuring solutions do not begin in the region of instability beyond Ω.

$\square$

In this example the non-uniformity (with respect to p) in the neighbourhood of pullback and forward stability occurs for $\theta_t p$ as $t \to \infty$ (that is, where the neighbourhood of stability vanishes). It is of interest to observe in the general case for a completely stable family $\hat{A}$, that non-uniformity of the stable neighbourhood with respect to p cannot take place for $\theta_{-t}p$ as $t \to \infty$ (that is, the neighbourhood of stability may not vanish for $\theta_{-t}p$ as $t \to \infty$).

To show that this is indeed the case, suppose $\hat{A}$ is completely stable, with a neighbourhood system of stability defined by $\hat{\delta}$ that vanishes for $\theta_{-t}p$ as $t \to \infty$ (that is, $\delta_{\theta_{-t}p} \to 0$, as $t \to \infty$).

However, assuming such a neighbourhood is neccesary contradicts pullback stability at some p as no constant $\delta > 0$ could be chosen to assure that forward evolution of solutions from $\theta_{-t}p$ remain within an ϵ-neighbourhood of $A(p)$ at p.

An additional lemma concerning pullback asymptotic stability concerns uniformity of the local neighbourhood of attraction for all $p^* > p$ given knowledge of the neighbourhood of attraction at p.

Lemma 2.1.5. *If $\hat{A} = \{A(p); p \in P\}$ is pullback asymptotically stable with a local neighbourhood of attraction at any $p \in P$ given by $\mathcal{N}_{\delta_p}(A(p))$ for some $\delta_p > 0$, then $\delta_{\theta_t p} = \delta_p$ for all $t > 0$.*

Proof: Let $p, p^* \in P$ and set $p^* > p$ with $t^* > 0$ such that $\theta_{t^*}p = p^*$. Since $\hat{A}$ is pullback asymptotically stable, $A(p)$ pullback attracts solutions within the δ_p - neighbourhood $\hat{\mathcal{N}}_{\delta_p,\hat{A}}$.

We proceed to show that $A(p^*)$ also pullback attracts solutions within the δ_p-neighbourhood $\hat{\mathcal{N}}_{\delta_p,\hat{A}}$ by contradiction.

Assume $A(p^*)$ does not pullback attract all solutions within $\hat{\mathcal{N}}_{\delta_p,\hat{A}}$. Then there exists an initial sequence $\hat{x}_{p^*} \in \hat{\mathcal{N}}_{\delta_p,\hat{A}}$ such that for any $\epsilon > 0$ small enough, there exists a sequence of values $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that for each $t_n, x(\theta_{-t_n}p^*) \in \hat{x}_{p^*}$,

$$\operatorname{dist}(\Phi_{(t_n,\theta_{-t_n}p^*)}(x(\theta_{-t_n}p^*)), A(p^*)) > \epsilon. \tag{2.10}$$

$\hat{A}$ is also pullback stable, hence for $\epsilon > 0$ as given above, there exists a $\delta_{p^*} > 0$ as defined for pullback stability at p^*.

$A(p)$ pullback attracts $\hat{x}_{p^*}$ and hence there exists $T = T(\hat{x}_{p^*}, \delta_{p^*})$ so that for all $t_n > T + t^*$, $x(\theta_{-(t_n)}p^*) \in \hat{x}_{p^*}$,

$$\Phi_{(t_n - t^*,\theta_{-t_n}p^*)}(x(\theta_{-t_n}p^*)) \in \mathcal{N}_{\delta_{p^*}}(A(p)).$$

Hence, by pullback stability of p^*,

$$\operatorname{dist}(\Phi_{(t^*,p)}(\Phi_{(t_n - t^*,\theta_{-t_n}p^*)}(x(\theta_{-t_n}p^*))), A(p^*)) < \epsilon.$$

But this contradicts (2.10). Hence given $\delta_p > 0$ for some $p \in P$, we may indeed choose $\delta_{\theta_t p} = \delta_p$ for all $t > 0$.

$\square$

Note that the converse is not necessarily true. Refer to Example 2.2.1 in Section 2.2.

2.1.6 Conclusions - Non-Autonomous Attractors

Examples 2.1.4, 2.1.3, 2.1.5, all examined attraction to a family of sets $\hat{A} = \{A(t); A(\mathtt{t}) = \tanh(\mathtt{t}/2)\}$. In each case, the family of sets $\hat{A}$ is Φ-Invariant. In addition, solutions converge to $\hat{A}$ although in each case the definition of attraction varies.

Invariance and attraction represent properties analogous to that of the semi-group attractor for autonomous systems. Thus in seeking an appropriate extension for this concept in non-autonomous systems, each of these examples possess uniquely differing properties that may be exhibited by a generic form of an attractor within a non-autonomous environment.

Φ-Invariance + Pullback Convergence Pullback convergence guarantees the attraction of solutions to an element of $\hat{A}$, although as seen in Figure 2.4, there is no guarantee that solutions will stay close to the family $\hat{A}$ afterwards. However this type of structure is useful in situations where only 'capture' of solutions at a particular time is required. Refer to [32]. It also has the additional advantage in that certain limit set results for classical autonomous theory are extendable to these objects (Arnold [2], Schmalfuss [29]). These results will be covered in more detail later in the chapter.

Φ-Invariance + Forward Convergence Solutions can only be guaranteed to be close to the attractor for large values in time. That is, it is characteristic of systems with stable 'eventual' characteristics. The example illustrated in Figure 2.3 shows that there is no attraction to the elements of $\hat{A}$ near $A(0)$. Hence, a good reason to not consider the whole family $\hat{A}$ as a valid attractor. However, many systems are only concerned with long time dynamics, and in these cases such analysis is useful. Note that the usual limit set theorems do not hold for time-varying structures that possess forward attraction.

Φ-Invariance + Complete Convergence Obviously complete convergence guarantees convergence to the 'attracting' family $\hat{A}$ in every way. Solutions are guaranteed to attract to each element of the family as well as ensuring they stay close and even converge to $\hat{A}$ thereafter.

The latter structure provides the ideal extension for the semi-group attractor to non-autonomous systems although the others are still equally valid and useful concepts.

The following section develops the notion of an attractor for non-autonomous systems in detail.

2.2 Attractors for Non-Autonomous Systems

The preceding examples characterise two essential features required of an attractor, that of Φ-Invariance and attraction. They also highlight clearly that the notion of attraction in non-autonomous systems is distinctly characterised in both a forward and pullback sense.

Hence the construction of non-autonomous attractors used throughout the thesis will be developed with relevance to forward/pullback/complete attraction, in both a local and global context, and with application to families of sets.

Before proceeding, a mention of the *cocycle attractors* developed by P.Kloeden, J.Lorenz, and B.Schmalfuss [19, 21, 22, 28] (later denoted *pullback attractors* by P.Kloeden, D.Cheban, B.Schmalfuss and V.Kozyakin [6, 7, 15, 16, 17, 18]) is necessary to explain the divergence of the definitions for pullback attraction used in this thesis in comparison with the original concepts used by the authors mentioned above.

2.2.1 Pullback Attractors

The form of the pullback attractor utilised by Kloeden et. al. was developed from early ideas originating within random dynamical systems using cocycle theory and as a result was initially referred to as a cocycle attractor. A change in the terminology by Kloeden et. al. to that of a pullback attractor was made after the investigations in Section 2.1 by Stonier were made known.

Initial theory for cocycle attractors was developed in the area of global attraction for bounded compact subsets of $\mathbb{R}^d$, although a more general formulation for local and parametric dependent regions of attraction was devised. Throughout the papers above [6, 7, 15, 16, 17, 18] the following construction is used:

Definition 2.2.1 (Pullback Attractor : Kloeden, Cheban, Schmalfuss)**.** *A Φ-invariant family of nonempty compact subsets $\hat{A} = \{A(p); p \in P\}$ will be called a pullback attractor with respect to a basin of attraction system $\mathcal{D}_{att}$ if it satisfies the pullback attraction property*

$$\lim_{t\to\infty} H^*(\Phi_{(t,\theta_{-t}p)}(D_{\theta_{-t}p}), A(p)) = 0,$$

for all $p \in P$ *and all* $\hat{D} = \{D_p; p \in P\}$ *belonging to a basin of attraction system* $\mathcal{D}_{att}$.

That is, a collection of families of sets $\hat{D} = \{D_p; p \in P\}$ *where* D_p *is compact in* $\mathbb{R}^d$ *for each* $p \in P$ *with the property that* $\hat{D}^1 = \{D_p^{(1)}; p \in P\} \in \mathcal{D}_{att}$ *if* $\hat{D}^2 = \{D_p^{(2)}; p \in P\} \in \mathcal{D}_{att}$ *and* $D_p^{(1)} \subseteq D_p^{(2)}$ *for all* $p \in P$.

Obviously $\hat{A} \in \mathcal{D}_{att}$.

In fact, $A(p) \subset int\mathcal{D}_{att}(p)$*, where* $\mathcal{D}_{att}(p) := \bigcup_{\hat{D}=\{D_p;p\in P\}\in\mathcal{D}_{att}} D_p$*, for each* $p \in P$.

Although utilising such a basin of attraction determined by $\mathcal{D}_{att}$ is perfectly feasible for the consideration of forward attractors, there arise several difficulties when defining pullback attractors as in Definition 2.2.1.

Without any loss in generality, we will consider these difficulties with an analysis of pullback attraction to a *constant* set A (for ease of illustration).

i) For a basin of attraction system $\mathcal{D}_{att}$ as described above, it is assumed that the boundaries of attraction are identical when investigating pullback attraction to the attractor for differing values of $p \in P$. In essence, the basin of attraction utilised in this way is uniform with respect to $p \in P$. For some pullback attractors this is not the case, see Example 2.2.1.

The concept of a uniform basin of attraction also contradicts existing definitions for asymptotic stability [4, 42], where the attracting neighbourhood is well known to be dependent on t_0.

ii) It is possible to construct objects that are pullback attractors by Definition 2.2.1 yet exhibit distinctly divergent behaviour. Refer to Example 2.2.2.

Example 2.2.1. *Consider the one dimensional dynamical system generated by*

$$\dot{x} = \begin{cases} -e^{\mathsf{t}}, & \textit{if } x > e^{3\mathsf{t}}; \\ -x^{1/3}, & \textit{if } -e^{3\mathsf{t}} \leq x \leq e^{3\mathsf{t}}; \\ e^{\mathsf{t}}, & \textit{if } x < -e^{3\mathsf{t}}. \end{cases}$$

Here the parameter set is simply represented by the set of all initial time values, that is, $P = R$*. Smoothness of the derivative across the boundaries* $s^+(\mathsf{t}) = e^{3\mathsf{t}}$

and $s^-(\mathsf{t}) = -e^{3\mathsf{t}}$ *guarantees existence and uniqueness of solutions except at the origin which finite attracts solutions in a local neighbourhood.*

Due to the finite convergence caused by the dynamics given by $\dot{x} = -(3/2)x^{1/3}$ *in a close neighbourhood of the origin, it is clear that the origin has attractive properties and is in fact a global forward attractor (A in Figure 2.7).*

It will also be shown that A pullback attracts solutions in a local neighbourhood of the origin. The size of the neighbourhood however, is dependent on the point in time from which the solutions are pulled back.

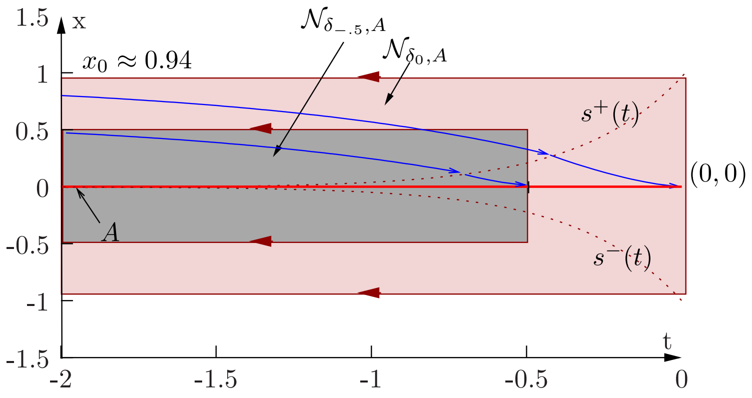

Figure 2.7: Variable Regions of Pullback Attraction

To investigate properties of pullback attraction, we first consider pullback attraction to A at $t_0 = 0$*, and* $t_0 = -0.5$*. Solutions in this region behave as depicted in Figure 2.7.*

Reverse integration of the positive solution from (0,0) (highlighted), indicates the trajectory approaches a limit at approximately $x_0 = 0.94$ *as shown. This value determines the maximum upper boundary for the neighbourhood of pullback attraction to A at* $t_0 = 0$ *since pulling back values* $x > x_0$ *means solutions do not traverse* $s^+(\mathsf{t})$ *early enough to completely finite attract to the origin (regardless of how far they have been pulled back).*

Now consider pullback attraction of x_0 *to A at* $t_0 = -0.5$ *as marked in Figure 2.7. Due to the nature of the system's dynamics, the trajectory originating from* x_0 *will never cross* $s^+(\mathsf{t})$*. Specifically, this implies that*

$$\Phi_{(t,-0.5-t)}(x_0) > s^+(-0.5), \qquad \forall t > 0.$$

Consequently A does not pullback attract x_0 to $t_0 = -0.5$.

Similarly, it can be shown that for any x_0 chosen, we can determine a time t_0 (far enough back) such that A cannot completely pullback attract x_0 regardless of how far the initial state is pulled back. Thus finding a suitable neighbourhood of pullback attraction to t_0 is strictly dependant on t_0 itself.

In conclusion, to properly define a local neighbourhood of pullback attraction for this example, we must choose a neighbourhood that shrinks as t_0 decreases. However, this involves constructing a neighbourhood that is not uniform with respect to t_0.

This then provides a counter-example of a dynamical system which clearly satisfies standard criteria for an attractor (in a pullback sense), yet does not have a fixed basin of attraction as required in Definition 2.2.1 (Kloeden et. al).

Example 2.2.2. *Consider the simple dynamics of $\dot{x} = x$. Solutions*

$$S_t(x_0) = x_0 e^t,$$

clearly diverge exponentially away from the origin, both in a forward and pullback sense, yet here we will show there exists a basin of attraction system that satisfies Definition 2.2.1, and as such defines the origin as a pullback attractor.

Let $P = \mathbb{R}$ be the parameter set for this example and propose $A = \{0\}$ as our pullback attractor.

Consider the basin of attraction system defined by $\mathcal{D}_{att}$ for which every family of sets $\hat{D} = \{D_\tau; \tau \in \mathbb{R}, D_\tau \subseteq [-e^{2\tau}, e^{2\tau}]\}$. Then for any t_0, and $\hat{D} \in \mathcal{D}_{att}$,

$$\lim_{t\to\infty} H^*(\Phi_{(t,t_0-t)}(D_{t_0-t}), A) = 0.$$

Thus by Definition 2.2.1, A must be a pullback attractor.

Interpreting A as a pullback attractor however, is obviously in contradiction with the basic nature of the dynamical system which exhibits no attractive properties whatsoever. Refer to Figure 2.8.

Conclusions

Definition 2.2.1 is valid in a global context, and it clearly attributes its construction to the extension of ideas developed for global cocycle attractors. However,

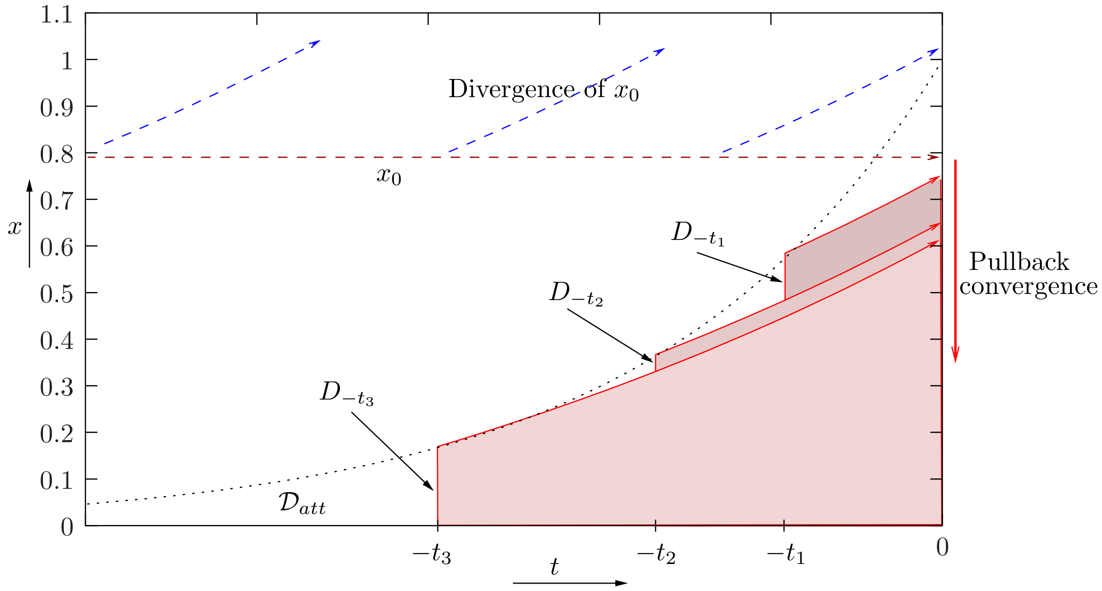

Figure 2.8: Improper Pullback Attractor

through the examples given it is clear that it inadequately defines the conditions required by local pullback attractors, and attractors with parametric dependent regions of attraction.

It was in part the problems with this construction that motivated the investigations of Section 2.1 and the redevelopment of several of the Theorems to follow throughout the remainder of this chapter.

The remainder of the thesis diverges from Kloeden's work, as it incorporates the redefined concepts of pullback attraction, and also makes use of the additional definitions and theorems developed for pullback stability and pullback asymptotic stability.

2.2.2 Defining the Non-Autonomous Attractor

The attractor definitions below are a modification of the original cocycle attractor concept and a natural extension of the ideas in Section 2.1. They are *valid and applicable in both a local or global context* and also combine the concepts of forward/pullback and complete attraction together in a comprehensive manner that identifies the uniqueness of each property and retain the underlying axioms that form the basis of classical asymptotic theory (refer to [4, 42]).

Definition 2.2.2 (Forward Attractor)**.** *A family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is called a* **forward attractor** *for the cocycle $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ on E if there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{\hat{\delta},\hat{A}} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$ the forward attractor $\hat{A}$ satisfies two properties,*

$$\Phi_{(t,p)}\left(A(p)\right) = A(\theta_t(p)) \qquad \text{for each} \quad t \in \mathbb{R}^+ \qquad (\Phi\text{-Invariance}) \tag{2.11}$$

$$\lim_{t\to\infty} H^*\left(\Phi_{(t,p)}(\mathcal{N}_{\delta_p}A(p), A(\theta_t p)\right) = 0 \qquad (\text{Forwards Convergence}) \tag{2.12}$$

Definition 2.2.3 (Pullback Attractor)**.** *A family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is called a* **pullback attractor** *for the cocycle $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ on E if there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{\hat{\delta},\hat{A}} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$ the pullback attractor $\hat{A}$ satisfies two properties,*

$$\Phi_{(t,p)}\left(A(p)\right) = A(\theta_t(p)) \qquad \text{for each} \quad t \in \mathbb{R}^+ \qquad (\Phi\text{-Invariance}) \tag{2.13}$$

$$\lim_{t\to\infty} H^*\left(\Phi_{(t,\theta_{-t}(p))}(\mathcal{N}_{\delta_p}A(\theta_{-t}(p)), A(p)\right) = 0 \qquad (\text{Pullback Convergence}) \tag{2.14}$$

Definition 2.2.4 (Complete Attractor)**.** *A family $\hat{A} = \{A(p); p \in P\}$ is called a* **complete attractor** *if it satisfies the properties for both a forward and pullback attractor.*

Remark 1: The Φ-Invariance property is equivalent in both a forwards and pullback context and is a generalisation of the invariance property for autonomous semi-group attractors.

Remark 2: Both the forward and pullback convergence properties (2.12, 2.14), are simply the requirement that a neighbourhood system $\hat{N}_{\hat{\delta},\hat{A}}$ exists that converges to the attractor $\hat{A}$ in either the forward, or pullback sense respectively. The rate of attraction is determined by $T = T(p, \epsilon)$ only, that is it is independent of the initialising state or sequence. In this respect the asymptotic attraction must necessarily be equi-asymptotic. It is this property that also assures the stability of a non-autonomous attractor.

The definition for all three variations of the attractor reduce to that of the definition introduced earlier (Definition 1.2.4) for a semi-group attractor when P is a singleton set (as is the case when the dynamical system is autonomous). Then the cocycle is in fact a semi-group, and the attractor $\hat{A}$, coincides with the semi-group attractor A_0 for each p. It is important to see here that the defining characteristics of these non-autonomous attractors implicitly retain all the properties and characteristics of semi-groups and their attractors when applied to autonomous systems.

The forward attractor also reduces to a form that retains all the classical characteristics of asymptotic behaviour presented in Section 1.3 when it is a constant set. That is $A(p) = A_0$, for all $p \in P$, or more briefly $\hat{A} = A_0$.

Also, if the forward attractor $\hat{A}$ attracts all bounded subsets $D \subset E$, then it is defined to be a **global forward attractor**. A similar definition can be made for **global pullback attractors** and **global complete attractors**.

2.2.3 Examples

The three examples in the previous section illustrating asymptotic behaviour, Examples 2.1.3, 2.1.4, and 2.1.5, all possess attractors. In each example $\hat{A} = \{A(t); A(\mathsf{t}) = \tanh(\mathsf{t}/2)\}$ was found to be forward (pullback/completely respectively) asymptotically stable. However, since $\hat{A}$ is also Φ-Invariant in each case, $\hat{A}$ is in an attractor too.

The following example constructs a complete attractor within a generalised dynamical system. It ensures tracking of solutions to a time dependent path through the state space.

Example 2.2.3. Consider the linearised differential equation with input control u,

$$\dot{\underset{\sim}{x}} = B(\mathsf{t})\,\underset{\sim}{x} + \underset{\sim}{u}\,. \tag{2.15}$$

Suppose we wish the system to converge to a time dependent path described parametrically by the vector $\underset{\sim}{c}(\mathsf{t})$, with $\mathsf{t} \in \mathbb{R}$. Given $\underset{\sim}{c} \in C^1$ we define a control by

$$\underset{\sim}{u} = k\,\underset{\sim}{c}(\mathsf{t}) + \dot{\underset{\sim}{c}}(\mathsf{t}) - (kI + B(\mathsf{t}))\,\underset{\sim}{x},$$

where $k > 0$ is some constant. Solutions to (2.15) with this control are

$$\Phi_{(t,t_0)}(\underset{\sim}{x}_0) = \underset{\sim}{c}(t) + e^{-kt}[\underset{\sim}{x}_0 - \underset{\sim}{c}(t_0)],$$

which are similar to those obtained previously in Example 2.1.5. Following a similar forward and pullback analysis of the problem we can see that here there exists a complete attractor $\hat{A} = \{A(\mathsf{t}); \mathsf{t} \in \mathbb{R}\}$ defined by

$$A(\mathsf{t}) = \underset{\sim}{c}(\mathsf{t}).$$

In addition, tracking onto the desired path will occur at an exponential rate.

The tracking problem was simulated for a periodic path in two dimensional space expressed in polar co-ordinates by

$$r(\mathsf{t}) = 1 + 0.2\cos(8\mathsf{t}),$$
$$\theta(\mathsf{t}) = \mathsf{t}.$$

The system was initialised at the origin, and the exponential convergence to the path can be seen to occur within a relatively short time, see Figure 2.9.

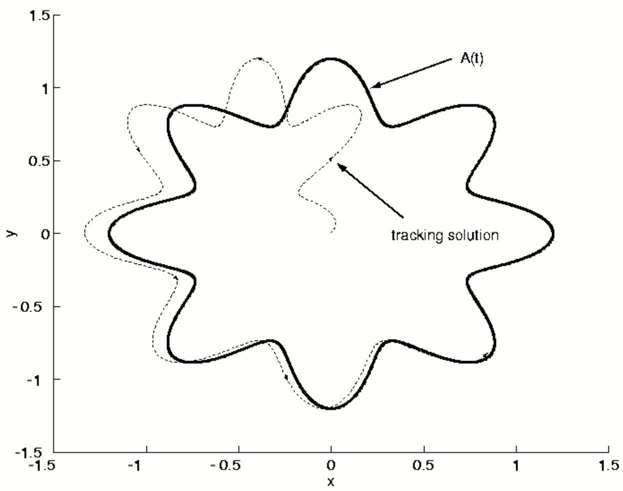

Figure 2.9: Complete Attractor in Tracking Control

□

2.2.4 Attractors and Asymptotic Stability

The following theorem and its lemma provide an equivalent counterpart in non-autonomous dynamical systems for the relationship between semi-group attractors and asymptotically stable sets in autonomous systems given in Theorem 1.2.1.

Theorem 2.2.1. *A pullback attractor is pullback equi-asymptotically stable.*

Proof: A pullback attractor by definition automatically satisfies the pullback attraction requirement for pullback equi-asymptotic stability, hence it is only required to show that it is pullback stable.

Assume that it is not pullback stable and consider some arbitrary value of $\epsilon > 0$, and any $p \in P$. Then there exists sequences $\delta_j \to 0$ and $t_j \to \infty$ as $j \to \infty$ such that for each j

$$\Phi_{(t,\theta_{-t}(p))}(\mathcal{N}_{\delta_j}(A(\theta_{-t}(p)), A(p) < \epsilon \qquad \forall t < t_j,$$

but with

$$\Phi_{(t_j,\theta_{-t_j}(p))}(\mathcal{N}_{\delta_j}(A(\theta_{-t_j}(p)), A(p) = \epsilon \tag{2.16}$$

However, $\hat{A}$ by definition pullback attracts an open neighbourhood system of itself, $\hat{\mathcal{N}}_{\hat{\delta},\hat{A}}$. Thus there exists a time $T = T(\epsilon, p)$ such that

$$\Phi_{(t,\theta_{-t}(p))}(\mathcal{N}_{\delta_p}(A(\theta_{-t}(p)), A(p)) < \epsilon \qquad \forall t > T.$$

Now for j large enough so that both $\delta_j < \delta_p$ and $t_j > T$ are satisfied, we have

$$\Phi_{(t_j,\theta_{-t_j}(p))}(\mathcal{N}_{\delta_j}(A(\theta_{-t_j}(p)), A(p)) < \epsilon$$

This provides the required contradiction with (2.16), and hence $\hat{A}$ must be pullback stable and consequently pullback asymptotically stable.

□

Lemma 2.2.1. *A forward (complete) attractor is forward (completely) equi-asymptotically stable.*

The proof for forward asymptotic stability follows along the same lines as that for the pullback result, and complete asymptotic stability (for a complete attractor) is simply a combination of the two results for the forward and pullback attractors respectively.

Note that a forward (pullback or complete) asymptotically stable set is not necessarily an attractor.

2.2.5 Comments

Some comment should be made here regarding the terminology used for the definitions of both asymptotic stability of sets and attractors. Much of the recent research into stability makes use of the terminology as presented above [6, 22, 28, 34].

However this is in contradiction with many older publications (most notably [4]). In these articles the situation is reversed. An attractor is defined as a set which simply attracts solutions and is not necessarily invariant and an asymptotically stable set is defined as an invariant set to which neighbouring solutions literally asymptote.

This latter terminology I believe is more apt as it characterises the very definitions of each. Nevertheless, this thesis will continue using the former to conform with that used in current publications.

2.3 Absorbing Neighbourhoods

It is possible to extend the concept of the absorbing set introduced in Definition 1.2.6 for autonomous dynamical systems to a broader classification of an absorbing neighbourhood for application within non-autonomous dynamical systems.

However, the applicability of using such absorbing neighbourhoods in a non-autonomous environment (compare with the Limit Set Theorem (1.2.3) for autonomous systems) is relevant only to objects possessing pullback asymptotically stable characteristics. In this situation, Theorem 1.2.3 may be extended with application to pullback attracting structures. A similar extension is not possible for non-autonomous systems with forward asymptotically stable properties. This is considered in more detail in Section 2.3.2.

It is also important to note that the use of absorbing neighbourhoods is identical to that of the original absorbing sets when applied to autonomous systems.

2.3.1 Pullback Absorbing Neighbourhoods

For generality we consider families of sets dependent on $p \in P$ that possess pullback absorbing properties. This allows us to conveniently describe motions around pullback attractors that may vary with $p \in P$. Thus we will refer to them as pullback absorbing neighbourhoods.

A set A is said to **pullback absorb** another set B at $p \in P$ if there exists a $T = T(p, B)$ such that

$$\Phi_{(t,\theta_{-t}(p))}(B) \subset A \qquad \forall t > T.$$

Similarly, a family of sets $\hat{A} = \{A(p) : p \in P\}$ is said to **pullback absorb** another family of sets $\hat{D} = \{D_p : p \in P\}$ at $p \in P$, if there exists a $T = T(p, \hat{D}) > 0$ such that,

$$\Phi_{(t,\theta_{-t}p)}(D_{\theta_{-t}(p)}) \subset A(p)) \qquad \forall t > T.$$

A *pullback absorbing neighbourhood* is then defined as

Definition 2.3.1 (Pullback Absorbing Neighbourhoods). *A family $\hat{B} = \{B(p); p \in P\}$ of uniformly bounded compact subsets of E, is called a* **Pullback Absorbing Neighbourhood** *for a cocycle $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ on E if it pullback absorbs a uniformly bounded δ - neighbourhood system of $\hat{B}$. That is, there exists an open δ-neighbourhood system $\hat{\mathcal{N}}_{\hat{\delta},\hat{B}}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ so that for each $p \in P$, there exists a $T_p > 0$ such that*

$$\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(B(\theta_{-t}(p))) \subset B(p) \qquad \forall t > T_p. \tag{2.17}$$

In many situations a single set B, for a dynamical system may be found which satisfies the requirements for a pullback absorbing neighbourhood for all p, that is $\hat{B} = B$. Indeed when this is the case many of the results associated with it become greatly simplified. The situation arising in Example 2.3.2 is such a case.

In certain cases it may be possible to find a pullback absorbing neighbourhood that is positively invariant, however it is usually easier and less restrictive to consider neighbourhoods which satisfy only the above definition. Note that a pullback absorbing neighbourhood automatically satisfies the following property.

Lemma 2.3.1. *If $\hat{B}$ is a pullback absorbing neighbourhood, then for each $p \in P$ there exists a $T_p > 0$ such that*

$$\Phi_{(t,\theta_{-t}(p))}(B(\theta_{-t}(p))) \subset B(p) \qquad \forall t > T_p.$$

Example 2.3.1. For this illustration we refer back to Example 1.4.1. A diagram illustrating the system's dynamics is repeated here for convenience (see Figure 2.10).

Finding a pullback absorbing neighbourhood that is positively invariant typically requires a detailed knowledge of the system's dynamics. Here however, a single constant set such as $B = [-2, 2]$, that is merely pullback absorbing for any $p \in P$, may be used to verify the existence of a pullback attractor. This will be shown shortly using Theorem 2.3.1, and is indeed a much simpler approach than finding a positively invariant family.

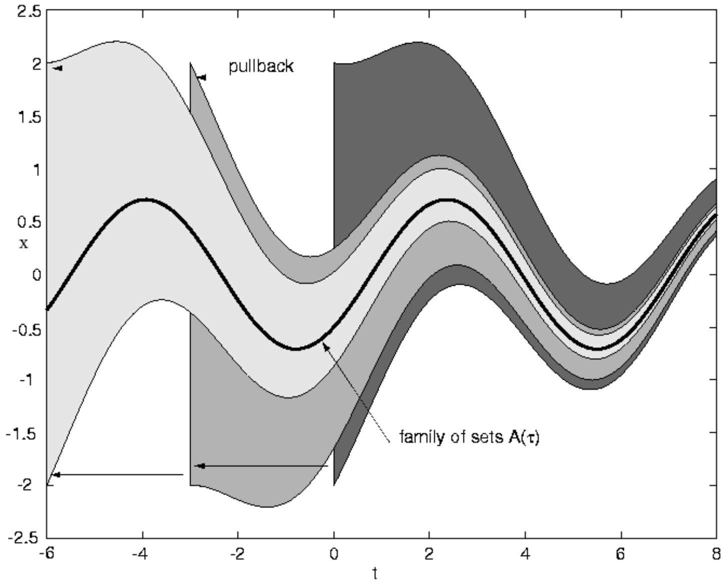

Figure 2.10: Introductory Example

□

Pullback absorbing neighbourhoods provide a means with which to identify pullback attractors in a similar fashion to that of absorbing sets for semi-group attractors.

Again, it is often much easier to find a pullback absorbing neighbourhood system rather than the pullback attractor itself. Indeed, in many systems the attractor cannot be found and written explicitly and an estimate based on the progression of an absorbing neighbourhood must be made.

The following theorem is an extension of Theorem 1.2.3 applied to pullback absorbing neighbourhoods for cocycles.

A similar theorem has been presented by B. Schmalfuss [29], however the construction for the pullback attractor differs slightly as was mentioned in Section 2.2. The key difference lies in the assumption of uniformity present in a neighbourhood of attraction for the attractor which (as has been shown) may not neccessarily exist. Positive invariance of the absorbing neighbourhood is also assumed [29]. Although technically more complicated, neither uniformity nor positive invariance are required here.

Theorem 2.3.1. *Let $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ be a cocycle of continuous mappings on E with a Pullback Absorbing Neighbourhood $\hat{B}$. Then there exists a pullback attractor $\hat{A} = \{A(p); p \in P\}$ uniquely determined for each $p \in P$ by*

$$A(p) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p))}. \tag{2.18}$$

Proof:

Proposing as our attractor the family of sets $\hat{A} = \{A(p); p \in P\}$ defined above we have

$$A(p) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p))}$$

Since $A(p)$ is an infinite intersection of nested compact sets, it must contain at least one accumulation point. Hence we may conclude $A(p)$ is non-empty. To show that this is indeed a pullback attractor, it must meet the requirements in Definition 2.2.3. This will be done in three steps.

i) *Uniform Boundedness and Compactness* : From (2.18) and using Lemma 2.3.1

$$\begin{aligned}
A(p) &= \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p))}, \\
&\subseteq \bigcap_{\tau \geq T_p} \overline{\bigcup_{t \geq \tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p))}, \\
&\subseteq \bigcap_{\tau \geq T_p} \overline{\bigcup_{t \geq \tau} B(p)}, \\
&= \overline{B(p)}.
\end{aligned}$$

Hence $\hat{A}$ is uniformly bounded since $\hat{B}$ is uniformly bounded with respect to p. The attractor set is also compact as it is an intersection of compact sets.

ii) *Pullback Property* : We need to find a pullback convergent neighbourhood system as in (2.14). First we shall show that $\hat{A}$ pullback attracts $\hat{B}$ at every $p \in P$. That is, for each $p \in P$

$$\lim_{t \to \infty} H^*(\Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p)), A(p)) = 0, \tag{2.19}$$

where $A(p)$ is defined as above.

Assume this is not the case. Then for some $\epsilon > 0$ there exists sequences $t_j \to \infty$ and $x_j \in \Phi_{(t_j,\theta_{-t_j}p)}(B(\theta_{-t_j}p))$ such that

$$\mathrm{dist}(x_j, A(p)) > \epsilon \qquad \forall j. \tag{2.20}$$

For large enough j, $x_j \in B(p)$ (Lemma 2.3.1). Now since $B(p)$ is compact, there exists a subsequence $t_{j'} \to \infty$ and an associated convergent subsequence, $x_{j'} \to x_0$ with $x_0 \in B(p)$. For any $\tau > 0$, the $x_{j'}$ satisfy

$$x_{j'} \in \bigcup_{t\geq\tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p)), \qquad t_{j'} > \tau.$$

As x_0 is the limit, we have for any $\tau \geq 0$,

$$x_0 \in \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p))},$$

and hence

$$x_0 \in \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-t}p)}(B(\theta_{-t}p))}.$$

That is, $x_0 \in A(p)$, which contradicts (2.20) and this implies $\mathrm{dist}(x_0, A(p)) \geq \epsilon$. As the choice of p was arbitrary, (2.19) holds true for all $p \in P$.

From (2.19), for any $\epsilon > 0$, and each $p \in P$, there exists a $t_1 = t_1(\epsilon, p)$ such that

$$H^*(\Phi_{(t_1,\theta_{-t_1}p)}(B(\theta_{-t_1}p), A(p)) < \epsilon.$$

Let us take $\hat{\mathcal{N}}_{\hat{\delta}^*,\hat{A}}$ as our neighbourhood system for the attractor, $\hat{A}$, defined by $\hat{\delta}^* = \{\delta_p^*; \delta_p^* = \delta_{\theta_{-t_1}(p)}, \delta_p \in \hat{\delta}\}$ where $\hat{\delta}$ defines $\hat{\mathcal{N}}_{\hat{\delta},\hat{B}}$, the associated $\hat{\delta}$-neighbourhood of $\hat{B}$. We need to show that $\hat{A}$ pullback attracts the system $\hat{\mathcal{N}}_{\hat{\delta}^*,\hat{A}}$.

Using the cocycle property, and because $B(p)$ is pullback absorbing, we can formulate attraction for elements of $\hat{\mathcal{N}}_{\hat{\delta}^*,\hat{A}}$

$$\begin{aligned}
&H^*(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p^*}(A(\theta_{-t}p)), A(p))\\
&\qquad \leq H^*(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p^*}(B(\theta_{-t}p)), A(p))\\
&\qquad = H^*(\Phi_{(t_1,\theta_{-t_1}p)} \circ \Phi_{(t-t_1,\theta_{-t}p)}(\mathcal{N}_{\delta_p^*}(B(\theta_{-t}p)), A(p))\\
&\qquad \leq H^*(\Phi_{(t_1,\theta_{-t_1}p)}(B(\theta_{-t_1}p), A(p))\\
&\qquad \leq \epsilon
\end{aligned}$$

for all $t > t_{(\theta_{-t_1}p)} + t_1$, where $t_{(\theta_{-t_1}p)}$ is the finite absorption time described in the definition of the pullback absorbing neighbourhood. Hence $\hat{A}$ satisfies the pullback property for each $p \in P$, that is,

$$\lim_{t\to\infty} H^*(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p^*}(A(\theta_{-t}p)), A(p)) = 0.$$

iii) Φ-*Invariance* : We are required to prove that the family of sets constituting $\hat{A}$ is equivariant as defined in (2.13). Consider an element of the family $A(\theta_{-t^*}p) \in \hat{A}$ for arbitrary p, and any $t^* > 0$

$$A(\theta_{-t^*}p) = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}. \tag{2.21}$$

First, we need to show

$$\begin{aligned}
&\Phi_{(t^*,\theta_{-t^*}p)}\left(\bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}\right),\\
&\qquad = \bigcap_{\tau\geq 0} \Phi_{(t^*,\theta_{-t^*}p)}\left(\overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}\right).
\end{aligned} \tag{2.22}$$

The inclusion "$\subset$" is trivial. To prove "$\supset$", let x be an element of the right hand side,

$$x \in \bigcap_{\tau\geq 0} \Phi_{(t^*,\theta_{-t^*}p)}\left(\overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p)))}\right).$$

Then for each $\tau > 0$, there exists

$$x_\tau \in \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))},$$

such that $x = \Phi_{(t^*,\theta_{-t^*}p)}(x_\tau)$. For τ large enough, $x_\tau \in B(\theta_{-t^*}p)$ due to Lemma 2.3.1. $B(\theta_{-t^*}p)$ is compact and hence there exists a subsequence $\tau' \to \infty$, and an associated convergent subsequence $x_{\tau'} \to x^*$ with $x^* \in B(\theta_{-t^*}p)$.

Now, given $\tau > 0$, we have

$$x_{\tau'} \in \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))} \qquad \forall \tau' > \tau.$$

Further because of closure,

$$x^* \in \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))},$$

for any $\tau > 0$. Hence

$$x^* \in \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}.$$

Finally, the limit and continuity of the cocycle implies that

$$\Phi_{(t^*,\theta_{-t^*}p)}(x^*) = x,$$

and so x also belongs to the left hand side. This verifies (2.22).

Returning to (2.21), using the result above and also making use of the cocycle property, we have

$$\begin{aligned}
&\Phi_{(t^*,\theta_{-t^*}p)}(A(\theta_{-t^*}(p))) \\
&\qquad = \Phi_{(t^*,\theta_{-t^*}p)}\left(\bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}\right), \\
&\qquad = \bigcap_{\tau\geq 0} \Phi_{(t^*,\theta_{-t^*}p)}\left(\overline{\bigcup_{t\geq\tau} \Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}\right), \\
&\qquad = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_{(t^*,\theta_{-t^*}p)}\Phi_{(t,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}, \\
&\qquad = \bigcap_{\tau\geq 0} \overline{\bigcup_{t\geq\tau} \Phi_{(t+t^*,\theta_{-(t+t^*)}p)}(B(\theta_{-(t+t^*)}p))}, \\
&\qquad = \bigcap_{\tau\geq t^*} \overline{\bigcup_{s\geq\tau} \Phi_{(s,\theta_{-s}p)}(B(\theta_{-s}p))},
\end{aligned}$$

where we have made the substitution $s = t + t^*$. Now, for all $\tau < t^*$,

$$\overline{\bigcup_{s\geq\tau} \Phi_{(s,\theta_{-s}p)}(B(\theta_{-s}p))} \supseteq \overline{\bigcup_{s\geq t^*} \Phi_{(s,\theta_{-s}p)}(B(\theta_{-s}p))}.$$

Hence

$$\begin{aligned}
&\Phi_{(t^*,\theta_{-t^*}p)}(A(\theta_{-t^*}p)) \\
&\qquad = \bigcap_{\tau\geq 0} \overline{\bigcup_{s\geq\tau} \Phi_{(s,\theta_{-s}p)}(B(\theta_{-s}p))} \\
&\qquad = A(p).
\end{aligned}$$

Thus the conditions for Φ-Invariance are satisfied.

□

Example 2.3.2. [*Attractor for a Perturbed Limit Cycle*]
Consider the two dimensional autonomous dynamical system

$$\begin{aligned}
\dot{x} &= y + x - x(x^2 + y^2), \\
\dot{y} &= -x + y - y(x^2 + y^2).
\end{aligned}$$

This system possesses an attractor A_0, being the stable limit cycle centred on the unit circle. Now, suppose the original system is perturbed slightly with a small non-autonomous perturbation resulting in the dynamical system defined by

$$\begin{aligned}
\dot{x} &= y + x - x(x^2 + y^2) + \epsilon\cos(2\mathsf{t}), \\
\dot{y} &= -x + y - y(x^2 + y^2) + \epsilon\cos(2\mathsf{t}).
\end{aligned}$$

A transformation to polar co-ordinates yields

$$\begin{aligned}
\dot{r} &= r(1 - r^2) + (\cos\theta + \sin\theta)\epsilon\cos(2\mathsf{t}), \\
\dot{\theta} &= -1 + \frac{1}{r^2}(\cos\theta - \sin\theta)\epsilon\cos(2\mathsf{t}).
\end{aligned}$$

If the perturbation parameter ϵ, is kept small, it could be expected that a pullback attractor may exist in the vicinity of the original attractor A_0. To begin our search for such a pullback attractor, let us start with the dynamics.

The θ dynamics remain rotating in the same direction if ϵ is small enough, so we need only be concerned with the radial variable. Similarly if we take ϵ small enough, let us say $\epsilon < 0.5$, then we have $\dot{r}$ positive for $r \leq 0.5$, and $\dot{r}$ negative for $r \geq 1.5$. Knowing this, we construct a toroidal shaped pullback absorbing neighbourhood for the non-autonomously perturbed system

$$B = \{(r, \theta); 0.5 \leq r \leq 1.5\}.$$

In this case, our pullback neighbourhood needs only a single set, and satisfies all the conditions required for a pullback absorbing neighbourhood.

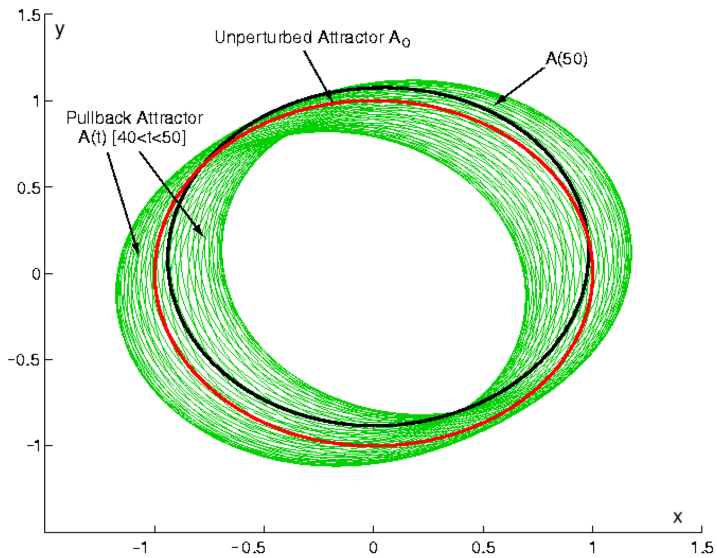

Figure 2.11: Pullback Attraction to a Non-Autonomously Perturbed Limit Cycle

By Theorem 2.3.1 we verify the existence of a pullback attractor, $\hat{A}$ contained within B. In fact, by numerically pullback integrating B to arrive at an estimation for (2.18), we obtain a good approximation for $\hat{A}$. Using this approach, initial states from the inner and outer boundaries of B were mapped (using a Runge-Kutta method) from a pulled back initial time towards a fixed final time.

This process was repeated for various values of the final time between $\mathtt{t} = 40$ and $\mathtt{t} = 50$, and the final approximations for $A(\mathtt{t})$ were plotted, see Figure 2.11. As can be seen the original fixed limit cycle has evolved with the non-autonomous perturbation to become a structure with similar characteristics, but now moving periodically (due to the nature of the $\cos(2t)$ perturbation) around the old limit cycle, A_0.

□

A converse result for the above theorem also holds under restricted conditions on the neighbourhood. If a non-autonomous dynamical system possesses a pullback attractor for which its pullback attracting neighbourhood system is known to be independent of p (that is, $\hat{\delta} = \delta$), then it can be shown that there exists an associated pullback absorbing neighbourhood. This condition effectively places a uniformity requirement on the pullback attractivity of the pullback attractor. This result is proved in [15], and reiterated in a similar fashion here for completeness.

Theorem 2.3.2. *Let $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ be a cocycle of continuous mappings on E with a pullback attractor $\hat{A} = \{A(p); p \in P\}$ that pullback attracts a δ-neighbourhood system $\hat{\mathcal{N}}_{\delta,\hat{A}}$ for some $\delta > 0$. Then there exists a Pullback Absorbing Neighbourhood $\hat{B}$, associated with $\hat{A}$.*

Proof:

By Theorem 2.2.1, the pullback attractor $\hat{A}$ is *pullback asymptotically stable*, and hence also *pullback stable*. If we take $\epsilon = \delta$ (where δ is such that $\hat{A}$ pullback attracts $\hat{\mathcal{N}}_{\delta,\hat{A}}$), then there exists a corresponding $\hat{\delta}^* = \{\delta_p^* \in \mathbb{R}^+; p \in P\}$ as in the definition for pullback stability.

We construct a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta}^*,\hat{A})}$ of $\hat{A}$ defined by $\hat{\delta}^*$ (note that this system will also serve as a pullback attracting neighbourhood system for the pullback attractor $\hat{A}$), and propose as our Pullback Absorbing Neighbourhood $\hat{B} = \{B(p); p \in P\}$ where

$$B(p) = \overline{\bigcup_{t \geq 0} \Phi_{(t,\theta_{-t}(p))}(\mathcal{N}_{\delta_p^*}(A(\theta_{-t}(p)))},$$

for each $p \in P$. To see that it is indeed a pullback absorbing neighbourhood for the pullback attractor $\hat{A}$ we must show it satisfies the conditions in Definition 2.3.1. This is accomplished in two stages.

i) *Boundedness and Compactness*: Given the pullback stability of $\hat{A}$ and the method of construction of $B(p)$, it is easy to see that $B(p) \subseteq \mathcal{N}_\delta(A(p))$ for every $p \in P$. Hence $\hat{B} \subseteq \mathcal{N}_{(\delta,\hat{A})}$ which is uniformly bounded and so $\hat{B}$ is also uniformly bounded. It is also compact due to the closure.

ii) *Pullback Absorption* : Note that $\hat{\mathcal{N}}_{\delta,\hat{A}}$ also qualifies as a δ - neighbourhood system for $\hat{B}$. We will proceed to show that $\hat{B}$ pullback absorbs this δ-neighbourhood system.

$\hat{A}$ pullback attracts $\hat{\mathcal{N}}_{\delta,\hat{A}}$, and hence for each $p \in P$, there exists a $T = T(\delta_p^*, p)$ such that

$$\begin{aligned}\Phi_{(t,\theta_{-t}(p))}(\mathcal{N}_\delta(A(\theta_{-t}(p))) &\subseteq \mathcal{N}_{\delta_p^*}(A(p)) \\ &\subseteq B(p),\end{aligned}$$

for all $t > T$. Thus $\hat{B}$ pullback absorbs the δ-neighbourhood system $\hat{\mathcal{N}}_{\delta,\hat{A}}$.

□

2.3.2 Forward Absorbing Neighbourhoods

A forward absorbing neighbourhood may be constructed in a similar manner to that above. However, analysis of a limiting object as in Theorem 2.3.1 cannot be undertaken in a similar fashion.

To see this, consider an appropriately defined neighbourhood with forward absorbing properties.

The set A is said to **forward absorb** another set B from $p \in P$ if there exists a $T = T(p, B)$ such that

$$\Phi_{(t,(p))}(B) \subset A \qquad \forall t > T.$$

Similarly, a family of sets $\hat{A} = \{A(p) : p \in P\}$ is said to **forward absorb** another family of sets $\hat{D} = \{D_p : p \in P\}$ at $p \in P$, if there exists a $T = T(p, \hat{D}) > 0$ such that,

$$\Phi_{(t,p)}(D(p)) \subset A(\theta_t p)) \qquad \forall t > T$$

A *forward absorbing neighbourhood* is then defined as

Definition 2.3.2 (Forward Absorbing Neighbourhood)**.** *A family $\hat{B} = \{B(p); p \in P\}$ of uniformly bounded compact subsets of E, is called a* **Forward Absorbing Neighbourhood** *for a cocycle $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ on E if it forward absorbs a uniformly bounded δ-neighbourhood system of $\hat{B}$. That is, there exists an open δ-neighbourhood system $\hat{\mathcal{N}}_{\hat{\delta},\hat{B}}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ so that for each $p \in P$, there exists a $T_p > 0$ such that*

$$\Phi_{(t,p)}(\mathcal{N}_{\delta_p}(B(p)) \subset B(\theta_t p) \qquad \forall t > T_p. \tag{2.23}$$

Example 2.3.3. Consider again Example 1.4.1. In this dynamical system the constant set $B = [-2, 2]$ satisfies the requirements for a forward absorbing neighbourhood. However, if the limiting set is calculated in a similar fashion as 2.18, we find

$$\begin{aligned} A &= \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \Phi_{(t,p)}(B)}, \\ &= [-1/\sqrt{2}, 1/\sqrt{2}]. \end{aligned}$$

A is not invariant and hence does not satisfy the conditions for a forward attractor.

□

It is not even possible to draw any conclusions regarding the existence of a forward attractor within a forward absorbing neighbourhood. This is illustrated in the following counter-example which possesses a forward absorbing neighbourhood that does not contain a forward attractor.

Example 2.3.4. Consider the non-autonomous dynamical system arising from the ODE

$$\dot{x} = \frac{10e^{-\mathsf{t}}}{1 + 10e^{-\mathsf{t}}}(-x + \tanh(\mathsf{t}/2)) + \frac{2e^{-\mathsf{t}}}{(1 + e^{-\mathsf{t}})^2}.$$

Its behaviour is similar to those examples presented in Section 2.1 and is shown in Figure 2.12.

We propose the set $B = [-1.5, 1.5]$ as an absorbing neighbourhood. By considering bounds on the derivative, it can be seen that $\dot{x}$ is always negative (and positive respectively) on the upper (and lower respectively) boundaries of the neighbourhood. Hence B is positively invariant and is both a pullback and forward absorbing neighbourhood. By Theorem 2.3.1, we can conclude there exists a pullback attractor contained within B. Determined analytically, the pullback attractor takes the form $A(\mathtt{t}) = \tanh(\mathtt{t}/2)$.

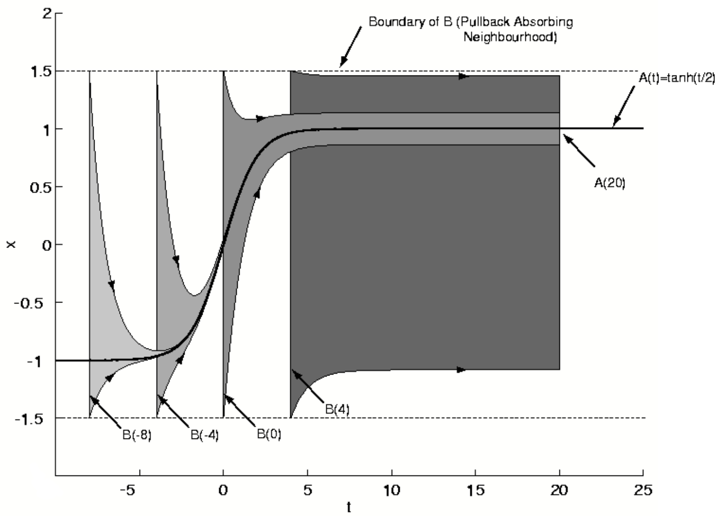

Figure 2.12: $\hat{A}$ is not a Forward Attractor

However, as seen in Figure 2.12, it is clear that B does not forward converge to a distinctly defined forward attractor, nor even to the pullback attractor $\hat{A}$.

Analytically this is clear upon investigation of the solutions expressed using a cocycle representation,

$$\Phi_{(t,t_0)}(x_0) = \tanh((t + t_0)/2) + \frac{1 + 10e^{-(t+t_0)}}{1 + 10e^{-t_0}}(x_0 - \tanh(t_0/2)).$$

Solutions originating at any (t_0, x_0) do not approach any closer than a distance of $(x_0 - \tanh(t_0/2))/(1 + 10e^{-t_0})$ from the pullback attractor.

□

The above result is in contrast with both pullback results and analogous results for autonomous systems. The reasons permitting attractor structures to exist within pullback absorbing neighbourhoods and not necessarily for forwards absorbing neighbourhoods, or of a method to guarantee existence of a forward attractor are open topics for further research.

2.4 Associated Attractor Results

In Section 2.3, a pullback absorbing neighbourhood was shown to guarantee the existence of a pullback attractor within a pullback absorbing neighbourhood. However, confirmation in a likewise manner for forward attractors cannot be achieved.

The only available approach to locate a forward attractor in a similar fashion is to locate a pullback attractor via an absorbing neighbourhood and determine its uniformity. The property of uniformity implies completeness of attraction (and hence forward convergence). This is formalised in the following lemmas which are an extension of Lemmas 2.1.1, 2.1.2, 2.1.3, 2.1.4.

Lemma 2.4.1. *If pullback attraction to a pullback attractor $\hat{A}$ is uniform, then $\hat{A}$ is a complete attractor.*

Proof: Pullback attraction to $\hat{A}$ is uniform. That is, given an $\epsilon > 0$, there exists a $\hat{\delta} = \{\delta\}$ only, and a $T = T(\epsilon)$ such that for every $p \in P$ and for all $t > T(\epsilon)$

$$H^*(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_\delta(A(\theta_{-t}p)), A(p)) < \epsilon.$$

We make the substitution $\theta_t p$ for p in the above equation, noting that the bounds δ and $T(\epsilon)$ are independent of p and thus still valid. Then for some $\epsilon > 0$, every $p \in P$ and $t > T(\epsilon)$,

$$H^*(\Phi_{(t,p)}(\mathcal{N}_\delta(A(p)), A(\theta_t p)) < \epsilon.$$

Thus $\hat{A}$ satisfies the property of forwards convergence and as a result is a *complete attractor*.

□

Lemma 2.4.2. *If forward attraction to a forward attractor $\hat{A}$ is uniform, then $\hat{A}$ is a complete attractor.*

Proof: The proof for the second Lemma follows similarly.

□

Chapter 3

Discrete Dynamics of Non-Autonomous Systems

Many of the results pertaining to continuous dynamical systems that have been presented in previous chapters are also relevant to discrete and numerically approximated dynamical systems. The theory of cocycles, pullback absorbing sets and stability translate under a slightly different notation for discretised systems.

3.1 Difference Equations

Discrete non-autonomous dynamical systems are often represented by difference equations of the form

$$x_{n+1} = f_n(x_n), \tag{3.1}$$

where each f_n is a Lipschitz continuous mapping on the state space $E \subset \mathbb{R}^d$ for all $n \in \mathbb{Z}^+$.

3.1.1 Constant Time-Step Discretisations

We will be interested in approximating continuous dynamical systems generated by non-autonomous ordinary differential equations,

$$\dot{x} = f(p, x), \tag{3.2}$$

which are known to possess unique solutions $\Phi_{(t,p_0)}(x_0)$ for the initial value problem (p_0, x_0) as introduced in Section 1.3.

Such dynamical systems are often approximated using a numerical scheme as in [33], [34]. This is often a Taylor Series or Runge-Kutta method, and the discretised system can then be represented with a difference equation in a similar form to that of (3.1). A one-step numerical scheme for a non-autonomous differential equation can be expressed in the form

$$\begin{aligned} x_{n+1} &= x_n + F_h(p_n, x_n), && (3.3) \\ &= \tilde{F}_h(p_n, x_n). && (3.4) \end{aligned}$$

Here $p_n = \theta_{nh} p_0$, and F_h is the increment function for the one-step method used.

It is important to note here that both the initial time and the step size are necessary in order to evaluate the solution because of the problem's non-autonomous nature. This will be of particular importance when considering attraction of the discretised system later. If the system is autonomous, then the initial time is not of concern and the problem reduces to that of the ordinary difference equation (3.1).

3.1.2 Variable Time-Step Discretisations

In place of using a constant time step we may discretise the dynamical system (3.2) with variable time steps $h_n > 0$. Such a discretisation is then expressed in the form

$$\begin{aligned} x_{n+1} &= x_n + F_{h_n}(p_n, x_n), && (3.5) \\ &= \tilde{F}_{h_n}(p_n, x_n). && (3.6) \end{aligned}$$

In order to analyse the system's stability we will utilise the structure defined below as a basis for formulating the variable time-step system.

Let $\rho > 0$, and (H^ρ, d_{H^ρ}) denote the compact metric space of all bi-infinite real sequences $\mathbf{h} = \{h_n\}_{n\in\mathbb{Z}}$, where $0 \le h_n \le \rho$ for all $n \in \mathbb{Z}$ with the metric

$$d_{H^\rho}(\mathbf{h}_{(1)}, \mathbf{h}_{(2)}) = \sum_{n=-\infty}^{\infty} 2^{-|n|}|h_n^{(1)} - h_n^{(2)}|.$$

The sequence $\mathbf{h}$ then represents the set of variable step sizes for a particular discretised system. A further property needed by the sequence $\mathbf{h}$ is reachability. That is any future time $\theta_t p$ is reachable by taking a summation of steps in $\mathbf{h}$. This may be expressed mathematically by the condition

$$\sum_{n=n_0}^{\infty} h_n = \infty,$$

for any $n_0 \in \mathbb{Z}$. Associating an initial value p_0 with such a sequence as a couple, written $(p_0, \mathbf{h}) \in P \times H^\rho$, we then define

$$\begin{aligned} p_1 &= \theta_{h_0} p_0, \\ p_2 &= \theta_{h_1} p_1, \\ &\cdots \\ p_n &= \theta_{h_{n-1}} p_{n-1}. \end{aligned}$$

The couple $(p_0, \mathbf{h})$ completely defines the sequence of discrete values that determines the discrete dynamical system. Thus the cross product space $P \times H^\rho$ serves as a parameter set for the cocycle representation of a variable time step discretisation.

3.1.3 Local Truncation Error

Errors between the numerical approximation and the actual state using a one-step numerical scheme are analysed through the use of a bound on the local error produced by the method over one iteration. This is known as the truncation error and is outlined below. (Refer to [14])

A one-step numerical scheme for a non-autonomous differential equation is known as r-th order scheme if its local (one-step) discretisation error satisfies a bound of the form,

$$||x_{n+1} - \Phi_{(h,p_n)}(x_n)|| \leq C_r h^{r+1}. \tag{3.7}$$

Similarly, if the numerical scheme uses variable time steps,

$$||x_{n+1} - \Phi_{(h_n,(p_n,\mathbf{h}))}(x_n)|| \leq C_r h_n^{r+1} \leq C_r \rho^{r+1}. \tag{3.8}$$

The order r for most commonly used one-step methods is generally dependent on the smoothness of the function f in (3.2) and the order of the Taylor series used. The truncation constant, C_r is dependent on the bounds of f and its derivatives (up to the order of the numerical scheme) over a finite interval of discretisation.

A note concerning the truncation constant needs to be mentioned here. In non-autonomous systems, it may become impossible to place bounds on f and its derivatives, even when the solutions are expected to remain in a bounded (possibly compact) region of the state space. This becomes a factor as we analyse asymptotic behaviour and consider discretisation on non-finite intervals of time. In these cases caution is required and more details will be given later. For autonomous systems this issue is not a concern if the state is guaranteed to remain within a bounded region.

For systems where a single truncation constant may not be appropriately defined, we alternatively define the truncation error with a variable bound that is modified from step to step. This issue is revisited later in this thesis.

3.2 Discrete Cocycle Representation

The generalised cocycle representation for a discrete dynamical system is presented below. It is analogous to that of continuous systems except that here the time set T is the set of positive integers, and the parameter set P is adjusted accordingly for the discrete problem.

Definition 3.2.1 (Discrete Cocycle Representation)**.** *Let $\{\theta_n, n \in \mathbb{Z}^+\}$ be a group of mappings on a nonempty parameter set P, that is $\theta_n : P \to P$ with $\theta_0 = id$ and $\theta_n \circ \theta_\eta = \theta_{n+\eta}$ for all $n, \eta \in \mathbb{Z}^+$.*

A family of mappings $\{\Phi_{(n,p)}, n \in \mathbb{Z}^+, p \in P\}$ with $\Phi_{(n,p)} : E \to E$ is called a **discrete cocycle** *on E if*

$$\begin{aligned} (i) \quad & \Phi_{(0,p)} = id, \\ (ii) \quad & \Phi_{(n+\eta,p)} = \Phi_{(n,\theta_\eta p)} \circ \Phi_{(\eta,p)}, \end{aligned}$$

for all $n, \eta \in \mathbb{Z}^+$ and $p \in P$.

3.2.1 Cocycle Representation for Difference Equations

The evolution of a solution to a difference equation such as (3.1), is dependent solely on the value of the iteration n. For this $P = \mathbb{Z}$, and the shift mapping $\theta_n : \mathbb{Z} \to \mathbb{Z}$ is defined by $\theta_n n_0 = n + n_0$. We then have

$$\Phi_{(n,n_0)}(x_{n_0}) = f_{n_0+n-1} \circ f_{n_0+n-2} \circ \cdots \circ f_{n_0}(x_{n_0}).$$

3.2.2 Cocycle Representation for Constant Time-Step Discretisations

If the discrete system is generated through a numerical scheme applied to a non-autonomous differential equation, as in (3.3), then the evolution of the set of discrete values is dependent on the step size h, and the initial value of p_0. In an autonomous differential equation it is solely dependent on the step size and the system's dynamics can be simply represented using a difference equation (3.1). A discrete cocycle representation for the non-autonomous case with constant time step, utilises the parameter set P used in the continuous case and the shift mapping defined by

$$\begin{aligned}\theta_n p_0 &= \theta^c_{nh} p_0,\\ &= p_n.\end{aligned}$$

where $\theta^c : P \to P$ is the shift mapping associated with the cocycle representation for the continuous non-autonomous system.

Solutions to (3.3), are then given by

$$x_n = \Phi^h_{(n,p_0)}(x_0) = \tilde{F}_h(p_{n-1}, \tilde{F}_h(p_{n-2}, \ldots, \tilde{F}_h(p_0, x_0 \ldots)). \tag{3.9}$$

where h is the step size used by the numerical scheme, x_0 corresponds to the initial state, and x_n the state at p_n.

In some cases it is convenient to define the parameter set P by the sequence of values $\{p_n\}_{n=1}^{\infty}$, however in a non-autonomous context where the initial time is a variable, it is important to leave the parameter set generally defined to take into account the fact that the discrete dynamics may differ significantly upon shifting the sequence marginally.

3.2.3 Cocycle Representation for Variable Time Step Discretisations

Let P^c be the parameter set, and θ^c the shift mapping associated with the cocycle representation for continuous solutions of 3.2, and recall the construction of the couple $(p_0, \mathbf{h}) \in P^c \times H^\rho$ in Section 3.1.2. Here we define the parameter set for the discrete system $P = P^c \times H^\rho$, and define the group mapping by $\theta_n(p_0, \mathbf{h}) = (p_n, \psi_n \mathbf{h})$, where

$$p_n = \theta^c_{h_{n-1}} \circ \cdots \circ \theta^c_{h_1} \circ \theta^c_{h_0} p_0,$$

and ψ is a shift operator on the sequence $\mathbf{h}$ so that the n_0-th element of the sequence $\psi_n \mathbf{h}$ is represented by

$$(\psi_n \mathbf{h})_{n_0} = h_{n+n_0}.$$

Using a discrete cocycle representation, solutions to (3.5), are then expressed as

$$\Phi^{\mathbf{h}}_{(n,(p_0,\mathbf{h}))}(x_0) = \tilde{F}_{h_{n-1}}(p_{n-1}, \tilde{F}_{h_{n-2}}(p_{n-2}, \ldots, \tilde{F}_{h_0}(p_0, x_0 \ldots)). \tag{3.10}$$

3.3 Discrete Stability and Absorbing Neighbourhoods

The following definitions and theorems for discrete systems are analogous to those presented in Section 2.3 and are listed below for completeness and as a reference. They are equally valid for representations of difference equations as well as for discretised continuous systems.

In the following we use the usual notation (for example, $\hat{A}$) to represent a family of sets. However to accurately distinguish the discrete family appropriate for discretisations of continuous systems in future we will use $\hat{A}^h$, $\hat{A}^{\mathbf{h}}$ for constant and variable time-step discretisations respectively.

Definition 3.3.1 (Discrete Forward Stability)**.** *A discrete family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is* **forward stable** *for the discrete cocycle $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ on E if for any $\epsilon > 0$ there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta},\hat{A})} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$*

$$H^*\left(\Phi_{(n,p)}(\mathcal{N}_{\delta_p}(A(p))), A(\theta_n p)\right) < \epsilon \qquad \forall n \geq 0.$$

Definition 3.3.2 (Discrete Pullback Stability)**.** *A discrete family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is* **pullback stable** *for the discrete cocycle $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ on E if for any $\epsilon > 0$ there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta},\hat{A})} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$,*

$$H^*\left(\Phi_{(n,\theta_{-n}p)}(\mathcal{N}_{\delta_p}(A(\theta_{-n}p))), A(p)\right) < \epsilon \qquad \forall n \geq 0.$$

Definition 3.3.3 (Discrete Forward Asymptotic Stability)**.** *A discrete family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is* **forward asymptotically stable** *for the discrete cocycle $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ on E if it is forward stable and if there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta},\hat{A})} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$ and any initial value $x_0 \in \mathcal{N}_{\delta_p}(A(p))$,*

$$\lim_{n\to\infty} \operatorname{dist}\left(\Phi_{(n,p)}(x_0), A(\theta_n p)\right) = 0. \tag{3.11}$$

Definition 3.3.4 (Discrete Pullback Asymptotic Stability). *A discrete family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is* **pullback asymptotically stable** *for the discrete cocycle $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ on E if it is pullback stable and if there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta},\hat{A})} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$ and any sequence of initial values $\hat{x}_p \in \mathcal{N}_{\delta_p,\hat{A}}$,*

$$\lim_{n\to\infty} \operatorname{dist}\left(\Phi_{(n,\theta_{-n}p)}(x(\theta_{-n}p)), A(p)\right) = 0. \tag{3.12}$$

Definition 3.3.5 (Discrete Forward Attractor). *A discrete family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is said to be a* **discrete forward attractor** *for the cocycle $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ on E if there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta},\hat{A})} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$ the forward attractor $\hat{A}$ satisfies two properties,*

$$\Phi_{(n,p)}\left(A(p)\right) = A(\theta_n p) \qquad \textit{for each} \quad n \in \mathbb{Z}^+. \tag{3.13}$$

$$\lim_{n\to\infty} H^*\left(\Phi_{(n,p)}(\mathcal{N}_{\delta_p}(A(p)), A(\theta_n p)\right) = 0. \tag{3.14}$$

Definition 3.3.6 (Discrete Pullback Attractor). *A discrete family $\hat{A} = \{A(p); p \in P\}$ of uniformly bounded compact subsets of E, is said to be a* **discrete pullback attractor** *for the cocycle $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ on E if there exists a δ-neighbourhood system $\hat{\mathcal{N}}_{(\hat{\delta},\hat{A})} = \{\hat{\mathcal{N}}_{\delta_p,\hat{A}}; \delta_p \in \hat{\delta}, p \in P\}$ defined by a delta set $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$, so that for each $p \in P$ the pullback attractor $\hat{A}$ satisfies two properties,*

$$\Phi_{(n,p)}\left(A(p)\right) = A(\theta_n p) \qquad \textit{for each} \quad n \in \mathbb{Z}^+. \tag{3.15}$$

$$\lim_{n\to\infty} H^*\left(\Phi_{(n,\theta_{-n}p)}(\mathcal{N}_{\delta_p}(A(\theta_{-n}p)), A(p)\right) = 0. \tag{3.16}$$

The same properties of *uniformity*, *equi-asymptotic stability* and *completeness* hold for the discrete definitions above as for continuous stability theory.

We may also construct a pullback absorbing neighbourhood to analyse pullback attractors as was done in Section 2.3 for continuous systems.

Definition 3.3.7 (Discrete Pullback Absorbing Neighbourhoods). *A family $\hat{B} = \{B(p); p \in P\}$ of uniformly bounded compact subsets of E, is said to* *be a* **Discrete Pullback Absorbing Neighbourhood** *for a discrete cocycle* $\{\Phi^h_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ *on* E *if it pullback absorbs a uniformly bounded* δ *- neighbourhood system of* $\hat{B}$*. That is, there exists an open* δ*-neighbourhood system* $\hat{\mathcal{N}}_{(\hat{\delta},\hat{B})}$ *defined by a delta set* $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ *so that for each* $p \in P$*, there exists a* $N_p > 0$ *such that*

$$\Phi^h_{(n,\theta_{-n}(p))}(\mathcal{N}_{\delta_p}(B(\theta_{-n}(p))) \subset B(p) \qquad \forall n > N_p, \tag{3.17}$$

The discrete pullback absorbing neighbourhood automatically satisfies the following property.

Lemma 3.3.1. *If* $\hat{B}$ *is a pullback absorbing neighbourhood, then for each* $p \in P$ *there exists a* $N_p > 0$ *such that*

$$\Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p))) \subset B(p) \qquad \forall n > N_p.$$

The following theorem is an extension of Theorem 2.3.1 to discrete dynamical systems. The proof follows in a similar manner to its continuous counterpart and is given for completeness.

Theorem 3.3.1. *Let* $\{\Phi_{(n,p)}; n \in \mathbb{Z}^+, p \in P\}$ *be a discrete cocycle on* E *with a Discrete Pullback Absorbing Neighbourhood* $\hat{B}$*. Then there exists a Discrete Pullback Attractor* $\hat{A} = \{A(p); p \in P\}$ *uniquely determined for each* $p \in P$ *by*

$$A(p) = \bigcap_{\eta \geq 0} \overline{\bigcup_{n \geq \eta} \Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p)))}. \tag{3.18}$$

Proof:

To show that $\hat{A}$ is indeed a discrete pullback attractor, it must meet the requirements given in Definition 3.3.6.

i) *Uniform Boundedness and Compactness* : From (3.18) and using Lemma 3.3.1

$$\begin{aligned} A(p) &= \bigcap_{\eta \geq 0} \overline{\bigcup_{n \geq \eta} \Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p)))}, \\ &\subseteq \bigcap_{\eta \geq N_p} \overline{\bigcup_{n \geq \eta} \Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p)))}, \\ &\subseteq \bigcap_{\eta \geq N_p} \overline{\bigcup_{n \geq \eta} B(p)}, \\ &= \overline{B(p)}. \end{aligned}$$

Hence $\hat{A}$ is uniformly bounded since $\hat{B}$ is uniformly bounded with respect to p. The attractor set is also compact as it is an intersection of compact sets.

ii) *Pullback Property* : We need to find a discrete pullback convergent neighbourhood system as in (3.3.6). We will firstly show that $\hat{A}$ pullback attracts $\hat{B}$. That is, for each $p \in P$

$$\lim_{n\to\infty} H^*(\Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p))), A(p)) = 0. \tag{3.19}$$

where $A(p)$ is defined as above.

Assume that this is not the case. Then for some $\epsilon > 0$ there exists sequences $n_j \to \infty$ and $x_j \in \Phi_{(n_j,\theta_{-n_j}(p))}(B(\theta_{-n_j}(p)))$ such that

$$H^*(x_j, A(p)) > \epsilon \qquad \forall j. \tag{3.20}$$

For large enough j, $x_j \in B(p)$ (Lemma 3.3.1). Now Since $B(p)$ is compact, there exists a subsequence $n_{j'} \to \infty$ and an associated convergent subsequence, $x_{j'} \to x_0$ with $x_0 \in B(p)$. Furthermore $H^*(x_0, A(p)) \geq \epsilon$. The $x_{j'}$ satisfy

$$x_{j'} \in \bigcup_{n\geq\eta} \Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p))), \qquad \forall\eta > 0 \quad \text{and} \quad n_{j'} > \eta.$$

As x_0 is the limit, we also have for any $\eta \geq 0$,

$$x_0 \in \overline{\bigcup_{n\geq\eta} \Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p)))},$$

and hence

$$x_0 \in \bigcap_{\eta\geq 0} \overline{\bigcup_{n\geq\eta} \Phi_{(n,\theta_{-n}(p))}(B(\theta_{-n}(p)))}.$$

That is $x_0 \in A(p)$, which contradicts (3.20). As the choice of p was arbitrary, (3.19) holds true for all $p \in P$.

From (3.19), for any $\epsilon > 0$, and each $p \in P$, there exists a $n_1 = n_1(\epsilon, p)$ such that

$$H^*(\Phi_{(n_1,\theta_{-n_1}(p))}(B(\theta_{-n_1}(p)), A(p)) < \epsilon.$$

Let us take $\hat{\mathcal{N}}_{\hat{\delta}^*,\hat{B}}$ as our neighbourhood system for the attractor, $\hat{A}$, defined by $\hat{\delta}^* = \{\delta_p^*; \delta_p^* = \delta_{\theta_{-n_1}(p)}, \delta_p \in \hat{\delta}\}$ where $\hat{\delta}$ defines $\hat{\mathcal{N}}_{\hat{\delta},\hat{B}}$, the associated $\hat{\delta}$-neighbourhood of $\hat{B}$. We need to show that $\hat{A}$ pullback attracts the system $\hat{\mathcal{N}}_{\hat{\delta}^*,\hat{B}}$.

Using the cocycle property, and because $B(p)$ is pullback absorbing, we can formulate attraction for elements of $\hat{\mathcal{N}}_{\hat{\delta}^*,\hat{B}}$

$$\begin{aligned}
&H^*(\Phi_{(n,\theta_{-n}(p))}(\mathcal{N}_{\delta_p^*}(A(\theta_{-n}(p))), A(p)) \\
&\quad \leq H^*(\Phi_{(n,\theta_{-n}(p))}(\mathcal{N}_{\delta_p^*}(B(\theta_{-n}(p))), A(p)) \\
&\quad = H^*(\Phi_{(n_1,\theta_{-n_1}(p))} \circ \Phi_{(n-n_1,\theta_{-n}(p))}(\mathcal{N}_{\delta_p^*}(B(\theta_{-n}(p))), A(p)) \\
&\quad \leq H^*(\Phi_{(n_1,\theta_{-n_1}(p))}(B(\theta_{-n_1}(p)), A(p)) \\
&\quad \leq \epsilon,
\end{aligned}$$

for all $n > n_{(\theta_{-n_1}(p))} + n_1$, where $n_{(\theta_{-n_1}(p))}$ is as defined in the definition for a pullback absorbing neighbourhood (the finite absorption time). Hence $\hat{A}$ satisfies the pullback property for each $p \in P$, i.e.

$$\lim_{n\to\infty} H^*(\Phi_{(n,\theta_{-n}(p))}(\mathcal{N}_{\delta_p^*}(A(\theta_{-n}(p))), A(p)) = 0.$$

iii) Φ-*Invariance* :
Consider pullback evolution of $A(\theta_{-n^*}(p)) \in \hat{A}$ for arbitrary p and any $t^* > 0$

$$\begin{aligned}
&\Phi_{(n^*,\theta_{-n^*}(p))}(A(\theta_{-n^*}(p))) \\
&\qquad = \Phi_{(n^*,\theta_{-n^*}p)}\left(\bigcap_{\eta\geq 0}\overline{\bigcup_{n\geq\eta}\Phi_{(n,\theta_{-n-n^*}(p)}(B(\theta_{-n-n^*}(p)))}\right), \\
&\qquad = \bigcap_{\eta\geq 0}\Phi_{(n^*,\theta_{-n^*}p)}\left(\overline{\bigcup_{n\geq\eta}\Phi_{(n,\theta_{-n-n^*}(p)}(B(\theta_{-n-n^*}(p)))}\right), \\
&\qquad = \bigcap_{\eta\geq 0}\overline{\bigcup_{n\geq\eta}\Phi_{(n^*,\theta_{-n^*}p)}\Phi_{(n,\theta_{-n-n^*}(p)}(B(\theta_{-n-n^*}(p)))}, \\
&\qquad = \bigcap_{\eta\geq 0}\overline{\bigcup_{n\geq\eta}\Phi_{(n+n^*,\theta_{-n-n^*}p)}(B(\theta_{-n-n^*}(p)))}, \\
&\qquad = \bigcap_{\eta\geq n^*}\overline{\bigcup_{u\geq\eta}\Phi_{(u,\theta_{-u}p)}(B(\theta_{-u}(p)))},
\end{aligned}$$

where we have made the substitution $u = n + n^*$. Now, for all $\eta < n^*$,

$$\overline{\bigcup_{u \geq \eta} \Phi_{(u,\theta_{-u}p)}(B(\theta_{-u}(p)))} \supseteq \overline{\bigcup_{u \geq n^*} \Phi_{(u,\theta_{-u}p)}(B(\theta_{-u}(p)))}.$$

Hence

$$\begin{aligned}
&\Phi_{(n^*,\theta_{-n^*}(p))}(A(\theta_{-n^*}(p))) \\
&\qquad = \bigcap_{\eta \geq 0} \overline{\bigcup_{u \geq \eta} \Phi_{(u,\theta_{-u}p)}(B(\theta_{-u}(p)))} \\
&\qquad = A(p).
\end{aligned}$$

The conditions for Φ-Invariance are satisfied.

$\square$

Chapter 4

Perturbed Autonomous Dynamical Systems

In this chapter we consider autonomous differential equations (both continuous and discrete) which are known to possess a semi-group attractor. In practice however, the use of an autonomous model will almost always be subject to small perturbations, and it is often desirable to determine under what conditions the perturbed system will retain an attracting object with similar characteristics to that of the original attractor. A Lyapunov approach involving the use of absorbing sets is applied here to guarantee the conditions needed to ensure the existence of a pullback attractor in the perturbed system with similar characteristics to that of the original semi-group attractor.

4.1 Continuous Autonomous Systems

The results that follow regarding continuously perturbed autonomous systems are a collection of those published by P.Kloedon and D.Stonier in [22].

4.1.1 The Non-Autonomous Perturbation

We consider autonomous dynamical systems generated by the ordinary differential equation

$$\dot{x} = f(x), \tag{4.1}$$

where $x \in E$, $E \subset \mathbb{R}^d$. We assume that the following conditions hold for the autonomous system:

P1) f is continuous and Lipschitz with respect to x.

P2) The autonomous system possesses a semi-group attractor A_0, as introduced in Definition 1.2.4.

We then subject the original ordinary differential equation to a non - autonomous perturbation $g : \mathbb{R} \times E \to \mathbb{R}^d$, to obtain a non-autonomous ordinary differential equation on E,

$$\dot{x} = f(x) + \epsilon g(\mathsf{t}, x), \tag{4.2}$$

where $\epsilon > 0$ is a small parameter.

P3) The perturbing function is uniformly bounded. That is

$$\sup_{(\mathsf{t},x)\in\mathbb{R}\times\mathbb{R}^d} ||g(\mathsf{t}, x)|| \leq K < \infty. \tag{4.3}$$

The non-autonomous system (4.2) generates a cocycle $\{\Phi^{\epsilon}_{(t,t_0)}; t \in \mathbb{R}^+, t_0 \in \mathbb{R}\}$ over the parameter set $P = \mathbb{R}$ with shift mapping $\theta_t(t_0) = t_0 + t$. The main theorem shows that the non-autonomously perturbed system possesses a pullback attractor $\hat{A}_\epsilon = \{A^\epsilon(\mathsf{t}); \mathsf{t} \in \mathbb{R}\}$.

4.1.2 Main Theorem

Before proceeding to the Main Theorem the following lemma is presented. It is an integral part of the proof for the main theorem that follows.

Lemma 4.1.1. *Let $A = e^{M|t_1 - t_0|}$ where M is the uniform Lipschitz constant of $G(\mathsf{t}, x) = f(x) + \epsilon g(\mathsf{t}, x)$ in x uniformly in $\mathsf{t} \in \mathbb{R}$. That is, for any $x_1, x_2 \in E$,*

$$||G(\mathsf{t}, x_1) - G(\mathsf{t}, x_2)|| \leq M||x_1 - x_2||,$$

for all $\mathtt{t} \in \mathbb{R}$*. Then*

$$A^{-1}||x-y|| \le ||\Phi^{\epsilon}_{(t_1-t_0,t_0)}(x) - \Phi^{\epsilon}_{(t_1-t_0,t_0)}(y)|| \le A||x-y||$$

Proof: Write $\Delta\Phi^{\epsilon}_{(t,t_0)} := \Phi^{\epsilon}_{(t,t_0)}(x) - \Phi^{\epsilon}_{(t,t_0)}(y)$ for $0 \le t \le t_1 - t_0$. Then

$$\Delta\Phi^{\epsilon}_{(t,t_0)} = (x-y) + \int_{t_0}^{t+t_0} [G(s,\Phi^{\epsilon}_{(s,t_0)}(x)) - G(s,\Phi^{\epsilon}_{(s,t_0)}(y))]ds.$$

Using the Lipschitz condition, we have

$$||\Delta\Phi^{\epsilon}_{(t,t_0)}|| \le ||x-y|| + M\int_{t_0}^{t+t_0} ||\Delta\Phi^{\epsilon}_{(s,t_0)}||ds$$

and the right hand inequality then follows by application of the Gronwall inequality (Lemma 5.1.2).

To obtain the left hand inequality, we repeat the argument starting at t_1 and integrate backwards to t_0 from the previously obtained endpoints at t_1.

□

Theorem 4.1.1 (Non-Autonomous Perturbation)**.** *For the perturbed autonomous dynamical system (4.2), suppose conditions* ***P1 - P3*** *hold.*

Then there exists a pullback attractor $\hat{A}_{\epsilon} = \{A^{\epsilon}(\mathtt{t}); \mathtt{t} \in \mathbb{R}\}$ *such that*

$$\lim_{\epsilon \to 0^+} H^*(A^{\epsilon}(t), A_0) = 0$$

for all $\mathtt{t} \in \mathbb{R}$*. In addition, the component sets* $A^{\epsilon}(\mathtt{t})$ *are continuous in* $\mathtt{t}$*, that is,*

$$\lim_{\mathtt{t} \to t_0} H(A^{\epsilon}(\mathtt{t}), A^{\epsilon}(t_0)) = 0 \qquad \forall t_0 \in \mathbb{R},$$

and have constant Hausdorff dimension

$$dim_H A^{\epsilon}(t_0) = dim_H A^{\epsilon}(t_1) \qquad \forall t_0, t_1 \in \mathbb{R}.$$

Proof: 1. Existence of a Pullback Absorbing Set:

The semi-group attractor A_0 is a uniformly asymptotically stable set, and by Theorem 1.3.3, there exists a Lyapunov function V on some neighbourhood, $\mathcal{N}_R(A_0)$, of the attractor which characterises this uniform asymptotic stability. This Lyapunov function will be used to construct a pullback absorbing set for the non-autonomous system.

We will consider the effect of the Lyapunov function on a subset of this neighbourhood, defined by $\mathcal{N}(A_0) = \{x \in \mathcal{N}_R(A_0); V(x) < a(R)\}$, where the function $a(\cdot)$ is the lower bounding class $\mathcal{K}$ function in Theorem 1.3.3 corresponding to the Lyapunov function $V(x)$. This neighbourhood is chosen to ensure that solutions pulled back from within this set remain within $\mathcal{N}_R(A_0)$ as will be seen later.

We now proceed to determine the rate of change of V for solutions in the perturbed system (4.2). For any $\mathsf{t} \in \mathbb{R}$ and $x \in \mathcal{N}_R(A_0)$,

$$\begin{aligned}
D^+_{(4.2)}V(x) &= \overline{\lim_{h\to 0^+}} \left\{ \frac{V(x + h(f(x) + \epsilon g(\mathsf{t}, x))) - V(x)}{h} \right\} \\
&= \overline{\lim_{h\to 0^+}} \left\{ \frac{V(x + h(f(x) + \epsilon g(\mathsf{t}, x))) - V(x + hf(x))}{h} \right. \\
&\qquad\qquad \left. + \frac{V(x + hf(x)) - V(x)}{h} \right\} \\
&\leq \overline{\lim_{h\to 0^+}} \left\{ \frac{Lh\epsilon||g(\mathsf{t}, x)||}{h} + \frac{V(x + hf(x)) - V(x)}{h} \right\} \\
&\leq L\epsilon||g(\mathsf{t}, x)|| + \overline{\lim_{h\to 0^+}} \frac{V(x + hf(x)) - V(x)}{h} \\
&\leq LK\epsilon + D^+_{(4.1)}V(x) \\
&\leq LK\epsilon - cV(x),
\end{aligned} \tag{4.4}$$

where L is the Lipschitz constant of $V(x)$. The result of Theorem 1.3.3 has been used on the last line.

Now for all $x \notin B^\epsilon$, where

$$B^\epsilon = \{x \in \mathcal{N}_R(A_0); V(x) \leq 2LK\epsilon/c\},$$

we have from (4.4)

$$D^+_{(4.2)}V(x) \leq -LK\epsilon. \tag{4.5}$$

If ϵ is small enough so that

$$\epsilon < ca(R)/2LK, \tag{4.6}$$

then B^ϵ is strictly a subset of $\mathcal{N}(A_0)$.

To see that the set B^ϵ is a pullback absorbing neighbourhood for the perturbed system, consider the Lyapunov function on solutions $x_0 \in \mathcal{N}(A_0)\backslash B^\epsilon$, pulled back from time t_0. Using (4.5), we have

$$V(\Phi_{(\tau,t_0-t)}(x_0)) \leq V(x_0) - LK\epsilon\tau,$$

for all $0 \leq \tau \leq t$ such that $\Phi_{(\tau,t_0-t)}(x_0) \notin B^\epsilon$. Hence there exists a $T(x_0) > 0$ such that $\Phi_{(\tau,t_0-t)}(x_0) \notin B^\epsilon$ for all $\tau < T(x_0)$, but $\Phi_{(T,t_0-t)}(x_0) \in B^\epsilon$.

Using similar reasoning it can be seen that B^ϵ is actually positively invariant for solutions of (4.2), and so for all $x_0 \in \mathcal{N}(A_0)$ we have

$$\Phi_{(t,t_0-t)}(x_0) \in B^\epsilon \qquad \forall t > T(x_0).$$

An upper bound T^* for $T(x_0)$ exists for all $x_0 \in \mathcal{N}(A_0)$ since the Lyapunov function is bounded above on $\mathcal{N}(A_0)$ by $a(R)$. Hence

$$\Phi_{(t,t_0-t)}(\mathcal{N}(A_0)) \in B^\epsilon \qquad \forall t > T^*.$$

Thus B^ϵ pullback absorbs a neighbourhood of itself, and is a pullback absorbing neighbourhood for the cocycle $\{\Phi_{(t,t_0)}; t \in \mathbb{R}^+, t_0 \in \mathbb{R}\}$ generated by the non-autonomous differential equation (4.2).

We then apply Theorem 2.3.1 to this cocycle and pullback absorbing neighbourhood to verify the existence of a pullback attractor, $\hat{A}_\epsilon$ contained within B^ϵ.

2. Approximation Property: For $x \in B^\epsilon$ we have

$$a(\mathrm{dist}(x, A_0)) \leq V(x) \leq 2LK\epsilon/c,$$

so $\mathrm{dist}(x, A_0) \leq a^{-1}(2LK\epsilon/c)$. Hence

$$H^*(B^\epsilon, A_0) \leq a^{-1}(2LK\epsilon/c),$$

and since $A^\epsilon(t_0) \subset B^\epsilon$, we have

$$H^*(A^\epsilon(t_0), A_0) \leq a^{-1}(2LK\epsilon/c).$$

Consequently,

$$H^*(A^\epsilon(t_0), A_0) \to 0^+ \qquad \text{as} \qquad \epsilon \to 0^+.$$

3. Continuity: The pullback attractor family is Φ^ϵ-Invariant. Further the continuity of the cocycle in all of its variables implies the continuity of $\Phi^\epsilon_{(\cdot,t_0)}(\cdot) : \mathbb{R}^+ \times \mathcal{H}(E) \to \mathcal{H}(E)$. Hence

$$H(A^\epsilon(t_0 + t), A^\epsilon(t_0)) = H(\Phi^\epsilon(A^\epsilon(t_0)), A^\epsilon(t_0)).$$

Consequently

$$H(A^\epsilon(t_0 + t), A^\epsilon(t_0)) \to 0^+ \qquad \text{as} \qquad t \to 0.$$

4. Hausdorff Dimension: Note that $\Phi^\epsilon_{(t_1-t_0,t_0)}(\cdot)$ for any $t_0 \leq t_1 \in \mathbb{R}$ is a bi-Lipschitz mapping by Lemma 4.1.1. So by Corollary 2.4, page 30 of [11] the sets $A^\epsilon(t_0)$ and $A^\epsilon(t_1)$ have the same Hausdorff dimension.

This completes the proof of Theorem 4.1.1.

$\square$

Remark 1. To guarantee the existence of a pullback attractor, the size of the perturbation is required to be restricted as given by inequality (4.6). To actually calculate the bound for ϵ is, in some cases not possible, as the form of the Lyapunov function may not be known, only that it exists, in which case the form of the function $a(\cdot)$ is also unknown.

Note that in the case that the original attractor A_0 is global, then restricting the size of the scalar value ϵ is unnecessary.

4.1.3 Time Periodic Perturbations

For time periodic perturbations, the cocycle and the pullback attractor components are also periodic with the same period as the perturbations and a forwards convergence property also holds. Hence the pullback attractor, is in fact a complete attractor.

Corollary 4.1.1 (Periodic Non-Autonomous Perturbation)**.**
If in addition, the perturbations $g(\cdot, x)$ are periodic with period T, then the cocycle is T- periodic, that is

$$\Phi^{\epsilon}_{(t,t_0)}(x_0) = \Phi^{\epsilon}_{(t,t_0+T)} \qquad \forall t \in \mathbb{R}^+, t_0 \in \mathbb{R}, x_0 \in E,$$

and the components of the pullback attractor $\hat{A}_\epsilon$ are also periodic with period T. Additionally, $\hat{A}$ is in fact a complete attractor.

Proof: 1. Periodicity: Let g be periodic in t with period T. Then $G(\mathsf{t}, x) = f(x) + g(\mathsf{t}, x)$ is also T-periodic in t, specifically $G(\mathsf{t} - T, x) = G(\mathsf{t}, x)$ for all $\mathsf{t} \in \mathbb{R}$ and $x \in E$.

Let x be the solution of the differential equation (4.2) with initial value $x(t_0) = x_0$. Consider the shifted solution X defined by $X(\mathsf{t}) = x(\mathsf{t} - T)$ with initial value $X(t_0 + T) = x_0$. It can be seen that it satisfies (4.2), since

$$\begin{aligned}
\frac{dX}{d\mathsf{t}}(\mathsf{t}) &= \frac{dx}{d\mathsf{t}}(\mathsf{t} - T) \\
&= \frac{dx}{d\tau}(\tau) \qquad\qquad \text{where} \qquad \tau = \mathsf{t} - T \\
&= G(\tau, x(\tau)) \\
&= G(\mathsf{t} - T, X(\mathsf{t})) = G(\mathsf{t}, X(\mathsf{t})), \qquad \text{by T-periodicity of G,}
\end{aligned}$$

and so is a solution of the differential equation. Hence by uniqueness we must have $\Phi_{(t,t_0)} = \Phi_{(t,t_0+T)}$, that is, T-periodicity of the cocycle.

To see that the pullback attractor $\hat{A}_\epsilon$ is also T-periodic, we replace the t_0 above with $t_0 - t$ where $t \geq 0$. We then have $\Phi^{\epsilon}_{(t,t_0-t)}(x_0) =$

$\Phi^{\epsilon}_{(t,t_0-t+T)}(x_0)$, and hence by (2.18),

$$A^{\epsilon}(t_0) = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\Phi^{\epsilon}_{(t,t_0-t)}(B^{\epsilon})} = \bigcap_{\tau\geq 0}\overline{\bigcup_{t\geq\tau}\Phi^{\epsilon}_{(t,t_0+T-t)}(B^{\epsilon})} = A^{\epsilon}(t_0+T).$$

Thus $\hat{A}_{\epsilon}$ is T-periodic.

2. Forwards Convergence: The rate of attraction governed by (4.2) is completely determined on the compact interval $[t_0, t_0+T]$. In fact the parameter space may be compactified so that $P = t_0 \bmod T$. As a result the rate of attraction will be uniform and since the attracting neighbourhood is also independent of the initial time, $\hat{A}^{\epsilon}$ a uniform attractor. Hence by Lemma 2.4.1, $\hat{A}^{\epsilon}$ is a complete attractor.

□

4.1.4 Asymptotically Vanishing Perturbations

For asymptotically vanishing perturbations, the cocycle and the pullback attractor components converge to the autonomous semi-group and its attractor, respectively.

Corollary 4.1.2 (Vanishing Perturbations)**.** *Consider the dynamical system (4.2) and suppose conditions* ***P1 - P3*** *hold. In addition, the perturbations satisfy the condition*

$$\sup_{x\in E}||g(\mathsf{t},x)|| \to 0 \qquad as \qquad \mathsf{t}\to\infty. \tag{4.7}$$

Then the cocycle $\{\Phi^{\epsilon}_{(t,t_0)}; t\in\mathbb{R}^+, t_0\in\mathbb{R}\}$ *converges uniformly for each* $t\geq 0$ *as* $t_0\to\infty$ *to the semi-group* $\{S_t; t\in\mathbb{R}^+\}$ *of the autonomous system (4.1). That is*

$$\sup_{x_0\in E}||\Phi^{\epsilon}_{(t,t_0)}(x_0) - S_t(x_0)|| \to 0 \qquad as \qquad t_0\to\infty,$$

for each $t\in\mathbb{R}^+$*. Also the pullback attractor components satisfy*

$$\lim_{\tau\to\infty} H^*(A^{\epsilon}(\tau), A_0) = 0,$$

where A_0 *is the semi-group attractor.*

Proof: 1. Asymptotic Convergence of the Cocycle to the Semi-Group: Due to the vanishing of the perturbations (4.7), given any $\gamma > 0$ there exists a $T(\gamma) \in \mathbb{R}^+$ such that

$$\sup_{x \in E} ||g(\mathsf{t}, x)|| \leq \gamma \qquad \forall \mathsf{t} \geq T(\gamma),$$

where $T(\gamma) \to \infty$ as $\gamma \to 0^+$. Let $S_t(x_0)$ be the solution of the autonomous equation (4.1), and $\Phi^{\epsilon}_{(t,t_0)}(x_0)$ be the corresponding solution of the non-autonomous equation (4.2), and define

$$\Delta(t, t_0, x_0) = ||\Phi^{\epsilon}_{(t,t_0)}(x_0) - S_t(x_0)||.$$

Writing the solutions to the differential equations in integral form we obtain for any $t_0 \geq T(\gamma)$, and $t \geq 0$,

$$\begin{aligned}
\Delta(t, t_0, x_0) &\leq || \int_0^t \left(f(\Phi_{(s,t_0)}(x_0)) - f(S_s(x_0)) \right) ds|| \\
&\qquad\qquad + \epsilon|| \int_0^t g(t_0 + s, x(s; t_0; x_0)) ds|| \\
&\leq \int_0^t ||f(\Phi_{(s,t_0)}(x_0)) - f(S_s(x_0))|| ds \\
&\qquad\qquad + \epsilon \int_0^t ||g(t_0 + s, \Phi_{(s,t_0)}(x_0))|| ds \\
&\leq M \int_0^t \Delta(s, t_0, x_0) ds + \epsilon\gamma t,
\end{aligned}$$

where M is the Lipschitz constant of f. The Gronwall inequality (Lemma 5.1.2) then gives

$$\Delta(t, t_0, x_0) \leq \frac{\epsilon\gamma}{M} t \exp(Mt).$$

In terms of the cocycle and the semi-group

$$||\Phi^{\epsilon}_{(t,t_0)}(x_0) - S_t(x_0)|| \leq \frac{\epsilon\gamma}{M} t \exp(Mt),$$

for $t_0 \geq T(\gamma)$, any $t \geq 0$ and any $x_0 \in E$. The asymptotic convergence as $t_0 \to \infty$ for fixed $t \geq 0$ (in fact, uniformly in t in bounded intervals) follows since for any $c > 0$ there exists a γ and hence a $T(\gamma)$ such that for all $t_0 > T(\gamma)$, we have $||\Phi^{\epsilon}_{(t,t_0)}(x_0) - S_t(x_0)|| < c$.

2. Asymptotic Convergence of the Pullback Attractor: With γ and $T(\gamma)$ as above, and from the Lyapunov inequality (4.4), we obtain

$$\begin{aligned} D^+_{(4.2)}V(x) &\leq L\epsilon||g(\mathsf{t},x)|| + D^+_{(4.1)}V(x) \\ &\leq L\epsilon||g(\mathsf{t},x)|| - cV(x) \\ &\leq L\epsilon\gamma - cV(x) \leq -L\epsilon\gamma, \end{aligned}$$

for all $x \notin B^{\epsilon,\gamma} = \{x \in \mathcal{N}_\delta(A_0); V(x) \leq 2L\epsilon\gamma/c\}$ and $\mathsf{t} \geq T(\gamma)$.

From this it can be seen that $B^{\epsilon,\gamma}$ is positively invariant with respect to the cocycle $\Phi_{(t,t_0)}$ for all $t_0 \geq T(\gamma)$ and $t \geq 0$.

Also trajectories starting outside of $B^{\epsilon,\gamma}$ enter it after a finite time. In particular, there exists a $T(\epsilon,\gamma) \geq 0$ such that

$$a(\mathrm{dist}(\Phi_{(t,t_0)}(x_0), A_0) \leq V(\Phi_{(t,t_0)}(x_0)) \leq 2L\epsilon\gamma/c,$$

for all $x_0 \in B^\epsilon$, $t_0 \geq T(\gamma)$ and $t \geq T(\epsilon,\gamma)$, from which we have

$$\mathrm{dist}(\Phi_{(t,t_0)}(x_0), A_0) \leq a^{-1}(2L\epsilon\gamma/c).$$

As $A^\epsilon(t_0) \subset B^\epsilon$, and since $\hat{A}_\epsilon$ is Φ-Invariant we have

$$H^*(A^\epsilon(t_0+t), A_0) \leq a^{-1}(2L\epsilon\gamma/c),$$

for $t_0 \geq T(\gamma)$ and $t \geq T(\epsilon,\gamma)$. This gives the desired asymptotic limit since $T(\gamma) \to \infty$ as $\gamma \to 0^+$.

$\square$

4.2 Example

4.2.1 Introduction

The pullback attractor arising from perturbation of a continuous autonomous system with a known semi-group attractor as in Section 4.1 was shown to be upper semi-continuous in its convergence with respect to the initial semi-group attractor it is derived from, that is,

$$\lim_{\epsilon \to 0^+} H^*(A^\epsilon(t_0), A_0) = 0, \qquad \forall t_0 \in \mathbb{R}.$$

Lower semi-continuity, that is

$$\lim_{\epsilon \to 0^+} H^*(A_0, A^\epsilon(t_0)) = 0,$$

does not necessarily hold, as the following counter-example shows.

4.2.2 Defining the Perturbed System

Consider the 2-dimensional autonomous dynamical system

$$\dot{x} = y - x(x^2 + y^2 - 1)^2, \tag{4.8}$$

$$\dot{y} = -x - y(x^2 + y^2 - 1)^2.$$

In polar co-ordinates this system is expressed by the equations,

$$\dot{r} = -r(r^2 - 1)^2, \tag{4.9}$$

$$\dot{\theta} = -1.$$

It possesses a semi-stable limit cycle at $r = 1$, with trajectories converging to the limit cycle from outside the unit circle, and trajectories converging to the origin from inside the unit circle. In this case the global semi-group attractor for the system is the disc at the origin with unit radius, as illustrated in Figure 4.1.

We now perturb the original system of equations (4.8) with a non-autonomous perturbation so that,

$$\dot{x} = y - x(x^2 + y^2 - 1)^2 - \epsilon x|\tanh(\mathtt{t})|, \tag{4.10}$$

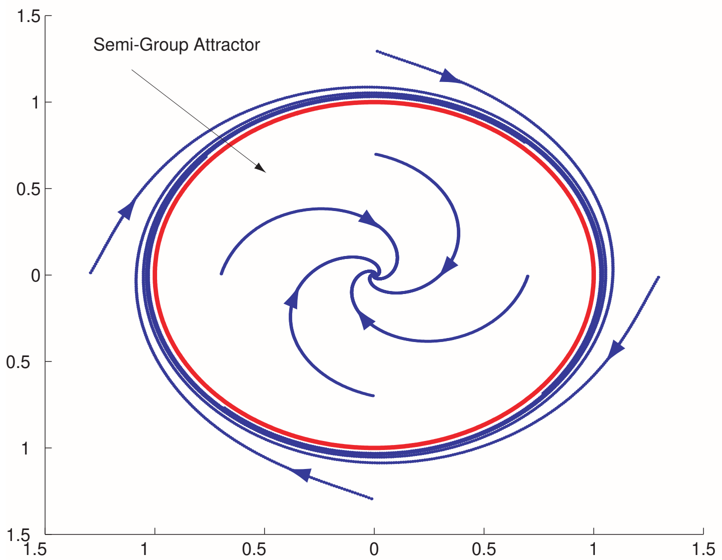

Figure 4.1: Semi-Group Attractor for (4.9)

$$\dot{y} = -x - y(x^2 + y^2 - 1)^2 - \epsilon y|\tanh(\mathtt{t})|.$$

The polar equations become:

$$\dot{r} = -r(r^2 - 1)^2 - \epsilon r|\tanh(\mathtt{t})|, \tag{4.11}$$

$$\dot{\theta} = -1 - \epsilon|\tanh(\mathtt{t})|\cos(2\theta).$$

According to Theorem 4.1.1, this perturbed system possesses a pullback attractor $\hat{A}$, which is upper semi-continuous in its convergence with respect to the semi-group attractor (the unit disk). However, the convergence is *not lower semi-continuous* as will be shown.

4.2.3 Pullback Behaviour of the Perturbed System

Let us consider the forwards asymptotic behaviour of trajectories. The θ dynamics remain rotating in the same direction if ϵ is small enough, so we need only be concerned with the radial variable. Let $r = r(s, s_0, r_0)$ where $s \geq 0$ is the time elapsed since the initial time s_0. Then

$$\frac{d}{ds}r^2 = -2r^2(1 - r^2)^2 - 2\epsilon r^2|\tanh(s + s_0)|.$$

We analyse pullback convergence of the system and use arbitrary initial values of t_0, and $r_0 > 1$. Substituting notation for pullback terms, we have $s_0 = t_0 - t$ with $0 \leq s \leq t$, so that

$$r = r(s, t_0 - t, r_0),$$
$$\frac{d}{ds} r^2 = -2r^2(1 - r^2)^2 - 2\epsilon r^2 |\tanh(s + t_0 - t)|. \tag{4.12}$$

Since $r_0 > 1$ we may choose δ small so that $(r_0^2 - 1)^2 \geq \delta^2$. Then while $r^2(s, t_0 - t, r_0) > 1 + \delta$ we have

$$\begin{aligned} \frac{d}{ds} r^2 &= -2r^2(1 - r^2)^2 - 2\epsilon r^2 |\tanh(s + t_0 - t)|, \\ &\leq -2r^2 \delta^2. \end{aligned}$$

Hence

$$\begin{aligned} r^2(s, t_0 - t, r_0) &\leq r_0^2 exp(-2\delta^2 s), \\ &\leq 1 + \delta, \end{aligned}$$

for all $s \geq \ln\left(r_0^2/(1 + \delta)\right)/2\delta^2$ (provided t is made large enough). For ease of notation, let $s_1(r_0, \delta) = \ln\left(r_0^2/(1 + \delta)\right)/2\delta^2$. Thus we have

$$r^2(s, t_0 - t, r_0) \leq 1 + \delta \qquad \forall s_1 \leq s \leq t.$$

Consequently, solutions will reach a neighbourhood of the unit circle within finite time.

4.2.4 Behaviour Near r = 1

We consider here the evolution of the trajectory across a neighbourhood of the unit circle. That is, within the neighbourhood defined by $r^2 \leq 1 + \delta$. Let $r_1 = r(s_1, t_0 - t, r_0)$.

If $r_1^2 \leq 1 - \delta$ we proceed automatically to the next step (Section 4.2.5). Otherwise, $1 - \delta \leq r_1^2 \leq 1 + \delta$, and we analyse the progress of trajectories represented by r across the δ-neighbourhood where

$$r = r(s_1 + s^*, t_0 - t, r_0).$$

s_1 is defined as above and $s = s_1 + s^*$. Using our original equation for the derivative of the system (4.12) we have

$$\begin{aligned}\frac{d}{ds^*}r^2 &= -2r^2(1-r^2)^2 - 2\epsilon r^2|\tanh(s^* + s_1 + t_0 - t)|, \\ &\leq -2\epsilon r^2|\tanh(s^* + s_1 + t_0 - t)|.\end{aligned}$$

If we choose t large enough so that $s^* \leq \frac{1}{2}t - s_1$ and $|\tanh(t_0 - \frac{1}{2}t)| \geq \frac{1}{2}$ then

$$\begin{aligned}\frac{d}{ds^*}r^2 &\leq -2\epsilon r^2|\tanh(t_0 - \frac{t}{2})|, \\ &\leq -2\epsilon r^2\frac{1}{2}, \\ &= -\epsilon r^2.\end{aligned}$$

Integrating,

$$\begin{aligned}r^2(s^* + s_1, t_0 - t, r_0) &\leq r_1^2 \exp\{-\epsilon s^*\}, \\ &\leq (1+\delta)\exp\{-\epsilon s^*\}, \\ &\leq 1 - \delta,\end{aligned}$$

for all $s^* \geq \ln((1+\delta)/(1-\delta))/\epsilon$. For ease of notation define the value $s_2 = \ln((1+\delta)/(1-\delta))/\epsilon$. Thus we now have

$$r^2(s, t_0 - t, r_0) \leq 1 - \delta \qquad \forall s_1 + s_2 \leq s \leq \frac{t}{2},$$

and

$$\begin{aligned}&1) \quad t \geq 2(s_1 + s_2). \\ &2) \quad |\tanh(t_0 - \frac{t}{2})| \geq \frac{1}{2}.\end{aligned} \tag{4.13}$$

4.2.5 Behaviour For r < 1

Consider the evolution of the trajectory within the unit circle. Let $r_2 = r(s_1 + s_2, t_0 - t, r_0)$ with s_1, s_2 and t defined as above. Here $r_2^2 \leq 1 - \delta$. Then we are interested in the behaviour of

$$r(s^* + s_1 + s_2, t_0 - t, r_0),$$

where $s = s^* + s_1 + s_2$.

Now $r^2(s^* + s_1 + s_2, t_0 - t, r_0) \leq r_2^2 \leq 1 - \delta$. Then from the original equation (4.12) we obtain

$$\begin{aligned} \frac{d}{ds^*} r^2 &= -2r^2(1-r^2)^2 - 2\epsilon r^2 |\tanh(s^* + s_1 + s_2 + t_0 - t)|, \\ &\leq -2r^2\delta^2. \end{aligned}$$

Integrating,

$$\begin{aligned} r^2(s^* + s_1 + s_2, t_0 - t, r_0) &\leq r_2^2 \exp\left\{-2\delta^2 s^*\right\}, \\ &\leq (1-\delta) \exp\left\{-2\delta^2 s^*\right\}, \\ r(s^* + s_1 + s_2, t_0 - t, r_0) &\leq \sqrt{1-\delta} \exp\left\{-\delta^2 s^*\right\}. \end{aligned}$$

4.2.6 The Pullback Attractor

Allowing the pullback term to run to completion, that is by letting $s^* + s_1 + s_2 = t$, and for t satisfying (4.13), we have

$$\begin{aligned} r(t, t_0 - t, r_0) &\leq \sqrt{1-\delta} \exp\left\{\delta^2(s_1 + s_2)\right\} \exp\left\{-\delta^2 t\right\}, \\ &= A(r_0, \delta, \epsilon) \exp\left\{-\delta^2 t\right\}, \end{aligned}$$

where A is a constant replacing and simplifying the expression on the previous line. The terms that constitute the definition of A are:

$$\begin{aligned} A(r_0, \delta, \epsilon) &= \sqrt{1-\delta} \exp\left\{\delta^2(s_1 + s_2)\right\}. \\ s_1 &= \frac{1}{2\delta^2} \ln\left(\frac{r_0^2}{1+\delta}\right). \\ s_2 &= \frac{1}{\epsilon} \ln\left(\frac{1+\delta}{1-\delta}\right). \end{aligned}$$

Therefore the trajectory of any initial state $x_0 \in \mathbb{R}^2$ pullback converges to the origin for any time t_0.

$$\lim_{t\to\infty} r(t, t_0 - t, r_0) \leq \lim_{t\to\infty} A(r_0, \delta, \epsilon) \exp\left\{-\delta^2 t\right\} = 0.$$

Hence,

$$A^\epsilon(t_0) = 0 \qquad \forall t_0 \in \mathbb{R}.$$

The motion of initial states for this system can be seen illustrated below for four points originating outside the unit circle. The motion follows a similar pattern to that of the original system, however as the state approaches the original attractor, it is pushed across the boundary by the small perturbation, falling into a region spiraling into the origin as in the original autonomous system. The perturbation has caused a collapse of the attractor, and here convergence to the original semi-group attractor is upper semi-continuous but not lower semi-continuous.

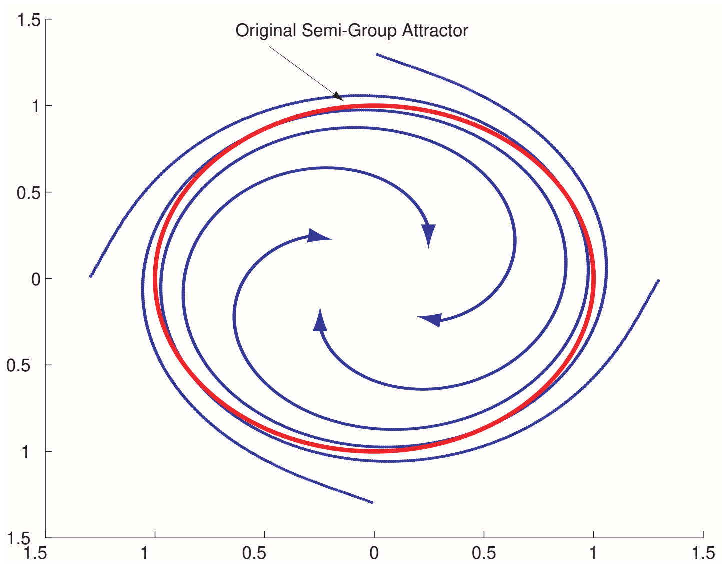

Figure 4.2: Pullback Attractor for a Perturbed Limit Cycle

4.3 Discretised Autonomous Systems

In the following we consider the same problem under circumstances where the solution has been numerically approximated using a one-step numerical scheme (refer to Section 3.1). The results here are supplementary to those presented in [22].

4.3.1 The Discretised Perturbed System

Again, we will consider the perturbed system

$$\dot{x} = f(x) + \epsilon g(\mathtt{t}, x), \tag{4.14}$$

where $\epsilon > 0$ is a small perturbation parameter. Solutions generated by a numerical scheme acting upon the above equation are discrete cocycle mappings for which we will use the notation (as in Definition 3.2.1)

$$x_{n+n_0} = \Phi^{\epsilon,h}_{(n,n_0)}(x_0).$$

where h is the step size (h for constant step sizes, and $\mathbf{h}$ for variable step sequences) for the numerical scheme. Also, as mentioned previously, the autonomous system has a local semi-group attractor, A_0.

In Section 4.1, it was shown that the perturbed system generates a corresponding continuous cocycle attractor, $A^\epsilon(t_0)$ which has components close to A_0 (for small ϵ), and is in fact, upper semi-continuous with respect to A_0.

We will proceed to show that the numerical scheme for (4.2) also generates a discretised cocycle attractor which is upper semi-continuous with respect to the original autonomous semi-group attractor.

4.3.2 Main Result

Theorem 4.3.1. *Suppose **P1-P3** hold for the non-autonomously perturbed dynamical system (4.2).*

Then a numerical scheme applied to the perturbed non-autonomous system (4.2), generates a discrete cocycle $\{\Phi^{\epsilon,h}_{(n,t_0)}, n \in \mathbb{Z}^+, t_0 \in \mathbb{R}\}$. *The discretised* *perturbed system possesses a discrete pullback attractor $\hat{A}^{\epsilon,h} = \{A^{\epsilon,h}(t); t \in \mathbb{R}\}$ (where h is the step size for the numerical scheme) such that*

$$\lim_{\epsilon,h\to 0^+} H^*(A^{\epsilon,h}(t_0), A_0) = 0.$$

Proof: 1. Existence: As in Section 4.1 we will consider the Lyapunov function V associated with A_0 in the autonomous system, and its nature on a neighbourhood of the semi-group attractor defined by $\mathcal{N}(A_0) = \{x \in \mathcal{N}_R(A_0); V(x) < a(R)\}$ within the context of the perturbed discretised system.

Recalling the inequality (4.4), for the upper left Dini Derivative of the Lyapunov function V for an arbitrary solution of the perturbed non-autonomous equation, it was found that

$$\begin{aligned} D^+_{(4.2)}V(x) &\leq LK\epsilon - cV(x), \\ &\leq -LK\epsilon, \end{aligned}$$

for all $x \notin B^\epsilon$, where B^ϵ was defined as

$$B^\epsilon = \{x \in \mathcal{N}_R(A_0) : V(x) \leq 2LK\epsilon/c\}.$$

B^ϵ was shown to be a pullback absorbing neighbourhood for solutions of the perturbed continuous system. Also recall from above, for all $x_0 \in \mathcal{N}(A_0)\backslash B^\epsilon$

$$V(\Phi_{(\tau,t_0-t)}(x_0)) \leq V(x_0) - LK\epsilon\tau,$$

where $0 \leq \tau \leq t$ such that $\Phi_{(\tau,t_0-t)}(x_0) \notin B^\epsilon$.

To construct a similar pullback absorbing neighbourhood for numerical solutions, we first note the behaviour of the Lyapunov function for a single step of the numerical scheme. Utilising the Lipschitz constant L associated with the Lyapunov function, and the local truncation error $C_p h^{p+1}$ for a single step in the numerical scheme, we obtain

$$\begin{aligned} V(x_{n+1}) &\leq |V(x_{n+1}) - V(\Phi_{(h,t_n)}(x_n))| + |V(\Phi_{(h,t_n)}(x_n))| \\ &\leq LC_p h^{p+1} + \begin{cases} V(x_n) - LK\epsilon h......(1) \\ 2LK\epsilon/c.................(2) \end{cases} \end{aligned}$$

(1) if $\Phi_{(\tau,t_n)}(x_n) \notin B^\epsilon \quad \forall \tau \leq h$.

(2) if $\Phi_{(\tau,t_n)}(x_n) \in B^\epsilon$ for some $\tau \leq h$. Recall that B^ϵ is a pullback absorbing neighbourhood for the continuous system and hence $\Phi_{(t,t_n)}(x_n) \in B^\epsilon \quad \forall t$ with $\tau \leq t \leq h$.

Now we propose as a pullback absorbing neighbourhood for the numerical scheme, namely:

$$B^{\epsilon,h} = \{x \in \Re^d : V(x) \leq 2LK\epsilon/c + LC_p h^{p+1}\}.$$

If ϵ and h are small enough so that

$$\epsilon < ca(R)/4LK \qquad \text{and} \qquad h^{p+1} < a(R)/2LC_p, \tag{4.15}$$

then $B^{\epsilon,h}$ is strictly a subset of the neighbourhood $\mathcal{N}(A_0)$.

To show that the set $B^{\epsilon,h}$ indeed forms a pullback absorbing neighbourhood for the numerical scheme, we need to show that it pullback absorbs $\mathcal{N}(A_0)$ in finite time.

Consider the pullback evolution of any point $x_0 \in \mathcal{N}(A_0)$ for the discretised system,

$$\Phi^{\epsilon,h}_{(n,\theta_{-n}t_0)}(x_0).$$

Let us firstly take $x_0 \notin B^{\epsilon,h}$. Then

$$V(\Phi^{\epsilon,h}_{(1,\theta_{-n}t_0)}(x_0)) \leq LC_p h^{p+1} + V(x_0) - LK\epsilon h.$$

Now if we choose a h_1 small enough so that $\forall h \leq h_1$, we have

$$h^p < K\epsilon/2C_p. \tag{4.16}$$

Then,

$$V(\Phi^{\epsilon,h}_{(1,\theta_{-n}t_0)}(x_0)) \leq V(x_0) - LK\epsilon h/2.$$

Correspondingly,

$$V(\Phi^{\epsilon,h}_{(\eta,\theta_{-n}t_0)}(x_0)) \leq V(x_0) - \eta LK\epsilon h/2,$$

for all $\eta < n^*_{x_0}$, and some $n^*_{x_0} < n$ such that $\Phi^{\epsilon,h}_{(\eta,\theta_{-n}t_0)}(x_0) \notin B^{\epsilon,h}$, and

$$\Phi^{\epsilon,h}_{(n^*_{x_0},\theta-nt_0)}(x_0) \in B^{\epsilon,h}. \tag{4.17}$$

$B^{\epsilon,h}$ **is positively invariant**. To see this note that either $x_n \in B^{\epsilon}$ with

$$V(x_{n+1}) \leq LC_p h^{p+1} + 2LK\epsilon/c,$$

and by definition x_{n+1} is automatically an element of $B^{\epsilon,h}$, or we have $x_n \in B^{\epsilon,h} \backslash B^{\epsilon}$, in which case

$$V(x_{n+1}) \leq V(x_n) + LC_p h^{p+1} - LK\epsilon h.$$

Then if $h \in [0, h_1]$, and since $x_n \in B^{\epsilon,h}$, we have $V(x_n) \leq 2LK\epsilon/c + LC_p h^{p+1}$, the following inequalities hold

$$\begin{aligned} V(x_{n+1}) &\leq V(x_n) + LC_p h^{p+1} - LK\epsilon h \\ &\leq 2LC_p h^{p+1} + 2LK\epsilon/c - LK\epsilon h \\ &\leq LC_p h^{p+1} + 2LK\epsilon/c - LK\epsilon h/2 \\ &\leq LC_p h^{p+1} + 2LK\epsilon/c. \end{aligned}$$

Hence, $x_{n+1} \in B^{\epsilon,h}$, and $B^{\epsilon,h}$ is positively invariant.

Returning to (4.17), and since $B^{\epsilon,h}$ is positively invariant,

$$\Phi^{\epsilon,h}_{(n,\theta_{-n}t_0)}(x_0) \in B^{\epsilon,h},$$

for all $n > n^*_{x_0}$ and $h < h_1$.

Finally, an upper bound n^* for $n^*_{x_0}$ exists for all $x_0 \in \mathcal{N}(A_0)$ since the Lyapunov function is bounded by $a(R)$ on $\mathcal{N}(A_0)$. Hence

$$\Phi^{\epsilon,h}_{(n,\theta-nt_0)}(\mathcal{N}(A_0)) \subset B^{\epsilon,h},$$

for all $n > n^*$ and $h < h_1$.

Thus our proposed set $B^{\epsilon,h}$ is indeed a pullback absorbing neighbourhood for solutions of the numerical scheme. We can then apply Theorem 3.3.1 to this to verify the existence of a discretised pullback attractor, $\hat{A}^{\epsilon,h}$, within $B^{\epsilon,h}$.

2. Approximation: For $x \in B^{\epsilon,h}$ we have

$$a(\mathrm{dist}(x, A_0)) \leq \ V(x) \leq 2LK\epsilon/c + LC_p h^{p+1},$$

so $\mathrm{dist}(x, A_0) \leq a^{-1} (2LK\epsilon/c + LC_p h^{p+1})$. Hence

$$H^*(B^{\epsilon,h}, A_0) \leq a^{-1} \left(2LK\epsilon/c + LC_p h^{p+1}\right).$$

Since $A^h_\epsilon(t_0) \subset B^{\epsilon,h}$, we have

$$H^*(A^{\epsilon,h}(t_0), A_0) \leq \alpha^{-1}(2LK\epsilon/c + LC_p h^{p+1}),$$

and

$$H^*(A^{\epsilon,h}(t_0), A_0) \to 0^+ \qquad \text{as} \quad \epsilon, h \to 0^+,$$

for arbitrary t_0.

This completes the proof of Theorem 4.3.1.

□

Remark 1. As in the result for the perturbed continuous system, the discrete pullback attractor is only guaranteed to exist under restricted values for both the size of the perturbation and step size, given by (4.15) and (4.16). If the original semi-group attractor A_0 is in fact a global attractor, then only the step size need be restricted by Equation 4.16.

4.3.3 Corollary: Upper Semi-Continuity

Theorems 4.1.1 and 4.3.1 derive the existence of a pullback attractor within the perturbed system (continuous and discretised) under certain conditions and make comparisons for them with the original semi-group attractor. Comparisons can also be made between the continuous and discretised pullback attractors, as outlined in the corollary below. It establishes that in the limit as $h \to 0^+$, every point on the discretised pullback attractor is arbitrarily close to a point on the continuous pullback attractor. This is known as *upper semi-continuity*.

A similar and more general result is achieved for semi-group attractors in *autonomous systems* by A.Stuart, [34], where it is noted that lower semi-continuity is impossible to achieve without placing strong conditions on the dynamical system.

The corresponding upper semi-continuity result for non-autonomously perturbed autonomous systems is provided below.

Corollary 4.3.1. *$\hat{A}^{\epsilon,h}$, is upper semi-continuous with respect to $\hat{A}^{\epsilon}$. That is for each $n \in \mathbb{Z}$, and corresponding $t_n \in \mathbb{R}$ where $t_n = t_0 + nh$,*

$$\lim_{h \to 0^+} H^*(A^{\epsilon,h}(t_n), A^{\epsilon}(t_n)) = 0. \tag{4.18}$$

Proof: Suppose the above statement is false. Given some arbitrary n_0 and corresponding t_0 (for which the result may be generalised for any value of n and t_n), there exists a sequence $\{h_j\}$ with $h_j \to 0$ as $j \to \infty$, and an $\epsilon_0 > 0$ such that

$$H^*(A^{\epsilon,h_j}(t_0), A^{\epsilon}(t_0)) \geq \epsilon_0,$$

for all j. Hence there exists a sequence $\{a_j\}$ with $a_j \in A^{\epsilon,h_j}(t_0)$ such that for each j

$$\operatorname{dist}(a_j, A^{\epsilon}(t_0)) \geq \epsilon_0. \tag{4.19}$$

As the discrete pullback attractor is invariant, for each j there exists a corresponding sequence of values $\{b_{j_n}; b_{j_n} \in A^{\epsilon,h_j}(\theta_{-n}t_0)\}$ such that

$$\Phi^{\epsilon,h_j}_{(n,\theta_{-n}t_0)}(b_{j_n}) = a_j.$$

Define c_{j_n}, the continuous image of each b_{j_n} at t_0 by

$$c_{j_n} = \Phi^{\epsilon}_{(t_n, t_0 - t_n)}(b_{j_n}),$$

where $t_n = nh$. Now $b_{j_n} \in \mathcal{N}(A_0)$ for each n, and since $\hat{A}^{\epsilon}$ pullback attracts $\mathcal{N}(A_0)$ there exists a $T(\epsilon_0) > 0$, and corresponding $N > 0$ (defined by $Nh > T$, and $(N-1)h < T$) such that

$$\begin{aligned}\operatorname{dist}(c_{j_N}, A^{\epsilon}(t_0) &\leq H^*(\Phi^{\epsilon}_{(t_N, t_0 - t_N)}(\mathcal{N}(A_0)), A^{\epsilon}(t_0)),\\ &< \epsilon_0/2.\end{aligned} \tag{4.20}$$

Also, given the cumulative numerical error arising between the continuous and numerical solutions, there exists a $J(\epsilon_0) > 0$ such that for all $j > J$,

$$\begin{aligned} \mathrm{dist}(a_j, c_{j_N}) &\leq NC_p h_j^{p+1}, \\ &< (T + h_j)C_p h_j^p, \\ &< \epsilon_0/2. \end{aligned} \tag{4.21}$$

Combining both equations (4.20) and (4.21),

$$\begin{aligned} \mathrm{dist}(a_j, A^\epsilon(t_0)) &\leq \mathrm{dist}(a_j, c_{j_N}) + \mathrm{dist}(c_{j_N}, A^\epsilon(t_0)), \\ &< \epsilon_0/2 + \epsilon_0/2, \\ &< \epsilon_0, \end{aligned}$$

for all $j > J$ as defined earlier. This contradicts the proposition (4.19), and hence the original statement is true.

$\square$

4.3.4 Corollary: Variable Time-Step Discretisation

A similar numerical process may be applied to the continuous perturbed system (4.2) using a variable time-step discretisation as outlined in Subsection 3.2.3. The variable time-step is represented as a bi-infinite real sequence $\mathbf{h} = \{h_n\}_{n\in\mathbb{Z}}$ bounded by some fixed constant $\rho > 0$ such that $\frac{1}{2}\rho \leq h_n \leq \rho$ for all n.

Corollary 4.3.2. *Consider a variable time-step numerical scheme applied to the perturbed system under the conditions existing for Theorem 4.3.1. The discretised system generates a cocycle* $\{\Phi^{\mathbf{h}}_{(n,(t_0,\mathbf{h}))}; n \in \mathbb{Z}, (t_0, \mathbf{h})) \in \mathbb{R} \times H^\rho\}$ *over the parameter set* $P = \mathbb{R} \times H^\rho$ *with shift mapping* $\theta_n(t_0, \mathbf{h}) = (t_n, \psi_n \mathbf{h})$. *Then the variable time-step discretisation possesses a discrete pullback attractor* $\hat{A}^{\mathbf{h}}_\epsilon$ *such that*

$$\lim_{\epsilon,\rho\to 0^+} H^*(A^{\mathbf{h}}_\epsilon(t_0, \mathbf{h}), A_0) = 0$$

Proof: The analysis follows similarly to the proof for Theorem 4.3.1, utilising the bound ρ on the variable time-step sequence.

1. Existence: Recalling the details concerning the Lyapunov function associated with the semi-group attractor, and using the definition for the pullback absorbing neighbourhood set B^ϵ of the continuous perturbed system we have

$$\begin{aligned} V(x_{n+1}) &\leq |V(x_{n+1}) - V(\Phi_{(h_n,t_n)}(x_n))| + |V(\Phi_{(h_n,t_n)}(x_n))| \\ &\leq LC_p h_n^{p+1} + |V(\Phi_{(h_n,t_n)}(x_n))| \\ &\leq LC_p \rho^{p+1} + \begin{cases} V(x_n) - LK\epsilon h_n ......(1) \\ 2LK\epsilon/c.................(2) \end{cases} \end{aligned}$$

(1) if $\Phi_{(\tau,t_n)}(x_n) \notin B^\epsilon \quad \forall \tau \leq h_n$.

(2) if $\Phi_{(\tau,t_n)}(x_n) \in B^\epsilon$ for some $\tau \leq h_n$. Recall that B^ϵ is a pullback absorbing neighbourhood for the continuous system and hence $\Phi_{(t,t_n)}(x_n) \in B^\epsilon \quad \forall t$ with $\tau \leq t \leq h_n$.

We propose as a discrete pullback absorbing neighbourhood for this system the set defined by

$$B^{\epsilon,\mathbf{h}} = \{x \in \mathbb{R}^d : V(x) \leq 2LK\epsilon/c + LC_p\rho^{p+1}\}.$$

First, if ϵ and ρ are chosen small enough so that

$$\epsilon < ca(R)/4LK \qquad \text{and} \qquad \rho^{p+1} < a(R)/2LC_p, \tag{4.22}$$

then $B^{\epsilon,\mathbf{h}}$ is strictly a subset of the neighbourhood $\mathcal{N}(A_0)$.

To show that it is pullback absorbing we consider pullback evolution of any point $x_0 \in \mathcal{N}(A_0)$ for the discretised system, with the restriction on the variable time-step bound ρ such that $\rho \leq \rho_1$ where

$$\rho_1^p = K\epsilon/2C_p.$$

The remainder of the proof follows automatically along the same lines as in Theorem 4.3.1 by showing that the set $B^{\epsilon,\mathbf{h}}$ is positively invariant, and then that it pullback absorbs the neighbourhood $x_0 \in \mathcal{N}(A_0)$. That is, there exists an $n^* > 0$ such that

$$\Phi^{\epsilon,\mathbf{h}}_{(n,\theta_{-n}(t_0,\mathbf{h})}(\mathcal{N}(A_0)) \subset B^{\epsilon,\mathbf{h}},$$

for all $n > n^*$ and $\rho < \rho_1$.

Finally, applying Theorem 3.3.1 we can verify the existence of the discretised pullback attractor $\hat{A}^{\mathbf{h}}_{\epsilon}$.

2. Approximation: For $x \in B^{\epsilon,\mathbf{h}}$ we have

$$a(\mathrm{dist}(x, A_0)) \leq \; V(x) \leq 2LK\epsilon/c + LC_p\rho^{p+1},$$

so $\mathrm{dist}(x, A_0) \leq a^{-1}\left(2LK\epsilon/c + LC_p\rho^{p+1}\right)$. Hence

$$H^*(B^{\epsilon,\mathbf{h}}, A_0) \leq a^{-1}\left(2LK\epsilon/c + LC_p\rho^{p+1}\right).$$

Since $A^{\mathbf{h}}_{\epsilon}((t_0, \mathbf{h}) \subset B^{\epsilon,\mathbf{h}}$, we have

$$H^*(A^{\mathbf{h}}_{\epsilon}(t_0, \mathbf{h}), A_0) \leq \alpha^{-1}(2LK\epsilon/c + LC_p\rho^{p+1}),$$

and consequently,

$$H^*(A^{\mathbf{h}}_{\epsilon}(t_0, \mathbf{h}), A_0) \to 0^+ \qquad \text{as} \quad \epsilon, \rho \to 0^+,$$

for arbitrary t_0.

$\square$

Chapter 5

Lyapunov Theory for Non-Autonomous Dynamical Systems

5.1 Introduction

Lyapunov functions provide an effective practical and theoretical tool in assisting in the analysis of a dynamical's system stability, either in verification of its stability or as a method of determining controls to ensure its stability. They have also been used theoretically to characterise a particular stability property, which has been useful in approximating asymptotically stable sets in perturbed autonomous systems (of which numerical approximations are included) [34].

Lyapunov functions were introduced earlier (Section 1.2.4, 1.3.3) to illustrate their use and importance in asserting the forward stability of sets that remain constant over time. In this chapter we will concern ourselves with *time-varying* families of sets, and Lyapunov functions associated with forward, complete and pullback stability results respectively and the difficulties arising with using such techniques for pullback behaviour.

To clearly state the problem we will initially define a few general assumptions concerning the dynamical system under investigation and the character of Lyapunov functions.

We consider the non-autonomous differential equation

$$\dot{x} = f(p, x), \tag{5.1}$$

and assume the following properties hold.

F1: The function $f(p, x)$ is continuous in both p and x.

F2: $f(p, \cdot)$ satisfies locally a Lipschitz condition with respect to x. That is, for any given $\delta > 0$, and x' such that $|x - x'| < \delta$, there exists a constant $L(p, \delta)$ satisfying

$$|f(p, x) - f(p, x')| < L(p, \delta)|x - x'|.$$

F3: There is a group of mappings $\{\theta_t, t \in \mathbb{R}^+\}$ with $\theta_t : P \to P$, continuous on P and satisfying $\theta_t \circ \theta_\tau = \theta_{t+\tau}$ for all t, $\tau \in T$.

A Lyapunov function $V(p, x)$ will be assumed to be a scalar continuous function that satisfies locally a Lipschitz condition with respect to x. The Dini Derivative is used to measure the rate of change of V for trajectories in (5.1). It is repeated here for ease of reference.

$$\overline{D_t}^{+} V(p, x) = \overline{\lim_{h \to 0}} \frac{V(\theta_h p, \Phi_{(h,p)}(x)) - V(p, x)}{h}. \tag{5.2}$$

5.1.1 Lemmas

The subsequent lemmas will be utilised throughout the rest of this chapter in the various Lyapunov proofs. The first ascertains a single neighbourhood for a uniformly forward asymptotically stable family, for which solutions from any state within the neighbourhood are guaranteed to remain close for all time, and to also attract asymptotically toward the sets in consideration.

Lemma 5.1.1. *If $\hat{A} = \{A(p); p \in P\}$ is uniformly forward asymptotically stable, then given any $\epsilon^* > 0$ there exists a $\delta = \delta(\epsilon^*)$ such that for any $\epsilon > 0$, and each $p \in P$, $x \in \mathcal{N}_\delta(A(p))$,*

$$\begin{array}{lll} i) & \mathrm{dist}(\Phi_{(t,p)}(x), A(\theta_t p)) < \epsilon^* & \textit{for all} \quad t > 0, \\ ii) & \mathrm{dist}(\Phi_{(t,p)}(x), A(\theta_t p)) < \epsilon & \textit{for all} \quad t > T(\epsilon), \end{array}$$

where $T = T(\epsilon)$ is as defined for uniform forward asymptotic stability. The same parameters then also hold for uniform pullback asymptotic stability.

Proof: Let $\delta = \min\{\delta_1, \delta_2\}$ where $\delta_1 = \delta_1(\epsilon^*)$ as defined for stability, and δ_2 as defined for asymptotic attraction. The results i), ii) follow immediately.

Due to uniformity, the equivalence of forward and pullback stability guarantees the same parameters will satisfy the conditions for pullback asymptotic stability.

□

The following two lemmas (the first being Gronwall's Lemma which can be found in one version or another in various texts, see for example [25]) provide bounds on differences in solutions beginning from initial values close to one another.

Lemma 5.1.2 (Gronwall's Lemma)**.** *Given x_0, x_1 such that $|\Phi_{(t,p)}(x_0) - \Phi_{(t,p)}(x_1))| < \epsilon$, for some $\epsilon > 0$, and any $t > 0$, then*

$$|\Phi_{(t,p)}(x_0) - \Phi_{(t,p)}(x_1)| \leq |x_0 - x_1| \exp\left(\int_0^t L(\theta_s p, \epsilon) ds\right).$$

Proof: We have

$$\begin{aligned} |\Phi_{(t,p)}(x_0) - &\Phi_{(t,p)}(x_1)| \\ &\leq |x_0 - x_1| + \int_0^t |f(s, \Phi_{(s,p)}(x_0)) - f(s, \Phi_{(s,p)}(x_1))| ds, \\ &\leq |x_0 - x_1| + \int_0^t L(\theta_s p, \epsilon) |\Phi_{(s,p)}(x_0) - \Phi_{(s,p)}(x_1))| ds. \end{aligned}$$

Taking the derivative with respect to t,

$$\frac{d|\Phi_{(t,p)}(x_0) - \Phi_{(t,p)}(x_1)|}{dt} \leq L(\theta_t p, \epsilon) |\Phi_{(t,p)}(x_0) - \Phi_{(t,p)}(x_1))|,$$

or

$$\frac{d}{dt}\left(|\Phi_{(t,p)}(x_0) - \Phi_{(t,p)}(x_1)| \exp\left(-\int_0^t L(\theta_s p, \epsilon) ds\right)\right) \leq 0,$$

Hence we arrive at the required result,

$$|\Phi_{(t,p)}(x_0) - \Phi_{(t,p)}(x_1)| \leq |x_0 - x_1| \exp\left(\int_0^t L(\theta_s p, \epsilon) ds\right).$$

□

The following lemma is an elementary result that provides a bound for differences of solutions that lie on the same trajectory. It is also feasible to consider the maximum over larger intervals, as will be assumed in Theorem 5.2.5.

Lemma 5.1.3. *For each p, x such that any solutions in future time exist and are unique, and any $t' < t$,*

$$|\Phi_{(t,p)}(x) - \Phi_{(t',p)}(x)| \leq \max_s\{|f(\theta_s p, x)|\}|t - t'|,$$

where the maximum is taken over the interval $t' < s < t$.

5.2 Forward Lyapunov Theory

At present there are several results concerning forwards stability of time varying families of sets (refer to Yoshizawa, [42]), though these results are infrequently used due to lack of information concerning attraction to the family of sets anywhere except approaching infinity. The results for Forward Lyapunov Theory for families of sets are similar, in general terms, to those of Yoshizawa. They are extended here to incorporate an analysis for local neighbourhoods, and to allow for general parameter sets P (as opposed to the construction used in [42] where $P = \mathbb{R}^+$).

5.2.1 Sufficiency Theorems

Theorem 5.2.1 (Forward Stability)**.**
Given a family of uniformly bounded compact sets $\hat{A} = \{A(p); p \in P\}$, suppose there exists a Lyapunov function $V : P \times \hat{\mathcal{N}}_{\epsilon,\hat{A}} \to \mathbb{R}$ for some $\epsilon > 0$ which satisfies the following conditions:

a) $V(p, x) = 0$ *for each* $p \in P$ *and* $x \in A(p)$*,*

b) $a(dist(x, A(p))) \le V(p, x)$*, where* $a \in \mathcal{K}$*,*

c) $\overline{D_t}^+ V(p, x) \le 0$*.*

d) $V(p, x)$ *is continuous in both variables and locally Lipschitz in* x*.*

Then $\hat{A}$ *is* ***forward stable****.*

Proof:
Let $p \in P$ be arbitrarily chosen. For any $\epsilon > 0$ we may choose a $\delta_p(\epsilon) > 0$ such that

$$\text{dist}(x_0, A(p)) < \delta_p \implies V(p, x_0) < a(\epsilon),$$

because of the continuity of V in the state variable. Now suppose that some solution $\Phi_{(t^*,p)}(x_0)$ with $\text{dist}(x_0, A(p)) < \delta_p$ satisfies

$\mathrm{dist}(\Phi_{(t^*,p)}(x_0), A(\theta_{t^*}p)) = \epsilon$ at some $t^* > 0$. We have by property *b)*,

$$\begin{aligned} a(\epsilon) &\leq V(\theta_{t^*}p, \Phi_{(t^*,p)}(x_0)), \\ &\leq V(p, x_0), \\ &< a(\epsilon). \end{aligned}$$

This is a contradiction. Hence all trajectories with initial values $x_0 \in \mathcal{N}_{\delta_p}(A(p))$ must remain within an epsilon neighbourhood of $\hat{A}$. Thus $\hat{A}$ is forward stable.

□

Theorem 5.2.2 (Uniform Forward Stability).
Given a family of uniformly bounded compact sets $\hat{A} = \{A(p); p \in P\}$, suppose there exists a Lyapunov function $V : P \times \hat{\mathcal{N}}_{\epsilon,\hat{A}} \to \mathbb{R}$ for some $\epsilon > 0$ which satisfies the following conditions:

a) $V(p,x) = 0$ *for each* $p \in P$ *and* $x \in A(p)$*,*

b) $a(dist(x, A(p))) \leq V(p,x) \leq b(dist(x, A(p)))$ *where* $a, b \in \mathcal{K}$*,*

c) $\overline{D_t}^+ V(p,x) \leq 0$*,*

d) $V(p,x)$ *is continuous in both variables and locally Lipschitz in* x*.*

Then $\hat{A}$ *is* ***uniformly forward stable****.*

Proof:

Choose $\delta > 0$, $\delta = \delta(\epsilon)$ so that $b(\delta) < a(\epsilon)$. Note that this is possible since a, b are both continuous and class $\mathcal{K}$.

We conclude that solutions originating from within $\mathcal{N}_\delta(A(p))$, for arbitrary $p \in P$, must remain within the ϵ - neighbourhood of $\hat{A}$ for all future times.

To see this, assume otherwise. That is, there exists some $p \in P$, $x \in \mathcal{N}_\delta(A(p))$, and a time $t^* > 0$ such that

$$dist(\Phi_{(t^*,p)}(x), A(\theta_{t^*}p)) = \epsilon.$$

Then we have,

$$a(\epsilon) \leq V(\theta_{t^*}p, \Phi_{(t^*,p)}(x)) \leq V(p,x) \leq b(\delta) < a(\epsilon),$$

which is a contradiction. Hence the original assertion must be valid, and $\hat{A}$ is uniformly forward stable.

□

Theorem 5.2.3 (Uniform Forward Asymptotic Stability).
Given a family of uniformly bounded compact sets $\hat{A} = \{A(p); p \in P\}$, suppose there exists a Lyapunov function $V : P \times \hat{\mathcal{N}}_{\epsilon_o,\hat{A}} \to \mathbb{R}$ for some $\epsilon_0 > 0$ which satisfies the following conditions:

a) $V(p,x) = 0$ for each $p \in P$ and $x \in A(p)$,

b) $a(dist(x, A(p))) \leq V(p,x) \leq b(dist(x, A(p)))$ where $a, b \in \mathcal{K}$,

c) $\overline{D_t}^+ V(p,x) \leq -cV(p,x)$ for some constant $c > 0$,

d) $V(p,x)$ is continuous in both variables and locally lipschitz in x.

*Then $\hat{A}$ is **uniformly forward asymptotically stable**.*

Proof:

By Theorem 5.2.1, $\hat{A}$ is uniformly forward stable. Thus there exists some $\delta_0 = \delta_0(\epsilon_0)$ such that for all $p \in P$, $x \in \mathcal{N}_{\delta_0}(A(p))$, solutions are guaranteed to exist and remain within the domain of definition for $V(p,x)$. That is, for all $t > 0$,

$$dist(\Phi_{(t,p)}(x), A(\theta_t p) < \epsilon_0.$$

Consider some $\epsilon > 0$ with $\epsilon \leq \delta_0$. Then there exists a $\delta = \delta(\epsilon)$ as defined for uniform forward stability. It will be shown that every solution from $x \in \mathcal{N}_{\delta_0}(A(p))$, satisfies

$$\mathrm{dist}(\Phi_{(t^*,p)}(x), A(\theta_{t^*}p)) < \delta(\epsilon),$$

at some time $t^* > 0$.

Assume that this is not the case. Then there exists some $x \in \mathcal{N}_{\delta_0}(A(p))$ such that

$$\operatorname{dist}(\Phi_{(t,p)}(x), A(\theta_t p)) > \delta(\epsilon), \qquad \forall t > 0.$$

Now by c),

$$V(\theta_t p, \Phi_{(t,p)}(x)) \leq e^{-ct} V(p, x),$$

for all $t > 0$. Let $T = -\ln[a(\delta)/b(\delta_0)]/c$. Then

$$a(\delta) \leq V(\theta_t p, \Phi_{(t,p)}(x)) \leq e^{-ct} V(p, x) < a(\delta),$$

for all $t > T$. Consequently, we have a contradiction. Thus there exists some $t^* \leq T$ such that $\operatorname{dist}(\Phi_{(t^*,p)}(x), A(\theta_{t^*} p)) < \delta(\epsilon)$, and hence for all $t > T$ we have (by definition of $\delta(\epsilon)$)

$$\operatorname{dist}(\Phi_{(t,p)}(x), A(\theta_t p)) < \epsilon.$$

Since this argument holds for each $p \in P$ and all $x \in \mathcal{N}_{\delta_0}(A(p))$ we have the required result. Note that T depends on ϵ only (through $\delta(\epsilon)$), as needed for uniformity.

□

Remark: The required asymptotic attraction can be achieved with alternative (usually slightly weaker) conditions on the Dini Derivative (for example, with $\overline{D_t}^+ V(p, x) < 0)$) . The manipulation of the proof follows in a similar manner.

5.2.2 Converse Theorems

Theorem 5.2.4 (Uniform Forward Asymptotic Stability).
Suppose the dynamical system (5.1) possesses a family $\hat{A} = \{A(p); p \in P\}$ that is uniformly forward asymptotically stable. Then there exists a Lyapunov function $V : P \times \mathcal{N}_{K,\hat{A}}$ defined on a neighbourhood of $\hat{A}$, $\mathcal{N}_{K,\hat{A}}$ which satisfies:

a) $V(p, x) = 0$ for each $p \in P$ and $x \in A(p)$,

b) $a(dist(x, A(p))) \leq V(p, x) \leq b(dist(x, A(p)))$ where $a, b \in \mathcal{K}$, *c)* $\overline{D_t}^+ V(p,x) \le -cV(p,x)$ *for some constant* $c > 0$*,*

d) $V(p,x)$ *is continuous in both variables and locally Lipschitz in* x*.*

Proof:

Since $\hat{A}$ is a uniformly asymptotically stable family of sets, given any $\epsilon^* > 0$, then there exists a $\delta(\epsilon^*) > 0$ that satisfies the properties of Lemma 5.1.1.

For this $\epsilon^* > 0$, we define:

$$L(p) = L(p, \epsilon^*),$$

as the Lipschitz constant for f on a ϵ^* bounded region of the state space;

$$T = T(\epsilon),$$

as defined in property ii) of Lemma 5.1.1;

$$F(p, \epsilon) = 1 + \max |f(\theta_t p, x)|,$$

where the maximum is taken over all $-T(\epsilon) \le t \le T(\epsilon)$ and $x \in \mathcal{N}_{\epsilon^*} A(\theta_t p)$;

$$\mathcal{A}(p,\epsilon) = e^{cT(\epsilon)} 2F(p,\epsilon) \exp\left(\int_0^{T(\epsilon)} L(\theta_s p) ds\right),$$

defined for any arbitrarily chosen $c > 0$.

Utilising a slightly modified form of the result by J. Massera, detailed in [42], there exists functions l, g satisfying $l(p) > 0$, $0 < g(\epsilon) \le 1$ for $\epsilon > 0$ and $g(0) = 0$, such that

$$g(\epsilon)\mathcal{A}(p,\epsilon) \le l(p). \tag{5.3}$$

Finally we begin composing the Lyapunov function for the required purpose. For $n = 1, 2 \ldots,$ we define $V_n(p, x)$ for each $p \in P$ and $x \in \mathcal{N}_\delta(A(p))$ by:

$$V_n(p,x) = g(1/n) \sup \left\{ D_n(dist(\Phi_{(\tau,p)}(x), A(\theta_\tau p)))e^{c\tau}; \tau \geq 0 \right\}.$$

Here the function $D_n(r)$ is defined as

$$D_n(r) = \begin{cases} r - 1/n & (r \geq 1/n), \\ 0 & (0 \leq r \leq 1/n). \end{cases}$$

i) From the definition of $V_n(p, x)$ and the invariance of $A(p)$ it is clear that for each $p \in P$,

$$V_n(p,x) = 0 \qquad \forall x \in A(p). \tag{5.4}$$

ii) Lower Bound - Define $a_n(r)$ by

$$a_n(r) = g(1/n)A_n(r),$$

where $A_n(r)$ is given by

$$A_n(r) = \begin{cases} 1/\left(n(n-1)\right) & (r \geq 1/(n-1)), \\ r - 1/n & (1/n \leq r \leq 1/(n-1)), \\ 0 & (0 \leq r \leq 1/n). \end{cases}$$

Note that $A_n(r) \leq D_n(r)$, and that a_n is a non-negative monotonically increasing function and is continuous with respect to r. Then if we set $r = dist(x, A(p))$, we have

$$\begin{aligned} V_n(p,x) &\geq g(1/n)D_n(dist(x, A(p))), \\ &\geq a_n(r). \end{aligned}$$

iii) Upper Bound - Note that

$$D_n(dist(\Phi_{(\tau,p)}(x), A(\theta_\tau p))) = 0$$

for all $\tau > T(1/n)$. Using this property together with that of uniform forward stability for $\hat{A}$ we have,

$$\begin{aligned}
V_n(p,x) &= g(1/n)\sup\{D_n(dist(\Phi_{(\tau,p)}(x), A(\theta_\tau p)))e^{c\tau}; \tau \geq 0\}, \\
&\leq g(1/n)\sup\{D_n(\epsilon^*)e^{c\tau}; 0 \leq \tau \leq T(1/n)\}, \\
&\leq g(1/n)e^{cT(1/n)}\epsilon^*(dist(x, A(p))), \\
&\leq g(1/n)\mathcal{A}(p^*, 1/n)\epsilon^*(dist(x, A(p))), \\
&\leq l(p^*)\epsilon^*(dist(x, A(p))),
\end{aligned}$$

for any arbitrarily chosen $p^* \in P$. Here $\epsilon^*(dist(x, A(p)))$ refers to the inverse function associated with the function $\delta = \delta(\epsilon^*)$ corresponding to the uniform forward stability of $\hat{A}$. We have also used the fact that $e^{cT(1/n)} < \mathcal{A}(p^*, 1/n)$ for arbitrarily chosen $p^* \in P$.

iv) Decrescence - Let $h > 0$ be some constant. Now for any $x \in \mathcal{N}_\delta(A(p))$, let x^* denote the state at some time h later. That is, $x^* = \Phi_{(h,p)}(x)$. Then

$$\begin{aligned}
V_n(\theta_h p, x^*) &= g(1/n)\sup\{D_n(dist(\Phi_{(\tau,\theta_h p)}(x^*), A(\theta_{\tau+h} p)))e^{c\tau}; \tau \geq 0\} \\
&= g(1/n)\sup\{D_n(dist(\Phi_{(\tau+h,p)}(x), A(\theta_{\tau+h} p)))e^{c\tau}; \tau \geq 0\} \\
&= g(1/n)\sup\{D_n(dist(\Phi_{(\tau,p)}(x), A(\theta_\tau p)))e^{c\tau}e^{-ch}; \tau \geq h\} \\
&\leq e^{-ch}V_n(p,x).
\end{aligned}$$

Taking the Dini derivative of $V_n(p,x)$ for solutions at p, x we obtain,

$$\begin{aligned}
\overline{D_t}^+ V_n(p,x) &= \overline{\lim_{h\to 0^+}} \frac{V_n(\theta_h p, x^*) - V_n(p,x)}{h}, \\
&\leq \overline{\lim_{h\to 0^+}} \frac{(e^{-ch} - 1)V_n(p,x)}{h}, \\
&= -cV_n(p,x).
\end{aligned}$$

v) Continuity - Let $p, p' \in P$, such that $\theta_{t^*}p' = p$ for some $t^* > 0$, and $x \in \mathcal{N}_\delta(A(p)), x' \in \mathcal{N}_\delta(A(p'))$. Then

$$\begin{aligned}
|V_n(p,x) - V_n(p',x')| &\leq g(1/n)|\sup\{e^{c\tau}D_n(\Phi_{(\tau,p)}(x)); \tau \geq 0\} \\
&\quad - \sup\{e^{c\tau}D_n(\Phi_{(\tau,p')}(x')); \tau \geq 0\}|, \\
&\leq g(1/n)\sup\{|\Phi_{(\tau,p)}(x) - \Phi_{(\tau,p')}(x')|e^{c\tau}; 0 \leq \tau \leq T\}, \\
&\leq g(1/n)e^{cT(1/n)}\sup\{|\Phi_{(\tau,p)}(x) - \Phi_{(\tau,p)}(X)| \\
&\qquad + |\Phi_{(\tau+t^*,p')}(x') - \Phi_{(\tau,p')}(x')|; 0 \leq \tau \leq T\},
\end{aligned}$$

where $X = \Phi_{(t^*,p')}(x')$.

Set

$$\mathcal{L}(p,1/n) = \exp\left(\int_0^T L(\theta_{-s}p)ds\right),$$

for ease of notation, noting that $\mathcal{L}(p,1/n) > 1$ for all p, n. Now from Lemma 5.1.2 we have

$$\begin{aligned}
|V_n(p,x) - V_n(p',x')| \leq g(1/n)e^{cT(1/n)}\{F(p,1/n)|p-p'| \\
+\mathcal{L}(p,1/n)|X-x|\}. \qquad (5.5)
\end{aligned}$$

Let t^* be small enough so that $t^* < T(1/n)$, then

$$\begin{aligned}
|X-x| &\leq (|X-x'| + |x'-x|), \\
&\leq (F(p,1/n) < p,p' > +|x'-x|.
\end{aligned}$$

Substituting back into (5.5), we have

$$\begin{aligned}
&|V_n(p,x) - V_n(p',x')| \\
&\quad\leq g(1/n)e^{cT(1/n)}\{F(p,1/n)\,(1+\mathcal{L}(p,1/n)) < p,p' > \\
&\qquad +\mathcal{L}(p,1/n)|x-x'|\}, \\
&\quad\leq g(1/n)e^{cT(1/n)}F(p,1/n)2\mathcal{L}(p,1/n)) \\
&\qquad (< p,p' > +|x-x'|), \\
&\quad\leq g(1/n)\mathcal{A}(p,1/n)(< p,p' > +|x-x'|), \\
&\quad\leq l(p)(< p,p' > +|x-x'|),
\end{aligned}$$

for all p' close enough to p. Hence each $V_n(p, x)$ is continuous and locally Lipschitz with respect to both p, and x.

Finally we define the Lyapunov function V by

$$V(p, x) = \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p, x).$$

Note that convergence of this is automatically ensured as a consequence of iii). Properties a) - d) will be verified sequentially.

a) Obviously from i) we have for each $p \in P$ and all $x \in A(p)$

$$V(p, x) = 0.$$

b-i) Lower Bound - From ii), if we set

$$a(r) = \sum_{n=1}^{\infty} \frac{1}{2^n} a_n(r),$$

we have $a(r) \in \mathcal{K}$. Clearly $a(0) = 0$. Also $a(0) > 0$ for $r > 0$ since for any r there exists an n such that $(1/n) < r$ and hence $a(r) > a_n(r) = g(1/n)(r - 1/n)$. Since each a_n is upper bounded by $1/n(n-1)$, and $g(\cdot) < 1$, the Weierstrass M-test can be used to conclude that the infinite series is continuous. Finally, to see that it is a lower bound for $V(p, x)$,

$$\begin{aligned} V(p, x) &= \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p, x), \\ &\geq \sum_{n=1}^{\infty} \frac{1}{2^n} a_n(\operatorname{dist}(x, A(p))), \\ &\geq a(\operatorname{dist}(x, A(p))). \end{aligned}$$

b-ii) Upper Bound - From iii) we have

$$\begin{aligned}
V(p,x) &= \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p,x), \\
&\leq \sum_{n=1}^{\infty} \frac{1}{2^n} l(p^*)\epsilon^*(\mathrm{dist}(x, A(p))), \\
&\leq l(p^*)\epsilon^*(\mathrm{dist}(x, A(p))), \\
&\leq b(\mathrm{dist}(x, A(p))).
\end{aligned}$$

where $b(r) = l(p^*)\epsilon^*(r)$. Note that the function $b \in \mathcal{K}$ since $\epsilon^*(\delta)$ is continuous and monotonically increasing from zero.

c) Decrescence - From iv),

$$\begin{aligned}
V(\theta_h p, \Phi_{(h,p)}(x)) &= \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(\theta_h p, \Phi_{(h,p)}(x)), \\
&\leq e^{-ch} \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p,x), \\
&\leq e^{-ch} V(p,x).
\end{aligned}$$

Again, taking the Dini derivative for $V(p,x)$, we arrive at the required result.

d) Continuity and Lipschitz Properties of V - This follows directly from the continuity and Lipschitzness of each V_n in v).

□

Theorem 5.2.5 (Forward Equi-Asymptotic Stability).
Suppose the dynamical system (5.1) possesses a family $\hat{A} = \{A(p); p \in P\}$ that is forward equi - asymptotically stable. Then there exists a Lyapunov function V defined on a neighbourhood of $\hat{A}$, $\mathcal{N}_{K,\hat{A}}$, satisfying

a) $V(p,x) = 0$ *for each* $p \in P$ *and* $x \in A(p)$*,*

b) $a(\mathrm{dist}(x, A(p))) \leq V(p,x)$ *where* $a \in \mathcal{K}$*,* *c)* $\overline{D_t}^+ V(p,x) \leq -cV(p,x)$ *for some constant* $c > 0$*,*

d) V *is continuous in both variables and locally Lipschitz in* x*.*

Proof:

The proof is similar to that of Theorem 5.2.4 where the rate of attraction is now dependent on $T = T(p,\epsilon)$ and the neighbourhoods $\delta = \delta(p,\epsilon)$. The resulting properties follow identically with the exception that the upper bounding function $b \in \mathcal{K}$, does not hold.

□

5.2.3 Complete Lyapunov Theory

Recalling that uniformity guarantees complete stability/asymptotic stability (Lemmas 2.1.1, 2.1.2, 2.1.3, 2.1.4) it suffices to show using Lyapunov functions that if a family of sets $\hat{A} = \{A(p); p \in P\}$ is uniformly forward stable/asymptotically stable, then its stability is complete. That is, it is also uniformly pullback stable/asymptotically stable. Similarly, the converse theorems will also hold for completely stable/asymptotically stable families of sets that are uniform.

The current development of *pullback attractors* ([10], [12], [13], [15]) generally involve structures that are uniform in nature. For these, use of forward stability theory and Lyapunov theory for such structures is perfectly applicable.

For completeness, the theorems may be applied for uniform complete stability/asymptotic stability and are referenced below.

Theorem 5.2.6 (Sufficiency)**.** *Given a family of uniformly bounded compact sets* $\hat{A} = \{A(p); p \in P\}$*, and a Lyapunov function* $V(p,x)$ *satisfying the conditions in either Theorem 5.2.1 or 5.2.3 respectively, implies uniform complete stability or uniform complete asymptotic stability of* $\hat{A}$ *respectively.*

Theorem 5.2.7 (Converse)**.** *If a family of uniformly bounded compact sets* $\hat{A} = \{A(p); p \in P\}$ *is uniformly completely asymptotically stable, then there exists a function* $V(p,x)$ *satisfying conditions a)-d) in Theorem 5.2.4.*

5.3 Pullback Lyapunov Theory

The difficulties in characterising the pullback behaviour of a dynamical system lie in the fact that pullback asymptotic behaviour is not determined by the behaviour along a trajectory, but characterised instead by the sensitivity of the function to changes in the initial time.

As a result, determining the decrescent nature of such a function isn't easily ascertained as in the forward case and finding a suitable function to determine the system's pullback properties will be difficult to realise. Consequently we will only deal with the converse theorem for pullback equi-asymptotic stability, a result which is useful in understanding the numerics or perturbations of such systems (recall the approach used for the simpler, perturbed autonomous problems of Chapter 4).

A further complication with pullback systems is that the rate of attraction at some $p_0 \in P$ is independent of the rate of attraction at a uniquely different point $p_1 \in P$. As a result, the generated Lyapunov function will necessarily have the form $V = V(p, t, x)$. This is in contrast to the forward case, where the rate of attraction is able to be determined with respect to the current state and no memory of the initial state is needed.

A Lyapunov function for cocycle (pullback) attractors is generated by P. Kloeden in [15]. However it does not possess a true decrescence property to characterise the rate of pullback attraction, and hence is expected to be of limited value for an in depth pullback analysis of non-autonomous dynamical systems. Although the function is used in [18], it is essentially a forward analysis of a uniform object, and it is the characterisation of the function along trajectories (forward behaviour) that guarantees the result.

Kloeden's result in [18] was also proved for this thesis (Theorem 6.1.1) both concurrently and independently using existing Lyapunov theory for uniform forward asymptotic stability, since forward Lyapunov functions characterise behaviour along trajectories in the same way as the Lyapunov function in [15].

An alternative Lyapunov-like function for pullback analysis is developed here that most importantly possesses the essential *decrescence property* characterising the rate of pullback attraction at any point in time. Alone it does not constitute a comprehensive Lyapunov-like pullback analysis, however it does fulfill the following points.

1) Explores the difficulty in establishing a Lyapunov theory for pullback dynamics.

2) Provides a useful tool for numerical purposes.

3) Forms the basis for further development of a Lyapunov-like theory for a pullback analysis of dynamical systems.

Before proceeding, a method of evaluating the rate of attraction as the initial state is pulled further back is needed. The Dini Derivative provided a means of identifying the rate of change of the Lyapunov function for non-smooth functions in the forward case, so we will make use of a slightly modified construction for the pullback case. For this we will use the notation $D^p_{(5.1)}V(p,t,x)$ and define it as

$$D^p_{(5.1)}V(p,t,x) = \overline{\lim_{h\to 0}}\frac{V(p,t+h,x)-V(p,t,x)}{h},$$

where the superscript p distinguishes it it as the rate of change function used for *pullback* systems. The theorem is finally presented as follows.

Theorem 5.3.1 (Pullback Equi-Asymptotic Stability).
Suppose the dynamical system (5.1) possesses a family $\hat{A} = \{A(p); p \in P\}$ that is pullback equi-asymptotically stable. Then there exists a Lyapunov function $V : P \times \mathbb{R}^+ \times \mathcal{N}_{K,\hat{A}}$ defined on a neighbourhood of $\hat{A}$, $\mathcal{N}_{K,\hat{A}}$, which satisfies for each $p \in P$, $t \geq 0$

a) $V(p,t,x) = 0$ *for each* $x \in A(p)$,

b) $a(dist(\Phi_{(t,\theta_{-t}p)}(x), A(p))) \leq V(p,t,x)$ *where* $a \in \mathcal{K}$,

c) $D^p_{(5.1)}V(p,t,x) \leq -cV(p,t,x)$ *for some constant* $c > 0$,

d) $V(p,\cdot,\cdot)$ *is continuous in* t *and locally Lipschitz in* x.

Proof:
Let $p \in P$ be arbitrary. Then by the same principles as in Lemma

5.1.1, there exists a $\delta > 0$ such that for any $\epsilon > 0$, and some $\epsilon^*(\delta) > 0$, and $T = T(p, \epsilon)$,

$$\begin{array}{lll} dist(\Phi_{(t,\theta_{-t}p)}(x), A(p)) < \epsilon^* & \text{for all} & t > 0, \\ dist(\Phi_{(t,\theta_{-t}p)}(x), A(p)) < \epsilon & \text{for all} & t > T(p, \epsilon). \end{array}$$

First we define some preliminary constants before proceeding with the construction of a valid Lyapunov function.

Let $F = F(p, \epsilon)$ where,

$$F(p, \epsilon) = 1 + \max_{t,x}\{|f(\theta_{-t}p, x)|; 0 < t < T(p, \epsilon), x \in \mathcal{N}_\delta(A(\theta_{-t}p)), \}$$

and for some constant $c > 0$, define the function $\mathcal{A} = \mathcal{A}(p, \epsilon)$ by

$$\mathcal{A}(p, \epsilon) = e^{cT(p,\epsilon)} \exp\left(\int_0^{T(p,\epsilon)} L(\theta_{-s}p)ds\right) F(p, \epsilon),$$

where $L(\cdot)$ is the Lipschitz constant for the function $f(p, x)$. By J. Massera ([42]), there exists functions l, g satisfying $l(p) > 0, 0 < g(\epsilon) \leq 1$ for $\epsilon > 0$ and $g(0) = 0$, such that

$$g(\epsilon)\mathcal{A}(p, \epsilon) \leq l(p).$$

We need to analyse the behaviour of initial states as they are pulled back in time. However, as $\hat{A}$ may be varying with p we make use of the notation introduced earlier whereby we consider sequences of initial states based on an initial state $x \in \mathcal{N}_\delta(A(p))$ defined by $\hat{x} = \{x_\tau; \tau \geq 0, \text{dist}(x_\tau, A(\theta_{-t}p) \leq \text{dist}(x, A(p))\}$. The set of all sequences $\hat{x}$ thus defined will be denoted by X_x.

Now, for $n = 1, 2, \ldots$, we define $V_n(p, t, x)$ for each $p \in P$, $t > 0$, and $x \in \mathcal{N}_\delta(A(p))$ by:

$$V_n(p, t, x) = g(1/n) \sup\{\sup_{\hat{x} \in X_x} D_n(dist(\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p))) \\ e^{c\tau}; \tau \geq 0\},$$

where $D_n(r)$ is a real valued function such that

$$D_n(r) = \begin{cases} r - 1/n & (r \geq 1/n), \\ 0 & (0 \leq r \leq 1/n). \end{cases}$$

i) Invariance - By invariance of the cocycle on $\hat{A}$ it is immediate that for each $p \in P$, and all $t > 0$, $x \in \mathcal{N}_\delta(A(p))$,

$$V_n(p,t,x) = 0.$$

ii) Lower Bound - Define $A_n(r)$ by

$$A_n(r) = \begin{cases} 1/n(n-1) & (r \geq 1/(n-1)), \\ r - 1/n & (1/n \leq r \leq 1/(n-1)), \\ 0 & (0 \leq r \leq 1/n), \end{cases}$$

and set

$$a_n(r) = g(1/n)A_n(r).$$

Here $A_n(r) \leq D_n(r)$, and that a_n is a non-negative monotonically increasing function and is continuous with respect to r. Denote $r = dist(\Phi_{(t,\theta_{-t}p)}(x_t), A(p))$ where $x_t \in \hat{x}$, then for any $\hat{x} \in X_x$ we have

$$\begin{aligned} V_n(p,t,x) &\geq g(1/n) \sup_{\hat{x} \in X_x} D_n(dist(\Phi_{(t,\theta_{-t}p)}(x_t), A(p))), \\ &\geq g(1/n)A_n(r), \\ &\geq a_n(r). \end{aligned}$$

iii) Upper Bound - $\hat{A}$ is pullback stable, hence for each $x \in \mathcal{N}_\delta(A(p))$ and all $\hat{x} \in X_x$,

$$\begin{aligned} &V_n(p,t,x) \\ &\quad = g(1/n) \sup_{\tau}\{ \sup_{\hat{x} \in X_x} D_n(dist(\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p)))e^{c\tau}\}, \\ &\quad \leq g(1/n)e^{cT(p,1/n)}D_n(\epsilon^*), \\ &\quad \leq l(p)\epsilon^*, \end{aligned}$$

where ϵ^* is as defined earlier for pullback stability. Note that this upper bound is dependent on p, as opposed to the upper bound generated for uniform equi-asymptotic stability.

iv) Decrescence - Let $h > 0$ be some small constant. Then

$$\begin{aligned}
&V_n(p, t+h, x) \\
&\quad = g(1/n) \sup_{\tau \geq 0} \sup_{\hat{x} \in X_x} D_n(dist(\Phi_{(\tau+t+h, \theta_{-(\tau+t+h)}p)}(x_{\tau+t+h}), A(p)))e^{c\tau}, \\
&\quad = g(1/n) \sup_{\tau \geq h} \sup_{\hat{x} \in X_x} D_n(dist(\Phi_{(\tau+t, \theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p)))e^{c\tau}e^{-ch}, \\
&\quad = e^{-ch}V_n(p, t, x).
\end{aligned}$$

Taking $D^p_{(5.1)}V(p, t, x)$,

$$\begin{aligned}
D^p_{(5.1)}V_n(p, t, x) &= \overline{\lim_{h \to 0^+}} \frac{V_n(p, t+h, x) - V_n(p, t, x)}{h}, \\
&\leq \overline{\lim_{h \to 0^+}} \frac{(e^{-ch} - 1)V_n(p, t, x)}{h}, \\
&\leq -cV_n(p, t, x).
\end{aligned}$$

v) Local Lipschitz condition in x- Let $x, x' \in \mathcal{N}_\delta(A(p))$. Without loss in generality we will assume that

$$\mathrm{dist}(x, A(p)) \geq \mathrm{dist}(x', A(p)).$$

Now for any t so that $t > T(1/n)$, $V_n(p, t, x) = V_n(p, t, x') = 0$, for which the fulfillment of Lipschitzness is trivial, hence we will only consider the situation for which $0 \leq t \leq T(1/n)$.

$$\begin{aligned}
&|V_n(p,t, x) - V_n(p, t, x')| \\
&\qquad = g(1/n)|\sup_{\tau}\{\sup_{\hat{x} \in X_x} e^{c\tau} D_n(dist(\Phi_{(\tau+t, \theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p)))\} \\
&\qquad\qquad - \sup_{\tau}\{\sup_{\hat{x}' \in X_{x'}} e^{c\tau} D_n(dist(\Phi_{(\tau+t, \theta_{-(\tau+t)}p)}(x'_{\tau+t}), A(p)))\}|, \\
&\qquad \leq g(1/n) \sup_{0 \leq \tau \leq T-t} \{e^{c\tau}| \sup_{\hat{x} \in X_x} \mathrm{dist}(\Phi_{(\tau+t, \theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p)) \\
&\qquad\qquad - \sup_{\hat{x}' \in X_{x'}} \mathrm{dist}(\Phi_{(\tau+t, \theta_{-(\tau+t)}p)}(x'_{\tau+t}), A(p))|\}.
\end{aligned}$$

Since $dist(\Phi_{(\tau+t, \theta_{-(\tau+t)}p)}(x), A(p)) < 1/n$ for all $\tau > (T - t)$, the supremum may be taken over the bounded interval indicated. If the supremum is obtained for both expressions by an element within $X_{x'}$, then the argument is trivial. Hence we will assume that the first expression has a supremum for some $\hat{x}^* \in X_x$ but for which $\mathrm{dist}(x^*_{\tau+t}, A(\theta_{-(\tau+t)}p)) > dist(x', A(p))$. Also, consider some $\hat{x}'^* \in X_{x'}$ chosen so that $|x'^*_{\tau+t} - x^*_{\tau+t}|$ is a minimum. Then

$$\begin{aligned}
&|V_n(p,t,x) - V_n(p,t,x')| \\
&\qquad \leq g(1/n) \sup_{0\leq\tau\leq T-t} \{e^{c\tau}|\mathrm{dist}(\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x^*_{\tau+t}), A(p)) \\
&\qquad\qquad - \mathrm{dist}(\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x'^*_{\tau+t}), A(p))|\}, \\
&\qquad \leq g(1/n) \sup_{0\leq\tau\leq T-t} \{e^{c\tau}|\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x^*_{\tau+t}) \\
&\qquad\qquad - \Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x'^*_{\tau+t})|\}.
\end{aligned}$$

Setting

$$\mathcal{L}(p,\epsilon) = \exp\left(\int_0^{T(p,\epsilon)} L(\theta_{-s}p)ds\right),$$

for ease of notation, then by Lemma 5.1.2,

$$\begin{aligned}
|V_n(p,t,x) - V_n(p,t,x')| &\leq g(p,1/n)e^{c(T(1/n)-t)}\mathcal{L}(p,1/n)|x^*_{\tau+t} - x'^*_{\tau+t}|, \\
&\leq e^{-ct}l(p)((\mathrm{dist}(x,A(p)) - \mathrm{dist}(x',A(p))), \\
&\leq e^{-ct}l(p)|x-x'|.
\end{aligned}$$

Hence each $V_n(p,t,x)$ is Lipschitz with respect to x.

vi) Continuity in t- Without loss of generality, let $0 < t' < t$, and

$x \in \mathcal{N}_\delta(A(p))$. Then

$$
\begin{aligned}
&|V_n(p,t,x) - V_n(p,t',x)| \\
&\quad = g(1/n)\Big|\sup\{e^{c\tau}\sup_{\hat{x}\in X_x} D_n(dist(\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p))); \tau \geq 0\} \\
&\qquad - \sup\{e^{c\tau}\sup_{\hat{x}\in X_x} D_n(dist(\Phi_{(\tau+t',\theta_{-(\tau+t')}p)}(x_{\tau+t'}), A(p))); \tau \geq 0\}\Big|, \\
&\quad \leq g(1/n)\sup\sup_{\hat{x}\in X_x}\{e^{c\tau}\Big|D_n(dist(\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x_{\tau+t}), A(p))) \\
&\qquad - D_n(dist(\Phi_{(\tau+t',\theta_{-(\tau+t')}p)}(x_{\tau+t'}), A(p)))\Big|; 0 \leq \tau \leq T - t'\}, \\
&\quad \leq g(1/n)\sup\sup_{\hat{x}\in X_x}\{e^{c\tau}|\Phi_{(\tau+t,\theta_{-(\tau+t)}p)}(x_{\tau+t}) \\
&\qquad\qquad - \Phi_{(\tau+t',\theta_{-(\tau+t')}p)}(x_{\tau+t'})|; 0 \leq \tau \leq T - t'\}, \\
&\quad \leq g(1/n)\sup\sup_{\hat{x}\in X_x}\{e^{c\tau}|\Phi_{(\tau+t',\theta_{-(\tau+t')}p)}(X_{\tau+t'}) \\
&\qquad\qquad - \Phi_{(\tau+t',\theta_{-(\tau+t')}p)}(x_{\tau+t'})|; 0 \leq \tau \leq T - t'\},
\end{aligned}
$$

where $X_{\tau+t'} = \Phi_{(t-t',\theta_{-(\tau+t)}p)}(x_{\tau+t})$. Recalling $\mathcal{L}(p,\epsilon)$ defined earlier, then by Lemma 5.1.2, and considering Lemma 5.1.3 on the bounded interval $[0, T(p,1/n)]$ we have,

$$
\begin{aligned}
&|V_n(p,t,x) - V_n(p,t',x)| \qquad (5.6) \\
&\qquad \leq g(1/n)\sup_{\tau}\sup_{\hat{x}\in X_x}\{e^{c\tau}\mathcal{L}(p,1/n)|X_{\tau+t'} - x_{\tau+t'}|; 0 \leq \tau \leq T - t'\}, \\
&\qquad \leq g(1/n)e^{cT((p,1/n)}e^{-ct'}\mathcal{L}(p,1/n)F(p,1/n)|t-t'|, \\
&\qquad = e^{-ct'}|t-t'|.
\end{aligned}
$$

(5.7)

Note for all $t, t' > T(p,1/n)$, the function $V_n = 0$, so we need only consider the difference on this set.

As a result each V_n is continuous and in fact Lipschitz with respect to t.

Finally, we define the Lyapunov function V by

$$V(p,t,x) = \sum_{n=1}^{\infty}\frac{1}{2^n}V_n(p,t,x).$$

Note that this series converges as a consequence of iii). Properties a) - d) will be verified sequentially.

a) Clearly from i) we have for each $p \in P$, $t \geq 0$, and all $x \in A(\theta_{-t}p)$

$$V(p,x) = 0.$$

b-i) Lower Bound - From ii), if we set

$$a(r) = \sum_{n=1}^{\infty} \frac{1}{2^n} a_n(r),$$

we have $a \in \mathcal{K}$. Clearly $a(0) = 0$. By the Weierstrass M-test it is a continuous function. Also $a(r) > 0$ for $r > 0$ since for any r there exists an n such that $(1/n) < r$ and hence $a(r) > a_n(r) = (r - 1/n)$. It is a lower bound for $V(p,t,x)$, where $r = dist(\Phi_{(t,\theta_{-t}p)}(x), A(p))$, as shown below.

$$\begin{aligned} V(p,t,x) &= \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p,t,x), \\ &\geq \sum_{n=1}^{\infty} \frac{1}{2^n} a_n(r), \\ &\geq a(r). \end{aligned}$$

c) Decrescence - From iv),

$$\begin{aligned} V(p,t+h,x) &= \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p,t+h,x), \\ &\leq e^{-ch} \sum_{n=1}^{\infty} \frac{1}{2^n} V_n(p,t,x), \\ &\leq e^{-ch} V(p,t,x). \end{aligned}$$

Again, taking the Dini derivative for V, we arrive at the required result.

d) Continuity and Lipschitz Properties of V - This follows directly from the continuity and Lipschitzness of each V_n in v).

□

Chapter 6

Discretisation of Uniform and Forward Asymptotic Behaviour

Throughout this chapter we analyse the effects of discretising a non - autonomous differential equation known to possess a uniform or forward asymptotically stable family. These results extend those of Stuart and Humphries [34, 35] to non-autonomous dynamical systems.

The problem is similar to that posed in Chapter 4, however it is considered here in a general context for non-autonomous dynamical systems of the form

$$\dot{x} = f(p, x),$$

where the usual continuity and Lipschitzness on f holds. Differentiability of solutions to the same order of the numerical scheme is also assumed in order to apply local truncation error bounds.

6.1 Uniform Asymptotic Stability

In this section we present theorems that deal with the discretisation of uniform objects, in particular, uniformly asymptotically stable families of sets, and uniform attractors. Both structures simultaneously exhibit forward and pullback properties (Section 2.1.5), and hence are completely asymptotically stable. For simplicity we will refer to them as uniformly asymptotically stable, or as uniform attractors respectively, and keep in mind that they possess both forward and pullback characteristics.

6.1.1 Bounded Systems

The following analysis assumes boundedness on f and its derivatives (up to the order of the numerical scheme) with respect to p. By consideration of the discretisation on a local neighbourhood of the uniformly asymptotically stable set, $\hat{A}$, it is assumed that f and its derivatives with respect to x also remain bounded on the interval of consideration. Hence the local truncation constant C_r is valid for all $p \in P$. The same argument also applies to the Lipschitz constant for the Lyapunov function V.

6.1.2 Unbounded Systems

If f is unbounded with respect to p, a modified truncation error is needed (refer to Subsection 3.1.3). As a result, the discrete analysis of these systems is somewhat more generalised and coincides with the analysis for discretisation of systems that possess objects that are only forward or pullback equi-asymptotically stable. Such systems are investigated in Section 6.2.

6.1.3 Main Result

The main result of this section concerns the discretisation of a bounded (in the sense described above) non-autonomous dynamical system that possesses a uniformly asymptotically stable family $\hat{A}$. If the step size is restricted to be small enough, then there is shown to exist a discrete uniformly asymptotically stable family, denoted $\hat{A}^h$ in the discretised dynamical system. Further, $\hat{A}^h$ is shown to be upper semi-continuous with respect to the original uniformly asymptotically stable family $\hat{A}$.

Remark 1: When approximating uniformly equi-asymptotically stable families, as with results found in autonomous systems, there is no guarantee of lower semi-continuity. Attracting and invariant objects may collapse under discretisation.

Remark 2: Since uniformly asymptotically stable objects are both forward and pullback asymptotically stable, it is perfectly reasonable to expect that a pullback or a forward analysis would establish the result for Theorem 6.1.1. A forward analysis is used to verify the result here.

It was also derived concurrently and independently in [18] by P. Kloeden and V. Kozyakin. Their approach attempts to solve the discretisation problem via a pullback analysis through the use of the pullback Lyapunov function constructed by Kloeden in [15]. The approach however is somewhat misleading since the Lyapunov function does not properly capture the pullback properties of attraction, but instead characterises the forward attraction of solutions to verify the result. This leads to an unneccessarily complicated result that can also be shown with a forward analysis using conventional Lyapunov theory (as is presented here). It is also uncertain that the techniques used in [18] would be useful when moving to an analysis of non-uniform (with respect to asymptotic stability) objects.

The method of approach by Kloeden and Kozyakin also utilises a variable time-step numerical scheme. This however is not technically necessary as boundedness of the dynamical system is assumed, and thus a constant time-step scheme suffices as is shown here (Theorem 6.1.1).

A variable time-step scheme however *is* necessary for unbounded systems, and this is covered in detail in Section 6.2.

As the proof of the main result is long, it will be established via a sequence of lemmas, with the declaration of the main result given at the end.

Preliminaries

We are given that $\hat{A}$ is uniformly asymptotically stable, and we assume a constant step size discretisation. Since $\hat{A}$ is uniformly asymptotically stable, there exists an associated Lyapunov function $V = V(p, x)$ (Theorem 5.2.5) that characterises $\hat{A}$.

Lemma 6.1.1 (B1)**.** *The family* $\hat{B} = \{B(p); p \in P\}$ *defined by*

$$B(p) = \{x; x \in \mathcal{N}_{\delta^*}(A(p)), V(p,x) < a(\delta_0)\}, \tag{6.1}$$

for some $\delta_0 > 0$*, is positively invariant under the discretisation.*

Proof: Since $\hat{A}$ is uniformly asymptotically stable, it is forward stable. Hence given $\epsilon^* > 0$, there exists a $\delta^* = \delta^*(\epsilon^*)$ such that

$$dist(\Phi_{(t,p)}(x), A(\theta_t p)) < \epsilon^* \qquad \forall t > 0.$$

From Theorem 5.2.5, the Lyapunov function $V = V(p,x)$ is well defined on this neighbourhood.

Choose $\delta_0 > 0$ so that $b(\delta_0) = a(\delta^*)$ (where a, b as characterised by Theorem 5.2.5) and consider the family $\hat{B} = \{B(p); p \in P\}$ defined by (6.1).

Let $x \in B(p)$ for arbitrary p. Then we have

$$a(\mathrm{dist}(x, A(p))) \leq V(p,x) < a(\delta_0).$$

Hence $B(p) \subset \mathcal{N}_{\delta_0}(A(p))$. Note that for each $p \in P$, $A(p) \subset B(p)$ since $V = 0$ on $A(p)$. Consequently $B(p)$ is a well defined neighbourhood of $A(p)$.

Consider any $x_0 \in B(p)$ and act on this state one step of the discretisation (with step size h), arriving at x_1 (here we have set $x_1 = \Phi_{(1,p)}(x_0)$ to simplify the notation). Utilising the Lipschitz property of the Lyapunov function we generate the inequality

$$\begin{aligned} V(\theta_h p, x_1) &\leq V(\theta_h p, \Phi_{(h,p)}(x_0)) + L|x_1 - \Phi_{(h,p)}(x_0)|, \\ &\leq e^{-ch} V(p,x) + LC_r h^{r+1}, \\ &\leq e^{-ch} a(\delta_0) + LC_r h^{r+1}, \end{aligned} \tag{6.2}$$

where $C_r h^{r+1}$ is the local truncation error bound for a one step numerical method of order r, and L the Lipschitz constant of V. Now let h be chosen small enough so that $h \in [0, h_1)$ with h_1 satisfying

$$LC_r h_1^{r+1}/(1 - e^{-ch_1}) \leq \frac{1}{4} a(\delta_1),$$

where

$$\delta_1 = \frac{1}{2} b^{-1}(a(\delta_0)). \tag{6.3}$$

The choice of δ_1 made here will become apparent later in the proof. By construction, $\delta_1 < \delta_0$ since

$$\begin{aligned} \delta_1 &= \frac{1}{2} b^{-1}(a(\delta_0)), \\ &\leq \frac{1}{2} a^{-1}(a(\delta_0)), \\ &< \delta_0. \end{aligned}$$

Hence

$$LC_r h_1^{r+1}/(1 - e^{-ch_1}) < \frac{1}{4} a(\delta_0).$$

Also note that $LC_r h_1^{r+1}/(1 - e^{-ch_1}) \to 0$ as $h_1 \to 0$. Hence restricting h as above is valid. Returning to (6.2),

$$\begin{aligned} V(\theta_h p, x_1) &\leq e^{-ch} a(\delta_0) + \frac{1}{4}(1 - e^{-ch}) a(\delta_0), \\ &\leq a(\delta_0). \end{aligned}$$

As a result of the above inequality, it is ensured that $x_1 \in B(\theta_h p)$. Since the choice of p and $x_0 \in B(p)$ was made arbitrarily, it is concluded that $\hat{B}$ is a positively invariant family under the discretisation.

□

Let $p_n = \theta_{nh} p_0$ denote elements in the discrete sequence of times generated by an initial value p_0 and a constant step size h.

Lemma 6.1.2 (B2)**.** *The discrete family* $\hat{B}^h = \{B^h(p_n); n \in \mathbb{Z}\}$ *defined by*

$$B^h(p_n) = B(p_n),$$

is a positively invariant family under the discretisation.

Proof: The proof follows directly from the properties of $\hat{B}$.

□

The following Lemmas construct a discrete family that forward attracts $\hat{B}^h$ in a finite number of steps.

Definition 6.1.1 (A1). *Define*

$$\gamma(h) = 2LC_r h^{r+1}/(1-e^{-ch}),$$

and $\hat{A}^h = \{A^h(p_n); n \in \mathbb{Z}\}$ *by*

$$A^h(p_n) = \{x; x \in B^h(p_n), V(p_n, x) \leq \gamma(h)\}, \tag{6.4}$$

for any $h \in [0, h_1)$.

We proceed to show that $\hat{A}^h$ is a discrete uniformly asymptotically stable family for the discretised dynamical system.

Note $\gamma(h) \leq a(\delta_0)$, and thus $A^h(p_n)$ is strictly a subset of $B^h(p_n)$ for each n.

Lemma 6.1.3 (A2). $A^h(p_n)$ *is bounded and compact for each* $n \in \mathbb{Z}$.

Proof: $A(p_n)$ is bounded for each $n \in \mathbb{Z}$. Consequently $A^h(p_n)$ must also be bounded since $A^h(p_n) \subset B^h(p_n) \subset \mathcal{N}_{\delta_0}(A(p_n))$. Compactness follows from the continuity of V.

□

Lemma 6.1.4 (A3). $\hat{A}^h$ *is positively invariant under the discretisation.*

Proof: Let $x_n \in A^h(p_n)$. It then follows that

$$\begin{aligned}
V(\theta_h p_n, x_n) &\leq V(\theta_h p_n, \Phi_{(h,p_n)}(x_n)) + L|x_{n+1} - \Phi_{(h,p_n)}(x_n)|,\\
&\leq e^{-ch} V(p_n, x_n) + LC_r h^{r+1},\\
&\leq e^{-ch}\gamma(h) + \frac{1}{2}\gamma(h)(1-e^{-ch}),\\
&\leq \gamma(h).
\end{aligned}$$

Hence $x_{n+1} \in A^h(p_{n+1})$.

□

Lemma 6.1.5 (A4)**.** *$\hat{A}^h$ uniformly attracts $\hat{B}$ in a finite number of steps.*

Proof: Let $x_0 \in B^h(p_0)\backslash A^h(p_0)$. Then $V(p_0, x_0) \geq \gamma(h)$ and we have

$$\begin{aligned} V(p_1, x_1) &\leq e^{-ch}V(p_0, x_0) + \frac{1}{2}\gamma(h)(1 - e^{-ch}), \\ &\leq \frac{1}{2}(1 + e^{-ch})V(p_0, x_0). \end{aligned}$$

If we define $h_2 = \min\{h_1, \psi\}$ where ψ satisfies $(1 + e^{-c\psi}) = 2e^{-c\psi/4}$, then for all $h \in [0, h_2)$,

$$(1 + e^{-ch}) \leq 2e^{-ch/4}.$$

Consequently,

$$V(p_1, x_1) \leq e^{-ch/4}V(p_0, x_0).$$

Suppose $x_j \in B(p_j)\backslash A^h(p_j)$ for $j = 1, \ldots, n-1$. Then

$$\begin{aligned} V(p_n, x_n) &\leq e^{-nch/4}V(p_0, x_0), \\ &\leq e^{-nch/4}a(\delta_0). \end{aligned}$$

If we define $N = N(h, \delta_0)$ by

$$N(h, \delta_0) = \frac{4}{ch}\ln\left(\frac{a(\delta_0)}{\gamma(h)}\right),$$

then for all $n > N$ (recalling that $\hat{A}^h$ is positively invariant under the discretisation),

$$V(p_n, x_n) \leq \gamma(h).$$

Hence $x_n \in A^h(p_n)$ for all $n > N$. Note that the attraction is uniform. That is, N is independent of p_0 and x_0.

□

To fulfill the requirements of uniform asymptotic stability, we construct an attracting δ-neighbourhod of $\hat{A}^h$, and also show that $\hat{A}^h$ is stable.

Lemma 6.1.6 (A5 - Attracting δ-Neighbourhood)**.** *Define*

$$\delta = \frac{1}{2} b^{-1}(a(\delta_0)).$$

Then $\hat{A}^h$ uniformly attracts the neighbourhood system $\mathcal{N}_{\delta,\hat{A}^h}$.

Proof: First we show $\mathcal{N}_\delta(A^h(p_n)) \subseteq B(p_n)$ for each $n \in \mathbb{Z}$.

For arbitrary n, consider any $x \in \mathcal{N}_\delta(A^h(p_n))$. Then

$$\begin{aligned}
\operatorname{dist}(x, A(p_n)) &\leq \operatorname{dist}(x, A^h(p_n)) + H^*(A^h(p_n), A(p_n)), \\
&\leq \frac{1}{2} b^{-1}(a(\delta_0)) + a^{-1}(\gamma(h)), \\
&\leq \frac{1}{2} b^{-1}(a(\delta_0)) + a^{-1}(\frac{1}{2} a(\delta_1)), \\
&\leq \frac{1}{2} b^{-1}(a(\delta_0)) + \delta_1, \\
&< b^{-1}(a(\delta_0)),
\end{aligned}$$

since $\delta_1 = \frac{1}{2} b^{-1}(a(\delta_0))$ from (6.3). As a result

$$V(p_n, x) \leq b(\operatorname{dist}(x, A(p_n))) < a(\delta_0).$$

Thus $x \in B(p_n)$. Since $\hat{A}^h$ uniformly attracts $\hat{B}$ and since $\mathcal{N}_{\delta,\hat{A}^h} \subseteq \hat{B}$, $\hat{A}^h$ uniformly attracts $\mathcal{N}_{\delta,\hat{A}^h}$.

□

Lemma 6.1.7 (A6 - Stability)**.** *$\hat{A}^h$ is a discrete uniformly stable family.*

Proof: Given any $\epsilon > 0$, take δ^* as defined earlier so that $\delta^* = \delta^*(\epsilon)$. Then for each p_n,

$$\mathcal{N}_\delta(A^h(p_n)) \subset B(p_n) \subset \mathcal{N}_\epsilon(A(p_n)) \subset \mathcal{N}_\epsilon(A^h(p_n)),$$

with the property that all discretised solutions originating from within $\mathcal{N}_\delta(A^h(p_n))$ will remain within the neighbourhood system $\mathcal{N}_{\epsilon,\hat{A}^h}$. Thus $\hat{A}^h$ is stable.

□

Lemma 6.1.8 (A7). *$\hat{A}^h$ is upper semi-continuous with respect to $\hat{A}$ in the variable h.*

Proof: Note that $\hat{A} \subseteq \hat{A}^h$,since V is continuous and $\gamma(h) > 0$. Also for each p_n and $x \in A^h(p_n)$,

$$\begin{aligned} \text{dist}(x, A(p_n)) &\leq a^{-1}(V(p_n, x_n)), \\ &\leq a^{-1}(\gamma(h)). \end{aligned}$$

Since $a^{-1}(\gamma(h)) \to 0$ as $h \to 0$, the required result follows. That is for each p_n,

$$H(A^h(p_n), A(p_n)) \to 0 \qquad \text{as} \qquad h \to 0.$$

□

Finally, by sequentially applying Lemmas and Definitions B1-B2, A1 - A7, we arrive at the main result:

Theorem 6.1.1 (Main Result). *Let $\{\Phi_{(t,p)}; t \in \mathbb{R} \cdot p \in P\}$ be a cocycle for* (5.1) *which possesses a uniformly asymptotically stable family of sets $\hat{A} = \{A(p); p \in P\}$. Then the discretisation of the system with a one-step numerical method possesses a discrete uniformly asymptotically stable family $\hat{A}^h = \{A^h(p_n); n \in \mathbb{Z}\}$ satisfying*

$$H(A^h(p_n), A(p_n)) \to 0 \qquad \textit{as} \qquad h \to 0. \tag{6.5}$$

for each $n \in \mathbb{Z}$. Here h is the step size of the discretisation used.

6.1.4 Uniform Attractors

The following corollary considers the numerical approximation of a non - autonomous system possessing a uniform attractor $\hat{A}$.

Corollary 6.1.1 (Discretisation of Uniform Attractors)**.** *Let* $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ *be a cocycle for (5.1) which possesses a uniform attractor* $\hat{A} = \{A(p); p \in P\}$*. Then the discretisation of the system with a one-step numerical method possesses a discrete uniform attractor* $\hat{\mathcal{A}}^h = \{\mathcal{A}^h(p_n); n \in \mathbb{Z}\}$ *satisfying*

$$H^*(\mathcal{A}^h(p_n), A(p_n)) \to 0 \qquad \text{as} \qquad h \to 0.$$

for each $n \in \mathbb{Z}$*. Here* h *is the step size of the discretisation used.*

Proof: From Theorem 2.2.1, and the associated Lemmas, $\hat{A}$ is uniformly asymptotically stable, hence Theorem 6.1.1 holds, and thus there exists a discrete uniformly asymptotically stable family of sets $\hat{A}^h$. Since $\hat{A}^h$ is a pullback absorbing neighbourhood, by Theorem 3.3.1 we may conclude that there exists a discrete uniform attractor. We will denote this attractor by $\hat{\mathcal{A}}^h$.

Upper semi-continuity with respect to h of discrete and continuous attractors holds. To see this, note that for each p_n

$$0 \le H^*(\mathcal{A}^h(p_n), A(p_n)) \le H^*(A^h(p_n), A(p_n)).$$

Hence

$$H^*(\mathcal{A}^h(p_n), A(p_n)) \to 0 \qquad \text{as} \qquad h \to 0.$$

□

6.2 Forward Equi-Asymptotic Stability

This section focuses on the effects of discretisation on a forward equi - asymptotically stable family of sets, $\hat{A} = \{A(p); p \in P\}$ that does not necessarily possesses uniformity of stability. It also incorporates analysis of the problem for which the differential dynamics is *unbounded* in nature (mentioned briefly in the preceding section).

We briefly discuss what effects such unbounded dynamics has on the numerics before continuing with the main result.

6.2.1 Unbounded Systems and a Numerical Analysis

A preliminary investigation of the long term numerical analysis is essential. For autonomous systems, that is for $\dot{x} = f(x)$, a numerical one-step discretisation method always yields a local truncation error as given by

$$||x_{n+1} - \Phi_{(h,p_n)}(x_n)|| \leq C_r h^{r+1}.$$

The truncation constant C_r is dependent on the magnitude of f and its derivatives, and thus in a compact neighbourhood, will always be bounded.

For non-autonomous systems f (or its derivatives) may become unbounded with respect to p, and hence a suitable truncation constant may remain valid only over a finite time interval on a compact neighbourhood of the object under investigation. As a result, it becomes necessary to define the truncation constant as a function of p, $C_r(p)$.

The elementary example below illustrates the difficulties encountered whilst analysing discrete asymptotic behaviour for an unbounded system.

Example 6.2.1. Consider the dynamical system generated by the NDE

$$\dot{x} = -2\mathtt{t}x.$$

It can be easily shown for this system that the origin is a forward attractor. If we discretise this system with an Euler scheme that employs a constant time step, it is obvious that the change in the state at each iterative step becomes large and possibly unstable when working in regions for which $\mathtt{t}$ is large.

For example, consider the initial value problems defined by $(x_0, t_0) = (1, 0)$, and also $(x_0, t_0) = (1, 100)$. Using a step size of $h = 0.1$, the ensuing iterated solutions after 50 steps are given by

$$\begin{aligned}(x_0, t_0) =&(1, 0) \qquad &\Rightarrow \qquad & x_{20} = 7.21 \times 10^{-3},\\ (x_0, t_0) =&(1, 100) \qquad &\Rightarrow \qquad & x_{20} = 4.68 \times 10^{25}.\end{aligned}$$

In the latter case, the iterated solution oscillates around the origin in an unstable fashion. This is due to the fact that the updating term in the Euler scheme, $hf(t_n, x_n)$, grows without bound as `t` increases. In fact, calculating x_1 for the latter problem yields $x_1 = 1 - 0.1 * (200)$. The error for the numerical scheme in this case is far greater than the rate of attraction to the origin.

$\square$

There are two approaches that may be used to compensate for this instability in the numerical method.

i) One may consider the discretisation on a decreasing neighbourhood of attraction system $\mathcal{N}_{\hat{\delta},\hat{A}}$ for which $hf(t_n, x_n)$ remains bounded, and then determine the attraction within the defined system.

The difficulty with this approach is that in order to consider the effects of discretisation, an explicit knowledge of the neighbourhoods are required, both for the bounds on f, and the neighbourhood upon which V is defined and bounded.

ii) Alternatively one may use a variable time step scheme for which the step sizes are restricted by the bounds of f in a local neighbourhood of the solution.

This ensures the error term remains bounded, and is viable for general systems where f does not approach zero as solutions approach the attractor.

A discerning question regarding the latter approach is reachability. That is, if the step sizes are restricted in such a fashion, can it be ensured that $t_n \to \infty$ as $n \to \infty$.

This approach is used when considering the discretisation problem for forward equi-asymptotically stable families, presented below.

6.2.2 Main Result

The main result of this section concerns the discretisation of a possibly unbounded (in the sense described above) non-autonomous dynamical system that possesses a forward equi-asymptotically stable family $\hat{A}$ with a variable time-step discretisation. This extends the results of Section 6.1 and those by P. Kloeden and V. Kozyakin [18] to cases that do not possess uniformity, nor boundedness of f. This introduces distinct changes in the approach needed, most notably the fact that a variable time-step scheme is *necessary*.

Again, as the proof of the entire procedure is long, it will be given via a sequences of Lemmas. It will be shown that if the step sizes are restricted appropriately, then there exists a discrete forward equi-asymptotically stable family $\hat{A}^{\mathbf{h}}$ in the discretised dynamical system. Further, $\hat{A}^{\mathbf{h}}$ is shown to be upper semi-continuous with respect to $\hat{A}$. The final declaration of the main result is established following the Lemmas.

Preliminaries

We consider a continuous non-autonomous dynamical system with cocycle mapping $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$. It is assumed to possess a forward equi-asymptotically stable family of sets $\hat{A} = \{A(p); p \in P\}$.

Constant Neighbourhood of Attraction

We assume that $\hat{A}$ possesses an attracting local neighbourhood which may be chosen uniformly with respect to p for analytical purposes. That is, there exists a $\delta^* > 0$ generating a constant local neighbourhood $\mathcal{N}_{\delta^*,\hat{A}}$ that is forward attracting to $\hat{A}$.

Note that this is true of the majority of dynamical systems, however it does not imply that the basin of attraction is constant with respect to p.

This assumption is important in investigating attraction to a discrete structure approximating $\hat{A}$ and ensuring the effects of the discretisation keep the state within a non-vanishing local neighbourhood (if the neighbourhood is varying, this is difficult to realise unless the boundaries of the neighbourhood are explicitly known).

Variable Step Sequence

We assume a variable time step sequence $\mathbf{h} \in \mathcal{H}^\rho$ where H^ρ is the compact metric space of all bi-infinite real sequences and ρ is a predefined upper bound such that $h_n \leq \rho$ for all $h_n \in \mathbf{h}$ as introduced in Section 3.1.2.

The Discrete Parameter Set P_d

The parameter space for the discretised system P_d is defined as the cross product space $P_d : P \times H^\rho$. Elements of P_d are represented by the couple $(p, \mathbf{h})$ and the shift mapping $\theta_n^d : P_d \to P_d$ is constructed as in Section 3.2.3. A discrete cocycle representation with variable time step is consequently defined appropriately.

To simplify the notation, the series' $\{x_n\}$ and $\{p_n\}$, are used to represent the sequences of states and time steps arising from discretisation of the initial value problem (p_0, x_0),

$$\begin{aligned} x_n &= \Phi^{\mathbf{h}}_{(n,(p_0,\mathbf{h}))}(x_0), \\ p_n &= \theta_{h_{n-1}} p_{n-1}. \end{aligned}$$

The actual composition of $\mathbf{h}$ will be restricted to guarantee asymptotic behaviour.

Truncation Error

Since f and its derivatives may become unbounded, the local truncation error is defined by

$$|x_{n+1} - \Phi_{(h_n,p_n)}(x_n)| \leq C_r^*(p_n)(h_n)^{r+1}.$$

The value $C_r^*(p_n)$ is defined as the appropriate truncation constant over the finite interval $p_n \to \theta_\rho p_n$. We will limit ourselves to systems for which this term may become unbounded only as $p \to \theta_t p$ for $t \to \pm\infty$.

Lipschitz Property of V

By Theorem 5.2.5, there exists an associated Lyapunov function $V = V(p, x)$ defined for each $p \in P$, and $x \in \mathcal{N}_{\delta^*}(A(p))$.

We also define a bound on the local Lipschitzness with respect to x of V by

$$L^*(p) = \sup_{0 \le h \le \rho} l(\theta_h p),$$

where $l(p)$ is the usual Lipschitz bound for the Lyapunov function V.

Attracting Neighbourhood - $\hat{B}$

Define $\delta_0 > 0$ by $\delta_0 = \delta^*/3.$ and a family of sets $\hat{B} = \{B(p); p \in P\}$, where

$$B(p) = \{x; x \in \mathcal{N}_{\delta^*}(A(p)), V(p, x) < a(\delta_0)\}.$$

From this continuous family of sets, a discrete family of sets will be constructed that is positively invariant under the discretisation for restricted step sizes.

To see that $\hat{B}$ is well defined, note that

$$a(\mathrm{dist}(x, A(p))) \le V(p, x) < a(\delta_0),$$

for each $p \in P$ and $x \in B(p)$. Consequently $B(p) \subset \mathcal{N}_{\delta_0}(A(p))$. Also, by construction and since V is continuous, it can be seen that $B(p)$ is open. Hence $\hat{B}$ is appropriately defined.

We show positive invariance of a discrete family through a sequence of Lemmas.

Define constants $L(p)$, and $C_r(p)$ so that

$$C_r(p) = \max\{a(\delta_0), C_r^*(p)\}, \tag{6.6}$$

$$L(p) = \max\{1, a(\delta_0)/\delta_0, L^*(p).\} \tag{6.7}$$

These are designed so that certain assertions may be made later. Note that these changes do not affect the local truncation error or Lipschitzness of V except to accommodate increased variation. Without loss in generality, we also define $\rho < 1$ (the upper bound for the variable-time step sequence).

Lemma 6.2.1 (B1)**.** *For each $p \in P$, $x_0 \in B(p)$, if $h(p) \in [0, h_1(p))$, where the bound $h_1(p)$ is defined as the largest value satisfying the inequality*

$$4L(p)C_r(p)h_1(p)^{r+1}/(1 - e^{-ch_1(p)}) < \rho^{r+1}a(\delta_0), \tag{6.8}$$

then $x_1 \in \mathcal{N}_{\delta^*}(A(\theta_{h(p)}p)$, and $V(\theta_{h(p)}p, x_1)$ is well defined.

Proof: For any $p \in P$, let $x_0 \in B(p)$. Then

$$\begin{aligned} a(\mathrm{dist}(\Phi_{(h,p)}(x_0), A(\theta_{h(p)}p)) &\leq V(\theta_{h(p)}p, \Phi_{(h,p)}(x_0)), \\ &\leq e^{-ch(p)}V(p, x_0), \\ &\leq a(\delta_0). \end{aligned} \tag{6.9}$$

Hence $\mathrm{dist}(\Phi_{(h,p)}(x_0), A(\theta_{h(p)}p) \leq \delta_0$. Thus, for x_1,

$$\begin{aligned} \mathrm{dist}(x_1, A(\theta_{h(p)}p)) &\leq \mathrm{dist}(\Phi_{(h,p)}(x_0), A(\theta_{h(p)}p)) + |x_1 - \Phi_{(h,p)}(x_0)|, \\ &\leq \delta_0 + C_r(p)h(p)^{r+1}, \\ &\leq \delta_0 + \frac{\rho^{r+1}a(\delta_0)(1 - e^{-ch(p)})}{4L(p)}, \\ &\leq \delta_0 + \delta_0, \\ &< \delta^*. \end{aligned}$$

Here we have used the fact that $L(p) \geq a(\delta_0)/\delta_0$.

Since $x_1 \in \mathcal{N}_{\delta^*}(A(\theta_{h(p)}p)$, $V(\theta_{h(p)}p, x_1)$ is well defined.

Also $h(p) \leq \rho$ for any $h(p) \in [0, h_1(p))$ as required for the variable time-step scheme. This can be seen from (6.6) and (6.7) where $1/C_r(p) \leq 1/a(\delta_0)$, and $1/L(p) \leq 1$. Noting this, and substituting into (6.8) we have

$$\begin{aligned} h_1(p)^{r+1} &< (1 - e^{-ch_1(p)})\rho^{r+1}a(\delta_0)/4L(p)C_r(p), \\ &< \rho^{r+1}. \end{aligned}$$

□

The preceding Lemma ensures that we may now investigate the rate of change of the Lyapunov function for steps in the discrete sequence.

Lemma 6.2.2 (B2). *If the variable step sequence* $\mathbf{h}$ *satisfies (6.8), then* $\hat{B}$ *is positively invariant under the discretisation.*

Proof: Let $p_n \in P$ and choose any $x_n \in B(p_n)$. Since $x_{n+1} \in \mathcal{N}_{\delta^*}(A(\theta_{h(p_n)}p_n)$, V is defined for x_{n+1} and we may use the properties of Theorem 5.2.5 to show

$$\begin{aligned} V(\theta_{h(p_n)}p_n, x_{n+1}) &\leq V(\theta_{h(p)}p_n, \Phi_{(h(p_n),p_n)}(x_n)) + \\ &\qquad\qquad L(p_n)|x_{n+1} - \Phi_{(h(p_n),p_n)}(x_n)|, \\ &\leq e^{-ch(p_n)}V(p_n, x_n) + L(p_n)C_r(p_n)h(p_n)^{r+1}, \\ &\leq e^{-ch(p_n)}a(\delta_0) + L(p_n)C_r(p_n)h(p_n)^{r+1}. \end{aligned}$$

If the step size at p_n is restricted so that $h(p_n) \in [0, h_1(p_n))$,

$$\begin{aligned} V(\theta_{h(p_n)}p_n, x_{n+1}) &\leq e^{-ch(p_n)}a(\delta_0) + \frac{1}{4}(1 - e^{-ch(p_n)})a(\delta_0), \\ &\leq a(\delta_0). \end{aligned}$$

Hence $x_{n+1} \in B(\theta_{h(p_n)}p_n)$.

□

By restricting the step size at each $p \in P$ in the preceding Lemma it is ensured that any increase in the Lyapunov value over one step due to the numerical approximation (characterised by the local truncation error) is negated by the rate of attraction (characterised by the exponential decrescence of the Lyapunov function).

Note also that the representation for $h(p)$ is defined continuously over p. This is necessary so that the required variable time step sequence may be generated for *any* initial value problem.

Discrete Attracting Neighbourhood - $\hat{B}^{\mathbf{h}}$

The discretisation of the initial value problem (p_0, x_0) is now stated as follows.

Let $\mathbf{h}$ be any variable time step sequence so that $h_n = h(p_n)$ satisfies (6.8) for each $n \in \mathbb{Z}$. Then solutions to the discrete initial value problem $(x_0, (p_0, \mathbf{h}))$ are expressed by

$$x_n = \Phi^{\mathbf{h}}_{(n,(p_0,\mathbf{h}))}(x_0). \tag{6.10}$$

From $\hat{B}$ we construct a discrete family of sets that is positively invariant. Recall that ψ_n is the shift operator acting on the variable time-step sequence (refer to Subsection 3.2.3).

Lemma 6.2.3 (B3)**.** *Define the discrete family* $\hat{B}^{\mathbf{h}} = \{B^{\mathbf{h}}(p_n, \psi_n\mathbf{h}); n \in \mathbb{Z}\}$, *by*

$$B^{\mathbf{h}}(p_n, \psi_n\mathbf{h})) = B(p_n).$$

Then $\hat{B}^{\mathbf{h}}$ *is a discrete, open and positively invariant family under the discretisation determined by the choice of* $\mathbf{h}$.

Proof: The proof follows by construction and from the results of the Lemmas **B1**, and **B2**.

□

The following Lemmas construct a discrete family that forward attracts $\hat{B}^{\mathbf{h}}$ in a finite number of steps.

Discrete Forward Equi-Asymptotically Stable Family - $\hat{A}^{\mathbf{h}}$

We propose as our discrete forward equi-asymptotically stable family:

Definition 6.2.1 (A1)**.** *Define* $\hat{A}^{\mathbf{h}} = \{A^{\mathbf{h}}(p_n, \psi_n\mathbf{h}); n \in \mathbb{Z}\}$ *by*

$$A^{\mathbf{h}}(p_n, \psi_n\mathbf{h}) = \{x; V(p_n, x) \leq \frac{1}{2}\rho^{r+1}a(\delta_0)\}.$$

Obviously the family $\hat{A}^{\mathbf{h}}$ is a strict subset of $\hat{B}^{\mathbf{h}}$. It is now shown via a sequence of Lemmas that $\hat{A}^{\mathbf{h}}$ is a discrete forward equi-asymptotically stable family of sets.

Lemma 6.2.4 (A2 - Boundedness and Compactness)**.** *$\hat{A}^{\mathbf{h}}$ is uniformly bounded and each element is compact.*

Proof:
The discrete family $\hat{A}^{\mathbf{h}}$ is uniformly bounded since $\hat{A}$ is uniformly bounded and

$$A^{\mathbf{h}}(p_n, \psi_n \mathbf{h}) \subset B(p_n) \subset \mathcal{N}_{\delta_0}(A(p_n)),$$

for each n. Compactness follows from the continuity of V.

□

Lemma 6.2.5 (A3 - Positive Invariance)**.** *$\hat{A}^{\mathbf{h}}$ is a positively invariant family.*

Proof: Given some $n \in \mathbb{Z}$, and any $x_n \in A^{\mathbf{h}}(p_n, \psi_n \mathbf{h})$, and consider one step of the discretisation with $h_n = h(p_n) \in [0, h_1(p_n))$. It follows that

$$\begin{aligned} V(p_{n+1}, x_{n+1}) &\leq V(p_{n+1}, \Phi_{(h_n, p_n)}(x_n)) + L(p_n) C_r(p_n) h_n^{r+1}, \\ &\leq e^{-ch_n} V(p_n, x_n) + L(p_n) C_r(p_n) h_n^{r+1}, \\ &\leq e^{-ch_n} \frac{1}{2} \rho^{r+1} a(\delta_0) + \frac{1}{4} \rho^{r+1} a(\delta_0)(1 - e^{-ch_n}), \\ &\leq \frac{1}{2} \rho^{r+1} a(\delta_0). \end{aligned}$$

Hence $x_{n+1} \in A^{\mathbf{h}}(p_{n+1}, \psi_{n+1} \mathbf{h})$. As the choice of n was arbitrary, it can be concluded that $\hat{A}^{\mathbf{h}}$ is positively invariant.

□

The following Lemma confirms that $\hat{A}^{\mathbf{h}}$ forward attracts $\hat{B}^{\mathbf{h}}$ under further restrictions on the sequence step bounds. We also need to guarantee that for the initial value problem (p_0, x_0) any future time is reachable by the variable-time step sequence $\mathbf{h}$. The restrictions required for these two assertions are as follows and will become relevant through the process of the following lemmas.

For each $p \in P$, define

i) $h_2(p)$ by

$$h_2(p) = \min\{h_1(p), \phi\},$$

where ϕ satisfies the equation $(1 + e^{-c\phi}) = 2e^{-c\phi/4}$,

ii) $h_{min1}(p)$ as the smallest value satisfying

$$\frac{1}{2}\rho^{r+1}a(\delta_0) < 4L(p)C_r(p)h_{min1}(p)^{r+1}/(1 - e^{-ch_{min1}(p)}), \tag{6.11}$$

iii) and $h_{min2}(p)$ by

$$h_{min2}(p) = \min\{h_{min1}(p), \phi/2\}.$$

We now restrict the variable-time step sequence $\mathbf{h}$ for the discretisation of the initial value problem (p_0, x_0) so that for each $n \in \mathbb{Z}$, $h_n = h(p_n)$ is chosen so that it satisfies

$$h_n \in (h_{min2}(p_n), h_2(p_n)). \tag{6.12}$$

Note that this is non-empty due to the manner of construction of the bounds given above.

Lemma 6.2.6 (A4 - Reachability)**.** *If the elements of* $\mathbf{h}$ *satisfy the bounds (6.12), then*

$$\lim_{N\to\infty} \sum_{j=0}^{N} h(p_j) \to \infty.$$

Proof: Assume otherwise, that is, there exists a $T > 0$ such that

$$\lim_{N\to\infty} \sum_{j=0}^{N} h_j < T.$$

Consider any $p_0 \in P$, then $L(p), C_r(p)$ are both bounded on the finite interval $p_0 \to \theta_T p_0$. If we denote these bounds by L, C, then for any $p \in [p_0, \theta_T p_0]$ by (6.11),

$$\frac{1}{8LC}\rho^{r+1}a(\delta_0) < h_{min1}(p)^{r+1}/(1 - e^{-ch_{min1}(p)}).$$

Hence $h_{min1}(p)$, and consequently $h_{min2}(p)$ is thus bounded below by some finite quantity $h^* > 0$, for every $p \in [p_0, \theta_T p_0]$. Let $N = N(T)$ be the first integer such that $Nh^* > T$. Then

$$T < Nh^* \leq \sum_{j=1}^{N} h_{min2}(p_j) \leq \sum_{j=1}^{N} h(p_j),$$

providing the required contradiction. Hence the original assertion is true.

□

We now verify that $\hat{A}^{\mathbf{h}}$ forward attracts $\hat{B}^{\mathbf{h}}$.

Lemma 6.2.7 (A5 - Forward Attracting)**.**
If the elements of $\mathbf{h}$ *satisfy the bounds (6.12), then* $\hat{A}^{\mathbf{h}}$ *forward attracts* $\hat{B}^{\mathbf{h}}$*.*

Proof: Let $p_0 \in P$, and consider any $x_0 \in B^{\mathbf{h}}(p_0, \mathbf{h}) \backslash A^{\mathbf{h}}(p_0, \mathbf{h})$. Then

$$\frac{1}{2}\rho^{r+1}a(\delta_0) \leq V(p_0, x_0)$$

and we have

$$\begin{aligned} V(p_1, x_1) &\leq e^{-ch_0}V(p_0, x_0) + \frac{1}{4}\rho^{r+1}a(\delta_0)(1 - e^{-ch_0}), \\ &\leq \frac{1}{2}(1 + e^{-ch_0})V(p_0, x_0). \end{aligned}$$

Now, for any $h_0 \in (h_{min2}(p_0), h_2(p_0))$,

$$(1 + e^{-ch_0}) \leq 2e^{-ch_0/4}.$$

This is due to the fact that $h_0 < \phi$ (as defined earlier) for which equality of the above expression occurs. Below this value the above inequality holds. Now

$$V(p_1, x_1) \leq e^{-ch_0/4}V(p_0, x_0).$$

Suppose $x_j \in B^{\mathbf{h}}(p_j, \psi_j\mathbf{h})) \backslash A^{\mathbf{h}}(p_j, \psi_j\mathbf{h}))$ for $j = 1, \ldots, n-1$. By extrapolating the above argument, we have

$$\begin{aligned} V(p_n, x_n) &\leq exp\left(-c\sum_{i=0}^{n-1} h_i/4\right)V(p_0, x_0), \\ &\leq exp\left(-c\sum_{i=0}^{n-1} h_i/4\right)a(\delta_0). \end{aligned}$$

Let $N = N(p_0, \mathbf{h})$ be the first integer satisfying

$$\sum_{i=0}^{N} h_i \geq \frac{4}{c} \ln\left(\frac{2}{\rho^{r+1}}\right),$$

then for all $n > N$ (and recalling that $\hat{A}^{\mathbf{h}}$ is positively invariant under the discretisation),

$$V(p_n, x_n) \leq \frac{1}{2}\rho^{r+1} a(\delta_0).$$

Hence $x_n \in A^{\mathbf{h}}(p_n, \psi_n \mathbf{h}))$ for all $n > N$.

□

Finally, to satisfy the definition for asymptotic attraction, we need to show $\hat{A}^{\mathbf{h}}$ forward attracts an appropriately defined neighbourhood system of itself and also verify its stability.

Lemma 6.2.8 (A6 - Forward Asymptotic Attraction)**.**
$\hat{A}^{\mathbf{h}}$ forward attracts the neighbourhood system $\mathcal{N}_{\hat{\delta},\hat{A}^h}$, where $\hat{\delta} = \{\delta_{(p_n,\psi_n\mathbf{h})}; \delta_{(p_n,\psi_n\mathbf{h})} > 0, n \in \mathbb{Z}\}$.

Proof: For any n, $B^{\mathbf{h}}(p_n, \psi_n \mathbf{h})$ is an open set for which $V(p_n, x_n) < a(\delta_0)$ for all $x_n \in B^{\mathbf{h}}(p_n, \psi_n \mathbf{h})$. Similarly $A^{\mathbf{h}}(p_n, \psi_n \mathbf{h})$ is a compact set whose elements are constrained by $V(p_n, x_n) \leq \frac{1}{2}\rho^{r+1} a(\delta_0)$.

Since V is a continuous function in x, the level curves given by $V_1 = \frac{1}{2}\rho^{r+1} a(\delta_0)$, and $V_2 = a(\delta_0)$ cannot meet. Hence there exists a minimum distance denoted by $\delta_m(p_n, \psi_n \mathbf{h}) > 0$ separating V_1 and V_2. Let $\delta_{(p_n,\psi\mathbf{h})} = \frac{1}{2}\delta_m(p_n, \psi_n \mathbf{h})$. Then $\mathcal{N}_{\hat{\delta},\hat{A}^h}$, where $\hat{\delta} = \{\delta_{(p_n,\psi_n\mathbf{h})}; n \in \mathbb{Z}\}$ forms an appropriate neighbourhood system.

This is valid since $\mathcal{N}_{\hat{\delta},\hat{A}^h} \subset \hat{B}^{\mathbf{h}}$ and as $\hat{A}^{\mathbf{h}}$ forward attracts $\hat{B}^{\mathbf{h}}$, it subsequently forward attracts $\mathcal{N}_{\hat{\delta},\hat{A}^h}$.

□

Lemma 6.2.9 (A7 - Forward Stability)**.** *$\hat{A}^{\mathbf{h}}$ is forwards stable.*

Proof: Given $\epsilon > 0$ small enough, take δ^* as defined earlier and let $\delta^* = \epsilon$. Then $\hat{\delta} = \{\delta_{(p_n,\psi_n\mathbf{h})}; \delta_{(p_n,\psi_n\mathbf{h})} > 0, n \in \mathbb{Z}\}$ as defined in Lemma 6.2.8 (note that by progressing through the proof that $\hat{\delta}$ is ultimately a function of δ^*) satisfies the requirements for forward stability of $\hat{A}^{\mathbf{h}}$.

To see this, consider any $x_n \in \mathcal{N}_{\delta_{(p_n,\psi_n\mathbf{h})}}(A(p_n,\psi_n\mathbf{h}))$. Then $x_n \in B^{\mathbf{h}}(p_n,\psi_n\mathbf{h})$. As $\hat{B}^{\mathbf{h}}$ is positively invariant, then $x_\eta \in B^{\mathbf{h}}(p_\eta,\psi_\eta\mathbf{h})$ for all $\eta > n$. Also

$$B(p_\eta,\psi_\eta\mathbf{h}) \subset \mathcal{N}_{\delta^*}(A(p_\eta)) \subset \mathcal{N}_{\delta^*}(A(p_\eta,\psi_\eta\mathbf{h})).$$

Hence $x_\eta \in \mathcal{N}_\epsilon(A(p_\eta,\psi_\eta\mathbf{h}))$ for all $\eta > n$. As the initial choice of n was arbitrary, we may conclude $\hat{A}^{\mathbf{h}}$ is forwards stable.

□

Lemma 6.2.10 (C1 - Upper Semi-Continuity). *$\hat{A}^{\mathbf{h}}$ is upper semi - continuous with respect to $\hat{A}$ in the variable step upper bound ρ.*

Proof: Note that $\hat{A}$ is contained within $\hat{A}^{\mathbf{h}}$, as V is continuous and $\frac{1}{2}\rho^{r+1}a(\delta_0) > 0$. For each n and $x_n \in A(p_n,\psi_n\mathbf{h})$,

$$\begin{aligned}\operatorname{dist}(x_n, A(p_n)) &\leq a^{-1}(V(p_n,x_n)),\\ &\leq a^{-1}(\frac{1}{2}\rho^{r+1}a(\delta_0)).\end{aligned}$$

Since $a^{-1}(\frac{1}{2}\rho^{r+1}a(\delta_0)) \to 0$ as $\rho \to 0$, the required result follows.

□

Applying Definitions and Lemmas B1-B3, A1-A7 and C1, we arrive at the main result.

Theorem 6.2.1 (Main Result). *Let $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ be a cocycle for (5.1) which contains a forward equi-asymptotically stable family of sets $\hat{A} = \{A(p); p \in P\}$ for which a constant neighbourhood of attraction can be defined. Then a variable-time step discretisation (with bound $\rho > 0$ and restrictions on* *the individual step sizes) of the initial value problem* (p_0, x_0) *with a one-step numerical method generates a discrete dynamical system possessing a discrete forward equi-asymptotically stable family* $\hat{A}^{\mathbf{h}} = \{A^{\mathbf{h}}(p_n, \psi_n \mathbf{h}); n \in \mathbb{Z}\}$ *satisfying*

$$H^*(A(p_n, \psi_n \mathbf{h}), A(p_n)) \to 0 \qquad \textit{as} \qquad \rho \to 0. \tag{6.13}$$

for each $n \in \mathbb{Z}$.

6.2.3 Forward Attractors

It is not possible to extrapolate the results of Theorem 6.2.1 for application to forward attractors as was done in [35] for semi-groups in ADE's, and in Corollary 6.1.1 for uniform attractors in NDE's. The reason is that no similar *limit set representation* (compare with Theorem 2.3.1) is available for forward attractors as there is for semi-group attractors in ADE's, and pullback attractors in NDE's.

6.2.4 Non-Constant Neighbourhoods of Attraction

The results of Theorem 6.2.1 may also apply to a family of sets, $\hat{A}$, that are forward equi-asymptotically stable for which the neighbourhood of attraction varies with $p \in P$.

For this, some knowledge of the δ-neighbourhood system is essential, as the attraction for the discrete system must occur faster than the rate at which the δ-neighbourhood changes. The rate of attraction however can only be estimated by the bound on the decrescence of the associated Lyapunov function. Due to the manner of construction of the Lyapunov function, this places an exponential bound on the rate of change for the neighbourhood of attraction.

Consequently this may be ensured if the associated Lyapunov function and δ-neighbourhood system are chosen so that for each $p \in P$ and all $t > 0$, the condition below is satisfied.

$$e^{-ct/2} a(\delta_p) \leq a(\delta_{\theta_t p}). \tag{6.14}$$

The theorem is stated as follows.

Theorem 6.2.2. *Let* $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ *be a cocycle for* (5.1) *which contains a forward equi-asymptotically stable family of sets* $\hat{A} = \{A(p); p \in P\}$. *If condition (6.14) is satisfied, then a variable-time step discretisation of the initial value problem* (p_0, x_0) *generates a discrete dynamical system possessing a discrete forward equi-asymptotically stable family* $\hat{A}^{\mathbf{h}} = \{A^{\mathbf{h}}(p_n, \psi_n \mathbf{h}); n \in \mathbb{Z}\}$ *satisfying*

$$H(A^{\mathbf{h}}(p_n, \psi_n \mathbf{h}), A(p_n)) \to 0 \qquad \text{as} \qquad \rho \to 0. \tag{6.15}$$

for each $n \in \mathbb{Z}$.

Proof: We assume condition (6.14) holds, with a neighbourhood of attraction defined by $\hat{\delta}^*$.

The proof follows closely the argument presented in Theorem 6.2.1 with the following amendments.

Construct $\hat{\delta}^0$ in a similar fashion to the procedure for δ_0 in Theorem 6.2.1. This redefines $\hat{B}$ and $\hat{A}$ accordingly. We also redefine

$$\begin{aligned} C_r(p) &= \max\{\sup_{0 \le h \le \rho} a(\delta^0_{\theta_h p}), C_r^*(p)\}, \\ L(p) &= \max\{1, \sup_{0 \le h \le \rho} a(\delta^0_{\theta_h p})/\delta^0_{\theta_h p}, L^*(p)\}, \end{aligned}$$

and $h_1(p)$ so that it is the maximal quantity satisfying the inequality

$$4e^{ch_1(p)/2} L(p) C_r(p) h_1(p)^{r+1}/(1 - e^{-ch_1(p)/2}) < \rho^{r+1} a(\delta^0_p).$$

Finally the procedures given in Lemmas B1-B3, A1-A7, C1 of Theorem 6.2.1 are repeated with the above bounds and applying the inequality (6.14) where necessary. For example in Lemma B1, inequality (6.14) must be applied at (6.9) to ensure the result remains smaller than $a(\delta^0_{\theta_{h(p)}p})$.

This then verifies the existence of a discrete forward equi - asymptotically stable family $\hat{A}^{\mathbf{h}}$ with the required properties.

□

Chapter 7

Discretisation of Pullback Asymptotic Behaviour

The focus of this chapter is to understand the effects of discretisation on pullback asymptotically stable families. An analysis of the pullback numerics however, is inherently more complicated than that of the forward numerics analysed in Chapter 6. Finding an approach to verify results analogous to Theorem 6.2.1 for pullback equi-asymptotically stable families is not as clear, and a completely analogous result is yet open to further research.

Before the problem is discretised, an investigation into the pullback asymptotic behaviour for a single $p \in P$ is undertaken by means of a transformation applied to the original dynamical system. Pullback asymptotic behaviour in the original system is characterised by forward asymptotic behaviour in the transformed problem. The transformation has the benefit of allowing conventional techniques to be applied to the problem of discretisation, as well as assisting in visualising the pullback attraction in the original dynamical system.

The theory is first introduced, followed by application of numerical methods that utilise the transformed problem to establish a numerical approximation of the original pullback behaviour.

7.1 Duality of Attraction

7.1.1 Introduction

We consider one particular aspect of pullback behaviour, and we refer to this as the *duality of attraction*. The procedure fixes a single $p \in P$, and transforms the dynamical system into one in which, as previously mentioned, the pullback behaviour at p is characterised by forward attraction. As far as the author is aware, no other research has currently been published in this direction.

7.1.2 Loci Dynamics for A

Pullback asymptotic stability to a fixed point $p \in P$ can be modelled by considering the dynamical system resulting from an analysis of the system's sensitivity to initial times. To illustrate the fundamental concept, we begin with the example below.

Example 7.1.1. Consider the dynamical system arising from the NDE

$$\dot{x} = 2\mathtt{t}x.$$

Solutions for this system may be expressed in cocycle form as (noting that in cocycle form t represents the time elapsed rather than actual time)

$$\Phi_{(t,t_0)}(x_0) = x_0 e^{(t+t_0)^2 - t_0^2},$$

where the parameter space $P = \mathbb{R}$ and $t_0 \in \mathbb{R}$. Alternatively, by considering pullback attraction to the point t_0, we may express solutions by

$$\Phi_{(t,t_0-t)}(x_0) = x_0 e^{t_0^2 - (t_0 - t)^2}.$$

It is easily proven that the origin for this system is a pullback attractor without forward convergence properties. This can be seen graphically in Figure 7.1 where the initial state is set at $x_0 = 1$ and pullback attraction to $t_0 = 1/2$ is considered.

The solutions at t_0 define a mapping with image $\Phi_{(t,t_0-t)}(x_0)$ associated with each $t \geq 0$. If the locus of points defined by $(t_0 - t, \Phi_{(t,t_0-t)}(x_0))$ is plotted

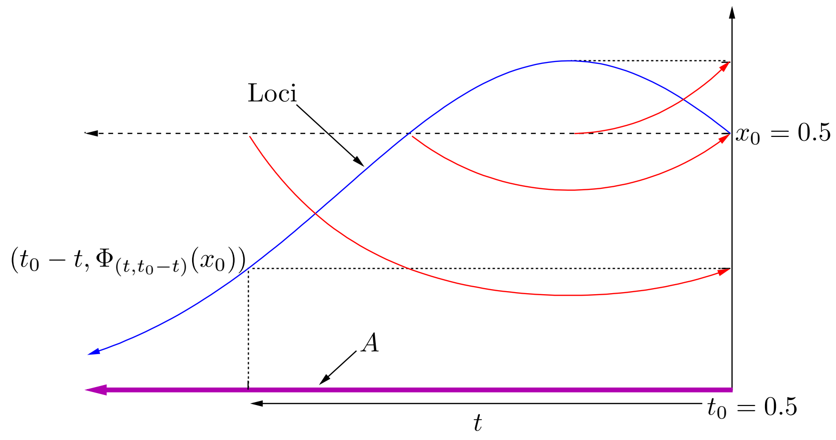

Figure 7.1: Locus of Transformation

as illustrated, the locus forms a continuous trajectory in reverse that asymptotes to the origin as t is increased. Similar loci may be plotted for each x_0, generating a system resembling a dynamical system with forward asymptotic properties.

The behaviour of the loci may in fact be calculated by determining the rate of change of the image $(\Phi_{(t,t_0-t)}(x_0))$ with respect to t. Hence we have

$$\begin{aligned}\frac{d}{dt}\left(\Phi_{(t,t_0-t)}(x_0)\right) &= \frac{d}{dt}\left(x_0 e^{t_0^2-(t_0-t)^2)}\right),\\ &= 2(t_0-t)\Phi_{(t,t_0-t)}(x_0).\end{aligned}$$

As x_0 is arbitrary, the dynamics for the loci is then simply governed by the equation $d\Phi/dt = 2(t_0-t)\Phi$, a dynamical system for which the origin is forward asymptotically stable.

$\square$

The dynamical system that will be referred to in the ideas that follow is generated by the non-autonomous differential equation $\dot{x} = f(p, x)$, and is assumed to possess a constant set A that is either pullback asymptotically stable or is a pullback attractor. These ideas are then extended to time-varying families, $\hat{A}$ that possess pullback behaviour in Section 7.2.

Since the pullback attracting object A is for the moment assumed to be a constant set, analysing the properties of pullback attraction by pulling back a single state x_0 is valid (x_0 is necessarily contained in the neighbourhood of attraction regardless of how far in time it is pulled back). Each locus then corresponds to a single initial state $x_0 \in \mathcal{N}_{\delta_p}(A)$, and the elements defining the locus are determined by the couple $(\theta_{-t}p, \Phi_{(t,\theta_{-t}p)}(x_0))$ for all $t \geq 0$.

Lemma 7.1.1. *Given any $p \in P$, then for each $x_0 \in \mathcal{N}_{\delta_p}(A)$, the loci defined by $(\theta_{-t}p, \Phi_{(t,\theta-tp)}(x_0))$ for $t \geq 0$ are continuous and unique.*

Proof: Continuity:

Let $\epsilon > 0$, and $x_0 \in \mathcal{N}_{\delta_p}(A)$ be arbitrary. Then

$$|\Phi_{(t+h,\theta_{-(t+h)}p)}(x_0) - \Phi_{(t,\theta_{-t}p)}(x_0)| = |\Phi_{(t,\theta_{-t}p)}(x_0^*) - \Phi_{(t,\theta_{-t}p)}(x_0)|,$$

where $x_0^* = \Phi_{(h,\theta_{-(t+h)}p)}(x_0)$. Using the same principle as in Lemma 5.1.2,

$$\begin{aligned}|\Phi_{(t+h,\theta_{-(t+h)}p)}(x_0) - \Phi_{(t,\theta_{-t}p)}(x_0)| &\leq |x_0^* - x_0| \exp\left(L(\theta_{-t}p)h\right), \\ &\leq hF(t) \exp\left(L(\theta_{-t}p)t\right).\end{aligned}$$

Here $F(t) = \sup\{f(\theta_{-\tau}p, x); t \leq \tau \leq t+h\}$ and $L(\theta_{-t}p)$ is the maximum local Lipschitz bound for f over the interval $(\theta_{-t}p, p)$.

Choose $h(t,\epsilon) \leq \epsilon/(F(t)\exp\left(L(\theta_{-t}p)t\right)$. Then

$$|\Phi_{(t+h,\theta_{-(t+h)}p)}(x_0) - \Phi_{(t,\theta_{-t}p)}(x_0)| \leq hF(t)\exp\left(L(\theta_{-t}p)\right) \leq \epsilon,$$

as required.

Uniqueness:

Assume the loci are not unique. That is, two loci cross paths at some $t^* > 0$. Since each loci is generated from distinct initial states, there exists $x_0, x_1 \in \mathcal{N}_{\delta_p}(A)$ with $x_0 \neq x_1$ such that $\Phi_{(t^*,\theta_{-t^*}p)}(x_0)) = \Phi_{(t^*,\theta_{-t^*}p)}(x_1))$. However, this contradicts the uniqueness of solutions for the original dynamical system. Hence each locus is necessarily unique.

□

Points along each locus may be defined by the mapping $\{\phi_{(t,t_0)}; t_0 \in \mathbb{R}^+, t \in [-t_0, \infty)\}$, with $\phi_{(t,t_0)} : E \to E$ and

$$\phi_{(t,t_0)}(\phi_0) = \Phi_{(t_0+t,\theta_{-(t_0+t)}p)}(\Phi_{(-t_0,p)}(\phi_0)).$$

E is the state space for the original system.

In terms of the original dynamical system, the loci for which (t_0, ϕ_0) is an element is simply a collection of images at p resulting from pulling back an initial value x_0 associated with ϕ_0. This association is determined by $x_0 = \Phi_{(-t_0,p)}(\phi_0)$. This can be seen graphically in Figure 7.2.

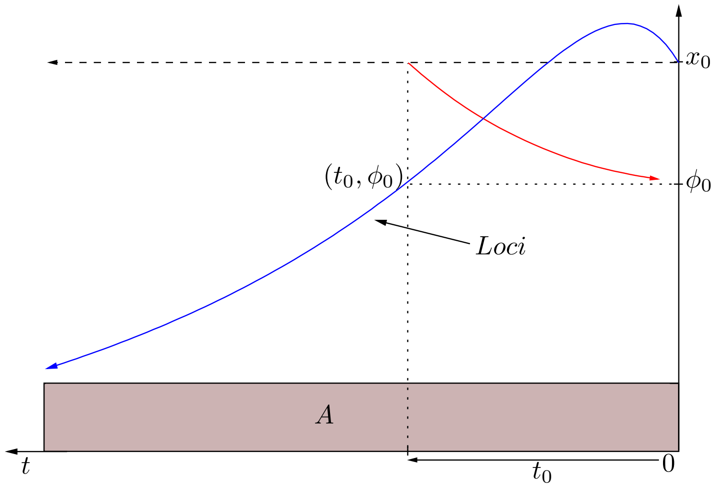

Figure 7.2: Loci in $\mathcal{D}_l$

Lemma 7.1.2. *The loci form a continuous dynamical system, $\mathcal{D}_l$, for which the group of mappings $\{\phi_{(t,t_0)}; t_0 \in \mathbb{R}^+, t \in [-t_0, \infty)\}$ with $\phi_{(t,t_0)} : E \to E$ forms a cocycle on E with respect to the group $\{\theta_t, t \in \mathbb{R}^+\}$, where $\theta_t t_0 = t_0 + t$.*

Proof: Identity:

$$\begin{aligned}\phi_{(0,t_0)}(\phi_0) &= \Phi_{(t_0,\theta_{-t_0}p)}((\Phi_{(-t_0,p)}(\phi_0)),\\ &= \phi_0.\end{aligned}$$

Cocycle Property:

$$
\begin{aligned}
\phi_{(t_1+t_2,t_0)}(\phi_0) &= \Phi_{(t_0+t_1+t_2,\theta_{-(t_0+t_1+t_2)}p)}(\Phi_{(-t_0,p)}(\phi_0)),\\
&= \Phi_{(t_0+t_1+t_2,\theta_{-(t_0+t_1+t_2)}p)}\\
&\qquad \left(\Phi_{(-(t_0+t_1),p)}(\Phi_{(t_0+t_1,\theta_{-(t_0+t_1)}p)}(\Phi_{(-t_0,p)}(\phi_0)))\right),\\
&= \Phi_{(t_0+t_1+t_2,\theta_{-(t_0+t_1+t_2)}p)}\left(\Phi_{(-(t_0+t_1),p)}(\phi_{(t_1,t_0)}(\phi_0))\right),\\
&= \phi_{(t_2,t_0+t_1)}(\phi_{(t_1,t_0)}(\phi_0)).
\end{aligned}
$$

Hence the cocycle property for ϕ is satisfied.

□

In some special cases the loci dynamics may be formulated (see Example 7.1.2), and if $P = \mathbb{R}$ a general expression for the ordinary differential equation which determines the loci dynamics may be formed.

Lemma 7.1.3 (Loci Dynamics)**.** *If f and its partial derivatives are continuous, and the parameter set $P = \mathbb{R}$, then the loci dynamics at each $t_0 \in \mathbb{R}$ are modelled by the ordinary differential equation with initial value $(0, \phi_0)$,*

$$\frac{d\phi}{dt} = \int_{t_0-t}^{t_0} \frac{\partial f^*}{\partial t}(\tau, t_0, t, \phi_0)d\tau + f(t_0 - t, \phi_0), \tag{7.1}$$

where $f^*(\tau, t_0, t, \phi_0) = f(\tau, \Phi_{(\tau-(t_0-t),t_0-t)}(\phi_0))$.

Proof: Note that

$$
\begin{aligned}
\phi_{(t,0)}(\phi_0) &= \Phi_{(t,t_0-t)}(\phi_0),\\
&= \phi_0 + \int_{t_0-t}^{t_0} f(\tau, \Phi_{(\tau-(t_0-t),t_0-t)}(\phi_0))d\tau.
\end{aligned}
$$

If f is continuous, and its partial derivative with respect to t exists and is continuous also, then application of the derivative arrives at the required result.

□

Example 7.1.2. [SDS - Loci Dynamics] Consider the NDE

$$\dot{x} = f(\mathsf{t})g(x),$$

for which we take $P = \mathbb{R}$, and assume $f, 1/g$ are continuous and bounded over the interval of consideration, and hence integrable.

We consider pullback attraction to an arbitrary choice of $t_0 \in \mathbb{R}$.

Equilibrium points, identified by $g(x) = 0$, are invariant in the original system, and hence are invariant in the loci dynamical system.

Elsewhere, let F, G denote the primitives of f, and $1/g$ respectively. Separating variables, and integrating from $t_0 - t \to t_0$:

$$G(\Phi_{(t,t_0-t)}(x_0)) = G(x_0) + F(t_0) - F(t_0 - t).$$

Finally, differentiating with respect to t, and rewriting solutions in terms of the cocycle mapping ϕ on $\mathcal{D}_l$, the loci dynamics are determined by

$$\frac{d\phi}{dt} = f(t_0 - t)g(\phi). \tag{7.2}$$

□

7.1.3 $\mathcal{D}_l$ - Asymptotic Behaviour

We now pursue the question of forward asymptotic attraction and stability within the dynamical system $\mathcal{D}_l$.

If A is locally pullback asymptotically stable, and x_0 lies in the local neighbourhood of pullback attraction at p (that is, $x_0 \in \mathcal{N}_{\delta_p}(A)$), then each point on the loci generated by x_0 must lie within the set $\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(A))$ for any given $t > 0$.

On the basis of this observation, we shall consider initial states for the dynamical system $\mathcal{D}_l$ at (t_0, ϕ_0) where $\phi_0 \in \Phi_{(t_0,\theta_{-t_0}p)}(\mathcal{N}_{\delta_p}(A))$. This is to ensure that the loci that passes through the point (t_0, ϕ_0) will reflect the original system's pullback asymptotic behaviour.

By continuity and uniqueness of the trajectories in the original system, there exists $x_0 \in \mathcal{N}_{\delta_p}(A)$ associated with any loci passing through ϕ_0 chosen in the fashion given above, so that

$$\Phi_{(t_0,\theta_{-t_0}p)}(x_0) = \phi_0,$$

or equivalently $x_0 = \Phi_{(-t_0,p)}(\phi_0)$. This was previously illustrated in Figure 7.2.

Then

$$\begin{aligned}\phi_{(t,t_0)}(\phi_0) &= \Phi_{(t+t_0,\theta_{-(t+t_0)}p)}(\Phi_{(-t_0,p)}(\phi_0)),\\ &= \Phi_{(t+t_0,\theta_{-(t+t_0)}p)}(x_0).\end{aligned}$$

Since $x_0 \in \mathcal{N}_{\delta_p}(A)$, then A pullback attracts x_0 and

$$\begin{aligned}\lim_{t\to\infty} \operatorname{dist}(\phi_{(t,t_0)}(\phi_0), A) &= \lim_{t\to\infty} \operatorname{dist}(\Phi_{(t+t_0,\theta_{-(t+t_0)}p)}(x_0), A),\\ &= 0.\end{aligned}$$

Hence for each t_0, A forward attracts any $\phi_0 \in \Phi_{(t_0,\theta_{-t_0}p)}(\mathcal{N}_{\delta_p}(A))$.

However, for A to be forward asymptotically stable, A must forward attract a local δ-neighbourhood system $\mathcal{N}_{\hat{\delta},A}$. The situation developed here allows for two possible scenarios:

i) Asymptotic Attraction - If

$$A \subset \operatorname{int}(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(A))),$$

for all $t \geq 0$, then there always exists a δ-neighbourhood of A in $\mathcal{D}_l$ defined by $\hat{\delta} = \{\delta_t; t \in \mathbb{R}^+\}$ where

$$\mathcal{N}_{\delta_t}(A) \subset \Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(A)).$$

Indeed if A is a pullback attractor, then this is automatically the case as solutions asymptote to the attractor.

Asymptotic pullback attraction is illustrated in Figure 7.3 where A is a pullback attractor, and solutions are pullback attracted to A in infinite time.

The following result is an automatic consequence of the asymptotic attraction of a forward attractor.

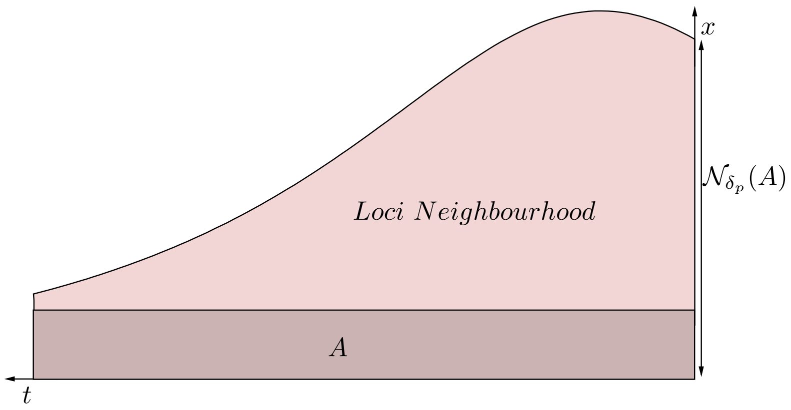

Figure 7.3: Region of Loci Generated by Asymptotic Attraction

Lemma 7.1.4 (Forward Attractors in $\mathcal{D}_l$)**.** *If A is a pullback attractor then A is a forward attractor in $\mathcal{D}_l$.*

Proof: Attraction and existence of an attracting neighbourhood have already been verified. Invariance follows from the invariance of A as a pullback attractor in the original system.

□

Remark: The neighbourhood system for forward attraction is not constant as it is bounded by pullback attraction of solutions from $\mathcal{N}_{\delta_p}(A)$ in the original system. As

$$\lim_{t\to\infty} \mathrm{dist}(\Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(A)), A) \to 0,$$

then the neighbourhood system for A in $\mathcal{D}_l$ also vanishes.

ii) Finite Attraction - A pullback absorbs some, or all states lying in a neighbourhood of A, that is $\mathcal{N}_{\delta_p}(A)$. In this case,

$$A \backslash \Phi_{(t,\theta_{-t}p)}(\mathcal{N}_{\delta_p}(A)) \neq \{0\},$$

for some $t > 0$.

As a result, all initial values (t_0, ϕ_0) in $\mathcal{D}_l$ chosen so that $\phi_0 \in \Phi_{(t_0,\theta_{-t_0}p)}(\mathcal{N}_{\delta_p}(A))$ may for some values of t_0 include only points that are contained in A. Consequently, forward attraction of points close to A in $\mathcal{D}_l$ at t_0 is unable to be determined without knowledge of the pullback attraction in the original system beyond a local neighbourhood of A.

This is illustrated in Figure 7.4 where the loci neighbourhood is displayed for a set A that pullback attracts solutions in the original dynamical system in finite time, t^*.

The behaviour of points outside this loci neighbourhood, in particular for any (t_0, ϕ_0) with $t_0 > t^*$, cannot be determined from the loci dynamics generated from the neighbourhood $\mathcal{N}_{\delta_p}(A)$. Hence there is no guarantee that A is a forward asymptotically stable set in $\mathcal{D}_l$.

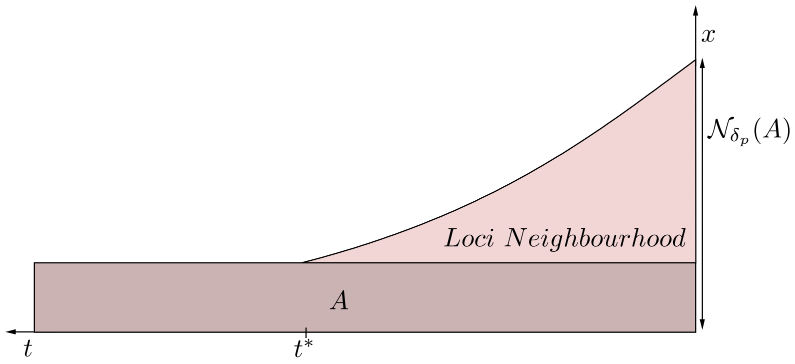

Figure 7.4: Region of Loci Generated by Finite Attraction

If however A is globally pullback asymptotically stable (and possibly finite pullback attracting) then an attracting neighbourhood of A in $\mathcal{D}_l$ may always be determined by a pullback analysis of all trajectories in the original dynamical system.

Lemma 7.1.5 (Eventual Forward Asympotic Stability in $\mathcal{D}_l$)**.** *If A is globally pullback asymptotically stable, then for each $p \in P$, A is eventually globally forward asymptotically stable in $\mathcal{D}_l$.*

Proof: Let $p \in P$ be arbitrarily chosen and consider any initial value in $\mathcal{D}_l$ defined by (t_0, ϕ_0).

Given (t_0, ϕ_0), there exists an x_0 such that in the original dynamical system

$$\Phi_{(-t_0, \theta_{-t_0} p)}(x_0) = \phi_0.$$

Since A is globally pullback attracting in the original system, it pullback attracts x_0. Hence there exists a loci originating at x_0 that is forward attracted by A in $\mathcal{D}_l$. Since ϕ_0 lies on this loci it is concluded that A forward attracts ϕ_0, and since the initial value (t_0, ϕ_0) was chosen arbitrarily, A must be globally forward attracting in $\mathcal{D}_l$.

□

Recall the reference to 'eventual asymptotic stability' in Section 1.3. Forward eventual asymptotic stability implies forward attraction of solutions, but not necessarily forward stability.

7.1.4 Loci Stability

To ensure forward stability of the loci in $\mathcal{D}_l$, stronger requirements on the pullback behaviour in the original dynamical are required.

Since equi-asymptotic attraction and invariance are sufficient requirements for stability (refer to [4]), the forward stability of A in $\mathcal{D}_l$ for pullback attractors is automatically guaranteed. As a result, Lemma 7.1.4 holds independently of any further requirements on the pullback stability of A in the original dynamical system.

Forward stability of A in $\mathcal{D}_l$ for globally pullback asymptotically stable sets however, is not necessarily true. The following example illustrates a case for which a globally pullback asymptotically stable set A is globally forward attracting (equivalently globally eventually forward asymptotically stable) but not forward stable in $\mathcal{D}_l$.

Example 7.1.3. Consider the NDE

$$\dot{x} = -(1/2 + 2\cos(3\mathtt{t}))x.$$

Set $P = \mathbb{R}$. Clearly the origin is a pullback attractor, however we will consider here the stability of $A = [-1, 1]$ in $\mathcal{D}_l$.

Solutions with initial value (t_0, x_0) are given by

$$x(\mathtt{t}) = x_0 \exp\left(\frac{1}{2}(t_0 - \mathtt{t}) + \frac{2}{3}(\sin(3t_0) - \sin(3\mathtt{t}))\right),$$

or using a cocycle representation analysing an initial state x_0 pulled back from t_0 by time t,

$$\Phi_{(t,t_0-t)}(x_0) = x_0 \exp\left(-\frac{1}{2}t + \frac{2}{3}(\sin(3(t_0 - t)) - \sin(3t_0))\right).$$

We shall investigate pullback attraction to $t_0 = 0$. From the above equation

$$\Phi_{(t,-t)}(x_0) = x_0 \exp\left(-\frac{1}{2}t - \frac{2}{3}\sin(3t)\right).$$

Hence the image at $t_0 = 0$ is monotonically decreasing as $(-\frac{1}{2}t - \frac{2}{3}\sin(3t)) < 0$ for all $t \geq 0$. As a result, A is pullback equi-asymptotically stable (and more importantly pullback stable) at $t_0 = 0$.

The loci trajectories in the associated loci dynamical system $\mathcal{D}_l$ for $t_0 = 0$ are of the form

$$\begin{aligned}\phi_{(t,0)}(x_0) &= \Phi_{(t,-t)}(x_0),\\ &= x_0 \exp\left(-\frac{1}{2}t - \frac{2}{3}\sin(3t)\right).\end{aligned}$$

Figure 7.5 illustrates the loci trajectory associated with initial state $x_0 = 2$. As the motion is symmetric around the time axis, the diagram is restricted to illustrate the behaviour of the loci for $x > 0$ only.

Here A forward attracts the initial state shown, but it is not forward stable. For example, let $\epsilon = 0.5$, and consider any δ-neighbourhood chosen whilst the loci is initially within A. Regardless of the choice of δ, points on this loci will emerge and travel beyond the ϵ-neighbourhood. Hence A is not forward stable in $\mathcal{D}_l$.

□

The critical difficulty with Example 7.1.3 is that the definition of pullback stability is not as *strong* as forward stability in terms of the behaviour characterised by the definitions.

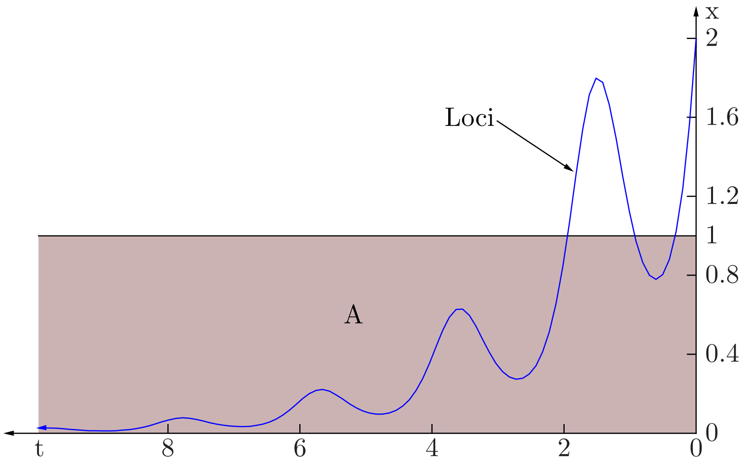

Figure 7.5: Non-stable Loci in $\mathcal{D}_l$

For example, suppose A is forward stable. Then the property of forward stability for A implies two defining features that are a natural consequence of its definition.

Properties of Stability

S1 - The trajectory of any solution that begins in close proximity to A remains in close proximity to A.

S2 - Any trajectory that is at some time caught in close proximity to A remains trapped and close to A thereafter.

Switching to a pullback analysis at some $p \in P$, the focus is on the behaviour of the images at p, or equivalently, the loci, as an initial state is pulled back rather than the trajectories themselves.

The definition of pullback stability automatically possesses the first property. That is, the image of any point close to A remains close to A as it is pulled back for all $t \geq 0$. However, it lacks the second property. As an initial state is pulled back, if at any time the image is caught within a small enough neighbourhood of A there is no guarantee that it remains trapped. Consequently, any loci that approach A in $\mathcal{D}_l$ are not guaranteed to remain close to A.

This reasoning leads to the conclusion that a stronger definition of pullback stability may be required. At the time of writing it is not immediately clear that it is essential, although redefining the structure of non-autonomous dynamical systems with an alternative definition for pullback stability that characterises the second property would be an interesting exercise, and is open for further work.

For the present task however, a stronger version of pullback stability is defined that will be an essential requirement for the ensuing problems involving discretisation.

Definition 7.1.1 (Loci Stability - A)**.** *A compact set A is said to be **Loci Stable** with respect to the cocycle $\{\Phi_{(t,p)}; t \in \mathbb{R}^+, p \in P\}$ on E if for any $\epsilon > 0$, there exists a $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ so that for any bounded and compact set B with the property*

$$\Phi_{(t^*,\theta_{-t^*}p)}(B) \subseteq \mathcal{N}_{\delta_p}(A(p)),$$

for some $t^* > 0$, and $p \in P$, then

$$H^*(\Phi_{(t,p)}(B), A) < \epsilon \qquad \forall t \geq t^*.$$

Remark 1: If A is loci stable, then it is pullback stable by the original definition. This can be seen by letting $B = \mathcal{N}_{\delta_p}(A(p))$, with $t^* = 0$ in the above definition.

Remark 2: The above definition characterises *both* properties **S1** and **S2**. In particular, if the image of a solution as it is pulled back (or equivalently its loci in $\mathcal{D}_l$) comes in close enough proximity to A, then the image as it is pulled further back (or the remainder of the loci's trajectory) remains trapped by an ϵ-neighbourhood of A thereafter.

Making use of this definition allows us to extend the result of Lemma 7.1.5 to guarantee the forward asymptotic stability of A in $\mathcal{D}_l$.

Theorem 7.1.1 (Forward Asympotic Stability in $\mathcal{D}_l$)**.** *If A is globally pullback attracting and loci stable, then A is globally forward asymptotically stable in $\mathcal{D}_l$.*

Proof: Forward attraction of solutions was shown in Lemma 7.1.5.
Thus it is only required to show forward stability.

Let $\epsilon > 0$, and consider the loci dynamical system generated at some $p \in P$. Finally, let $\delta_p = \delta_p(\epsilon)$ be chosen so that it satisfies the conditions for loci stability of A at p.

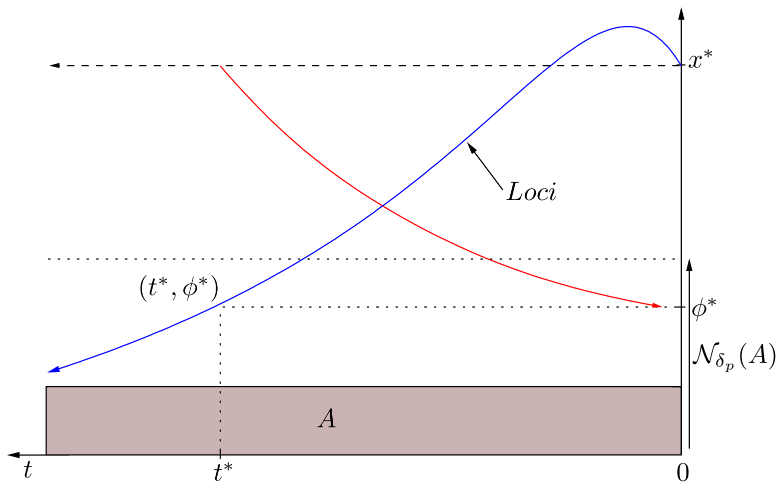

Figure 7.6: Forward Stability of Loci in $\mathcal{D}_l$

Consider any (t^*, ϕ^*) with $\phi^* \in \mathcal{N}_{\delta_p}(A)$. ϕ^* consequently lies on a loci trajectory in $\mathcal{D}_l$ that travels within a δ_p-neighbourhood of A at time t^*. Under an analysis of the original system, $\phi^* \in \mathcal{N}_{\delta_p}(A)$ at p, and is the image of some point x^* that has been pulled back in time t^*. Refer to Figure 7.6

Since A is loci stable in the original dynamical system, then

$$\begin{aligned} \text{dist}(\phi_{(t,t^*)}(\phi^*), A) &= \text{dist}(\Phi_{(t+t^*, \theta_{-(t+t^*)}p)}(x^*), A), \\ &< \epsilon, \end{aligned}$$

for all $t > 0$, confirming that A is forward stable in $\mathcal{D}_l$.

□

Remark: Understanding the effect of attraction to a single point $p \in P$ by analysis of the loci dynamical system $\mathcal{D}_l$ does not take into account the pullback attraction to A for all $p \in P$. This may be a limiting factor in observing completely the behaviour of a non-autonomous system possessing a pullback attractor, and in particular understanding the effects of discretisation.

7.2 Loci Dynamics for $\hat{A}$

7.2.1 Construction of $\mathcal{D}_l$

Earlier it was noted that the generation of the system of loci $\mathcal{D}_l$ for constant pullback attractors/pullback asymptotically stable families A could be extended to time-varying families $\hat{A}$.

The difficulty lies in tracing a single loci analysing attraction to $p \in P$ for an initial state x_0. Using the previous method, each point on the loci is generated by the image of the solution at p from the initial state x_0 pulled back in time. However, to obtain meaningful results regarding the attractive properties of the dynamical system requires that x_0 remains within the local neighbourhood of attraction as it is pulled back. This is not always possible for a time-varying attractor with a time-varying neighbourhood of attraction.

Such a point might also traverse repeatedly across the attractor as it is pulled back, destroying the illusion of a forward attracting object in $\mathcal{D}_l$.

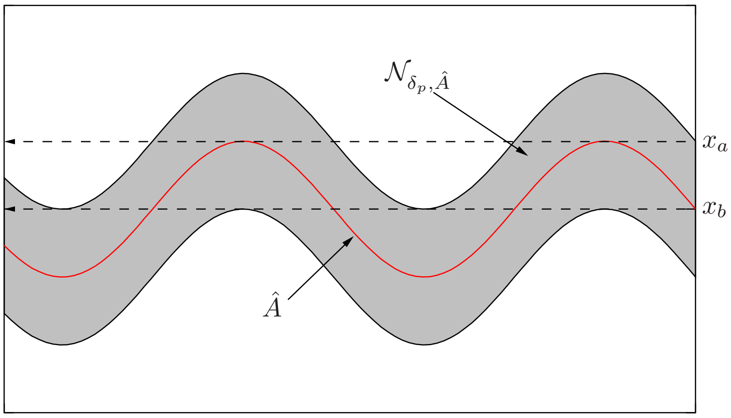

Figure 7.7: Loci Analysis of $\hat{A}$ - I

This is illustrated in Figures 7.7 and 7.8 for a sinusoidally varying pullback attractor which pullback attracts solutions exponentially. In the first diagram, analysis of x_a by pulling it straight back in time is clearly not possible as it may fall outside of the local neighbourhood of pullback attraction. In addition, applying the same process to a point on the attractor, x_b generates a loci that

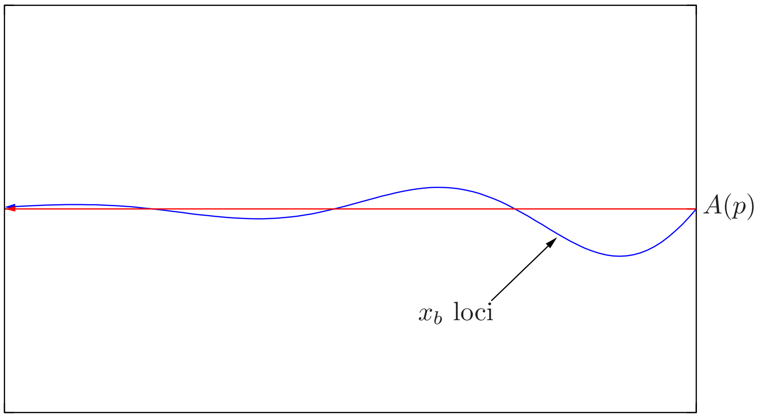

Figure 7.8: Loci Analysis of $\hat{A}$ - II

repeatedly traverses across $A(p)$ in $\mathcal{D}_l$. As a result, with this approach, $A(p)$ is not invariant, nor even stable in $\mathcal{D}_l$.

An alternative means for generating the loci may be achieved by using a sequence of uniformly bounded initial states contained within the neighbourhood system of pullback attraction that are associated with each $x_0 \in \mathcal{N}_{\delta_p}(A(p))$. These sequences are denoted by $\hat{x}(x_0) = \{x_0(\theta_{-t}p) \in \mathcal{N}_{\delta_p}(A(\theta_{-t}p)); t \geq 0\}$ for each $x_0 \in \mathcal{N}_{\delta_p}(A(p))$ and defined in the same manner as in the definition for pullback asymptotic stability (Definition 2.1.8).

This process is illustrated in Figures 7.9 and 7.10 for the sinusoidally varying attractor with exponential pullback attraction used to generate Figures 7.7 and 7.8. Here $A(p)$ is both invariant and forward asymptotically stable in $\mathcal{D}_l$.

To ensure continuity and uniqueness of the loci, (essential in requiring that $\mathcal{D}_l$ is an appropriately defined dynamical system) the initial sequences $\hat{x}(x_0)$, must also possess continuity and uniqueness. To guarantee uniqueness, the initial sequences must be constructed so that for all $t \geq 0$, each element $x_0(\theta_{-t}p)$ is unique to one and only one initial value $x_0 \in \mathcal{N}_{\delta_p}(A(p))$. That is, if there exists two sequences $\hat{x}(x_0), \hat{x}(x_1)$ with $x_0(\theta_{-t}p) = x_1(\theta_{-t}p)$ for some $t \geq 0$, then $x_0 = x_1$, and $\hat{x}(x_0) = \hat{x}(x_1)$.

Finally, the loci mapping may be generated as before, where each loci is associated with an initial sequence $\hat{x}(x_0) \in \mathcal{N}_{\delta_p,\hat{A}}$ with $x_0 \in \mathcal{N}_{\delta_p}(A(p))$. Each point on the loci is represented by the couple $(\theta_{-t}p, \Phi_{(t,\theta_{-t}p)}(x_0(\theta_{-t}p))$ parameterised

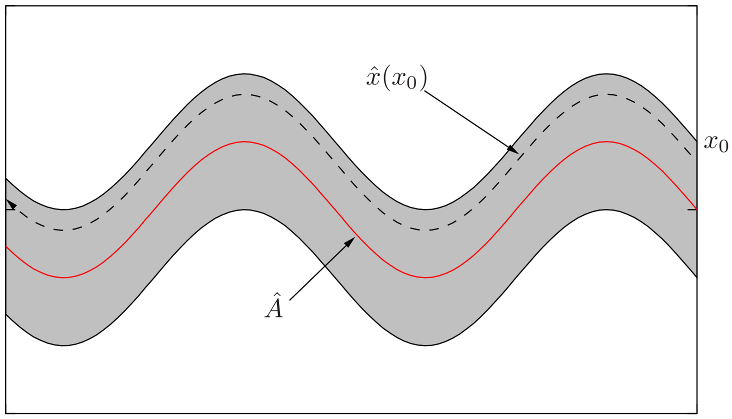

Figure 7.9: Time Varying Loci Analysis of $\hat{A}$ - I

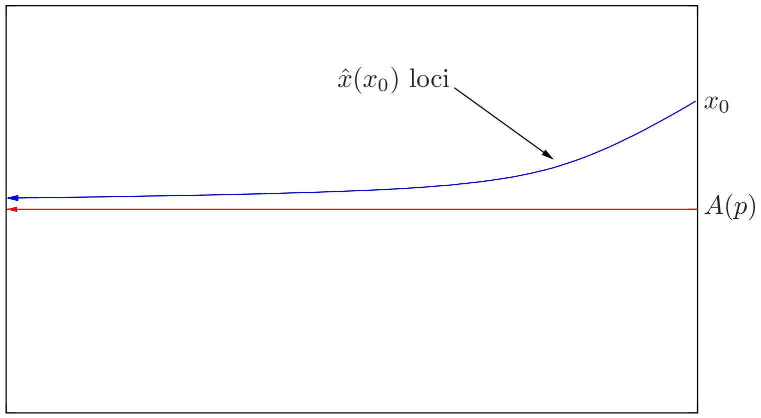

Figure 7.10: Time Varying Loci Analysis of $\hat{A}$ - II

for any $t \geq 0$.

A cocycle representation of the form

$$\phi_{(t,0)}(x_0) = \Phi_{(t,\theta_{-t}p)}(x_0(\theta_{-t}p)). \tag{7.3}$$

is used to define loci trajectories associated with the initial value $(0, \hat{x}(x_0))$.

Lemma 7.2.1 (The Loci Dynamical System $\mathcal{D}_l$)**.** *The loci mappings defined by (7.3) form a dynamical system $\mathcal{D}_l$.*

Proof. Utilising the above construction, the proof follows similarly to that for Lemma 7.1.2. □

Example 1: Initial sequences may be formulated by introducing a co-ordinate system on the local neighbourhood system of $\hat{A}$. A two dimensional system for example, could utilise a neighbourhood co-ordinate represented by the couple (α, φ), based on the co-ordinate distance (α) of a point in the neighbourhood away from an invariant solution passing through $\hat{A}$, and an angle of rotation (φ) centred on the invariant solution. Each initial sequence $\hat{x}(x_0)$ may then be uniquely defined by the co-ordinate value of x_0, and the subsequent loci dynamical system generated based on these values.

Example 2: If the elements of $\hat{A}$ constitute a single point in $\mathbb{R}^d$ for each $p \in P$, then the transformation defined by $x_0(\theta_{-t}p) = A(\theta_{-t}p) + (x_0 - A(p))$ provides a suitable construction for defining continuous and unique initial sequences, and the generation of the loci dynamical system $\mathcal{D}_l$.

Remark: The manner of generation of the initial sequences that determine a pullback analysis is abstractly arbitrary. In fact, what reasoning determines that single states are pulled straight back in time for a pullback analysis of systems with a constant attracting object A? The only restrictions placed on the sequences defined above is that of uniform boundedness and containment within a local neighbourhood. This is to ensure initial sequences cannot be chosen so that the rate of pullback attraction isn't nullified by a quickly diverging sequence, and as a consequence, any confirmation of pullback asymptotic behaviour erroneously missed.

Much of the pullback asymptotic theory given so far concerns a pullback analysis on δ-neighbourhoods (ideal for a local analysis), and this automatically restricts the construction of initial sequences required if pullback analysis of single states, or bounded sets is to be undertaken. Since the use of δ-neighbourhoods as a tool to correctly capture the effects of pullback attraction has proven useful, any subsequent conditions we will place on the construction of the initial sequences to correctly determine a pullback analysis, will endeavour to reflect δ-neighbourhood pullback asymptotic theory and avoid any conflicts that may arise.

7.2.2 Eventual Forward Asymptotic Stability in $\mathcal{D}_l$

As mentioned, the actual composition of the uniformly bounded initialising sequences as defined above is arbitrary, and although it provides the means to properly establish the loci dynamical system $\mathcal{D}_l$, it does not guarantee that properties of invariance or stability are transferred to the loci dynamical system.

Nevertheless, arbitrarily chosen, they still ensure $A(p)$ retains properties of eventual asymptotic stability in $\mathcal{D}_l$. This is analogous to the result given in Lemma 7.1.5 for constant sets A.

Lemma 7.2.2 (Eventual Asymptotic Stability in $\mathcal{D}_l$)**.** *If $\hat{A}$ is globally pullback asymptotically stable, then for each $p \in P$, $A(p)$ is eventually globally forward asymptotically stable in $\mathcal{D}_l$.*

Proof: The proof follows similarly to the proof for Lemma 7.1.5 with a pullback analysis based upon pulling back the initialising sequences rather than a fixed point.

□

7.2.3 Pullback Analysis of Bounded Sets

Before proceeding to extend the notion of Loci Stability for time varying sets $\hat{A}$, a means to analyse pullback properties of bounded and compact sets (compare with the pullback analysis of the bounded set B in Definition 7.1.1) is required.

Consider any bounded and compact set B in a suitable neighbourhood of $\hat{A}$ at $\theta_{-t^*}p$ for some $t^* > 0$.

An essential requirement for pullback analysis of B is that the initialising sequences originating from B correspondingly define a family of bounded, compact and *connected* sets associated with B. We establish this as a condition required of the initialising sequences.

A1 - For each $p \in P$, and any bounded, compact and connected set B_0 at p, the initial sequences defined by $\hat{B}(B_0)$ generate a family of bounded, compact and connected sets defined by $\hat{B}(B_0) = \{B_0(\theta_{-t}p); t \geq 0\}$ where

$$B_0(\theta_{-t}p) = \bigcup_{x_0 \in B_0} x_0(\theta_{-t}p).$$

This concept extends also to allow generation of a family of bounded, compact and connected sets associated with some B at t^* by tracing the initial sequences passing through B. This family is denoted by $\hat{B}(B_{t^*}) = \{B_{t^*}(\theta_{-\tau}p); \tau \geq t^*\}$ where

$$B_{t^*}(\theta_{-\tau}p) = \bigcup_{x \in B} x_0(\theta_{-\tau}p).$$

where each $x \in B$ is the $(\theta_{-t^*}p)$'th element of some initial sequence $\hat{x}(x_0)$.

Note that for a pullback analysis of B from t^* only elements further back in time are considered.

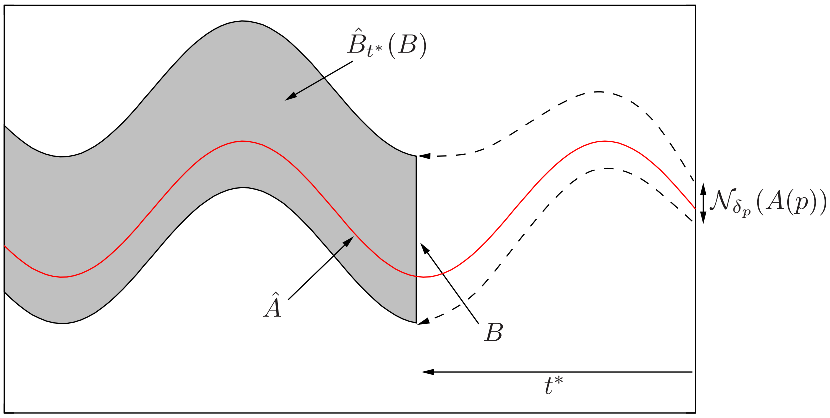

Figure 7.11: Pullback Analysis of Bounded Sets

Essentially this procedure is automatically satisfied for the usual pullback analysis of a δ-neighbourhood of $\hat{A}$, although some extra care is needed for general sets B since the appropriate construction isn't neatly defined by a δ value.

An actual physical illustration of what is required is simpler to grasp, and should be immediately obvious. An example of this is presented in Figure 7.11 with the additional marking of B's image at p which will be discussed shortly in connection with loci stability.

7.2.4 Loci Stability

If the initial sequences for a pullback analysis are generated in compliance with **A1**, then loci stability for time varying $\hat{A}$ may be defined as follows.

Definition 7.2.1 (Loci Stability - $\hat{A}$)**.** *A uniformly bounded family of compact sets $\hat{A} = \{A(p); p \in P\}$ is said to be* **Loci Stable** *if a pullback analysis in compliance with **A1** can be made so that for any $\epsilon > 0$, there exists a $\hat{\delta} = \{\delta_p \in \mathbb{R}^+; p \in P\}$ that satisfies the following condition for any bounded and compact set $B \subset \mathbb{R}^d$. If B satisfies the property*

$$\Phi_{(t^*,\theta_{-t^*}p)}(B) \subseteq \mathcal{N}_{\delta_p}(A(p)),$$

for some $t^* > 0$ and $p \in P$, then

$$H^*(\Phi_{(t,p)}(B_{t^*}(\theta_{-t}p)), A) < \epsilon \qquad \forall t \geq t^*.$$

Remark : Note that not all choices of initial sequences in compliance with **A1** for a pullback analysis of loci stability will be valid. It is only required that a single choice is available. A finite pullback asymptotically stable set A for example, may be loci stable if solutions are pulled straight back, yet it is easy to construct sequences that cause the images (or loci) to criss cross A in $\mathcal{D}_l$. Nevertheless, the fact that a particular choice exists is enough to characterise the properties of loci stability. That is, once the image is close, it stays relatively close.

7.2.5 Forward Asymptotic Stability in $\mathcal{D}_l$

Pullback asymptotic stability of $\hat{A}$ together with loci stability as defined above ensure attraction *and* stability of the loci around $A(p)$ in $\mathcal{D}_l$ in a similar manner to the construction in Section 7.1.3.

If the initial sequences are constructed as above, the proof for the theorem below then follows similarly to the proof for Theorem 7.1.1.

Theorem 7.2.1 (Forward Asymptotic Stability in $\mathcal{D}_l$)**.** *If $\hat{A}$ is globally pullback attracting and loci stable, then for each $p \in P$, $A(p)$ is globally forward asymptotically stable in $\mathcal{D}_l$.*

7.2.6 Forward Attractors in $\mathcal{D}_l$

One further condition is placed on the construction of the initial sequences for the loci mappings if $\hat{A}$ is a pullback attractor:

A2 - For every $x_0 \in A(p)$, the associated initial sequence is the invariant solution passing through $\hat{A}$ backwards in time. That is, $\hat{x}(x_0) = \{x_0(\theta_{-t}p) \in A(\theta_{-t}p); \Phi_{(t,\theta_{-t}p)}(x_0(\theta_{-t}p)) = x_0, t \geq 0\}$.

This condition ensures invariance of the forward attractor $A(p)$ in $\mathcal{D}_l$.

Lemma 7.2.3 (Forward Attractors in $\mathcal{D}_l$)**.** *Let $\hat{A}$ be a pullback attractor for the non-autonomous dynamical system (5.1). Then for each $p \in P$, if the loci are formulated in compliance with **A2**, $A(p)$ is a forward attractor in $\mathcal{D}_l$.*

This concludes an analysis of the dynamical system $\mathcal{D}_l$. The following section proceeds with an investigation on the effects of discretising the original dynamical system as observed in $\mathcal{D}_l$.

7.3 Discretisation of $\mathcal{D}_l$

Here we consider a the discretisation of a globally pullback equi-asymptotically stable family $\hat{A}$, and observe the effects of the discretisation from within the loci dynamical system $\mathcal{D}_l$. Since $\mathcal{D}_l$ characterises pullback dynamical behaviour at some fixed $p \in P$, we discuss two approaches that may be used to numerically approximate an element of the original pullback equi-asymptotically stable family $A(p)$ at $p \in P$.

7.3.1 Discretisation of the Loci Dynamics

We first examine the discretisation of the loci dynamics. This approach is not always possible as the loci dynamics (7.1) may only be explicitly formulated for certain special cases (as for instance in Example 7.1.2).

If however the loci dynamics can be generated, then for each $p \in P$ we may discretise the loci dynamics and apply the results of Theorem 6.2.1. This then verifies the existence of a discrete forward equi-asymptotically stable set $A_p^{\mathbf{h}}$ that serves as an approximation for $A(p)$.

Remark: Generating a numerical approximation in this manner, does not give any indication of the effect of discretisation on the original system, but merely serves as a technique to approximate the original pullback behaviour.

A discretisation using this approach is illustrated in the following example.

Example 7.3.1. [SDS - Discretisation of $\mathcal{D}_l$] Consider again the NDE introduced in Example 7.1.2

$$\dot{x} = f(\mathsf{t})g(x).$$

for which there exists a globally pullback equi-asymptotically stable set A that is also loci stable.

To analyse pullback attraction to some $t_0 \in \mathbb{R}$, recall that the loci dynamics are determined by the NDE

$$\frac{d\phi}{dt} = f(t_0 - t)g(\phi), \tag{7.4}$$

and that A was shown to be a global forward equi-asymptotically stable set for the loci dynamical system $\mathcal{D}_l$.

Discretising (7.4) with a variable time-step scheme and applying Theorem 6.2.1 will verify the existence of a discrete forward equi-asymptotically stable set that approximates A at t_0.

$\square$

7.3.2 Discretisation of the Original Dynamics

We now examine the case for discretisation of the original dynamics. With this approach the original pullback dynamics are discretised and the evolution of the images at p, or equivalently, the corresponding numerically approximated loci in $\mathcal{D}_l$ are observed.

The discretisation problem is formalised as follows:

As in Section 6.2, we consider a numerical scheme (possibly with a variable time-step construction) applied to a non-autonomous dynamical system defined by $\dot{x} = f(p, x)$ possessing a global pullback equi-asymptotically stable family $\hat{A}$. The discretisation is used to approximate pullback attraction to some $p \in P$.

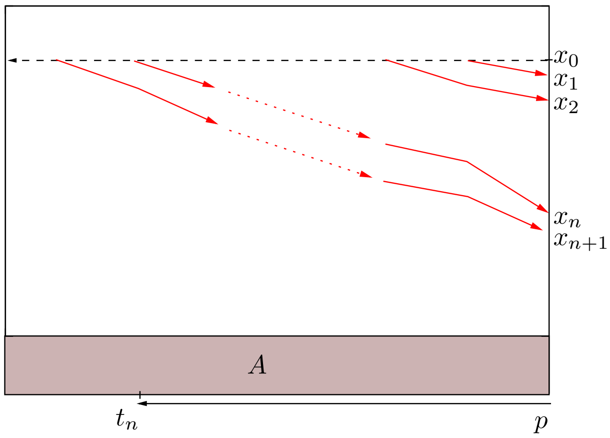

Figure 7.12: The Discrete Pullback Sequence

The numerical scheme generates a discrete cocycle $\{\Phi^{\mathbf{h}}_{(n,(p,\mathbf{h}))}, n \in \mathbb{Z}^+, (p, \mathbf{h}) \in$

$P_d\}$ as used in Section 6.2. To analyse pullback attraction of an initial sequence $\hat{x}_0 \in \mathcal{N}_{\delta_p,\hat{A}}$ to p, a discrete sequence of the images at p is generated, denoted by $\{x_n\}$ and defined by

$$\begin{aligned} x_n &= \Phi^{\mathbf{h}}_{(n,\theta_{-n}(p,\mathbf{h}))}(x_0(\theta_{-n}(p,\mathbf{h}))), \\ &= \Phi^{\mathbf{h}}_{(n,(p_{-n},\psi_{-n}\mathbf{h}))}(x_0(p_{-n},\psi_{-n}\mathbf{h})). \end{aligned} \tag{7.5}$$

If $\hat{A}$ is not time varying, that is $\hat{A} = A$, then pullback attraction of a single initial state x_0 rather than initial sequences may be used. For clarity of illustration we have assumed $\hat{A}$ is not time varying in all the associated figures.

The **discrete sequence** $\{x_n\}$ represents a series of approximated images at p that are each individually generated by an n-step discretisation. This is illustrated in Figure 7.12.

Each element in the discrete sequence is generated from a chain of discrete points according to the numerical method applied. The set of n points associated with the discrete image x_n will be referred to as the **n-th discrete chain**, and each element in the chain denoted by the notation $\{x_n^i\}$ for $i = 1, \cdots, n$ (see Figure 7.13). Subsequently, the notation for x_n and x_n^n are equivalent, and the choice of which is used should hereafter be relevant to the situation.

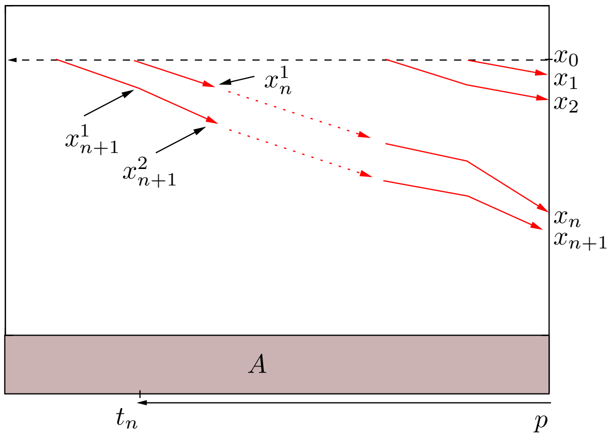

Figure 7.13: Discrete Chains

If plotted in $\mathcal{D}_l$, the discrete sequence traces out a discrete trajectory of points (t_n, x_n) that approximates the continuous loci in exactly the same way that a numerical discretisation typically approximates a trajectory in the forward sense. This is shown in Figure 7.14.

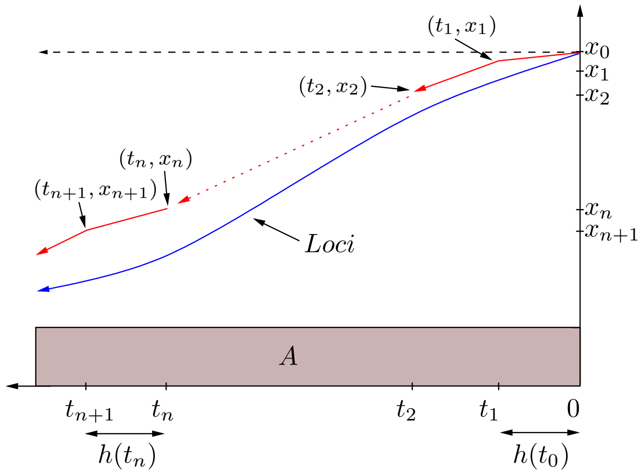

Figure 7.14: The Discrete Sequence in $\mathcal{D}_l$

7.3.3 The Numerical Algorithm in $\mathcal{D}_l$

To understand the effects of the numerical method applied to the original system as seen in $\mathcal{D}_l$, it is necessary to derive the *numerical algorithm* that defines the discrete sequence $\{x_n\}$. If the numerical algorithm can be derived then an analysis of the discrete pullback dynamics may be restricted solely to $\mathcal{D}_l$.

The numerical algorithm defining the discrete sequence $\{x_n\}$ takes the general form

$$x_{n+1} = x_n + h(t_n)F(t_n, x_n, x_0, n) + O(h(t_n)^2), \tag{7.6}$$

where $h(t_n) \in \mathbf{h}$ denotes the variable time-step from $\theta_{(-t_{n+1})}p$ to $\theta_{(-t_n)}p$ in the original system (see Figure 7.14).

For consistency and clarity throughout the remainder of the Chapter, (7.6 will be referred to as the **numerical algorithm** in $\mathcal{D}_l$ derived from the **numerical method** applied to the original dynamical system.

Typically (in an analysis of numerical methods applied in a forward sense) the dependence of F is limited to the variables t_n and x_n. Any knowledge of the sequence up to that point is unnecessary.

However, except for specific cases, the pullback numerical approximation is intrinsically more complex and takes the form given by (7.6). Its dependence on x_0 is illustrated with the example below.

Example 7.3.2. The dynamical system generated by the NDE

$$\dot{x} = 2\mathsf{t}x^3,$$

possesses a constant pullback attractor defined by $A = \{0\}$. We consider discrete pullback attraction to $t_0 = 0$ using an Euler method, investigating the dependence of the numerical algorithm (7.6) on the initial state x_0.

Let the step size $h = 0.1$ for the first few steps of any generated discrete sequence, and set $x_1 = 1$ as the second element in a discrete sequence generated from some unknown x_0.

We wish to calculate x_2, however this requires more information than simply the requisite knowledge of the previous element $x_1 = 1$.

Considering the discrete problem from the perspective of the original dynamical system (see Figure 7.15) generating x_2 requires the precise value of x_0. The difficulty however is that the discrete chains are not reversibly unique.

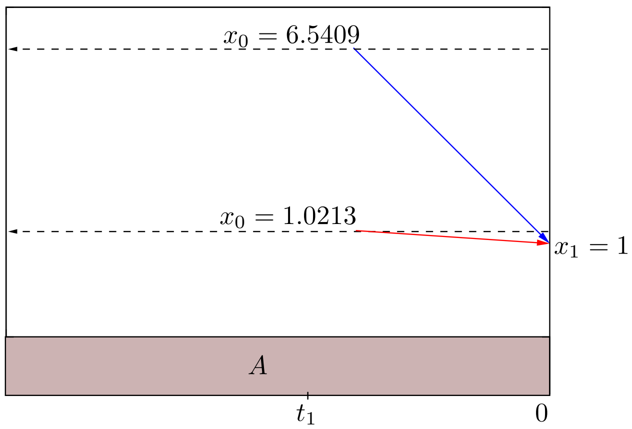

Figure 7.15: x_0-Dependence

For instance, given $x_1 = 1$ with an Euler method and step size of $h = 0.1$ we have

$$\begin{aligned} x_1 &= x_0 + 2h(t_0 - t_1)(x_0)^3, \\ 1 &= x_0 - 0.02(x_0)^3. \end{aligned}$$

Approximating solutions to the cubic polynomial yields the approximate solutions $x_0 \approx 1.0213$ or $x_0 \approx 6.5049$. Using these initial states to determine the discrete image after two steps gives $x_2 \approx 0.96$ or $x_2 \approx -2.64$ This is illustrated in Figure 7.16 as observed from within $\mathcal{D}_l$.

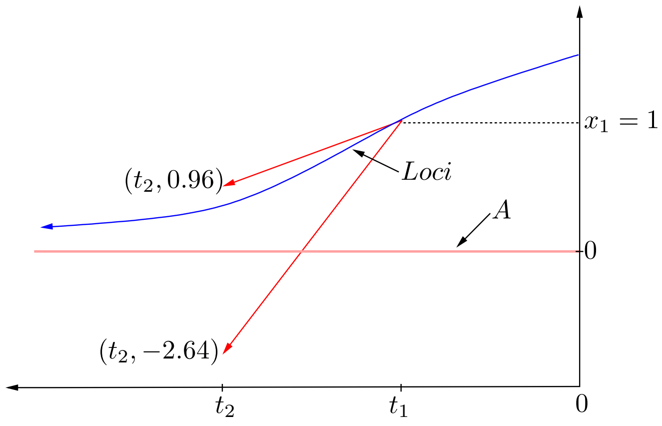

Figure 7.16: x_0-Dependence in $\mathcal{D}_l$

As a result, it is concluded that the numerical algorithm (7.6) generating the discrete sequences for this problem possesses an explicit dependence on x_0.

□

The above example illustrates the effect of an example for which the numerical method is not uniquely reversible, and outlines the factors influencing a general case example.

If however, the numerical method is uniquely reversible, then the numerical algorithm (7.6) may be simplified so that it is independent of x_0. Unfortunately, the class of discrete problems for which this occurs is relatively small.

Lemma 7.3.1. *If the numerical method is uniquely reversible, then the numerical algorithm (7.6) is independent of* x_0*.*

Proof: For a numerical method of any order, the $(n+1)$-th element in the discrete sequence, x_{n+1} may be expressed as a function of x_0.

However, x_n is also a function of x_0 and since it is uniquely reversible, x_0 may be expressed as a function of x_n. Consequently, by substitution, x_{n+1} may be expressed as a function of x_n.

$\square$

In addition, the following example illustrates the explicit dependence of (7.6) on the number of steps taken in $\mathcal{D}_l$ to reach some time t_n.

Example 7.3.3. The singleton set $A = \{0\}$ is a pullback attractor for the dynamical system generated by the NDE

$$\dot{x} = 2t\left[sgn(x)x^2\right].$$

For the dynamical system given, a discrete pullback analysis to $t_0 = 0$ with an Euler method is made for two uniquely distinct initial points a_0 and b_0.

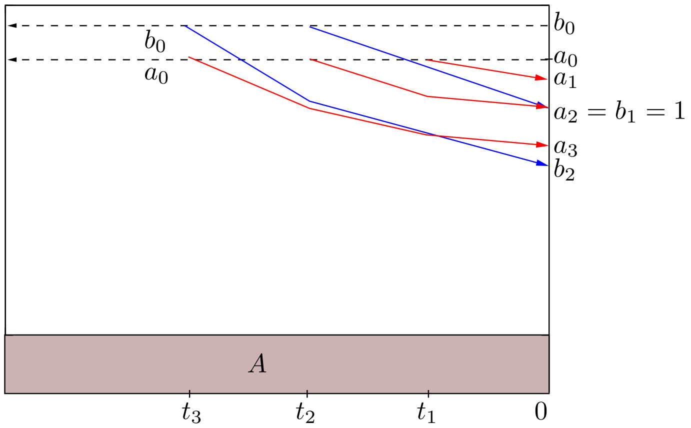

Figure 7.17: n-Dependence

The first three elements of the sequence for $\{a_n\}$ are calculated using a constant step size of 0.05, whereas the sequence for $\{b_n\}$ uses step sizes 0.1 and 0.05 to generate b_1 and b_2. Additionally, we set $a_2 = b_1 = 1$. This is shown in Figure 7.17.

In setting $a_2 = b_1 = 1$ the effect of discretisation from a particular point on a loci in $\mathcal{D}_l$ at a specified time may be observed for two sequences which differ only in the number of steps taken to arrive there (refer to Figure 7.18).

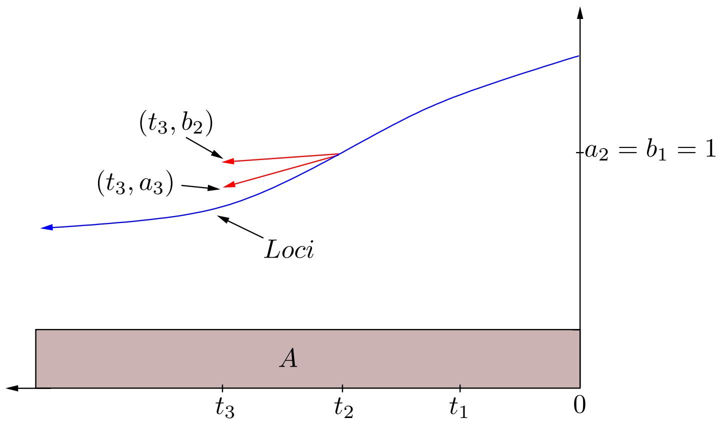

Figure 7.18: n-Dependence in $\mathcal{D}_l$

Since the discrete chains for this example are reversibly unique, given $a_2 = b_1 = 1$ it is possible to explicitly calculate a_0 and b_0, and consequently the progressive steps a_3 and b_2.

$\mathbf{a_0}$	1.0154		
$\mathbf{a_1}$	1.0102	$\mathbf{b_0}$	1.0208
$\mathbf{a_2}$	1.0000	$\mathbf{b_1}$	1.0000
$\mathbf{a_3}$	0.9849	$\mathbf{b_2}$	0.9800

Note that $a_3 \neq b_2$ as expected. As the only variable effecting the difference between a_3 and b_2 is the number of steps taken, it is clear the numerical algorithm that generates the discrete sequences for this example is explicitly dependent on n.

$\square$

Definition 7.3.1. *A numerical method is said to be* invariant under the loci mapping *if the numerical algorithm (7.6) is equivalent to the application of the same numerical method directly applied to the loci dynamics.*

It is interesting to observe that for linear non-autonomous dynamical systems, an r-th order Taylor series numerical method is indeed invariant under the loci mapping.

Lemma 7.3.2 (LDS 1 - Numerical Algorithm)**.** *Given the linear non-autonomous dynamical system*

$$\dot{x} = f(\mathsf{t})x, \tag{7.7}$$

where f is C^r continuous, then an r-th order Taylor series numerical method with variable time steps applied to the original dynamical system is invariant under the loci mapping.

Proof: Here $P = \mathbb{R}$, and consider pullback attraction to an arbitrarily chosen $t_0 \in \mathbb{R}$.

The Taylor series method utilises a variable time-step sequence denoted by $\mathbf{h} = \{h(t_0), h(t_1), \ldots h(t_n), \ldots\}$ where each $h(t_i)$ corresponds to the discrete step length between $t_0 - t_{i+1}$ and $t_0 - t_i$, and each $t_i = \sum_{j=1}^{i} h(t_{j-1})$.

The discrete sequence $\{x_n\}$ is then generated as follows

$$\begin{aligned} x_n^n &= x_n^{n-1}\left(1 + \sum_{i=1}^{r} \frac{h(t_0)^i}{i!} f^{(i-1)}(t_0 - t)\right), \\ &= x_0 \prod_{j=1}^{n}\left(1 + \sum_{i=1}^{r} \frac{h(t_{j-1})^i}{i!} f^{(i-1)}(t_0 - t_j)\right). \end{aligned} \tag{7.8}$$

The difference at each stage between elements in the $(n+1)$-th and n-th discrete chains is of the form

$$
\begin{aligned}
x_{n+1}^{1} - x_0 &= x_0 \sum_{i=1}^{r} \frac{h(t_n)^i}{i!} f^{(i-1)}(t_0 - t_{n+1}), \\
x_{n+1}^{2} - x_n^{1} &= (x_{n+1}^{1} - x_0) - (x_{n+1}^{1} - x_0) \sum_{i=1}^{r} \frac{h(t_{n-1})^i}{i!} f^{(i-1)}(t_0 - t_n), \\
&= x_0 \left(\sum_{i=1}^{r} \frac{h(t_n)^i}{i!} f^{(i-1)}(t_0 - t_{n+1}) \right) \\
&\qquad \left(1 + \sum_{i=1}^{r} \frac{h(t_{n-1})^i}{i!} f^{(i-1)}(t_0 - t_n) \right).
\end{aligned}
$$

Recursing the procedure, and substituting in (7.8), the numerical algorithm (7.6) is then formally defined by

$$
\begin{aligned}
x_{n+1}^{n+1} &= x_n^n + x_0 \left(\sum_{i=1}^{r} \frac{h(t_n)^i}{i!} f^{(i-1)}(t_0 - t_{n+1}) \right) \\
&\qquad \prod_{j=1}^{n} \left(1 + \sum_{i=1}^{r} \frac{h(t_{j-1})^i}{i!} f^{(i-1)}(t_0 - t_j) \right), \\
&= x_n^n + x_n^n \left(\sum_{i=1}^{r} \frac{h(t_n)^i}{i!} f^{(i-1)}(t_0 - t_{n+1}) \right). \qquad (7.9)
\end{aligned}
$$

Recall the loci dynamics (7.2) for a separable non-autonomous dynamical system. Since the linear system is a special case, the loci dynamics here is of the form

$$\dot{\phi} = f(t_0 - t)\phi.$$

Comparing this with the form of the numerical algorithm (7.9), it is clear that applying the numerical method to the original dynamical system is equivalent to applying the same r-th order Taylor series numerical method directly to the loci dynamical system $\mathcal{D}_l$. Hence the r-th order Taylor series method is invariant under the loci mapping.

□

Note that the numerical algorithm for the linear case is independent of both x_0 and n. The first is due to the fact that the discrete chains are uniquely reversible, while the latter is due to the numerical method for this example possessing properties similar to 'associativity'. Each element in the n-th discrete chain is simply a progressive application of n scaling factors that can be rearranged or combined in any alternate fashion. As a result the final image of the discrete chain associated with x_0 that has been pulled back a specified time is independent of the number of steps taken.

Lemma 7.3.3 (SDS 1 - Numerical Algorithm)**.** *Given the separable non-autonomous dynamical system*

$$\dot{x} = f(\mathsf{t})g(x), \tag{7.10}$$

where f is C^r continuous, then an Euler numerical method with variable step sequence $\mathbf{h}$ *applied to the original dynamical system generates a numerical algorithm of the form*

$$x_{n+1} - x_n = h(t_n)(a_0 + e_0)\prod_{j=1}^{n}(a_j + e_j), \tag{7.11}$$

where

$$\begin{aligned}
F'(a,b) &= (f(b) - f(a))/(b-a),\\
G'(x,y) &= (g(y) - g(x))/(y-x),\\
a_0 &= f(t_0 - t_n)g(x_0),\\
a_i &= 1 + h(t_{n-i})f(t_0 - t_{n+1-i})G'(x_n^i, x_n^{i-1}),\\
e_0 &= h(t_n)F'(t_0 - t_{n+1}, t_0 - t_n)g(x_0),\\
e_i &= h(t_{n-i})f(t_0 - t_{n+1-i})[G'(x_{n+1}^i, x_n^{i-1}) - G'(x_n^i, x_n^{i-1})].
\end{aligned}$$

Proof: The substitutions F', G', a_i, e_i are used for clarity of expression and to present the numerical algorithm in a form that allows calculation of the truncation error later in the chapter.

Let $P = \mathbb{R}$, and consider discrete pullback attraction to an arbitrarily chosen $t_0 \in \mathbb{R}$ using an Euler method with variable step sequence $\mathbf{h}$ defined as in Lemma 7.3.2.

Generating the difference between elements in the $(n+1)$-th and n-th chains in a similar fashion to that resolved in the linear case, we have

$$\begin{aligned}
x_{n+1}^1 - x_0 &= h(t_n)f(t_0 - t_{n+1})g(x_0),\\
x_{n+1}^2 - x_n^1 &= (x_{n+1}^1 - x_0) + h(t_{n-1})f(t_0 - t_n)(g(x_{n+1}^1) - g(x_0),\\
&= (x_{n+1}^1 - x_0)\left(1 + h(t_{n-1})f(t_0 - t_n)\left[\frac{g(x_{n+1}^1) - g(x_0)}{x_{n+1}^1 - x_0}\right]\right),
\end{aligned}$$

Making substitutions for F', G', a_i, e_i,

$$\begin{aligned}
x_{n+1}^1 - x_0 &= h(t_n)\left[f(t_0 - t_n)g(x_0)\right.\\
&\qquad\qquad \left.+h(t_n)\left((f(t_0 - t_{n+1}) - f(t_0 - t_n))/h\right)g(x_0)\right],\\
&= h(t_n)(a_0 + e_0),\\
x_{n+1}^2 - x_n^1 &= (x_{n+1}^1 - x_0)(1 + h(t_{n-1})f(t_0 - t_n)G'(x_{n+1}^1, x_0)),\\
&= (x_{n+1}^1 - x_0)\left[1 + h(t_{n-1})f(t_0 - t_n)G'(x_n^1, x_0)+\right.\\
&\qquad \left.h(t_{n-1})f(t_0 - t_n)(G'(x_{n+1}^1, x_0) - G'(x_n^1, x_0))\right],\\
&= h(t_n)(a_0 + e_0)(a_1 + e_1).
\end{aligned}$$

Consequently

$$x_{n+1}^{i+1} - x_n^i = h(t_n)(a_0 + e_0)\prod_{j=1}^{i}(a_j + e_j).$$

The numerical algorithm for the discrete sequence $\{x_n\}$ is then of the form

$$x_{n+1} - x_n = h(t_n)(a_0 + e_0)\prod_{j=1}^{n}(a_j + e_j),$$

□

Once the numerical algorithm (7.6) for the loci dynamical system is determined it is then possible to restrict an analysis to $\mathcal{D}_l$ and the study of the discrete forward behaviour it possesses. If feasible, this approach is advantageous as we may then utilise the results of Chapter 6 where applicable.

The results in Chapter 6 however, rely on the assumption that the local truncation error is of order h^{r+1}. Unfortunately this property is not necessarily transferred to the discrete sequence in $\mathcal{D}_l$, and needs to be carefully addressed.

7.3.4 Local Truncation Error in $\mathcal{D}_l$

Consider some initial state (t_n, x_n) in the loci dynamical system $\mathcal{D}_l$ where x_n is the n-th step in the discrete sequence originating from some x_0. Since $A(p)$ is a global forward equi-asymptotically stable family, then the point (t_n, x_n) lies on some continuous loci (not necessarily originating at x_0) in $\mathcal{D}_l$.

The error arising from x_n between the continuous evolution of the loci, that is $\phi_{(h(t_n),t_n)}(x_n)$, and the discretisation, x_{n+1}, over a single step form the local truncation error for forward analysis on $\mathcal{D}_l$. Refer to Figure 7.19.

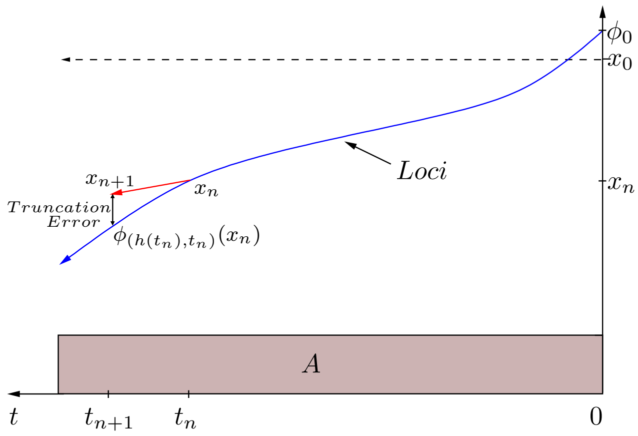

Figure 7.19: Local Truncation Error in $\mathcal{D}_l$

Recall the loci dynamics (7.1) are governed by an ODE

$$\frac{d\phi}{dt} = \int_{t_0-t}^{t_0} \frac{\partial f^*}{\partial t}(\tau, t_0, t, \phi_0)d\tau + f(t_0 - t, \phi_0),$$

where $f^*(\tau, t_0, t, \phi_0) = f(\tau, \Phi_{(\tau-(t_0-t),t_0-t)}(\phi_0))$.

Expanding $\phi_{(h(t_n,t_n)}(x_n)$ as a Taylor series around t_n, we have

$$\phi_{(h(t_n),t_n)}(x_n) = x_n + h(t_n)\frac{d\phi}{dt}(t_n) + O(h(t_n)^2). \tag{7.12}$$

Combining (7.6) and (7.12), the **local truncation error**, $|\phi_{(h(t_n),t_n)}(x_n)-x_{n+1}|$ is then given by

$$|\phi_{(h(t_n),t_n)}(x_n) - x_{n+1}| = h(t_n)\left|\frac{d\phi}{dt}(t_n, x_n) - F(t_n, x_n, x_0, n))\right| + O(h(t_n)^2). \tag{7.13}$$

As already mentioned, the intention is to apply Theorem 6.2.1 in order to verify the existence of a discrete set that approximates $A(p)$ in $\mathcal{D}_l$. Application of Theorem 6.2.1 however, relies on the assumption that the local truncation error is at least $O(h(t_n)^2)$ and the truncation bound C_r is a function of t_n only. In general this is not true as the linear components of x_{n+1} and $\phi_{(h(t_n,t_n)}(x_n)$ do not equate. Additionally, the numerical algorithm is also dependent on n and x_0.

Nevertheless, the linear case is an exception to the general rule and the numerical dynamics obey a suitable local truncation error bound that allows application of Theorem 6.2.1. The local truncation error for the linear case is generated below, and its numerical approximation discussed later in the chapter.

Lemma 7.3.4 (LDS 2 - Truncation Error)**.**
Given the linear non-autonomous dynamical system

$$\dot{x} = f(\mathsf{t})x,$$

the local truncation error in $\mathcal{D}_l$ for a discrete pullback analysis with an r-th order Taylor series numerical method is of the form

$$|\phi_{(h(t_n),t_n)}(x_n) - x_{n+1}| \leq C_r(t_n)h(t_n)^{r+1}. \tag{7.14}$$

Proof: Lemma 7.3.2 verifies that the numerical method is invariant under the loci mapping and hence the numerical algorithm in $\mathcal{D}_l$ is equivalent to an r-th order Taylor series method applied directly to the loci dynamics. Since a Taylor series method generates a local truncation error of the order (7.14), the result follows.

□

What form then does the local truncation error take in the general case? A literal interpretation of (7.13) would imply that it is merely of $O(h(t_n))$. Nevertheless, the linear components of (7.13) are far from randomly chosen, and in fact, $F(t_n, x_n, x_0, n)$ should provide a rough approximation to $\frac{d\phi}{dt}(t_n, x_n)$. As a result, it is likely that the local truncation error is stronger than simply $O(h(t_n))$.

Keeping in mind that the numerical algorithm (7.6) possesses an explicit dependence on n it is reasonable to conclude that the local truncation error at any point must also reflect a dependence on the number of steps taken, or equivalently, on ρ, the variable time-step upper bound.

Although the actual formulation of the local truncation error will be different for each numerical method, the following analysis for separable dynamical systems highlights the relevance of ρ and provides an illustrative perspective on what form the local truncation error may take in general.

Lemma 7.3.5 (SDS 2 - Truncation Error). *Given the separable non-autonomous dynamical system*

$$\dot{x} = f(\mathsf{t})g(x),$$

the local truncation error in $\mathcal{D}_l$ for a discrete pullback analysis with an Euler method using a variable time-step sequence $\mathbf{h}$ *is of the form*

$$|\phi_{(h(t_n),t_n)}(x_n) - x_{n+1}| \leq C_r(t_n)h(t_n)\gamma(\rho). \tag{7.15}$$

where each time step satisfies $\rho/2 < h(t_i) < \rho$ for some $\rho > 0$, $C(t_n)$ is the local truncation function, and $\lim_{\rho\to 0}\gamma(\rho) \to 0$.

Proof: Let $P = \mathbb{R}$, and consider discrete pullback attraction to an arbitrarily chosen $t_0 \in \mathbb{R}$ using an Euler method with variable time step sequence $\mathbf{h} = \{h(t_0), h(t_1), \ldots h(t_n), \ldots\}$ where each $h(t_i)$ corresponds to the discrete step length between $t_0 - t_{i+1}$ and $t_0 - t_i$, and each $t_i = \sum_{j=1}^{i} h(t_{j-1})$. The variable time-step sequence is upper bounded by $\rho > 0$.

Two conditions are postulated here that are essential to the remainder of the proof.

A1 - Given any x_0 and $t^* > 0$, then any discrete chain linking $(x_0, t_0 - t^*)$ and (x_n^n, t_0) remains uniformly bounded (with respect to n) within the state space. This is easily verified. Note that any continuous solution for the problem is ultimately bounded (this can be shown by applying the Lemmas and Theorems on ultimate boundedness in [23, 41]. As a result, any discrete chain possessing a finite number of steps will obviously remain bounded. If this were otherwise, then the difference between the discrete chain and the continuous solution from x_0 would become larger than any accumulated numerical error which over the finite interval of length t^* is guaranteed to be bounded.

A2 - For any $t^* > 0$ and n-step discrete chain linking $(x_0, t_0 - t^*)$ and (x_n^n, t_0),

$$\lim_{n\to\infty} (x_{n+1}^{i+1} - x_n^i) \to 0,$$

holds for each $0 < i < n$.

The loci dynamics for a separable differential equation is of the form

$$\dot{\phi} = f(t_0 - t)g(\phi).$$

Given any (t_n, x_n) and expressing $\phi_{(h(t_n),t_n)}(x_n)$ as a Taylor series expansion around (t_n, x_n),

$$\phi_{(h(t_n),t_n)}(x_n) = x_n + h(t_n)f(t_0 - t_n)g(x_n) + O(h(t_n)^2). \quad (7.16)$$

Throughout the remainder of the proof it is essential to keep in mind that t_n is a fixed point on the time axis. This is made clear here as it may become confused as discrete chains with a varying number of steps will be considered. Each discrete chain however is the same length (fixed by t_n), and varying the number of steps affects only the step sizes.

Using the notation introduced in Lemma 7.3.3,

$$\begin{aligned} g(x_n^n) &= g(x_n^{n-1}) + (x_n^n - x_n^{n-1})G'(x_n^n, x_n^{n-1}), \\ &= g(x_n^{n-1})(1 + h(t_0)f(t_0 - t_1)G'(x_n^n, x_n^{n-1})), \\ &= g(x_n^{n-1})a_n. \end{aligned}$$

Repeating the same process through to $g(x_0)$ we have

$$g(x_n^n) = g(x_0) \prod_{j=1}^{n} a_j.$$

Substituting back in 7.16,

$$\phi_{(h(t_n),t_n)}(x_n) = x_n + h(t_n) \left(a_0 \prod_{j=1}^{n} a_j \right) + O(h(t_n)^2). \qquad (7.17)$$

Recall that the numerical algorithm (7.11) generating the discrete sequence $\{x_n\}$ for a separable dynamical system is of the form

$$x_{n+1} - x_n = h(t_n)(a_0 + e_0) \prod_{j=1}^{n} (a_j + e_j).$$

The local truncation error for the Euler method is then of the form

$$\left| x_{n+1} - \phi_{(h(t_n),t_n)}(x_n) \right| = h(t_n) \left| (a_0 + e_0) \prod_{j=1}^{n} (a_j + e_j) - a_0 \prod_{j=1}^{n} a_j \right| + O(h(t_n)^2).$$

Given the above expression the local truncation error is of at least order $h(t_n)$. We proceed to show that it is in fact of order $h(t_n)\gamma(\rho)$ where $\gamma(\rho)$ is some function such that

$$\lim_{\rho \to 0} \gamma(\rho) \to 0.$$

To see this, note that $n \to \infty$ as $\rho \to 0$ and consider the limit

$$\lim_{n \to \infty} \left| (a_0 + e_0) \prod_{j=1}^{n} (a_j + e_j) - a_0 \prod_{j=1}^{n} a_j \right|. \qquad (7.18)$$

Assuming **A1** holds so that any discrete chain lies within some bounded compact set B, define

$$\bar{g}' = \sup_{x \in B} g'(x).$$

Also let

$$\begin{aligned}
\bar{f} &= \sup_{0\le t\le t_n} f(t_0 - t),\\
\bar{a} &= 1 + \rho \bar{f}\bar{g}\,',\\
\Delta G' &= \max_{i=1,\ldots,n}\{G'(x^i_{n+1}, x^{i-1}_n) - G'(x^i_n, x^{i-1}_n)\},\\
\bar{e} &= \rho \bar{f}\Delta G'.
\end{aligned}$$

Note that $|a_i| \le \bar{a}$ and $|e_i| \le \bar{e}$ for each $1 \le i \le n$. Returning to (7.18),

$$\begin{aligned}
\lim_{n\to\infty} &\left|(a_0 + e_0)\prod_{j=1}^{n}(a_j + e_j) - a_0\prod_{j=1}^{n} a_j\right|,\\
&\le \lim_{n\to\infty}\left[(|a_0| + |e_0|)(\bar{a} + \bar{e}\,)^n - |a_0|(\bar{a})^n\right],\\
&\le \lim_{n\to\infty}\left[|a_0|\left(\sum_{j=1}^{n}\tbinom{n}{j}(\bar{a})^{n-j}(\bar{e}\,)^j\right) + |e_0|(\bar{a})^n\right]. \qquad (7.19)
\end{aligned}$$

To resolve the above limit, consider the latter term first. Note that

$$\begin{aligned}
(\bar{a})^n &= (1 + \rho\bar{f}\bar{g}\,')^n,\\
&\le (1 + 2t_n\bar{f}\bar{g}\,'/n)^n.
\end{aligned}$$

Taking the limit as $n \to \infty$ and applying L'Hopital's rule,

$$\lim_{n\to\infty}(\bar{a})^n \le \exp(2t_n\bar{f}\bar{g}\,').$$

Consequently,

$$\begin{aligned}
\lim_{n\to\infty}|e_0|(\bar{a})^n &= \lim_{n\to\infty} hF'(t_0 - t_{n+1}, t_0 - t_n)g(x_0)(\bar{a})^n,\\
&\le (0)\left(\exp(2t_n\bar{f}\bar{g}\,')\right),\\
&= 0, \qquad (7.20)
\end{aligned}$$

since $h \to 0$ as $n \to \infty$.

Taking the first term in (7.19),

$$\begin{aligned}
\lim_{n\to\infty} |a_0| \sum_{j=1}^{n} & \left(\binom{n}{j} (\bar{a})^{n-j} (\bar{e}\,)^j \right) \\
& \leq \lim_{n\to\infty} |a_0| \sum_{j=1}^{\infty} \left(\frac{n^j}{j!} (\bar{a})^n (\bar{e})^j \right), \\
& \leq \lim_{n\to\infty} |a_0| (\bar{a})^n \sum_{j=1}^{\infty} \left(\frac{(nh\bar{f}\Delta G')^j}{j!} \right), \\
& \leq \lim_{n\to\infty} |a_0| (\bar{a})^n \sum_{j=1}^{\infty} \left(\frac{(t_n\bar{f}\Delta G')^j}{j!} \right), \\
& \leq |a_0| \exp(2t_n\bar{f}\bar{g}\,') \lim_{n\to\infty} \left(\exp(t_n\bar{f}\Delta G') - 1 \right). \quad (7.21)
\end{aligned}$$

As $n \to \infty$, $h \to 0$, and for each $i = 1, \ldots, n$,

$$\begin{aligned}
\lim_{n\to\infty} \Delta G' &= \lim_{n\to\infty} [G'(x_{n+1}^i, x_n^{i-1}) - G'(x_n^i, x_n^{i-1})], \\
&= g'(x_n^{i-1}) - g'(x_n^{i-1}), \\
&= 0,
\end{aligned}$$

since $x_{n+1}^i \to x_n^{i-1}$ and $x_n^i \to x_n^{i-1}$ as $\rho \to 0$ (due to both **A2** and the nature of the Euler method). Consequently (7.21) $\to 0$ as $\rho \to 0$ (or equivalently $n \to \infty$). Substituting the evaluation of these limits back into (7.19) we find

$$\begin{aligned}
\lim_{n\to\infty} & \left| (a_0 + e_0) \prod_{j=1}^{n} (a_j + e_j) - a_0 \prod_{j=1}^{n} a_j \right| \\
& = \lim_{\rho\to 0} \left| (a_0 + e_0) \prod_{j=1}^{n} (a_j + e_j) - a_0 \prod_{j=1}^{n} a_j \right|, \\
& = 0.
\end{aligned}$$

From the above analysis, it may be concluded that the local truncation error for a separable dynamical system is bounded by the inequality

$$\left| x_{n+1} - \phi_{(h(t_n),t_n)}(x_n) \right| \leq C_r(t_n) h(t_n) \gamma(\rho),$$

where $\lim_{\rho\to 0}\gamma(h) = 0$ and $C(t_n)$ is some function determined by the bounds of f, g on the interval of consideration defined by t_n.

□

Remark 1: The local truncation error at t_n given by (7.15) is in fact applicable for any $t < t_n$. Thus for a discretisation over the finite interval $[0, t_n]$, (7.15) evaluated at t_n is applicable for each point in the discrete sequence on that finite interval.

Remark 2: It is strongly suspected that the local truncation error in $\mathcal{D}_l$ for the general case is always $O(h(t_n)\rho)$. However, verification of this, even for individual cases, is extremely difficult and as yet remains an open problem.

7.3.5 The Numerical Approximation

Here we determine the actual nature of any discrete attracting structures that characterise the pullback asymptotic stability present in the original dynamical system. Since the local truncation error in $\mathcal{D}_l$ for the linear case resolves itself simply, we shall first examine the nature of any discrete structures that arise in the loci dynamical system $\mathcal{D}_l$ under such conditions. We then proceed to the problem of discretisation for a broader class of dynamical systems that possess a local truncation error of a specified form.

Linear Non-Autonomous Dynamical Systems

Theorem 7.3.1 (LDS 3 - Numerical Approximation)**.** *Assume the origin, $A = \{0\}$, is a globally pullback equi-asymptotically stable set that is also loci stable for the linear non-autonomous dynamical system*

$$\dot{x} = f(\mathsf{t})x.$$

Then for each $t_0 \in \mathbb{R}$, a variable step discretisation (with bound $\rho > 0$ and restrictions on the step sizes) of the system with a Taylor series numerical method generates a discrete dynamical system on $\mathcal{D}_l$ which possesses a discrete *global forward equi-asymptotically stable family* $\hat{A}^{\mathbf{h}} = \{A(t_n, \psi_n \mathbf{h}); n \in \mathbb{Z}\}$ *such that*

$$H(A(t_n, \psi_n \mathbf{h}), A) \to 0 \qquad as \qquad \rho \to 0,$$

for each n.

Proof: Let $t_0 \in \mathbb{R}$ be chosen arbitrarily and consider the dynamics of the associated loci dynamical system $\mathcal{D}_l$.

By Lemma 7.3.2, the Taylor series numerical method is invariant under the loci mapping, and thus satisfies the local truncation error condition given in Lemma 7.3.4.

Consequently, the effects of the discretisation of the original system in $\mathcal{D}_l$ satisfy the assumptions for Theorem 6.2.1, which may be applied under restrictions on the variable time-step sequence to verify the existence of a discrete globally forward equi - asymptotically stable family in $\mathcal{D}_l$ that approximates A at t_0 with the required properties.

$\square$

Nonlinear Dynamical Systems

The numerical approximation for non-linear systems is significantly less trivial since the local truncation error is not of the form required by Theorem 6.2.1 (that is, not $O(h(t_n)^2)$). As a result, a modified approach is needed.

This is undertaken assuming only that the discretisation fulfills a local truncation error of the form

$$\left| x_{n+1} - \phi_{(h(t_n),t_n)}(x_n) \right| \leq C_r(t_n) h(t_n) \gamma(\rho), \tag{7.22}$$

where $\gamma(\rho) \to 0$ as $\rho \to 0$. Note that by Lemma 7.3.5, an Euler discretisation of a separable dynamical system automatically satisfies (7.22). The theorem itself however, is presented in a general context to allow for analysis of other examples which possess a local truncation error of the same form.

Preliminaries

Let $\hat{A}$ be a pullback equi-asymptotically stable family that is also loci stable for the non-autonomous dynamical system

$$\dot{x} = f(p, x),$$

and consider the loci dynamics generated by pullback attraction to some $p \in P$. By Theorem 7.2.1, $A(p)$ is a globally forward equi-asymptotically stable family in $\mathcal{D}_l$. As a consequence of Theorem 5.2.5 there exists a Lyapunov function $V = V(t, x)$ (with $t \geq 0$) that characterises the forward equi-asymptotic stability of $A(p)$ in $\mathcal{D}_l$.

Following closely the outline of the proof for Theorem 6.2.1, let $\delta^* > 0$ define a forward attracting neighbourhood in $\mathcal{D}_l$ (since attraction is global any predefined value satisfies the requirements for δ^*) and set $\delta_0 = \delta^*/3$.

Similarly, we also set

$$\begin{aligned}
L^*(t) &= \sup_{0 \leq \tau \leq t} l(\tau), \\
L(t) &= \max\{1, a(\delta_0)/\delta_0, L^*(t)\}, \\
C^*(t) &= \sup_{0 \leq \tau \leq t} C_r(\tau), \\
C(t) &= \max\{a(\delta_0), C^*(t)\},
\end{aligned}$$

where l is the Lipschitz function associated with V, and C_r is the local truncation error function. Note that these do not change the local truncation error or Lipschitzness of V except to accommodate increased variation.

The Discretisation

A numerical approximation is made by applying an Euler method with variable step sequence $\mathbf{h}$ to the original dynamical system, the bounds of which will be determined to ensure discrete forward attraction in $\mathcal{D}_l$.

The key difference between the proof for Theorem 6.2.1 and the proof detailed here is in the restrictions made to ensure that discrete attraction occurs. In

Theorem 6.2.1 this is achieved by guaranteeing at each step that $h(t_n)$ remains small enough. Due to the form of the local truncation error (7.22) here, it requires restricting the step sizes used throughout the entire sequence over the interval considered.

With this in mind, we approach the problem by fixing a time t, and determining the required bound $\rho(t)$ on the variable step sizes to ensure discrete attraction occurs over the finite interval $[0, t]$. As t is increased, $\rho(t)$ is adjusted and the discrete sequence reconstructed. Finally, we consider behaviour of the discrete sequences in the limit as $t \to \infty$. This is formalised as follows.

Let λ be some constant such that $0 < \lambda < 1$ and define $\rho_1(t)$ as the largest value satisfying

$$4L(t)C(t)\rho_1(t)\gamma(\rho_1(t))/(1 - e^{-c\rho_1(t)}) \leq \lambda^2 a(\delta_0), \tag{7.23}$$

for each $t \geq 0$. Note that due to the definition of $L(t)$ and $C(t)$, $\rho(t)$ is a monotonically decreasing function in t. We also have

$$\begin{aligned} \rho_1(t)^2 &\leq (1 - e^{-c\rho_1(t)})\lambda^2 a(\delta_0)/4L(t)C(t), \\ &\leq \lambda^2 a(\delta_0)/4a(\delta_0), \\ &\leq \lambda^2, \end{aligned}$$

and consequently $\rho_1(t)$ is bounded above by λ for all $t \geq 0$. The variable time step sequence over the interval $[0, t]$ is subsequently defined so that

$$\rho_1(t)/2 \leq h(t_n) \leq \rho_1(t) \qquad\qquad \forall t_n \leq t. \tag{7.24}$$

Hence the local truncation error for any single step x_n to x_{n+1} on this interval may be expressed in the form

$$\begin{aligned} |x_{n+1} - \phi_{(h(t_n),t_n)}(x_n)| &\leq C(t)h(t_n)\gamma(\rho_1(t)), \\ &\leq C(t)\rho_1(t)\gamma(\rho_1(t)). \end{aligned} \tag{7.25}$$

The Attracting Neighbourhood - $\hat{B}$

Definition 7.3.2 (B1 - Continuous Attracting Neighbourhood)**.** *Define* $\hat{B} = \{B(t); t \geq 0\}$ *in* $\mathcal{D}_l$ *by*

$$B(t) = \{x : x \in \mathcal{N}_{\delta^*}(A(p)), V(t,x) \leq a(\delta_0)\}.$$

Lemma 7.3.6 (B2 - Positive Invariance)**.** *If the variable step sequence* **h** *satisfies the bounds determined by (7.24) over the finite interval* $[0,t]$*, then* $\hat{B}$ *is a* **positively invariant** *family under the discretisation over that interval.*

Proof: Consider any point in the discrete sequence on the interval $[0,t]$ with $x_n \in B(t_n)$, $t_n \leq t$ and $\rho_1(t)/2 \leq h(t_n) \leq \rho_1(t)$. Then

$$\begin{aligned} V(t_{n+1}, x_{n+1}) &\leq V(t_n, \phi_{(h(t_n),t_n)}(x_n)) + L(t)|x_{n+1} - \phi_{(h(t_n),t_n)}(x_n)|, \\ &\leq e^{-ch(t_n)} V(t_n, x_n) + L(t)C(t)\rho_1(t)\gamma(\rho_1(t)), \\ &\leq e^{-ch(t_n)} a(\delta_0) + \frac{1}{4}(1 - e^{-c\rho_1(t)}) a(\delta_0), \\ &\leq a(\delta_0). \end{aligned}$$

Consequently, $\hat{B}$ is positively invariant over the finite interval $[0,t]$ under the discretisation as constructed above.

□

The following definition creates a discrete family that is positively invariant under the discretisation, the proof for which follows immediately since it is derived directly from $\hat{B}$.

Lemma 7.3.7 (B3 - Discrete Attracting Neighbourhood)**.** *Define the discrete attracting family* $\hat{B}^{\mathbf{h}} = \{B^{\mathbf{h}}(t_n, \psi_n \mathbf{h}); n \in \mathbb{Z}^+\}$ *by*

$$B^{\mathbf{h}}(t_n, \psi_n \mathbf{h}) = B(t_n), \tag{7.26}$$

for all $n \in \mathbb{Z}^+$*. If the variable step sequence* **h** *satisfies the bounds determined by (7.24) over the finite interval* $[0,t]$*, then* $\hat{B}^{\mathbf{h}}$ *is* ***positively invariant*** *under the discretisation over that interval.*

Discrete Asymptotic Structure $\hat{A}^{\mathbf{h}}$

Definition 7.3.3 (A1)**.** *Define* $\hat{A}^{\mathbf{h}} = \{A^{\mathbf{h}}(t_n, \psi_n \mathbf{h}); n \in \mathbb{Z}^+\}$ *by*

$$A^{\mathbf{h}}(t_n, \psi_n \mathbf{h}) = \{x; V(t_n, x) \leq \frac{1}{2}\lambda^2 a(\delta_0)\}.$$

We propose $\hat{A}^{\mathbf{h}}$ as a discrete structure that approximates $A(p)$ in $\mathcal{D}_l$.

Lemma 7.3.8 (A2 - Boundedness and Compactness)**.** $\hat{A}^{\mathbf{h}}$ *is uniformly bounded and each element is compact.*

Proof: The discrete family $\hat{A}^{\mathbf{h}}$ is uniformly bounded since $A(p)$ is bounded and

$$A^{\mathbf{h}}(t_n, \psi_n \mathbf{h}) \subset B(t_n) \subset \mathcal{N}_{\delta_0}(A(p)),$$

for each n. Compactness follows from the continuity of V.

□

Lemma 7.3.9 (A3 - Positive Invariance)**.** *If the variable step sequence* $\mathbf{h}$ *satisfies the bounds determined by (7.24) on the finite interval* $[0, t]$*, then* $\hat{A}^{\mathbf{h}}$ *is a positively invariant family under the discretisation on that interval.*

Proof: Consider any point in the discrete sequence on the interval $[0, t]$ with $x_n \in A^{\mathbf{h}}(t_n, \psi_n \mathbf{h})$ with $t_n \leq t$ and $\rho_1(t)/2 \leq h(t_n) \leq \rho_1(t)$. Then

$$\begin{aligned} V(t_{n+1}, x_{n+1}) &\leq V(t_{n+1}, \phi_{(h(t_n), t_n)}(x_n)) + L(t)C(t)\rho_1(t)\gamma(\rho_1(t)), \\ &\leq e^{-ch(t_n)} V(t_n, x_n) + L(t)C(t)\rho_1(t)\gamma(\rho_1(t)), \\ &\leq e^{-ch(t_n)} \frac{1}{2}\lambda^2 a(\delta_0) + \frac{1}{4}\lambda^2 a(\delta_0)(1 - e^{-ch(t_n)}), \\ &\leq \frac{1}{2}\lambda^2 a(\delta_0). \end{aligned}$$

Hence $x_{n+1} \in A^{\mathbf{h}}(t_{n+1}, \psi_{n+1} \mathbf{h})$, and $\hat{A}^{\mathbf{h}}$ is positively invariant over the defined interval under the discretisation specified.

□

The following lemma verifies that $\hat{A}^{\mathbf{h}}$ forward attracts $\hat{B}^{\mathbf{h}}$ on the interval $[0,t]$ if the bound on the step sizes is made correspondingly small enough.

For this we define $\rho(t)$ so that it satisfies

$$\rho(t) = \min\{\rho_1(t), \rho_2\}, \tag{7.27}$$

where ρ_2 satisfies the equation $(1+e^{-c\rho_2}) = 2e^{-c\rho_2/4}$. The variable step sequence $\mathbf{h}$ on $[0,t]$ is then defined so that

$$\rho(t)/2 \leq h(t_n) \leq \rho(t), \qquad \forall t_n \leq t. \tag{7.28}$$

Lemma 7.3.10 (A4 - Forward Absorbing)**.** *If the variable step sequence* $\mathbf{h}$ *satisfies the bounds determined by (7.28) over the interval* $[0,t]$*, then in the limit as* $t \to \infty$*,* $\hat{A}^{\mathbf{h}}$ *forward absorbs* $\hat{B}^{\mathbf{h}}$ *in finite time.*

Proof: Let $t > 0$ and consider any $x_0 \in B^{\mathbf{h}}(0,\mathbf{h}) \backslash A^{\mathbf{h}}(0,\mathbf{h})$. Then

$$\frac{1}{2}\lambda^2 a(\delta_0) < V(0,x_0),$$

and we have

$$\begin{aligned} V(t_1,x_1) &\leq e^{-ch(t_0)}V(0,x_0) + \frac{1}{4}\rho(t)\gamma(\rho(t))a(\delta_0)(1-e^{-ch(t_0)}), \\ &\leq \frac{1}{2}(1+e^{-ch(t_0)})V(0,x_0). \end{aligned}$$

Now if $h(t_0)$ satisfies (7.28), then

$$(1+e^{-ch(t_0)}) \leq 2e^{-ch(t_0)/4},$$

and consequently,

$$\begin{aligned} V(t_1,x_1) &\leq e^{-ch(t_0)/4}V(0,x_0), \\ &\leq e^{-ct_1/4}V(0,x_0), \end{aligned}$$

If we extrapolate the argument presented above for the remainder of the discrete sequence on $[0,t]$ there are two possible cases that arise.

i) The discrete sequence is absorbed into $\hat{A}^{\mathbf{h}}$ at some point on the interval $[0,t]$. That is, there exists a j such that $x_j \in B^{\mathbf{h}}(t_j, \psi_j \mathbf{h}) \setminus A^{\mathbf{h}}(t_j, \psi_j \mathbf{h})$ and $x_{j+1} \in A^{\mathbf{h}}(t_{j+1}, \psi_{j+1}\mathbf{h})$ with $t_{j+1} \leq t$. Since $\hat{A}^{\mathbf{h}}$ is positively invariant under the discretisation on $[0,t]$, then $x_i \in A^{\mathbf{h}}(t_i, \psi_i \mathbf{h})$ for all $i > j$.

ii) The discrete sequence never enters $\hat{A}^{\mathbf{h}}$ in which case the Lyapunov value of the final element in the discrete sequence on $[0,t]$, denoted by x_n with $t_n \leq t$ and $t_{n+1} > t$, is bounded by

$$\begin{aligned} V(t_n, x_n) &\leq e^{-ct_n/4} V(0, x_0), \\ &\leq e^{-c(t-\rho(t))/4} a(\delta_0), \\ &\leq e^{c/4} e^{-ct/4} a(\delta_0). \end{aligned}$$

Note that if we only consider the problem of discretisation on finite intervals $[0,t]$ for values of t such that

$$t \geq \frac{4}{c} \ln(2/\lambda^2) + 1, \tag{7.29}$$

then the discrete sequence must at some point be absorbed by $\hat{A}^{\mathbf{h}}$, and case i) always occurs for any $x_0 \in B^{\mathbf{h}}(0, \mathbf{h})$. To see this, assume otherwise. That is, it is not absorbed by $\hat{A}^{\mathbf{h}}$ for any $t_n < t$ (case ii)). Then

$$\begin{aligned} V(t_n, x_n) &\leq e^{c/4} e^{-ct/4} a(\delta_0), \\ &\leq \frac{1}{2} \lambda^2 a(\delta_0), \end{aligned}$$

where x_n is defined as previously as the last element in the discrete sequence on $[0,t]$. Consequently $x_n \in A^{\mathbf{h}}(t_n, \psi_n \mathbf{h})$ which contradicts ii). Hence i) must occur if t satisfies (7.29).

Thus we may conclude that in the limit as $t \to \infty$, $\hat{A}^{\mathbf{h}}$ forward absorbs $B^{\mathbf{h}}(0, \mathbf{h})$ in finite time. That is, for all j such that $t_j \geq \frac{4}{c}\ln(2/\lambda^2) + 1$, then

$$\phi_{(j,(0,\mathbf{h}))}(B^{\mathbf{h}}(0, \mathbf{h})) \subseteq A^{\mathbf{h}}(t_j, \psi_j \mathbf{h}).$$

□

Due to the fact that the discretisation is only considered on finite intervals it is not possible to define $\hat{A}^{\mathbf{h}}$ as a discrete forward equi-asymptotically family. Nevertheless, it is still possible to obtain results regarding the attractive properties of the discrete sequences on these intervals. This is desirable as it translates directly to an understanding of the discrete pullback analysis of solutions within a neighbourhood of $A(p)$ in the original dynamical system.

Lemma 7.3.11 (A5 - δ-Neighbourhood of Forward Attraction)**.** *If the variable step sequence* $\mathbf{h}$ *satisfies the bounds determined by (7.28) over the interval* $[0, t]$*, then in the limit as* $t \to \infty$*,* $\hat{A}^{\mathbf{h}}$ *forward absorbs* $\mathcal{N}_\delta(A^{\mathbf{h}}(0, \mathbf{h}))$ *for some* $\delta > 0$*.*

Proof: Since V is continuous and elements of $\hat{B}^{\mathbf{h}}$ and $\hat{A}^{\mathbf{h}}$ are bounded in the state space by the level Lyapunov curves $V_B = a(\delta_0)$ and $V_A = \frac{1}{2}\lambda^2 a(\delta_0)$, it must be that V_B and V_A are separated by some minimum distance $\delta > 0$. This δ then forms an appropriate local δ-neighbourhood that is forward absorbed by $\hat{A}^{\mathbf{h}}$ in finite time if the appropriate conditions on the variable time-step sequence are met.

□

Lemma 7.3.12 (A6 - Boundedness of the Discrete Sequence)**.** *For any* $x_0 \in \mathcal{N}_\delta(A^{\mathbf{h}}(0, \mathbf{h}))$*, the resulting discrete sequence on* $[0, t]$ *is bounded so that*

$$x_j \in \mathcal{N}_{\delta^*}(A(t_j, \psi_j \mathbf{h})),$$

for all j *such that* $t_j < t$*.*

Proof: $\hat{B}$ is positively invariant under the discretisation on $[0, t]$, hence each $x_j \in B(t_j)$ since $x_0 \in B(0)$. Also note that

$$B(t_j) \subset \mathcal{N}_{\delta^*}(A(p)) \subset \mathcal{N}_{\delta^*}(A^{\mathbf{h}}(t_j, \psi_j \mathbf{h})),$$

and the desired result follows.

□

Finally, we may verify that the numerical approximation converges upper semi-continuously to $A(p)$ by decreasing the bound on each $\rho(t)$. This is achieved by letting $\lambda \to 0$ since λ provides a suitable bound for each $\rho(t)$.

Lemma 7.3.13 (C1 - Upper Semi-Continuity).
$\hat{A}^{\mathbf{h}}$ is upper semi-continuous with respect to $A(p)$ and the bound λ.

Proof: Note that $A(p)$ is contained within every element of $\hat{A}^{\mathbf{h}}$ since V is continuous and $\frac{1}{2}\lambda^2 a(\delta_0) > 0$. Then for any variable time step sequence $\mathbf{h}$ upper bounded by $\rho(t)$, and any $x_j \in A^{\mathbf{h}}(t_j, \psi_j \mathbf{h})$,

$$\begin{aligned} \mathrm{dist}(x_j, A(p)) &\leq a^{-1}(V(t_j, x_j)), \\ &\leq a^{-1}(\frac{1}{2}\lambda^2 a(\delta_0)). \end{aligned}$$

Since $a^{-1}(\frac{1}{2}\lambda^2 a(\delta_0)) \to 0$ as $\lambda \to 0$, the required result follows.

□

Combining the results of Lemmas and Definitions B1-B3, A1-A6, and C1 we obtain a meaningful analysis of the effects of discretisation on the original dynamical system.

Theorem 7.3.2 (Main Result). *Let $\hat{A}$ be a pullback equi-asymptotically stable family that is also loci stable for the dynamical system*

$$\dot{x} = f(p, x).$$

For any $p \in P$, if a discrete pullback analysis is made over a finite interval $[0, t]$ using a numerical method possessing a local truncation error in $\mathcal{D}_l$ of the form (7.22) using suitable restrictions on the variable time-step sequence as required in Lemmas B1-B6, and A1-A6, then there exists a discrete family $\hat{A}^{\mathbf{h}} = \{A^{\mathbf{h}}(t_n, \psi_n \mathbf{h}); n \in \mathbb{Z}^+\}$ in the loci dynamical system that approximates $A(p)$ and possesses the following properties:

i) *$\exists\delta > 0$, $T > 0$, such that on the finite interval $[0, t]$ in $\mathcal{D}_l$, $\hat{A}^{\mathbf{h}}$ forward absorbs all solutions originating from within $\mathcal{N}_\delta(A^{\mathbf{h}}(0, \mathbf{h}))$ within finite time.*

ii) *The discrete trajectories in $\mathcal{D}_l$ originating from within $\mathcal{N}_\delta(A^{\mathbf{h}}(0, \mathbf{h}))$ are bounded.*

iii) *If λ is the uniform (on any finite interval of analysis) step size upper bound for the variable time step sequences, then for any appropriately defined variable step sequence $\mathbf{h}$,*

$$\lim_{\lambda \to 0} H^*(A^{\mathbf{h}}(t_n, \psi_n \mathbf{h}), A(p)) = 0.$$

The properties i) - iii) translate to meaningful properties in the original dynamical system.

i) This property ensures that any discrete sequence originating within a local neighbourhood of $\hat{A}$ converges to a structure that approximates $A(p)$ under the appropriate conditions on the step sizes.

ii) The δ-neighbourhood constructed here can be utilised to generate a neighbourhood of $\hat{A}$ that is pullback stable with respect to $A(p)$ under the discretisation.

iii) This ensures convergence of the attracting structure created by numerical method to the continuous family $\hat{A}$.

7.3.6 Conclusion

In summary it may be concluded that the effects of numerical approximation on systems possessing properties of pullback asymptotic stability are largely dependent on the nature of the actual dynamical system and the numerical method used. As a result, a case by case analysis is needed, due primarily to the resulting nature of the local truncation error in $\mathcal{D}_l$. Investigations were made here specifically for linear and separable dynamical systems, but of more importance, the initial theory for loci dynamics and a clear interpretation of the local truncation error is laid down to provide a basis for which further analysis of individual cases may be undertaken.

Conclusions

The nature of stability and numerical approximation over non-finite intervals for non-autonomous dynamical systems has been a topic left relatively unexplored until only recently.

Research primarily by P. Kloeden and B.Schmalfuss initiated an investigation into attractive structures within a non-autonomous context, and this has been an initiative taken up by several other authors since (notably D.Cheban, P.Flandoli and V.Kozyakin among others). The results generated however, primarily reflect the use of pullback attractors in their field of interest, and a comprehensive analysis of non-autonomous stability was still incomplete.

The initial chapters of this thesis were written to provide as comprehensively as possible, the fundamentals of non-autonomous stability. In scope, it introduces several new concepts, but also incorporates the work on pullback attractors by P.Kloeden et. al, as well as retaining existing classical asymptotic stability theory as an integral component. It also introduces a preliminary Lyapunov theory for pullback stability, although its usefulness as a tool may well be restricted to the Theorems of the converse nature.

A comprehensive stability theory for non-autonomous dynamical systems as composed here should provide the essential basis for which further research in control, chaos theory and numerical approximation of non-autonomous dynamical systems will benefit.

The latter half of the thesis devotes its attention to the numerical approximation of non-autonomous dynamical systems over non-finite intervals, a topic initially explored by A. Stuart for autonomous dynamical systems with relevance to understanding computer models that approximated real time dynamics. Obtaining equivalent results for non-autonomous dynamical systems however is increasingly difficult as one diverges from properties of uniformity. Results for uniform and non-uniform attraction were found for forward asymptotic theory, however the case for pullback asymptotic approximations are shown to be much more complicated. Developing the loci theory as an aid for understanding pullback asymptotic behaviour proved to be useful in determing numerically approximated behaviour of such systems on a case by case basis. As such, this leaves it as a basis for application to further problems.

Bibliography

[1] Tom Apostol, *Mathematical Analysis* Addison-Wesley Publishing Company, 1974.

[2] L.Arnold, *Random Dynamical Systems*, Springer-Verlag, New York Inc., 1998.

[3] D.Arrowsmith & C.Place, *An introduction to Dynamical Systems*, Cambridge University Press, Cambridge, 1990.

[4] N.P. Bhatia & G.P. Szegö, *Dynamical Systems: Stability theory and application*, Lecture Notes in Mathematics, No.37, Springer, Berlin, 1967.

[5] Julyan Cartwright & Oreste Piro, *The Dynamics of Runge-Kutta Methods*, School of Mathematical Sciences, University of London, 'http://lec.ugr.es/ julyan/papers/rkpaper/root.html'.

[6] D.Cheban, P.Kloeden & B.Schmalfuss, *Pullback Attractors in Dissipative Nonautonomous Differential Equations under Discretization*, In Proc. of EQUADIFF 99, 1024-1029, Berllin, World Scientific, Singapore, 2000.

[7] D.Cheban, P.Kloeden & B.Schmalfuss, *The Relationship Between Pullback, Forwards and Global Attractors of Nonautonomous Dynamical Systems*, Nonlinear Dynamics and Systems Theory, 2000.

[8] D.Cheban, Peter Kloeden and B. Schmalfuss, *Pullback Attractors in Dissipative Nonautonomous Differential Equations Under Discretization* , J. Dynamical Systems and Differential Equations 13, 185-213, 2001.

[9] H. Crauel, F. Flandoli & A. Debussche, *Random Attractors*, Journal of Dynamics and Differential Equations, Vol. 9, No. 2, 1997.

[10] H. Crauel & F. Flandoli, *Attractors for Random Dynamical Systems*, Probability Theory and Related Fields 100, 1095-1113, 1994.

[11] K.Falconer, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley and Sons, Chichester, 1990.

[12] F. Flandoli & B.Schmalfuss, *Random Attractors of the 3D Navier Stokes Equation with Multiplicative White Noise*, Stochastics and Stochastics Reports 59, 21-45, 1996.

[13] Fang Shuhong, *Global Attractor for General Nonautonomous Dynamical Systems*, Nonlinear World. 2, 191-216, 1995.

[14] L.Johnson & D.Riess, *Numerical Analysis*, Addison-Wesley Publishing Company, Philippines, 1982.

[15] Peter Kloeden, *Construction of a Lyapunov Function for Cocycle Attractors in Non-Autonomous Difference Equations*, Electronic Journal of Differential Equations, Conference 05, 91-102, 2000.

[16] P.Kloeden & V.Kozyakin, *The Inflation of Attractors and their Discretization: The Autonomous Case*, Nonlinear Analysis, 333-343, TMA 40, 2000.

[17] P.Kloeden & V.Kozyakin, *The Inflation of Nonautonomous Systems and their Pullback Attractors*, Izvestiya of hte Russian Academy of Natural Sciences, Series MMMIU, 4, No. 1-2, 2000.

[18] P.Kloeden & V.Kozyakin, *Uniform Nonautonomous Attractors under Discretization*, DANSE-Preprint 36/2000, Discrete Continuous Dynamical Systems (to appear).

[19] Peter Kloeden & J. Lorenz, *Stable Attracting Sets in Dynamical Systems and their One-Step Discretisations*, SIAM J. Numer. Anal. 23, 986-995, 1986.

[20] Peter Kloeden & B. Schmalfuss, *Nonautonomous Systems, Cocycle Attractors and Varable Time-Step Discretization*, Annals of Numerical Mathematics.

[21] Peter Kloeden & B. Schmalfuss, *Lyapunov Functions and Attractors Under Variable Time-Step Discretisation*, Discrete and Continuous Dynamical Systems 2, 163-172, 1996.

[22] Peter Kloeden & Daniel Stonier, *Cocycle Attractors in Nonautonomous Perturbed Differential Equations*, Dyanmics of Discrete, Continuous and Impulsive Systems, **4**, 211-226, 1998.

[23] Joseph La Salle & Solomon Lefschetz, *Stability by Liapunov's Direct Method*, Academic Press, New York, 1967.

[24] R.Moore, *Introduction to Differential Equations*, Allyn and Bacon, Boston, 1962.

[25] Lawrence Perko, *Differential Equations and Dynamical Systems*, Springer-Verlag, New York Inc., 1991.

[26] Albert Reinstein, *Introduction to Ordinary Differential Equations*, Academic Press, 1972.

[27] N.Rouche, P.Habets & M.Laloy, *Stability Theory by Liapunov's Direct Method*, Springer-Verlag, New York, 1977.

[28] B.Schmalfuss, *The Stochastic Attractor of the Stochastic Lorenz System*, Nonlinear Dynamics: Attractor Approximation and Global Behaviour, Proc. ISAM 92, TU Dresden, 185-192, 1992.

[29] B.Schmalfuss, *Attractors for the Nonautonomous Dynamical Systems*, In Proc. of EQUADIFF 99, Berlin, World Scientific, Singapore, 1024-1029, 2000.

[30] G.R.Sell, *Nonautonomous Differential Equations and Topological Dynamics I, II*, Trans. Amer. Math. Soc. 127, 241-283, 1967.

[31] G.R.Sell, *Lectures on Topological Dynamics and Differential Equations*, Van Nostrand-Reinbold, London, 1971.

[32] J.Skowronski, *Control of Nonlinear Mechanical Systems*, Plenum Press, New York, 1991.

[33] H.J.Stetter, *Analysis of Discretisation Methods for Ordinary Differential Equations*, Springer-Verlag, Heidelberg, 1973.

[34] Andrew Stuart, *Analysis of Numerical Dynamical Systems*, Acta Numerica 2, 1994.

[35] Andrew Stuart & A.R. Humphries, *Numerical Analysis and Dynamical Systems*, Cambridge University Press, Cambridge, 1996.

[36] Daniel Stonier, *Pullback Stability and Attraction for Non-Autonomus Dynamical Systems*, Journal of Nonlinear Science(submitted).

[37] Daniel Stonier, *Discretisation of Asymptotically Stable Non-Autonomous Dynamical Systems*, Journal of Nonlinear Science(submitted).

[38] Daniel Stonier, *Duality of Attraction*, Journal of Nonlinear Science(submitted).

[39] R. Teman, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, Springer-Verlag, Heidelberg, 1988.

[40] M.I. Vishik, *Asymptotic Behaviour of Solutions of Evolution Equations*, Cambridge University Press, Cambridge, 1992.

[41] T.Yoshizawa, *Liapunov's Function and Boundedness of Solutions*, Funkcialaj Ekvacioj, 2, 1959.

[42] T.Yoshizawa, *Stability Theory By Liapunov's Second Method*, Mathematical Society of Japan, 1966.