THE PROFINITE SHAPE EQUIVALENCE

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ABSTRACT. We show that, for every ∞-topos $\mathcal{X}$, the profinite shape induces a canonical equivalence between the ∞-category of pro-objects in locally constant constructible sheaves of anima in $\mathcal{X}$ and the slice ∞-category of profinite anima over the profinite shape $\widehat{\Pi}_\infty(\mathcal{X})$. We deduce this profinite shape equivalence from the monodromy classification theorem by establishing a continuous version of the straightening–unstraightening equivalence for profinite anima. In addition, we explain how the monodromy classification theorem recovers its classical counterparts concerning locally constant sheaves of finite sets and of (abelian) groups.

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0. INTRODUCTION

Profinite shape theory attaches to an ∞-topos $\mathcal{X}$ a profinite anima $\widehat{\Pi}_\infty(\mathcal{X})$ distilling all of the cohomological information of locally constant constructible sheaves in $\mathcal{X}$ into a homotopy type. For example, for a scheme S, the profinite shape of its étale ∞-topos $S_{\text{ét}}$ is the profinite *étale homotopy type* of S: a invariant in arithmetic geometry that has been used by, e.g., Schmidt and Stix in [SS16] to resolve one of the main conjectures of Grothendieck's anabelian programme.

Explicitly computing and controlling $\widehat{\Pi}_\infty(\mathcal{X})$ is generally very difficult. One reason for this is that many techniques from classical homotopy theory are not available — for instance, explicitly manipulating paths. Instead, a central tool is the *monodromy classification theorem*, which asserts a canonical equivalence

$$\mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Ani}_\pi) \simeq \mathrm{lcc}(\mathcal{X})$$

between the ∞-categories of continuous local systems of π-finite anima on $\widehat{\Pi}_\infty(\mathcal{X})$ and locally constant constructible sheaves in $\mathcal{X}$ (see Theorem 4.7 for a more precise formulation and [BH20, (10.1)] as well as [BGH20, (4.4.18)] for proofs). This equivalence often allows to translate problems concerning $\widehat{\Pi}_\infty(\mathcal{X})$ into sheaf-theory.

The purpose of this note is to prove the following consequence of the monodromy classification theorem:

Theorem A (profinite shape equivalence)**.** *Let $\mathcal{X}$ be an ∞-topos. The profinite shape functor induces an equivalence of ∞-categories*

$$\widehat{\Pi}_\infty : \ \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \to \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\,\widehat{\Pi}_\infty(\mathcal{X})}.$$

See Theorem 4.10 for a more precise formulation that also spells out an explicit inverse as well as its functoriality under geometric morphisms, and Theorem 4.15 for a variant for the Σ-completed shape functor for some full class of finite groups Σ.

Theorem A is a formal consequence of the monodromy classification theorem combined with our second main result, a continuous analogue of the straightening–unstraightening equivalence for profinite anima:

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Theorem B (continuous (un-)straightening)**.** *Let B be a profinite anima. There is a canonical adjoint equivalence of ∞-categories*

$$\mathrm{Pro}(\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_\pi)) \underset{\partial_B}{\overset{\int^B}{\rightleftarrows}} \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}.$$

Again, we give a more general pro-Σ variant and also describe the functors $\int^B \dashv \partial_B$ in Theorem 3.7.

To deduce Theorem A from Theorem B, one needs to unpack the monodromy equivalence explicitly. Since in the existing literature, the functors furnishing the monodromy equivalence are left mostly implicit, we take this opportunity to explicate them. To round things up, we also discuss how the monodromy classification theorem relates to more classical notions of local systems of finite sets and (abelian) groups considered in e.g. [SGA1].

Theorem A constitutes the main formal input in forthcoming joint work with Schmidt and Stix generalising Friedlander's fibre sequence in étale homotopy theory of [Fri73].

Linear overview. In §1, we recall some background on profinite anima and filtered colimits of ∞-categories; the familiar reader can safely skip this section. Next we study continuous local systems on profinite anima in §2. We then prove Theorem B, the continuous analogue of the straightening-unstraightening equivalence for profinite anima, in §3. Finally, in §4, we deduce the profinite shape equivalence, Theorem A, and discuss some of its consequences.

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1. Background

In this section, we will collect some background on profinite anima and filtered colimits of ∞-categories.

1.1. Profinite anima.

Notation 1.1 (pro-categories)**.** Given an ∞-category $\mathscr{C}$, we write $\mathrm{Pro}(\mathscr{C})$ for the ∞-category of pro-objects in $\mathscr{C}$ obtained by formally adjoining cofiltered limits to $\mathscr{C}$. The existence of $\mathrm{Pro}(\mathscr{C})$ is a special case of (the dual of) [HTT, 5.3.6.2]. Given a functor $F : \mathscr{C} \to \mathscr{D}$, we simply write $F : \mathrm{Pro}(\mathscr{C}) \to \mathrm{Pro}(\mathscr{D})$ for the cofiltered-limit-preserving extension of F.

Recall that a *full class of finite groups* is a collection of finite groups that is closed under taking subgroups, homomorphic images, and extensions.

Definition 1.2 ((pro-)finite anima)**.** Let B be an anima and Σ a nonempty full class of finite groups.

(1) The anima B is said to be Σ-*finite* if it is truncated, has finitely many connected components, and if for all points $b \in B$ and all $i \geq 1$ the homotopy group $\pi_i(B, b)$ is in Σ.
(2) We write $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}$ for the full subcategory spanned by Σ-finite anima.
(3) We call $\mathrm{Pro}(\mathbf{Ani}_\Sigma)$ the ∞-category of *pro-Σ anima*.
(4) We write π for the class consisting of all finite groups and call $\mathrm{Pro}(\mathbf{Ani}_\pi)$ the ∞-category of *profinite anima*.

Recollection 1.3 (profinite & pro-Σ completion)**.** Let Σ denote a nonempty full class of finite groups.

(1) The canonical inclusions $\mathrm{Pro}(\mathbf{Ani}_\Sigma) \subset \mathrm{Pro}(\mathbf{Ani}_\pi) \subset \mathrm{Pro}(\mathbf{Ani})$ admit left adjoints

$$\mathrm{Pro}(\mathbf{Ani}) \xrightarrow{(-)^\wedge_\pi} \mathrm{Pro}(\mathbf{Ani}_\pi) \xrightarrow{(-)^\wedge_\Sigma} \mathrm{Pro}(\mathbf{Ani}_\Sigma)$$

called *profinite completion* and *pro-Σ completion*, respectively.

(2) The localisations $(-)^\wedge_\pi$ and $(-)^\wedge_\Sigma$ preserve cofiltered limits and are locally cartesian, see [HHW24, Prop. 3.18]. The latter means that, for every map $K \to B$ of pro-Σ anima and every map $E \to B$ of pro-anima, the natural map

$$\big(K \times_B E\big)^\wedge_\Sigma \xrightarrow{\sim} K \times_B E^\wedge_\Sigma$$

is an equivalence (see [HHW24, §3; GK17, §1.2; Hoy17b, §3.2]).

An important feature of profinite anima is that equivalences can be detected on homotopy groups.

Theorem 1.4 (profinite Whitehead theorem, [SAG, E.3.1.6])**.** *The following are equivalent for a map $f : E \to B$ of profinite anima.*

(1) The map f is an equivalence of profinite anima.
(2) The induced map on connected components $f_* :\ \pi_0 E \to \pi_0 B$ is bijective and for every point $e \in E$ with image $b = f(e) \in B$ and every $k \geq 1$ the induced homomorphism

$$f_* :\ \pi_k(E, e) \to \pi_k(B, b)$$

is an isomorphism of profinite groups.

As a consequence, we have the following useful criterion for detecting pullback squares of profinite anima.

Corollary 1.5 (fibrewise criterion of pullbacks)**.** *The following are equivalent for a commutative square*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ \downarrow & & \downarrow \\ B' & \longrightarrow & B \end{array}$$

of anima or of profinite anima, respectively:

(1) The square is cartesian.
(2) For every point $p \in E \times_B B'$, the fibre $\mathrm{fib}_p(E' \to E \times_B B')$ is contractible.
(3) For every point $b' \in B'$ with image $b \in B$, the induced map

$$\mathrm{fib}_{b'}(E' \to B') \longrightarrow \mathrm{fib}_b(E \to B)$$

is an equivalence.

Proof. If (2) holds, then the homotopy sequence for $E' \to E \times_B B'$ and every point $e' \in E'$ with image p in $E \times_B B'$ yields isomorphisms $\pi_i(E', e') \to \pi_i(E \times_B B', p)$ and all $i \in \mathbb{N}$, so that (1) follows from Theorem 1.4. The implication (1) $\Rightarrow$ (3) is formal.

For the remaining direction (3) $\Rightarrow$ (2), we consider an arbitrary point $p \in E \times_B B'$ and its images $e \in E$, $b' \in B'$ and $b \in B$. A diagram chase in

$$\begin{array}{ccccc} \mathrm{fib}_{b'}(E' \to B') & \longrightarrow & E' & & \\ \downarrow & \lrcorner & \downarrow & & \\ \mathrm{fib}_b(E \to B) & \longrightarrow & E \times_B B' & \longrightarrow & E \\ \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ * & \xrightarrow{b'} & B' & \longrightarrow & B. \end{array}$$

shows that $\mathrm{fib}_p(E' \to E \times_B B')$ is equivalent to the fibre of $\mathrm{fib}_{b'}(E' \to B') \to \mathrm{fib}_b(E \to B)$ in the point induced by e. Since the fibre of an equivalence is contractible, this proves the claim. □

Lemma 1.6. *Let B be a profinite anima. There exists a level-representation $B = \text{«lim»}_{\lambda \in \Lambda} B_\lambda$ with the property that any of the projection maps $B \to B_\mu$ is surjective on connected components.*

Proof. Choose any level-representation $B = \text{«lim»}_{\lambda\in\Lambda} B_\lambda$. Given any $\mu \in \Lambda$, write $p_\mu : B \to B_\mu$ for the projection and $T_\mu \coloneqq \mathrm{im}(\pi_0(p_\mu)) \subset \pi_0(B_\mu)$. The limit over $\mu \in \Lambda$ of the pullback diagrams

$$\begin{array}{ccc} B'_\mu & \longrightarrow & B_\mu \\ \downarrow & \lrcorner & \downarrow \\ T_\mu & \hookrightarrow & \pi_0(B_\mu) \end{array}$$

yields a pullback diagram

$$\begin{array}{ccc} \text{«lim»}_{\lambda\in\Lambda} B'_\lambda & \longrightarrow & B \\ \downarrow & \lrcorner & \downarrow \\ \text{«lim»}_{\lambda\in\Lambda} T_\lambda & \longrightarrow & \pi_0(B) \end{array}$$

in $\mathrm{Pro}(\mathbf{Ani}_\pi)$. By construction, the map $\text{«lim»}_{\lambda\in\Lambda} T_\lambda \to \pi_0(B)$ is an equivalence. Hence so is $\text{«lim»}_{\lambda\in\Lambda} B'_\lambda \to B$, and moreover, for each $\mu \in \Lambda$ the projection $\text{«lim»}_{\lambda\in\Lambda} B'_\lambda \to B'_\mu$ is surjective on connected components. □

1.2. **Filtered colimits of ∞-categories.**

Recollection 1.7 (filtered colimits of ∞-categories)**.** Let Λ be a filtered ∞-category. and $\mathscr{C}_\bullet : \Lambda \to \mathbf{Cat}_\infty$ a diagram.

(1) For any $\lambda \in \Lambda$, the forgetful functor $\Lambda_{\lambda/} \to \Lambda$ is colimit-cofinal, see [Ker, Tag 06D0] and the dual of [Ker, Tag 02XR].
(2) The ∞-category $\mathbf{Cat}_\infty$ of (small) ∞-categories admits all (small) limits and colimits.
(3) An ∞-category $\mathscr{C}$ is *finite* if it is contained in the smallest full subcategory of $\mathbf{Cat}_\infty$ containing $\emptyset, \Delta^0$, and Δ^1 that is closed under pushouts.
(4) The compact objects in $\mathbf{Cat}_\infty$ are precisely the retracts of finite ∞-categories, see [Ker, Tag 068L, Tag 068E]. In particular, finite ∞-categories are compact in $\mathbf{Cat}_\infty$.
(5) As a consequence of (4), any finite diagram $K \to \mathrm{colim}_{\lambda\in\Lambda} \mathscr{C}_\lambda$ lifts to some finite stage $\mu \in \Lambda$.
(6) Given two objects $c = [c_\lambda]$ and $c' = [c'_\lambda]$, the canonical map
$$\mathrm{colim}_{\lambda\to\mu} \mathrm{Map}_{\mathscr{C}_\mu}(c_\mu, c'_\mu) \to \mathrm{Map}_{\mathrm{colim}_\Lambda \mathscr{C}_\bullet}(c, c')$$
is an equivalence. Here, we write c_μ for the image of c_λ under the transition functor $\mathscr{C}_\lambda \to \mathscr{C}_\mu$ and similarly for c'_μ.

Lemma 1.8. *Let Λ be a filtered ∞-category, $\mathscr{C}_\bullet, \mathscr{D}_\bullet : \Lambda \to \mathbf{Cat}_\infty$ two diagrams, and $f_\bullet : \mathscr{C}_\bullet \to \mathscr{D}_\bullet$ a natural transformation.*

(1) If for each $\lambda \in \Lambda$, the functor f_λ is essentially surjective, so is $\mathrm{colim}_{\lambda\in\Lambda} f_\lambda$.
(2) If for each $\lambda \in \Lambda$, the functor f_λ is fully faithful, so is $\mathrm{colim}_{\lambda\in\Lambda} f_\lambda$.

Proof. By [Ker, Tag 068G], the various functors $\{\mathscr{C}_\lambda \to \mathrm{colim}_{\lambda\in\Lambda} \mathscr{C}_\lambda\}_{\lambda\in\Lambda}$ are jointly surjective, and similarly for $\mathscr{D}_\bullet$. Since for each $\lambda \in \Lambda$ the diagram

$$\begin{array}{ccc} \mathscr{C}_\lambda & \xrightarrow{f_\lambda} & \mathscr{D}_\lambda \\ \downarrow & & \downarrow \\ \mathrm{colim}_{\lambda\in\Lambda} \mathscr{C}_\lambda & \xrightarrow[\mathrm{colim}_{\lambda\in\Lambda} f_\lambda]{} & \mathrm{colim}_{\lambda\in\Lambda} \mathscr{D}_\lambda \end{array}$$

commutes, this shows (1). A proof of assertion (2) can be found in [HRS25, Prop. 2.1]. □

Lemma 1.9. *Let $\mathscr{C}_\bullet : \Lambda \to \mathbf{Cat}_\infty, \lambda \mapsto \mathscr{C}_\lambda$ be a filtered diagram of ∞-categories, and $K, \mathscr{E}$ any other ∞-categories. For a map $\mu \to \lambda \in \Lambda$, let $f_{\mu\to\lambda} : \mathscr{C}_\mu \to \mathscr{C}_\lambda$ denote the transition functor.*

(1) Assume that $\mathscr{C}_\lambda$ admits K-indexed colimits for each $\lambda \in \Lambda$ and that all transition functors preserve K-indexed colimits.
  (a) The ∞-category $\mathrm{colim}_\Lambda \mathscr{C}_\bullet$ admits K-indexed colimits.
  (b) For each $\lambda \in \Lambda$ the canonical functor $i_\lambda : \mathscr{C}_\lambda \to \mathrm{colim}_\Lambda \mathscr{C}_\bullet$ preserves K-indexed colimits.

(c) *A functor* $\mathrm{colim}_\Lambda \mathscr{C}_\bullet \to \mathscr{E}$ *preserves* K*-indexed colimits if and only if the composition*

$$\mathscr{C}_\lambda \to \mathrm{colim}_\Lambda \mathscr{C}_\bullet \to \mathscr{E}$$

does for any $\lambda \in \Lambda$.

(2) *Assume that* $\mathscr{C}_\lambda$ *admits* K*-indexed limits for each* $\lambda \in \Lambda$*, and that all transition functors preserve* K*-indexed limits. Moreover, assume that* K *is* finite.
   (a) *The* ∞*-category* $\mathrm{colim}_\Lambda \mathscr{C}_\bullet$ *admits* K*-indexed limits.*
   (b) *For each* $\lambda \in \Lambda$ *the canonical functor* $i_\lambda : \mathscr{C}_\lambda \to \mathrm{colim}_\Lambda \mathscr{C}_\bullet$ *preserves* K*-indexed limits.*
   (c) *A functor* $\mathrm{colim}_\Lambda \mathscr{C}_\bullet \to \mathscr{E}$ *preserves* K*-indexed limits if and only if the composition*

$$\mathscr{C}_\lambda \to \mathrm{colim}_\Lambda \mathscr{C}_\bullet \to \mathscr{E}$$

does for any $\lambda \in \Lambda$.

Proof. The case of colimits is a combination of [Ker, Tag 0632] and [Ker, Tag 0635]. For the rest of the proof, we assume that we are in situation (2). Since Λ is assumed to be filtered, any finite diagram in $\mathrm{colim}_\Lambda \mathscr{C}_\bullet$ lifts to some finite stage $\lambda \in \Lambda$ by [Ker, Tag 068G]. This shows that both (a) and (c) follow from (b). We now show that $i_\lambda : \mathscr{C}_\lambda \to \mathrm{colim}_\Lambda \mathscr{C}_\bullet$ preserves K-indexed limits. Let $K \to \mathscr{C}_\lambda, k \mapsto c_{\lambda,k}$ be a diagram and $c_\Lambda = [c_{\lambda'}] \in \mathrm{colim}_\Lambda \mathscr{C}_\bullet$ any object. Since Λ is filtered, we can find some $0 \in \Lambda$ and maps $\lambda \to 0 \leftarrow \lambda'$ so that $c_\Lambda = [c_0]$ with $c_0 = f_{\lambda' \to 0}(c_{\lambda'})$. Then, by Recollection 1.7 (6), we have

$$\begin{aligned} \mathrm{Map}_{\mathrm{colim}_\Lambda \mathscr{C}_\bullet}(c_\Lambda, i_\lambda(\lim_{k\in K} c_{\lambda,k})) &= \mathrm{colim}_{0\to\mu} \mathrm{Map}_{\mathscr{C}_\mu}(f_{0\to\mu}(c_0), f_{\lambda\to 0\to\mu}(\lim_{k\in K} c_{\lambda,k})) \\ &= \mathrm{colim}_{0\to\mu} \lim_{k\in K} \mathrm{Map}_{\mathscr{C}_\mu}(f_{0\to\mu}(c_0), f_{\lambda\to 0\to\mu}(c_{\lambda,k})), \end{aligned}$$

where the last equality holds by the assumption that each of the transition functors $f_{\lambda\to 0\to\mu}$ preserves K-indexed limits. The claim now follows from the fact that filtered colimits commute with finite limits in **Ani**, see [Ker, Tag 05XS]. □

Lemma 1.10. *Let* $\mathscr{C}_\bullet : \Lambda \to \mathbf{Cat}_\infty, \lambda \mapsto \mathscr{C}_\lambda$ *be a filtered diagram of* ∞*-categories with colimit* $\mathscr{C}_\infty := \mathrm{colim}_\Lambda \mathscr{C}_\bullet$*. Let* $c_\infty \in \mathscr{C}_\infty$ *be obtained as the colimit of a compatible system of objects* $c_\lambda : \Delta^0 \to \mathscr{C}_\lambda$*. The canonical map*

$$\mathrm{colim}_{\lambda\in\Lambda}(\mathscr{C}_\lambda)_{c_\lambda/} \to (\mathscr{C}_\infty)_{c_\infty/}$$

is an equivalence of ∞*-categories.*

Proof. Recall that, given an ∞-category $\mathscr{C}$ and an object $e \in \mathscr{C}$, the slice $\mathscr{C}_{e/}$ is given by the pullback in $\mathbf{Cat}_\infty$

$$\begin{array}{ccc} \mathscr{C}_{e/} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathscr{C}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \Delta^0 & \xrightarrow{e} & \mathscr{C}, \end{array}$$

where s denotes the source functor. Thus, as Δ^1 is compact in $\mathbf{Cat}_\infty$ by Recollection 1.7 (4), the claim follows since filtered colimits commute with finite limits in $\mathbf{Cat}_\infty$. □

2. Continuous local systems

In this section, we study continuous local systems on profinite anima.

Recollection 2.1 (continuous functors)**.** Let $\mathscr{C}, \mathscr{D}$ and $\mathscr{E}$ be ∞-categories.

(1) There is a canonical equivalence of ∞-categories

$$\mathrm{Fun}(\mathscr{C} \times \mathscr{D}, \mathscr{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathscr{C}, \mathrm{Fun}(\mathscr{D}, \mathscr{E}))$$

given by $F \mapsto (c \mapsto F(c, -))$.

(2) Pre- and postcomposition induce a functor

$$\mathrm{Fun} : \mathbf{Cat}_\infty^{\mathrm{op}} \times \mathbf{Cat}_\infty \to \mathbf{Cat}_\infty, (\mathscr{C}, \mathscr{D}) \mapsto \mathrm{Fun}(\mathscr{C}, \mathscr{D}).$$

(3) Since $\mathbf{Cat}_\infty$ is cocomplete, so is $\mathrm{Fun}(\mathbf{Cat}_\infty, \mathbf{Cat}_\infty)$. Therefore, the functor
$$\mathbf{Cat}_\infty^{\mathrm{op}} \to \mathrm{Fun}(\mathbf{Cat}_\infty, \mathbf{Cat}_\infty)$$
corresponding under (1) to the functor in (2) extends along filtered colimits to a functor
$$\mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} = \mathrm{Ind}(\mathbf{Cat}_\infty^{\mathrm{op}}) \to \mathrm{Fun}(\mathbf{Cat}_\infty, \mathbf{Cat}_\infty).$$
(4) We write
$$\mathrm{Fun}^{\mathrm{cts}} : \mathrm{Pro}(\mathbf{Cat}_\infty)^{\mathrm{op}} \times \mathbf{Cat}_\infty \to \mathbf{Cat}_\infty$$
for the functor corresponding under (1) to the functor constructed in (3). Given a map $f \in \mathrm{Pro}(\mathbf{Cat}_\infty)$, we write $f^* = (-\circ f)$ for the functor induced by precomposition. Given a functor g, we write $g_* = (g\circ -)$ for the functor induced by postcomposition.
(5) By construction, if $\mathscr{C} = \text{«lim»}_{\lambda\in\Lambda}\mathscr{C}_\lambda \in \mathrm{Pro}(\mathbf{Cat}_\infty)$, then
$$\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{D}) = \mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Fun}(\mathscr{C}_\lambda, \mathscr{D}),$$
where the transition functors are given by precomposition. We call $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{D})$ the *∞-category of continuous functors from $\mathscr{C}$ to $\mathscr{D}$*.

The inclusion $\mathbf{Ani} \subset \mathbf{Cat}_\infty$ induces an inclusion $\mathrm{Pro}(\mathbf{Ani}) \subset \mathrm{Pro}(\mathbf{Cat}_\infty)$, allowing us to make the following definition:

Definition 2.2 (continuous local systems)**.** Let B be any profinite anima, and $\mathscr{C}$ an ∞-category.
(1) We call $\mathrm{Fun}^{\mathrm{cts}}(B, \mathscr{C})$ the ∞-category of *continuous $\mathscr{C}$-valued local systems* on B.
(2) We simply refer to $\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_\pi)$ as the ∞-category of *continuous local systems* on B.

2.1. **(Co)Limits of continuous local systems.**

Proposition 2.3. *Let $f : \mathscr{C} = \text{«lim»}_{\lambda\in\Lambda}\mathscr{C}_\lambda \to \text{«lim»}_{\lambda\in\Lambda}\mathscr{C}'_\lambda = \mathscr{C}'$ be a map in $\mathrm{Pro}(\mathbf{Cat}_\infty)$ and $g : \mathscr{E} \to \mathscr{E}'$ a left-exact functor of ∞-categories with finite limits. Given any $\lambda \in \Lambda$, write $p_\lambda : \mathscr{C} \to \mathscr{C}_\lambda$ and $p'_\lambda : \mathscr{C}' \to \mathscr{C}'_\lambda$ for the projection maps. Let K be an ∞-category.*
(1) Assume that K is finite and that $\mathscr{E}$ admits K-indexed limits.
  (a) The ∞-category $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$ admits K-indexed limits.
  (b) The functor $p_\lambda^* : \mathrm{Fun}(\mathscr{C}_\lambda, \mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$ preserves K-indexed limits.
  (c) The functor $f^* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}', \mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$ preserves K-indexed limits.
  (d) If g preserves K-indexed limits, so does $g_* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}')$.
(2) Assume that $\mathscr{E}$ admits K-indexed colimits.
  (a) The ∞-category $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$ admits K-indexed colimits.
  (b) The functor $p_\lambda^* : \mathrm{Fun}(\mathscr{C}_\lambda, \mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$ preserves K-indexed colimits.
  (c) The functor $f^* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}', \mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$ preserves K-indexed colimits.
  (d) If g preserves K-indexed colimits, so does $g_* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}')$.

Proof. Assertions (1)(a) and (b) are special cases of Lemma 1.9 (2)(a) and (b), respectively. Next, we verify (1)(c) and (d). By construction, the diagrams
$$\begin{array}{ccc} \mathrm{Fun}(\mathscr{C}'_\lambda, \mathscr{E}) & \xrightarrow{f^*} & \mathrm{Fun}(\mathscr{C}_\lambda, \mathscr{E}) \\ {\scriptstyle (p'_\lambda)^*}\downarrow & & \downarrow{\scriptstyle p_\lambda^*} \\ \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}', \mathscr{E}) & \xrightarrow[f^*]{} & \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}) \end{array} \qquad \begin{array}{ccc} \mathrm{Fun}(\mathscr{C}_\lambda, \mathscr{E}) & \xrightarrow{g_*} & \mathrm{Fun}(\mathscr{C}_\lambda, \mathscr{E}') \\ {\scriptstyle p_\lambda^*}\downarrow & & \downarrow{\scriptstyle p_\lambda^*} \\ \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}) & \xrightarrow[g_*]{} & \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}') \end{array}$$
commute for each $\lambda \in \Lambda$. Therefore, by Lemma 1.9 (2)(b) and (c), we are reduced to the analogous statements for ordinary functor categories. Here, the result is obvious because (co)limits in functor categories are computed pointwise, see (the dual of) [Ker, Tag 02X9]. The argument for (2) is analogous. □

Lemma 2.4. *Let $L : \mathscr{E} \to \mathscr{E}'$ be a functor and $\mathscr{C} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$.*

(1) *If R is right-adjoint to L, then the pair of functors*

$$\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}) \underset{R_*}{\overset{L_*}{\rightleftarrows}} \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}')$$

induced by postcomposition are again adjoint to each other.
(2) *If L is fully faithful, so is L_* :* $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}')$.

Proof. Choose a cofiltered diagram $\mathscr{C}_\bullet : \Lambda \to \mathbf{Cat}_\infty, \lambda \mapsto \mathscr{C}_\lambda$ representing $\mathscr{C}$ and a unit $u : \mathrm{id}_{\mathscr{C}} \to R\circ L$ exhibiting R as right adjoint to L. Write $i_\lambda : \mathrm{Fun}(\mathscr{C}_\lambda,\mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})$ for the canonical inclusion functor. Assertion (1) holds for the ordinary functor categories on each finite stage $\lambda \in \Lambda$. More precisely, u induces a unit $u^\lambda : \mathrm{id}_{\mathrm{Fun}(\mathscr{C}_\lambda,\mathscr{E})} \to R_*\circ L_*$ with components $u^\lambda_F = (u_{F(c_\lambda)})_{c_\lambda \in \mathscr{C}_\lambda}$ exhibiting $R_* : \mathrm{Fun}(\mathscr{C}_\lambda,\mathscr{E}') \to \mathrm{Fun}(\mathscr{C}_\lambda,\mathscr{E})$ as right adjoint of the corresponding L_*. Moreover, since the transition functors in $\{\mathrm{Fun}(\mathscr{C}_\lambda,\mathscr{E})\}_{\lambda\in\Lambda}$ are given via *pre*composition, the various functors induced by *post*composition with L assemble into a natural transformation $L_\bullet : \mathrm{Fun}(\mathscr{C}_\bullet,\mathscr{E}) \to \mathrm{Fun}(\mathscr{C}_\bullet,\mathscr{E}')$, and similarly for R. By definition, L_* (resp. R_*) is the colimit of $L_\bullet$ (resp. $R_\bullet$), and

$$(i_\lambda u^\lambda)_{\lambda\in\Lambda} \in \lim_{\lambda\in\Lambda} \mathrm{Fun}(\mathrm{Fun}(\mathscr{C}_\lambda,\mathscr{E}),\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}_\lambda,\mathscr{E})) = \mathrm{Fun}(\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}),\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}))$$

defines a unit exhibiting that $R_* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}') \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})$ is right adjoint to L_* as claimed. Assertion (2) follows from the fact that the analogous statement is true for ordinary functor categories, together with Lemma 1.8 (2). □

2.2. **Truncations of continuous local systems.**

Notation 2.5. Let $\mathscr{C}$ be an ∞-category and $k \geq -1$. We write $\mathscr{C}_{\leq k}$ for the full subcategory of $\mathscr{C}$ spanned by the k-truncated objects.

Notation 2.6. Let $\mathscr{C} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$ and $\mathscr{E}$ an ∞-category.

(1) An *object* of $\mathscr{C}$ is a map $c : \Delta^0 \to \mathscr{C}$ in $\mathrm{Pro}(\mathbf{Cat}_\infty)$. We usually simply write $c \in \mathscr{C}$ in this case.
(2) Let $c \in \mathscr{C}$ be an object. We write $\mathrm{ev}_c := c^* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\Delta^0,\mathscr{E}) = \mathscr{E}$ and refer to ev_c as the *evaluation at c*.
(3) Let $e \in \mathscr{E}$ be an object. We write $\mathrm{const}_e \in \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})$ for the image of e under the functor $\mathscr{E} = \mathrm{Fun}^{\mathrm{cts}}(\Delta^0,\mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})$ induced by precomposition with the unique map $\mathscr{C} \to \Delta^0 \in \mathrm{Pro}(\mathbf{Cat}_\infty)$ and refer to const_e as the *constant diagram with value e*.

Lemma 2.7. *Let $B \in \mathrm{Pro}(\mathbf{Ani}_\pi)$, $n \geq 0$ and $\mathscr{E}$ an n-category. Precomposition with the n-truncation map $B \to \tau_{\leq n}B$ induces an equivalence* $\mathrm{Fun}^{\mathrm{cts}}(\tau_{\leq n}B,\mathscr{E}) \to \mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E})$.

Proof. Write $B = \text{«lim»}_{\lambda\in\Lambda} B_\lambda$ as a cofiltered limit of π-finite anima. Then $B \to \tau_{\leq n}B$ is the cofiltered limit of the n-truncation maps $B_\lambda \to \tau_{\leq n}B_\lambda$. We may thus assume that $B \in \mathbf{Ani}_\pi$ and thus $\mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E}) = \mathrm{Fun}(B,\mathscr{E})$. Since $\mathscr{E}$ is an n-category, it is an n-truncated object of $\mathbf{Cat}_\infty$. The claim now follows from the fact that the inclusion $\mathbf{Ani} \subset \mathbf{Cat}_\infty$ preserves n-truncations. □

Proposition 2.8. *Let $\mathscr{C} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$, $B \in \mathrm{Pro}(\mathbf{Ani}_\pi)$, $\mathscr{E}$ an ∞-category and $k \geq -1$ and integer.*

(1) *The inclusion $\mathscr{E}_{\leq k} \subset \mathscr{E}$ induces a functor*

$$\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}_{\leq k}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})$$

that is fully faithful with essential image contained in $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})_{\leq k}$.
(2) *We have* $\mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E}_{\leq k}) = \mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E})_{\leq k}$.
(3) *If the inclusion $\mathscr{E}_{\leq k} \subset \mathscr{E}$ admits a left-adjoint $\tau_{\leq k} : \mathscr{E} \to \mathscr{E}_{\leq k}$, then $(\tau_{\leq k}\circ -)$ is left-adjoint to the inclusion* $\mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E})_{\leq k} \subset \mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E})$.

Proof. By Lemma 2.4 (2), the only thing left to show in (1) is that $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E}_{\leq k}) \subset \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C},\mathscr{E})_{\leq k}$. This follows from Proposition 2.3 (1)(d) combined with [HTT, 5.5.6.5] and [HTT, 5.5.6.16]. To verify (2), pick a level-representation $B = \text{«lim»}_{\lambda\in\Lambda} B_\lambda$ with the property that each of the canonical projections $p_\lambda : B \to B_\lambda$ is surjective on connected components as in Lemma 1.6. Let $L \in \mathrm{Fun}^{\mathrm{cts}}(B,\mathscr{E})_{\leq k}$. Then there exists some $\lambda \in \Lambda$ and $L_\lambda \in \mathrm{Fun}(B_\lambda,\mathscr{E})$ such that $L = L_\lambda\circ p_\lambda$. It therefore suffices to

conclude that L_λ takes values in $\mathscr{C}_{\leq k}$. Let $b_\lambda \in B_\lambda$ and choose some $b \in B$ with the property that $p_\lambda(b)$ lies in the same connected component of B_λ as b_λ. Then $L_\lambda(b_\lambda) \cong \mathrm{ev}_b(L) \in \mathscr{C}$. Therefore, in order to show that L_λ takes values in $\mathscr{C}_{\leq k} \subset \mathscr{C}$, it suffices to show that $\mathrm{ev}_b(L) \in \mathscr{C}_{\leq k}$ for each $b \in B$. This follows from Proposition 2.3 (1)(c) combined with [HTT, 5.5.6.5] and [HTT, 5.5.6.16]. Finally, (3) follows from (2) combined with Lemma 2.4 (1). □

2.3. **Relation with classical local systems.** In this section, we relate continuous local systems of π-finite anima with continuous versions of classical local systems.

Lemma 2.9. *Let $\mathscr{C} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$, $\mathscr{E}$ an ∞-category, $e \in \mathscr{E}$ an object, and write $\mathrm{const}_e \in \mathrm{Fun}(\mathscr{C}, \mathscr{E})$ for the constant diagram as in Notation 2.6 (3). Postcomposition with the forgetful functor $V : \mathscr{E}_{e/} \to \mathscr{E}$ induces an equivalence of ∞-categories $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}_{e/}) \to \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})_{\mathrm{const}_e/}$.*

Proof. Since $\mathrm{const}_{\mathrm{id}_e}$ defines an initial object of $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}_{e/})$ and $V(\mathrm{const}_{\mathrm{id}_e}) = \mathrm{const}_e$, we get the induced functor. To see that it is an equivalence, note that Lemma 1.10 reduces us to the case where $\mathscr{C} \in \mathbf{Cat}_\infty$. Now recall that both diagrams

$$\begin{array}{ccc} \mathscr{E}_{e/} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathscr{E}) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \Delta^0 & \xrightarrow[e]{} & \mathscr{E} \end{array} \qquad \begin{array}{ccc} \mathrm{Fun}(\mathscr{C}, \mathscr{E})_{\mathrm{const}_e/} & \longrightarrow & \mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathscr{C}, \mathscr{E})) \\ \downarrow & \lrcorner & \downarrow{\scriptstyle s} \\ \Delta^0 & \xrightarrow[\mathrm{const}_e]{} & \mathrm{Fun}(\mathscr{C}, \mathscr{E}) \end{array}$$

are cartesian and similarly for $\mathrm{Fun}(\mathscr{C}, \mathscr{E})_{\mathrm{const}_e/}$. The claim now follows from applying and carefully unwinding the chain of equivalences $\mathrm{Fun}(\Delta^1, \mathrm{Fun}(\mathscr{C}, \mathscr{E})) \simeq \mathrm{Fun}(\Delta^1 \times \mathscr{C}, \mathscr{E}) \simeq \mathrm{Fun}(\mathscr{C}, \mathrm{Fun}(\Delta^1, \mathscr{E}))$. □

Notation 2.10. Let $\mathscr{C}$ be an ∞-category with terminal object $* \in \mathscr{C}$. We write $\mathscr{C}_* := \mathscr{C}_{*/}$ for the ∞-category of pointed objects in $\mathscr{C}$.

Proposition 2.11. *Let $\mathscr{C} \in \mathrm{Pro}(\mathbf{Cat}_\infty)$ and $\mathscr{E}$ an ∞-category.*

(1) If $* \in \mathscr{E}$ is a terminal object, the forgetful functor $V : \mathscr{E}_* \to \mathscr{E}$ determines an equivalence of ∞-categories $V_* : \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E}_*) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})_*$.

(2) If $\mathscr{E}$ admits pullbacks, we have a canonical equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathbf{Grpd}(\mathscr{E})) \xrightarrow{\sim} \mathbf{Grpd}(\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})).$$

(3) If $\mathscr{E}$ admits finite products, we have a canonical equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathbf{Grp}(\mathscr{E})) \xrightarrow{\sim} \mathbf{Grp}(\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})).$$

Proof. Since $* \in \mathscr{C}_*$ is initial, so is $\mathrm{const}_* \in \mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{C}_*)$. Moreover, $V(\mathrm{const}_*)$ is a terminal object of $\mathrm{Fun}^{\mathrm{cts}}(\mathscr{C}, \mathscr{E})$. Assertion (1) is thus an immediate consequence of Lemma 2.9. Assertions (2) and (3) are straightforward consequences of [SAG, A.8.2.5]. □

In summary, we have the following relation between $\mathbf{Ani}_\pi$-valued and classical local systems:

Theorem 2.12. *Let B be a profinite anima.*

(1) Postcomposition with the inclusion $\mathbf{Ani}_{\pi,\leq n} \subset \mathbf{Ani}_\pi$ induces a fully faithful embedding

$$\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_{\pi,\leq n}) \subset \mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_\pi)$$

with essential image given by the n-truncated objects. Moreover, this inclusion admits a left-adjoint obtained by postcomposition with the usual n-truncation functor $\tau_{\leq n} : \mathbf{Ani}_\pi \to \mathbf{Ani}_{\pi,\leq n}$.

(2) The inclusion $\mathbf{Set}_{\mathrm{fin}} = \mathbf{Ani}_{\pi,\leq 0} \subset \mathbf{Ani}_\pi$ induces an equivalence

$$\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Set}_{\mathrm{fin}}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_\pi)_{\leq 0}.$$

(3) There is a canonical equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_{\pi,*}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_\pi)_*.$$

(4) We have a canonical equivalence of 1-categories

$$\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Grp}_{\mathrm{fin}}) \xrightarrow{\sim} \mathbf{Grp}(\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Set}_{\mathrm{fin}})).$$

(5) *We have a canonical equivalence of* 1*-categories*

$$\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ab}_{\mathrm{fin}}) \xrightarrow{\sim} \mathbf{Ab}(\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Set}_{\mathrm{fin}})).$$

Proof. Assertion (1) is a special case of Proposition 2.8, (2) is a special case of (1), (3) is a special case of Proposition 2.11 (1), (4) is a special case of Proposition 2.11 (1), and, since $\mathbf{Ab} = \mathbf{Grp}(\mathbf{Grp})$, (5) follows by applying Proposition 2.11 (1) twice. □

3. Continuous (un-)straightening

The goal of this section is to prove Theorem 3.7, a continuous analogue of the following.

Recollection 3.1. Let $B \in \mathbf{Ani}$. There is an adjoint equivalence of ∞-categories

$$\int^B : \mathrm{Fun}(B, \mathbf{Ani}) \rightleftarrows \mathbf{Ani}_{/B} : \partial_B$$

called the *straightening-unstraightening equivalence*. Here, $\int^B := \mathrm{colim}_B$ denotes the colimit functor and $\partial_B(f : E \to B)$ is the local system of fibres $b \mapsto \mathrm{fib}_b(f)$.

Lurie shows in [SAG, Appendix E] that the colimit functor

$$\mathrm{colim}_B : \mathrm{Fun}(B, \mathrm{Pro}(\mathbf{Ani}_\pi)) \to \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}$$

is an equivalence if B is π-finite, but not for general profinite anima (see [SAG, E.6.0.7, E.6.0.9]). In order to get an equivalence for profinite anima, one has to replace $\mathrm{Fun}(B, \mathrm{Pro}(\mathbf{Ani}_\pi))$ with the ∞-category $\mathrm{Pro}(\mathrm{Fun}^{\mathrm{cts}}(B, \mathbf{Ani}_\pi))$.

In fact, we will prove a continuous relative pro-Σ variant of Recollection 3.1. To this end, we introduce the following notion.

Definition 3.2. Let Σ denote a full class of finite groups and $f : E \to B$ a map of π-finite anima.

(1) The map f is said to be Σ-*finite* if for every point $b \in B$, we have that $\mathrm{fib}_b(f) \in \mathbf{Ani}_\Sigma$.
(2) We write $\mathbf{Ani}_\Sigma \downarrow B$ for the full subcategory of $\mathbf{Ani}_{\pi/B}$ spanned by the Σ-finite maps.

Lemma 3.3 (Σ-finite straightening-unstraightening)**.** *Let Σ denote a nonempty full class of finite groups.*

(1) If $B \in \mathbf{Ani}_\Sigma$, we have $\mathbf{Ani}_{\Sigma/B} = \mathbf{Ani}_\Sigma \downarrow B \subset \mathbf{Ani}_{\pi/B}$. In particular $\mathbf{Ani}_\pi \downarrow B = \mathbf{Ani}_{\pi/B}$.
(2) The straightening-unstraightening equivalence of Recollection 3.1 restricts to an adjoint equivalence of ∞-categories

$$\mathrm{Fun}(B, \mathbf{Ani}_\Sigma) \underset{\partial_B}{\overset{\int_B}{\rightleftarrows}} \mathbf{Ani}_\Sigma \downarrow B.$$

(3) For any map of π-finite anima $g : B' \to B$, the diagram

$$\begin{array}{ccc} \mathrm{Fun}(B, \mathbf{Ani}_\Sigma) & \xleftarrow{\partial_B} & \mathbf{Ani}_\Sigma \downarrow B \\ {\scriptstyle -\circ g}\downarrow & & \downarrow{\scriptstyle -\times_B B'} \\ \mathrm{Fun}(B', \mathbf{Ani}_\Sigma) & \xleftarrow[\partial_{B'}]{} & \mathbf{Ani}_\Sigma \downarrow B'. \end{array}$$

commutes.

Proof. The inclusion $\mathbf{Ani}_{\Sigma/B} \subset \mathbf{Ani}_\Sigma \downarrow B$ is an immediate consequence of the fact that $\mathbf{Ani}_\Sigma \subset \mathbf{Ani}$ is closed under finite limits. If $E \in \mathbf{Ani}_\Sigma \downarrow B$, both B and all fibres $E_b = E \times_B b$ are Σ-finite. By inspecting the induced long exact sequences on homotopy groups for varying $b \in B$ and $e \in E_b$, one readily checks that $E \in \mathbf{Ani}_\Sigma$. This shows (1). Assertion (2) is an immediate consequence of (1), and (3) follows from Corollary 1.5. □

Next, we want to extend Definition 3.2 to maps of profinite anima. The following definition and lemma are essentially a variant of [Cla25, Lemma 4.13].

Definition 3.4 (Σ-(pro-)finite maps)**.** Let Σ be a full class of finite groups and $f : E \to B$ a map of profinite anima.

(1) The map $f : E \to B$ is called Σ-*finite* if there exists a cartesian square

$$\begin{array}{ccc} E & \longrightarrow & E_\pi \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f_\Sigma} \\ B & \longrightarrow & B_\pi \end{array}$$

in $\mathrm{Pro}(\mathbf{Ani}_\pi)$ with π-finite anima E_π, B_π and $f_\Sigma \in \mathbf{Ani}_\Sigma \downarrow B_\pi$. We write $\mathbf{Ani}_\Sigma \downarrow B \subset \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}$ for the full subcategory spanned by the Σ-finite maps.

(2) The map $f : E \to B$ is called Σ-*profinite* if for any point $b \in B$, we have that $\mathrm{fib}_b(f) \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$. We write $\mathrm{Pro}(\mathbf{Ani}_\Sigma) \downarrow B \subset \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}$ for the full subcategory spanned by the Σ-profinite maps.

Proposition 3.5. *Let $B = \text{«lim»}_{\lambda\in\Lambda} B_\lambda$ be a profinite anima.*

(1) The various functors $\mathbf{Ani}_\Sigma \downarrow B_\lambda \to \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}$ given by $E_\lambda \mapsto B \times_{B_\lambda} E_\lambda$ assemble into a fully faithful functor

$$\mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}}(\mathbf{Ani}_\Sigma \downarrow B_\lambda) \to \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}$$

with essential image $\mathbf{Ani}_\Sigma \downarrow B$.

(2) If $B \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$, then $\mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B = \mathrm{Pro}(\mathbf{Ani}_\Sigma)_{/B}$.

(3) Extending along cofiltered limits yields an inclusion of ∞-categories

$$\mathrm{Pro}(\mathbf{Ani}_\Sigma \downarrow B) \subset \mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B.$$

that is an equivalence if $B \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$.

Proof. Since for any map $\mu \to \lambda \in \Lambda$, the diagram

$$\begin{array}{ccc} \mathbf{Ani}_\Sigma \downarrow B_\lambda & \xrightarrow{-\times_{B_\lambda} B_\mu} & \mathbf{Ani}_\Sigma \downarrow B_\mu \\ & {\scriptstyle -\times_{B_\lambda} B} \searrow \quad \swarrow {\scriptstyle -\times_{B_\mu} B} & \\ & \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B} & \end{array}$$

commutes, we get the functor in (1). To see fully faithfulness, let $[E_0 \to B_0]$ and $[E_0' \to B_0]$ be two objects in $\mathbf{Ani}_\Sigma \downarrow B$ that we may assume to be represented at some common level 0 since Λ is cofiltered. Then, by Recollection 1.7 (6), we have

$$\begin{aligned} \mathrm{Map}([E_0'], [E_0]) &= \mathrm{colim}_{0\to\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Map}_{B_\lambda}(E_0' \times_{B_0} B_\lambda, E_0 \times_{B_0} B_\lambda) \\ &= \mathrm{colim}_{0\to\lambda\in\Lambda^{\mathrm{op}}} \mathrm{Map}_{B_0}(E_0' \times_{B_0} B_\lambda, E_0) \\ &= \mathrm{Map}_{\mathrm{Pro}(\mathbf{Ani}_\pi)_{/B_0}}(\text{«lim»}_{\lambda\to 0\in\Lambda}(E_0' \times_{B_0} B_\lambda), E_0) \\ &= \mathrm{Map}_{\mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}}(\text{«lim»}_{\lambda\to 0\in\Lambda}(E_0' \times_{B_0} B_\lambda), E_0 \times_{B_0} B) \end{aligned}$$

which, since $\text{«lim»}_{\lambda\to 0\in\Lambda}(E_0' \times_{B_0} B_\lambda) = E_0' \times_{B_0} B$ by the dual of Recollection 1.7 (1), shows fully faithfulness. Clearly, the image is contained in $\mathbf{Ani}_\Sigma \downarrow B$. To see equality, let $f : E \to B$ be Σ-finite and choose a cartesian diagram

$$\begin{array}{ccc} E & \longrightarrow & E_\pi \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle f_\Sigma} \\ B & \longrightarrow & B_\pi \end{array}$$

exhibiting the Σ-finiteness of f. Since $B_\pi \in \mathbf{Ani}_\pi$, the map $B \to B_\pi$ factors through one the B_λ's. Then $E_\lambda := E_\pi \times_{f_\Sigma, B_\pi} B_\lambda \to B_\lambda \in \mathbf{Ani}_\Sigma \downarrow B_\lambda$ pulls back to f. This completes the proof of (1). Next, we verify (2). That $\mathrm{Pro}(\mathbf{Ani}_\Sigma)_{/B} \subset \mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B$ follows from the fact that $\mathrm{Pro}(\mathbf{Ani}_\Sigma) \subset \mathrm{Pro}(\mathbf{Ani}_\pi)$ is closed under pullbacks. To see equality, let $f : E \to B$ be a map of profinite anima in $\mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow$

B. We need to show that $E \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$. To this end, let $b \in B$ be any point and consider the corresponding map of fibre sequences

$$\begin{array}{ccccc} F_b & \longrightarrow & E & \xrightarrow{f} & B \\ \downarrow & & \downarrow & & \downarrow \cong \\ F_b(\Sigma) & \longrightarrow & E^\wedge_\Sigma & \xrightarrow[f^\wedge_\Sigma]{} & B^\wedge_\Sigma. \end{array}$$

Since $B \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$, the right vertical map is an equivalence. Moreover, we have

$$F_b \xrightarrow{\sim} (F_b)^\wedge_\Sigma = (b \times_B E)^\wedge_\Sigma \xrightarrow{\sim} b \times_{B^\wedge_\Sigma} E^\wedge_\Sigma = F_b(\Sigma),$$

where the first isomorphism comes from the assumption that $f \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B$, and the second isomorphism follows from Recollection 1.3 (2) together with the assumption that $B \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$. Since this holds for arbitrary $b \in B$, we conclude that $E \to E^\wedge_\Sigma$ is an equivalence by Theorem 1.4. Next, we verify (3). That $\mathbf{Ani}_\Sigma \downarrow B \subset \mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B$ is an immediate consequence of Corollary 1.5. To see that $\mathrm{Pro}(\mathbf{Ani}_\Sigma \downarrow B) \to \mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B$ is fully faithful, it is enough to show that every object $E \in \mathbf{Ani}_\Sigma \downarrow B$ is cocompact in $\mathrm{Pro}(\mathbf{Ani}_\pi)_{/B}$. To this end, simply note that for $E = E_\lambda \times_{B_\lambda} B$, we have that

$$\mathrm{Map}_B(-, E) = \mathrm{Map}_{B_\lambda}(-, E_\lambda) = \mathrm{Map}(-, E_\lambda) \times_{\mathrm{Map}(-, B_\lambda)} *$$

takes cofiltered limits to filtered colimits since $E_\lambda, B_\lambda \in \mathbf{Ani}_\pi$ are cocompact in $\mathrm{Pro}(\mathbf{Ani}_\pi)$ and since colimits are universal in $\mathbf{Ani}$. To finish the proof of (3), the only thing left to show is that any map of profinite anima $f : E \to B$ in $\mathrm{Pro}(\mathbf{Ani}_\Sigma) \downarrow B$ can be written as a cofiltered limit of maps $f_{(\lambda)} \in \mathbf{Ani}_\Sigma \downarrow B$ if $B \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$. With B, also $E \in \mathrm{Pro}(\mathbf{Ani}_\Sigma)$ by (2). We can thus write $f = \text{«lim»}_{\lambda\in\Lambda} f_\lambda$ with $f_\lambda : E_\lambda \to B_\lambda$ a map in $\mathbf{Ani}_{\Sigma/B_\lambda} = \mathbf{Ani}_\Sigma \downarrow B_\lambda$. Since $f = \lim_\lambda (f_\lambda \times_{B_\lambda} B)$, we conclude. □

Corollary 3.6. *Let $f : E' \to E$ and $g : E \to B$ be maps of profinite anima. If $g \in \mathbf{Ani}_\pi \downarrow B$, then $f \in \mathbf{Ani}_\pi \downarrow E$, if and only if $(g \circ f) \in \mathbf{Ani}_\pi \downarrow B$.*

Proof. Write $E' \xrightarrow{f} E \xrightarrow{g} B$ as a cofiltered limit of maps $E'_\lambda \xrightarrow{f_\lambda} E_\lambda \xrightarrow{g_\lambda} B_\lambda$ over some cofiltered category $\lambda \in \Lambda$ and choose a pullback square

$$\begin{array}{ccc} E & \longrightarrow & E_\pi \\ {\scriptstyle g}\downarrow & \lrcorner & \downarrow{\scriptstyle g_\pi} \\ B & \longrightarrow & B_\pi \end{array}$$

exhibiting the π-finiteness of g. Since B_π is π-finite, there exists some $\lambda \in \Lambda$ such that $B \to B_\pi$ factors through $B \to B_\lambda$. Similarly, since E_π is π-finite and Λ is cofiltered, we can find some $\mu \to \lambda \in \Lambda$ such that $E \to E_\pi \times_{B_\pi} B_\lambda$ factors through E_μ. Replacing Λ with $\Lambda_{/\mu}$, we may hence assume that for any $\lambda \in \Lambda$ we have a commutative diagram

$$\begin{array}{ccccc} E' & \longrightarrow & E'_\lambda & & \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_\lambda} & & \\ E & \longrightarrow & E_\lambda & \longrightarrow & E_\pi \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_\lambda} & & \downarrow{\scriptstyle g_\pi} \\ B & \longrightarrow & B_\lambda & \longrightarrow & B_\pi \end{array}$$

in which the lower horizontal rectangle is cartesian. Replacing g_λ by the projection $E_\pi \times_{B_\pi} B_\lambda \to B_\lambda$ and noting that

$$\text{«lim»}_{\lambda\in\Lambda}(E_\pi \times_{B_\pi} B_\lambda) = E_\pi \times_{B_\pi} (\text{«lim»}_{\lambda\in\Lambda} B_\lambda) = E,$$

we may also assume that the lower right square is cartesian. Consequently, for every $\lambda \in \Lambda$, also the lower left square is cartesian. If f is π-finite, applying the same reductions to f, we may assume that also the upper square is cartesian. Hence so is the vertical rectangle, exhibiting the π-finiteness

of $g\circ f$. If, one the other hand, $(g\circ f)$ is assumed to be π-finite, a similar reduction lets us assume that the vertical rectangle is cartesian. Hence so is the upper square by cancellation of pullbacks, exhibiting the π-finiteness of f. □

Finally, we can prove continuous (un-)straightening.

Theorem 3.7 (continuous straightening-unstraightening)**.** *Let B be a profinite anima and Σ a nonempty full class of finite groups.*

(1) The straightening-unstraightening equivalences of the various π-finite constituents of B assemble into an adjoint equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{cts}}(B,\mathbf{Ani}_\Sigma) \underset{\partial_B}{\overset{\int^B}{\rightleftarrows}} \mathbf{Ani}_\Sigma\downarrow B.$$

(2) If B is in $\mathrm{Pro}(\mathbf{Ani}_\Sigma)$, there is an induced adjoint equivalence of ∞-categories

$$\mathrm{Pro}(\mathrm{Fun}^{\mathrm{cts}}(B,\mathbf{Ani}_\Sigma)) \underset{\partial_B}{\overset{\int^B}{\rightleftarrows}} \mathrm{Pro}(\mathbf{Ani}_\Sigma)\downarrow B$$

obtained by passing to pro-categories.

Proof. Choose a level representation $B = \text{«lim»}_{\lambda\in\Lambda} B_\lambda$. Combining Lemma 3.3 (2) and (3) with Lemma 1.8, the straightening-unstraightening equivalences of the various B_λ assemble into an adjoint equivalence of ∞-categories

$$\mathrm{Fun}^{\mathrm{cts}}(B,\mathbf{Ani}_\Sigma) \underset{\partial_B}{\overset{\int^B}{\rightleftarrows}} \mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}}(\mathbf{Ani}_\Sigma\downarrow B_\lambda)$$

on the colimits. Here, we write $\int^B \coloneqq \mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}}(\int^{B_\lambda})$ and $\partial_B \coloneqq \mathrm{colim}_{\lambda\in\Lambda^{\mathrm{op}}}(\partial_{B_\lambda})$ for the induced functors on the colimits. Proposition 3.5 (1) identifies the right-hand side with $\mathbf{Ani}_\Sigma\downarrow B$, showing (1). In view of Proposition 3.5 (3), assertion (2) follows from (1) by passing to pro-categories. □

Remark 3.8. Let B be a π-finite anima. The functor

$$\mathrm{Pro}(\mathrm{Fun}(B,\mathbf{Ani}_\pi)) \to \mathrm{Fun}(B,\mathrm{Pro}(\mathbf{Ani}_\pi))$$

induced by postcomposing with the inclusion $\mathbf{Ani}_\pi \subset \mathrm{Pro}(\mathbf{Ani}_\pi)$ and then extending along cofiltered limits is an equivalence of ∞-categories by [SAG, E.6.1.3]. Moreover, the diagram

$$\begin{array}{ccc} \mathrm{Pro}(\mathrm{Fun}(B,\mathbf{Ani}_\pi)) & \longrightarrow & \mathrm{Fun}(B,\mathrm{Pro}(\mathbf{Ani}_\pi)) \\ & {\scriptstyle\int^B}\searrow & \downarrow{\scriptstyle \mathrm{colim}_B} \\ & & \mathrm{Pro}(\mathbf{Ani}_\pi)_{/B} \end{array}$$

commutes. Restricted to π-finite anima, our continuous straightening-unstraightening equivalence therefore coincides with Lurie's variant [SAG, E.6.0.7].

Corollary 3.9. *Let Σ be a full class of finite groups and B a profinite anima. Precomposition with the pro-Σ completion map $u: B \to B^\wedge_\Sigma$ induces a fully faithful embedding*

$$u^*: \mathrm{Fun}^{\mathrm{cts}}(B^\wedge_\Sigma,\mathbf{Ani}_\Sigma) \hookrightarrow \mathrm{Fun}^{\mathrm{cts}}(B,\mathbf{Ani}_\Sigma).$$

Proof. Under Theorem 3.7, the functor u^* gets identified with

$$-\times_{B^\wedge_\Sigma} B: \mathbf{Ani}_\Sigma\downarrow B^\wedge_\Sigma \to \mathbf{Ani}_\Sigma\downarrow B.$$

The latter functor is readily seen to be fully faithful by Recollection 1.3 (2). □

Lemma 3.10. *Let*

$$\begin{array}{ccc} E' & \longrightarrow & E \\ {\scriptstyle f'}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ B' & \underset{p}{\longrightarrow} & B \end{array}$$

be a pullback square in $\mathrm{Pro}(\mathbf{Ani}_\pi)$ *with* $f \in \mathbf{Ani}_\pi \downarrow B$. *Then* $\partial_{B'}(f')$ *is given by the continuous functor*

$$B' \xrightarrow{p} B \xrightarrow{\partial_B(f)} \mathbf{Ani}_\pi .$$

Proof. Unwinding the definitions, this follows immediately from Corollary 1.5. □

4. Applications to profinite shape theory

In this section, we finally prove Theorem A of the introduction, see Theorem 4.10, and derive some consequences.

4.1. The profinite shape.

Definition 4.1. Let $\mathcal{X}$ be an ∞-topos with terminal object $1_\mathcal{X}$ and Σ a full class of finite groups.

(1) A sheaf $F \in \mathcal{X}$ is called *locally Σ-constant constructible* if there exists an effective epimorphism $\sqcup_{i=1}^n U_i \twoheadrightarrow 1_\mathcal{X}$ and Σ-finite anima $K_1, \dots, K_n$ together with equivalences $F \times U_i \simeq \Gamma^* K_i \times U_i$ in $\mathcal{X}_{/U_i}$.
(2) We write $\mathrm{lcc}(\mathcal{X}, \mathbf{Ani}_\Sigma) \subset \mathcal{X}$ for the full subcategory spanned by the locally Σ-constant constructible sheaves.
(3) When $\Sigma = \pi$, we simply say *locally constant constructible* and write $\mathrm{lcc}(\mathcal{X}) \coloneqq \mathrm{lcc}(\mathcal{X}, \mathbf{Ani}_\pi)$.
(4) A morphism $f : F' \to F$ in $\mathcal{X}$ is called *locally constant constructible* if $f \in \mathrm{lcc}(\mathcal{X}_{/F})$.

Recollection 4.2. Let $f_* : \mathcal{X} \to \mathcal{S}$ be a geometric morphism of ∞-topoi.

(1) The subcategory $\mathrm{lcc}(\mathcal{X}) \subseteq \mathcal{X}$ is closed under finite limits and finite coproducts, see [SAG, E.2.5.2] and [SAG, E.2.6.8].
(2) The pullback functor $f^* : \mathcal{S} \to \mathcal{X}$ preserves being locally Σ-constant constructible.

Lemma 4.3. *Let* $f_* : \mathcal{X} \to \mathcal{S}$ *and* $g_* : \mathcal{Y} \to \mathcal{X}$ *be geometric morphisms of* ∞*-topoi.*

(1) Each functor in the commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f^*} & \mathcal{X} \\ \uparrow & & \uparrow \\ \mathrm{lcc}(\mathcal{S}) & \xrightarrow[f^*]{} & \mathrm{lcc}(\mathcal{X}) \end{array}$$

has a pro-left adjoint and the resulting diagram

$$\begin{array}{ccc} \mathrm{Pro}(\mathcal{X}) & \xrightarrow{f_!} & \mathrm{Pro}(\mathcal{S}) \\ {\scriptstyle (-)^\wedge_{\mathrm{lcc}}}\downarrow & & \downarrow{\scriptstyle (-)^\wedge_{\mathrm{lcc}}} \\ \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) & \xrightarrow[f_\sharp]{} & \mathrm{Pro}(\mathrm{lcc}(\mathcal{S})) \end{array}$$

commutes.

(2) There is a natural equivalence $(f \circ g)_\sharp \simeq f_\sharp \circ g_\sharp$.

Proof. Assertion (1) is a formal consequence of Recollection 4.2 (1) and (2). Assertion (2) follows from $(f \circ g)^* \simeq g^* \circ f^*$ by passing to left adjoints. □

Definition 4.4 (profinite shape)**.** Let $f = (f^* \dashv f_*) : \mathcal{X} \to \mathcal{S}$ be a geometric morphism of ∞-topoi.

(1) The functor
$$f_\sharp : \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \to \mathrm{Pro}(\mathrm{lcc}(\mathcal{S}))$$
is called the *(relative) profinite shape functor of* f. The sheaf $f_\sharp F$ is called the *profinite shape of* F *relative to* f.
(2) The sheaf
$$\widehat{\Pi}_\infty(\mathcal{X}/\mathcal{S}) \coloneqq \widehat{\Pi}_\infty(f) \coloneqq f_\sharp 1_\mathcal{X}$$
is called the *relative profinite shape of* f.

(3) The functor
$$\widehat{\Pi}_\infty \coloneqq \Gamma_{\mathcal{X},\sharp} : \ \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \to \mathrm{Pro}(\mathbf{Ani}_\pi)$$
is called the *profinite shape functor of* $\mathcal{X}$. The profinite anima $\widehat{\Pi}_\infty(F)$ is called the *profinite shape* of $F \in \mathrm{Pro}(\mathrm{lcc}(\mathcal{X}))$.

(4) The profinite anima
$$\widehat{\Pi}_\infty(\mathcal{X}) \coloneqq \widehat{\Pi}_\infty(1_{\mathcal{X}})$$
is called the *profinite shape of* $\mathcal{X}$.

Remark 4.5. Let $f_* : \mathcal{X} \to \mathcal{S}$ be geometric morphism of ∞-topoi and $\Gamma_* : \mathcal{S} \to \mathbf{Ani}$ the unique geometric morphism.

(1) An anima K is a locally constant constructible object of $\mathbf{Ani}$ if and only if it is π-finite. Consequently, lcc-completion $(-)^\wedge_{\mathrm{lcc}}$ on $\mathrm{Pro}(\mathbf{Ani})$ agrees with profinite completion $(-)^\wedge_\pi$.

(2) By Lemma 4.3, we see that
$$\widehat{\Pi}_\infty(\mathcal{X}) = (\Gamma_! 1_{\mathcal{X}})^\wedge_\pi$$
recovers the usual definition of the profinite shape.

(3) By definition, $\widehat{\Pi}_\infty(\mathcal{X}/\mathcal{S})$ has the following universal property: Functorially in $F \in \mathrm{lcc}(\mathcal{S})$, there is an equivalence
$$\mathrm{Map}_{\mathrm{Pro}(\mathrm{lcc}(\mathcal{S}))}(\widehat{\Pi}_\infty(\mathcal{X}/\mathcal{S}), F) \simeq \Gamma_*(\mathcal{X}, f^*F).$$
That is: $\widehat{\Pi}_\infty(\mathcal{X}/\mathcal{S})$ encodes the cohomology of locally constant constructible sheaves that arise via pullback from $\mathcal{S}$.

Lemma 4.6. *Let $\mathcal{X}$ be an ∞-topos.*

(1) The adjunction $\widehat{\Pi}_\infty : \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \rightleftarrows \mathrm{Pro}(\mathbf{Ani}_\pi) : \Gamma^*$ *induces an adjunction*
$$\widehat{\Pi}_{\infty/\mathcal{X}} : \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \rightleftarrows \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{X})} : \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}.$$
Here, $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}$ *is given as the composition of the sliced pullback functor* $\Gamma^* : \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{X})} \to \mathrm{Pro}(\mathrm{lcc}(\mathcal{X}))_{/\Gamma^*\widehat{\Pi}_\infty(\mathcal{X})}$ *followed by pulling back along the canonical section* $1_{\mathcal{X}} \to \Gamma^*\widehat{\Pi}_\infty(\mathcal{X})$ *given by the unit of the adjunction* $\widehat{\Pi}_\infty \dashv \Gamma^*$.

(2) The functor $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}$ *restricts to a functor* $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})} : \mathbf{Ani}_\pi \downarrow \widehat{\Pi}_\infty(\mathcal{X}) \to \mathrm{lcc}(\mathcal{X})$.

Proof. Assertion (1) is formal, and (2) is a direct consequence of Recollection 4.2 (1). □

4.2. **The monodromy equivalence.** For a geometrically pointed connected qcqs scheme (X, x), the category of locally constant constructible étale sheaves of sets on X is equivalent to finite sets with a continuous action by the étale fundamental group $\hat{\pi}_1^{\text{ét}}(X, x)$. Under this equivalence, a sheaf F is mapped to the monodromy action of $\hat{\pi}_1^{\text{ét}}(X, x)$ on the stalk F_x. More canonically, and without connectivity assumptions on X, the locally constant constructible sheaf F can be recovered from a continuous local system on $\widehat{\Pi}^{\text{ét}}_{\leq 1}(X)$, the *monodromy representation* $\mathrm{Mon}(F) : \widehat{\Pi}^{\text{ét}}_{\leq 1}(X) \to \mathbf{Set}_{\mathrm{fin}}$ of the étale fundamental groupoid of X. There is an equivalence of 1-categories
$$\mathrm{Mon} : \ \mathrm{lcc}(X_{\text{ét}}, \mathbf{Set}) \longrightarrow \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}^{\text{ét}}_{\leq 1}(X), \mathbf{Set}_{\mathrm{fin}}).$$
The fundamental theorem of profinite shape theory, due to Lurie [SAG, Appendix E], Bachmann–Hoyois [BH20, Prop. 10.1] and Barwick–Glasman–Haine [BGH20, Prop. 4.4.18], refines the above equivalence to one capturing the whole of $\mathrm{lcc}(X_{\text{ét}})$.

Theorem 4.7 (monodromy equivalence)**.** *Let $\mathcal{X}$ be an ∞-topos. The composition* $\mathrm{Mon} = \partial_{\widehat{\Pi}_\infty(\mathcal{X})} \circ \widehat{\Pi}_{\infty/\mathcal{X}}$ *is an equivalence of ∞-categories*
$$\mathrm{Mon} : \ \mathrm{lcc}(\mathcal{X}) \longrightarrow \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Ani}_\pi)$$

that carries a locally constant constructible sheaf F to its monodromy representation $\mathrm{Mon}(F)$. *Its inverse is given by the composition*

$$\mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Ani}_\pi) \xrightarrow{\int^{\widehat{\Pi}_\infty(\mathcal{X})}} \mathbf{Ani}_\pi \downarrow \widehat{\Pi}_\infty(\mathcal{X}) \xrightarrow{\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}} \mathrm{lcc}(\mathcal{X}).$$

Proof. We refer the reader to [BGH20, Prop. 4.4.18] and [BH20, Prop. 10.1] (together with [Hoy17a, Thm. 4.3]). Note that both references just state the existence of "canonical" equivalences. However, upon inspecting the proofs in either of the above cases, one sees the explicit description we stated. □

Remark 4.8. Let $\mathcal{X}$ be an ∞-topos.

(1) In view of Theorem 2.12, the monodromy equivalence restricts to an equivalence

$$\mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Set}_{\mathrm{fin}}) \xrightarrow{\sim} \mathrm{lcc}(\mathcal{X}, \mathbf{Set})$$

that in turn induces equivalences

$$\mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Grp}_{\mathrm{fin}}) \xrightarrow{\sim} \mathrm{lcc}(\mathcal{X}, \mathbf{Grp}_{\mathrm{fin}}) \quad \text{and} \quad \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Ab}_{\mathrm{fin}}) \xrightarrow{\sim} \mathrm{lcc}(\mathcal{X}, \mathbf{Ab}_{\mathrm{fin}}).$$

(2) If X is a connected scheme, the choice of a geometric point $\bar{x} \to X$ determines an equivalence

$$\mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}^{\text{ét}}_\infty(X), \mathbf{Set}_{\mathrm{fin}}) = \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}^{\text{ét}}_{\leq 1}(X), \mathbf{Set}_{\mathrm{fin}}) \simeq \hat{\pi}^{\text{ét}}_1(X, \bar{x})\text{-}\mathbf{Set}_{\mathrm{fin}},$$

where the right-hand side denotes the category of finite sets equipped with a continuous $\hat{\pi}^{\text{ét}}_1(X, \bar{x})$-action. Under this equivalence, (1) recovers [Stacks, Tag 0DV5].

In view of Corollary 3.9, we are led to the following notion.

Definition 4.9 (monodromy in Σ)**.** Let $\mathcal{X}$ be an ∞-topos, Σ a full class of finite groups, and $F \in \mathrm{lcc}(\mathcal{X}, \mathbf{Ani}_\Sigma)$.

(1) The sheaf F has *monodromy in* Σ if its monodromy representation $\mathrm{Mon}(F)$ lies in the essential image of the inclusion $\mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma, \mathbf{Ani}_\Sigma) \hookrightarrow \mathrm{Fun}^{\mathrm{cts}}(\widehat{\Pi}_\infty(\mathcal{X}), \mathbf{Ani}_\Sigma)$ of Corollary 3.9.

(2) We write $\mathrm{lcc}_\Sigma(\mathcal{X})$ for the full subcategory of $\mathrm{lcc}(\mathcal{X}, \mathbf{Ani}_\Sigma)$ spanned by the locally Σ-constant constructible sheaves with monodromy in Σ.

4.3. **The profinite shape equivalence.** The monodromy equivalence has the following fairly straightforward consequence, that appears to not have been recorded in the literature so far.

Theorem 4.10 (profinite shape equivalence)**.** *Let $\mathcal{X}$ be an ∞-topos.*

(1) The profinite shape adjunction

$$\mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \underset{\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}}{\overset{\widehat{\Pi}_{\infty/\mathcal{X}}}{\rightleftarrows}} \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{X})}$$

is an adjoint equivalence of ∞-categories.

(2) Let $f = (f^*, f_*) \colon \mathcal{X} \to \mathcal{S}$ be a geometric morphism of ∞-topoi. In the diagram

$$\begin{array}{ccc} \mathrm{Pro}(\mathrm{lcc}(\mathcal{S})) & \underset{f^*}{\overset{f_\sharp}{\rightleftarrows}} & \mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) \\ \Big\downarrow{\scriptstyle \widehat{\Pi}_{\infty/\mathcal{S}}} \Big\uparrow{\scriptstyle \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{S})}} & & \Big\downarrow{\scriptstyle \widehat{\Pi}_{\infty/\mathcal{X}}} \Big\uparrow{\scriptstyle \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}} \\ \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{S})} & \underset{-\times_{\widehat{\Pi}_\infty(\mathcal{S})}\widehat{\Pi}_\infty(\mathcal{X})}{\overset{\widehat{\Pi}_\infty(f)\circ -}{\rightleftarrows}} & \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{X})} \end{array}$$

both the subdiagrams of left-adjoints and of right-adjoints commute.

Proof. By Theorem 3.7 (1), the unstraightening functor $\int^{\widehat{\Pi}_\infty(\mathcal{S})}$ is an equivalence. Therefore Theorem 4.7 implies that also $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{S})} \colon \mathbf{Ani}_\pi \downarrow \widehat{\Pi}_\infty(\mathcal{S}) \to \mathrm{lcc}(\mathcal{S})$ is an equivalence. Passing to pro-categories, we see that also

$$\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{S})} \colon \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{S})} \to \mathrm{Pro}(\mathrm{lcc}(\mathcal{S}))$$

is an equivalence by Proposition 3.5 (3). By construction, $\widehat{\Pi}_{\infty/\mathcal{S}}$ is left-adjoint to $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{S})}$, hence also an equivalence. This shows (1). That the subdiagram of right-adjoints in (2) commutes is obvious. Hence so does the one of left-adjoints by uniqueness of adjoints. □

Remark 4.11. Alternatively, it should be possible to deduce Theorem 4.10 by carefully tracing the proof of [SAG, A.8.0.5] in the case $\mathcal{C} = \mathrm{lcc}(\mathcal{X})$.

Corollary 4.12. *Let $\mathcal{X}$ be an ∞-topos. The profinite shape equivalence restricts to an adjoint equivalence of ∞-categories* $\widehat{\Pi}_{\infty/\mathcal{X}} : \mathrm{lcc}(\mathcal{X}) \rightleftarrows \mathbf{Ani}_\pi \downarrow \widehat{\Pi}_\infty(\mathcal{X}) : \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}$.

Proof. Since both $\mathrm{lcc}(\mathcal{X})$ and $\mathbf{Ani}_\pi \downarrow \widehat{\Pi}_\infty(\mathcal{X})$ are essentially small and idempotent complete, this follows from Theorem 4.10 (1) and Proposition 3.5 (3) by passing to cocompact objects, see [HTT, 5.4.2.4] and [HTT, 5.4.3.6]. □

Corollary 4.13. *Let $\mathcal{X}$ be an ∞-topos and $f : F \to G$ as well as $g : G \to H$ two morphisms in $\mathcal{X}$.*

(1) If $F \in \mathrm{lcc}(\mathcal{X})$, the canonical diagram

$$\begin{array}{ccc} F & \longrightarrow & \Gamma^*\widehat{\Pi}_\infty(F) \\ \downarrow & \lrcorner & \downarrow \\ 1_{\mathcal{X}} & \longrightarrow & \Gamma^*\widehat{\Pi}_\infty(\mathcal{X}) \end{array}$$

is a pullback in $\mathrm{Pro}(\mathrm{lcc}(\mathcal{X}))$.

(2) If both f and g are locally constant constructible, so is $g \circ f$.

Proof. If $F \in \mathrm{lcc}(\mathcal{X})$, we have $F \simeq \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}\widehat{\Pi}_\infty(F)$. Unwinding the definition of $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}$, this precisely gives the pullback square of (1). To see (2), note that we may assume $H = 1_{\mathcal{X}}$ by replacing $\mathcal{X}$ with $\mathcal{X}_{/H}$. Then by (1), the squares

$$\begin{array}{ccc} F & \xrightarrow{f} & G \\ {\scriptstyle (u_F, f)}\downarrow & & \downarrow{\scriptstyle (u_G, \mathrm{id}_G)} \\ \Gamma^*\widehat{\Pi}_\infty(F) \times G & \xrightarrow[\Gamma^*\widehat{\Pi}_\infty(f) \times \mathrm{id}_G]{} & \Gamma^*\widehat{\Pi}_\infty(G) \times G \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{g} & 1_{\mathcal{X}} \\ {\scriptstyle u_G}\downarrow & & \downarrow{\scriptstyle u_{\mathcal{X}}} \\ \Gamma^*\widehat{\Pi}_\infty(G) & \xrightarrow[\Gamma^*\widehat{\Pi}_\infty(g)]{} & \Gamma^*\widehat{\Pi}_\infty(\mathcal{X}). \end{array}$$

are cartesian in $\mathrm{Pro}(\mathrm{lcc}(\mathcal{X}_{/G}))$ respectively $\mathrm{Pro}(\mathrm{lcc}(\mathcal{X}))$, hence both are cartesian in $\mathrm{Pro}(\mathcal{X})$. It follows that also

$$\begin{array}{ccc} F & \xrightarrow{u_F} & \Gamma^*\widehat{\Pi}_\infty(F) \\ {\scriptstyle g\circ f}\downarrow & \lrcorner & \downarrow{\scriptstyle \Gamma^*\widehat{\Pi}_\infty(g\circ f)} \\ 1_{\mathcal{X}} & \xrightarrow[u_H]{} & \Gamma^*\widehat{\Pi}_\infty(\mathcal{X}) \end{array}$$

is cartesian in $\mathrm{Pro}(\mathcal{X})$. We conclude by combining Corollary 3.6, Recollection 4.2 (1), and Corollary 4.12. □

Corollary 4.14. *Let $f = (f^*, f_*) : \mathcal{X} \to \mathcal{S}$ be a geometric morphism of ∞-topoi, Σ a full class of finite groups, and $F \in \mathrm{lcc}(\mathcal{S})$.*

(1) The canonical diagram

$$\begin{array}{ccc} \widehat{\Pi}_\infty(f^*F) & \longrightarrow & \widehat{\Pi}_\infty(F) \\ \downarrow & \lrcorner & \downarrow \\ \widehat{\Pi}_\infty(\mathcal{X}) & \xrightarrow[\widehat{\Pi}_\infty(f)]{} & \widehat{\Pi}_\infty(\mathcal{S}) \end{array}$$

is a pullback square in $\mathrm{Pro}(\mathbf{Ani}_\pi)$.

(2) The monodromy representation of f^*F is given as the continuous functor

$$\widehat{\Pi}_\infty(\mathcal{X}) \xrightarrow{\widehat{\Pi}_\infty(f)} \widehat{\Pi}_\infty(\mathcal{S}) \xrightarrow{\mathrm{Mon}(F)} \mathbf{Ani}_\pi\,.$$

(3) If $F \in \mathrm{lcc}_\Sigma(\mathcal{S})$, then $f^*F \in \mathrm{lcc}_\Sigma(\mathcal{X})$.

Proof. Assertion (1) is a formal consequence of Theorem 4.10, since the vertical adjunctions are equivalences: Since $F = \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{S})}\widehat{\Pi}_\infty(F)$, we have that $f^*F = \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}(\widehat{\Pi}_\infty(F) \times_{\widehat{\Pi}_\infty(\mathcal{S})} \widehat{\Pi}_\infty(\mathcal{X}))$ by Theorem 4.10 (2). The result follows by applying $\widehat{\Pi}_\infty(-)$. Assertion (2) follows from (1) and Lemma 3.10. Finally, (3) follows from (2) by considering the commutative diagram of continuous functors

$$\begin{array}{ccccc}
\widehat{\Pi}_\infty(\mathcal{X}) & \xrightarrow{\widehat{\Pi}_\infty(f)} & \widehat{\Pi}_\infty(\mathcal{S}) & \xrightarrow{\mathrm{Mon}(F)} & \mathbf{Ani}_\Sigma\,. \\
\downarrow & & \downarrow & \overset{\exists?}{\nearrow} & \\
\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma & \xrightarrow[\widehat{\Pi}_\infty(f)^\wedge_\Sigma]{} & \widehat{\Pi}_\infty(\mathcal{S})^\wedge_\Sigma & &
\end{array}$$

□

Finally, we also have the following pro-Σ variant of the profinite shape equivalence:

Theorem 4.15 (pro-Σ shape equivalence)**.** *Let $\mathcal{X}$ be an ∞-topos and Σ full class of finite groups. The pro-Σ shape induces an adjoint equivalence of ∞-categories*

$$\widehat{\Pi}_\infty(-)^\wedge_\Sigma : \mathrm{Pro}(\mathrm{lcc}_\Sigma(\mathcal{X})) \rightleftarrows \mathrm{Pro}(\mathbf{Ani}_\Sigma)_{/\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma} : \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma},$$

where $\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma} = \Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})} \circ (- \times_{\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma} \widehat{\Pi}_\infty(\mathcal{X}))$.

Proof. Combining Recollection 4.2 (1) with Corollary 4.14 (3) and observing that $\mathbf{Ani}_\Sigma = \mathrm{lcc}_\Sigma(\mathbf{Ani}) \subset \mathrm{lcc}(\mathbf{Ani}) = \mathbf{Ani}_\pi$, we see that all of the functors in the commutative diagram on the left

$$\begin{array}{ccc}
\mathrm{lcc}(\mathcal{X}) & \xleftarrow{\Gamma^*} & \mathbf{Ani}_\pi \\
\uparrow & & \uparrow \\
\mathrm{lcc}_\Sigma(\mathcal{X}) & \xleftarrow[\Gamma^*]{} & \mathbf{Ani}_\Sigma
\end{array}
\qquad
\begin{array}{ccc}
\mathrm{Pro}(\mathrm{lcc}(\mathcal{X})) & \underset{\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})}}{\overset{\widehat{\Pi}_\infty}{\rightleftarrows}} & \mathrm{Pro}(\mathbf{Ani}_\pi)_{/\widehat{\Pi}_\infty(\mathcal{X})} \\
(-)^\wedge_\Sigma \downarrow \uparrow & & (-)^\wedge_\Sigma \downarrow \uparrow -\times_{\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma}\widehat{\Pi}_\infty(\mathcal{X}) \\
\mathrm{Pro}(\mathrm{lcc}_\Sigma(\mathcal{X})) & \underset{\Gamma^*_{\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma}}{\overset{\Gamma_{\sharp,\Sigma}}{\rightleftarrows}} & \mathrm{Pro}(\mathbf{Ani}_\Sigma)_{/\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma}
\end{array}$$

admit pro-left adjoints, inducing the commutative diagram of sliced adjoints on the right. By Theorem 4.10, the top horizontal adjunction is an equivalence. Unwinding the definition of $\mathrm{lcc}_\Sigma(\mathcal{X})$, we moreover see that the image of $\mathrm{Pro}(\mathrm{lcc}_\Sigma(\mathcal{X}))$ under $\widehat{\Pi}_\infty$ coincides with the image of the pullback functor $(- \times_{\widehat{\Pi}_\infty(\mathcal{X})^\wedge_\Sigma} \widehat{\Pi}_\infty(\mathcal{X}))$. Since $\Gamma_{\sharp,\Sigma} = (-)^\wedge_\Sigma \circ \widehat{\Pi}_\infty$, the claim follows. □

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