# Applications of the étale fundamental gerbe in anabelian geometry

GAUS-AG WS23/24

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## Introduction

Grothendieck's "anabelian geometry", as first laid out in his letter [Gro83] to Faltings, is concerned with the question to which extent arithmetic-geometric properties of a variety $X$ over a field $k$ are determined by the *fundamental exact sequence* of étale fundamental groups

$$1 \to \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x}) \to \pi_1^{\text{ét}}(X, \bar{x}) \to G_k \to 1.$$

For example, (a $\pi_1^{\text{ét}}(X_{\bar{k}})$–conjugacy class of) a section of this sequence is called a *Galois section.* Any rational point $a \in X(k)$ gives rise to such a Galois section

$$[s_a] \in \operatorname{Hom}^{\text{out}}_{G_k}(G_k, \pi_1^{\text{ét}}(X, \bar{x})) := \operatorname{Hom}_{G_k}(G_k, \pi_1^{\text{ét}}(X, \bar{x}))/\pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$$

by means of the functoriality of $\pi_1^{\text{ét}}(-)$, and Galois sections of this form are called *geometric*. Among other things, Grothendieck now conjectured the following:

**Section Conjecture.** Let $X$ be a proper hyperbolic curve over a field $k$ that is finitely generated over $\mathbb{Q}$. Then the Kummer map

$$\kappa_X : X(k) \to \operatorname{Hom}^{\text{out}}_{G_k}(G_k, \pi_1^{\text{ét}}(X, \bar{x})) =: \mathscr{S}^{\text{ét}}(X/k), \quad a \mapsto [\pi_1^{\text{ét}}(a)]$$

is a bijection.

While injectivity of $\kappa_X$ was already known to Grothendieck, the question of whether every Galois section is geometric, i.e. whether $\kappa_X$ is surjective, is still wide open. The aim of this seminar to obtain an overview of Bresciani's work concerning applications of Borne-Vistoli's *étale fundamental gerbe* $\Pi^{\text{ét}}_{X/k}$ introduced in [BV12] to the section conjecture. While working with $\Pi^{\text{ét}}_{X/k}$ is essentially equivalent to working with $\pi_1^{\text{ét}}(X, \bar{x})$, both formalisms have different strengths: The fundamental group $\pi_1^{\text{ét}}(X, \bar{x})$ is, by its very definition, entirely determined by covering spaces of $X$, and thus amenable to geometric arguments. The fundamental gerbe $\Pi^{\text{ét}}_{X/k}$ on the other hand should (as will hopefully become clear in the seminar) be thought of as the *space of Galois sections* of $X$ over $k$, and is hence very useful for studying phenomena related to these. Another advantage of $\Pi^{\text{ét}}_{X/k}$ is that it is intrinsically base point free, which makes it a very convenient object for studying specializations in different fibres of algebraic families.

After gaining some familiarity with the abstract formalism of stacks & gerbes, we will go on to introduce the étale fundamental gerbe and discuss its role as space of Galois sections. We will go on by discussing the notion of *printability* and how it can be used to show that the Section Conjecture implies another one of Grothendieck's anabelian conjectures, namely:

**Hom Conjecture.** Let $X$ be a proper hyperbolic curve over a field $k$ that is finitely generated over $\mathbb{Q}$. Then for every connected variety[1] $T$, the canonical map

$$\mathrm{Hom}_k(T, X) \to \mathrm{Hom}^{\mathrm{out}}_{G_k}(\pi_1^{\text{ét}}(T), \pi_1^{\text{ét}}(X))$$

is a bijection.

Next, we will show that in order to prove the section conjecture over fields that are finitely generated over $\mathbb{Q}$, it suffices to prove it for number fields.

Finally, we will discuss Bresciani's most recent preprint [Bre23a]. Recall that a Galois section $s \in \mathscr{S}^{\text{ét}}(X/k)$ is said to be *birationally liftable* if it's in the image of the canonical map $\mathscr{S}^{\text{ét}}(k(X)/k) \to \mathscr{S}^{\text{ét}}(X/k)$. Bresciani now proves that a Galois section $s \in \mathscr{S}^{\text{ét}}(X/k)$ is *geometric* if and only if its basechange $s_{k(t)}$ along the field extension $k(t) \mid k$ is birationally liftable. From this, he furthermore deduces that the Section Conjecture is in fact equivalent to the so-called *cuspidalization conjecture* stating that for any non-empty open $U \subset X$, the induced map $\mathscr{S}^{\text{ét}}(U/k) \to \mathscr{S}^{\text{ét}}(X/k)$ is surjective (which is a known consequence of the Section Conjecture).

## Time and Place

- We'll meet in person roughly every two weeks on *Wednesday* in either Heidelberg (Mathematikon, SR A) or Frankfurt (Robert-Mayer Str. 6-8, SR 310).
- Usually, a session will consist of three talks:

| # | Heidelberg | Frankfurt |
|---|---|---|
| 1. | 13:15 - 14:15 | 13:30 - 14:30 |
| 2. | 14:30 - 15:30 | 14:45 - 15:45 |
| ☺ | coffee break | coffee break |
| 3. | 16:00 - 17:00 | 16:15 - 17:15 |

[1] We refrain from giving a precise definition of what "variety" means here; come join the seminar if you want to find out!

## Schedule

### Overview, Stacks & Gerbes *(Frankfurt, 18.10.2023)*

**Talk 0. Overview**. *(Speaker: Tim/Marcin/Jakob).*
Provide an overview of the seminar and distribute the topics.

(i) Mention fundamental exact sequence;

(ii) Overview of anabelian conjectures;

(iii) State the main theorems.

(iv) Make sure that Birational SC (as in [Bre23a, Intro]) is stated in the Intro talk.

**Talk 1. Stacks**. *(Speaker: Jaro).* Recall the notion of a *fibered category* (over some $\mathbf{C}$) [BL, Def. 2.2.6] and explain how the choice of a *cleavage* [BL, Def. 2.3.3] lets one extract a pseudofunctor $\mathbf{C}^{\mathrm{op}} \to \mathrm{Cat}$ from a fibered category over $\mathbf{C}$ [BL, Thm. 2.3.5]. Mention how fibered categories generalize presheaves [BL, §2.5]. Define the *category of objects with descent data* of a fibered category over a site $\mathbf{C}$ with cleavage [BL, Def. 2.6.2-2.6.3] and explain its connection to the sheaf condition [BL, §2.6]. Recall the notions of *prestack* and *stack* as in the paragraph preceding [BL, Def. 2.6.3]. Explain the universal property of the *stackification* of a fibered category [Giraud, Def. II.2.1.1][2] and state the Théorème d'existence [Giraud, Thm. II.2.1.3]. Cf. [Stacks, §02ZM].

Mention classical examples: vector bundles [BL, Ex. 2.4.1] and, if time permits, the fibered category of quasi-coherent sheaves [Vis07, §3.2.1] + [Vis07, Thm. 4.23]. Mention what it means to be an Artin / Deligne–Mumford stack [Stacks, §026N]. Introduce the *classifying stack* $\mathbf{B}G$ of $G$ [Stacks, Tag 036Z][3] and explain in what sense it classifies $G$-torsors (using the 2-categorical Yoneda lemma [Vis07, §3.6.2]).

Discuss representability, products and notions like proper, smooth and geometrically connected in the context of (DM) stacks. You can consider moving examples to the beginning and explain the definitions in terms of the example along the way.

**Talk 2. Gerbes**. *(Speaker: Jaro).* Define a *gerbe over a site* $S$ following [Rivano, §III.2.1], and show that $\mathbf{B}G$ is a gerbe. Discuss that all gerbes *locally* are of this form, and show that a gerbe is *neutral* precisely if it equivalent to $\mathbf{B}G$ for some $G$ [Rivano, §III.2.1.1.1].

For sheaves of groups $G, G'$, introduce the following:

(i) the stack $\mathbf{Grps}(S) \to S$ of *sheaves of groups on* $S$ [Giraud, §II.3.4.1+3.4.1.1] and, given any stack (in groupoids) $F$, construct the canonical morphism of stacks $F \to \mathbf{Grps}(S)$ [Giraud, §II.3.5.2].

(ii) the fibered category $\mathrm{Li}(S) \to S$ of *bands on* $S$ (fr. "lien") as well as its stackification $\mathbf{Lien}(S)$, construct the canonical morphism of stacks lien: $\mathbf{Grps}(S) \to \mathbf{Lien}(S)$ [Giraud,

[2] Seems that at the beginning of this definition, Giraud writes stack when he means a fibered category, be careful!

[3] Note that $\mathbf{B}G$ is simply called "*G-Torsors*" in the stacks project, and that the stacks project unnecessarily restricts their definition to the *fppf* topology on schemes only.

§IV.1.1.5] and use it to define *representable* bands [Giraud, Def. IV.1.2.1]. Following [Giraud, Prop. IV.1.1.7], if time allows, sketch the proof that

$$\mathrm{Hom}(\mathrm{lien}(G), \mathrm{lien}(G')) = \mathrm{Hom}^{\mathrm{out}}(G, G').$$

Define what it means for a stack $F$ to be equipped with an *action* by a band $L$ [Giraud, Def. IV.2.1.4].

Prove [Giraud, Prop. IV.2.2.1] to finally define the band $L$ of a gerbe $G$ [Giraud, Def. IV.2.2.2]. Mention the more explicit characterisation of the band of a gerbe given in [Giraud, §IV.2.2.2.2]. Define what it means for a gerbe $G$ to be *representable* (resp. *affine, flat, ...*) as well as the notion of a *tannakian gerbe* (what [BV12] refers to as "*affine fpqc gerbes*") [Rivano, §III.2.2]. Cf. also [DM12, Appendix].

## The étale fundamental gerbe and the Section Conjecture *(Heidelberg, 15.11.2023)*

**Talk 3. The étale fundamental gerbe**. *(Speaker: Jon).*[⊙] Following [Bre21, Appendix] (which updates [BV12, §8]), introduce the étale fundamental gerbe: discuss briefly the (geometric) connectedness of fibered categories [Bre21, §A.1 + A.2] (structure sheaf of a category fibered is defined e.g. in [Stacks, Def. 06TV]) and projective limits of gerbes / fibered categories ([BV12, §3] + [Bre21, Rmk. A.12]) with a view towards preparation for the existence result proof later on. Define the *étale fundamental gerbe* $\Pi^{\text{ét}}_{X/k}$ [Bre21, A.13] and comment on its uniqueness ([Bre21, Lm. A.14]). Then state and prove the existence ([Bre21, Thm. A.17]) for geometrically connected categories. This involves following the proof of [BV12, Thm. 5.7] with necessary changes. If needed, say a few words about finite gerbes [BV12, §4]. If time allows, mention briefly the injectivity in **SC** of [BV12, Prop. 9.6] (likely to be skipped to save time).

Very briefly comment, how the definition/existence looks if we drop "étale" [BV12, Defn. 5.1, 5.3 + Thm. 5.7] (we only need the étale one later on). State the base-change along field extensions, [Bre21, Prop. A.18 + A.23], and sketch its proof (at least the proof of [Bre21, Prop. A.18] following [BV12, Prop. 6.1], then [Bre21, Prop. A.23] if time allows).

**Talk 4. The section conjecture in the language of gerbes**. *(Speaker: Jakob).*[⊙] Explain the reformulation of the section conjecture via étale fundamental gerbes:

State that the points of $\Pi^{\text{ét}}_{X/k}$ correspond to sections of $\pi_1^{\text{ét}}$: [Bre21, Prop. A.19] + [BV12, Prop. 9.3] and try to give a reasonably detailed proof. For the proof, a reinterpretation of $\Pi^{\text{ét}}_{X/k}$ in terms of the Deligne's relative fundamental gerbe [BV12, 8.1 - 8.3] might be useful.

State the **SC** 's reformulation in [BV12, Conj. 9.5 + 9.8].

State [Bre21, Lm. 2.6] that fits the discussion and should be useful later on.

Finally, introduce the *relative* étale fundamental gerbe following [Bre23b, §2] and [Bre23a, §2.1]. Combine [Bre23b, Lm. 2.2] and [Bre23a, Lm. 1] into a single statement.

**Talk** ☺. Backup slot (if one of the earlier talks needs more time) / discussion session / having a walk / ending earlier.

[⊙]It is encouraged to use the backup slot and make this talk 20-30 minutes longer!

## Printability *(Frankfurt, 29.11.2023)*

**Talk 5. Applications of the fundamental gerbe.** *(Speaker: Marius)* Overview the "Stacky going up and going down theorems" from [Bre21, §2] – state the results from there and, if time allows, sketch some of the proof ingredients.

Then discuss the behavior of $\Pi^{\text{ét}}_{X/k}$ for finite étale covers of $X$, [Bre21, Prop. A.25]. Sketch the proof.

Finally, define *printability* and *fundamentally full faithfulness (fff)* [Bre21, §1.2, Defn. 3.2], then state the characterization of printability from [Bre21, Prop. 3.1] and sketch its proof.

**Talk 6. Printability: intro and tools**. *(Speaker: Jakob/Marcin)*. State two of the main theorems involving printability from [Bre21, §1.3]: [Bre21, Thm. A] (i.e. printability is a geometric notion) and [Bre21, Thm. C] (i.e. $X$ printable implies strong hom form of anabelian conjectures). The goal of this and the next talk is to prove them.

State and prove a result on existence of finite étale covers of fff DM stacks by algebraic spaces: [Bre21, Prop. 5.1].

Next, cover [Bre21, §6]: state the Lemmas, try to include a proof of [Bre21, Lm. 6.5], and finally prove [Bre21, Thm. A (=Thm. 6.6)].

**Talk 7. Printability: proofs of the main results**. *(Speaker: Jakob/Marcin)*. The main task of this talk is to state and prove [Bre23b, Thm. 9.1 (=Thm. C)].

In the remaining time: cover printability in dimension 0 ([Bre21, Lm. 3.4 + Cor. 3.5]). Then, mention the results [Bre21, 3.6 - 3.8]. End by stating and proving [Bre21, Cor. 9.3]. Only if time premits: discuss [Bre21, Prop. 9.4].

## Number fields vs. absolutely finitely generated fields *(Heidelberg, 13.12.2023)*

**Talk 8. Section conjecture for orbicurves and affine curves via root stacks**. *(Speaker: Morten)*.[⊙]

Following [BE14, §2.2], define infinite root stacks and orbicurves.[4] Discuss the results concerning orbicurves from [Bre21, §7, 8].[5] Then use these notions to discuss the section conjecture for affine curves [Bre21, §8] and the specialization of sections via the infinite root stack following [Bre23a, §3]. In this last reference, make sure to state and explain in detail [Bre23a, Lm. 8] – it will be quite useful later on.

In these two talks we want to tackle the main result of [Bre23b], which shows that **SC** over number fields implies **SC** over fields of finite type over $\mathbb{Q}$ under some (seemingly) mild extra condition.

**Talk 9. Spreading out sections and coherence property**. *(Speaker: Marcin)*.[⊙] Very briefly introduce (verbally) the goal [Bre23b, Thm. B] of today's sessions, the precise statement will be during the next talk. The point of this first talk is to prepare the necessary tools:

[⊙] It is encouraged to use the backup slot and make this talk 20-30 minutes longer!

[4] Mention that Borne and Emsalem conjectured [BE14, Conjecture 2] that the section conjecture holds for them.

[5] Define the "degrees of residual gerbes" that appear in the statement of [Bre21, Prop. 7.1] and are introduced above it.

[⊙] It is encouraged to use the backup slot and make this talk 20-30 minutes longer!

(i) make sure that [Bre23b, Lm. 2.2] was covered earlier, recall it if needed.
(ii) Discuss spreading out sections of [Bre23b, §3], state and prove [Bre23b, Cor. 3.4].
(iii) Discuss the coherence property of [Bre23b, §4]. Skip [Bre23b, §4.1], make sure to cover [Bre23b, Cor. 4.9] and the Lemmas it requires, with proofs.

**Talk** ☺ Backup slot (if one of the earlier talks needs more time) / discussion session / having a walk / ending earlier.

### Finishing SC over fields of finite type. Moving on to birationally liftable sections *(Frankfurt, 17.01.2024)*

**Talk 10. SC over fields of finite type: main theorems**. *(Speaker: Amine)*. Introduce the main *unconditional* theorem [Bre23b, Thm. B]. Extract from [Bre23b, Cor. A+B] examples of curves that satisfy the condition of the main theorem. Briefly comment how this generalizes previous results of e.g. Saïdi (as mentioned in the Intro of *loc. cit.*). Prove [Bre23b, Thm. B].

**Talk 11. Birational liftability: intro**. *(Speaker: Tim)*. Elements of [Bre23a, Intro]: definition of *birationally liftable* and $t$*-birationally liftable*. State the alternative formulation of Birational **SC** . State the main theorems: [Bre23a, Thm. A + Thm. B].

Make sure that [Bre23a, Lm. 8] was stated and proven by this point (it is a part of Talk 9).

Now, cover in detail the properties of $t$-birationally liftable sections in [Bre23a, §4]. All the lemmas and corollaries there should be stated. Try to include the proofs.

**Talk 12. Birational liftability: main tool**. *(Speaker: Leonie)*. State and prove the main tool of the paper that will allow to prove the main theorems quickly: [Bre23a, Prop. 17].

### Reduction to number fields and proofs of the main theorems *(Heidelberg, 31.01.2024)*

**Talk 13. Birational liftability: reduction to number fields**. *(Speaker: Magnus)*. Cover the reduction to number fields [Bre23a, §6]: state and prove [Bre23a, Prop. 22] and the two [Bre23a, Lemmas 19, 20] needed in the proof. This also uses many results from previous talks: [Bre23b, Cor. 3.4, Defn. 4.1, Cor. 4.9], [Bre21, Lm. 6.5, Thm. 7.2].[6]

**Talk 14. Birational liftability: proving main theorems**. *(Speaker: Katharina)*. Put the tools together and finish the proofs of the main theorems: [Bre23a, Thm. A and B] following [Bre23a, §7].

**Talk** ☺. Backup / discussions / hike.

## References

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[6] You can use the external references that appear there – like [Tam97, Prop. 2.8 (iv)] or [Sti15, Thm. B] – without proofs.

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