Computing Schneider p-adic heights on hyperelliptic Mumford curves

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joint work in progress with Marc Masdeu,
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Winter Workshop Chabauty–Kim 2024
Heidelberg University
February 16, 2024

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For a prime number p, a p-adic height pairing is a function

$$A(F) \times \tilde{A}(F) \to \mathbb{Q}_p$$

which can be regarded as a p-adic analogue of the Néron–Tate height pairing.

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Main goal

Present an algorithm to compute the Schneider p-adic height pairing on (Jacobians of) hyperelliptic Mumford curves.

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is defined as

$$\langle \cdot, \cdot \rangle_{\mathrm{CG}} = \sum_{q \in \{\text{prime numbers}\}} \langle \cdot, \cdot \rangle_q = \langle \cdot, \cdot \rangle_p + \sum_{q \neq p} \langle \cdot, \cdot \rangle_q.$$

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The local components away from p are described using "arithmetic intersection theory", and

$$\langle D, E \rangle_p := {}^{\mathrm{Vol}}\!\!\int_E \omega_D$$

where

- ω_D is a "canonical" differential form attached to D, and
- ${}^{\mathrm{Vol}}\!\!\int$ is the Vologodsky integration.

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which satisfy the expected properties. We call this integral the **Berkovich–Coleman integral**.

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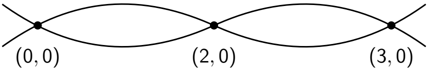

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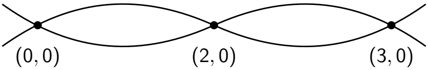

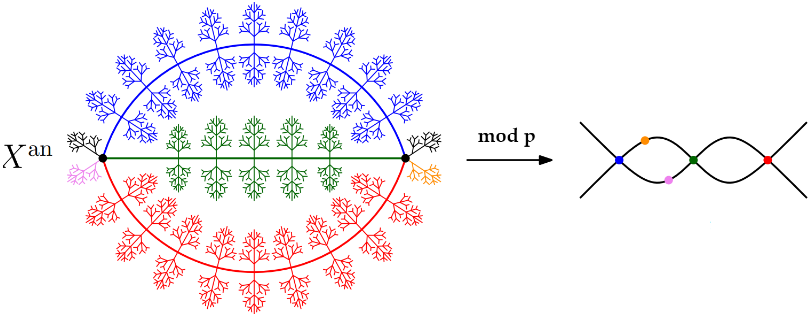

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- If X has good reduction, then ${}^{\mathrm{Vol}}\!\int_x^y \omega = {}^{\mathrm{BC}}\!\int_\gamma \omega$.

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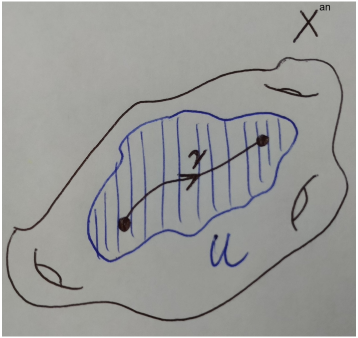

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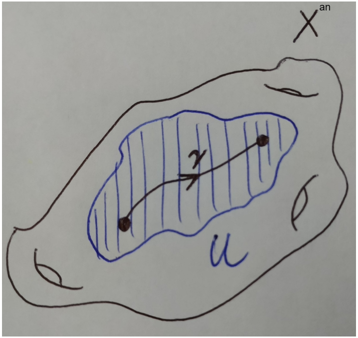

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Theorem (Katz–K, K.)

We have

$$ {}^{\mathrm{Vol}}\!\int_P^Q \omega = {}^{\mathrm{BC}}\!\int_\gamma \omega - \sum_i \left(c_i \cdot {}^{\mathrm{BC}}\!\int_{\gamma_i} \omega \right) $$

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Algorithm (Katz–K, K.)

Compute Vologodsky integrals on hyperelliptic curves using this formula and the fact that the Berkovich–Coleman integral is local.

Come back to the main goal... Overview

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For a finite prime $\mathfrak{p}$ of F that lies over p, a formula for

$$(\cdot,\cdot)_{\mathfrak{p}}$$

was given by Werner in the case where C is a *Mumford* curve at $\mathfrak{p}$.

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Question: What is Ω_Γ/Γ?

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Remark: For any Schottky group Γ, X_Γ has *split degenerate* reduction: it has a semistable $\mathcal{O}_K$-model $\mathfrak{X}$ such that

- all irreducible components of $\mathfrak{X}_k$ are isomorphic to $\mathbb{P}^1_k$, and
- all double points are k-rational with two k-rational branches,

where k is the residue field.

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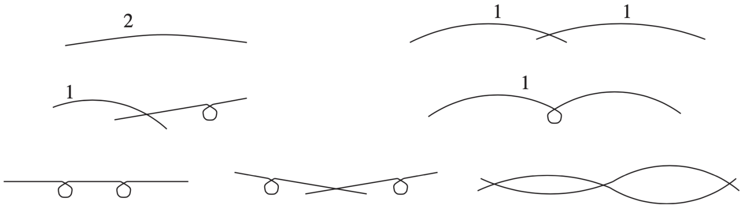

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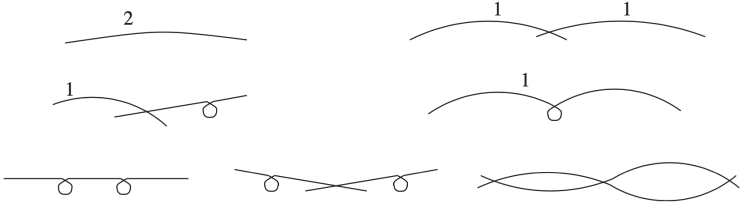

A genus 2 curve has split degenerate reduction precisely when the special fiber of its stable model is one of the three pictures at the bottom (picture taken from Liu's **Algebraic Geometry and Arithmetic Curves** book).

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- Γ is a Schottky group, freely generated by $\gamma_i := s_i s_0$, $i = 1, \ldots, g$.
- Γ and $\Gamma^{'}$ have the same set of ordinary points, call it Ω.
- The following map has degree 2:

$$\Omega/\Gamma \to \Omega/\Gamma^{'}, \quad a\Gamma \mapsto a\Gamma^{'};$$

so the Mumford curve $X_\Gamma = \Omega/\Gamma$ is a double cover of $\Omega/\Gamma^{'}$.

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- Γ is a Schottky group, freely generated by $\gamma_i := s_i s_0$, $i = 1, \ldots, g$.
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For suitably chosen $a, b \in \Omega$, the (theta) function

$$F(z) := F_{a,b}(z) := \prod_{\gamma \in \Gamma'} \frac{z - \gamma(a)}{z - \gamma(b)}, \quad z \in \Omega$$

is Γ'-invariant and induces an isomorphism $\Omega/\Gamma' \simeq \mathbb{P}^1$.

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Remarks:

- The group Γ is called a **(p-adic) Whittaker** group.
- Every hyperelliptic Mumford curve can be parametrized by a Whittaker group in this way.

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Result: Not only Mumford curves, but also their Jacobians have nice reduction types.

§3.1. Theta functions

K - $F_{\mathfrak{p}}$; a finite extension of $\mathbb{Q}_p$
X - $C \otimes K$; a curve over K of genus g

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It's an automorphic form with constant factors of automorphy: for all $\gamma \in \Gamma$ and all $z \in \Omega$,

$$\Theta(a, b; z) = c(a, b, \gamma) \cdot \Theta(a, b; \gamma(z))$$

for some $c(a, b, \gamma) \in K^\times$.

§3.2. Werner's formula for $(\cdot,\cdot)_{\mathfrak{p}}$

Now let $\rho\colon K^\times \to \mathbb{Q}_p$ be a non-trivial continuous homomorphism. In practice, it will be $\log_p \circ N_{K/\mathbb{Q}_p}$ where $\log_p$ is the branch of the p-adic logarithm that sends p to 0.

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Take $D, E \in \mathrm{Div}^0(X)$. Since the pairing $(\cdot,\cdot)_{\mathfrak{p}}$ is additive in both arguments, we can assume that

$$D = (x) - (y) \quad \text{and} \quad E = (z) - (w)$$

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Theorem (Werner): Choose preimages x', y', z', w' in Ω. We then have

$$(D, E)_{\mathfrak{p}} = \rho\left(\frac{\Theta(x', y'; z')}{\Theta(x', y'; w')}\right) - \begin{array}{c}\text{another function}\\ \text{in terms of } \Theta.\end{array}$$

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C - a hyperell. curve over F of genus $g \geq 1$ s.t. for every finite prime $\mathfrak{p}$ of F above p, C is a Mumford curve at $\mathfrak{p}$.

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Remark: Another method due to Masdeu–Xarles allows us to compute this function in a comparatively faster way.

§4.2. Γ: determining the Schottky group Γ

To find Γ, it suffices to compute

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Answer: Of course not. But, thanks to Kadziela's approximation theorem, we can simultaneously approximate both S and F such that

$$F(S) = R.$$

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We may assume that $S = \{0, b_0, a_1, b_1, \ldots, a_{g-1}, b_{g-1}, 1, \infty\}$. Then the parameters $a = 0$ and $b = 1$ are suitable.

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For $z \in S \setminus \{0, 1, \infty\}$, we have

- *$F(z) \equiv 0 \mod \pi$,*
- $$F(z) \equiv \begin{cases} -4b_0 \prod_{i=1}^{g-1} \left(1 - \left(\frac{a_i - b_i}{a_i + b_i}\right)^2\right) \mod \pi^2 & \text{if } z = b_0, \\ -2z \prod_{i=1}^{g-1} \left(1 + \frac{(a_i - b_i)^2}{(a_i + b_i)(2z - a_i - b_i)}\right) \mod \pi^2 & \text{otherwise,} \end{cases}$$

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- *$F(z) \mod \pi^t = \prod_{m=0}^{t-2} F_m(z \mod \pi^t)$ for $t \geq 3$,*

where π is a uniformizer in K.

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Recall that R consists the roots of the defining polynomial of X. We may assume that

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In other words, we guess the elements in S digit by digit using the approximation theorem.

§4.2. Γ: determining the Schottky group Γ

Recall that R consists the roots of the defining polynomial of X. We may assume that

$$R = \{0, r_0, r_1, \ldots, r_{2g-2}, 1, \infty\}.$$

If we know correctly the elements z in $S \setminus \{0, 1, \infty\}$ mod π^t, and use them to approximate

- the elliptic matrices s_i,
- the group Γ',
- the theta function $F(z)$,

then the images $F(z)$ will also correctly approximate the roots points in R mod π^t.

In other words, we guess the elements in S digit by digit using the approximation theorem. This algorithm is a brute force algorithm but works quite well when g and p are small.

§4.2. Ω: lifting points from the curve to Ω

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$$\begin{array}{ccc} \Omega & & \\ \downarrow & \searrow & \\ \Omega/\Gamma & \rightarrow & \Omega/\Gamma' \\ \downarrow & & \downarrow \\ X & \rightarrow & \mathbb{P}^1 \end{array}$$

where the isomorphism $\Omega/\Gamma' \simeq \mathbb{P}^1$ is induced by $F = F_{a,b} : \Omega \rightarrow \mathbb{P}^1$ for parameters $a, b \in \Omega$.

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Question: But... How do we distinguish?

§4.2. Ω: lifting points from the curve to Ω

Theorem (K.–Masdeu–Müller–van der Put)

Set $\gamma := \gamma_1 \cdots \gamma_g$, *and*

$$H(z) := \Theta(a, \gamma(a); z) \cdot \prod_{i=0}^{g} \Theta(a_i, b; z) \cdot \Theta(b_i, s_0(b); z), \quad z \in \Omega.$$

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- *The curve* $X = \Omega/\Gamma$ *is parametrized by* $z \in \Omega \mapsto (F(z), H(z))$.

Corollary: If $H(z) = y$, then z is a lift of P. Else $s_0(z)$ is a lift of P.

§5. Numerical example

Consider the hyperelliptic curve $C/\mathbb{Q}$ given by

$$y^2 = x^5 - 326x^4 + 1052 \cdot 5^2 x^3 - 5914 \cdot 5^2 x^2 + 39 \cdot 5^5 x.$$

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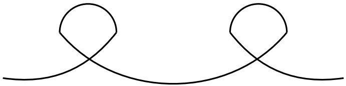
The prime $p = 5$ is a prime of bad reduction. Moreover, the corresponding (stable) reduction is a projective line with two ordinary double points:



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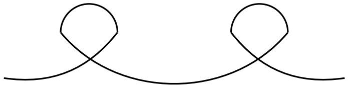

Set $D = (x) - (y)$ and $E = (z) - (w)$, where

$$x = (7,\ 1+3\cdot5+4\cdot5^2+5^5+5^6+O(5^7)), \qquad y = (12,\ 1+2\cdot5+3\cdot5^2+5^5+4\cdot5^6+O(5^7)),$$

$$z = (-3,\ 1+2\cdot5^2+4\cdot5^4+2\cdot5^5+5^6+O(5^7)), \quad w = (-18,\ 1+3\cdot5+2\cdot5^3+5^4+5^5+2\cdot5^6+O(5^7)).$$

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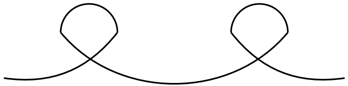

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Goal

Compute the local height $(D, E)_p$.

§5. Numerical example

We have:

$$a_0 = 0, \qquad b_0 = 3 \cdot 5^3 + 3 \cdot 5^4 + 3 \cdot 5^5 + 3 \cdot 5^6 + O(5^7),$$
$$a_2 = 1, \qquad a_1 = 5 + 2 \cdot 5^2 + 2 \cdot 5^3 + 3 \cdot 5^4 + 5^6 + O(5^7),$$
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The function $(\cdot,\cdot)_p$ is symmetric, and we have

$$(E,D)_5 = 3\cdot 5 + 2\cdot 5^2 + 4\cdot 5^3 + 2\cdot 5^5 + O(5^6). \quad :-)$$

From Marc's talk...

Danke Schön!

- *Basic Notions of Rigid Analytic Geometry* - Schneider
- *Non-archimedean Uniformization and Monodromy Pairing* - Papikian
- *Schottky Groups and Mumford Curves* - Gerritzen–van der Put
- *Rigid Geometry of Curves and Their Jacobians* - Lütkebbohmert

- *p-adic Height Pairings I* - Schneider
- *Local Heights on Mumford Curves* - Werner

- *Algorithms for Mumford Curves* - Morrison–Ren
- *Rigid Analytic Uniformization of Hyperelliptic Curves* - Kadziela
- *Algorithms for Heights On Mumford Curves* (to be modified) - K.–Masdeu–Müller–van der Put