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LINEAR AND QUADRATIC CHABAUTY
FOR AFFINE HYPERBOLIC CURVES

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$X/\mathbb{Q}$ smooth proj. curve of genus $g \geq 2$

p prime of good red., $U^{ét} \twoheadrightarrow U$ $G_{\mathbb{Q}}$-equiv. quotient

Chabauty – Kim diagram

$$\begin{tikzcd} X(\mathbb{Q}) \arrow[r] \arrow[d, "j"] & X(\mathbb{Q}_p) \arrow[d, "j_p"] \\ \mathrm{Sel}_U(X) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U) \end{tikzcd}$$

Chabauty – Kim locus

$$X(\mathbb{Q}_p)_U := \{ x \in X(\mathbb{Q}_p) : j_p(x) \in \mathrm{loc}_p(\mathrm{Sel}_U(X)) \}$$

$$X(\mathbb{Q}) \subseteq X(\mathbb{Q}_p)_U \subseteq X(\mathbb{Q}_p)$$

<u>Conj</u>: $\# X(\mathbb{Q}_p)_U < \infty$ for U sufficiently large

- Linear Chabauty: $U = U_1^{ét} = (U^{ét})^{ab} \rightsquigarrow X(\mathbb{Q}_p)_1$

<u>Thm</u>: Let $r_p := \mathrm{rk}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(\mathrm{Jac}_X)$ ($= r$ under Tate–Shafarevich conj.).

If $g - r_p > 0$, then $\# X(\mathbb{Q}_p)_1 < \infty$.

- Quadratic Chabauty: $U = U_{QC}$ = certain intermediate quotient $U_2^{ét} \twoheadrightarrow U_{QC} \rightarrow U_1^{ét}$

$\rightsquigarrow X(\mathbb{Q}_p)_{QC}$

<u>Thm</u>: (Balakrishnan – Dogra)

Let $\rho_f := \mathrm{rk}\, NS(\mathrm{Jac}_X) + \mathrm{rk}\, NS(\mathrm{Jac}_{X_{\bar{\mathbb{Q}}}})^{\sigma = -1}$, σ = complex conj.

If $g + \rho_f - 1 - r_p > 0$, then $\# X(\mathbb{Q}_p)_{QC} < \infty$.

Aim: generalise this to <u>affine</u> hyperbolic curves

Setup: $Y/\mathbb{Q}$ affine hyperbolic curve, $Y = X \setminus D$ with X projective, $n := \#D(\overline{\mathbb{Q}})$

Ex: • $Y = \mathbb{P}^1 \setminus \{0,1,\infty\}$,

• $Y = E \setminus \{0\}$ with E elliptic curve,

• Y: $y^2 = f(x)$ affine hyperell. curve

S finite set of primes

$\mathbb{Z}_S = \{ x \in \mathbb{Q} : v_\ell(x) \geq 0 \ \forall \ell \in S \}$ ring of S-integers

$\mathcal{Y} = \mathcal{X} \setminus \mathcal{D}$ regular model of $Y = X \setminus D$ over $\mathbb{Z}_S$

Thm: (Siegel, Faltings)

$\# \mathcal{Y}(\mathbb{Z}_S) < \infty.$

Variant of Chabauty–Kim for S-integral points:

$p \notin S$ s.t. $\mathcal{X}_{\mathbb{F}_p}$ and $\mathcal{D}_{\mathbb{F}_p}$ smooth, choose base point in $\mathcal{Y}(\mathbb{Z}_S)$ for $U^{\text{ét}}$,

$U^{\text{ét}} \twoheadrightarrow U$ $G_\mathbb{Q}$-equiv. quotient.

Chabauty–Kim diagram:

$$\begin{tikzcd} \mathcal{Y}(\mathbb{Z}_S) \arrow[r, hook] \arrow[d, "j"] & \mathcal{Y}(\mathbb{Z}_p) \arrow[d, "j_p"] \\ \mathrm{Sel}_{S,U}(\mathcal{Y}) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U) \end{tikzcd}$$

Chabauty–Kim locus:

$$\mathcal{Y}(\mathbb{Z}_p)_{S,U} := \{ y \in \mathcal{Y}(\mathbb{Z}_p) : j_p(y) \in \mathrm{loc}_p(\mathrm{Sel}_{S,U}(\mathcal{Y})) \}$$

• Linear Chabauty: $U = U_1^{\text{ét}} \rightsquigarrow \mathcal{Y}(\mathbb{Z}_p)_{S,1}$

• Quadratic Chabauty: construct suitable intermediate quotient

$U_2^{\text{ét}} \twoheadrightarrow U_{QC} \rightarrow U_1^{\text{ét}} \rightsquigarrow \mathcal{Y}(\mathbb{Z}_p)_{S,QC}$

Invariants attached to (Y, S, p):

- $r_p = \mathrm{rk}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(\mathrm{Jac}_X)$ p^∞-Selmer rank
- g = genus of X
- $\rho_f = \mathrm{rk}\, NS(\mathrm{Jac}_X) + \mathrm{rk}\, NS(\mathrm{Jac}_{X_{\bar{\mathbb{Q}}}})^{\sigma=-1}$
- $n = \# D(\bar{\mathbb{Q}})$ number of geom. pts at infinity

  $= n_1 + 2n_2$ with $n_1 = \# D(\mathbb{R})$, $2n_2 = \#(D(\mathbb{C}) \setminus D(\mathbb{R}))$
- $b := \#|D| + n_2 - 1$ (≥ 0 since Y affine)
- $s := \# S$

Theorem A: (Leonhardt – L. – Müller)

(1) If $\alpha_1 := g - r_p - b - s > 0$, then $\# \mathcal{Y}(\mathbb{Z}_p)_{S,1} < \infty$.

(2) If $\alpha_2 := \alpha_1 + \rho_f > 0$, then $\# \mathcal{Y}(\mathbb{Z}_p)_{S,QC} < \infty$.

Theorem B: (LLM)

(1) If $\beta_1 := \frac{1}{2} g(g+3) - \frac{1}{2} r(r+3) + b - s > 0$, then

$$\# \mathcal{Y}(\mathbb{Z}_p)_{S,1} \leq \kappa_p \cdot \prod_{\ell \in S} (n_\ell + n) \cdot \prod_{\ell \notin S} n_\ell \cdot \# \mathcal{Y}(\mathbb{F}_p) \cdot (4g + 2n - 2)^2 (g+1).$$

(2) If $\beta_2 := \beta_1 + \rho_f > 0$, same bound for $\mathcal{Y}(\mathbb{Z}_p)_{S,QC}$.

Here: $\kappa_p := 1 + \frac{p-1}{(p-2)\log(p)}$ (p odd), $\kappa_2 := 2 + \frac{2}{\log(2)}$.

n_ℓ := # irred. cpts of the mod-ℓ special fibre (of $\mathcal{X}_{\mathbb{F}_\ell}$ (if $\ell \notin S$) resp. of the minimal regular normal crossings model of (X, D) over $\mathbb{Z}_\ell$ (if $\ell \in S$))

Rmks:
- Have similar results for $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$: Set

$$h_{BK} := \dim_{\mathbb{Q}_p} H^1_f(G_{\mathbb{Q}}, \mathrm{Hom}(\Lambda^2 V_p \mathrm{Jac}_X, \mathbb{Q}_p(1)))$$

(= 0 conjecturally by Bloch – Kato)

C. If $g^2 - r_p + \rho + b - s > 0$, then $\# \mathcal{Y}(\mathbb{Z}_p)_{S,2} < \infty$

D. If $\frac{1}{2} g(3g+1) - \frac{1}{2} r_p (r_p + 3) + \rho + b - s - h_{BK} > 0$,

then $\mathcal{Y}(\mathbb{Z}_p)_{S,2}$ satisfies bound from Thm B.

- "Balakrishnan-Dogra trick" $\rightsquigarrow$ CK loci $\mathcal{Y}(\mathbb{Z}_S)^{BD}_{S,U}$ where Theorems hold with r instead of r_p
- Betts-Corwin-Leonhardt: Bound on $\mathcal{Y}(\mathbb{Z}_S)_\infty$ assuming Tate-Shafarevich + Bloch-Kato.

1. Refined Selmer schemes

For all ℓ have diagram

$$\begin{array}{ccc} \mathcal{Y}(\mathbb{Z}_S) & \longrightarrow & Y(\mathbb{Q}_\ell) \\ \downarrow j & & \downarrow j_\ell \\ H^1(G_{\mathbb{Q}}, U) & \xrightarrow{\mathrm{loc}_\ell} & H^1(G_\ell, U) \end{array}$$

can replace $Y(\mathbb{Q}_\ell)$ with $\mathcal{Y}(\mathbb{Z}_\ell)$ for $\ell \notin S$

Selmer functor: $R \mapsto \left\{ \alpha \in H^1(G_{\mathbb{Q}}, U) : \mathrm{loc}_\ell(\alpha) \in \begin{cases} j_\ell(\mathcal{Y}(\mathbb{Z}_\ell))^{\mathrm{Zar}}, & \ell \notin S, \\ j_\ell(Y(\mathbb{Q}_\ell))^{\mathrm{Zar}}, & \ell \in S. \end{cases} \right\}$

This is representable by the (refined) Selmer scheme $\mathrm{Sel}_{S,U}(\mathcal{Y})$.

$\rightsquigarrow$ CK diagram

Say $x, y \in \mathcal{Y}(\mathbb{Z}_S)$ have same <u>reduction type</u> if

$\forall \ell$: mod-ℓ reductions on the same irred. cpt. <u>or</u> (if $\ell \in S$) the same cuspidal point.

reduction type: $\Sigma = (\Sigma_\ell)_{\ell \text{ prime}}$

Selmer scheme is union $\mathrm{Sel}_{S,U}(\mathcal{Y}) = \bigcup_{\Sigma} \mathrm{Sel}_{\Sigma,U}(\mathcal{Y})$,

corresponding to $\mathcal{Y}(\mathbb{Z}_S) = \coprod_{\Sigma} \mathcal{Y}(\mathbb{Z}_S)_\Sigma$ $\leftarrow$ points of red. type Σ

$$\rightsquigarrow \quad \mathcal{Y}(\mathbb{Z}_p)_{S,U} = \bigcup_{\Sigma} \mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$$

Σ-refined CK diagram

$$\begin{tikzcd} \mathcal{Y}(\mathbb{Z}_S)_\Sigma \arrow[r, hook] \arrow[d, "j"] & \mathcal{Y}(\mathbb{Z}_p) \arrow[d, "j_p"] \\ \mathrm{Sel}_{\Sigma,U}(\mathcal{Y}) \arrow[r, "\mathrm{loc}_p"] & H^1_f(G_p, U) \end{tikzcd}$$

Strategy for finiteness of CK loci:

<u>If</u> $\dim \mathrm{Sel}_{\Sigma,U}(\mathcal{Y}) < \dim H^1_f(G_p, U)$

$\Rightarrow$ loc_p not dominant

$\Rightarrow$ $\exists\, 0 \neq f$ s.t. $f \circ \mathrm{loc}_p = 0$

$\Rightarrow$ $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma} \subseteq V(f \circ j_p)$ finite

Can compute dimensions using (abelian) Galois cohomology.

2. Weight filtrations on Selmer schemes

Betts: filtration $\ldots \subseteq W_{-2}U \subseteq W_{-1}U = U$ by subgroup schemes

s.t. $[W_{-i}U, W_{-j}U] \subseteq W_{-(i+j)}U \quad \forall\, i,j \geq 1$.

$\Rightarrow \mathrm{gr}^W_{-k} U := W_{-k}U / W_{-k-1}U$ rep'n of $G_\mathbb{Q}$ on fin. dim. $\mathbb{Q}_p$-v.s.

$$\mathrm{Sel}_{\Sigma,U} \hookrightarrow \prod_{k=1}^{\infty} \underbrace{H^1_f(G_\mathbb{Q}, \mathrm{gr}^W_{-k} U)}_{\substack{\text{crystalline at } p, \\ \text{unram. away from } p}} \times \prod_{\ell \in S} \underbrace{\mathfrak{S}_{\Sigma_\ell}}_{\dim \leq 1} \qquad \text{(non-canonically)}$$

$$\Rightarrow \dim \mathrm{Sel}_{\Sigma,U} \leq s + \sum_{k=1}^{\infty} \dim_{\mathbb{Q}_p} H^1_f(G_\mathbb{Q}, \mathrm{gr}^W_{-k} U),$$

$$H^1_f(G_p, U) \cong \prod_{k=1}^{\infty} H^1_f(G_p, \mathrm{gr}^W_{-k}) \qquad \text{(non-canonically)}$$

$$\Rightarrow \dim H^1_f(G_p, U) = \sum_{k=1}^{\infty} \dim_{\mathbb{Q}_p} H^1_f(G_p, \mathrm{gr}^W_{-k})$$

- Linear Chabauty: $U = U_1^{\text{ét}} = U_Y^{ab}$

$Y \hookrightarrow X$ induces $U_Y^{ab} \twoheadrightarrow U_X^{ab} = V_p \mathrm{Jac}_X := \left(\varprojlim_n \mathrm{Jac}_X(\bar{\mathbb{Q}})[p^n] \right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$

$$1 \longrightarrow \underbrace{\mathbb{Q}_p(1)^{D(\bar{\mathbb{Q}})} / \mathbb{Q}_p(1)}_{gr_{-2}^W =: I} \longrightarrow U_Y^{ab} \longrightarrow \underbrace{V_p \mathrm{Jac}_X}_{gr_{-1}^W} \longrightarrow 1$$

Can compute local and global Galois coh:

wt -1: $\dim H^1_f(G_{\mathbb{Q}}, V_p \mathrm{Jac}_X) = r_p$

$\dim H^1_f(G_p, V_p \mathrm{Jac}_X) = g$

wt -2: $\dim H^1_f(G_{\mathbb{Q}}, I) = n_1 + n_2 - \#|D|$

$\dim H^1_f(G_p, I) = n - 1$

If $\alpha_1 > 0$ in Thm A

$\Rightarrow \dim \mathrm{Sel}_{\Sigma, U} < \dim H^1_f(G_p, U)$

$\Rightarrow \# \mathcal{Y}(\mathbb{Z}_p)_{S,1} < \infty$

- Quadratic Chabauty: Construction of U_{QC}

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Lambda^2 V_p \mathrm{Jac}_X \oplus I & \longrightarrow & U_Y / W_{-3} U_Y & \longrightarrow & V_p \mathrm{Jac}_X & \longrightarrow & 1 \\ & & \Big\downarrow \scriptstyle{\text{max. Artin-Tate quotient}} & & \Big\downarrow & & \Big\| & & \\ 1 & \longrightarrow & \left(\mathbb{Q}_p \otimes NS(\mathrm{Jac}_{X_{\bar{\mathbb{Q}}}})^\vee(1) \right) \oplus I & \longrightarrow & U_{QC} & \longrightarrow & V_p \mathrm{Jac}_X & \longrightarrow & 1 \end{array}$$

can compute its Galois coh.

3. Bounding Ck loci

Weight filtration on $\mathcal{O}(\mathrm{Sel}_{\Sigma, U})$ and $\mathcal{O}(H^1_f(G_p, U))$ by fin. dim. subspaces, preserved by $\mathrm{loc}_p^\#: \mathrm{Sel}_{\Sigma, U} \to H^1_f$

<u>Assume</u> $\dim W_m \mathcal{O}(\mathrm{Sel}_{\Sigma, U}) < \dim W_m \mathcal{O}(H^1_f)$

$\Rightarrow \exists\, 0 \neq f \in W_m \mathcal{O}(H^1_f)$ s.t. $\mathrm{loc}_p^\# f = 0$

$\Rightarrow f \circ j_p : \mathcal{Y}(\mathbb{Z}_p) \longrightarrow \mathbb{Q}_p$ is "Coleman algebraic function of weight $\leq m$"

Betts: bound on number of zeroes in each residue disc

$\Rightarrow$ get bound on $\mathcal{Y}(\mathbb{Z}_p)_{S,U,\Sigma}$ and $\mathcal{Y}(\mathbb{Z}_p)_{S,U}$

In our theorems we get weight 2 functions

$\rightsquigarrow$ sum of double and single Coleman integrals and rat. functions

Compute $\dim W_m \mathcal{O}(Z)$ via Hilbert series.

$$HS_Z(t) := \sum_{i=0}^{\infty} \dim gr_i^W \mathcal{O}(Z)\, t^i \in \mathbb{N}_0 [\![t]\!]$$

Betts: $$HS_{Sel_{\Sigma,U}}(t) \preccurlyeq (1-t^2)^{-s} \prod_{k=1}^{\infty} (1-t^k)^{-\dim H^1_f(G_{\mathbb{Q}},\, gr^W_{-k} U)} =: HS_{glob}(t)$$

$$HS_{H^1_f}(t) = \prod_{k=1}^{\infty} (1-t^k)^{-\dim H^1_f(G_p,\, gr^W_{-k} U)} =: HS_{loc}(t)$$

For $U = U_1^{\text{ét}}$:

$$HS_{glob}(t) = 1 + r_p t + (s + n_1 + n_2 - \#|D| + \tfrac{1}{2} r_p (r_p + 1))\, t^2 + \dots$$

$$HS_{loc}(t) = 1 + gt + (n - 1 + \tfrac{1}{2} g(g+1))\, t^2 + \dots$$